Lesson M4.6: Proof Via Generating Function

Lesson M4.6: Proof Via Generating Function

Learning Goals

- Use generating functions together with combinatorial proofs to show equality

Textbook References

Grimaldi: 9.3

Keller and Trotter: 8.5

Levin: Not Covered

Lesson

Part 1: Detailed Manipulations

So far, we have learned that $f(x) = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$ is the ordinary generating function for the sequence $(p(k))_{k\geq 0}$ that gives the total number of partitions of each integer.

We saw in the last lesson that we can limit ourselves to count only partitions with summands of size at most n by only using factors $\frac{1}{1-x^i}$ for values of i that are between 1 and n- but we can generalize this strategy to count partitions using *any* collection of summands.

Proposition: The product

$$\prod_{s\in S} \frac{1}{1-x^s} = \frac{1}{1-x^{s_1}} \frac{1}{1-x^{s_2}} \cdots \frac{1}{1-x^{s_n}} \cdots$$

is the OGF for the sequence whose i^{th} term is the number of partitions of i using only summands s belonging to a set S.

Example: The OGF for the sequence consisting of the number of partitions of n into parts of size 2, 3, and/or 5 is $\frac{1}{1-x^2}\frac{1}{1-x^3}\frac{1}{1-x^5}$.

The OGF that gives the number of partitions of n into only odd summands is

$$\frac{1}{1-x}\frac{1}{1-x^3}\frac{1}{1-x^5}\cdots = \prod_{i=1}^{\infty}\frac{1}{1-x^{2i-1}}.$$

Problem: Find the OGF for the number of partitions of n into only even summands.

We proceed similarly, but use only the even powers of x to represent even summands: if $p_e(n)$ is the number of partitions of n into even summands, the sequence $(p_e(n))_{n\geq 0}$ is generated by

$$f(x) = \prod_{i=1}^{\infty}\frac{1}{1-x^{2i}}.$$

In addition to more nuanced manipulation of which summands could appear in the partitions we're counting, we might wish to limit the number of times any particular summand is used in a partition: we do this by changing the term $\frac{1}{1-x^i}$ corresponding to that summand i.

Recall that $\frac{1}{1-x^i} = 1 + x^i + x^{2i} + x^{3i} + \cdots$ represents the contributions of summands of size i to the total sum (the number we're trying to partition) by using a term in the sum for each possible number of is in the partition. That is, 1 represents 0 is, x^i represents one i, x^{ki} represents k is, and so on.

Example: Find the OGF giving the number $p_d(n)$ of partitions of n into distinct parts.

In this set of partitions we can use any given summand at most once (otherwise if we used a summand twice, they would not be distinct). That means each potential summand i (any positive integer, since that's not limited in this problem) can be used 0 or 1 times: to represent this we just use the polynomial $1 + x^i$.

Thus the OGF we are looking for has one of these factors for each potential summand i: it is $f(x) = \prod_{i=1}^{\infty}(1 + x^i)$.

Problem: Find the OGF giving the number of partitions of n where each summand is used at most twice.

If each summand is used at most twice, it can be used 0, 1, or 2 times: thus each potential summand i is represented in our OGF by a factor of $1 + x^i + x^{2i}$. Our OGF is then $f(x) = \prod_{i=1}^{\infty}(1 + x^i + x^{2i})$.

It looks slightly more complex, but we can easily combine the ideas of limiting the repetition of each summand with limiting which summands are allowed. We just limit repetition using the term representing a particular summand, and limit which summands may be used by limiting which terms we include in our product.

Example: Find the OGF whose coefficients give the number of partitions of n into odd summands, each of which appears an odd number of times or not at all.

We know each possible summand i can appear 0, 1, 3, 5, 7... times, so is represented by $1 + x^i + x^{3i} + x^{5i} + x^{7i} + \cdots = 1 + \sum_{k=0}^{\infty}(x^i)^{2k+1}$. *Note since 0 doesn't fit the pattern of the rest of the numbers, we add that term separately.*

Now we need to ensure that we only use odd parts: to be sure i is odd, we can iterate through $i = 2m + 1$ for $m \geq 0$.

Overall, then, our OGF is

$$\prod_{m=1}^{\infty} 1 + \sum_{k=0}^{\infty}(x^{2m+1})^{2k+1} = \prod_{m=1}^{\infty} 1 + \sum_{k=0}^{\infty} x^{(2m+1)(2k+1)}.$$

Proficiency in writing generating functions for different types of partitions, along with a couple of basic arithmetic facts, allows us to easily prove some interesting and advanced equalities that would otherwise be quite difficult.

Part 2: Proofs About Partitions

Before we dive in, let's establish a few crucial properties.

Proposition A: $1 + x + x^2 + x^3 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$, and more generally $1 + x^i + x^{2i} + \cdots x^{ni} = \frac{1-x^{(n+1)i}}{1-x^i}$.

These equalities can be derived from algebraic manipulations (the general case follows in the same way):

- Method 1: Separate the right side of the equation into two fractions, and expand: $\frac{1-x^{n+1}}{1-x} = \frac{1}{1-x} - \frac{x^{n+1}}{1-x} = (1 + x + x^2 \cdots) - (x^{n+1} + x^{n+2} + \cdots) = 1 + x + x^2 + x^3 + \cdots$
- Method 2: Multiply both sides by $1 - x$: $(1 + x + \cdots x^n) - (x + x^2 + \cdots x^{n+1}) = 1 - x^{n+1}$.

The other two propositions we need won't be proved, but hopefully are somewhat intuitive.

Proposition B: If the sequences $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ are generated by the same function, then $a_n = b_n$ for all $n \geq 0$.

Proposition C: If two sequences of terms $a_i = b_i$ are equal for each $i \geq 0$, then the products $\prod_{i=1}^{\infty} a_i$ and $\prod_{i=1}^{\infty} b_i$ are also equal.

Putting all of these ideas together, we can create the following proof.

Proposition: For any positive integer n, the number of partitions of n into odd parts is equal to the number of partitions into distinct parts.

Proof: First, recall that the number of partitions of n into odd summands is given by

$$f_o(x) = \prod_{i=1}^{\infty} \frac{1}{1 - x^{2i-1}},$$

and the number of partitions of n into distinct summands is given by

$$f_d(x) = \prod_{i=1}^{\infty}(1+x^i).$$

By proposition B, if we understand the number of partitions of n into odd or distinct parts as the n^{th} term of the sequences generated by the functions $f_o(x)$ and $f_d(x)$, then to prove the number of partitions is equal all we need to do is show $f_o(x) = f_d(x)$.

Further, proposition C guarantees that since both functions consist of a product from $i = 1$ to ∞, if we can prove each term of the two products is equal, we know the whole products (and thus the functions) are equal.

Finally, then, all we need to show is that $\frac{1}{1-x^{2i-1}} = (1+x^i)$- but this is just the result from substituting $n = 1$ into the equality we proved in proposition A. $\diamondsuit$

As we wrap up this unit, let's reflect for a moment on what we were just able to do: we know in general it's quite difficult to even count any type of partition, but we were able to efficiently use generating functions to show that there are the same number of partitions following two very different and specific patterns!

In general proofs of this type are quite sophisticated- there are tons of identities like the one we just proved, and most of them are much more complex to show! First-class mathematicians have been working on research related to this type of identity for the last century- the most famous example are the "Rogers-Ramanujan Identities:"

- The number of partitions of an integer n where all parts differ by at least 2 is equal to the number of partitions of n where all parts are congruent to 1 or 4 (mod 5).
- The number of partitions of n where parts differ by at least 2 and the smallest part is at least 2 equals the number of partitions of n with all parts congruent to 2 or 3 (mod 5).

These identities were discovered and proved by Leonard James Rogers and (independently) Srinivasa Ramanujan in the early 1900s (you may have heard of **Ramanujan (https://en.wikipedia.org/wiki/Srinivasa_Ramanujan)** - he was a phenomenal Indian mathematician who made huge contributions to several areas of math despite very little formal training and a tragically short life). Many identities of this type have been proven in the years since, and are still an active area of research today.