LEBESGUE INTEGRAL THEORY

Measure Theory, Integration, and Hilbert spaces

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Chapter 1

Measure Theory

The goal of this course, especially in this chapter, is to provide a notion of "size" for subsets of $\mathbb{R}^d$ by constructing an appropriate function

$$m : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{\geq 0} \cup \{\infty\}$$

satisfying the following properties:

(1) $m(\emptyset) = 0$

(2) $m([b, a]) = b - a$

(3) If $E = \bigsqcup_{n=1}^{\infty} E_n$, then $m(E) = \sum_{n=1}^{\infty} m(E_n)$.

(4) $m(E + h) = m(E)$ for all $h \in \mathbb{R}$

The third and fourth properties are called **countable additivity** and **translation invariance**, respectively. By regarding m as a function that measures the size of sets, it seems very natural that m should satisfy the above properties. However, this is just our intuition! As a math student, the natural follow-up question is as follows:

Does m exist? If it does, is m unique?

In fact, such a function m does not exist. Then, what can we do next? One way to resolve this problem is to restrict the domain of m. By removing certain pathological sets, we can define a function m which satisfies some nice properties. This domain will be called "measurable sets" and the function m will be called the "Lebesgue measure".

1.1 Preliminaries

Recall that the collection $\{E_\alpha\}_{\alpha \in \mathcal{A}}$ of subsets in $\mathbb{R}^d$ is said to be disjoint if $E_\alpha \cap E_\beta = \emptyset$ whenver $\alpha \neq \beta$. We define the slightly weaker concept "almost disjoint" as follows:

Definition 1.1.1

Let $\{E_\alpha\}_{\alpha \in \mathcal{A}}$ be a collection subsets in $\mathbb{R}^d$. Then $\{E_\alpha\}_{\alpha \in \mathcal{A}}$ is said to be **almost disjoint** if $\{\mathrm{Int}(E_\alpha)\}_{\alpha \in \mathcal{A}}$ is disjoint.

Note that a (closed) **rectangle** R in $\mathbb{R}^d$ is given by the product of d one-dimensional closed and bounded intervals

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d],$$

where $a_j \leq b_j$ are real numbers, $j = 1, 2, \ldots, d$. The **volume** of the rectangle R is denoted by $|R|$, and is defined to be

$$|R| = (b_1 - a_1) \cdots (b_d - a_d).$$

Also, a **cube** is a rectangle for which $b_1 - a_1 = b_2 - a_2 = \cdots = b_d - a_d$. So if $Q \subseteq \mathbb{R}^d$ is a cube of common side length l, then $|Q| = l^d$.

Lemma 1.1.1

If a rectangle is the almost disjoint union of finitely many other rectangles, say $R = \bigcup_{k=1}^N R_k$, then

$$|R| = \sum_{k=1}^N |R_k|.$$

Proof. Consider the grid formed by extending indefinitely the sides of all rectangles $R_1, \ldots, R_N$. This gives finitely many rectangles $\tilde{R}_1, \ldots, \tilde{R}_M$, and a partition $J_1, \ldots, J_N$ of integers between 1 and M such that

$$R = \bigcup_{j=1}^M \tilde{R}_j \quad \text{and} \quad R_k = \bigcup_{j \in J_k} \tilde{R}_j \quad \text{for} \quad 1 \leq k \leq M$$

are almost disjoint.

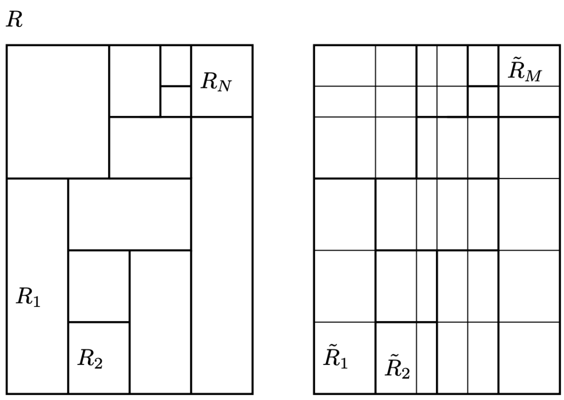

Figure 1.1: The grid formed by the rectangles R_k

For the rectangle R, it is clear that $|R| = \sum_{j=1}^{M} |\tilde{R}_j|$. By the same reason, we also have $|R_k| = \sum_{j\in J_k} |\tilde{R}_j|$. Therefore, one can obtain

$$|R| = \sum_{j=1}^{M} |\tilde{R}_j| = \sum_{k=1}^{N} \sum_{j\in J_k} |\tilde{R}_j| = \sum_{k=1}^{N} |R_k|.$$

□

Lemma 1.1.2

If $R, R_1, \ldots, R_N$ are rectangles, and $R \subseteq \bigcup_{k=1}^{N} R_k$, then

$$|R| \leq \sum_{k=1}^{N} |R_k|.$$

Proof. As we did in Lemma 1.1.1, we can find smaller rectangles $\tilde{R}_1, \ldots, \tilde{R}_M$. Note that there exist $J_0 \subseteq \{1, \ldots, M\}$ such that $R = \bigcup_{j\in J_0} \tilde{R}_j$. Let $R' := \bigcup_{j=1}^{M} \tilde{R}_j$. Again, we can find a partition $J_1, \ldots, J_N$ of $\{1, \ldots, M\}$ such that

$$R_k = \bigcup_{j\in J_k} \tilde{R}_j \quad \text{for all} \quad 1 \leq k \leq N.$$

Note that $J_0, J_1, \ldots, J_N$ need not be disjoint.

Since $R \subseteq \bigcup_{k=1}^{N} R_k$, we have $J_0 \subseteq \bigcup_{k=1}^{N} J_k$. Therefore, we can obtain

$$|R| = \sum_{j\in J_0} |\tilde{R}_j| \leq \sum_{k=1}^{N} \sum_{j\in J_k} |\tilde{R}_j| = \sum_{k=1}^{N} |R_k|.$$

This completes the proof. □

Theorem 1.1.3

Every open set $U \subseteq \mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals.

Proof. For each $x \in U$, there exists an open interval, which contains x, contained in U. If $a_x = \inf\{a < x : (a, x) \subseteq U\}$ and $b_x = \sup\{b > x : (x, b) \subseteq U\}$, then

$$x \in I_x := (a_x, b_x) \subseteq U.$$

Hence, we get $U = \bigcup_{x\in U} I_x$. Moreover, note that I_x is the largest open interval, which contains x, contained in U.

Suppose that two intervals I_x and I_y intersect. Then we have

$$x \in I_x \cup I_y \subseteq U;$$

by the maximality of I_x, we conclude that $(I_x \cup I_y) \subseteq I_x$. By the same argument, since $(I_x \cup I_y) \subseteq I_y$, we finally have $I_x = I_y$. Therefore, any two distinct intervals in $\mathcal{C} := \{I_x\}_{x \in U}$ is disjoint.

We claim that $\mathcal{C}$ is countable, i.e., it has countably many disjoint open intervals. Since each I_x contains a rational number, any two distinct intervals must contain distinct rationals. Hence, the collection $\mathcal{C}$ must be countable. □

However, in $\mathbb{R}^d$ for $d \geq 2$, the direct analogue of Theorem 1.1.3 is not valid (see Exercise 1.12). Instead, there is a substitute result as follows:

Theorem 1.1.4

Every open subset $\mathcal{O} \subseteq \mathbb{R}^d$, $d \geq 1$, can be written as a countable union of almost disjoint closed cubes.

Proof. □

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MAS441: Lebesgue Intergral Theory Homework Problems

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4.1 Chapter 1

Exercise 1.5

Suppose E is a given set, and $\mathcal{O}_n$ is the open set:

$$\mathcal{O}_n = \{x : d(x, E) < 1/n\}.$$

Show:

(a) If E is compat, then $m(E) = \lim_{n\to\infty} m(\mathcal{O}_n)$.

(b) However, the conclusion in (a) may be false for E closed and unbounded; or E open and bounded.

Proof. (a) We first prove that $\mathcal{O}_n$ is open for all $n \in \mathbb{Z}^+$. For any $x \in \mathcal{O}_n$, since $d(x, E) < \frac{1}{n}$, there exists $\epsilon > 0$ such that $d(x, E) + \epsilon < \frac{1}{n}$. For any $y \in B_d(x, \epsilon)$, by the triangle inequality,

$$\begin{aligned} d(y, E) &= \inf\{d(y, z) : z \in E\} \\ &\le \inf\{d(y, x) + d(x, z) : z \in E\} \\ &< \epsilon + \inf\{d(x, z) : z \in E\} \\ &= \epsilon + d(x, E) < \frac{1}{n}. \end{aligned}$$

Therefore, we conclude that $x \in B_d(x, \epsilon) \subseteq \mathcal{O}_n$, that is, $\mathcal{O}_n$ is open.

Recall that all open sets are measurable and the countable intersection of measurable sets is measurable. Then

$$\mathcal{O} := \bigcap_{n=1}^{\infty} \mathcal{O}_n = \{x : d(x, E) = 0\}$$

is measurable. It is clear that $E \subseteq \mathcal{O}$, by definition. However, since the compactness of E implies that $\bar{E} = E$, note that

$$\begin{aligned} d(x, E) = 0 \quad &\Longleftrightarrow \quad \forall \epsilon > 0,\ \exists y \neq x \text{ such that } y \in E \cap B_d(x, \epsilon) \\ &\Longleftrightarrow \quad x \in \bar{E} \\ &\Longleftrightarrow \quad x \in E \quad (\because E \text{ is compact.}) \end{aligned}$$

Thus, we get $\mathcal{O} = E$. Moreover, since $\mathcal{O}_n \supseteq \mathcal{O}_{n+1}$, for all $n \in \mathbb{Z}^+$, we have $\mathcal{O}_n \searrow \mathcal{O}$.

One more thing we need to check is whether $m(\mathcal{O}_1)$ is finite or not. Note that the compactness of E implies that it is bounded, i.e., there exists $N > 0$ such that $E \subseteq B_d(\mathbf{0}, N)$ where $\mathbf{0}$ is the origin. Then, the triangle inequality implies that $\mathcal{O}_1 \subseteq B_d(\mathbf{0}, N+1)$; therefore,

$$m(\mathcal{O}_1) \le m\big(B_d(\mathbf{0}, N+1)\big) < \infty.$$

Then, by Corollary 1.3.3 in the Textbook, since $\{\mathcal{O}_n\}_{n=1}^{\infty}$ are the measurable sets satisfying $\mathcal{O}_n \searrow E$ and $m(\mathcal{O}_1) < \infty$, we conclude that

$$m(E) = \lim_{n\to\infty} m(\mathcal{O}_n).$$

(b) We fisrt consider the case when E is closed and unbounded. Let $E = \mathbb{Z} \subseteq \mathbb{R}$, then it is closed but it is unbounded. Note that

$$\mathcal{O}_n = \bigcup_{x\in\mathbb{Z}} \left(x - \frac{1}{n}, x + \frac{1}{n}\right) \quad \text{for all} \quad n \in \mathbb{Z}^+.$$

Although each interval $(x - 1/n, x + 1/n)$ has length (or measure) $2/n$, but since there are infinitely many integers, $m(\mathcal{O}_n)$ is infinite for all $n \in \mathbb{Z}^+$.

On the other hand, since $\mathbb{Z}$ is the countable union of singletons, $\{x\}$ with $x \in \mathbb{Z}$, and since each singleton has measure zero, it follows that $\mathbb{Z}$ is measurable. Moreover, since these singletons are disjoint measurable sets, we have

$$m(\mathbb{Z}) = m\left(\bigcup_{x\in\mathbb{Z}} \{x\}\right) = \sum_{x\in\mathbb{Z}} m(\{x\}) = 0.$$

Therefore, we can obtain $m(E) \neq \lim_{n\to\infty} m(\mathcal{O}_n)$, in this case.

Likewise, let's consider the case when E is open and bounded. Since $[0,1]\cap\mathbb{Q}$ contains countably many elements, let's label them r_i where $i \in \mathbb{Z}^+$ so that $\{r_1, r_2, \dots\} = [0,1]\cap\mathbb{Q}$. Under this notation, define

$$E \coloneqq \bigcup_{k=1}^{\infty} \left(r_k - \frac{1}{2^{k+2}}, r_k + \frac{1}{2^{k+2}}\right)$$

which is open by being countable union of open intervals.

Recall that $\mathcal{O}_n = \{x : d(x,E) < \frac{1}{n}\}$. Hence, we clearly have $\mathcal{O}_n \supseteq \mathcal{O}_{n+1}$ for all $n \in \mathbb{Z}^+$; moreover, each $\mathcal{O}_n$ is measurable by being an open set. Let

$$\mathcal{O} \coloneqq \bigcap_{n=1}^{\infty} \mathcal{O}_n = \{x : d(x,E) = 0\},$$

then, as we did before, we have

$$\begin{aligned} d(x,E) = 0 \quad &\iff \quad \forall \epsilon > 0,\ \exists y \neq x \text{ such that } y \in E \cap B_d(x,\epsilon) \\ &\iff \quad x \in \bar{E}. \end{aligned}$$

Hence, we conclude that $\mathcal{O} = \bar{E}$. This implies that $\mathcal{O}_n \searrow \bar{E}$ where $m(\mathcal{O}_1) < \infty$ (since E is bounded, so is $\mathcal{O}_1$); therefore, we get

$$\lim_{n\to\infty} m(\mathcal{O}_n) = m(\bar{E}).$$

Here, since $\bar{E}$ is a well-defined closed set, so it is measurable. Hence, this proves the existence of $\lim_{n\to\infty} m(\mathcal{O}_n)$.

For any $x \in [0,1]$ and $n \geq 1$, if $x + \frac{1}{n} \leq 1$, then there exists $m \in \mathbb{Z}^+$ such that $r_m \in (x, x + \frac{1}{n})$ so that $x \in \mathcal{O}_n$; likewise, if $x + \frac{1}{n} > 1$, then there exists $m \in \mathbb{Z}^+$ such that $r_m \in (x - \frac{1}{n}, x)$ so that $x \in \mathcal{O}_n$. Hence, we have $[0,1] \subseteq \mathcal{O}_n$ for all $n \in \mathbb{Z}^+$. Then the monotonicity gives

$$1 = m\big([0,1]\big) \leq m(\mathcal{O}_n) \quad \text{for all} \quad n \geq 1.$$

Therefore, we conclude that $\lim_{n\to\infty} m(\mathcal{O}_n) \geq 1$.

However, from the definition of E, the countable subadditivity gives that

$$m(E) \leq \sum_{k=1}^{\infty} m\left(\left(r_k - \frac{1}{2^{k+2}}, r_k + \frac{1}{2^{k+2}}\right)\right) = \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} = \frac{1}{2}.$$

Hence, we proved that

$$m(E) \neq \lim_{n\to\infty} m(\mathcal{O}_n).$$

$\square$

Exercise 1.8

Suppose L is a linear transformation on $\mathbb{R}^d$. Show that if E is measurable subset of $\mathbb{R}^d$, then so is $L(E)$, by proceeding as follows:

(a) Note that if E is compact, so is $L(E)$. Hence if E is an F_σ set, so is $L(E)$.

(b) Because L automatically satisfies the inequality

$$|L(x) - L(x')| \le M|x - x'|$$

for some M, we can see that L maps any cube of side length l into a cube of side length $c_d Ml$, with $c_d = 2\sqrt{d}$. Now if $m(E) = 0$, there is a collection of cubes $\{Q_j\}$ such that $E \subset \bigcup_j Q_j$, and $\sum_j m(Q_j) < \epsilon$. Thus $m_*(L(E)) \le c'\epsilon$, and hence $m(L(E)) = 0$. Finally, use Corollary 3.5.

One can show that $m(L(E)) = |\det L| m(E)$; see Problem 4 in the next chapter.

Proof. (a) Since we know that every linear transformation on $\mathbb{R}^d$ is Lipschitz, thus L is a continuous map. Hence, it maps a compact set E to a compact set, that is, $L(E)$ is compact.

Assume that E is an F_σ set, i.e. $E = \bigcup_{n=1}^\infty F_n$ where $\{F_n\}_{n=1}^\infty$ are closed sets in $\mathbb{R}^d$. We now claim that $L(E)$ is also compact. To prove this, we first prove the following lemma:

Lemma. Every closed in $\mathbb{R}^d$ is a countable union of compact sets.

Proof. Let F be a closed set in $\mathbb{R}^d$. For each $m \in \mathbb{Z}^+$, let

$$K_m \coloneqq F \cap \overline{B_d(\mathbf{0}, m)}.$$

Since K_m is closed and bounded in $\mathbb{R}^d$, thus it is compact. Also, it is clear that

$$F = \bigcup_{m=1}^\infty K_m;$$

therefore, any closed set F can be written as a countable union of compact sets. □

By lemma, each closed set F_n can be written as

$$F_n = \bigcup_{m=1}^\infty K_{n,m}$$

where each $K_{n,m}$ is compact. Thus we get

$$E = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} K_{n,m}.$$

Then, $L(E)$ can be expressed as

$$L(E) = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} L(K_{n,m})$$

where each $L(K_{n,m})$ is compact, by the above conclusion. Therefore, this implies that $L(E)$ is a countable union of compact sets (which are closed in $\mathbb{R}^d$) so that it is a F_σ set.

(b) If $m(E) = 0$, then there exist a collection of closed cubes $\{Q_j\}_{j=1}^{\infty}$ such that

$$E \subseteq \bigcup_{j=1}^{\infty} Q_j \quad \text{and} \quad 0 \leq \sum_{j=1}^{\infty} |Q_j| < \epsilon.$$

Let l_j be the side length of Q_j, for each j, then $L(Q_j)$ is contained in a cube Q'_j of side length $c_d M l_j$ where $c_d = 2\sqrt{d}$. Since

$$L(E) \subseteq \bigcup_{j=1}^{\infty} L(Q_j) \subseteq \bigcup_{j=1}^{\infty} Q'_j$$

thus

$$m_*(L(E)) \leq \sum_{j=1}^{\infty} |Q'_j| = (c_d M)^d \sum_{j=1}^{\infty} l_j^d < \epsilon (c_d M)^d.$$

This proves that $m_*(L(E)) = 0$.

Note that since E is measurable, by Corollary 1.3.5, there exist an F_σ set F such that

$$m(E \triangle F) = 0.$$

Then, by part (a), $L(F)$ is also F_σ set and since we have

$$L(E) \triangle L(F) \subseteq L(E \triangle F),$$

in general, thus

$$0 \leq m_*\big(L(E) \triangle L(F)\big) \leq m_*\big(L(E \triangle F)\big).$$

Since $m(E \triangle F) = 0$ implies that $m_*\big(L(E \triangle F)\big) = 0$, we get

$$m_*\big(L(E) \triangle L(F)\big) = 0.$$

This implies that $L(E)$ differs from an F_σ set $L(F)$ by a set of measure zero; therefore, $L(E)$ is measurable and we conclude that $m(L(E)) = 0$. $\square$

Exercise 1.11

Let A be the subset of $[0,1]$ which consists of all numbers which do not have the digit 4 appearing in their decimal expansion. Find $m(A)$.

Proof. Let A_n denote the set

$$A_n \coloneqq \{x \in [0,1] : \text{first } n \text{ digits of } x \text{ do not contain } 4\}.$$

for all $n \in \mathbb{Z}^+$. Then $A_{n+1} \subseteq A_n$, for all n, and

$$A = \bigcap_{n=1}^{\infty} A_n.$$

This implies that $A_n \searrow A$.

Note that each A_n is a finite union of intervals. For instance, since A_1 is

$$A_1 = \left[0, \frac{4}{10}\right) \cup \left[\frac{5}{10}, 1\right],$$

thus it is measurable and has measure $9/10$. Similarly, A_2 is obtained from A_1 by removing 9 subintervals of length $1/100$, corresponding to numbers whose second digit is 4. Therefore, we get $m(A_2) = (9/10)^2$. By repeating this procedure, we have

$$m(A_n) = \left(\frac{9}{10}\right)^n \quad \text{for all} \quad n \in \mathbb{Z}^+.$$

By Corollary 1.3.3, since $\{A_n\}_{n=1}^{\infty}$ are measurable sets with $A_n \searrow A$ and $m(A_1) < \infty$, thus we have

$$m(A) = \lim_{n\to\infty} m(A_n) = \lim_{n\to\infty} \left(\frac{9}{10}\right)^n = 0.$$

$\square$

Exercise 1.12

Theorem 1.3 states that every open set in $\mathbb{R}$ is the disjoint of open intervals. The analogue in $\mathbb{R}^d$, $d \geq 2$, is generally false. Prove the following:

(a) An open disc in $\mathbb{R}^2$ is not the disjoint union of open rectangles.

[Hint: What happens to the boundary of any of these rectangles?]

(b) An open connected set Ω is the disjoint union of open rectangles if and only if Ω is itself an open rectangle.

Proof. (a) Suppose not. Let $\mathcal{C}$ be the collection of disjoint open rectangles whose union is an open disc in $\mathbb{R}^2$. Consider $R \in \mathcal{C}$ and $x \in \mathrm{bd}(R)$ which does not lie on the boundary of disc. Since $x \in D$, where D denotes the disc, there exist $R' \in \mathcal{C}$ such that $x \in R'$. Hence, there exist a basis element B (which is an ϵ-ball centered at x) such that

$$x \in B \subseteq R'.$$

Since $x \in \mathrm{bd}(R) = \overline{R} \cap \overline{(\mathbb{R}^d \setminus R)}$ and B is a neighborhood of x, so B intersects R at some point y other than x. Since $y \in B \subseteq R'$ and $y \in R$, we have $R \cap R' = \emptyset$ which is a contradiction. Therefore, D cannot be the disjoint union of open rectangles.

(b) Suppose that $\Omega = \bigcup_{U \in \mathcal{C}} U$ where the elements of $\mathcal{C}$ are disjoint open rectangles. Let $\emptyset \neq \mathcal{A} \subsetneq \mathcal{C}$. Then

$$\Omega = \left(\bigcup_{U \in \mathcal{A}} U \right) \cup \left(\bigcup_{U \in \mathcal{C} \setminus \mathcal{A}} U \right)$$

gives a separation of Ω which contradicts to the connectedness of Ω. Therefore, such $\mathcal{A}$ cannot exist, i.e., $|\mathcal{C}| = 1$ so that Ω is an open rectangle. The converse is trivial; hence, the proof is done. □

Exercise 1.16

Suppose $\{E_k\}_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that

$$\sum_{k=1}^{\infty} m(E_k) < \infty.$$

Let

$$\begin{aligned} E &= \{x \in \mathbb{R}^d : x \in E_k, \text{ for infinitely many } k\} \\ &= \limsup_{k\to\infty} (E_k). \end{aligned}$$

(a) Show that E is measurable.

(b) Prove $m(E) = 0$.

[Hint: Write $E = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k$.]

Proof. (a) Recall that $\limsup_{k\to\infty}(E_k)$ is obtained by

$$\limsup_{k\to\infty}(E_k) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k.$$

This is because

$$\begin{aligned} x \in E_k \text{ for infinitely many } k \quad &\iff \quad x \in \bigcup_{k=n}^{\infty} E_k \text{ for all } n \in \mathbb{Z}^+ \\ &\iff \quad x \in \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k. \end{aligned}$$

Since each E_k is measurable, so is $\bigcup_{k=n}^{\infty} E_k$. Likewise, this implies that $\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$ is also measurable. Therefore, we conclude that E is measurable.

(b) $\sum_{k=1}^{\infty} m(E_k) < \infty$ implies that, for any $\epsilon > 0$, there exists $N > 0$ such that

$$\sum_{k=N}^{\infty} m(E_k) < \epsilon.$$

Then, by the countable subadditivity, we have

$$m\left(\bigcup_{k=N}^{\infty} E_k\right) \leq \sum_{k=N}^{\infty} m(E_k) < \epsilon.$$

However, since the monotonicity implies that

$$m(E) = m\left(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k\right) \leq m\left(\bigcup_{k=N}^{\infty} E_k\right) < \epsilon,$$

we conclude that $m(E) = 0$. $\square$

Exercise 1.17

Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$ with $|f_n(x)| < \infty$ for a.e x. Show that there exists a sequence c_n of positive real numbers such that

$$\frac{f_n(x)}{c_n} \to 0 \quad \text{a.e.} \quad x$$

[Hint: Pick c_n such that $m(\{x : |f_n(x)/c_n| > 1/n\}) < 2^{-n}$, and apply the Borel-Cantelli lemma.]

Proof. The fact that $|f_n(x)| < \infty$ for a.e x implies that

$$m\big(\{x : |f_n(x)| = \infty\}\big) = 0.$$

Let A denote the set $\{x : |f(x)| = \infty\}$. If we let $A_k := \{x : |f_n(x)| > k/n\}$ for all $k \in \mathbb{Z}^+$, then we have $A_k \searrow A$ since

$$A_{k+1} \subseteq A_k \quad \text{for all} \quad k \in \mathbb{Z}^+$$

and

$$\bigcap_{k=1}^{\infty} A_k = \{x : |f_n(x)| = \infty\}.$$

Therefore, we get

$$\lim_{k\to\infty} m(A_k) = m(A) = 0.$$

This implies that there exists $c_n \in \mathbb{Z}^+$ such that $m(A_{c_n}) < 2^{-n}$, i.e.

$$m\left(\left\{x : |f_n(x)| > \frac{c_n}{n}\right\}\right) < 2^{-n}. \tag{$*$}$$

Define, for all $n \in \mathbb{Z}^+$,

$$E_n = \left\{x : \left|\frac{f_n(x)}{c_n}\right| > \frac{1}{n}\right\}.$$

Then $(*)$ implies that $m(E_n) < 2^{-n}$ for all n. Note that $\{E_n\}_{n=1}^{\infty}$ is a countable family of measurable sets with

$$\sum_{n=1}^{\infty} m(E_n) < \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 < \infty;$$

therefore, by the Borel-Cantelli lemma, we get $m(E) = 0$ where

$$E := \limsup_{n\to\infty} (E_n).$$

For any $x \in E^c$, there exists $N > 0$ such that $x \notin E_n$ for all $n \geq N$. This implies that

$$\lim_{n\to\infty}\left|\frac{f_n(x)}{c_n}\right| = 0 \quad \text{for all} \quad x \in E^c.$$

Therefore, $m(E) = 0$ implies that

$$\frac{f_n(x)}{c_n} \to 0 \quad \text{a.e.} \quad x.$$

$\square$

Exercise 1.20

Show that there exist closed sets A and B with $m(A) = m(B) = 0$, but $m(A+B) > 0$:

(a) In $\mathbb{R}$, let $A = \mathcal{C}$ (the Cantor set), $B = \mathcal{C}/2$. Note that $A + B \supset [0, 1]$.

(b) In $\mathbb{R}^2$, observe that if $A = I \times \{0\}$ and $B = \{0\} \times I$ (where $I = [0, 1]$), then $A + B = I \times I$.

Proof. (a) It is clear that every number in $[0, 1]$ has a ternary expansion

$$x = \sum_{k=1}^{\infty} a_k 3^{-k}, \quad \text{where} \quad a_k = 0, 1, \text{or } 2.$$

The algorithm for finding a_k is as following: If $x = 1$, then $a_k = 2$ for all $k \in \mathbb{Z}^+$; if $x < 1$, take $a_1 = \lfloor 3x \rfloor$ where $\lfloor \cdot \rfloor$ denotes the greatest integer less than or equal to x. Similarly, we take $a_2 = \lfloor 3(3x - a_1) \rfloor$ and by repeating this procedure, we have

$$a_n = \left\lfloor 3^n x - \sum_{k=1}^{n-1} a_k 3^{n-k} \right\rfloor \quad \text{for all} \quad n \geq 2.$$

Note that the ternary expansion is not unique since, for example, $1/3 = \sum_{k=2}^{\infty} 2/3^k$.

Since $A = \mathcal{C}$, the Cantor set, A consists of all numbers which have a ternary expansion with $a_k \in \{0, 2\}$ for all $k \in \mathbb{Z}^+$. By the same reason, since $B = \mathcal{C}$, it consists of all numbers which have a ternary expansion with $a_k \in \{0, 1\}$.

Now we claim that, for any $x \in [0, 1]$, there exist $a \in A$ and $b \in B$ such that $x = a + b$. Let $x = \sum_{k=1}^{\infty} a_k 3^{-k}$, where $a_k \in \{0, 1, 2\}$, be a ternary expansion of x. Likewise, let

$$a = \sum_{k=1}^{\infty} a_k 3^{-k} \quad \text{and} \quad b = \sum_{k=1}^{\infty} b_k 3^{-k}$$

be ternary expansions of a and b, respectively. By letting,

$$a_k = \begin{cases} 2, & \text{if } x_k = 2 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad b_k = \begin{cases} 1, & \text{if } x_k = 1 \\ 0, & \text{otherwise} \end{cases}$$

we clearly have $x = a + b$ (since $x_k = a_k + b_k$ for all $k \in \mathbb{Z}^+$). Therefore,

$$[0, 1] \subseteq A + B.$$

(b) Let $x = (x_1, x_2) \in A + B$, then $x = a + b$ for some $a \in A$ and $b \in B$. Since a and b are of the form $a = (a', 0)$ and $b = (0, b')$ where $a', b' \in I$. Then

$$(x_1, x_2) = (a', 0) + (0, b') \quad \Longrightarrow \quad x_1 = a' \quad \text{and} \quad x_2 = b';$$

therefore, $(x_1, x_2) \in I \times I$.

Conversely, assume that $x = (x_1, x_2) \in I \times I$. Since each x_1 and x_2 belongs to I, it is clear that $(x_1, 0) \in A$, $(0, x_2) \in B$, and

$$x = (x_1, 0) + (0, x_2) \in A + B.$$

Therefore, we get $I \times I \subseteq A + B$ and this completes the proof.

In part (a), note that both A and B are closed with measure zero, but $A + B$ has positive measure since $A + B \supset [0, 1]$. Likewise, note that A and B in part (b) are also closed with measure zero, but $A + B = I \times I$ has positive measure, obviously. $\square$

Exercise 1.21

Prove that there is a continuous function that maps a Lebesgue measurable set to a non-measurable set.
[Hint: Consider a non-measurable subset of $[0, 1]$, and its inverse image in $\mathcal{C}$ by the function F in Exercise 2.]

Proof. Let $\mathcal{C}$ denote the Cantor set. Define a map $f : \mathcal{C} \to [0, 1]$ to because

$$f(x) = \sum_{k=1}^{\infty} \frac{b_k}{2^k} \qquad \text{if } x = \sum_{k=1}^{\infty} a_k 3^{-k}, \text{ where } b_k = a_k/2.$$

In this definition, we choose the expansion of x in which $a_k = 0$ or 2. We first claim that f is well-defined and continuous on $\mathcal{C}$, and moreover it is surjective.

(Well-definedness). Since we proved that every number in $[0, 1]$ has at least one ternary expansion, the only problem is that some numbers (such as $1/3$) have more than one ternary expansions. However, such a number can have a unique ternary expansion that consists of all 0's and 2's, because the only problem arises when one representation terminates while the other doesn't. If a representation terminates, then it must end with 2 (if it contains only 0's and 2's). However, then the other expansion ends with $1222\cdots$; therefore, it contains 1 in its expansion. This proves that each element of $\mathcal{C}$ has the unique ternary expansion consists of only 0's and 2's; hence, the map f is well-defined.

(Continuity). For any $\epsilon > 0$, there exists $n \in \mathbb{Z}^+$ such that $1/2^n < \epsilon$. Let $\delta = 1/3^n$, then for any $x, y \in \mathcal{C}$ with $|x - y| < \delta$, first n digits of their ternary expansion must be the same. Otherwise, one must contain 1 in its expansion or their difference must be greater than or equal to δ. Therefore, for such x and y, we have

$$|f(x) - f(y)| < \sum_{k=n+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^n} < \epsilon.$$

Therefore, we conclude that f is continuous.

(Surjectivity). As we did in Exercise 1.20, every number in $[0, 1]$ also has a binary expansion

$$x = \sum_{k=1}^{\infty} b_k 2^{-k}, \quad \text{where} \quad b_k = 0 \text{ or } 1.$$

If $x < 1$, then $b_k = 1$ for all $k \in \mathbb{Z}^+$; if $x < 1$, then $b_1 = \lfloor 2x \rfloor$ and

$$b_n = \left\lfloor 2^n x - \sum_{k=1}^{n-1} a_k 2^{n-k} \right\rfloor \quad \text{for all} \quad n \geq 2.$$

Hence, for any $y \in [0,1]$, it has a binary expression

$$y = \sum_{k=1}^{\infty} b_k 2^{-k}.$$

Let $a_k := 2b_k$ for all $k \in \mathbb{Z}^+$. Since a_k if either 0 or 2, for all k,

$$x := \sum_{k=1}^{\infty} a_k 3^{-k}$$

belongs to $\mathcal{C}$. Moreover, it is clear that $f(x) = y$; therefore, f is surjective.

Note that the set $\mathcal{N}$ defined in the Theorem 1.3.6 is non-measurable subset of $[0,1]$. Let $E := f^{-1}(\mathcal{N}) \subseteq \mathcal{C}$. Since E is a subset of the measure zero set $\mathcal{C}$, we have $m(E) = 0$. This implies that E is measurable. Therefore, f is a continuous function which maps the Lebesgue measurable set E to a non-measurable set $\mathcal{N}$. $\square$

Exercise 1.22

Let $\chi_{[0,1]}$ be the characteristic function of $[0,1]$. Show that there is no everywhere continuous function f on $\mathbb{R}$ such that

$$f(x) = \chi_{[0,1]}(x) \quad \text{almost everywhere.}$$

Proof. Suppose that such a continuous function f exists. We first claim that

$$f(x) = 0 \quad \text{for all} \quad x \in (-\infty, 0).$$

Assume that there exists $x_0 \in (-\infty, 0)$ such that $f(x_0) \neq 0$. Then, since f is continuous, for any $0 < \epsilon < |f(x_0)|$, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \quad \Longrightarrow \quad |f(x) - f(x_0)| < \epsilon.$$

Then, for any $x \in (x_0 - \delta, x_0]$, the triangle inequality implies that

$$|f(x)| \geq |f(x_0)| - |f(x) - f(x_0)| > |f(x_0)| - \epsilon > 0.$$

Thus, $(x_0 - \delta, x_0] \subseteq \{x : f(x) \neq \chi_{[0,1]}(x)\}$ so that

$$m\left(\{x : f(x) \neq \chi_{[0,1]}(x)\}\right) \geq \delta > 0$$

which is a contradiction. Therefore, we have $f|_{(-\infty,0)} = 0$.

By the same argument, we can check that $f|_{(0,1)} = 1$. If there exists $x_0 \in (0,1)$ such that $f(x_0) \neq 1$, then for any $0 < \epsilon < |f(x_0) - 1|$, there exists $\delta > 0$ such that

$$B_d(x_0, \delta) \subseteq (0,1) \qquad \text{and} \qquad |f(x) - f(x_0)| < \epsilon \quad \text{for all} \quad x \in B_d(x_0, \delta).$$

Therefore, for any $x \in B_d(x_0, \delta)$, the triangle inequality gives

$$|f(x) - 1| \geq |f(x_0) - 1| - |f(x) - f(x_0)| > |f(x_0) - 1| - \epsilon > 0.$$

Thus, $B_d(x_0, \delta) \subseteq \{x : f(x) \neq \chi_{[0,1]}(x)\}$ so that

$$m\left(\{x : f(x) \neq \chi_{[0,1]}(x)\}\right) \geq 2\delta > 0$$

which is again a contradiction. Therefore, we conclude that $f|_{(0,1)} = 1$.

However, since $f|_{(-\infty,0)} = 0$ and $f|_{(0,1)} = 1$, it follows that f cannot be continuous at 0, no matter what value is assigned to $f(0)$. This contradicts to the assumption that f is a continuous function on $\mathbb{R}$. Therefore, we conclude that there is no everywhere continuous function f on $\mathbb{R}$ such that $f(x) = \chi_{[0,1]}(x)$ almost everywhere. $\square$

Exercise 1.26

Suppose $A \subset E \subset B$, where A and B are measurable sets of finite measure. Prove that if $m(A) = m(B)$, then E is measurable.

Proof. Since A and B are measurable, so is $B \setminus A$. Note that A and $B \setminus A$ are disjoint measurable sets. Therefore, we have

$$m(B) = m\big(A \cup (B \setminus A)\big) = m(A) + m(B \setminus A),$$

and since $m(A) = m(B)$, we get $m(B \setminus A) = 0$.

Note that $E \setminus A \subseteq B \setminus A$, thus we have

$$m_*(E \setminus A) \leq m_*(B \setminus A) = 0.$$

Since $m_*(E \setminus A) = 0$, it is measurable. Then, since $E = A \cup (E \setminus A)$ is a union of two measurable set, thus we conclude that E is also measurable. $\square$

Exercise 1.28

Let E be a subset of $\mathbb{R}$ with $m_*(E) > 0$. Prove that for each $0 < \alpha < 1$, there exists an open interval I so that

$$m_*(E \cap I) \geq \alpha m_*(I).$$

Loosely speaking, this estimate shows that E contains almost a whole interval. [Hint: Choose an open set $\mathcal{O}$ that contains E, and such that $m_*(E) \geq \alpha m_*(\mathcal{O})$. Write $\mathcal{O}$ as the countable union of disjoint open intervals, and show that one of these intervals must satisfy the desired property.]

Proof. For any $0 < \alpha < 1$, let $\mathcal{O}$ be an open set containing E such that $m_*(E) \geq \alpha m_*(\mathcal{O})$. Note that $\mathcal{O}$ can be written as a countable union of disjoint open intervals so that

$$E = \bigcup_{k=1}^{\infty} I_k$$

where $\{I_k\}_{k=1}^{\infty}$ are disjoint open intervals. Then we have

$$E = E \cap \mathcal{O} = \bigcup_{k=1}^{\infty} (E \cap I_k);$$

hence, by the countable subadditivity,

$$m_*(E) \leq \sum_{k=1}^{\infty} m_*(E \cap I_k).$$

If there is no open interval I satisfying $m_*(E \cap I) \geq \alpha m_*(I)$, since $\{I_k\}_{k=1}^{\infty}$ are disjoint open sets, we have

$$m_*(E) < \alpha \sum_{k=1}^{\infty} m_*(I_k) = \alpha m_*(\mathcal{O}) \leq m_*(E)$$

which is a contradiction. Therefore, there exist $k \in \mathbb{Z}^+$ such that

$$m_*(E \cap I_k) \geq \alpha m_*(I_k)$$

where I_k is an open interval in $\mathbb{R}$. $\square$

Exercise 1.30

If E and F are measurable, and $m(E) > 0,\ m(F) > 0$, prove that

$$E + F = \{x + y : x \in E,\ y \in F\}$$

contains an interval.

Proof. Note that $-F = \{-x : x \in F\}$ is also measurable with $m(-F) > 0$ since F is measurable. By the Exercise 1.28, for any $0 < \alpha < 1$, there exist open intervals I_1 and I_2 such that

$$m(E \cap I_1) \geq \alpha m(I_1) \quad \text{and} \quad m(-F \cap I_2) \geq \alpha m(I_2).$$

Without loss of generality, assume that $m(I_2) \leq m(I_1)$. Then, since they are intervals, there exists $h \in \mathbb{R}$ such that $I_2 + h \subseteq I_1$. (Here, note that $I_2 + h \coloneqq \{x + h : x \in I_2\}$.) For convenience, let $l = m(I_2) > 0$. Then, for any $0 < |r| < l$, we have

$$m\big(I_1 \cap (I_2 + h + r)\big) \geq l - |r| > 0.$$

This implies that

$$I_1 \cap (I_2 + h + r) \neq \emptyset;$$

hence, there exist $e \in E$, $f \in F$ such that

$$e = -f + h + r.$$

In other words, for any $0 < |r| < l$,

$$h + r = e + f$$

for some $e \in E$ and $f \in F$. Therefore, we conclude that

$$(h - l, h + l) \subseteq E + F,$$

that is, $E + F$ contains an interval. $\square$

Exercise 1.37

Suppose Γ is a curve $y = f(x)$ in $\mathbb{R}^2$, where f is continuous. Show that $m(\Gamma) = 0$. [Hint: Cover Γ by rectangles, using the uniform continuity of f.]

Proof. For each $n \in \mathbb{Z}$, define

$$\Gamma_n \coloneqq \{(x, f(x)) : x \in [n, n+1]\} \subseteq \mathbb{R}^2.$$

Since f is continuous and $[n, n+1]$ is compact, thus f is uniformly continuous on $[n, n+1]$. This implies that, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x - y| < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \epsilon.$$

Note that there exists $N \in \mathbb{Z}^+$ such that $1/N < \delta$. Let

$$I_i \coloneqq \left[n + \frac{i-1}{N}, n + \frac{i}{N}\right]$$

for each $1 \leq i \leq N$. Since, for any $x, y \in I_i$, we have $|f(x) - f(y)| < \epsilon$; thus

$$f(x) \in \left[f\left(n + \frac{i-1}{N}\right) - \epsilon,\ f\left(n + \frac{i-1}{N}\right) + \epsilon\right]$$

for all $x \in I_i$. Let

$$R_i \coloneqq \left[f\left(n + \frac{i-1}{N}\right) - \epsilon,\ f\left(n + \frac{i-1}{N}\right) + \epsilon\right] \times I_i$$

for each $1 \leq i \leq N$. Then, we have

$$\Gamma_n \subseteq \bigcup_{i=1}^{N} R_i \quad \Longrightarrow \quad m_*(\Gamma_n) \leq \sum_{i=1}^{N} m_*(R_i) = N\left(2\epsilon \cdot \frac{1}{N}\right) = 2\epsilon.$$

Since $\epsilon > 0$ is chosen arbitrarily, we have $m_*(\Gamma_n) = 0$, that is, Γ_n is measurable and $m(\Gamma_n) = 0$. Finally, since we have

$$\Gamma = \bigcup_{n \in \mathbb{Z}^+} \Gamma_n,$$

Γ is measurable and by countable subadditivity, we get

$$m(\Gamma) \leq \sum_{n \in \mathbb{Z}} m(\Gamma_n) = 0$$

□

4.2 Section 2