Ph 1b Recitation Notes
Section 7

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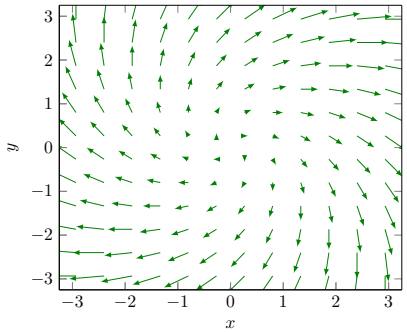
1 Vector Calculus

1.1 Vector Fields

Vector calculus is effectively the study of derivatives and integrals of multivariable functions instead of single variable functions that you are used to. The basic structure is the "vector field", a function $\mathbf{F}$ with multiple components (denoted $\mathrm{F_i}$) coupled to some vector basis. In physics, we usually use three component vector functions $\mathbf{F}(x, y, z)$ with some basis such as the Cartesian coordinate axes $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$).[1] For example, in Cartesian coordinates,

$$\mathbf{F}(x, y, z) = F_x\hat{\mathbf{i}} + F_y\hat{\mathbf{j}} + F_z\hat{\mathbf{k}} \tag{1}$$

These vector equations can be visualized by assigning to every point on the x-y plane a vector direction and magnitude corresponding to the vector field evaluated at that point. For example, we could represent $\mathbf{F} = (x+y)\hat{\mathbf{i}}+(y-x)\hat{\mathbf{j}}$ with:



[1] Note that a vector can almost be thought of as just the collection of three functions, each with dimension three. However, vectors (at least in physics) also have the additional property of having some level of coupling to the space that they are in. Namely, vectors depend on the basis you use and thus changes in a predictable way when changing coordinate systems (i.e. all vectors transform in the same way when changing your coordinate system from, say, Cartesian to spherical polar). This is also true of 4-vectors.

You can also express these vectors in different coordinate systems. Every coordinate system is defined by a set of variables (e.g. x, y, z) and a basis (e.g. $\{\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}\}$)[2]. To transform a vector to new set of coordinates, you simply need to define the relations between the variables and the basis and substitute. For example, spherical coordinates (r, θ, ϕ)[3] can be converted to and from Cartesian coordinates using

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \sin\theta\cos\phi\,\hat{\mathbf{r}} + \cos\theta\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\theta\sin\phi\,\hat{\mathbf{r}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \end{cases} \tag{2}$$

$$\begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \tan^{-1}(\sqrt{x^2+y^2}/z) \\ \phi = \tan^{-1}(y/x) \end{cases} \qquad \begin{cases} \hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\phi}} = -\sin\theta\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \end{cases} \tag{3}$$

And cylindrical coordinates (s, ϕ, z) can be converted to and from Cartesian coordinates with

$$\begin{cases} x = s\cos\phi \\ y = s\sin\phi \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{x}} = \cos\phi\,\hat{\mathbf{s}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{y}} = \sin\phi\,\hat{\mathbf{s}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases} \tag{4}$$

$$\begin{cases} s = \sqrt{x^2+y^2} \\ \phi = \tan^{-1}(y/x) \\ z = z \end{cases} \qquad \begin{cases} \hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} = -\sin\theta\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}} \\ \hat{\mathbf{z}} = \hat{\mathbf{z}} \end{cases} \tag{5}$$

Note that 2D polar coordinates relations can be obtained by using cylindrical coordinates and ignoring the z component[4]. There are a lot of useful formulae like this, for example, in the appendices of Purcell and Morin (2013) or at the back of Griffiths (2017) or Jackson (1998). These three coordinate systems (Cartesian, spherical and cylindrical) are the easiest to use since they all describe some common symmetry.[5]

[2]Note that sometimes $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ are written as $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$.

[3]Using the physics convention, where θ is the polar angle, ranging from 0 to π, and ϕ is the azimuthal angle, ranging from 0 to 2π.

[4]If you want to memorize these for some reason, you only need remember the spherical coordinate transformation and let $\theta \to \pi/2$ to get the cylindrical coordinate transformation. This makes $\hat{\mathbf{r}} \to \hat{\mathbf{s}}$ and leaves $\hat{\boldsymbol{\phi}}$ and $\hat{\mathbf{z}}$ unchanged.

[5]There are, of course, other convenient coordinates to use when you have other symmetries. It is common when you have some toroidal symmetry (e.g. when calculating the inductance of a toroid, or studying a torodial magnetic field around a supermassive black hole) to use toroidal coordinates.

1.2 Partial derivatives

In order to describe the change of multivariable functions, we need to define derivatives. If you want to find how a single variable function changes with its parameter, it is the total derivative. Namely, the derivative of $f(x)$ is

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \tag{6}$$

Scalar fields (multivariable functions that return a single value) have multiple variables that they depend on. Therefore, we will have many different partial derivatives (one for each parameter of the function) and so the definition of the derivative must be updated. The limit definition of the x partial derivative of a three variable scalar field $T(x, y, z)$ is

$$\frac{\partial T}{\partial x} = \lim_{h\to 0} \frac{T(x+h, y, z) - T(x, y, z)}{h} \tag{7}$$

And similarly $\frac{\partial T}{\partial y}$ and $\frac{\partial T}{\partial z}$ can be defined in this way[6]. This effectively means that you take the derivative of the function in the direction of the variable to which you took the derivative of (and thus ignore the other variables). You can generalize this to any possible direction $\mathbf{v}$ by scaling these sort of "basis derivatives" using the directional derivative

$$\nabla_{\mathbf{v}} T(x, y, z) = v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \tag{8}$$

This measures the change in the scalar field T with respect to the direction $\mathbf{v}$. This is useful for example if you want to find how much the temperature changes in a particular direction around a point[7].

1.3 Gradient, Divergence and Curl

Now, we move on to derivatives of vector fields. There are three kinds of derivatives in vector calculus: 1) One that takes a scalar field (function taking x, y, z and returning a single value) and returns a vector field, 2) One that takes a vector field and returns a scalar field, and 3) One that takes a vector field and returns a vector field. They are called the gradient, the divergence and the curl, respectively.

We define the **gradient** of a scalar field T in Cartesian coordinates as

$$\nabla T \equiv \frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{9}$$

[6] Sometimes the partial derivative notation is shortened to just $\partial_x T$

[7] Note that there are many other notations used for directional derivatives such as $D_{\mathbf{v}} T$ or $\partial_{\mathbf{v}} T$.

The gradient measures the direction of maximum change in T at a given point (effectively the maximum directional derivative at that point). Note that this changes depending on your coordinate system not only because the new coordinate values depend on the variables (so there will be chain rules), but also that the vector bases depend on the variables (see e.g. equation 3). In this way, scalar fields are different from vector fields, because vector fields have additional transformation under coordinate changes. In spherical coordinates, it is

$$\nabla T = \frac{\partial T}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} \tag{10}$$

And in cylindrical coordinates, it is

$$\nabla T = \frac{\partial T}{\partial s}\hat{\mathbf{s}} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\boldsymbol{\phi}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}} \tag{11}$$

It is convenient to define the nabla operator as

$$\nabla \equiv \frac{\partial}{\partial x}\hat{\mathbf{x}} + \frac{\partial}{\partial y}\hat{\mathbf{y}} + \frac{\partial}{\partial z}\hat{\mathbf{z}} \tag{12}$$

where whenever it is placed in front of a scalar function, it takes the corresponding derivatives. Then, we can easily define the **divergence** of a vector field $\mathbf{v}$, in Cartesian coordinates, as

$$\nabla \cdot \mathbf{v} \equiv \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \tag{13}$$

The divergence of a vector field measures the amount of sources or sinks in the field that there are. If the divergence is zero at a point, there are no sources or sinks at that point, if it is positive it is a source, and if it is negative it is a sink. In spherical coordinates, the divergence is instead

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} \tag{14}$$

And in cylindrical coordinates, it is

$$\nabla \cdot \mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \tag{15}$$

The **curl** of a vector field $\mathbf{v}$ is defined, in Cartesian coordinates, as

$$\nabla \times \mathbf{v} \equiv \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\mathbf{z}} \tag{16}$$

The curl can be interpreted as measuring the amount of swirling of the vector field $\mathbf{v}$ around a given point. The returned vector defines the normal to the motion of the swirl. In spherical coordinates, it is

$$\begin{aligned}\nabla \times \mathbf{v} = &\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right)\hat{\mathbf{r}} \\ &+ \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(rv_\phi)\right)\hat{\boldsymbol{\theta}} \\ &+ \frac{1}{r}\left(\frac{\partial}{\partial r}(rv_\theta) - \frac{\partial v_r}{\partial\theta}\right)\hat{\boldsymbol{\phi}}\end{aligned} \tag{17}$$

The cylindrical coordinate version is

$$\begin{aligned}\nabla \times \mathbf{v} = &\left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} \\ &+ \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} \\ &+ \frac{1}{s}\left(\frac{\partial}{\partial s}(sv_\phi) - \frac{\partial v_s}{\partial\phi}\right)\hat{\mathbf{z}}\end{aligned} \tag{18}$$

The product rules also transfer to vector calculus. The product rules in vector calculus, for scalar fields f, g, and vector fields $\mathbf{A}$, $\mathbf{B}$ are

$$\nabla(fg) = f(\nabla g) + g(\nabla f) \tag{19}$$
$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} \tag{20}$$
$$\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f) \tag{21}$$
$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{B}\cdot(\nabla\times\mathbf{A}) \tag{22}$$
$$\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f) \tag{23}$$
$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) \tag{24}$$

There are also second derivatives. It is useful to know that

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0 \tag{25}$$
$$\nabla\times(\nabla f) = 0 \tag{26}$$

And that there is a common second derivative operator called the Laplacian defined as $\nabla^2 \equiv \nabla\cdot\nabla$. in Cartesian coordinates, the Laplacian acting on a scalar function T is

$$\nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \tag{27}$$

In spherical coordinates, it is

$$\nabla^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} \tag{28}$$

In cylindrical coordinates, it is

$$\nabla^2 T = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial T}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 T}{\partial\phi^2} + \frac{\partial^2 T}{\partial z^2} \tag{29}$$

Now, we will move on to integrals of vector fields.

1.4 Line, Surface and Volume Integrals

Just as there are multiple types of derivatives, there are multiple types of integrals. These integrals occur over all points on a line (1D), surface (2D) or volume (3D). The line integral of a vector field $\mathbf{v}$ over a line with differential tangent vector $\mathrm{d}\mathbf{l}$ (tangent vector at each point along the curve) from point $\mathbf{A}$ to $\mathbf{B}$ is

$$\int_{\mathbf{A}}^{\mathbf{B}} \mathbf{v}\cdot\mathrm{d}\mathbf{l} \tag{30}$$

Line integrals that have closed paths (where $\mathbf{A}=\mathbf{B}$) are denoted with a circle around the integral sign ($\oint$). Work is an example of a line integral. It is the line integral over the path you move through a force (vector) field $\mathbf{F}$, namely of $\mathbf{F}\cdot\mathrm{d}\mathbf{l}$. You can calculate a line integral by parameterizing the line (i.e. finding relations between e.g. $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$ and some parameter $\mathrm{d}\lambda$) and then integrating using the corresponding line element (listed Cartesian, spherical and cylindrical below)

$$\mathrm{d}\mathbf{l} = \mathrm{d}x\,\hat{\mathbf{x}} + \mathrm{d}y\,\hat{\mathbf{y}} + \mathrm{d}z\,\hat{\mathbf{z}} \tag{31}$$

$$\mathrm{d}\mathbf{l} = \mathrm{d}r\,\hat{\mathbf{r}} + r\mathrm{d}\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\mathrm{d}\phi\,\hat{\boldsymbol{\phi}} \tag{32}$$

$$\mathrm{d}\mathbf{l} = \mathrm{d}s\,\hat{\mathbf{s}} + s\mathrm{d}\phi\,\hat{\boldsymbol{\phi}} + \mathrm{d}z\,\hat{\mathbf{z}} \tag{33}$$

There is a fundamental theorem for line integrals which relates it conceptually to gradients. Such fundamental theorems (there is one for each type of integral) allow you to reduce the number of integrals you perform if the object you are integrating has some special property. In this case, you can remove the integral if the vector field is a gradient of some scalar field (this is not always true). This theorem is called the gradient theorem where

$$\int_{\mathbf{A}}^{\mathbf{B}} (\nabla T)\cdot\mathrm{d}\mathbf{l} = T(\mathbf{B}) - T(\mathbf{A}) \tag{34}$$

Line integrals that are depend on the path are called path-dependent. If they are path independent, then the vector field that generates the line integral $\mathbf{v}$ is called conserved and can be written as the gradient of some scalar field $\mathbf{v} = \nabla T$. This implies that any closed integral will be zero by the fundamental theorem of calculus (or that any general integral will only depend on its end points).

Surface integrals are taken over a surface with differential area normal of $\mathrm{d}\mathbf{a}$. The area normal is simply the normal vector pointing out from a patch of the surface at a given point. The surface integral of a vector field $\mathbf{v}$ over a surface S with differential normal area vector $\mathrm{d}\mathbf{a}$ is

$$\int_{\mathbf{A}}^{\mathbf{B}} \mathbf{v} \cdot \mathrm{d}\mathbf{l} \tag{35}$$

To calculate a surface integral, you typically break the surface into parts where you know the explicit form of the differential area normal (you can deduce them intuition and using the differentials from equations 31, 32 and 33) and sum each integrated part. There is a fundamental theorem associated to surface integrals. It is called the curl theorem (also known as Stokes' theorem or Kelvin–Stokes theorem) and it relates surface integrals of the curl of a vector field $\mathbf{v}$ (over a surface S) to line integrals of $\mathbf{v}$ over their closed boundary (denoted ∂S). Namely,

$$\iint_S (\nabla \times \mathbf{v}) \cdot \mathrm{d}\mathbf{a} = \oint_{\partial S} \mathbf{v} \cdot \mathrm{d}\mathbf{l} \tag{36}$$

Volume integrals are taken over a volume with differential volume element $\mathrm{d}\tau$.[8] The volume integral of a scalar field T over a volume V with differential volume $\mathrm{d}\tau$ is

$$\iiint_V T \mathrm{d}\tau \tag{37}$$

You can calculate a volume integral using the differential volume elements obtained by finding the Jacobian (see any vector calculus textbook for how to do this). The common ones (Cartesian, spherical and cylindrical, respectively) are

$$\mathrm{d}\tau = \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \tag{38}$$

$$\mathrm{d}\tau = r^2 \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi \tag{39}$$

$$\mathrm{d}\tau = s\, \mathrm{d}s\, \mathrm{d}\phi\, \mathrm{d}z \tag{40}$$

[8] The use of τ is to prevent confusing with the electric potential V.

There is also a fundamental theorem associated to volume integrals called the Divergence theorem (also known as Gauss's theorem[9] or Ostrogradsky's theorem). It reduces volume integrals to (closed) surface integrals when integrating over the divergence of a vector field $\mathbf{v}$. Namely,

$$\iiint_V (\nabla \cdot \mathbf{v}) \mathrm{d}\tau = \oiint_{\partial V} \mathbf{v} \cdot \mathrm{d}\mathbf{a} \tag{41}$$

That is all the vector calculus we need. If you want examples on how to use vector calculus in detail, see the many examples in Schey (1996).

1.5 Aside: Dirac Delta Function

Consider the vector field $\mathbf{v} = \frac{\hat{\mathbf{r}}}{r^2}$ (in spherical coordinates). This vector field is commonly encountered since it describes a point source field with inverse-square strength drop off as you move away from the source. The divergence of the field is

$$\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial 1}{\partial r^2} \left(r^2 \frac{1}{r^2} \right) = 0$$

This seems to somehow suggest that there is no source. This implies that the integral of $\nabla \cdot \mathbf{v}$ over the volume of a sphere of radius R centered at the origin should be zero. To get a better look, let's find the corresponding surface integral at the boundary of the volume.

$$\oint \mathbf{v} \cdot \mathrm{d}\mathbf{a} = \int \left(\frac{1}{R^2} \hat{\mathbf{r}} \right) \cdot \left(R^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi \, \hat{\mathbf{r}} \right) = 4\pi$$

But we expected this to be zero by the divergence theorem. What is going wrong? The issue arises from the point of discontinuity at $r = 0$. By defining the vector field as $\mathbf{v} = \frac{1}{r^2}\hat{\mathbf{r}}$, we implicitly assumed that $r \neq 0$. Normally, this would not matter since a single point should not contribute much to an integral over a large region, however in this case, the only contribution to the integral *is* from the central point. Somehow, there is a sort of infinite divergence at $r = 0$ so that when summing it over the volume we should get 4π.

This is very annoying, since we have to be very careful when taking derivatives like this. The way to get around this is with Dirac delta functions $\delta(x)$. A Dirac delta function is infinite at $x = 0$ and zero everywhere else, but has the unique property that when integrating over you get a finite value. Specifically,

$$\delta(x) = \begin{cases} \infty, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases} \tag{42}$$

[9] Not to be confused with Gauss's law, or the million other things named after him.

but

$$\int_{-\infty}^{\infty} \delta(x)\mathrm{d}x = 1 \tag{43}$$

Of course, we can shift the place of infinity to $x = a$ by simply using $\delta(x - a)$. The definition also allows the Dirac delta to pick out a value of a function when integrated over. In summary,

$$\int_{-\infty}^{\infty} \delta(x - a)f(x)\mathrm{d}x = f(a) \tag{44}$$

This can be generalize to three dimensions with the three dimensional Dirac delta $\delta^3(\mathbf{x}) \equiv \delta(x)\delta(y)\delta(z)$, as

$$\int_{\mathbb{R}^3} \delta^3(\mathbf{x} - \mathbf{a})T(\mathbf{x})\mathrm{d}\tau = T(\mathbf{a}) \tag{45}$$

If we do everything again carefully using the three-dimensional Dirac delta function, we would find that the divergence of our original vector function is

$$\nabla \cdot \left(\frac{\hat{\boldsymbol{\imath}}}{\imath^2} \right) = 4\pi\delta^3(\boldsymbol{\imath}) \tag{46}$$

Note that the four pi comes from the integral over the volume in spherical coordinates.

2 Electrostatics

2.1 Coulomb's Law

When charges do not move or are sufficiently slow, we are in the electrostatic regime. In this regime, we can use the famous Coulomb's law. Namely, the force due to a charge q_1 acting on a charge q_2 is

$$\mathbf{F_2} = k\frac{q_1 q_2}{r_{21}^2}\hat{\mathbf{r}}_{21} \tag{47}$$

where $\mathbf{r}_{21} \equiv r_{21}\hat{\mathbf{r}}_{21}$ is the displacement vector pointing from charge q_1 to charge q_2.[10] The constant $k = 8.988 \times 10^9$ $\mathrm{Nm^2C^{-2}}$ is Coulomb's constant. Coulomb's constant can be written in terms of a fundamental constant of the universe, the permittivity of free space $\epsilon_0 \equiv \frac{1}{4\pi k} = 8.854 \times 10^{-12}$ $\mathrm{N^{-1}m^{-2}C^2}$.

[10] This is an experimentally proven theory (for the electrostatic regime). We will soon replace it with Maxwell's equations, a more complete theory.

We can calculate the force due to a collection of N charges $\{q_i\}$ on a charge q with the sum of all the forces on q. Namely,

$$\mathbf{F} = kq \sum_{i=1}^{N} \frac{q_i}{r_i^2} \hat{\mathbf{r}}_i \tag{48}$$

2.2 Electric Potential Energy

Since we defined a force, we can also define a potential energy. The potential energy of a system of charges U is obtained by simply finding the work it takes to bring each of the charges sequentially in from infinity (against the Coulomb force field produced by the other charges). Recall that the work required to move *against*[11] a force $\mathbf{F}$ from A to B is defined as

$$W \equiv -\int_A^B \mathbf{F} \cdot \mathrm{d}\mathbf{s} \tag{49}$$

This allows us to find that the potential energy stored in a system of static charges $\{q_i\}$ is

$$U = k \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j>i}}^{N} \frac{q_i q_j}{r_{ij}} \hat{\mathbf{r}}_{ij} \tag{50}$$

2.3 Electric Field

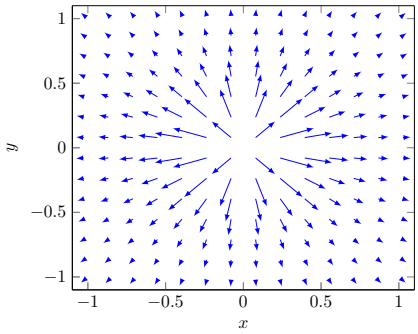
Going back to equation 48, you might notice that the the charge q was taken outside of the sum. In fact, with that, the complicated sum of many vectors is described only in terms of the other charges, evaluated at the position of charge q. This means that we can treat it as a vector field that is generated by the sources, and we simply need to evaluate it wherever we would like to place a "test" charge q. We call this object the electric field $\mathbf{E}$. In electrostatics, it is

$$\mathbf{E}(x, y, z) = k \sum_{i=1}^{N} \frac{q_i}{r_i^2} \hat{\mathbf{r}}_i \tag{51}$$

where $\mathbf{r} \equiv r_i \hat{\mathbf{r}}_i$ is the displacement from the charge q_i to the charge q. So, the force on a charge is proportional to the electric field $F \equiv q\mathbf{E}$. The electric field permeates all of space, encoding all the information of the sources in it. This allows us to therefore easily consider how small "test charges" move through space using the electric field using $\mathbf{F} = m\ddot{\mathbf{x}}$. We can therefore visualize an electric field as a vector field that contains some scaled version of the acceleration

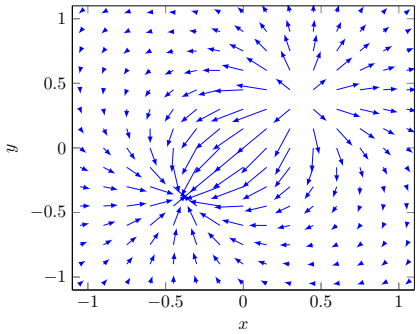
[11] To make it the work done by the force, just remove the minus sign.

that a small test charge would experience at any given location. The electric field of a single positive source charge centered at the origin looks like[12]



This can be interpreted as there being a repulsive force to all positive test charges (if it was a negative test charge it would be attractive because of the sign of $F \equiv q\mathbf{E}$). If you were to then put a positive charge near the origin, it would experience a large force repelling it. This would cause it to increase in velocity and then ultimately move further from the origin (from $\mathbf{F} = m\ddot{\mathbf{x}}$). However, after it moves away, it will experience a smaller force then it previously, causing it to accelerate less fast. This is due to the inverse-square scaling of the force. If the test charge was negative, then the source would act similar to gravity (note the similarities between equation 47 and Newton's law of gravity).

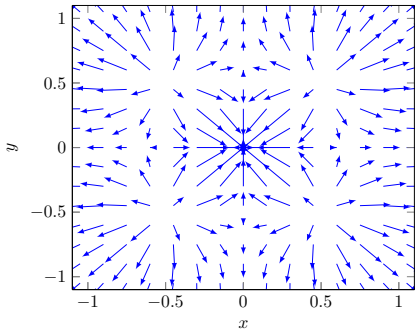
Now imagine that there is a positive and a negative source charge positioned some distance from each other. The electric field will then look like



This means that any positive test charge near the positive source charge will eventually find its way to the negative source charge. These electric field plots

[12]Note that I have actually smoothed the distribution a little bit to make it look nice. This just means that near the center, the vector magnitudes should be larger, so much so that they would lie off of the plot.

are fun to make[13]. The following is a plot of the electric field corresponding to four positive source charges around a negative source charge. Note that this can effectively prevent a test charge outside of the system from entering it naturally (without putting some work in from another force).



2.4 Charge Distributions

Imagine we have many (like $\sim 10^{23}$) charges. It will be difficult to use equation 51 as it stands, since we will have a very large sum. Instead, we will take the continuum limit of the series, turning it into an integral. Specifically, we can consider a vector field quantity that describes the charge density $\rho(x, y, z)$ at any given point along the charge distribution. Then, to obtain the total electric field of all components of the field, we simply integrate over the whole volume. Namely, equation 51 is replaced with

$$\mathbf{E}(x, y, z) = k \iiint \frac{\rho(x', y', z')}{\boldsymbol{\imath}^2} \hat{\boldsymbol{\imath}} \, \mathrm{d}x' \mathrm{d}y' \mathrm{d}z' \tag{52}$$

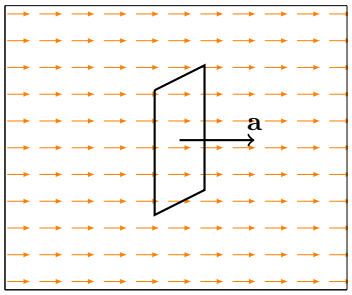
where $\boldsymbol{\imath} \equiv \imath \hat{\boldsymbol{\imath}}$ points from the location on the charge distribution (x', y', z') to the general electric field location (x, y, z). In other words, if $\mathbf{r'}$ is the displacement from the origin to a point on the source and $\mathbf{r}$ is the location of a point of the field, then $\boldsymbol{\imath} \equiv \mathbf{r} - \mathbf{r'}$. Note that by convention, we usually write $\rho(\mathbf{r})$ to mean the charge density of a volume charge distribution. If it is a surface charge distribution, we use $\sigma(\mathbf{r})$ and integrate over the surface instead, and if it is a line charge, then we use $\lambda(\mathbf{r})$ and integrate over the line. Thus, in principle we could find now the electric field of any charge distribution.

2.5 Flux

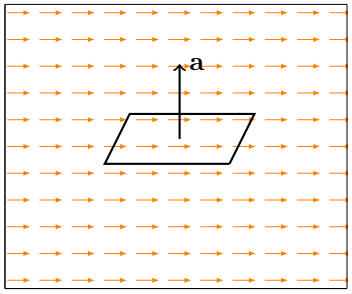
In physics, flux simply means the amount of something going through an given area over a given time. Imagine you have some uniform light source flowing in

[13] You can make them yourself too with code if you want!

some direction $\mathbf{v}$, and you place a light-collecting plane with normal $\mathbf{n}$ perpendicular to the incoming light. It will have a corresponding area-scaled normal of $\mathbf{a} \equiv a\mathbf{n}$, where a is the the area of the plane. It will look like



All of the light will be intercepted, so the number of photons that flow through this given area in a given time will be $\Phi = |\mathbf{v}||\mathbf{a}|$. We call this quantity the flux of $\mathbf{v}$ through the surface $\mathbf{a}$ and it is the rate of flow of photons through the surface. Now consider tilting the plane so that it is perfectly parallel to the light flow.



Then, there will be no flow of photons through $\mathbf{a}$ (they all go around it), so the flux is $\Phi = 0$. In fact, if you consider a plane tilted at some angle θ with respect to the flow v, then the portion of the plane that is perpendicular to the flow is $a\cos\theta$. Thus, the flux is $\Phi = |\mathbf{v}||\mathbf{a}|\cos\theta = \mathbf{v}\cdot\mathbf{a}$. We can construct this for any surface by simply integrating over the surface (treating each infinitesimal area piece as a plane), to get that the general flux of $\mathbf{v}(x, y, z)$ through the surface S is

$$\Phi = \iint_S \mathbf{v}\cdot \mathrm{d}\mathbf{a} \tag{53}$$

This can be applied to the electric field to get the electric flux through a surface S. Namely,

$$\Phi_E = \iint_S \mathbf{E}\cdot \mathrm{d}\mathbf{a} \tag{54}$$

This quantity roughly measures the number of electric field lines that pass through S.

2.6 Gauss's Law

Now we have all of the formalism needed to understand how to use Gauss's law (one of the Maxwell equations). Gauss's law relates the flux of the electric field through a closed surface to the enclosed charge. It is

$$\oiint_S \mathbf{E} \cdot \mathrm{d}\mathbf{a} = \frac{1}{\epsilon_0} Q_{\text{enc}} \tag{55}$$

It gives a way to find the source charges Q_{enc} enclosed in the closed surface S from the electric field $\mathbf{E}$.[14] This is the integral form, but you can take the derivative and get its vector form also. Since $Q_{\text{enc}} = \int \rho(\mathbf{r}')dV'$ (where you integrate over the volume of the enclosed surface[15]), then using the Divergence Theorem (Gauss's) gives

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho \tag{56}$$

Actually applying Gauss's law, is not immediately evident. Usually we have the enclosed charge or charge distribution and we want to find the corresponding electric field (the other way around is straight forward). The way to do this is using "Gauss surfaces", surfaces that preserve some symmetry of the system which the electric field should be perpendicular to. In spherical symmetry, use a concentric sphere. In cylindrical symmetry, use a coaxial cylinder and in plane symmetry, use a prism. When there is a symmetry, you know that $\mathbf{E}$ will be parallel to the normal of the surface, so $\mathbf{E} \cdot \mathrm{d}\mathbf{a} = |\mathbf{E}|\mathrm{d}a$. Since the magnitude of $|\mathbf{E}|$ is constant along the surface, the flux through a Gaussian surface is simply $\Phi_{E,\text{Gaussian surface}} = |\mathbf{E}|A$. We will see some examples of this in action.

2.7 Example: Uniform Spherical Charge

Suppose we have a uniform spherical charge density ρ, with radius R. Outside of the sphere, what is the electric? We expect it to be described by Coulomb's law. To solve this, choose a Gaussian surface outside of the sphere at a radius $r > R$. Then, the enclosed charge is the full charge $q \equiv \int \rho \mathrm{d}\tau$, so we can use Gauss's Law in the form

$$|\mathbf{E}|A = |\mathbf{E}|4\pi r^2 = \frac{q}{\epsilon_0} \implies |\mathbf{E}| = k\frac{q}{r^2}$$

Since the Gaussian surface normal points in the same direction as $\mathbf{E}$ (radially), then $\mathbf{E} = k\frac{q}{r^2}\hat{\mathbf{r}}$, which is indeed Coulumb's law.

[14] Note that often people drop the multiple integral signs, since it is understood that we are integrating over a surface from the differential element. However, it is not as common to drop the closed symbol (the circle through the integral symbol), since it carries extra (important) meaning.

[15] I.e. $S = \partial V$.

What about inside the sphere? If the charge is uniformly distributed inside the sphere, then the amount of charge in a spherical Gaussian surface with radius $r < R$ is $Q_{enc} = \int_{GS} \rho \mathrm{d}\tau = q(4\pi r^3/3)/(4\pi R^3/3) = q(r/R)^3$. So, inside the sphere, the electric field is $\mathbf{E} = k\frac{qr}{R^3}\hat{\mathbf{r}}$. Thus, the electric field grows as you travel outwards from the center of the sphere.

2.8 Example: Coaxial Cable

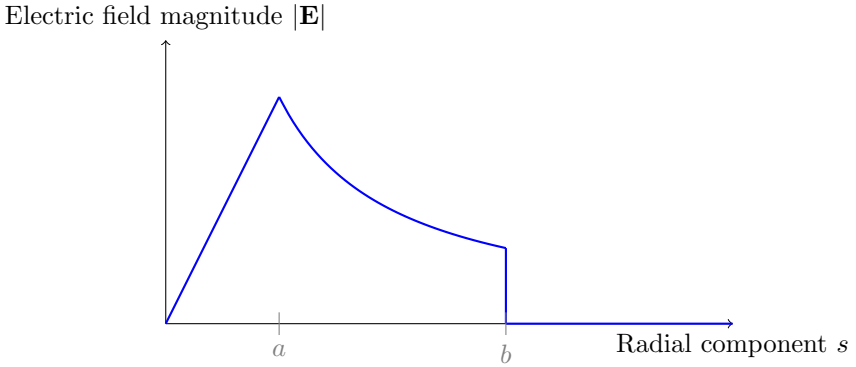
This is problem 2.16 from Griffiths (2017). Consider a long coaxial cable (cylinder) with a uniform volume charge density ρ and radius a and an outer cylindrical shell of radius $b > a$ with uniform surface charge density σ. The inner (filled) cylinder has total charge $Q > 0$ and the outer cylinder has total charge $-Q < 0$. Find the electric field in the regions $s < a$ (inside the inner cylinder), $a < s < b$ (in between the cylinders) and $s > b$ (outside the cylinders).

We start by noticing that we have cylindrical symmetry[16], so we can choose a cylindrical Gaussian surface with radius s and height h. Inside this Gaussian surface (when $s < a$), the enclosed charge is $Q_{enc} = \rho(\pi s^2 h)$ since the density is uniform. We therefore have, by Gauss' law

$$|\mathbf{E}|A = |\mathbf{E}|2\pi sh = \frac{q}{\epsilon_0}\rho\pi s^2 h \implies |\mathbf{E}| = \frac{\rho s}{2\epsilon_0}$$

So, the electric charge inside the cylinder ($r < a$) is $\mathbf{E} = \frac{\rho s}{2\epsilon_0}\hat{\mathbf{s}}$. In between the cylinders, the enclosed charge is simply the charge of the inner cylinder $Q \equiv \pi\rho R^2 h$. So, in between the cylinders ($a < r < b$), the electric field is $\mathbf{E} = \frac{\rho a^2}{2\epsilon_0 s}\hat{\mathbf{s}}$. Finally, outside of the cylinder ($r > b$), there is no enclosed charge since the charges of the two cylinders cancel, and you get that $\mathbf{E} = \mathbf{0}$.

If we plot the electric field magnitude as a function of the radial component s (for arbitrary $a = 1$, $b = 3$, $\rho = 1$, $\epsilon_0 = 1$), we get



[16] It is a long cylinder, so there are no edge effects.

It appears that the electric field is continuous across the boundaries of the inner cylinder, but not the outer cylinder.

2.9 Electric Potential

In electrostatics, you might notice that $\nabla \times \mathbf{E} = 0$ always[17]. This means that the $\mathbf{E}$ field can be expressed as the gradient of some function $-V$ ($\mathbf{E} \equiv -\nabla V$)[18]. This definition of the electric potential is equivalent (through the gradient theorem) to

$$V(\mathbf{r}) \equiv -\int_{\mathbf{O}}^{\mathbf{r}} \mathbf{E} \cdot \mathrm{d}\mathbf{l} \tag{57}$$

where $\mathbf{O}$ is some predefined origin point that determines the gauge (you can always arbitrarily add a scalar to the potential). Typically, we try to make this reference point lie at infinity, but this only works if there is a vanishing electric field at infinity[19]. Just like the electric field, the potential obeys the superposition principle (namely, $\mathbf{E}_{\text{tot}} = \sum_i \mathbf{E}_i$ and $\mathbf{V}_{\text{tot}} = \sum_i \mathbf{V}_i$).

The potential V is useful, because it is easy to get the electric field once we have it. We need, therefore, a formula that we can use to find V. Using Gauss' law, we can derive an equation for the potential

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} \tag{58}$$

This is called Poisson's equation. In regions without charge, it reduces to Laplace's equation $\nabla^2 V = 0$. We can get a general form for V in electrostatics through Coulumb's law to be

$$V(\mathbf{r}) \equiv \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\mathbf{r}')}{\mathscr{r}} \mathrm{d}\tau' \tag{59}$$

This looks similar to equation 52 for the electric field, except now it is a scalar field, so the annoying vectors are gone, making the calculation much easier. To apply this formula, you simply have to find a formula for the distance from any point on the charge distribution to any point in $\mathbb{R}^3$ (namely, $\mathscr{r}$) and integrate. Then, to find the electric field, we simply take the negative gradient.

[17] This arises from another one of the Maxwell equations $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$. But in the electrostatic regime, there are no changing magnetic fields.

[18] Note that the negative sign is just a convention, but we could have defined it to be ∇V.

[19] This therefore does not work for an infinite plane.

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