Lesson M4.4: The Twelvefold Way, Part III: Integer Partitions

The Lesson M4.4: The Twelvefold Way, Part III: Integer Partitions

Learning Goals

- Define and give examples of integer partitions

Textbook References

Grimaldi: 9.3

Keller and Trotter: 8.5

Levin: Not Covered

Lesson

We return at last to the two final entries yet to be solved in our Twelvefold Way table- let's remind ourselves of our current progress. The number of ways to place k balls into n baskets is:

Balls	Baskets	Total solutions	At most 1 ball in each basket	At least 1 ball in each basket
Distinct	Distinct	A: n^k	B: $P(n,k)$	C: $S(k,n) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)^k$
Identical	Distinct	D: $\binom{n+k-1}{k-1}$	E: $\binom{n}{k}$	F: $\binom{k-1}{n-1}$
Distinct	Identical	G: $\left\{ {k \atop 1} \right\} + \left\{ {k \atop 2} \right\} + \cdots + \left\{ {k \atop n} \right\}$	H: 1 if $k \leq n$; 0	I: $\left\{ {k \atop n} \right\} = \frac{S(k,n)}{n!}$

Balls	Baskets	Total solutions	At most 1 ball in each basket	At least 1 ball in each basket
			otherwise	
Identical	Identical	J: Unknown right now	K: 1 if $k \leq n$; 0 otherwise	L: Unknown right now; related to J

What remains to be discovered is the number of ways to place indistinct balls into indistinct baskets- with the tools we had last time we considered this question, we lacked the ability to do this really carefully and in a reasonably manageable way. It turns out, though, that generating functions are the trick we need to come up with a really nice solution.

Before we can solve this problem, though, we need to translate it. First, a definition:

Definition: For any positive integer k, a *partition* of k is a grouping of k "ones" into positive, unordered summands (or *parts*). The number of partitions of k is denoted $p(k)$.

If we want to specify the number of positive summands involved in a partition, we use a subscript: $p_n(k)$ is the number of partitions of k into exactly n positive summands.

Example: We see below the number of partitions of the integers 0-4, based on the actual partitions possible (which are listed):

$$
\begin{aligned}
&p(0) = 1 \text{ by convention} \\
&p(1) = 1 : 1 \\
&p(2) = 2 : 2, 1+1 \\
&p(3) = 3 : 3, 2+1, 1+1+1 \\
&p(4) = 5 : 4, 3+1, 2+2, 2+1+1, 1+1+1+1
\end{aligned}
$$

Our first order of business is to note that this is an equivalent problem: we know the "ones" in any number are indistinct from each other, and the summands are unordered (remember this is equivalent to unlabeled, or indistinct). Thus if "ones" are "balls" and "summands" are "baskets," then the number of partitions of k into exactly n summands is precisely the number of ways to distribute k identical balls into n identical baskets so that no basket is empty- which is exactly the quantity we're looking for in box L of our table.

Similarly, if we don't care how many of the baskets are actually used, we just need the number of partitions of k into at most n parts; we can find this by finding the number of partitions of k into i

parts for $1 \leq i \leq n$, often written as $p_{\leq n}(k) = p_1(k) + p_2(k) + \cdots + p_n(k)$. This is the number of ways to place k indistinct balls into n indistinct baskets with no further conditions- or quantity J.

Thus our completed table is as follows:

Balls	Baskets	Total solutions	At most 1 ball in each basket	At least 1 ball in each ba
Distinct	Distinct	A: n^k	B: $P(n,k)$	C: $S(k,n) = \sum_{i=0}^{n}(-1)^i \binom{n}{i}($
Identical	Distinct	D: $\binom{n+k-1}{k-1}$	E: $\binom{n}{k}$	F: $\binom{k-1}{n-1}$
Distinct	Identical	G: $\left\{ {k \atop 1} \right\} + \left\{ {k \atop 2} \right\} + \cdots + \left\{ {k \atop n} \right\}$	H: 1 if $k \leq n$; 0 otherwise	I: $\left\{ {k \atop n} \right\} = \frac{S(k,n)}{n!}$
Identical	Identical	J: $p_{\leq n}(k) = p_1(k) + p_2(k) + \cdots + p_n(k)$	K: 1 if $k \leq n$; 0 otherwise	L: $p_n(k)$

We are not quite done yet though- we've translated a little bit and defined a concise notation for the quantities we wanted in boxes J and L, and have given a few very small examples, but in general we have no idea how to actually find $p_n(k)$ without just listing out every possible partition- which gets to be a daunting task very quickly!

We might try the same strategy we used to get quantity I from quantity C: can we somehow use stars and bars/binomial coefficients to count the number of ways to put identical objects into distinct groups, then divide to "unlabel" the baskets? This doesn't quite work-- this strategy worked before because we counted every possibility a constant number of times, but this is not the situation we currently have. For example, if we count ordered groupings of 15, then $1+2+3+4+5$ is counted a very different number of times than $3+3+3+3+3$.

It turns out that there is not a particularly nice way of calculating $p(k) = p_{\leq k}(k)$, $p_n(k)$, or any simply related quantity. A strategy that is successful, though, is to find a generating function for the sequence $(p(k))_{k \geq 0}$ — if we have this, then anytime we want $p(k)$, all we have to do is ask a computer for the coefficient of x^k in our OGF. In the next lesson, we'll look at ways to manipulate this OGF to give us particular kinds of partitions.

We can construct this function the same way we've been doing: we can think of a particular partition of k as a non-negative integer solution to the equation $(1)x_1 + 2x_2 + 3x_3 + \cdots + kx_k = k$, where x_i is the number of summands of size i that appear in the given partition. Note the sum on the left side of the equation contains the terms it does since each summand of a partition of k must be between (or equal to) 1 and k.

So now we use our generating functions to encode possible contributions of each term toward the total sum. One helpful thing to notice is that since our plan is to create an OGF that contains a particular coefficient, we actually don't need to worry about the possibility of making a function that gets "too big:" any possibilities that make the overall sum too big (such as $x_1 = k + 1$) can exist without hurting anything- they will just not actually contribute to the value of the coefficient of x^k.

We know now our summands of size 1 can contribute any number toward a sum of k (i.e., x_1 can be any value), so this contribution is modeled by

$$1 + x + x^2 + x^3 + x^4 + x^5 + \cdots = \frac{1}{1-x}.$$

Similarly, x_2 can take on any non-negative value (we can theoretically include any number of 2s in a partition of k), but their overall contribution has to be an even number so is modeled by

$$1 + x^2 + x^4 + x^6 + x^8 + \cdots = \frac{1}{1-x^2}.$$

We have to get a multiple of 3 from summands of size 3 (so $1 + x^3 + x^6 + x^9 + \cdots = \frac{1}{1-x^3}$ represents the contributions to the sum from summands of size 3), and so on.

So we see that, in general, the ways summands of size i can contribute to the total can be represented by the OGF $f_i(x) = \frac{1}{1-x^i}$.

Now we have an OGF that models the possible contributions of each individual potential summand in our partition, and we need to combine all those possibilities into one OGF to allow for partitions using non-unique summands: again, it's okay if we include possibilities where the summands are larger than k, since the terms obtained from actually using those factors will all have larger powers of x than the one we care about. Thus we can simply take an infinite product, and find that

Theorem: The number of partitions $p(k)$ of a non-negative integer k is the coefficient of x^k in the OGF

$$f(x) = \prod_{i=1}^{\infty} \frac{1}{1-x^i}.$$

If we use what we know about $f_i(x)$ as defined above, we see that the number of partitions of k into summands of size at most n is given by the product of only the first n terms of the function in our theorem, that is, $\prod_{i=1}^{n} \frac{1}{1-x^i}$ is the OGF for partitions of this type.

In particular, then, since we know partitions of k may have summands of size at most k, we may safely truncate the given infinite product and find the coefficient of x^k in $\prod_{i=1}^{k} \frac{1}{1-x^i}$.