< Situation >

Assume $X_1, \cdots, X_n \sim N(\mu, 1)$. If only the values of (1) $Z_i = \lfloor X_i \rfloor$, $i = 1, \cdots, n$ are available (only the integer part), (2) $Z_i = \frac{\lfloor 10 X_i \rfloor}{10}$, $i = 1, \cdots, n$ are available (only up to the first decimal point). how the MLEs are calculated? Construct the code to compare those estimators's variance with Cramer-Rao bound when whole X_i values are available.

Sol) For $Z_i = \lfloor X_i \rfloor$, the distribution of Z_i becomes discrete. The probability mass function of Z_i is calculated as follows.

$$P(Z_i = z) = P(\lfloor X_i \rfloor = z) = P(X_i < z+1) - P(X_i < z) = \Phi(z+1-\mu) - \Phi(z-\mu) \quad \text{for } z \in \mathbb{Z}$$
$$= 0 \quad \text{for } z \notin \mathbb{Z}.$$

$$L(\mu; z_i) = \prod_{i=1}^{n} P(Z_i = z_i)$$
$$\ell(\mu) \equiv \log L(\mu; z_i) = \sum_{i=1}^{n} \log\left(\Phi(z_i+1-\mu) - \Phi(z_i-\mu)\right)$$

$$\frac{\partial \ell}{\partial \mu} = \sum_{i=1}^{n} \frac{-\phi(z_i+1-\mu) + \phi(z_i-\mu)}{\Phi(z_i+1-\mu) - \Phi(z_i-\mu)}.$$ To find the solution of $\frac{\partial \ell}{\partial \mu} = 0$ iteratively.

first. set $\mu_0 = \bar{Z} + 0.5$. (since $E(X_i) = \mu$ and $X_i = Z_i + \{X_i\}$, assume $E(\{X_i\})$ to be approximately 0.5.)

Next, using the iterative relation $\mu_k = \mu_{k-1} + \left(\sum_{i=1}^{n} \frac{-\phi(z_i+1-\mu_{k-1}) + \phi(z_i-\mu_{k-1})}{\Phi(z_i+1-\mu_{k-1}) - \Phi(z_i-\mu_{k-1})}\right) \cdot \Delta$

($k \geq 1$, Δ : step size), solve μ numerically until it converges.

($|\mu_k - \mu_{k-1}| <$ threshold; ex. threshold : 10^{-10})

Similarly, for the case (2), we can obtain μ with slightly modified relation :

$$\mu_k = \mu_{k-1} + \left(\sum_{i=1}^{n} \frac{-\phi(z_i + 0.1 - \mu_{k-1}) + \phi(z_i - \mu_{k-1})}{\Phi(z_i + 0.1 - \mu_{k-1}) - \Phi(z_i - \mu_{k-1})} \right) .\ \Delta$$

On the other hand, when intact information X_i $(i=1, \cdots, n)$ are provided. MLE (and UMVUE) of μ is given as $\overline{X}$ with the distribution $\overline{X} \sim N(\mu, \frac{1}{n})$. Cramer–Rao bound is calculated as follows. $\left(= \frac{1}{n}\right)$

$$\ulcorner\ X_i \sim N(\mu, 1) \longrightarrow f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2}\right) \qquad \log f_X(x) = -\frac{(x-\mu)^2}{2} - \log(\sqrt{2\pi})$$

$$E_\mu\left(-\frac{\partial^2 \log f}{\partial \mu^2}\right) = 1 \quad \therefore \quad I(\mu) = 1. \quad (n I(\mu))^{-1} = \frac{1}{n}. \lrcorner$$