1.4 Continuous Functions

We define continuous functions and discuss a few of their basic properties. The class of continuous functions will play a central role later.

Definition 1.14. *Let f be a function and c a point in its domain. The function is said to be continuous at c if for all $\epsilon > 0$ there exists a $\delta > 0$, such that $|f(c) - f(x)| < \epsilon$ whenever x belongs to the domain of f and $|x - c| < \delta$. A function f is continuous if it is continuous at all points in its domain.*

In most cases the condition in Definition 1.14 says that

$$\lim_{x \to c} f(x) = f(c). \tag{1.7}$$

In fact, this equation holds whenever there are points in the domain of f arbitrarily close to c. See the footnote to Proposition 1.2. If c is an isolated point in the domain of f, i.e., there are no other points in the domain of f arbitrarily close to c, then the function is always continuous at c.

Polynomials, rational functions, and trigonometric functions are continuous. One can produce many more continuous functions through standard operations on functions.

Proposition 1.15. *Let f and g be continuous functions. Then $f + g$, $f \cdot g$, f/g and $f \circ g$ are continuous, wherever these functions are defined.*

The clarify the remark about the domain in the proposition, we note that the function $(f + g)(x) = f(x) + g(x)$ is defined for those x for which both f and g are defined. The same statement holds for $(f \cdot g)(x) = f(x) \cdot g(x)$. To determine the domain of f/g one needs to exclude those points where g is zero. For the composition $(f \circ g)(x) = f(g(x))$ on needs that g takes values in the domain of f.

One may also reverse the order of applying a continuous function and calculating a limit:

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right), \tag{1.8}$$

provided the natural technical assumption hold, i.e., g is defined at points arbitrarily close to c, f is defined for all $g(x)$ where x is in the domain of g and close to c, and f is continuous at $\lim_{x \to c} g(x)$.

Theorem 1.16 (Intermediate Value Theorem). *Suppose that f is defined and continuous on the closed interval $[a, b]$. If C is in between $f(a)$ and $f(b)$, then there exists a $c \in [a, b]$, such that $f(c) = 0$.*

E.g., suppose that $p(x) = x^3 - x^2 + 2x - 1$. The polynomial is certainly a continuous function, $p(0) = -1$ and $p(1) = 1$. According to the theorem there exists some $c \in (0, 1)$, such that $p(c) = 0$.

Theorem 1.17 (Extreme Value Theorem). *Let f be defined and continuous on the closed interval $[a, b]$. Then there exist points c and d in $[a, b]$, such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$.*

Expressed in words, the theorem says that a continuous function on a closed interval assumes a smallest and largest value.

The Intermediate Value and Extreme Value theorem are typically proved in an introductory analysis course. They are equivalent to the completeness of the real line. We mentioned this property of the real numbers in Section 1.1.

1.5 Lines

In general, a line consists of the points (x, y) in the plane which satisfy the equation

$$ax + by = c \tag{1.9}$$

for some given real numbers a, b and c, where it is assumed that a and b are not both zero. The line is vertical if and only if $b = 0$. If $b \neq 0$ we may rewrite the equation as

$$y = -\frac{a}{b}x + \frac{c}{b} = mx + B. \tag{1.10}$$

The number m is called the *slope* of the line, and B is the point in which the line intersects the y-axis, also called the *y-intercept*. Given any two points (x_1, y_1) and (x_2, y_2) in the plane, the line through them has slope

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

For our purposes, the most useful version of the equation of a line is its *point-slope formula*. The equation of a line with slope m through the point (x_1, y_1) is

$$y = m(x - x_1) + y_1. \tag{1.11}$$

1.6 Tangent Lines and the Derivative

We like to introduce the concept of tangent lines. To be able to express ourselves concisely, let us say

Definition 1.18. *A point c is an* interior point *of a subset B of $\mathbb{R}$ if there is an open interval I, such that $c \in I \subseteq B$.*

We give a first definition for a tangent line.

Definition 1.19. *Suppose $f(x)$ is a function and c is an interior point of its domain. We call a line $t(x)$ the* tangent line *to the graph of $f(x)$ at $x = c$ if $t(x)$ is the best linear approximation of $f(x)$ on some open interval around c, i.e., the line $t(x)$ is closer to the graph of $f(x)$ than any other line for all x in some open interval around c.*

For a given function and an interior point c in its domain there may or may not be a tangent line, but it there is a tangent line, then it is unique.

Although the term *'best linear approximation near c'* gives an excellent intuitive picture what a tangent line is, this definition is hard to work with. It is easier to work with a more concrete definition.

Definition 1.20. *Suppose $f(x)$ is a function and c is an interior point of its domain. We call a line $t(x)$ the* tangent line *to the graph of $f(x)$ at $x = c$ if*

$$\lim_{x \to c} \frac{f(x) - t(x)}{x - c} = 0. \tag{1.12}$$

The equation in (1.12) expresses in a precise form in which sense the tangent line is close to the graph of $f(x)$ near c. Not only does $f(x) - t(x)$ converge to zero as x approaches c, it does so even when divided by $x - c$.

We use tangent lines to define the concept of differentiability and the derivative.

Definition 1.21. *Suppose $f(x)$ is a function and c is an interior point of its domain, and assume that there is a tangent line to the graph of $f(x)$ at $x = c$. Then we say that $f(x)$ is* differentiable *at c. We call the slope of the tangent line the* derivative *of $f(x)$ at c, and we denote it by $f'(c)$.*

Utilizing the notation in the previous definition we can write down the equation of the tangent line to the graph of $f(x)$ at $x = c$ in point-slope form:

$$t(x) = f'(c)(x - c) + f(c). \tag{1.13}$$