LEBESGUE INTEGRAL THEORY

Measure Theory, Integration, and Hilbert spaces

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Chapter 1

Measure Theory

The goal of this course, especially in this chapter, is to provide a notion of "size" for subsets of $\mathbb{R}^d$ by constructing an appropriate function

$$m : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^{\geq 0} \cup \{\infty\}$$

satisfying the following properties:

(1) $m(\emptyset) = 0$

(2) $m([b, a]) = b - a$

(3) If $E = \bigsqcup_{n=1}^{\infty} E_n$, then $m(E) = \sum_{n=1}^{\infty} m(E_n)$.

(4) $m(E + h) = m(E)$ for all $h \in \mathbb{R}$

The third and fourth properties are called **countable additivity** and **translation invariance**, respectively. By regarding m as a function that measures the size of sets, it seems very natural that m should satisfy the above properties. However, this is just our intuition! As a math student, the natural follow-up question is as follows:

Does m exist? If it does, is m unique?

In fact, such a function m does not exist. Then, what can we do next? One way to resolve this problem is to restrict the domain of m. By removing certain pathological sets, we can define a function m which satisfies some nice properties. This domain will be called "measurable sets" and the function m will be called the "Lebesgue measure".

1.1 Preliminaries

Recall that the collection $\{E_\alpha\}_{\alpha \in \mathcal{A}}$ of subsets in $\mathbb{R}^d$ is said to be "disjoint" if $E_\alpha \cap E_\beta = \emptyset$ whenver $\alpha \neq \beta$. We define the slightly weaker concept "almost disjoint" as follows:

Definition 1.1.1

Let $\{E_\alpha\}_{\alpha\in\mathcal{A}}$ be a collection subsets in $\mathbb{R}^d$. Then $\{E_\alpha\}_{\alpha\in\mathcal{A}}$ is said to be **almost disjoint** if $\{\mathrm{Int}(E_\alpha)\}_{\alpha\in\mathcal{A}}$ is disjoint.

Note that a closed **rectangle** R in $\mathbb{R}^d$ is given by

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d],$$

where $a_j \leq b_j$ for all $1 \leq j \leq d$. The **volume** of the rectangle R is denoted by $|R|$, and is defined to be

$$|R| = (b_1 - a_1)(b_2 - a_2)\cdots(b_d - a_d).$$

Moreover, a **cube** is a rectangle satisfying $b_1 - a_1 = b_2 - a_2 = \cdots = b_d - a_d$.

Lemma 1.1.1

If a rectangle R can be written as a union of finitely many almost disjoint rectangles $R_1, \ldots, R_N$, then

$$|R| = \sum_{k=1}^{N} |R_k|.$$

Proof. Consider the grid formed by extending the sides of all rectangles $R_1, \ldots, R_N$. This gives finitely many samller rectangles $\tilde{R}_1, \ldots, \tilde{R}_M$ (they are also almost disjoint), and there exists a partition $J_1, \ldots, J_N$ of integers between 1 and M such that

$$R = \bigcup_{j=1}^{M} \tilde{R}_j \quad \text{and} \quad R_k = \bigcup_{j\in J_k} \tilde{R}_j \quad \text{for} \quad 1 \leq k \leq M.$$

For the rectangle R, it is clear that $|R| = \sum_{j=1}^{M} |\tilde{R}_j|$. By the same reason, we also have $|R_k| = \sum_{j\in J_k} |\tilde{R}_j|$. Therefore, one can obtain

$$|R| = \sum_{j=1}^{M} |\tilde{R}_j| = \sum_{k=1}^{N} \sum_{j\in J_k} |\tilde{R}_j| = \sum_{k=1}^{N} |R_k|.$$

□

Lemma 1.1.2

If $R, R_1, \ldots, R_N$ are rectangles satisfying $R \subseteq \bigcup_{k=1}^{N} R_k$, then

$$|R| \leq \sum_{k=1}^{N} |R_k|.$$

Proof. As we did in Lemma 1.1.1, we can find smaller rectangles $\tilde{R}_1, \ldots, \tilde{R}_M$. Note that there exist $J_0 \subseteq \{1, \ldots, M\}$ such that $R = \bigcup_{j \in J_0} \tilde{R}_j$. Let $R' := \bigcup_{j=1}^M \tilde{R}_j$. Again, we can find a partition $J_1, \ldots, J_N$ of $\{1, \ldots, M\}$ such that

$$R_k = \bigcup_{j \in J_k} \tilde{R}_j \quad \text{for all} \quad 1 \leq k \leq N.$$

Note that $J_0, J_1, \ldots, J_N$ need not be disjoint.

Since $R \subseteq \bigcup_{k=1}^N R_k$, we have $J_0 \subseteq \bigcup_{k=1}^N J_k$. Therefore, we can obtain

$$|R| = \sum_{j \in J_0} |\tilde{R}_j| \leq \sum_{k=1}^N \sum_{j \in J_k} |\tilde{R}_j| = \sum_{k=1}^N |R_k|.$$

This completes the proof. □

Theorem 1.1.3

Every open set $U \subseteq \mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals.

Proof. For each $x \in U$, there exists an open interval, which contains x, contained in U. If $a_x = \inf\{a < x : (a, x) \subseteq U\}$ and $b_x = \sup\{b > x : (x, b) \subseteq U\}$, then

$$x \in I_x := (a_x, b_x) \subseteq U.$$

Hence, we get $U = \bigcup_{x \in U} I_x$. Moreover, note that I_x is the largest open interval, which contains x, contained in U.

Suppose that two intervals I_x and I_y intersect. Then we have

$$x \in I_x \cup I_y \subseteq U;$$

by the maximality of I_x, we conclude that $(I_x \cup I_y) \subseteq I_x$. By the same argument, since $(I_x \cup I_y) \subseteq I_y$, we finally have $I_x = I_y$. Therefore, any two distinct intervals in $\mathcal{C} := \{I_x\}_{x \in U}$ is disjoint.

We claim that $\mathcal{C}$ is countable, i.e., it has countably many disjoint open intervals. Since each I_x contains a rational number, any two distinct intervals must contain distinct rationals. Hence, the collection $\mathcal{C}$ must be countable. □

For $\mathbb{R}^d$ with $d \geq 2$, the direct analogue of Theorem 1.1.3 does not holds in general. However, we have a following alternative result:

Theorem 1.1.4

For any $d \geq 1$, every open set $U \subseteq \mathbb{R}^d$ can be written as a countable union of almost disjoint closed cubes.

Proof. Consider the grid generated by $\mathbb{Z}^d$. If a unit cube C in the grid is entirely contained in U, then we collect such C in a collection $\mathcal{C}$. If C intersects both U and U^c, then tentatively accept it; if C entirely contained in U^c, then we reject it.

As a second step, we divide the tentatively accepted cubes into 2^d cubes of side length $1/2$ and repeat the same procedure to collect the cubes entirely contained in U. This will be repeated indefinitely, and the resulting collection $\mathcal{C}$ of accepted cubes is countable and the elements are almost disjoint.

It is clear that, for any $x \in U$, there exists $N \in \mathbb{Z}^+$ such that some cube of side length 2^{-N} contains x and is contained in U. Either this cube is in $\mathcal{C}$ or it is contained in some cube in $\mathcal{C}$. Therefore, we have $U = \bigcup_{C \in \mathcal{C}} C$ which completes the proof. □

We now define a special set in $\mathbb{R}$ which is called the "Cantor set". Let $C_0 = [0, 1]$ and let C_1 be the set obtained from C_0 by removing the middle third open interval of it. Thus we have

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right].$$

On top of that, we define C_2 as the set obtained from C_1 by removing the middle third of each of its intervals. Hence

$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

Repeating this procedure indefinitely, we can obtain a sequence $\{C_n\}_{n=0}^{\infty}$ of subsets in $\mathbb{R}$. We now define the Cantor set as following:

Definition 1.1.2

The **Cantor set** $\mathcal{C}$ is defined to be

$$\mathcal{C} := \bigcap_{n=0}^{\infty} C_n.$$

The Cantor set plays a significant role in various fields, particularly in analysis. We will later consider some examples involving this set.

1.2 The Exterior Measure

This is an intermediate step towards constructing a measure that we will use throughout this course. We first define the "exterior measure" as follows.

Definition 1.2.1

For any set $E \subseteq \mathbb{R}^d$, the **exterior measure** $m_*(E)$ of E is defied to be

$$m_*(E) = \inf \sum_{j=1}^{\infty} |Q_j|$$

where the infimum is taken over all countable coverings $\{Q_j\}_{j=1}^{\infty}$ of E by closed cubes. (Note that they are not required to be disjoint.)

The above definition of $m_*(E)$ is well-defined because every $E \subseteq \mathbb{R}^d$ has at least one trivial covering $\{Q_j\}_{j=1}^{\infty}$ defined by

$$Q_j = [-j, j]^d \quad \text{for all} \quad j \in \mathbb{Z}^+,$$

and the completeness axiom of $\mathbb{R}$ implies that the infimum should exists.

The one direct observation is as following: If $R = [a_1, b_1] \times \cdots \times [a_d, b_d] \subseteq \mathbb{R}^d$, then

$$m_*(R) = \prod_{i=1}^{d} (b_i - a_i) = \mathrm{Vol}(R).$$

where $\mathrm{Vol}(R)$ means the volume of R.

There are some facts and nice properties of exterior measure as follows.

Proposition 1.2.1

For every $E \subseteq \mathbb{R}^d$, $0 \leq m_*(E) \leq \infty$.

Proof. trivial. □

Proposition 1.2.2

If E is a singleton (one point set), then $m_*(E) = 0$.

Proof. (Omitted). □

Proposition 1.2.3

The exterior measure of $\mathbb{R}^d$ is infinte, i.e., $m_*(\mathbb{R}^d) = \infty$.

Proof. Note that any covering $\{Q_j\}_{j=1}^\infty$ of $\mathbb{R}^d$ also covers any closed cube Q in $\mathbb{R}^d$ so that

$$|Q| = m_*(Q) \leq \sum_{j=1}^\infty |Q_j|.$$

Then, by taking an infimum, we have $|Q| \leq m_*(\mathbb{R}^d)$. Since Q is chosen arbitrarily, we conclude that $m_*(\mathbb{R}^d) = \infty$. □

Theorem 1.2.4 (Monotonicity)

If $E_1 \subseteq E_2$, then $m_*(E_1) \leq m_*(E_2)$.

Proof. As we did in the proof of Proposition 1.2.3, note that any covering $\{Q_j\}_{j=1}^\infty$ of E_2 also covers E_1 so that

$$m_*(E_1) \leq \sum_{j=1}^\infty |Q_j|.$$

Since $m_*(E_1)$ is constant, by taking infimum, we get $m_*(E_1) \leq m_*(E_2)$. □

Corollary 1.2.5

If $E \subseteq \mathbb{R}^d$ is bounded, then $m_*(E) < \infty$.

Proof. Since E is bounded, there exists $N > 0$ such that

$$E \subseteq [-N, N]^d.$$

By the monotonicity (Theorem 1.2.4), we conclude that

$$m_*(E) \leq m_*([-N, N]^d) = (2N)^d < \infty.$$

□

Theorem 1.2.6 (Countable subadditivity)

If $E = \bigcup_{j=1}^\infty E_j$, where the sets E_j are not necessarily disjoint, then

$$m_*(E) \leq \sum_{j=1}^\infty m_*(E_j).$$

Proof. For any $\epsilon > 0$, let $\{Q_{k,j}\}_{k=1}^{\infty}$ be a covering of E_j, for each $j \in \mathbb{Z}^+$, such that

$$\sum_{k=1}^{\infty} |Q_{k,j}| < m_*(E_j) + \frac{\epsilon}{2^j}.$$

Since we have $E \subseteq \bigcup_{j,k} Q_{k,j}$, thus

$$m_*(E) \leq \sum_{j,k} |Q_{k,j}| < \sum_{j=1}^{\infty} m_*(E_j) + \epsilon.$$

This implies that $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$, as desired. $\square$

Theorem 1.2.7

For any $E \subseteq \mathbb{R}^d$, $m_*(E) = \inf m_*(\mathcal{O})$ where the infimum is taken over all open sets containing E.

Proof. By the monotonicity, it is clear that $m_*(E) \leq \inf m_*(\mathcal{O})$. We now claim that the opposite inclusion is also true. For any $\epsilon > 0$, there exists a covering $\{Q_j\}_{j=1}^{\infty}$ of E by closed cubes such that

$$\sum_{j=1}^{\infty} |Q_j| \leq m_*(E) + \epsilon.$$

Let Q_j^0 denote a open cube containing Q_j such that

$$|Q_j^0| \leq |Q_j| + \frac{\epsilon}{2^j} \quad \text{for all} \quad j \in \mathbb{Z}^+.$$

Let $\mathcal{O} := \bigcup_{j=1}^{\infty} Q_j^0$, then it is an open set containing E. Then, by countable subadditivity (Theorem 1.2.6), we have

$$\begin{aligned} m_*(\mathcal{O}) \leq \sum_{j=1}^{\infty} m_*(Q_j^0) &= \sum_{j=1}^{\infty} |Q_j^0| \\ &= \sum_{j=1}^{\infty} \left(|Q_j| + \frac{\epsilon}{2^j}\right) \\ &\leq m_*(E) + 2\epsilon. \end{aligned}$$

This proves that $\inf m_*(\mathcal{O}) \leq m_*(E)$; hence, they are the same. $\square$

Theorem 1.2.8

Let $E = E_1 \cup E_2$ where E_1 and E_2 are disjoint with $\operatorname{dist}(E_1, E_2) > 0$. Then

$$m_*(E) = m_*(E_1) + m_*(E_2).$$

Proof. Let $\delta > 0$ be a constant such that $0 < \delta < \text{dist}(E_1, E_2)$. For any $\epsilon > 0$, there exists a covering $\{Q_j\}_{j=1}^{\infty}$ of E by closed cubes such that

$$\sum_{j=1}^{\infty} |Q_j| \le m_*(E) + \epsilon.$$

By dividing every cube Q_j into smaller pieces whose diameter is strictly less than δ, we can find another covering $\{\tilde{Q}_j\}_{j=1}^{\infty}$ of E such that $\text{diam}(\tilde{Q}_j) < \delta$. Since $\text{dist}(E_1, E_2) > \delta$, each $\tilde{Q}_j$ intersect at most one of E_1 and E_2.

Let J_1 and J_2 be the collection of indices j for which $\tilde{Q}_j$ intersect E_1 and E_2, respectively. Then, it is clear that $\emptyset \neq J_1 \cap J_2 \subseteq \mathbb{Z}^+$. Therefore,

$$m_*(E_1) + m_*(E_2) \le \sum_{j \in J_1} |\tilde{Q}_j| + \sum_{j \in J_2} |\tilde{Q}_j| \le \sum_{j=1}^{\infty} |Q_j| \le m_*(E) + \epsilon.$$

□

Theorem 1.2.9

Let E be the countable union of almost disjoint closed cubes $\{Q_j\}_{j=1}^{\infty}$, then

$$m_*(E) = \sum_{j=1}^{\infty} |Q_j|.$$

Proof. By the monotonicity, $m_*(E) \le \sum_{j=1}^{\infty} |Q_j|$ is trivial. Note that, for each j, there exists an open cube $\tilde{Q}_j$ such that $\overline{\tilde{Q}_j} \subseteq \text{Int}(Q_j)$ such that

$$|Q_j| \le |\tilde{Q}_j| + \frac{\epsilon}{2^j}.$$

Fix some $N \in \mathbb{Z}^+$, then $\tilde{Q}_1, \ldots, \tilde{Q}_N$ are all disjoint with a positive distance to each other. Then, by Theorem 1.2.8, we have

$$m_*\left(\bigcup_{j=1}^{N} \tilde{Q}_j\right) = \sum_{j=1}^{N} |\tilde{Q}_j| \ge \sum_{j=1}^{N} \left(|Q_j| - \frac{\epsilon}{2^j}\right).$$

Since E contains $\bigcup_{j=1}^{N} \tilde{Q}_j$, the monotonicity gives that

$$m_*(E) \ge m_*\left(\bigcup_{j=1}^{N} \tilde{Q}_j\right) \ge \sum_{j=1}^{N} |Q_j| - \epsilon.$$

Since N is chosen arbitrarily, as $N \to \infty$, we obtain

$$\sum_{j=1}^{\infty} |Q_j| \le m_*(E) + \epsilon.$$

Therefore, we conclude that $\sum_{j=1}^{\infty} |Q_j| \le m_*(E)$ and this completes the proof. □

1.3 Measurable Sets and the Lebesgue Measure

1.4 Measurable Functions

Chapter 2

Integration Theory

2.1 The Lebesgue Integral: Basic Properties and Convergence Theorems

2.2 The Space L^1 of Integrable Functions

2.3 Fubini's theorem

2.4 A Fourier Inversion Formula

Chapter 3

Differentiation and Integration

3.1 Differentiation of the Integral

3.2 Good Kernels and Approximations to the Identity

3.3 Differentiability of Functions

3.4 Rectifiable Curves and the Isoperimetric Inequality

Chapter 4

Hilbert Spaces: An Introduction

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4.2 Hilbert Space

4.3 Fourier Series and Fatou's Theorem

4.4 Closed Subpaces and Orthogonal Projections

4.5 Linear Transformations

4.6 Compact Operators

Chapter 5

Solution for exercises

MAS441: Lebesgue Intergral Theory Homework Problems

Solutions for the selected exercises (To be updated)

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