Lesson M4.5: Ferrers diagrams and Calculating $p_n(k)$

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Learning Goals

- Use generating functions to model the number of integer partitions satisfying a set of conditions

Textbook References

Grimaldi: 9.3

Keller and Trotter: 8.5

Levin: Not Covered

Lesson

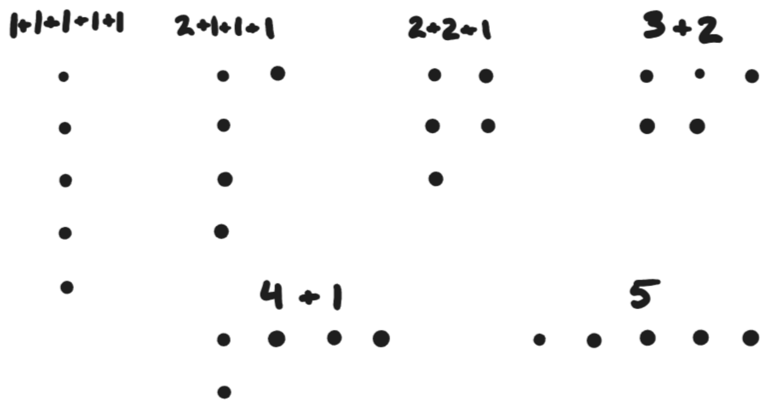
We learned in the previous lesson that $f(x) = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$ is the ordinary generating function for the sequence $(p(k))_{k\geq 0}$.

While it's great to be able to find the total number of partitions of k, it's also important to be able to count particular subsets of those partitions. In particular, since it's relevant to actually finding the values in our twelvefold way table, we want to know how to calculate $p_n(k)$ and $p_{\leq n}(k)$.

In order to better understand the manipulations we do to calculate these quantities, we define another way to visually represent partitions:

Definition: The *Ferrers diagram* of a partition of k displays each summand of size i as a row of i dots, arranged so that each row contains at least as many dots as the row below it. (*This is equivalent to a convention of listing summands in order of size*). Note the total number of dots in a diagram is always equal to k.

Example: The partitions of 5, along with their associated Ferrers diagrams, are given below.



Ferrers diagrams help us to visualize partitions, but they also provide a key insight that allows us to actually find the quantities we want for our table.

Recall from the last lesson that the number of partitions of k into summands of size at most n is given by the coefficient of x^k in the OGF $\prod_{i=1}^{n} \frac{1}{1-x^i}$.

While it is conventional to think of Ferrers diagrams as displaying each summand as a row (thus the number of rows is the number of parts in a particular partition), it really would be equally reasonable to assign this role to columns instead (each diagram would just correspond to a different partition).

If we visualize swapping rows and columns, we'll end up with the same set of diagrams- and really all we've done is swapped "number of parts" (# of rows) with "size of the largest part" (# of columns). That is, we have proved combinatorially that:

Proposition: The number of partitions of k into at most n parts is equal to the number of partitions of k into parts of size $\leq n$.

Aha- this is a quantity we already know!

Result 1: So the total number of ways to place k identical balls into n identical baskets is $p_{\leq n}(k)$, which is the coefficient of x^k in the OGF $\prod_{i=1}^{n} \frac{1}{1-x^i}$.

Example: Suppose we know that

$$\prod_{i=1}^{5} \frac{1}{1-x^i} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 10x^6 + 13x^7 + 18x^8 + 23x^9 + 30x^{10} + \cdots$$

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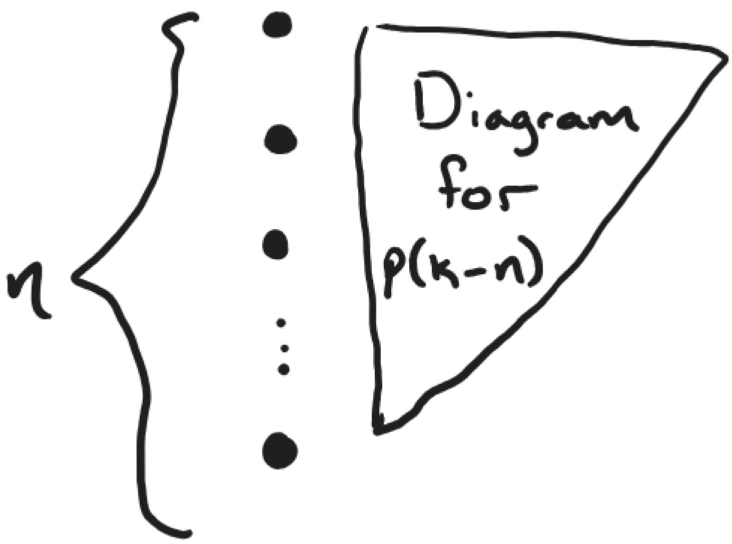
We just look for the coefficient of x^8.

We just look for the coefficient of x^{10}, thanks to our proposition.

Now all we need to do is figure out how many partitions of k consist of *exactly* n parts- it turns out Ferrers diagrams can help us here too. Suppose we want to draw a Ferrers diagram for a partition of k that consists of exactly n summands. By definition, the diagram needs to have exactly n rows- so the first column needs to consist of n dots.



Once we have placed those n dots though, we can build the rest of the diagram however we'd like, using as many or as few of the established rows as desired. That is, we can get any Ferrers diagram for a partition of k that has exactly k parts by appending a column of n dots to any Ferrers diagram representing a partition of $k - n$ into at most n parts.

Because Ferrers diagrams correspond exactly to partitions (they just represent them visually), we conclude that the number of partitions of k into exactly n parts is equal to the number of partitions of $k - n$ into at most n parts.

Result 2: Thus the number of ways to place k identical balls into n non-empty baskets is $p_n(k)$, the coefficient of x^{k-n} in in the OGF $\prod_{i=1}^{n} \frac{1}{1-x^i}$.

Example: Again, recall that

$$\prod_{i=1}^{5} \frac{1}{1-x^i} = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 10x^6 + 13x^7 + 18x^8 + 23x^9 + 30x^{10} + \cdots$$

23:

We check that the number of parts we want is equal to the number of factors we've multiplied out in our product, and both are 5 so we proceed. We then need the coefficient of $x^{14-5} = x^9$, or 23.

This calculation fails the "initial check" we did in the other part. We need to find the number of partitions of 14-6=8 into at most 6 parts, which is the number of partitions of 8 into parts of size at most 6- and this function only counts the [illegible]es with parts