Lesson M4.3: Combining Generating Functions

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Learning Goals

- Use addition to form a generating function for the sum of two sequences
- Use multiplication to form a generating function for the convolution of two sequences
- Consistently use correct vocabulary and notation, with sufficient work, to discuss and work with generating functions

Textbook References

Grimaldi: Relevant parts of section 9.2

Keller and Trotter: 8.1

Levin: 5.1

Lesson

So far we have learned the basic building blocks of generating functions, and several ways of manipulating those generating functions to give us other similar ones. The last collection of tools we'll learn today allow us to create even more interesting generating functions by combining our building blocks, rather than just changing them.

We begin by recalling that the two nicest ways of combining any pair of functions are addition and multiplication (composition combined with power series is not something we want to try right now). In this lesson we investigate what the addition and multiplication of generating functions does to the sequences they generate.

Addition:

Addition of generating functions has the rather predictable effect of adding together the sequences they generate. This should make intuitive sense, as we can define the addition of two sequences by adding the terms in each index.

More carefully, if $f_a(x)$ generates $(a_i)_{i\geq 0}$ and $f_b(x)$ generates $(b_i)_{i\geq 0}$, then by summation rules we immediately have

$$\sum_{i=0}^{\infty} a_i x^i + \sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} (a_i + b_i) x^i.$$

In other words, the function $f(x) = f_a(x) + f_b(x)$ generates the sequence $(a_i + b_i)_{i\geq 0}$.

<u>Example:</u> Suppose

$$\begin{aligned} (a_n)_{n\geq 0} &= (1, 3, 3^2, 3^3, 3^4, \ldots) \\ (b_n)_{n\geq 0} &= (1, 3, 3, 1, 0, 0, \ldots) \\ (c_n)_{n\geq 0} &= \left(2, 2, \frac{2}{2!}, \frac{2}{3!}, \frac{2}{4!}, \ldots\right). \end{aligned}$$

Then the corresponding generating functions are, respectively, $f_a(x) = \frac{1}{1-3x}$, $f_b(x) = (1+x)^3$, and $f_c(x) = 2e^x$.

(a) We see that the function $f_a(x) + f_b(x) = \frac{1}{1-3x} + (1+x)^3$ generates the sequence $(1+1, 3+3, 3^2+3, 3^3+1, 3^4, 3^5, \ldots)$.

Note that this sequence looks like the sequence of increasing powers of 3, except for the first few terms- adding *any* polynomial to a generating function is a really nice way to change just a couple terms of a sequence that otherwise follows a more predictable pattern.

(b) Similarly, the sequence $\left(\frac{2}{i!} + 3^i\right)_{i\geq 0}$ is generated by $\frac{1}{1-3x} + 2e^x = f_a(x) + f_c(x)$.

<u>Problem:</u> Find the sequence generated by $2e^x - (1+x)^3$.

We recognize this function as $f_c(x) - f_b(x)$ and we know subtraction works the same way as addition- so we just need to subtract the corresponding terms in the sequences generated by each of the terms in our function.

That is, the sequence we want is $\left(\frac{2}{i!} - \binom{3}{i}\right)_{i\geq 0} = \left(2-1, 2-3, \frac{2}{2!} - 3, \frac{2}{3!} - 1, \frac{2}{4!}, \ldots\right)$.

Multiplication:

Like addition, multiplication of *closed form* generating functions is relatively straightforward. The more interesting aspect of this process, though, is what happens with the corresponding power series expansions (and thus the sequences these functions generate). Multiplication of power series and sequences is not quite so straightforward.

One could, of course, try defining the product of two sequences by just multiplying corresponding terms- and we wouldn't run into anything weird right away. The reason we don't do this, though, is that it would completely break the way our generating functions work- and that's something we want to avoid.

For example, we know $(1+x)^2$ generates $(1, 2, 1, 0, 0, \ldots)$ and $(1+x)^3$ generates $(1, 3, 3, 1, 0, 0, \ldots)$- so multiplying these series together term by term would give us $(1, 2\cdot 3, 1\cdot 3, 0\cdot 1, 0, 0, \ldots) = (1, 6, 3, 0, 0\ldots)$.

But we also know that

$$(1+x)^2(1+x)^3 = (1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$$

, so by our original definition, this generating function has to generate the sequence $(1, 5, 10, 10, 5, 1, 0, 0, 0, \ldots)$. What we see here, then, is that multiplying term-by-term *doesn't work.*

The answer to this discrepancy is that we need a different definition of what it means to "multiply" two sequences together (and consequently, power series). The fact that we have power series to think about is actually helpful though- a power series is really just a pretentiously named infinite polynomial, and we know how to multiply polynomials together. So if we generalize that process, we see that all we need to do to multiply two series together is multiply every term in each series by every term in the other series, and add the results.

We do want to be careful about the order in which we multiply our terms, though: we'll get the same answer no matter what we do, but if we want to multiply together $\sum_{i=0}^{\infty} a_i$ and $\sum_{i=0}^{\infty} b_i$, if we start with $a_0b_0 + a_0b_1x + a_0b_2x^2 + a_0b_3x^3 + \cdots$, we'll never get to what happens with the further terms of our first series, and won't be able to calculate what happens beyond scaling our second series by a_0 .

Instead, then, we think about all the possible terms of our product, and we sort by what power of x will be produced- this allows us to look at the whole picture for a finite number of terms. That is, we can write

$$\sum_{i=0}^{\infty} a_i \sum_{i=0}^{\infty} b_i = (a_0b_0) + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \cdots.$$

Since the resulting sequence doesn't behave quite the way we usually imagine products working, we use a different word for it: combining two sequences in the way that corresponds to power series multiplication is called *convoluting.*

Definition: The sequence $(c_i)_{i\geq 0}$ generated by the product $f_a(x)f_b(x)$ is called the *convolution* of the sequences $(a_i)_{i\geq 0}$ and $(b_i)_{i\geq 0}$ i.e.,

$$f_a(x)f_b(x) = \sum_{i=0}^{\infty} a_i x^i \sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} c_i x^i,$$

where $c_i = \sum_{i=0}^{n} a_i b_{n-i}$.

In practice, finding the convolution of two functions by looking at this formula is not what we want to do- we really want to exploit the fact that we know the generating functions for each separate sequence whenever we can. Not only does this save us a lot of tedious calculations, it gives us a much better chance of being able to write down the general pattern of the resulting sequence rather than just the first few terms- and **this is always our goal.**

Example: Find the convolution of the sequences $(a_i)_{i\geq 0} = (1, 1, 1, 1, \ldots)$ and $(b_i)_{i\geq 0} = (-1^i)_{i\geq 0} = (1, -1, 1, -1, \ldots)$.

Solution 1: If we'd like, we could try the method we were just advised against. Using the definition above, we know the convolution $(c_i)_{i\geq 0}$ is generated by $\sum_{i=0}^{\infty} c_i x^i$, where

$$c_i = \sum_{n=0}^{i} a_n b_{i-n} = \sum_{n=0}^{i} a_{i-n} b_n = \sum_{n=0}^{i} 1^{i-n}(-1)^n = \sum_{n=0}^{i} (-1)^n.$$

Now we happen to be able to reason through what happens if we take a sum of a particular length of positive and negative 1s: the result is 1 if i is even, and 0 if i is odd. That is, the convolution we want is $1, 0, 1, 0, 1, 0, \ldots) = \left(\frac{1+(-1)^i}{2}\right)_{i \geq 0}$.

Note: this second fancy way of writing an alternating sequence as a fraction using an alternating sum in the numerator is really nothing more than a fancy trick- you might find it convenient, but you will not be tested on your ability to do this. You just need to be sure you fully describe the pattern of a sequence in a way that shows you understand what's going on beyond the first few terms.

Solution 2: Our preferred method of solution here is to remember that we actually know the generating functions for each of these sequences: $f_a(x) = \frac{1}{1-x}$, and $f_b(x) = \frac{1}{1+x}$.

The convolution of these two sequences, then, is generated by the product of their generating functions: $f(x) = f_a(x) f_b(x) = \frac{1}{1-x} \cdot \frac{1}{1+x} = \frac{1}{1-x^2} = \sum_{i=0}^{\infty} x^{2i}$.

We get this last step from what we know about geometric sequences, and it allows us to read off the convolution as $(1, 0, 1, 0, \ldots)$, or c_i equal to 1 when i is even and 0 when i is odd, as expected- but without the tricks in the middle.

The last technique we need to think about is how to transform a convolution into something nicer to handle. Suppose we want to find the convolution of two sequences, and we know their generating functions in closed form, then all we need to do is multiply those functions together and figure out what sequence that OGF generates. This is great if the product we have happens to be exceptionally nice, but that's not usually what happens!

So instead we think: wouldn't it be great if we were adding instead of multiplying? That's so much easier. But the good news is that we know how to transform multiplication of fractions into addition- we just need a partial fraction decomposition.

*Note: We won't go over the basics of this process in detail, since you know it from Calculus, and it's essentially the process of finding a common denominator, but run in reverse. If you want a refresher, check out **Paul's Online Notes** (https://tutorial.math.lamar.edu/classes/calcii/partialfractions.aspx) - this is a calculus-based treatment, but it's a decent review nevertheless.*

Example: Find the sequence generated by $f(x) = \frac{1}{(x-3)(x+2)}$.

First, let's use partial fractions to turn this product into a sum. We set up our equation:

$$\frac{1}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}.$$

Now we know these quantities are equal, so if we find a common denominator on the right side of the equation, we end up with

$$\frac{1}{(x-3)(x+2)} = \frac{A(x+2)}{(x-3)(x+2)} + \frac{B(x-3)}{(x-3)(x+2)}.$$

We can multiply both sides by the denominator to make things simpler:

$$Ax + 2A + Bx - 3B = 1.$$

Now we just need to remember what makes two polynomials equal: the coefficients of each power of our variable need to be equal. Comparing these, we get $A + B = 0$ (from the coefficients of x), and $2A - 3B = 1$ from the constant term.

The first equation tells us $A = -B$, and substituting into the second we find that $A = 1/5$ and subsequently $B = -1/5$. It follows that $f(x) = \frac{1}{5(x-3)} - \frac{1}{5(x+2)}$.

Now we have transformed out problem into two smaller ones- we just need to sort out these geometric sequences. Be careful here though- we can only read off the generated sequence if a function is in the form $\frac{A}{1-B}$- and these aren't, so we need to do a bit of creative rewriting. This isn't too bad though- really all we need to do is remember (1) we can change the order of a subtraction by factoring out -1, and (2) we can always divide through numerator and denominator by the same number without changing the value of a fraction.

This gives us:

$$\begin{aligned} \frac{1}{5(x-3)} &= \frac{1}{5} \cdot \frac{-1}{3-x} \\ &= \frac{1}{5} \cdot \frac{-1/3}{1-(x/3)} \\ &= \frac{-1}{15} \cdot \frac{1}{1-(x/3)} \end{aligned}$$

and

$$\begin{aligned}\frac{1}{5(x+2)} &= \frac{1}{5} \cdot \frac{1}{2-(-x)} \\ &= \frac{1}{5} \cdot \frac{1/2}{1-(-x/2)} \\ &= \frac{1}{10} \cdot \frac{1}{1-(-x/2)}\end{aligned}$$

Putting this all together then,

$$f(x) = \frac{-1}{15} \cdot \frac{1}{1-(x/3)} - \frac{1}{10} \cdot \frac{1}{1-(-x/2)}$$

We see that this generates the sequence $\left(\frac{-1}{15} \cdot \left(\frac{x}{3}\right)^n - \frac{1}{10} \cdot \left(\frac{-x}{2}\right)^n\right)_{n \geq 0}$.