Final Summary with THMS + Insight from Midterms

This is just guidline of THMS and Insight that I need for final.
More specific terms, information and methods will be ode.pdf.

IMPORTANT!

Cite all the theorem whenever I use it.
Be specific and check all the conditions correctly.
Understand concept profoundly.
Understand question correctly.

Theorem

If A is continuos on I, then theer are two linearly independent solutions X_1 and X_2.
Then the set X_1, X_2 is called **fundamental set of solutions** to $X' = A(t)X$ on I.
General Sol'n will be $X(t) = c_1X_1(t) + c_2X_2(t)$

IMPORTANT!

Test for L.I
This can be done by using Wronskian.
Two vectors solutions X_1 and X_2 are linearly independent if and only if $W(X_1, X_2)(t) \neq 0$ for all $t \in I$

- Deducing the autonomou or not autonomous ODE
- Deducing linear or non-linear ODE
- Deducing Homogeneous or non-homogeneous ODE
- Deduing separable or not separable ODE

This seems hard to do, but it is not.
Following is the step-by-step process to deduce the ODE.

Definition

Autonomouse ODE is an ODE that does not depend on the independent variable explicitly.
Generally, look at if t is in the equation. If t is not in the equation, then it is autonomouse ODE.

- Example: $\frac{dy}{dt} = 2y$
- Example: $\frac{dy}{dt} = y^2$

Definition

Non-autonomouse ODE is an ODE that depend on the independent variable explicitly.
Generally, look at if t is in the equation. If t is in the equation, then it is non-autonomouse ODE.

For dealing with autonomous ODE, we can use the following method.

IMPORTANT!

As $y' = f(y)$ doesn't depend on t, the **slope of direction field for autonomous ODE only vary in vertical direction.**
Therefore, the **slope of each horizontal line is constant as** $y = c$ where c is constant.

Or in shorten, as time goes, if horizontal slope doesn't change, then it is autonomous ODE. (as it doesn't depend on time)

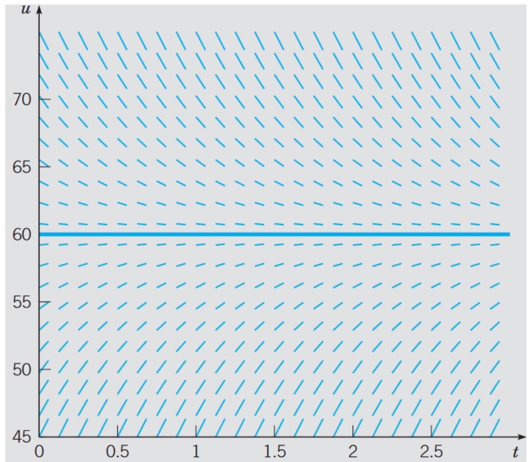


Figure 1: Autonomous ODE

As you can see from figure 1, the slope of each horizontal line is constant, and vertical line is not constant. Therefore, this is autonomous ODE.

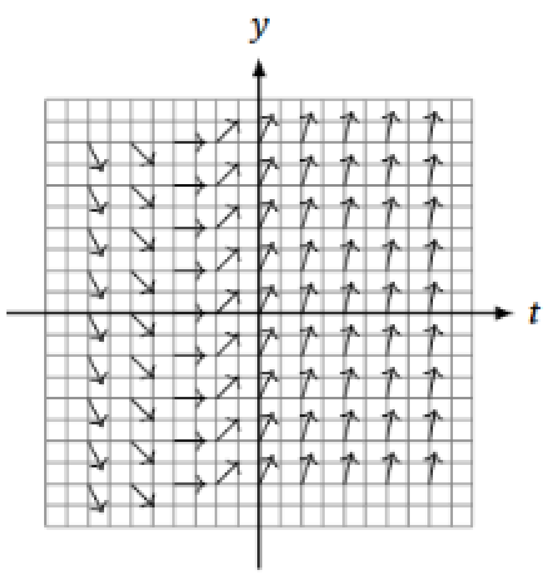


Figure 2: Non-Autonomous ODE

As you can see from figure 2, the slope of each horizontal line is not constant, and vertical line is constant. Therefore, this is non-autonomous ODE.

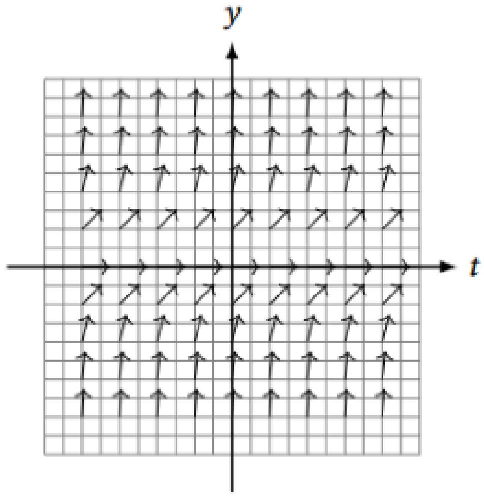


Figure 3: Homogeneous and autonomous ODE

As you can see from figure 3, the slope of each horizontal is constant, this is autonomous ODE.
Also, at y = 0, slope is 0, therefore, this is homogeneous.

Definition

Linear ODE is the equation that can be reduced to the form $Ly = f$ where L is linear operator.
There are few examples with explanation.

- Example: $y'' + y = 0$ is linear
- Example: $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} = sinx$ is linear. The reason behind this is dependent variable and its derivatives are linear.
- Example: $y' + y^2 = 0$ is non-linear. The reason behind this is dependent variable and its derivatives are not linear. (which is y^2)
- Example: $yy'' + y = 0$ is non-linear. The reason behind this is dependent variable and its derivatives are not linear. (which is yy'')
- Example: $xy'' + y = 0$ is linear, as x is not dependent variable.
- Example: $\frac{d^2y}{dx^2} + sin(y) = 0$ is non - linear, as $siny$ is not linear.

Example

During midterm, there was few equations that are linear or non-linear.

- Example: $x'(t) + expt^2 x(t) = sin(t^3)$. This is linear, as $x'(t)$ and $x(t)$ are linear.
- Example: $x'(t) = x^2 sin(t) + exptx(t)$. This is non-linear, as $x^2 sin(t)$ and $exptx(t)$ are not linear.
- Example: $\frac{dx_1}{dt} = -x_1 + 2x_2 + t$. This is linear, as x_1 and x_2 are linear.
- Example: $\frac{dx_2}{dt} = 3x_1 - x_2 + t^2$. This is linear, as x_1 and x_2 are linear.

IMPORTANT!

In first ODE, if there is **monotonic is shown in direction fields, then it is linear.** (also constant) through column.
If in the direction field, there is no monotonic, then it is non-linear.

IN SHORTEN, if the direction field doesn't depend on y, then it is linear.

By doing that we can deduce the linear or non-linear ODE from 2 and 3.
In figure 2, we can see that slope is constant through out the column, therefore, this is linear.
In figure 3, we can see that slope is increasing and then decreasing thorughout the column, therefore, this is non-linear.

Definition

The homogenous ODE is when $A\frac{du^2}{dx^2} + B\frac{du}{dx} + Fu = G(x)$ where $G(x) = 0$

The non-homogenous ODE is when $A\frac{du^2}{dx^2} + B\frac{du}{dx} + Fu = G(x)$ where $G(x) \neq 0$

IMPORTANT!

To deduce the homogenous ODE, we can see the directoin field.
Based on the what we have defined homogenous, if $G(x) = 0$, then it is homogenous.
Therefore, if it is autonomous, then it is homogenous.
However, if it is non-autonomous, then it is non-homogenous.

This can be shown in figure 2 and 3.

Definition

if $\frac{\partial A}{\partial y} = \frac{\partial B}{\partial t}$, then the DE is **exact**.
Exact tells that the existence of a function f s.t. $\frac{\partial f(t,y)}{\partial t} = A(t,y)$ and $\frac{\partial f(t,y)}{\partial y} = B(t,y)$

- Integrate the first one, we get $f(t,y) = \int A(t,y)dt + a(y)$ where $a(y)$ is arbitary function of y which is independent
- This is similar to calculate the potential function in vector field which can be solved easily.

IMPORTANT!

- If we use integrating factor, we can say that is also exact.
- ANY SEPERABLE ODE IS EXACT, **BUT EXACT ODE IS NOT NECESSARILY SEPERABLE**
- This is trival (Implicit function theorem), therefore, we can say as f(t,y) = C, this is implicit solution.

Example

Exact tells that the existence of a function f s.t. $\frac{\partial f(t,y)}{\partial t} = A(t,y)$ and $\frac{\partial f(t,y)}{\partial y} = B(t,y)$
Integrating $A(t,y)$, the it gives the $f(t,y) = \int A(t,y)dt + a(y)$
Then, $a'(y) = B(t,y) - \frac{\partial}{\partial y}(\int A(t,y)dt)$
As $f(t,y) = \int A(t,y)dt + a(y)$
This gives, $f(t,y) = \int A(t,y)dt + \int B(t,y) - (\int A(t,y)dt)$

Theorem

There are 4 types of behaviors of sol'n curves near e.q points.

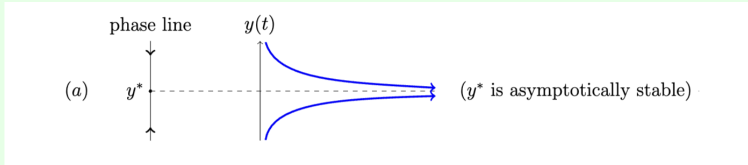


Figure 4: y^* is Asymptotically Stable / Attractor

- Figure 6 is when $\lim_{t\to\infty} y(t) = y^*$

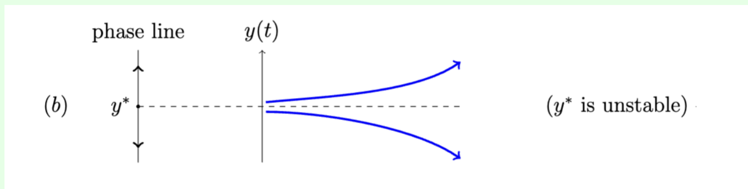


Figure 5: y^* is Unstable / Repeller

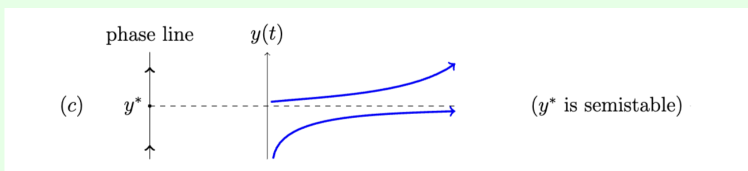


Figure 6: y^* is Semi-Stable

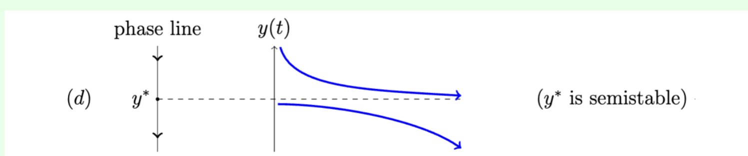


Figure 7: y^* is Semi-Stable

IMPORTANT!

Figure 6 and 7 shows both attractor and repeller at the same time.

Theorem

Let y^* be e.q point of autonomous ODE $y' = f(y)$ and assume that f has a continuous first order derivative in neighbourhood of y^*.
Then,

- If $f'(y^*) < 0$, then y^* is asymptotically stable.
- If $f'(y^*) > 0$, then y^* is unstable.

Example

This can be applied to disease example.
As $f(y) = ky(N+1-y)$, $f'(y) = k(N+1-2y)$
When $y^* = 0, f'(y^*) = k(N+1) > 0$, therefore, y^* is unstable.
When $y^* = N+1, f'(y^*) = -k(N+1) < 0$, therefore, y^* is asymptotically stable.

This is corresponding to the phase portrait.

IMPORTANT!

Sometimes, you need to check the two different interval to see the behaviour of the sol'n curve before drawing the phase portrait.
Example can be found for $y' = f(y) = (y-1)^2$ which can be found in lec note 5.

IMPORTANT!

Solution curve has no discontinuity.

- $\frac{dy}{dt} = f(t,y)$ in interval I of R
- Then y is differentiable, therefore, continuous.

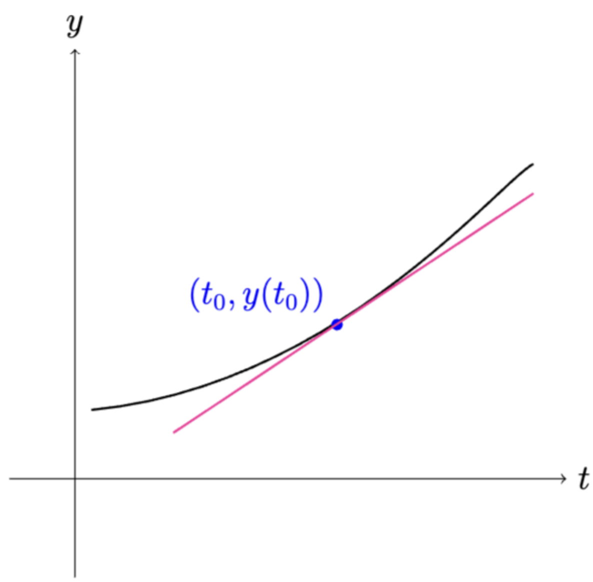


Figure 8: Non-autonomous ODE sol'n curve

Definition

Direction field of $\frac{dy}{dt} = f(t,y)$ is the collection of all tangent lines in rectangular grid of the plane, passing at each point (t,y) of the grid with slope $f(t,y)$.

Step-By-Step

- **Process of modeling**
- Identify the variables (dependent, independent) and the parameters likely to change the phenomenon (physical, sociological,economic,...) we wish to describe.
- Make reasonable assumptions and hypotheses which are as close as possible to the observations (use empirical laws). In many situations, we want to understand the rate of change of the principal dependent variable with respect to one or more independent variables. This involves derivatives and leads to differential equations.
- Solve the resulting equation (analytically, graphically) and verify that the properties of solutions agree with the model predictions.

- Solving $y' = f(t, y)$ may lead to one parameter family of solutions, then what happen when we have initial condition?
- Following is the geometrical interpretation of IVP.

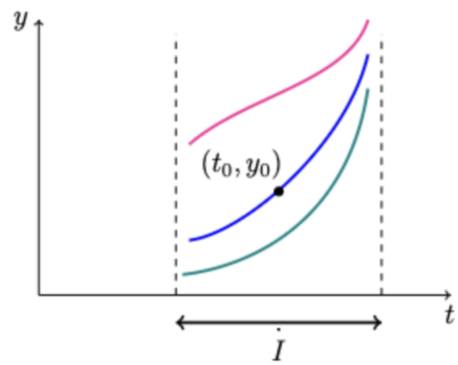


Figure 9: Geometrical INterpretation of IVP

Theorem

Non-Linear Existance and Uniquness THM
Let R be rectangular in the $t - y$ plane defined $t \in (\alpha, \beta)$ and $y \in (\mu, \gamma)$ and containing the point (t_0, y_0). If $f(t, y)$ **and** $\frac{\partial f}{\partial y}$ **are continuous** in R, then there exists $h > 0$ such that $I_0 = (t_0 - h, t_0 + h)$ is contained in (α, β) and there exists a unique solution $y(t)$ of the IVP $y' = f(t, y)$, $y(t_0) = y_0$ for $t_0 \in I$ on I_0.

IMPORTANT!

- This is for general case, including, the non-linear case.
- Two sol'ns cannot intersect each other.
- **Even though this theorem fails, we don't know if there is unique, infinite, or no sol'n.**
- A.K.A **Sufficent but not necessary condition**

Theorem

Linear Existance and Uniquness THM
$f(t, y) = a(t)y(t) + b(t)$ If $I \in (\alpha, \beta)$ is that a and b are continuous on I, then there exists a unique solution $y(t)$ of the IVP $y' = f(t, y)$, $y(t_0) = y_0$ for $t_0 \in I$ on I_0.

IMPORTANT!

- This is for linear case.
- Two sol'ns cannot intersect each other.
- **Even though this theorem fails, we don't know if there is unique, infinite, or no sol'n.**
- A.K.A **Sufficent but not necessary condition**

Now, this is the stuff that after midterm.
In here, we will cover Second Order Non-Homogeneous ODE and Laplace Transform.

Example

Method of Undetermined Coefficients

$a(t)y'' + b(t)y' + c(t)y = f(t)$
Assume y_1 and y_2 are two sol'ns to equation above.
Then, $a(t)y'' + b(t)y' + c(t)y = 0$

Theorem

THe general solution to the *non-homogeneous* equation is given by
$y(t) = y_h(t) + y_p(t)$
y_h is general solution to the *homogeneous* equation
y_p is particular solution to the *non-homogeneous* equation.

Example

$f(t)$	A	$e^{\alpha t}$	$a \sin \beta t$ or $b \cos \beta t$	n^{th} degree polynomial
$y_p(t)$	B	$Ce^{\alpha t}$	$D \sin \beta t + E \cos \beta t$	$A_n t^n + A_{n-1} t^{n-1} + \cdots + A_0$

Figure 10: Common Guess of particular solution

where $a, b, \alpha, \beta, A, B, C, D, E, A_0, A_1, ..., A_n$ are all constants.

Example

if $f(t)$ is the product of n^th degree polynomial and e^{at}
Then, we can guess $y_p(t) = (A_n t^n + A_{n-1} t^{n-1} + ... + A_1 t + A_0)e^{at}$

If $f(t)$ is the product of n^th degree polynomial, $e^{at}, sin\beta(t), cos\beta(t)$
Then, the guess will be formed as $y_p(t) = (A_n t^n + A_{n-1} t^{n-1} + ... + A_1 t + A_0)e^{at} cos\beta(t) + (B_n t^n + B_{n-1} t^{n-1} + ... + B_0)e^{at} sin\beta(t)$

IMPORTANT!

For example, there is $4y'' + y = e^{-t} cos2t + t^3 sin2t$
One can use superposition principle to solve this.
$y_{1p} = (Acos2t + Bsin2t)e^{-t}$
and $y_{2p} = (Ct^3 + Dt^2 + Et + F)sin2t + (Gt^3 + Ht^2 + It + J)cos2t$
y_{1p} will be solution to $4y'' + y = e^{-t} cos2t$
and y_{2p} will be solution to $4y'' + y = t^3 sin2t$

Step-By-Step

- Find the general solution y_h of the corresponding homogeneous equation. (complementary function)
- Use the form of non-homo term f to guess a particular solution y_p to equation.
- Substitute the educated guess into the equation for making particular sol'n form.
- This is significant to do derivitaves of that educated guess equation until u get order of what u desired in general equation.
- Then, match the educated guess with the first equation, then make that equal to non-homogenous euqation part (which will be located at RHS).
- Solve for unknowns, and plug that back into particular sol'n form that we found before.
- Write the gernal solution as $y = y_h + y_p$

IMPORTANT!

- if r = 2,3 then general sol'n will be $y(t) = c_1e^{r_1t} + c_2e^{r_2t}$
- if r is repeated, then the general sol'n will be $y(t) = c_1e^{rt} + c_2e^{rt}$
- if $r = \alpha \pm \beta i$, then the general sol'n will be $y(t) = e^{\alpha t}(C_1cos(\beta t) + C_2sin(\beta t))$
- if $r^n = 0$, then the general sol'n will be (as this is repeated eigenvalue), $y(t) = C_1 + C_2t + ... + C_nt^{n-1}$

Example

Three distinct cases to consider in here for damping case.
Let $\omega^2 = \frac{k}{m}$, $\lambda = \frac{2\gamma}{m}$.

- When $\lambda^2 - \omega^2 > 0$, then this is overdamped case.
  Therefore, $x(t) = e^{-\lambda t}(\alpha_1 e^{\sqrt[2]{\lambda^2-\omega^2 t}} + \alpha_2 e^{\sqrt[2]{\lambda^2-\omega^2 t}})$
- When $\lambda^2 - \omega^2 = 0$, then this is critically damped case.
  Therefoer, this is simliar to PHY 293 course, which will be $x(t) = e^{-\lambda t}(\alpha_1 + \alpha_2 t)$
  Mass will pass through e.q point only once.
- When $\lambda^2 - \omega^2 < 0$, then this is underdamped case.
  Therefore, this will be $x(t) = e^{-\lambda t}(\alpha_1 cos(\sqrt[2]{\omega^2 - \lambda^2}t) + \alpha_2 sin(\sqrt[2]{\omega^2 - \lambda^2}t))$

Theorem

Variation of Parameters

A *particular solution* to system like $X' = A(t)X + F(t)$ where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $F = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$ is given by the formula $X_p(t) = \psi(t) \int \psi^{-1}(t)F(t)dt$
where $\psi(t)$ is the fundamental matrix of the corresponding $X' = A(t)X$
In addition, if the column vectors of this matrix are $X_1(t)$ and $X_2(t)$, then
the general solution will be $X(t) = c_1X_1(t) + c_2X_2(t) + X_p(t)$

There is another theorem.

Theorem

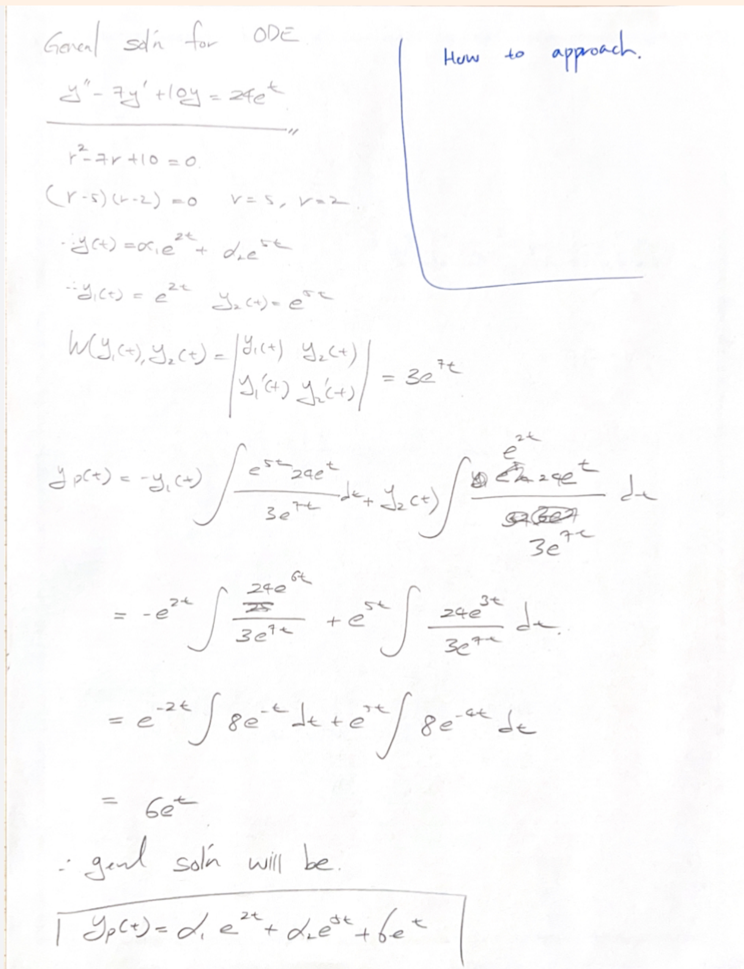
If y_1, y_2 is a fundmental set of solution to the homogenous ODE $y'' + p(t)y' + q(t)y = 0$
Then, the particular solution to the non-homogenous ODE $y'' + p(t)y' + q(t)y = f(t)$ is given by
$y_p(t) = -y_1(t) \int \frac{y_2(t)f(t)}{W(y_1,y_2)}dt + y_2(t) \int \frac{y_1(t)f(t)}{W(y_1,y_2)}dt$
where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2
Then, the general solution will be $y(t) = c_1y_1(t) + c_2y_2(t) + y_p(t)$

In this case Wronskian is

Definition

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t)$$

Example



General sol'n for ODE

$y'' - 7y' + 10y = 24e^t$

$r^2 - 7r + 10 = 0$

$(r-5)(r-2) = 0 \quad r = 5,\ r = 2$

$\therefore y(t) = \alpha_1 e^{2t} + \alpha_2 e^{5t}$

$\therefore y_1(t) = e^{2t} \quad y_2(t) = e^{5t}$

$$W(y_1(t), y_2(t)) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = 3e^{7t}$$

$$y_p(t) = -y_1(t)\int \frac{e^{5t}\,24e^t}{3e^{7t}}\,dt + y_2(t)\int \frac{e^{2t}\,24e^t}{3e^{7t}}\,dt$$

$$= -e^{2t}\int \frac{24e^{6t}}{3e^{7t}} + e^{5t}\int \frac{24e^{3t}}{3e^{7t}}\,dt$$

$$= e^{-2t}\int 8e^{-t}\,dt + e^{7t}\int 8e^{-4t}\,dt$$

$$= 6e^t$$

$\therefore$ gen'l sol'n will be

$$\boxed{y_p(t) = \alpha_1 e^{2t} + \alpha_2 e^{5t} + 6e^t}$$

How to approach.

Figure 11: Variation of Parameters

IMPORTANT!

The approach itself is same. This will help you to find particula solution for non-homogenous ODE.
Then you can just add up the complementary solution of homogenous ODE and particular solution of non-homogenous ODE.
Then you will get the general solution.

Definition

Laplace Transform
$L(f(t)) = \int_0^\infty e^{-st} f(t)dt$
If this converges, then it is called Laplace Transform.
* Will only converge for certain values of s.
** f(t) usually represents input(external force) as time starts from 0.
*** This is always important to have condition denoted for s to converge.

Example

For convention,
$L(f(t)) = F(s)$
$L(g(t)) = G(s)$
$L(h(t)) = H(s)$

Definition

$Lf(t)$ is linearity. Therefore, if $L(f(t)) = F(s)$ and $L(g(t)) = G(s)$ and α, β are constants, and $s > a$, $s > a'$, then

$L(\alpha f(t) + \beta g(t)) = \alpha F(s) + \beta G(s)$ for $s > max(a, a')$

Theorem

Comparing
Let a, A be positive constants and f be piecewise continuous on $[a, \infty]$
s.t. $|f(t)| \leq g(t)$ for every $t \geq A$
If $\int_A^\infty g(t)dt$ exists then $\int_a^\infty f(t)dt$ also converges.

Theorem

Exponential Order
A function f is said to be of exponential order if there exist c in real number, and $M > 0$ and $T \geq 0$ such that $|f(t)| \leq Me^{ct}$ for all $t \geq T$

This just mean f(t) doesnt grow faster than exponential function.

Theorem

If the function f is *piecewise continuous* on $[0, \infty)$ and of exponential order, then the Laplace transform $L(f(t))$ exists for all $s > c$

IMPORTANT!

This is important that those two condition is only **sufficient** condition.
For example, $f(t) = \frac{1}{\sqrt{t}}$ is not piecewise continuous, however, one can still do laplace transform.

Theorem

Let f be a *piecewise continous fuction on* $[0, \infty)$ and of exponential order.
Then $lim_{s \to \infty} L(f(t)) = 0$

This is the asymptotic behaviour of Laplace transform.
This theorem can be checked to see if the given function F in the s-domain is the Laplace transformation of a function f which is peicewise continuous and of exponential order.
In shorten, this is only for checking if the f is piecewise continuous and of exponential order.
important: As I menteiond, that this is only sufficient condition (piecewise continuous and of exponential order)

Example

There is no f, piecewise continous and of exponential order s.t. $L(f(t)) = \frac{3s}{2s+1}$
Since the slmit doesnt approach to 0 as s goes to infinity.

Theorem

Assume that f and all its derivitives of order up to n-1 are continuou on $[0, \infty)$ and of exponential order.
If $f^{(n)}$ is piecewise continuous and of exponential order, then
$L(f^{(n)}(t)) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - ... - f^{(n-1)}(0)$

This THM is useful since the ODE
$y'' + ay' + by = f(t)$ can be reduced to $s^2 Y(s) - sy(0) - y'(0) + a(sY(s) - y(0)) + bY(s) = F(s)$

Theorem

If $L(f(t)) = F(s)$ exist for $s > 0$ and a is real positive constant, then $L(e^{at} f(t)) = F(s-a)$, $s > a$

This is literally saying that translation by in s-domain is equal to multiplication by e^{at} in t-domain.

Theorem

If $F(s) = L(f(t))$ and n = 1,2,3,..., then $L(t^n f(t)) = (-1)^n F^{(n)}(s)$ or $(-1)^n \frac{d^n}{ds^n} F(s)$

Example

As $L1 = \frac{1}{s}$, we find that $L(t^n) = \frac{n!}{s^{n+1}}$

There should be conditions that need to ensure that the inverse Laplace transform is *unique*.

Theorem

Assume f is piecewise continuos and of exponential order on $[0, \infty)$ and $L(f(t)) = F(s)$.
We say that f(t) is the inverse Laplace of F(s) then
$L^{-1}(F(s)) = f(t)$

Example

Just wrote down each e.gs from my cheat sheet.

Example

IVP with Laplace Transform

Example

Heaviside function is defined as $U(t-a) = \begin{cases} 0 & t < a \\ 1 & t > a \end{cases}$

This usefule for constructing many other discontinous functions.

Theorem

Translation in t-domain Let F(s) be the Laplace transform of f(t) and a >0. Then $L(U(t-a)f(t-a)) = e^{-as}F(s)$

Theorem

It suffices to integrate over only one period to find the laplace transform of periodic function.

Assume f is a piecewise continuous function on $[0, \infty)$ and of exponential order and periodic with period T¿ 0

Then $L(f(t)) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t) dt$

Example

In here we will look to see the certain input singal (i.e. f).

There is *dirac delta function* which is defined as

$\delta(t-a) = \begin{cases} 0 & t \neq a \\ \infty & t = a \end{cases}$

and $\int_0^\infty \delta(t) dt = 1$

Example

its laplce makes sens in s-domain and can be evaluated.

$L(\delta_n(t-t_0)) = \frac{e^{ns}-e^{-ns}}{2nse^{st_0}}$

ALso, one may use this $\delta_n(t-t_0) = \frac{1}{2n}[U(t-(t_0-n)) - U(t-(t_0+n))]$

Theorem

Let $t_0 > 0$. One has $L(\delta(t-t_0)) = e^{-st_0}$

This shows that $L(\delta(t)) = 1$

As $L(\delta(t)) - > 0 ass - > \infty$, this is not asymptotic behaviour.

Definition

Given two functions f and g continous on $[0, \infty)$ and of exponential order, the convolution of f and g which is $f * g$is defined as

$f * g = \int_0^t f(t-\tau) g(\tau) d\tau$

Some properties are

Example

- $f * g = g * f$
- $f * (g * h) = (f * g) * h$
- $f * (g + h) = f * g + f * h$

Theorem

Assume f and g are continous on $[0, \infty)$ and of exponential order.
let $F(s) = L(f(t))$ and $G(s) = L(g(t))$
Then $L(f * g) = F(s)G(s)$
In other words, the laplace transform of convolution of two function is the product of their corresponding laplace transform.
When g = 1, then $L(\int_0^t f(t - \tau)d\tau) = \frac{1}{s}F(s)$ for $s > 0$

If this is piece-wise continuous and of exponential order,
then inverse laplace transform of $L^{-1}(F(s)G(s)) = f * g$