LEBESGUE INTEGRAL THEORY

Measure Theory, Integration, and Hilbert spaces

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Contents

Chapter 1

Measure Theory

The goal of this course, especially in this chapter, is to provide a notion of "size" for subsets of $\mathbb{R}^d$ by constructing an appropriate function

$$m : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{\geq 0} \cup \{\infty\}$$

satisfying the following properties:

(1) $m(\emptyset) = 0$

(2) $m([b, a]) = b - a$

(3) If $E = \bigsqcup_{n=1}^{\infty} E_n$, then $m(E) = \sum_{n=1}^{\infty} m(E_n)$.

(4) $m(E + h) = m(E)$ for all $h \in \mathbb{R}$

The third and fourth properties are called **countable additivity** and **translation invariance**, respectively. By regarding m as a function that measures the size of sets, it seems very natural that m should satisfy the above properties. However, this is just our intuition! As a math student, the natural follow-up question is as follows:

Does m exist? If it does, is m unique?

In fact, such a function m does not exist. Then, what can we do next? One way to resolve this problem is to restrict the domain of m. By removing certain pathological sets, we can define a function m which satisfies some nice properties. This domain will be called "measurable sets" and the function m will be called the "Lebesgue measure".

1.1 Preliminaries

Recall that the collection $\{E_\alpha\}_{\alpha \in \mathcal{A}}$ of subsets in $\mathbb{R}^d$ is said to be "disjoint" if $E_\alpha \cap E_\beta = \emptyset$ whenver $\alpha \neq \beta$. We define the slightly weaker concept "almost disjoint" as follows:

Definition 1.1.1

Let $\{E_\alpha\}_{\alpha\in\mathcal{A}}$ be a collection subsets in $\mathbb{R}^d$. Then $\{E_\alpha\}_{\alpha\in\mathcal{A}}$ is said to be **almost disjoint** if $\{\mathrm{Int}(E_\alpha)\}_{\alpha\in\mathcal{A}}$ is disjoint.

Note that a closed **rectangle** R in $\mathbb{R}^d$ is given by

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d],$$

where $a_j \le b_j$ for all $1 \le j \le d$. The **volume** of the rectangle R is denoted by $|R|$, and is defined to be

$$|R| = (b_1 - a_1)(b_2 - a_2)\cdots(b_d - a_d).$$

Moreover, a **cube** is a rectangle satisfying $b_1 - a_1 = b_2 - a_2 = \cdots = b_d - a_d$.

Lemma 1.1.1

If a rectangle R can be written as a union of finitely many almost disjoint rectangles $R_1, \dots, R_N$, then

$$|R| = \sum_{k=1}^{N} |R_k|.$$

Proof. Consider the grid formed by extending the sides of all rectangles $R_1, \dots, R_N$. This gives finitely many samller rectangles $\tilde{R}_1, \dots, \tilde{R}_M$ (they are also almost disjoint), and there exists a partition $J_1, \dots, J_N$ of integers between 1 and M such that

$$R = \bigcup_{j=1}^{M} \tilde{R}_j \quad \text{and} \quad R_k = \bigcup_{j\in J_k} \tilde{R}_j \quad \text{for} \quad 1 \le k \le M.$$

For the rectangle R, it is clear that $|R| = \sum_{j=1}^{M} |\tilde{R}_j|$. By the same reason, we also have $|R_k| = \sum_{j\in J_k} |\tilde{R}_j|$. Therefore, one can obtain

$$|R| = \sum_{j=1}^{M} |\tilde{R}_j| = \sum_{k=1}^{N}\sum_{j\in J_k} |\tilde{R}_j| = \sum_{k=1}^{N} |R_k|.$$

□

Lemma 1.1.2

If $R, R_1, \dots, R_N$ are rectangles satisfying $R \subseteq \bigcup_{k=1}^{N} R_k$, then

$$|R| \le \sum_{k=1}^{N} |R_k|.$$

Proof. As we did in Lemma 1.1.1, we can find smaller rectangles $\tilde{R}_1, \dots, \tilde{R}_M$. Note that there exist $J_0 \subseteq \{1, \dots, M\}$ such that $R = \bigcup_{j \in J_0} \tilde{R}_j$. Let $R' := \bigcup_{j=1}^{M} \tilde{R}_j$. Again, we can find a partition $J_1, \dots, J_N$ of $\{1, \dots, M\}$ such that

$$R_k = \bigcup_{j \in J_k} \tilde{R}_j \quad \text{for all} \quad 1 \leq k \leq N.$$

Note that $J_0, J_1, \dots, J_N$ need not be disjoint.

Since $R \subseteq \bigcup_{k=1}^{N} R_k$, we have $J_0 \subseteq \bigcup_{k=1}^{N} J_k$. Therefore, we can obtain

$$|R| = \sum_{j \in J_0} |\tilde{R}_j| \leq \sum_{k=1}^{N} \sum_{j \in J_k} |\tilde{R}_j| = \sum_{k=1}^{N} |R_k|.$$

This completes the proof. □

Theorem 1.1.3

Every open set $U \subseteq \mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals.

Proof. For each $x \in U$, there exists an open interval, which contains x, contained in U. If $a_x = \inf\{a < x : (a, x) \subseteq U\}$ and $b_x = \sup\{b > x : (x, b) \subseteq U\}$, then

$$x \in I_x := (a_x, b_x) \subseteq U.$$

Hence, we get $U = \bigcup_{x \in U} I_x$. Moreover, note that I_x is the largest open interval, which contains x, contained in U.

Suppose that two intervals I_x and I_y intersect. Then we have

$$x \in I_x \cup I_y \subseteq U;$$

by the maximality of I_x, we conclude that $(I_x \cup I_y) \subseteq I_x$. By the same argument, since $(I_x \cup I_y) \subseteq I_y$, we finally have $I_x = I_y$. Therefore, any two distinct intervals in $\mathcal{C} := \{I_x\}_{x \in U}$ is disjoint.

We claim that $\mathcal{C}$ is countable, i.e., it has countably many disjoint open intervals. Since each I_x contains a rational number, any two distinct intervals must contain distinct rationals. Hence, the collection $\mathcal{C}$ must be countable. □

For $\mathbb{R}^d$ with $d \geq 2$, the direct analogue of Theorem 1.1.3 does not holds in general. However, we have a following alternative result:

Theorem 1.1.4

For any $d \geq 1$, every open set $U \subseteq \mathbb{R}^d$ can be written as a countable union of almost disjoint closed cubes.

Proof. Consider the grid generated by $\mathbb{Z}^d$. If a unit cube C in the grid is entirely contained in U, then we collect such C in a collection $\mathcal{C}$. If C intersects both U and U^c, then tentatively accept it; if C entirely contained in U^c, then we reject it.

As a second step, we divide the tentatively accepted cubes into 2^d cubes of side length $1/2$ and repeat the same procedure to collect the cubes entirely contained in U. This will be repeated indefinitely, and the resulting collection $\mathcal{C}$ of accepted cubes is countable and the elements are almost disjoint.

It is clear that, for any $x \in U$, there exists $N \in \mathbb{Z}^+$ such that some cube of side length 2^{-N} contains x and is contained in U. Either this cube is in $\mathcal{C}$ or it is contained in some cube in $\mathcal{C}$. Therefore, we have $U = \bigcup_{C \in \mathcal{C}} C$ which completes the proof. □

We now define a special set in $\mathbb{R}$ which is called the "Cantor set". Let $C_0 = [0, 1]$ and let C_1 be the set obtained from C_0 by removing the middle third open interval of it. Thus we have

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

On top of that, we define C_2 as the set obtained from C_1 by removing the middle third of each of its intervals. Hence

$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

Repeating this procedure indefinitely, we can obtain a sequence $\{C_n\}_{n=0}^{\infty}$ of subsets in $\mathbb{R}$. We now define the Cantor set as following:

Definition 1.1.2

The **Cantor set** $\mathcal{C}$ is defined to be

$$\mathcal{C} := \bigcap_{n=0}^{\infty} C_n.$$

The Cantor set plays a significant role in various fields, particularly in analysis. We will later consider some examples involving this set.

1.2 The Exterior Measure

This is an intermediate step towards constructing a measure that we will use throughout this course. We first define the "exterior measure" as follows.

Definition 1.2.1

For any set $E \subseteq \mathbb{R}^d$, the **exterior measure** $m_*(E)$ of E is defied to be

$$m_*(E) = \inf \sum_{j=1}^{\infty} |Q_j|$$

where the infimum is taken over all countable coverings $\{Q_j\}_{j=1}^{\infty}$ of E by closed cubes. (Note that they are not required to be disjoint.)

The above definition of $m_*(E)$ is well-defined because every $E \subseteq \mathbb{R}^d$ has at least one trivial covering $\{Q_j\}_{j=1}^{\infty}$ defined by

$$Q_j = [-j, j]^d \quad \text{for all} \quad j \in \mathbb{Z}^+,$$

and the completeness axiom of $\mathbb{R}$ implies that the infimum should exists.

The one direct observation is as following: If $R = [a_1, b_1] \times \cdots \times [a_d, b_d] \subseteq \mathbb{R}^d$, then

$$m_*(R) = \prod_{i=1}^{d} (b_i - a_i) = \mathrm{Vol}(R).$$

where $\mathrm{Vol}(R)$ means the volume of R.

There are some facts and nice properties of exterior measure as follows.

Proposition 1.2.1

For every $E \subseteq \mathbb{R}^d$, $0 \leq m_*(E) \leq \infty$.

Proof. trivial. □

Proposition 1.2.2

If E is a singleton (one point set), then $m_*(E) = 0$.

Proof. (Omitted). □

Proposition 1.2.3

The exterior measure of $\mathbb{R}^d$ is infinte, i.e., $m_*(\mathbb{R}^d) = \infty$.

Proof. Note that any covering $\{Q_j\}_{j=1}^\infty$ of $\mathbb{R}^d$ also covers any closed cube Q in $\mathbb{R}^d$ so that

$$|Q| = m_*(Q) \leq \sum_{j=1}^\infty |Q_j|.$$

Then, by taking an infimum, we have $|Q| \leq m_*(\mathbb{R}^d)$. Since Q is chosen arbitrarily, we conclude that $m_*(\mathbb{R}^d) = \infty$. □

Theorem 1.2.4 (Monotonicity)

If $E_1 \subseteq E_2$, then $m_*(E_1) \leq m_*(E_2)$.

Proof. As we did in the proof of Proposition 1.2.3, note that any covering $\{Q_j\}_{j=1}^\infty$ of E_2 also covers E_1 so that

$$m_*(E_1) \leq \sum_{j=1}^\infty |Q_j|.$$

Since $m_*(E_1)$ is constant, by taking infimum, we get $m_*(E_1) \leq m_*(E_2)$. □

Corollary 1.2.5

If $E \subseteq \mathbb{R}^d$ is bounded, then $m_*(E) < \infty$.

Proof. Since E is bounded, there exists $N > 0$ such that

$$E \subseteq [-N, N]^d.$$

By the monotonicity (Theorem 1.2.4), we conclude that

$$m_*(E) \leq m_*([-N, N]^d) = (2N)^d < \infty.$$

□

Theorem 1.2.6 (Countable subadditivity)

If $E = \bigcup_{j=1}^\infty E_j$, where the sets E_j are not necessarily disjoint, then

$$m_*(E) \leq \sum_{j=1}^\infty m_*(E_j).$$

Proof. For any $\epsilon > 0$, let $\{Q_{k,j}\}_{k=1}^{\infty}$ be a covering of E_j, for each $j \in \mathbb{Z}^+$, such that

$$\sum_{k=1}^{\infty} |Q_{k,j}| < m_*(E_j) + \frac{\epsilon}{2^j}.$$

Since we have $E \subseteq \bigcup_{j,k} Q_{k,j}$, thus

$$m_*(E) \leq \sum_{j,k} |Q_{k,j}| < \sum_{j=1}^{\infty} m_*(E_j) + \epsilon.$$

This implies that $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$, as desired. $\square$

Theorem 1.2.7

For any $E \subseteq \mathbb{R}^d$, $m_*(E) = \inf m_*(\mathcal{O})$ where the infimum is taken over all open sets containing E.

Proof. By the monotonicity, it is clear that $m_*(E) \leq \inf m_*(\mathcal{O})$. We now claim that the opposite inclusion is also true. For any $\epsilon > 0$, there exists a covering $\{Q_j\}_{j=1}^{\infty}$ of E by closed cubes such that

$$\sum_{j=1}^{\infty} |Q_j| \leq m_*(E) + \epsilon.$$

Let Q_j^0 denote a open cube containing Q_j such that

$$|Q_j^0| \leq |Q_j| + \frac{\epsilon}{2^j} \quad \text{for all} \quad j \in \mathbb{Z}^+.$$

Let $\mathcal{O} \coloneqq \bigcup_{j=1}^{\infty} Q_j^0$, then it is an open set containing E. Then, by countable subadditivity (Theorem 1.2.6), we have

$$\begin{aligned} m_*(\mathcal{O}) \leq \sum_{j=1}^{\infty} m_*(Q_j^0) &= \sum_{j=1}^{\infty} |Q_j^0| \\ &= \sum_{j=1}^{\infty} \left(|Q_j| + \frac{\epsilon}{2^j} \right) \\ &\leq m_*(E) + 2\epsilon. \end{aligned}$$

This proves that $\inf m_*(\mathcal{O}) \leq m_*(E)$; hence, they are the same. $\square$

Theorem 1.2.8

Let $E = E_1 \cup E_2$ where E_1 and E_2 are disjoint with $\operatorname{dist}(E_1, E_2) > 0$. Then

$$m_*(E) = m_*(E_1) + m_*(E_2).$$

Proof. Let $\delta > 0$ be a constant such that $0 < \delta < \text{dist}(E_1, E_2)$. For any $\epsilon > 0$, there exists a covering $\{Q_j\}_{j=1}^{\infty}$ of E by closed cubes such that

$$\sum_{j=1}^{\infty} |Q_j| \leq m_*(E) + \epsilon.$$

By dividing every cube Q_j into smaller pieces whose diameter is strictly less than δ, we can find another covering $\{\tilde{Q}_j\}_{j=1}^{\infty}$ of E such that $\text{diam}(\tilde{Q}_j) < \delta$. Since $\text{dist}(E_1, E_2) > \delta$, each $\tilde{Q}_j$ intersect at most one of E_1 and E_2.

Let J_1 and J_2 be the collection of indices j for which $\tilde{Q}_j$ intersect E_1 and E_2, respectively. Then, it is clear that $\emptyset \neq J_1 \cap J_2 \subseteq \mathbb{Z}^+$. Therefore,

$$m_*(E_1) + m_*(E_2) \leq \sum_{j \in J_1} |\tilde{Q}_j| + \sum_{j \in J_2} |\tilde{Q}_j| \leq \sum_{j=1}^{\infty} |Q_j| \leq m_*(E) + \epsilon.$$

□

Theorem 1.2.9

Let E be the countable union of almost disjoint closed cubes $\{Q_j\}_{j=1}^{\infty}$, then

$$m_*(E) = \sum_{j=1}^{\infty} |Q_j|.$$

Proof. By the monotonicity, $m_*(E) \leq \sum_{j=1}^{\infty} |Q_j|$ is trivial. Note that, for each j, there exists an open cube $\tilde{Q}_j$ such that $\overline{\tilde{Q}_j} \subseteq \text{Int}(Q_j)$ such that

$$|Q_j| \leq |\tilde{Q}_j| + \frac{\epsilon}{2^j}.$$

Fix some $N \in \mathbb{Z}^+$, then $\tilde{Q}_1, \ldots, \tilde{Q}_N$ are all disjoint with a positive distance to each other. Then, by Theorem 1.2.8, we have

$$m_*\left(\bigcup_{j=1}^{N} \tilde{Q}_j\right) = \sum_{j=1}^{N} |\tilde{Q}_j| \geq \sum_{j=1}^{N} \left(|Q_j| - \frac{\epsilon}{2^j}\right).$$

Since E contains $\bigcup_{j=1}^{N} \tilde{Q}_j$, the monotonicity gives that

$$m_*(E) \geq m_*\left(\bigcup_{j=1}^{N} \tilde{Q}_j\right) \geq \sum_{j=1}^{N} |Q_j| - \epsilon.$$

Since N is chosen arbitrarily, as $N \to \infty$, we obtain

$$\sum_{j=1}^{\infty} |Q_j| \leq m_*(E) + \epsilon.$$

Therefore, we conclude that $\sum_{j=1}^{\infty} |Q_j| \leq m_*(E)$ and this completes the proof. □

1.3 Measurable Sets and the Lebesgue Measure

Definition 1.3.1

A subset $E \subseteq \mathbb{R}^d$ is said to be **measurable** if for any $\epsilon > 0$, there exists an open set $\mathcal{O}$ containing E such that

$$m_*(\mathcal{O} - E) < \epsilon.$$

Recall that the exterior measure was defined for all subsets of $\mathbb{R}^d$ in Section 1.2. Now, we define the **Lebesgue measure**, denoted by m, as the restriction of m_* to the collection of all measurable sets.

The above definition clearly implies the following properties:

Proposition 1.3.1

(1) Every open set $E \subseteq \mathbb{R}^d$ is measurable.

(2) If $m_*(E) = 0$, then E is measurable.

(3) A countable union of measurable sets is measurable.

(4) Every closed set is measurable.

(5) The complement of a measurable set is measurable.

(6) A countable intersection of measurable sets is measurable.

Proof. (1) Taking $\mathcal{O} = E$ completes the proof.

(2) By Theorem 1.2.7, since $m_*(E) = 0$, for any $\epsilon > 0$, there exists an open set $\mathcal{O}$ containing E such that $m_*(\mathcal{O}) < \epsilon$. Since we have $\mathcal{O} - E \subseteq \mathcal{O}$, by the monotonicity,

$$m_*(\mathcal{O} - E) \leq m_*(\mathcal{O}) < \epsilon.$$

Therefore, E is measurable.

(3) Let $E = \bigcup_{j=1}^{\infty} E_j$ where each E_j is measurable. For any $\epsilon > 0$, there exists an open set $\mathcal{O}_j$ containing E_j such that

$$m_*(\mathcal{O}_j - E_j) < \frac{\epsilon}{2^j}$$

for all $j \in \mathbb{Z}^+$. Let $\mathcal{O} := \bigcup_{j=1}^{\infty} \mathcal{O}_j$, then $\mathcal{O}$ is an open set containing E. Moreover, since

$$\mathcal{O} - E \subseteq \bigcup_{j=1}^{\infty} (\mathcal{O}_j - E_j),$$

we have

$$m_*(\mathcal{O}-E) \leq \sum_{j=1}^{\infty} m_*(\mathcal{O}_j - E_j) < \sum_{j=1}^{\infty} \frac{\epsilon}{2^j} = \epsilon.$$

Therefore, E is also measurable.

(4) We first claim that any compact set $F \subseteq \mathbb{R}^d$ is measurable. Let $\mathcal{O}$ be an open set containing F such that $m_*(\mathcal{O}-F) < \epsilon$ so that

$$m_*(\mathcal{O}) = m_*\big(F \cup (\mathcal{O}-F)\big) \leq m_*(F) + \epsilon.$$

Since $\mathcal{O}-F$ is open in $\mathbb{R}^d$, by Theorem 1.1.4, it can be written as a countable union of almost disjoint closed cubes, that is,

$$\mathcal{O}-F = \bigcup_{j=1}^{\infty} Q_j$$

where $\{Q_j\}_{j=1}^{\infty}$ is a collection of almost disjoint closed cubes. For a fixed $N \in \mathbb{Z}^+$, let $K := \bigcup_{j=1}^{N} Q_j$ so that K is compact and $K \cap F = \emptyset$. Therefore, by Lemma 1.3.2, we have $\operatorname{dist}(F, K) > 0$. Hence, the Theorem 1.2.8 implies that

$$m_*(\mathcal{O}) \geq m_*(F \cup K) = m_*(F) + m_*(K) = m_*(F) + \sum_{j=1}^{N} m_*(Q_j).$$

Since we have $m_*(\mathcal{O}) \leq m_*(F) + \epsilon$, the above inequality implies that

$$\sum_{j=1}^{N} m_*(Q_j) \leq \epsilon.$$

Since N is chosen arbitrarily, we have $\sum_{j=1}^{\infty} m_*(Q_j) \leq \epsilon$; therefore,

$$m_*(\mathcal{O}-F) = \sum_{j=1}^{\infty} m_*(Q_j) \leq \epsilon$$

by Theorem 1.2.9. This proves that F is measurable.

For the general case that the closed set F is not compact, note that

$$F = \bigcup_{k=1}^{\infty} \left[F \cap [-k,k]^d\right]$$

where $F \cap [-k,k]^d$ is compact for all $k \in \mathbb{Z}^+$. Since F is a countable union of measurable sets, by part (3), it is also measurable.

(5) Note that, for any $n \in \mathbb{Z}^+$, there exists an open set $\mathcal{O}_n$ containing E such that

$$m_*(\mathcal{O}_n - E) \le \frac{1}{n}.$$

Let $S := \bigcup_{n=1}^{\infty} \mathcal{O}_n^c$, then we have $S \subseteq E^c$. Since each $\mathcal{O}_n^c$ is closed, so is S and this implies that S is measurable by part (4). Note that $E^c - S \subseteq \mathcal{O}_n - E$ holds for all n, thus

$$m_*(E^c - S) \le m_*(\mathcal{O}_n - E) \le \frac{1}{n}.$$

As $n \to \infty$, we obtain $m_*(E^c - S) = 0$, that is, $E^c - S$ is measurable by part (2). Therefore, we conclude that E^c is also measurable since it is the union of two measurable sets $E^c - S$ and S.

(6) Recall that $\bigcap_{i=1}^{\infty} E_j = \left(\bigcup_{j=1}^{\infty} E_j^c\right)^c$ and apply the part (3) and (5). □

Lemma 1.3.2

If $F \subseteq \mathbb{R}^d$ is closed, $K \subseteq \mathbb{R}^d$ is compact, and $F \cap K = \emptyset$, then $\operatorname{dist}(F, K) > 0$.

Proof. Pick any $x \in K$. Since $x \in F$ and F is closed, there exists $\delta_x > 0$ such that $B_d(x, \delta_x) \subseteq F^c$ so that $\operatorname{dist}(x, F) \ge \delta_x$. Let

$$\mathcal{C} = \left\{B_d\left(x, \delta_x/2\right) \mid x \in K\right\}$$

then $\mathcal{C}$ is an open covering of K. Since K is compact, we can find some finite subcover of $\mathcal{C}$, that is, there exists $x_1, \dots, x_N \in K$ such that $\{B_d(x_i, \delta_{x_i}/2)\}_{i=1}^N$ covers K. Let

$$\delta := \min\left\{\frac{\delta_{x_i}}{2} \;\middle|\; 1 \le i \le N\right\}.$$

Then, for any $x \in K$, there exists $i \in \{1, \dots, N\}$ such that $x \in B_d(x_i, \delta_{x_i}/2)$; hence, for any $y \in F$, we can obtain

$$d(y, x) \ge d(y, x_i) - d(x_i, x) \ge 2\delta - \delta = \delta > 0.$$

Therefore, we conclude that

$$d(F, K) = \inf_{\substack{x \in K \\ y \in F}} d(y, x) \ge \delta > 0.$$

□

Theorem 1.3.3

If $\{E_j\}_{j=1}^{\infty}$ are disjoint measurable sets, then their union is measurable and

$$m\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} m(E_j).$$

Proof. Note that the union is measurable by Proposition 1.3.1 and

$$m\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} m(E_j)$$

by countable subadditivity. We now claim that the opposite inclusion also holds.

We first prove the claim for the case when each E_j is bounded so that $m_*(E_j) < \infty$. Since each E_j is measurable, so is E_j^c. Hence, for any $\epsilon > 0$, there exists an open set $\mathcal{O}_j$ containing E_j^c such that

$$m_*(\mathcal{O}_j - E_j^c) < \frac{\epsilon}{2^j}.$$

Since $\mathcal{O}_j^c$ is a closed set contained in E_j with $m_*(E_j - \mathcal{O}_j^c) < \epsilon/2^j$, Thus

$$m_*(E_j) \leq m_*(\mathcal{O}_j^c) + m_*(E_j - \mathcal{O}_j^c) \leq m_*(\mathcal{O}_j^c) + \frac{\epsilon}{2^j}.$$

$\square$

1.4 Measurable Functions

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Solution for exercises

MAS441: Lebesgue Intergral Theory Homework Problems

Solutions for the selected exercises (To be updated)

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