5 Particle Behaviour Near a Capacitor

Imagine that you have a particle with charge $q > 0$ and mass m inside a sufficiently large parallel plate capacitor with a negatively charged plate at $x = 0$ and a positively charged plate at $x = d$.

(a) Justify using the following potential function and find α in terms of q and the electric field E.

$$V(x) = \begin{cases} \infty & \text{if } x \geq d \\ \alpha x & \text{if } 0 < x < d \\ \infty & \text{if } x \leq 0 \end{cases} \tag{12}$$

Hint: Recall that the electric force on a particle of charge q is $F = q\boldsymbol{E}$ and that the $\boldsymbol{E}$ field inside a sufficiently large capacitor is constant.

(b) Will this potential yield bound states, scattering states, or both? Explain.

(c) Using the time-independent Schrödinger equation, find a general expression for the wavefunction of a quantum system in this potential. *Hint: Write your answer in terms of "Airy functions" $Ai(z)$ and $Bi(z)$ which are the two linearly independent solutions of $y'' - zy = 0$. See Figure 1 for a diagram.*

$$\psi_n(x) = c_1 \mathrm{Ai}\left[\left(\frac{2m\alpha}{\hbar^2}\right)^{\frac{1}{3}}\left(x - \frac{E_n}{\alpha}\right)\right] + c_2 \mathrm{Bi}\left[\left(\frac{2m\alpha}{\hbar^2}\right)^{\frac{1}{3}}\left(x - \frac{E_n}{\alpha}\right)\right]$$

(d) Now, assume that the particle is close enough the capacitor at $x = 0$ for it to be a good approximation to ignore the far plate barrier (i.e. $x \ll d$), giving the simpler potential

$$V(x) = \begin{cases} \alpha x & \text{if } x > 0 \\ \infty & \text{if } x \leq 0 \end{cases} \tag{13}$$

Note that your previous answers will remain correct. Knowing that $\mathrm{Ai}(x \to \infty) = 0$ and $\mathrm{Bi}(x \to \infty) = \infty$, enforce the boundary condition that $\psi(x \to \infty) = 0$. Then, enforce $\psi_n(0) = 0$ to find an implicit expression for the energy levels E_n using the $x \to -\infty$ airy function approximations:

$$\mathrm{Ai}(-z) \approx \frac{1}{\sqrt{\pi}\ z^{1/4}}\ \sin\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right) \tag{14}$$

$$\mathrm{Bi}(-z) \approx \frac{1}{\sqrt{\pi}\ z^{1/4}}\ \cos\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right) \tag{15}$$

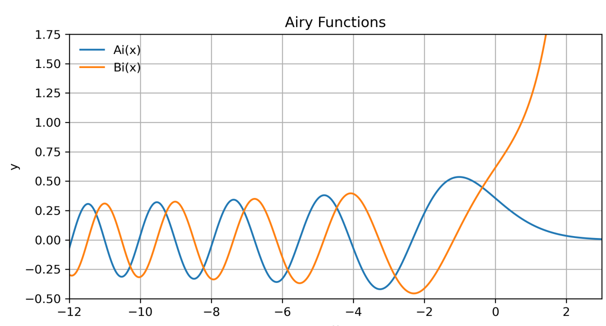

Figure 1: Airy functions of the first and second kind.