Lesson M4.2: Closed Forms

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Learning Goals

Textbook References

Grimaldi: Relevant parts of section 9.2

Keller and Trotter: 8.1-8.3

Levin: 5.1; 2.1 and 2.2 may also be a helpful refresher on sequences.

Lesson

In our first lesson on generating functions, we learned that we can use a power series $\sum_{i=0}^{\infty} a_i x^i$ as a convenient indexed container for storing a sequence $(a_i)_{i \geq 0}$. Today we want to remember a fun fact from Calculus II: some power series converge nicely to a closed-form function, and these are very often nicer to work with.

Note: Since we are using power series strictly as a storage facility for the sequences we care about (and will not ever evaluate them at specific x values), we do not need to worry about where they converge- the fact that they do anywhere is good enough for us.

Basic Functions:

Most of the generating functions we'll deal with this semester can be obtained from three basic "building block," functions- we'll look at those first, then move on to the various ways we can manipulate them to get other functions.

Function 1: We already know from our earlier studies that Binomial Coefficients work really nicely as coefficients (hence the name)- and we even have an expression that fits the format we are looking for:

$$(1+x)^n = \sum_{i=0}^{\infty} \binom{n}{i} x^i.$$

We may be used to seeing the summation on the right side terminate at n, but since all larger index terms are equal to 0, we can extend the sum with no consequences. We recognize this summation as one of the form we used to define a generating function, and therefore determine that the closed form function $(1+x)^n$ generates the sequence $\left(\binom{n}{i}\right)_{i\geq 0} = \left(\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}, 0, 0, \ldots\right)$.

<u>Function 2:</u> We learned in Calculus about the convergence of geometric series: in particular, we know that

$$\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i.$$

That is, $f(x) = \frac{1}{1-x}$ is a closed form function whose power series representation has a coefficient of 1 for every power of x- so it generates the sequence $(1, 1, 1, 1, 1, 1, \ldots)$.

<u>Function 3:</u> Similarly, we know that $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!} = \sum_{i=0}^{\infty} \frac{1}{i!} x^i$, so e^x is the closed form expression for the power series whose coefficients are $\frac{1}{i!}$- or in other words, e^x generates the sequence $\left(\frac{1}{i!}\right)_{i\geq 0}$.

Before we move on, here is a chart to summarize these generating functions: you will need to know these correspondences.

Function	Generates
$(1+x)^n = \sum_{i=0}^{\infty} \binom{n}{i} x^i$	$\left(\binom{n}{i}\right)_{i\geq 0}$
$\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$	$(1, 1, 1, 1, 1, \ldots)$
$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$	$\left(\frac{1}{i!}\right)_{i\geq 0}$

Manipulation Techniques:

Almost all of the techniques we will discuss are consequences of algebraic manipulations that can be done to power series. We may not need to be proficient in doing all of the detailed derivations carefully, but this perspective can help us to understand where the rules come from.

1. Multiply by a Constant
   To multiply every term of a sequence by a constant c, we can use the fact that factoring still works nicely in power series.
   That is, if we know that $f(x)$ generates the sequence $(a_i)_{i\geq 0}$ and we'd like a generating function for $(c \cdot a_i)_{i\geq 0}$, we just multiply:

$$\sum_{i=0}^{\infty} ca_i x^i = c\sum_{i=0}^{\infty} a_i x^i = c \cdot f(x).$$

Problem: Find the ordinary generating function for the sequence $(2, 2, 2, 2, \ldots)$.

$$\sum_{i=0}^{\infty} 2x^n = 2\sum_{i=0}^{\infty} x^n = \frac{2}{1-x} = 2 \cdot \frac{1}{1-x}.$$

So we can derive the generating function $\frac{2}{1-x}$ from the definition, or can recognize our sequence as $2(1, 1, 1, 1, \ldots)$ and multiply the function we already know by 2 to get the same answer.

2. Shift by an index
   To keep the same list of numbers, but shift them by one index to the right, we can multiply through by x to shift the alignment of coefficients with powers of x.

   That is, if $(a_i)_{i\geq 0} = (a_0, a_1, a_2, \ldots)$ is the sequence generated by $f(x) = \sum_{i=0}^{\infty} a_i x^i$, then the function $xf(x) = x\sum_{i=0}^{\infty} a_i x^i = \sum_{i=0}^{\infty} a_i x^{i+1} = a_0 x + a_1 x^2 + a_2 x^3 + \cdots$ generates the sequence $(0, a_0, a_1, a_2, \ldots)$.

Problem: Find the function that generates the sequence $(0, 0, 1, 1, \frac{1}{2!}, \frac{1}{3!}, \ldots)$.

We recognize that this sequence has denominators with factorials (so we might want to start with e^x), but we have shifted to get two 0s at the beginning. The sequence we want can be generated by the function $\sum_{i=0}^{\infty} \frac{x^{i+2}}{i!} = x^2 e^x$.

3. Multiply by the index

Suppose we know as a starting place that $f(x) = \sum_{i=0}^{\infty} a_i x^i$ generates $(a_i)_{i \geq 0}$, and we want to find the OGF for $(ia_i)_{i \geq 0}$.

The trick to this manipulation is to remember that differentiation has the effect of multiplying x^i by i: we proceed then by differentiating both sides of the equality we already have.

$$f(x) = \sum_{i=0}^{\infty} a_i x^i$$
$$f'(x) = \sum_{i=0}^{\infty} ia_i x^{i-1}$$

This gives us the coefficients ia_i that we wanted, but they don't line up with the correct index anymore (i.e., we want ia_i in index i of our sequence, but right now it's in spot $i - 1$. We can fix this though- all we have to do is multiply by x to realign. Thus $xf_a'(x) = \sum_{i=0}^{\infty} ia_i x^i$ is the generating function we wanted.

Problem: Find the sequence generated by the OGF $\frac{2x}{(1-x)^2}$.

Hint: What is the derivative of $\frac{1}{1-x}$*?*

We know $\frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$, and differentiating both sides gives us

$$\frac{1}{(1-x)^2} = \sum_{i=0}^{\infty} ix^{i-1}.$$

This is almost the function we have, but we need to multiply by $2x$:

$$\frac{2x}{(1-x)^2} = \sum_{i=0}^{\infty} 2ix^i.$$

Extracting the coefficients, we now see that the sequence generated is $(2i)_{i\geq 0} = (0, 2, 4, 6, 8, \ldots)$.

4. Substitution
   As with any other algebraic expression, we can replace x with any other variable quantity in any generating function. Most basically, this has the effect of scaling coefficients or exponents.
   For example:
   (a) $\frac{1}{1+4x} = \sum_{i=0}^{\infty}(-4x)^i = \sum_{i=0}^{\infty}(-4)^i x^i$, which generates the sequence $(1, -4, 4^2, -4^3, \ldots)$.
   (b) $e^{x^2} = \sum_{i=0}^{\infty} \frac{(x^2)^i}{i!} = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots$, which generates the sequence $(1, 0, \frac{1}{1!}, 0, \frac{1}{2!}, 0, \frac{1}{3!}, 0, \ldots)$.

Problem: Find the sequence generated by the OGF $\frac{1}{1-x^2}$.

We recognize this function as our standard geometric series, with x^2 substituted for x. Thus we know $\frac{1}{1-x^2} = \sum_{i=0}^{\infty} (x^2)^i = \sum_{i=0}^{\infty} x^{2i} = 1 + x^2 + x^4 + x^6 + x^8 + \cdots$.

Reading off the coefficients, then, we see that the sequence generated is $(1, 0, 1, 0, 1, 0, \ldots)$.

Don't worry if this feels like a lot right now- using all of these tools takes lots of practice!