Mathematics

Mathematics Test Overview

The math portion of the SAT consists of two 70-minute modules with 22 questions each. This gives you approximately 1.59 minutes to solve each problem.

Concepts Covered

SAT math questions fall into four categories:

- Algebra
- Advanced Math
- Problem-Solving and Data Analysis
- Geometry and Trigonometry

The table below gives a complete breakdown of questions:

Combined Modules	**Number of Questions**	**% of Test**
Total Questions	*44*	*100%*
Multiple Choice	28–32	75%
Student-Produced Response	8–12	25%
Content Categories		
Algebra	13–15	35%
Advanced Math	13–15	35%
Problem-Solving and Data Analysis	5–7	15%
Geometry and Trigonometry	5–7	15%

Use the Practice Tests

The best thing you can do to prepare for the SAT is to take several practice tests and review all your wrong answers very carefully. Work back through those problems until you understand how the answer was derived and you are confident you could answer a similar problem on your own.

This guide includes 3 practice tests with answer keys and explanations. Examples are also available on the College Board website. If you feel uncertain about a particular concept or problem type, use these tests to practice.

Introductory Information

Number Basics

Classifications of Numbers

Numbers are the basic building blocks of mathematics. Specific features of numbers are identified by the following terms:

Integer – any positive or negative whole number, including zero. Integers do not include fractions $\left(\frac{1}{3}\right)$, decimals (0.56), or mixed numbers $\left(7\frac{3}{4}\right)$.

Prime number – any whole number greater than 1 that has only two factors, itself and 1; that is, a number that can be divided evenly only by 1 and itself.

Composite number – any whole number greater than 1 that has more than two different factors; in other words, any whole number that is not a prime number. For example: The composite number 8 has the factors of 1, 2, 4, and 8.

Even number – any integer that can be divided by 2 without leaving a remainder. For example: 2, 4, 6, 8, and so on.

Odd number – any integer that cannot be divided evenly by 2. For example: 3, 5, 7, 9, and so on.

Decimal number – any number that uses a decimal point to show the part of the number that is less than one. Example: 1.234.

Decimal point – a symbol used to separate the ones place from the tenths place in decimals or dollars from cents in currency.

Decimal place – the position of a number to the right of the decimal point. In the decimal 0.123, the 1 is in the first place to the right of the decimal point, indicating tenths; the 2 is in the second place, indicating hundredths; and the 3 is in the third place, indicating thousandths.

The **decimal**, or base 10, system is a number system that uses ten different digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9). An example of a number system that uses something other than ten digits is the **binary**, or base 2, number system, used by computers, which uses only the numbers 0 and 1. It is thought that the decimal system originated because people had only their 10 fingers for counting.

Rational numbers include all integers, decimals, and fractions. Any terminating or repeating decimal number is a rational number.

Irrational numbers cannot be written as fractions or decimals because the number of decimal places is infinite and there is no recurring pattern of digits within the number. For example, pi (π) begins with 3.141592 and continues without terminating or repeating, so pi is an irrational number.

Real numbers are the set of all rational and irrational numbers.

Review Video: Classification of Numbers
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Review Video: Prime and Composite Numbers
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Numbers in Word Form and Place Value

When writing numbers out in word form or translating word form to numbers, it is essential to understand how a place value system works. In the decimal or base-10 system, each digit of a number represents how many of the corresponding place value—a specific factor of 10—are contained in the number being represented. To make reading numbers easier, every three digits to the left of the decimal place is preceded by a comma. The following table demonstrates some of the place values:

Power of 10	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}
Value	1,000	100	10	1	0.1	0.01	0.001
Place	thousands	hundreds	tens	ones	tenths	hundredths	thousandths

For example, consider the number 4,546.09, which can be separated into each place value like this:

4: thousands
5: hundreds
4: tens
6: ones
0: tenths
9: hundredths

This number in word form would be *four thousand five hundred forty-six and nine hundredths.*

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RATIONAL NUMBERS

The term **rational** means that the number can be expressed as a ratio or fraction. That is, a number, r, is rational if and only if it can be represented by a fraction $\frac{a}{b}$ where a and b are integers and b does not equal 0. The set of rational numbers includes integers and decimals. If there is no finite way to represent a value with a fraction of integers, then the number is **irrational**. Common examples of irrational numbers include: $\sqrt{5}$, $(1+\sqrt{2})$, and π.

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Review Video: Ordering Rational Numbers
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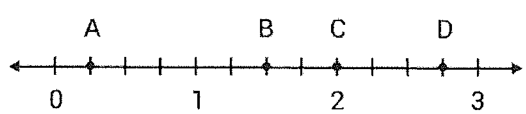
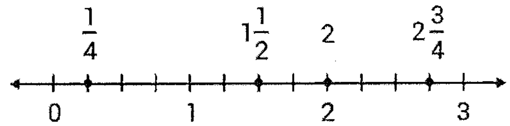
NUMBER LINES

A number line is a graph to see the distance between numbers. Basically, this graph shows the relationship between numbers. So a number line may have a point for zero and may show negative numbers on the left side of the line. Any positive numbers are placed on the right side of the line. For example, consider the points labeled on the following number line:



We can use the dashed lines on the number line to identify each point. Each dashed line between two whole numbers is $\frac{1}{4}$. The line halfway between two numbers is $\frac{1}{2}$.



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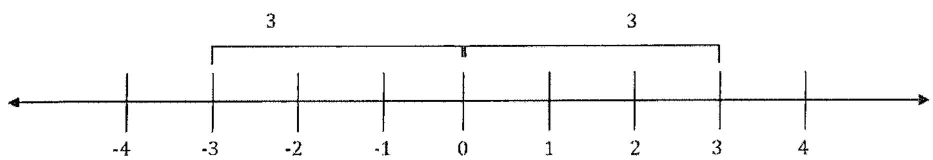
Absolute Value

A precursor to working with negative numbers is understanding what **absolute values** are. A number's absolute value is simply the distance away from zero a number is on the number line. The absolute value of a number is always positive and is written $|x|$. For example, the absolute value of 3, written as $|3|$, is 3 because the distance between 0 and 3 on a number line is three units. Likewise, the absolute value of -3, written as $|-3|$, is 3 because the distance between 0 and -3 on a number line is three units. So $|3| = |-3|$.



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Operations

An **operation** is simply a mathematical process that takes some value(s) as input(s) and produces an output. Elementary operations are often written in the following form: *value operation value*. For instance, in the expression $1 + 2$ the values are 1 and 2 and the operation is addition. Performing the operation gives the output of 3. In this way we can say that $1 + 2$ and 3 are equal, or $1 + 2 = 3$.

Addition

Addition increases the value of one quantity by the value of another quantity (both called **addends**). Example: $2 + 4 = 6$ or $8 + 9 = 17$. The result is called the **sum**. With addition, the order does not matter, $4 + 2 = 2 + 4$.

When adding signed numbers, if the signs are the same simply add the absolute values of the addends and apply the original sign to the sum. For example, $(+4) + (+8) = +12$ and $(-4) + (-8) = -12$. When the original signs are different, take the absolute values of the addends and subtract the smaller value from the larger value, then apply the original sign of the larger value to the difference. Example: $(+4) + (-8) = -4$ and $(-4) + (+8) = +4$.

Subtraction

Subtraction is the opposite operation to addition; it decreases the value of one quantity (the **minuend**) by the value of another quantity (the **subtrahend**). For example, $6 - 4 = 2$ or $17 - 8 = 9$. The result is called the **difference**. Note that with subtraction, the order does matter, $6 - 4 \neq 4 - 6$.

For subtracting signed numbers, change the sign of the subtrahend and then follow the same rules used for addition. Example: $(+4) - (+8) = (+4) + (-8) = -4$

Multiplication

Multiplication can be thought of as repeated addition. One number (the **multiplier**) indicates how many times to add the other number (the **multiplicand**) to itself. Example: $3 \times 2 = 2 + 2 + 2 = 6$. With multiplication, the order does not matter, $2 \times 3 = 3 \times 2$ or $3 + 3 = 2 + 2 + 2$, either way the result (the **product**) is the same.

If the signs are the same, the product is positive when multiplying signed numbers. Example: $(+4) \times (+8) = +32$ and $(-4) \times (-8) = +32$. If the signs are opposite, the product is negative. Example: $(+4) \times (-8) = -32$ and $(-4) \times (+8) = -32$. When more than two factors are multiplied together, the sign of the product is determined by how many negative factors are present. If there are an odd number of negative factors then the product is negative, whereas an even number of negative factors indicates a positive product. Example: $(+4) \times (-8) \times (-2) = +64$ and $(-4) \times (-8) \times (-2) = -64$.

Division

Division is the opposite operation to multiplication; one number (the **divisor**) tells us how many parts to divide the other number (the **dividend**) into. The result of division is called the **quotient**. Example: $20 \div 4 = 5$. If 20 is split into 4 equal parts, each part is 5. With division, the order of the numbers does matter, $20 \div 4 \neq 4 \div 20$.

The rules for dividing signed numbers are similar to multiplying signed numbers. If the dividend and divisor have the same sign, the quotient is positive. If the dividend and divisor have opposite signs, the quotient is negative. Example: $(-4) \div (+8) = -0.5$.

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Parentheses

Parentheses are used to designate which operations should be done first when there are multiple operations. Example: $4 - (2 + 1) = 1$; the parentheses tell us that we must add 2 and 1, and then subtract the sum from 4, rather than subtracting 2 from 4 and then adding 1 (this would give us an answer of 3).

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Exponents

An **exponent** is a superscript number placed next to another number at the top right. It indicates how many times the base number is to be multiplied by itself. Exponents provide a shorthand way to write what would be a longer mathematical expression, Example: $2^4 = 2 \times 2 \times 2 \times 2$. A number with an exponent of 2 is said to be "squared," while a number with an exponent of 3 is said to be "cubed." The value of a number raised to an exponent is called its power. So 8^4 is read as "8 to the 4th power," or "8 raised to the power of 4."

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Roots

A **root**, such as a square root, is another way of writing a fractional exponent. Instead of using a superscript, roots use the radical symbol ($\sqrt{\ }$) to indicate the operation. A radical will have a number underneath the bar, and may sometimes have a number in the upper left: $\sqrt[n]{a}$, read as "the n^{th} root of a." The relationship between radical notation and exponent notation can be described by this equation:

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

The two special cases of $n = 2$ and $n = 3$ are called square roots and cube roots. If there is no number to the upper left, the radical is understood to be a square root ($n = 2$). Nearly all of the roots you encounter will be square roots. A square root is the same as a number raised to the one-half power. When we say that a is the square root of b ($a = \sqrt{b}$), we mean that a multiplied by itself equals b: ($a \times a = b$).

A **perfect square** is a number that has an integer for its square root. There are 10 perfect squares from 1 to 100: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100 (the squares of integers 1 through 10).

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Review Video: Perfect Squares and Square Roots
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Word Problems and Mathematical Symbols

When working on word problems, you must be able to translate verbal expressions or "math words" into math symbols. This chart contains several "math words" and their appropriate symbols:

Phrase	Symbol
equal, is, was, will be, has, costs, gets to, is the same as, becomes	$=$
times, of, multiplied by, product of, twice, doubles, halves, triples	$\times$
divided by, per, ratio of/to, out of	$\div$
plus, added to, sum, combined, and, more than, totals of	$+$
subtracted from, less than, decreased by, minus, difference between	$-$
what, how much, original value, how many, a number, a variable	x, n, etc.

Examples of Translated Mathematical Phrases

- The phrase four more than twice a number can be written algebraically as $2x + 4$.
- The phrase half a number decreased by six can be written algebraically as $\frac{1}{2}x - 6$.
- The phrase the sum of a number and the product of five and that number can be written algebraically as $x + 5x$.

- You may see a test question that says, "Olivia is constructing a bookcase from seven boards. Two of them are for vertical supports and five are for shelves. The height of the bookcase is twice the width of the bookcase. If the seven boards total 36 feet in length, what will be the height of Olivia's bookcase?" You would need to make a sketch and then create the equation to determine the width of the shelves. The height can be represented as double the width. (If x represents the width of the shelves in feet, then the height of the bookcase is $2x$. Since the seven boards total 36 feet, $2x + 2x + x + x + x + x + x = 36$ or $9x = 36$; $x = 4$. The height is twice the width, or 8 feet.)

SUBTRACTION WITH REGROUPING

A great way to make use of some of the features built into the decimal system would be regrouping when attempting longform subtraction operations. When subtracting within a place value, sometimes the minuend is smaller than the subtrahend, **regrouping** enables you to 'borrow' a unit from a place value to the left in order to get a positive difference. For example, consider subtracting 189 from 525 with regrouping.

First, set up the subtraction problem in vertical form:

$$\begin{array}{r} 525 \\ -\ 189 \\ \hline \end{array}$$

Notice that the numbers in the ones and tens columns of 525 are smaller than the numbers in the ones and tens columns of 189. This means you will need to use regrouping to perform subtraction:

$$\begin{array}{cccc} & 5 & 2 & 5 \\ - & 1 & 8 & 9 \\ \hline \end{array}$$

To subtract 9 from 5 in the ones column you will need to borrow from the 2 in the tens columns:

$$\begin{array}{cccc} & 5 & 1 & 15 \\ - & 1 & 8 & 9 \\ \hline & & & 6 \end{array}$$

Next, to subtract 8 from 1 in the tens column you will need to borrow from the 5 in the hundreds column:

$$\begin{array}{cccc} & 4 & 11 & 15 \\ - & 1 & 8 & 9 \\ \hline & & 3 & 6 \end{array}$$

Last, subtract the 1 from the 4 in the hundreds column:

$$\begin{array}{cccc} & 4 & 11 & 15 \\ - & 1 & 8 & 9 \\ \hline & 3 & 3 & 6 \end{array}$$

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ORDER OF OPERATIONS

The **order of operations** is a set of rules that dictates the order in which we must perform each operation in an expression so that we will evaluate it accurately. If we have an expression that includes multiple different operations, the order of operations tells us which operations to do first. The most common mnemonic for the order of operations is **PEMDAS**, or "Please Excuse My Dear Aunt Sally." PEMDAS stands for parentheses, exponents, multiplication, division, addition, and subtraction. It is important to understand that multiplication and division have equal precedence, as do addition and subtraction, so those pairs of operations are simply worked from left to right in order.

For example, evaluating the expression $5 + 20 \div 4 \times (2 + 3)^2 - 6$ using the correct order of operations would be done like this:

- **P:** Perform the operations inside the parentheses: $(2 + 3) = 5$
- **E:** Simplify the exponents: $(5)^2 = 5 \times 5 = 25$
  - The expression now looks like this: $5 + 20 \div 4 \times 25 - 6$
- **MD:** Perform multiplication and division from left to right: $20 \div 4 = 5$; then $5 \times 25 = 125$
  - The expression now looks like this: $5 + 125 - 6$
- **AS:** Perform addition and subtraction from left to right: $5 + 125 = 130$; then $130 - 6 = 124$

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PROPERTIES OF EXPONENTS

The properties of exponents are as follows:

Property	Description
$a^1 = a$	Any number to the power of 1 is equal to itself
$1^n = 1$	The number 1 raised to any power is equal to 1
$a^0 = 1$	Any number raised to the power of 0 is equal to 1
$a^n \times a^m = a^{n+m}$	Add exponents to multiply powers of the same base number
$a^n \div a^m = a^{n-m}$	Subtract exponents to divide powers of the same base number
$(a^n)^m = a^{n \times m}$	When a power is raised to a power, the exponents are multiplied
$(a \times b)^n = a^n \times b^n$ $(a \div b)^n = a^n \div b^n$	Multiplication and division operations inside parentheses can be raised to a power. This is the same as each term being raised to that power.
$a^{-n} = \frac{1}{a^n}$	A negative exponent is the same as the reciprocal of a positive exponent

Note that exponents do not have to be integers. Fractional or decimal exponents follow all the rules above as well. Example: $5^{\frac{1}{4}} \times 5^{\frac{3}{4}} = 5^{\frac{1}{4}+\frac{3}{4}} = 5^1 = 5$.

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FACTORS AND MULTIPLES

FACTORS AND GREATEST COMMON FACTOR

Factors are numbers that are multiplied together to obtain a **product**. For example, in the equation $2 \times 3 = 6$, the numbers 2 and 3 are factors. A **prime number** has only two factors (1 and itself), but other numbers can have many factors.

A **common factor** is a number that divides exactly into two or more other numbers. For example, the factors of 12 are 1, 2, 3, 4, 6, and 12, while the factors of 15 are 1, 3, 5, and 15. The common factors of 12 and 15 are 1 and 3.

A **prime factor** is also a prime number. Therefore, the prime factors of 12 are 2 and 3. For 15, the prime factors are 3 and 5.

The **greatest common factor** (GCF) is the largest number that is a factor of two or more numbers. For example, the factors of 15 are 1, 3, 5, and 15; the factors of 35 are 1, 5, 7, and 35. Therefore, the greatest common factor of 15 and 35 is 5.

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Review Video: Greatest Common Factor and Least Common Multiple
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MULTIPLES AND LEAST COMMON MULTIPLE

Often listed out in multiplication tables, **multiples** are integer increments of a given factor. In other words, dividing a multiple by the factor will result in an integer. For example, the multiples of 7 include: $1 \times 7 = 7, 2 \times 7 = 14, 3 \times 7 = 21, 4 \times 7 = 28, 5 \times 7 = 35$. Dividing 7, 14, 21, 28, or 35 by 7 will result in the integers 1, 2, 3, 4, and 5, respectively.

The least common multiple (**LCM**) is the smallest number that is a multiple of two or more numbers. For example, the multiples of 3 include 3, 6, 9, 12, 15, etc.; the multiples of 5 include 5, 10, 15, 20, etc. Therefore, the least common multiple of 3 and 5 is 15.

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Algebra

SLOPE

On a graph with two points, (x_1, y_1) and (x_2, y_2), the **slope** is found with the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$; where $x_1 \neq x_2$ and m stands for slope. If the value of the slope is **positive**, the line has an *upward direction* from left to right. If the value of the slope is **negative**, the line has a *downward direction* from left to right. Consider the following example:

A new book goes on sale in bookstores and online stores. In the first month, 5,000 copies of the book are sold. Over time, the book continues to grow in popularity. The data for the number of copies sold is in the table below.

# of Months on Sale	1	2	3	4	5
# of Copies Sold (In Thousands)	5	10	15	20	25

So, the number of copies that are sold and the time that the book is on sale is a proportional relationship. In this example, an equation can be used to show the data: $y = 5x$, where x is the number of months that the book is on sale. Also, y is the number of copies sold. So, the slope of the corresponding line is $\frac{\text{rise}}{\text{run}} = \frac{5}{1} = 5$.

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LINEAR EQUATIONS

Equations that can be written as $ax + b = 0$, where $a \neq 0$, are referred to as **one variable linear equations**. A solution to such an equation is called a **root**. In the case where we have the equation $5x + 10 = 0$, if we solve for x we get a solution of $x = -2$. In other words, the root of the equation is -2. This is found by first subtracting 10 from both sides, which gives $5x = -10$. Next, simply divide both sides by the coefficient of the variable, in this case 5, to get $x = -2$. This can be checked by plugging -2 back into the original equation $(5)(-2) + 10 = -10 + 10 = 0$.

The **solution set** is the set of all solutions of an equation. In our example, the solution set would simply be -2. If there were more solutions (there usually are in multivariable equations) then they would also be included in the solution set. When an equation has no true solutions, it is referred to as an **empty set**. Equations with identical solution sets are **equivalent equations**. An **identity** is a term whose value or determinant is equal to 1.

Linear equations can be written many ways. Below is a list of some forms linear equations can take:

- **Standard Form**: $Ax + By = C$; the slope is $\frac{-A}{B}$ and the y-intercept is $\frac{C}{B}$
- **Slope Intercept Form**: $y = mx + b$, where m is the slope and b is the y-intercept
- **Point-Slope Form**: $y - y_1 = m(x - x_1)$, where m is the slope and (x_1, y_1) is a point on the line
- **Two-Point Form**: $\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$, where (x_1, y_1) and (x_2, y_2) are two points on the given line
- **Intercept Form**: $\frac{x}{x_1} + \frac{y}{y_1} = 1$, where $(x_1, 0)$ is the point at which a line intersects the x-axis, and $(0, y_1)$ is the point at which the same line intersects the y-axis

Review Video: Slope-Intercept and Point-Slope Forms
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Review Video: Linear Equations Basics
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Solving Equations

Solving One-Variable Linear Equations

Multiply all terms by the lowest common denominator to eliminate any fractions. Look for addition or subtraction to undo so you can isolate the variable on one side of the equal sign. Divide both sides by the coefficient of the variable. When you have a value for the variable, substitute this value into the original equation to make sure you have a true equation. Consider the following example:

Kim's savings are represented by the table below. Represent her savings, using an equation.

X (Months)	Y (Total Savings)
2	\$1,300
5	\$2,050
9	\$3,050
11	\$3,550
16	\$4,800

The table shows a function with a constant rate of change, or slope, of 250. Given the points on the table, the slopes can be calculated as $\frac{(2{,}050-1300)}{(5-2)}$, $\frac{(3{,}050-2{,}050)}{(9-5)}$, $\frac{(3{,}550-3{,}050)}{(11-9)}$, and $\frac{(4{,}800-3{,}550)}{(16-11)}$, each of which equals 250. Thus, the table shows a constant rate of change, indicating a linear function. The slope-intercept form of a linear equation is written as $y = mx + b$, where m represents the slope and b represents the y-intercept. Substituting the slope into this form gives $y = 250x + b$. Substituting corresponding x- and y-values from any point into this equation will give the y-intercept, or b. Using the point, $(2, 1{,}300)$, gives $1{,}300 = 250(2) + b$, which simplifies as $b = 800$. Thus, her savings may be represented by the equation, $y = 250x + 800$.

Rules for Manipulating Equations

Like Terms

Like terms are terms in an equation that have the same variable, regardless of whether or not they also have the same coefficient. This includes terms that *lack* a variable; all constants (i.e., numbers without variables) are considered like terms. If the equation involves terms with a variable raised to different powers, the like terms are those that have the variable raised to the same power.

For example, consider the equation $x^2 + 3x + 2 = 2x^2 + x - 7 + 2x$. In this equation, 2 and -7 are like terms; they are both constants. $3x$, x, and $2x$ are like terms, they all include the variable x raised to the first power. x^2 and $2x^2$ are like terms, they both include the variable x, raised to the second power. $2x$ and $2x^2$ are not like terms; although they both involve the variable x, the variable is not raised to the same power in both terms. The fact that they have the same coefficient, 2, is not relevant.

Review Video: Rules for Manipulating Equations
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Carrying Out the Same Operation on Both Sides of an Equation

When solving an equation, the general procedure is to carry out a series of operations on both sides of an equation, choosing operations that will tend to simplify the equation when doing so. The reason why the same operation must be carried out on both sides of the equation is because that leaves the meaning of the equation unchanged, and yields a result that is equivalent to the original equation. This would not be the case if we carried out an operation on one side of an equation and not the other. Consider what an equation means: it is a statement that two values or expressions are equal. If we carry out the same operation on both sides of the equation—add 3 to both sides, for example—then the two sides of the equation are changed in the same way, and so remain equal. If we do that to only one side of the equation—add 3 to one side but not the other—then that wouldn't be true; if we change one side of the equation but not the other then the two sides are no longer equal.

ADVANTAGE OF COMBINING LIKE TERMS

Combining like terms refers to adding or subtracting like terms—terms with the same variable—and therefore reducing sets of like terms to a single term. The main advantage of doing this is that it simplifies the equation. Often, combining like terms can be done as the first step in solving an equation, though it can also be done later, such as after distributing terms in a product.

For example, consider the equation $2(x + 3) + 3(2 + x + 3) = -4$. The 2 and the 3 in the second set of parentheses are like terms, and we can combine them, yielding $2(x + 3) + 3(x + 5) = -4$. Now we can carry out the multiplications implied by the parentheses, distributing the outer 2 and 3 accordingly: $2x + 6 + 3x + 15 = -4$. The $2x$ and the $3x$ are like terms, and we can add them together: $5x + 6 + 15 = -4$. Now, the constants 6, 15, and –4 are also like terms, and we can combine them as well: subtracting 6 and 15 from both sides of the equation, we get $5x = -4 - 6 - 15$, or $5x = -25$, which simplifies further to $x = -5$.

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CANCELING TERMS ON OPPOSITE SIDES OF AN EQUATION

Two terms on opposite sides of an equation can be canceled if and only if they *exactly* match each other. They must have the same variable raised to the same power and the same coefficient. For example, in the equation $3x + 2x^2 + 6 = 2x^2 - 6$, $2x^2$ appears on both sides of the equation and can be canceled, leaving $3x + 6 = -6$. The 6 on each side of the equation *cannot* be canceled, because it is added on one side of the equation and subtracted on the other. While they cannot be canceled, however, the 6 and –6 are like terms and can be combined, yielding $3x = -12$, which simplifies further to $x = -4$.

It's also important to note that the terms to be canceled must be independent terms and cannot be part of a larger term. For example, consider the equation $2(x + 6) = 3(x + 4) + 1$. We cannot cancel the x's, because even though they match each other they are part of the larger terms $2(x + 6)$ and $3(x + 4)$. We must first distribute the 2 and 3, yielding $2x + 12 = 3x + 12 + 1$. Now we see that the terms with the x's do not match, but the 12s do, and can be canceled, leaving $2x = 3x + 1$, which simplifies to $x = -1$.

PROCESS FOR MANIPULATING EQUATIONS

ISOLATING VARIABLES

To **isolate a variable** means to manipulate the equation so that the variable appears by itself on one side of the equation, and does not appear at all on the other side. Generally, an equation or inequality is considered to be solved once the variable is isolated and the other side of the equation or inequality is simplified as much as possible. In the case of a two-variable equation or inequality, only one variable needs to be isolated; it will not usually be possible to simultaneously isolate both variables.

For a linear equation—an equation in which the variable only appears raised to the first power—isolating a variable can be done by first moving all the terms with the variable to one side of the equation and all other terms to the other side. (*Moving* a term really means adding the inverse of the term to both sides; when a term is *moved* to the other side of the equation its sign is flipped.) Then combine like terms on each side. Finally, divide both sides by the coefficient of the variable, if applicable. The steps need not necessarily be done in this order, but this order will always work.

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Equations with More Than One Solution

Some types of non-linear equations, such as equations involving squares of variables, may have more than one solution. For example, the equation $x^2 = 4$ has two solutions: 2 and -2. Equations with absolute values can also have multiple solutions: $|x| = 1$ has the solutions $x = 1$ and $x = -1$.

It is also possible for a linear equation to have more than one solution, but only if the equation is true regardless of the value of the variable. In this case, the equation is considered to have infinitely many solutions, because any possible value of the variable is a solution. We know a linear equation has infinitely many solutions if when we combine like terms the variables cancel, leaving a true statement. For example, consider the equation $2(3x + 5) = x + 5(x + 2)$. Distributing, we get $6x + 10 = x + 5x + 10$; combining like terms gives $6x + 10 = 6x + 10$, and the $6x$-terms cancel to leave $10 = 10$. This is clearly true, so the original equation is true for any value of x. We could also have canceled the 10s leaving $0 = 0$, but again this is clearly true—in general if both sides of the equation match exactly, it has infinitely many solutions.

Equations with No Solution

Some types of non-linear equations, such as equations involving squares of variables, may have no solution. For example, the equation $x^2 = -2$ has no solutions in the real numbers, because the square of any real number must be positive. Similarly, $|x| = -1$ has no solution, because the absolute value of a number is always positive.

It is also possible for an equation to have no solution even if does not involve any powers greater than one, absolute values, or other special functions. For example, the equation $2(x + 3) + x = 3x$ has no solution. We can see that if we try to solve it: first we distribute, leaving $2x + 6 + x = 3x$. But now if we try to combine all the terms with the variable, we find that they cancel: we have $3x$ on the left and $3x$ on the right, canceling to leave us with $6 = 0$. This is clearly false. In general, whenever the variable terms in an equation cancel leaving different constants on both sides, it means that the equation has no solution. (If we are left with the *same* constant on both sides, the equation has infinitely many solutions instead.)

Features of Equations That Require Special Treatment

Linear Equations

A linear equation is an equation in which variables only appear by themselves: not multiplied together, not with exponents other than one, and not inside absolute value signs or any other functions. For example, the equation $x + 1 - 3x = 5 - x$ is a linear equation; while x appears multiple times, it never appears with an exponent other than one, or inside any function. The two-variable equation $2x - 3y = 5 + 2x$ is also a linear equation. In contrast, the equation $x^2 - 5 = 3x$ is *not* a linear equation, because it involves the term x^2. $\sqrt{x} = 5$ is not a linear equation, because it involves a square root. $(x - 1)^2 = 4$ is not a linear equation because even though there's no exponent on the x directly, it appears as part of an expression that is squared. The two-variable equation $x + xy - y = 5$ is not a linear equation because it includes the term xy, where two variables are multiplied together.

Linear equations can always be solved (or shown to have no solution) by combining like terms and performing simple operations on both sides of the equation. Some non-linear equations can be solved by similar methods, but others may require more advanced methods of solution, if they can be solved analytically at all.

Solving Equations Involving Roots

In an equation involving roots, the first step is to isolate the term with the root, if possible, and then raise both sides of the equation to the appropriate power to eliminate it. Consider an example equation, $2\sqrt{x+1} - 1 = 3$. In this case, begin by adding 1 to both sides, yielding $2\sqrt{x+1} = 4$, and then dividing both sides by 2, yielding $\sqrt{x+1} = 2$. Now square both sides, yielding $x + 1 = 4$. Finally, subtracting 1 from both sides yields $x = 3$.

Squaring both sides of an equation may, however, yield a spurious solution—a solution to the squared equation that is *not* a solution of the original equation. It's therefore necessary to plug the solution back into the original equation to make sure it works. In this case, it does: $2\sqrt{3+1} - 1 = 2\sqrt{4} - 1 = 2(2) - 1 = 4 - 1 = 3$.

The same procedure applies for other roots as well. For example, given the equation $3 + \sqrt[3]{2x} = 5$, we can first subtract 3 from both sides, yielding $\sqrt[3]{2x} = 2$ and isolating the root. Raising both sides to the third power yields $2x = 2^3$; i.e., $2x = 8$. We can now divide both sides by 2 to get $x = 4$.

Review Video: Solving Equations Involving Roots
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Solving Equations with Exponents

To solve an equation involving an exponent, the first step is to isolate the variable with the exponent. We can then take the appropriate root of both sides to eliminate the exponent. For instance, for the equation $2x^3 + 17 = 5x^3 - 7$, we can subtract $5x^3$ from both sides to get $-3x^3 + 17 = -7$, and then subtract 17 from both sides to get $-3x^3 = -24$. Finally, we can divide both sides by -3 to get $x^3 = 8$. Finally, we can take the cube root of both sides to get $x = \sqrt[3]{8} = 2$.

One important but often overlooked point is that equations with an exponent greater than 1 may have more than one answer. The solution to $x^2 = 9$ isn't simply $x = 3$; it's $x = \pm 3$ (that is, $x = 3$ or $x = -3$). For a slightly more complicated example, consider the equation $(x - 1)^2 - 1 = 3$. Adding 1 to both sides yields $(x - 1)^2 = 4$; taking the square root of both sides yields $x - 1 = 2$. We can then add 1 to both sides to get $x = 3$. However, there's a second solution. We also have the possibility that $x - 1 = -2$, in which case $x = -1$. Both $x = 3$ and $x = -1$ are valid solutions, as can be verified by substituting them both into the original equation.

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Solving Equations with Absolute Values

When solving an equation with an absolute value, the first step is to isolate the absolute value term. We then consider two possibilities: when the expression inside the absolute value is positive or when it is negative. In the former case, the expression in the absolute value equals the expression on the other side of the equation; in the latter, it equals the additive inverse of that expression—the expression times negative one. We consider each case separately and finally check for spurious solutions.

For instance, consider solving $|2x - 1| + x = 5$ for x. We can first isolate the absolute value by moving the x to the other side: $|2x - 1| = -x + 5$. Now, we have two possibilities. First, that $2x - 1$ is positive, and hence $2x - 1 = -x + 5$. Rearranging and combining like terms yields $3x = 6$, and hence $x = 2$. The other possibility is that $2x - 1$ is negative, and hence $2x - 1 = -(-x + 5) = x - 5$. In this case, rearranging and combining like terms yields $x = -4$. Substituting $x = 2$ and $x = -4$ back into the original equation, we see that they are both valid solutions.

Note that the absolute value of a sum or difference applies to the sum or difference as a whole, not to the individual terms; in general, $|2x - 1|$ is not equal to $|2x + 1|$ or to $|2x| - 1$.

Spurious Solutions

A **spurious solution** may arise when we square both sides of an equation as a step in solving it or under certain other operations on the equation. It is a solution to the squared or otherwise modified equation that is *not* a solution of the original equation. To identify a spurious solution, it's useful when you solve an equation involving roots or absolute values to plug the solution back into the original equation to make sure it's valid.

Choosing Which Variable to Isolate in Two-Variable Equations

Similar to methods for a one-variable equation, solving a two-variable equation involves isolating a variable: manipulating the equation so that a variable appears by itself on one side of the equation, and not at all on the other side. However, in a two-variable equation, you will usually only be able to isolate one of the variables; the other variable may appear on the other side along with constant terms, or with exponents or other functions.

Often one variable will be much more easily isolated than the other, and therefore that's the variable you should choose. If one variable appears with various exponents, and the other is only raised to the first power, the latter variable is the one to isolate: given the equation $a^2 + 2b = a^3 + b + 3$, the b only appears to the first power, whereas a appears squared and cubed, so b is the variable that can be solved for: combining like terms and isolating the b on the left side of the equation, we get $b = a^3 - a^2 + 3$. If both variables are equally easy to isolate, then it's best to isolate the dependent variable, if one is defined; if the two variables are x and y, the convention is that y is the dependent variable.

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Cross Multiplication

Finding an Unknown in Equivalent Expressions

It is often necessary to apply information given about a rate or proportion to a new scenario. For example, if you know that Jedha can run a marathon (26.2 miles) in 3 hours, how long would it take her to run 10 miles at the same pace? Start by setting up equivalent expressions:

$$\frac{26.2 \text{ mi}}{3 \text{ hr}} = \frac{10 \text{ mi}}{x \text{ hr}}$$

Now, cross multiply and solve for x:

$$26.2x = 30$$
$$x = \frac{30}{26.2} = \frac{15}{13.1}$$
$$x \approx 1.15 \text{ hrs } or \text{ } 1 \text{ hr } 9 \text{ min}$$

So, at this pace, Jedha could run 10 miles in about 1.15 hours or about 1 hour and 9 minutes.

Review Video: Cross Multiplying Fractions
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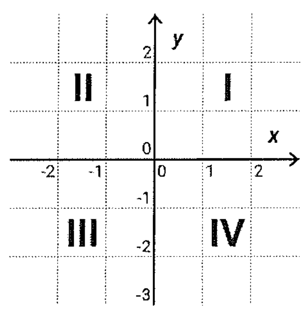
Graphing Equations

Graphical Solutions to Equations

When equations are shown graphically, they are usually shown on a **Cartesian coordinate plane**. The Cartesian coordinate plane consists of two number lines placed perpendicular to each other and intersecting at the zero point, also known as the origin. The horizontal number line is known as the x-axis, with positive values to the right of the origin, and negative values to the left of the origin. The vertical number line is known as the y-axis, with positive values above the origin, and negative values below the origin. Any point on the plane can be identified by an ordered pair in the form (x, y), called coordinates. The x-value of the coordinate is called the abscissa, and the y-value of the coordinate is called the ordinate. The two number lines divide the plane into **four quadrants**: I, II, III, and IV.



Note that in quadrant I $x > 0$ and $y > 0$, in quadrant II $x < 0$ and $y > 0$, in quadrant III $x < 0$ and $y < 0$, and in quadrant IV $x > 0$ and $y < 0$.

Recall that if the value of the slope of a line is positive, the line slopes upward from left to right. If the value of the slope is negative, the line slopes downward from left to right. If the y-coordinates are the same for two points on a line, the slope is 0 and the line is a **horizontal line**. If the x-coordinates are the same for two points on a line, there is no slope and the line is a **vertical line**. Two or more lines that have equivalent slopes are **parallel lines**. **Perpendicular lines** have slopes that are negative reciprocals of each other, such as $\frac{a}{b}$ and $\frac{-b}{a}$.

Review Video: Cartesian Coordinate Plane and Graphing
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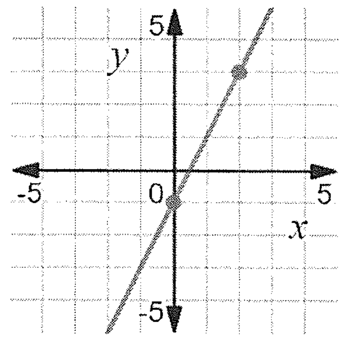
Graphing Equations in Two Variables

One way of graphing an equation in two variables is to plot enough points to get an idea for its shape and then draw the appropriate curve through those points. A point can be plotted by substituting in a value for one variable and solving for the other. If the equation is linear, we only need two points and can then draw a straight line between them.

For example, consider the equation $y = 2x - 1$. This is a linear equation—both variables only appear raised to the first power—so we only need two points. When $x = 0, y = 2(0) - 1 = -1$. When $x = 2, y = 2(2) - 1 = 3$. We can therefore choose the points $(0, -1)$ and $(2, 3)$, and draw a line between them:



Inequalities

Working with Inequalities

Commonly in algebra and other upper-level fields of math you find yourself working with mathematical expressions that do not equal each other. The statement comparing such expressions with symbols such as $<$ (less than) or $>$ (greater than) is called an *inequality*. An example of an inequality is $7x > 5$. To solve for x, simply divide both sides by 7 and the solution is shown to be $x > \frac{5}{7}$. Graphs of the solution set of inequalities are represented on a number line. Open circles are used to show that an expression approaches a number but is never quite equal to that number.

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Conditional inequalities are those with certain values for the variable that will make the condition true and other values for the variable where the condition will be false. **Absolute inequalities** can have any real number as the value for the variable to make the condition true, while there is no real number value for the variable that will make the condition false. Solving inequalities is done by following the same rules for solving equations with the exception that when multiplying or dividing by a negative number the direction of the inequality sign must be flipped or reversed. **Double inequalities** are situations where two inequality statements apply to the same variable expression. Example: $-c < ax + b < c$.

Review Video: Conditional and Absolute Inequalities
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Determining Solutions to Inequalities

To determine whether a coordinate is a solution of an inequality, you can substitute the values of the coordinate into the inequality, simplify, and check whether the resulting statement holds true.

ADD POLYNOMIALS

To add polynomials, you need to add like terms. These terms have the same variable part. An example is $4x^2$ and $3x^2$ have x^2 terms. To find the sum of like terms, find the sum of the coefficients. Then, keep the same variable part. You can use the distributive property to distribute the plus sign to each term of the polynomial. For example:

$$(4x^2 - 5x + 7) + (3x^2 + 2x + 1) =$$
$$(4x^2 - 5x + 7) + 3x^2 + 2x + 1 =$$
$$(4x^2 + 3x^2) + (-5x + 2x) + (7 + 1) =$$
$$7x^2 - 3x + 8$$

SUBTRACT POLYNOMIALS

To subtract polynomials, you need to subtract like terms. To find the difference of like terms, find the difference of the coefficients. Then, keep the same variable part. You can use the distributive property to distribute the minus sign to each term of the polynomial. For example:

$$(-2x^2 - x + 5) - (3x^2 - 4x + 1) =$$
$$(-2x^2 - x + 5) - 3x^2 + 4x - 1 =$$
$$(-2x^2 - 3x^2) + (-x + 4x) + (5 - 1) =$$
$$-5x^2 + 3x + 4$$

Review Video: Adding and Subtracting Polynomials
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MULTIPLYING POLYNOMIALS

In general, multiplying polynomials is done by multiplying each term in one polynomial by each term in the other and adding the results. In the specific case for multiplying binomials, there is a useful acronym, FOIL, that can help you make sure to cover each combination of terms. The **FOIL method** for $(Ax + By)(Cx + Dy)$ would be:

F	Multiply the *first* terms of each binomial	$(\overset{first}{Ax} + By)(\overset{first}{Cx} + Dy)$	ACx^2
O	Multiply the *outer* terms	$(\overset{outer}{Ax} + By)(Cx + \overset{outer}{Dy})$	$ADxy$
I	Multiply the *inner* terms	$(Ax + \overset{inner}{By})(\overset{inner}{Cx} + Dy)$	$BCxy$
L	Multiply the *last* terms of each binomial	$(Ax + \overset{last}{By})(Cx + \overset{last}{Dy})$	BDy^2

Then, add up the result of each and combine like terms: $ACx^2 + (AD + BC)xy + BDy^2$.

For example, using the FOIL method on binomials $(x + 2)$ and $(x - 3)$:

First: $(\boxed{x} + 2)(\boxed{x} + (-3)) \rightarrow (x)(x) = x^2$
Outer: $(\boxed{x} + 2)(x + \boxed{(-3)}) \rightarrow (x)(-3) = -3x$
Inner: $(x + \boxed{2})(\boxed{x} + (-3)) \rightarrow (2)(x) = 2x$
Last: $(x + \boxed{2})(x + \boxed{(-3)}) \rightarrow (2)(-3) = -6$

This results in: $(x^2) + (-3x) + (2x) + (-6)$

Combine like terms: $x^2 + (-3 + 2)x + (-6) = x^2 - x - 6$

Review Video: Multiplying Terms Using the FOIL Method
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DIVIDING POLYNOMIALS

Use long division to divide a polynomial by either a monomial or another polynomial of equal or lesser degree.

When **dividing by a monomial**, divide each term of the polynomial by the monomial.

When **dividing by a polynomial**, begin by arranging the terms of each polynomial in order of one variable. You may arrange in ascending or descending order, but be consistent with both polynomials. To get the first term of the quotient, divide the first term of the dividend by the first term of the divisor. Multiply the first term of the quotient by the entire divisor and subtract that product from the dividend. Repeat for the second and successive terms until you either get a remainder of zero or a remainder whose degree is less than the degree of the divisor. If the quotient has a remainder, write the answer as a mixed expression in the form:

$$\text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

For example, we can evaluate the following expression in the same way as long division:

$$\frac{x^3 - 3x^2 - 2x + 5}{x - 5}$$

$$
\begin{array}{rl}
 & \ x^2 \quad + 2x \quad + 8 \\
x - 5 & \overline{)\ x^3 - 3x^2 \ - 2x \ + 5} \\
 & \underline{-(x^3 - 5x^2)} \\
 & \qquad 2x^2 - 2x \\
 & \qquad \underline{-(2x^2 - 10x)} \\
 & \qquad\qquad 8x + 5 \\
 & \qquad\qquad \underline{-(8x - 40)} \\
 & \qquad\qquad\quad 45
\end{array}
$$

$$\frac{x^3 - 3x^2 - 2x + 5}{x - 5} = x^2 + 2x + 8 + \frac{45}{x - 5}$$

When **factoring** a polynomial, first check for a common monomial factor, that is, look to see if each coefficient has a common factor or if each term has an x in it. If the factor is a trinomial but not a perfect trinomial square, look for a factorable form, such as one of these:

$$x^2 + (a + b)x + ab = (x + a)(x + b)$$
$$(ac)x^2 + (ad + bc)x + bd = (ax + b)(cx + d)$$

For factors with four terms, look for groups to factor. Once you have found the factors, write the original polynomial as the product of all the factors. Make sure all of the polynomial factors are prime. Monomial factors may be *prime* or *composite*. Check your work by multiplying the factors to make sure you get the original polynomial.

Below are patterns of some special products to remember to help make factoring easier:

- Perfect trinomial squares: $x^2 + 2xy + y^2 = (x + y)^2$ or $x^2 - 2xy + y^2 = (x - y)^2$
- Difference between two squares: $x^2 - y^2 = (x + y)(x - y)$
- Sum of two cubes: $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$
    - Note: the second factor is *not* the same as a perfect trinomial square, so do not try to factor it further.
- Difference between two cubes: $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
    - Again, the second factor is *not* the same as a perfect trinomial square.
- Perfect cubes: $x^3 + 3x^2y + 3xy^2 + y^3 = (x + y)^3$ and $x^3 - 3x^2y + 3xy^2 - y^3 = (x - y)^3$

Rational Expressions

Rational expressions are fractions with polynomials in both the numerator and the denominator; the value of the polynomial in the denominator cannot be equal to zero. Be sure to keep track of values that make the denominator of the original expression zero as the final result inherits the same restrictions. For example, a denominator of $x - 3$ indicates that the expression is not defined when $x = 3$ and, as such, regardless of any operations done to the expression, it remains undefined there.

To **add or subtract** rational expressions, first find the common denominator, then rewrite each fraction as an equivalent fraction with the common denominator. Finally, add or subtract the numerators to get the numerator of the answer, and keep the common denominator as the denominator of the answer.

When **multiplying** rational expressions, factor each polynomial and cancel like factors (a factor which appears in both the numerator and the denominator). Then, multiply all remaining factors in the numerator to get the numerator of the product, and multiply the remaining factors in the denominator to get the denominator of the product. Remember: cancel entire factors, not individual terms.

To **divide** rational expressions, take the reciprocal of the divisor (the rational expression you are dividing by) and multiply by the dividend.

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Simplifying Rational Expressions

To simplify a rational expression, factor the numerator and denominator completely. Factors that are the same and appear in the numerator and denominator have a ratio of 1. For example, look at the following expression:

$$\frac{x - 1}{1 - x^2}$$

The denominator, $(1 - x^2)$, is a difference of squares. It can be factored as $(1 - x)(1 + x)$. The factor $1 - x$ and the numerator $x - 1$ are opposites and have a ratio of -1. Rewrite the numerator as $-1(1 - x)$. So, the rational expression can be simplified as follows:

$$\frac{x-1}{1-x^2} = \frac{-1(1-x)}{(1-x)(1+x)} = \frac{-1}{1+x}$$

Note that since the original expression is only defined for $x \neq \{-1, 1\}$, the simplified expression has the same restrictions.

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QUADRATICS

SOLVING QUADRATIC EQUATIONS

Quadratic equations are a special set of trinomials of the form $y = ax^2 + bx + c$ that occur commonly in math and real-world applications. The **roots** of a quadratic equation are the solutions that satisfy the equation when $y = 0$; in other words, where the graph touches the x-axis. There are several ways to determine these solutions including using the quadratic formula, factoring, completing the square, and graphing the function.

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Review Video: Solutions of a Quadratic Equation on a Graph
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QUADRATIC FORMULA

The **quadratic formula** is used to solve quadratic equations when other methods are more difficult. To use the quadratic formula to solve a quadratic equation, begin by rewriting the equation in standard form $ax^2 + bx + c = 0$, where a, b, and c are coefficients. Once you have identified the values of the coefficients, substitute those values into the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Evaluate the equation and simplify the expression. Again, check each root by substituting into the original equation. In the quadratic formula, the portion of the formula under the radical $(b^2 - 4ac)$ is called the **discriminant**. If the discriminant is zero, there is only one root: $-\frac{b}{2a}$. If the discriminant is positive, there are two different real roots. If the discriminant is negative, there are no real roots; you will instead find complex roots. Often these solutions don't make sense in context and are ignored.

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FACTORING

To solve a quadratic equation by factoring, begin by rewriting the equation in standard form, $x^2 + bx + c = 0$. Remember that the goal of factoring is to find numbers f and g such that $(x + f)(x + g) = x^2 + (f + g)x + fg$, in other words $(f + g) = b$ and $fg = c$. This can be a really useful method when b and c are integers. Determine the factors of c and look for pairs that could sum to b.

For example, consider finding the roots of $x^2 + 6x - 16 = 0$. The factors of -16 include, -4 and 4, -8 and 2, -2 and 8, -1 and 16, and 1 and -16. The factors that sum to 6 are -2 and 8. Write these factors as the product of two binomials, $0 = (x-2)(x+8)$. Finally, since these binomials multiply together to equal zero, set them each equal to zero and solve each for x. This results in $x - 2 = 0$, which simplifies to $x = 2$ and $x + 8 = 0$, which simplifies to $x = -8$. Therefore, the roots of the equation are 2 and -8.

Review Video: Factoring Quadratic Equations
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Completing the Square

One way to find the roots of a quadratic equation is to find a way to manipulate it such that it follows the form of a perfect square $(x^2 + 2px + p^2)$ by adding and subtracting a constant. This process is called **completing the square**. In other words, if you are given a quadratic that is not a perfect square, $x^2 + bx + c = 0$, you can find a constant d that could be added in to make it a perfect square:

$$x^2 + bx + c + (d - d) = 0;\ \{\text{Let } b = 2p \text{ and } c + d = p^2\}$$

then:

$$x^2 + 2px + p^2 - d = 0 \text{ and } d = \frac{b^2}{4} - c$$

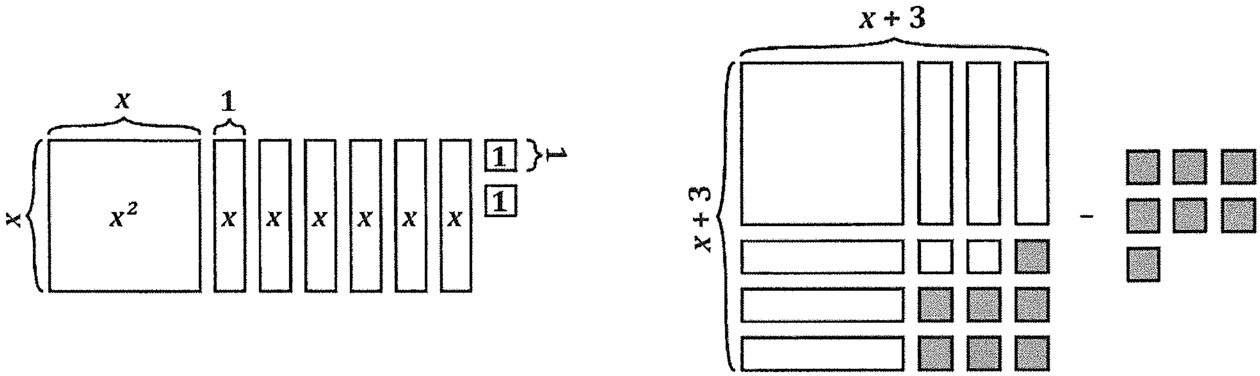
Once you have completed the square you can find the roots of the resulting equation:

$$\begin{aligned} x^2 + 2px + p^2 - d &= 0 \\ (x+p)^2 &= d \\ x + p &= \pm\sqrt{d} \\ x &= -p \pm \sqrt{d} \end{aligned}$$

It is worth noting that substituting the original expressions into this solution gives the same result as the quadratic formula where $a = 1$:

$$x = -p \pm \sqrt{d} = -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} - c} = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2} = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Completing the square can be seen as arranging block representations of each of the terms to be as close to a square as possible and then filling in the gaps. For example, consider the quadratic expression $x^2 + 6x + 2$:



$$x^2 + 6x + 2 \quad = \quad (x+3)^2 - 7$$

Review Video: Completing the Square
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Using Given Roots to Find Quadratic Equation

One way to find the roots of a quadratic equation is to factor the equation and use the **zero product property**, setting each factor of the equation equal to zero to find the corresponding root. We can use this technique in reverse to find an equation given its roots. Each root corresponds to a linear equation which in turn corresponds to a factor of the quadratic equation.

For example, we can find a quadratic equation whose roots are $x = 2$ and $x = -1$. The root $x = 2$ corresponds to the equation $x - 2 = 0$, and the root $x = -1$ corresponds to the equation $x + 1 = 0$.

These two equations correspond to the factors $(x - 2)$ and $(x + 1)$, from which we can derive the equation $(x - 2)(x + 1) = 0$, or $x^2 - x - 2 = 0$.

Any integer multiple of this entire equation will also yield the same roots, as the integer will simply cancel out when the equation is factored. For example, $2x^2 - 2x - 4 = 0$ factors as $2(x - 2)(x + 1) = 0$.

Problem-Solving and Data Analysis

Fractions, Decimals, and Percentages

Fractions

A **fraction** is a number that is expressed as one integer written above another integer, with a dividing line between them $\left(\frac{x}{y}\right)$. It represents the **quotient** of the two numbers "x divided by y." It can also be thought of as x out of y equal parts.

The top number of a fraction is called the **numerator**, and it represents the number of parts under consideration. The 1 in $\frac{1}{4}$ means that 1 part out of the whole is being considered in the calculation. The bottom number of a fraction is called the **denominator**, and it represents the total number of equal parts. The 4 in $\frac{1}{4}$ means that the whole consists of 4 equal parts. A fraction cannot have a denominator of zero; this is referred to as "*undefined*."

Fractions can be manipulated, without changing the value of the fraction, by multiplying or dividing (but not adding or subtracting) both the numerator and denominator by the same number. If you divide both numbers by a common factor, you are **reducing** or simplifying the fraction. Two fractions that have the same value but are expressed differently are known as **equivalent fractions**. For example, $\frac{2}{10}, \frac{3}{15}, \frac{4}{20}$, and $\frac{5}{25}$ are all equivalent fractions. They can also all be reduced or simplified to $\frac{1}{5}$.

When two fractions are manipulated so that they have the same denominator, this is known as finding a **common denominator**. The number chosen to be that common denominator should be the least common multiple of the two original denominators. Example: $\frac{3}{4}$ and $\frac{5}{6}$; the least common multiple of 4 and 6 is 12. Manipulating to achieve the common denominator: $\frac{3}{4} = \frac{9}{12}; \frac{5}{6} = \frac{10}{12}$.

Review Video: Overview of Fractions
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Proper Fractions and Mixed Numbers

A fraction whose denominator is greater than its numerator is known as a **proper fraction**, while a fraction whose numerator is greater than its denominator is known as an **improper fraction**. Proper fractions have values *less than one* and improper fractions have values *greater than one*.

A **mixed number** is a number that contains both an integer and a fraction. Any improper fraction can be rewritten as a mixed number. Example: $\frac{8}{3} = \frac{6}{3} + \frac{2}{3} = 2 + \frac{2}{3} = 2\frac{2}{3}$. Similarly, any mixed number can be rewritten as an improper fraction. Example: $1\frac{3}{5} = 1 + \frac{3}{5} = \frac{5}{5} + \frac{3}{5} = \frac{8}{5}$.

Review Video: Proper and Improper Fractions and Mixed Numbers
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Adding and Subtracting Fractions

If two fractions have a common denominator, they can be added or subtracted simply by adding or subtracting the two numerators and retaining the same denominator. If the two fractions do not already have the same denominator, one or both of them must be manipulated to achieve a common denominator before they can be added or subtracted. Example: $\frac{1}{2} + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} = \frac{3}{4}$.

Review Video: Adding and Subtracting Fractions
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Multiplying Fractions

Two fractions can be multiplied by multiplying the two numerators to find the new numerator and the two denominators to find the new denominator. Example: $\frac{1}{3} \times \frac{2}{3} = \frac{1\times 2}{3\times 3} = \frac{2}{9}$.

Dividing Fractions

Two fractions can be divided by flipping the numerator and denominator of the second fraction and then proceeding as though it were a multiplication problem. Example: $\frac{2}{3} \div \frac{3}{4} = \frac{2}{3} \times \frac{4}{3} = \frac{8}{9}$.

Review Video: Multiplying and Dividing Fractions
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Multiplying a Mixed Number by a Whole Number or a Decimal

When multiplying a mixed number by something, it is usually best to convert it to an improper fraction first. Additionally, if the multiplicand is a decimal, it is most often simplest to convert it to a fraction. For instance, to multiply $4\frac{3}{8}$ by 3.5, begin by rewriting each quantity as a whole number plus a proper fraction. Remember, a mixed number is a fraction added to a whole number and a decimal is a representation of the sum of fractions, specifically tenths, hundredths, thousandths, and so on:

$$4\frac{3}{8} \times 3.5 = \left(4 + \frac{3}{8}\right) \times \left(3 + \frac{1}{2}\right)$$

Next, the quantities being added need to be expressed with the same denominator. This is achieved by multiplying and dividing the whole number by the denominator of the fraction. Recall that a whole number is equivalent to that number divided by 1:

$$= \left(\frac{4}{1} \times \frac{8}{8} + \frac{3}{8}\right) \times \left(\frac{3}{1} \times \frac{2}{2} + \frac{1}{2}\right)$$

When multiplying fractions, remember to multiply the numerators and denominators separately:

$$= \left(\frac{4 \times 8}{1 \times 8} + \frac{3}{8}\right) \times \left(\frac{3 \times 2}{1 \times 2} + \frac{1}{2}\right)$$
$$= \left(\frac{32}{8} + \frac{3}{8}\right) \times \left(\frac{6}{2} + \frac{1}{2}\right)$$

Now that the fractions have the same denominators, they can be added:

$$= \frac{35}{8} \times \frac{7}{2}$$

Finally, perform the last multiplication and then simplify:

$$= \frac{35 \times 7}{8 \times 2} = \frac{245}{16} = \frac{240}{16} + \frac{5}{16} = 15\frac{5}{16}$$

Comparing Fractions

It is important to master the ability to compare and order fractions. This skill is relevant to many real-world scenarios. For example, carpenters often compare fractional construction nail lengths when preparing for a project, and bakers often compare fractional measurements to have the correct ratio of ingredients. There are three commonly used strategies when comparing fractions.

These strategies are referred to as the common denominator approach, the decimal approach, and the cross-multiplication approach.

Using a Common Denominator to Compare Fractions

The fractions $\frac{2}{3}$ and $\frac{4}{7}$ have different denominators. $\frac{2}{3}$ has a denominator of 3, and $\frac{4}{7}$ has a denominator of 7. In order to precisely compare these two fractions, it is necessary to use a common denominator. A common denominator is a common multiple that is shared by both denominators. In this case, the denominators 3 and 7 share a multiple of 21. In general, it is most efficient to select the least common multiple for the two denominators.

Rewrite each fraction with the common denominator of 21. Then, calculate the new numerators as illustrated below.

$$\frac{2}{3} \xrightarrow[\times 7]{\times 7} \frac{\mathbf{14}}{21} \qquad \frac{4}{7} \xrightarrow[\times 3]{\times 3} \frac{\mathbf{12}}{21}$$

$$\frac{2}{3} = \frac{14}{21} \qquad \frac{4}{7} = \frac{12}{21}$$

For $\frac{2}{3}$, multiply the numerator and denominator by 7. The result is $\frac{14}{21}$.

For $\frac{4}{7}$, multiply the numerator and denominator by 3. The result is $\frac{12}{21}$.

Now that both fractions have a denominator of 21, the fractions can accurately be compared by comparing the numerators. Since 14 is greater than 12, the fraction $\frac{14}{21}$ is greater than $\frac{12}{21}$. This means that $\frac{2}{3}$ is greater than $\frac{4}{7}$.

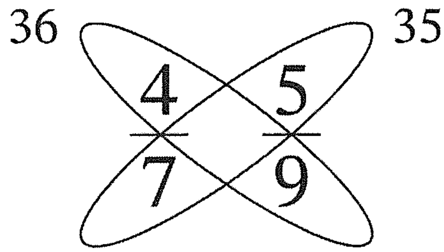
Using Decimals to Compare Fractions

Sometimes decimal values are easier to compare than fraction values. For example, $\frac{5}{8}$ is equivalent to 0.625 and $\frac{3}{5}$ is equivalent to 0.6. This means that the comparison of $\frac{5}{8}$ and $\frac{3}{5}$ can be determined by comparing the decimals 0.625 and 0.6. When both decimal values are extended to the thousandths place, they become 0.625 and 0.600, respectively. It becomes clear that 0.625 is greater than 0.600 because 625 thousandths is greater than 600 thousandths. In other words, $\frac{5}{8}$ is greater than $\frac{3}{5}$ because 0.625 is greater than 0.6.

Using Cross-Multiplication to Compare Fractions

Cross-multiplication is an efficient strategy for comparing fractions. This is a shortcut for the common denominator strategy. Start by writing each fraction next to one another. Multiply the numerator of the fraction on the left by the denominator of the fraction on the right. Write down the result next to the fraction on the left. Now multiply the numerator of the fraction on the right by the denominator of the fraction on the left. Write down the result next to the fraction on the right. Compare both products. The fraction with the larger result is the larger fraction.

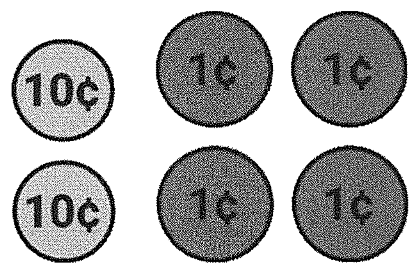
Consider the fractions $\frac{4}{7}$ and $\frac{5}{9}$.



36 is greater than 35. Therefore, $\frac{4}{7}$ is greater than $\frac{5}{9}$.

Decimals

Decimals are one way to represent parts of a whole. Using the place value system, each digit to the right of a decimal point denotes the number of units of a corresponding *negative* power of ten. For example, consider the decimal 0.24. We can use a model to represent the decimal. Since a dime is worth one-tenth of a dollar and a penny is worth one-hundredth of a dollar, one possible model to represent this fraction is to have 2 dimes representing the 2 in the tenths place and 4 pennies representing the 4 in the hundredths place:



To write the decimal as a fraction, put the decimal in the numerator with 1 in the denominator. Multiply the numerator and denominator by tens until there are no more decimal places. Then simplify the fraction to lowest terms. For example, converting 0.24 to a fraction:

$$0.24 = \frac{0.24}{1} = \frac{0.24 \times 100}{1 \times 100} = \frac{24}{100} = \frac{6}{25}$$

Review Video: Decimals
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Operations with Decimals

Adding and Subtracting Decimals

When adding and subtracting decimals, the decimal points must always be aligned. Adding decimals is just like adding regular whole numbers. Example: $4.5 + 2.0 = 6.5$.

If the problem-solver does not properly align the decimal points, an incorrect answer of 4.7 may result. An easy way to add decimals is to align all of the decimal points in a vertical column visually. This will allow you to see exactly where the decimal should be placed in the final answer. Begin adding from right to left. Add each column in turn, making sure to carry the number to the left if a column adds up to more than 9. The same rules apply to the subtraction of decimals.

Review Video: Adding and Subtracting Decimals
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Multiplying Decimals

A simple multiplication problem has two components: a **multiplicand** and a **multiplier**. When multiplying decimals, work as though the numbers were whole rather than decimals. Once the final product is calculated, count the number of places to the right of the decimal in both the multiplicand and the multiplier. Then, count that number of places from the right of the product and place the decimal in that position.

For example, 12.3×2.56 has a total of three places to the right of the respective decimals. Multiply 123×256 to get 31,488. Now, beginning on the right, count three places to the left and insert the decimal. The final product will be 31.488.

Review Video: How to Multiply Decimals
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Dividing Decimals

Every division problem has a **divisor** and a **dividend**. The dividend is the number that is being divided. In the problem $14 \div 7$, 14 is the dividend and 7 is the divisor. In a division problem with decimals, the divisor must be converted into a whole number. Begin by moving the decimal in the divisor to the right until a whole number is created. Next, move the decimal in the dividend the same number of spaces to the right. For example, 4.9 into 24.5 would become 49 into 245. The decimal was moved one space to the right to create a whole number in the divisor, and then the same was done for the dividend. Once the whole numbers are created, the problem is carried out normally: $245 \div 49 = 5$.

Review Video: Dividing Decimals
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Review Video: Dividing Decimals by Whole Numbers
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Percentages

Percentages can be thought of as fractions that are based on a whole of 100; that is, one whole is equal to 100%. The word **percent** means "per hundred." Percentage problems are often presented in three main ways:

- Find what percentage of some number another number is.
  - Example: What percentage of 40 is 8?
- Find what number is some percentage of a given number.
  - Example: What number is 20% of 40?
- Find what number another number is a given percentage of.
  - Example: What number is 8 20% of?

There are three components in each of these cases: a **whole** (W), a **part** (P), and a **percentage** (%). These are related by the equation: $P = W \times \%$. This can easily be rearranged into other forms that may suit different questions better: $\% = \frac{P}{W}$ and $W = \frac{P}{\%}$. Percentage problems are often also word problems. As such, a large part of solving them is figuring out which quantities are what. For example, consider the following word problem:

In a school cafeteria, 7 students choose pizza, 9 choose hamburgers, and 4 choose tacos. What percentage of student choose tacos?

To find the whole, you must first add all of the parts: $7 + 9 + 4 = 20$. The percentage can then be found by dividing the part by the whole $\left(\% = \frac{P}{W}\right)$: $\frac{4}{20} = \frac{20}{100} = 20\%$.

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Converting Between Percentages, Fractions, and Decimals

Converting decimals to percentages and percentages to decimals is as simple as moving the decimal point. To *convert from a decimal to a percentage*, move the decimal point **two places to the right**. To *convert from a percentage to a decimal*, move it **two places to the left**. It may be helpful to remember that the percentage number will always be larger than the equivalent decimal number. Example:

$$0.23 = 23\% \quad 5.34 = 534\% \quad 0.007 = 0.7\%$$
$$700\% = 7.00 \quad 86\% = 0.86 \quad 0.15\% = 0.0015$$

To convert a fraction to a decimal, simply divide the numerator by the denominator in the fraction. To convert a decimal to a fraction, put the decimal in the numerator with 1 in the denominator. Multiply the numerator and denominator by tens until there are no more decimal places. Then simplify the fraction to lowest terms. For example, converting 0.24 to a fraction:

$$0.24 = \frac{0.24}{1} = \frac{0.24 \times 100}{1 \times 100} = \frac{24}{100} = \frac{6}{25}$$

Fractions can be converted to a percentage by finding equivalent fractions with a denominator of 100. Example:

$$\frac{7}{10} = \frac{70}{100} = 70\% \quad \frac{1}{4} = \frac{25}{100} = 25\%$$

To convert a percentage to a fraction, divide the percentage number by 100 and reduce the fraction to its simplest possible terms. Example:

$$60\% = \frac{60}{100} = \frac{3}{5} \quad 96\% = \frac{96}{100} = \frac{24}{25}$$

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Review Video: Converting Decimals to Fractions and Percentages
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Review Video: Converting Decimals, Improper Fractions, and Mixed Numbers
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PROPORTIONS AND RATIOS

PROPORTIONS

A proportion is a relationship between two quantities that dictates how one changes when the other changes. A **direct proportion** describes a relationship in which a quantity increases by a set amount for every increase in the other quantity, or decreases by that same amount for every decrease in the other quantity. Example: Assuming a constant driving speed, the time required for a car trip increases as the distance of the trip increases. The distance to be traveled and the time required to travel are directly proportional.

An **inverse proportion** is a relationship in which an increase in one quantity is accompanied by a decrease in the other, or vice versa. Example: the time required for a car trip decreases as the speed increases and increases as the speed decreases, so the time required is inversely proportional to the speed of the car.

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RATIOS

A **ratio** is a comparison of two quantities in a particular order. Example: If there are 14 computers in a lab, and the class has 20 students, there is a student to computer ratio of 20 to 14, commonly written as $20:14$. Ratios are normally reduced to their smallest whole number representation, so $20:14$ would be reduced to $10:7$ by dividing both sides by 2.

Review Video: Ratios
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CONSTANT OF PROPORTIONALITY

When two quantities have a proportional relationship, there exists a **constant of proportionality** between the quantities. The product of this constant and one of the quantities is equal to the other quantity. For example, if one lemon costs $0.25, two lemons cost $0.50, and three lemons cost $0.75, there is a proportional relationship between the total cost of lemons and the number of lemons purchased. The constant of proportionality is the **unit price**, namely $0.25/lemon. Notice that the total price of lemons, t, can be found by multiplying the unit price of lemons, p, and the number of lemons, n: $t = pn$.

WORK/UNIT RATE

Unit rate expresses a quantity of one thing in terms of one unit of another. For example, if you travel 30 miles every two hours, a unit rate expresses this comparison in terms of one hour: in one hour you travel 15 miles, so your unit rate is 15 miles per hour. Other examples are how much one ounce of food costs (price per ounce) or figuring out how much one egg costs out of the dozen (price per 1 egg, instead of price per 12 eggs). The denominator of a unit rate is always 1. Unit rates are used to compare different situations to solve problems. For example, to make sure you get the best deal when deciding which kind of soda to buy, you can find the unit rate of each. If soda #1 costs $1.50 for a 1-liter bottle, and soda #2 costs $2.75 for a 2-liter bottle, it would be a better deal to buy soda #2, because its unit rate is only $1.375 per 1-liter, which is cheaper than soda #1. Unit rates can also help determine the length of time a given event will take. For example, if you can paint 2 rooms in 4.5 hours, you can determine how long it will take you to paint 5 rooms by solving for the unit rate per room and then multiplying that by 5.

Review Video: Rates and Unit Rates
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METRIC AND CUSTOMARY MEASUREMENTS

METRIC MEASUREMENT PREFIXES

Giga-	One billion	1 *giga*watt is one billion watts
Mega-	One million	1 *mega*hertz is one million hertz
Kilo-	One thousand	1 *kilo*gram is one thousand grams
Deci-	One-tenth	1 *deci*meter is one-tenth of a meter
Centi-	One-hundredth	1 *centi*meter is one-hundredth of a meter
Milli-	One-thousandth	1 *milli*liter is one-thousandth of a liter
Micro-	One-millionth	1 *micro*gram is one-millionth of a gram

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MEASUREMENT CONVERSION

When converting between units, the goal is to maintain the same meaning but change the way it is displayed. In order to go from a larger unit to a smaller unit, multiply the number of the known amount by the equivalent amount. When going from a smaller unit to a larger unit, divide the number of the known amount by the equivalent amount.

For complicated conversions, it may be helpful to set up conversion fractions. In these fractions, one fraction is the **conversion factor**. The other fraction has the unknown amount in the numerator. So, the known value is placed in the denominator. Sometimes, the second fraction has the known value from the problem in the numerator and the unknown in the denominator. Multiply the two fractions to get the converted measurement. Note that since the numerator and the denominator of the factor are equivalent, the value of the fraction is 1. That is why we can say that the result in the new units is equal to the result in the old units even though they have different numbers.

It can often be necessary to chain known conversion factors together. As an example, consider converting 512 square inches to square meters. We know that there are 2.54 centimeters in an inch and 100 centimeters in a meter, and we know we will need to square each of these factors to achieve the conversion we are looking for.

$$\frac{512\text{ in}^2}{1} \times \left(\frac{2.54\text{ cm}}{1\text{ in}}\right)^2 \times \left(\frac{1\text{ m}}{100\text{ cm}}\right)^2 = \frac{512\,\cancel{\text{in}^2}}{1} \times \left(\frac{6.4516\,\cancel{\text{cm}^2}}{1\,\cancel{\text{in}^2}}\right) \times \left(\frac{1\text{ m}^2}{10{,}000\,\cancel{\text{cm}^2}}\right) = 0.330\text{ m}^2$$

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Common Units and Equivalents

Metric Equivalents

1000 μg (microgram)	1 mg
1000 mg (milligram)	1 g
1000 g (gram)	1 kg
1000 kg (kilogram)	1 metric ton
1000 mL (milliliter)	1 L
1000 μm (micrometer)	1 mm
1000 mm (millimeter)	1 m
100 cm (centimeter)	1 m
1000 m (meter)	1 km

Distance and Area Measurement

Unit	Abbreviation	US equivalent	Metric equivalent
Inch	in	1 inch	2.54 centimeters
Foot	ft	12 inches	0.305 meters
Yard	yd	3 feet	0.914 meters
Mile	mi	5280 feet	1.609 kilometers
Acre	ac	4840 square yards	0.405 hectares
Square Mile	sq. mi. or mi.^{2}	640 acres	2.590 square kilometers

Capacity Measurements

Unit	Abbreviation	US equivalent	Metric equivalent
Fluid Ounce	fl oz	8 fluid drams	29.573 milliliters
Cup	c	8 fluid ounces	0.237 liter
Pint	pt.	16 fluid ounces	0.473 liter
Quart	qt.	2 pints	0.946 liter
Gallon	gal.	4 quarts	3.785 liters
Teaspoon	t or tsp.	1 fluid dram	5 milliliters
Tablespoon	T or tbsp.	4 fluid drams	15 or 16 milliliters
Cubic Centimeter	cc or cm^3	0.271 drams	1 milliliter

Weight Measurements

Unit	Abbreviation	US equivalent	Metric equivalent
Ounce	oz	16 drams	28.35 grams
Pound	lb	16 ounces	453.6 grams
Ton	tn.	2,000 pounds	907.2 kilograms

Volume and Weight Measurement Clarifications

Always be careful when using ounces and fluid ounces. They are not equivalent.

1 pint = 16 fluid ounces	1 fluid ounce $\neq$ 1 ounce
1 pound = 16 ounces	1 pint $\neq$ 1 pound

Having one pint of something does not mean you have one pound of it. In the same way, just because something weighs one pound does not mean that its volume is one pint.

In the United States, the word "ton" by itself refers to a short ton or a net ton. Do not confuse this with a long ton (also called a gross ton) or a metric ton (also spelled *tonne*), which have different measurement equivalents.

$$1 \text{ US ton} = 2000 \text{ pounds} \quad \neq \quad 1 \text{ metric ton} = 1000 \text{ kilograms}$$

PROBABILITY

Probability is the likelihood of a certain outcome occurring for a given event. An **event** is any situation that produces a result. It could be something as simple as flipping a coin or as complex as launching a rocket. Determining the probability of an outcome for an event can be equally simple or complex. As such, there are specific terms used in the study of probability that need to be understood:

- **Compound event**—an event that involves two or more independent events (rolling a pair of dice and taking the sum)
- **Desired outcome** (or success)—an outcome that meets a particular set of criteria (a roll of 1 or 2 if we are looking for numbers less than 3)
- **Independent events**—two or more events whose outcomes do not affect one another (two coins tossed at the same time)
- **Dependent events**—two or more events whose outcomes affect one another (two cards drawn consecutively from the same deck)
- **Certain outcome**—probability of outcome is 100% or 1
- **Impossible outcome**—probability of outcome is 0% or 0
- **Mutually exclusive outcomes**—two or more outcomes whose criteria cannot all be satisfied in a single event (a coin coming up heads and tails on the same toss)
- **Random variable**—refers to all possible outcomes of a single event which may be discrete or continuous.

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SAMPLE SPACE

The total set of all possible results of a test or experiment is called a **sample space**, or sometimes a universal sample space. The sample space, represented by one of the variables S, Ω, or U (for universal sample space) has individual elements called outcomes. Other terms for outcome that may be used interchangeably include elementary outcome, simple event, or sample point. The number of outcomes in a given sample space could be infinite or finite, and some tests may yield multiple unique sample sets. For example, tests conducted by drawing playing cards from a standard deck would have one sample space of the card values, another sample space of the card suits, and a third sample space of suit-denomination combinations. For most tests, the sample spaces considered will be finite.

An **event**, represented by the variable E, is a portion of a sample space. It may be one outcome or a group of outcomes from the same sample space. If an event occurs, then the test or experiment will generate an outcome that satisfies the requirement of that event. For example, given a standard deck of 52 playing cards as the sample space, and defining the event as the collection of face cards, then the event will occur if the card drawn is a J, Q, or K. If any other card is drawn, the event is said to have not occurred.

For every sample space, each possible outcome has a specific likelihood, or probability, that it will occur. The probability measure, also called the **distribution**, is a function that assigns a real number probability, from zero to one, to each outcome. For a probability measure to be accurate, every outcome must have a real number probability measure that is greater than or equal to zero and less than or equal to one. Also, the probability measure of the sample space must equal one, and the probability measure of the union of multiple outcomes must equal the sum of the individual probability measures.

Probabilities of events are expressed as real numbers from zero to one. They give a numerical value to the chance that a particular event will occur. The probability of an event occurring is the sum of the probabilities of the individual elements of that event. For example, in a standard deck of 52 playing cards as the sample space and the collection of face cards as the event, the probability of drawing a specific face card is $\frac{1}{52} = 0.019$, but the probability of drawing any one of the twelve face cards is $12(0.019) = 0.228$. Note that rounding of numbers can generate different results. If you multiplied 12 by the fraction $\frac{1}{52}$ before converting to a decimal, you would get the answer $\frac{12}{52} = 0.231$.

THEORETICAL AND EXPERIMENTAL PROBABILITY

Theoretical probability can usually be determined without actually performing the event. The likelihood of an outcome occurring, or the probability of an outcome occurring, is given by the formula:

$$P(A) = \frac{\text{Number of acceptable outcomes}}{\text{Number of possible outcomes}}$$

Note that $P(A)$ is the probability of an outcome A occurring, and each outcome is just as likely to occur as any other outcome. If each outcome has the same probability of occurring as every other possible outcome, the outcomes are said to be equally likely to occur. The total number of acceptable outcomes must be less than or equal to the total number of possible outcomes. If the two are equal, then the outcome is certain to occur and the probability is 1. If the number of acceptable outcomes is zero, then the outcome is impossible and the probability is 0. For example, if there are 20 marbles in a bag and 5 are red, then the theoretical probability of randomly selecting a red marble is 5 out of 20, $\left(\frac{5}{20} = \frac{1}{4}, 0.25, \text{or } 25\%\right)$.

If the theoretical probability is unknown or too complicated to calculate, it can be estimated by an experimental probability. **Experimental probability**, also called empirical probability, is an estimate of the likelihood of a certain outcome based on repeated experiments or collected data. In other words, while theoretical probability is based on what *should* happen, experimental probability is based on what *has* happened. Experimental probability is calculated in the same way as theoretical probability, except that actual outcomes are used instead of possible outcomes. The more experiments performed or datapoints gathered, the better the estimate should be.

Theoretical and experimental probability do not always line up with one another. Theoretical probability says that out of 20 coin-tosses, 10 should be heads. However, if we were actually to toss 20 coins, we might record just 5 heads. This doesn't mean that our theoretical probability is incorrect; it just means that this particular experiment had results that were different from what was predicted. A practical application of empirical probability is the insurance industry. There are no set functions that define lifespan, health, or safety. Insurance companies look at factors from hundreds of thousands of individuals to find patterns that they then use to set the formulas for insurance premiums.

Review Video: Empirical Probability
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Objective and Subjective Probability

Objective probability is based on mathematical formulas and documented evidence. Examples of objective probability include raffles or lottery drawings where there is a pre-determined number of possible outcomes and a predetermined number of outcomes that correspond to an event. Other cases of objective probability include probabilities of rolling dice, flipping coins, or drawing cards. Most gambling games are based on objective probability.

In contrast, **subjective probability** is based on personal or professional feelings and judgments. Often, there is a lot of guesswork following extensive research. Areas where subjective probability is applicable include sales trends and business expenses. Attractions set admission prices based on subjective probabilities of attendance based on varying admission rates in an effort to maximize their profit.

Complement of an Event

Sometimes it may be easier to calculate the possibility of something not happening, or the **complement of an event**. Represented by the symbol $\bar{A}$, the complement of A is the probability that event A does not happen. When you know the probability of event A occurring, you can use the formula $P(\bar{A}) = 1 - P(A)$, where $P(\bar{A})$ is the probability of event A not occurring, and $P(A)$ is the probability of event A occurring.

Addition Rule

The **addition rule** for probability is used for finding the probability of a compound event. Use the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, where $P(A \cap B)$ is the probability of both events occurring to find the probability of a compound event. The probability of both events occurring at the same time must be subtracted to eliminate any overlap in the first two probabilities.

Conditional Probability

Given two events A and B, the **conditional probability** $P(A|B)$ is the probability that event A will occur, given that event B has occurred. The conditional probability cannot be calculated simply from $P(A)$ and $P(B)$; these probabilities alone do not give sufficient information to determine the conditional probability. It can, however, be determined if you are also given the probability of the intersection of events A and B, $P(A \cap B)$, the probability that events A and B both occur. Specifically, $P(A|B) = \frac{P(A \cap B)}{P(B)}$. For instance, suppose you have a jar containing two red marbles and two blue marbles, and you draw two marbles at random. Consider event A being the event that the first marble drawn is red, and event B being the event that the second marble drawn is blue. If we want to find the probability that B occurs given that A occurred, $P(B|A)$, then we can compute it using the fact that $P(A)$ is $\frac{1}{2}$, and $P(A \cap B)$ is $\frac{1}{3}$. (The latter may not be obvious, but may be determined by finding the product of $\frac{1}{2}$ and $\frac{2}{3}$). Therefore $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/3}{1/2} = \frac{2}{3}$.

CONDITIONAL PROBABILITY IN EVERYDAY SITUATIONS

Conditional probability often arises in everyday situations in, for example, estimating the risk or benefit of certain activities. The conditional probability of having a heart attack given that you exercise daily may be smaller than the overall probability of having a heart attack. The conditional probability of having lung cancer given that you are a smoker is larger than the overall probability of having lung cancer. Note that changing the order of the conditional probability changes the meaning: the conditional probability of having lung cancer given that you are a smoker is a very different thing from the probability of being a smoker given that you have lung cancer. In an extreme case, suppose that a certain rare disease is caused only by eating a certain food, but even then, it is unlikely. Then the conditional probability of having that disease given that you eat the dangerous food is nonzero but low, but the conditional probability of having eaten that food given that you have the disease is 100%!

Review Video: Conditional Probability
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INDEPENDENCE

The conditional probability $P(A|B)$ is the probability that event A will occur given that event B occurs. If the two events are independent, we do not expect that whether or not event B occurs should have any effect on whether or not event A occurs. In other words, we expect $P(A|B) = P(A)$.

This can be proven using the usual equations for conditional probability and the joint probability of independent events. The conditional probability $P(A|B) = \frac{P(A\cap B)}{P(B)}$. If A and B are independent, then $P(A \cap B) = P(A)P(B)$. So $P(A|B) = \frac{P(A)P(B)}{P(B)} = P(A)$. By similar reasoning, if A and B are independent then $P(B|A) = P(B)$.

MULTIPLICATION RULE

The **multiplication rule** can be used to find the probability of two independent events occurring using the formula $P(A \cap B) = P(A) \times P(B)$, where $P(A \cap B)$ is the probability of two independent events occurring, $P(A)$ is the probability of the first event occurring, and $P(B)$ is the probability of the second event occurring.

The multiplication rule can also be used to find the probability of two dependent events occurring using the formula $P(A \cap B) = P(A) \times P(B|A)$, where $P(A \cap B)$ is the probability of two dependent events occurring and $P(B|A)$ is the probability of the second event occurring after the first event has already occurred.

Use a **combination of the multiplication** rule and the rule of complements to find the probability that at least one outcome of the element will occur. This is given by the general formula $P(\text{at least one event occurring}) = 1 - P(\text{no outcomes occurring})$. For example, to find the probability that at least one even number will show when a pair of dice is rolled, find the probability that two odd numbers will be rolled (no even numbers) and subtract from one. You can always use a tree diagram or make a chart to list the possible outcomes when the sample space is small, such as in the dice-rolling example, but in most cases it will be much faster to use the multiplication and complement formulas.

Review Video: Multiplication Rule
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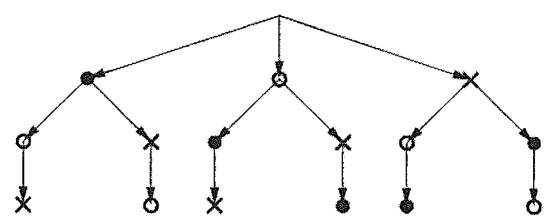
Union and Intersection of Two Sets of Outcomes

If A and B are each a set of elements or outcomes from an experiment, then the **union** (symbol $\cup$) of the two sets is the set of elements found in set A or set B. For example, if $A = \{2, 3, 4\}$ and $B = \{3, 4, 5\}$, $A \cup B = \{2, 3, 4, 5\}$. Note that the outcomes 3 and 4 appear only once in the union. For statistical events, the union is equivalent to "or"; $P(A \cup B)$ is the same thing as $P(A \text{ or } B)$. The **intersection** (symbol $\cap$) of two sets is the set of outcomes common to both sets. For the above sets A and B, $A \cap B = \{3, 4\}$. For statistical events, the intersection is equivalent to "and"; $P(A \cap B)$ is the same thing as $P(A \text{ and } B)$. It is important to note that union and intersection operations commute. That is:

$$A \cup B = B \cup A \text{ and } A \cap B = B \cap A$$

Tree Diagrams

For a simple sample space, possible outcomes may be determined by using a **tree diagram** or an organized chart. In either case, you can easily draw or list out the possible outcomes. For example, to determine all the possible ways three objects can be ordered, you can draw a tree diagram:



Review Video: Tree Diagrams
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You can also make a chart to list all the possibilities:

First object	Second object	Third object
●	x	o
●	o	x
o	●	x
o	x	●
x	●	o
x	o	●

Either way, you can easily see there are six possible ways the three objects can be ordered.

If two events have no outcomes in common, they are said to be **mutually exclusive**. For example, in a standard deck of 52 playing cards, the event of all card suits is mutually exclusive to the event of all card values. If two events have no bearing on each other so that one event occurring has no influence on the probability of another event occurring, the two events are said to be independent. For example, rolling a standard six-sided die multiple times does not change that probability that a particular number will be rolled from one roll to the next. If the outcome of one event does affect the probability of the second event, the two events are said to be dependent. For example, if cards are drawn from a deck, the probability of drawing an ace after an ace has been drawn is different than the probability of drawing an ace if no ace (or no other card, for that matter) has been drawn.

In probability, the **odds in favor of an event** are the number of times the event will occur compared to the number of times the event will not occur. To calculate the odds in favor of an event, use the formula $\frac{P(A)}{1-P(A)}$, where $P(A)$ is the probability that the event will occur. Many times, odds in favor is given as a ratio in the form $\frac{a}{b}$ or $a\colon b$, where a is the probability of the event occurring and b is the complement of the event, the probability of the event not occurring. If the odds in favor are given as 2:5, that means that you can expect the event to occur two times for every 5 times that it does not occur. In other words, the probability that the event will occur is $\frac{2}{2+5} = \frac{2}{7}$.

In probability, the **odds against an event** are the number of times the event will not occur compared to the number of times the event will occur. To calculate the odds against an event, use the formula $\frac{1-P(A)}{P(A)}$, where $P(A)$ is the probability that the event will occur. Many times, odds against is given as a ratio in the form $\frac{b}{a}$ or $b\colon a$, where b is the probability the event will not occur (the complement of the event) and a is the probability the event will occur. If the odds against an event are given as 3:1, that means that you can expect the event to not occur 3 times for every one time it does occur. In other words, 3 out of every 4 trials will fail.

Two-Way Frequency Tables

If we have a two-way frequency table, it is generally a straightforward matter to read off the probabilities of any two events A and B, as well as the joint probability of both events occurring, $P(A \cap B)$. We can then find the conditional probability $P(A|B)$ by calculating $P(A|B) = \frac{P(A\cap B)}{P(B)}$. We could also check whether or not events are independent by verifying whether $P(A)P(B) = P(A\cap B)$.

For example, a certain store's recent T-shirt sales:

	Small	Medium	Large	Total
Blue	25	40	35	100
White	27	25	22	74
Black	8	23	15	46
Total	60	88	72	220

Suppose we want to find the conditional probability that a customer buys a black shirt (event A), given that the shirt he buys is size small (event B). From the table, the probability $P(B)$ that a customer buys a small shirt is $\frac{60}{220} = \frac{3}{11}$. The probability $P(A \cap B)$ that he buys a small, black shirt is $\frac{8}{220} = \frac{2}{55}$. The conditional probability $P(A|B)$ that he buys a black shirt, given that he buys a small shirt, is therefore $P(A|B) = \frac{2/55}{3/11} = \frac{2}{15}$.

Similarly, if we want to check whether the event a customer buys a blue shirt, A, is independent of the event that a customer buys a medium shirt, B. From the table, $P(A) = \frac{100}{220} = \frac{5}{11}$ and

$P(B) = \frac{88}{220} = \frac{4}{10}$. Also, $P(A \cap B) = \frac{40}{220} = \frac{2}{11}$. Since $\left(\frac{5}{11}\right)\left(\frac{4}{10}\right) = \frac{20}{110} = \frac{2}{11}$, $P(A)P(B) = P(A \cap B)$ and these two events are indeed independent.

Introduction to Statistics

Statistics is the branch of mathematics that deals with collecting, recording, interpreting, illustrating, and analyzing large amounts of **data**. The following terms are often used in the discussion of data and **statistics**:

- **Data** – the collective name for pieces of information (singular is datum)
- **Quantitative data** – measurements (such as length, mass, and speed) that provide information about quantities in numbers
- **Qualitative data** – information (such as colors, scents, tastes, and shapes) that cannot be measured using numbers
- **Discrete data** – information that can be expressed only by a specific value, such as whole or half numbers. (e.g., since people can be counted only in whole numbers, a population count would be discrete data.)
- **Continuous data** – information (such as time and temperature) that can be expressed by any value within a given range
- **Primary data** – information that has been collected directly from a survey, investigation, or experiment, such as a questionnaire or the recording of daily temperatures. (Primary data that has not yet been organized or analyzed is called **raw data**.)
- **Secondary data** – information that has been collected, sorted, and processed by the researcher
- **Ordinal data** – information that can be placed in numerical order, such as age or weight
- **Nominal data** – information that *cannot* be placed in numerical order, such as names or places

Data Collection

Population

In statistics, the **population** is the entire collection of people, plants, etc., that data can be collected from. For example, a study to determine how well students in local schools perform on a standardized test would have a population of all the students enrolled in those schools, although a study may include just a small sample of students from each school. A **parameter** is a numerical value that gives information about the population, such as the mean, median, mode, or standard deviation. Remember that the symbol for the mean of a population is μ and the symbol for the standard deviation of a population is σ.

Sample

A **sample** is a portion of the entire population. Whereas a parameter helped describe the population, a **statistic** is a numerical value that gives information about the sample, such as mean, median, mode, or standard deviation. Keep in mind that the symbols for mean and standard deviation are different when they are referring to a sample rather than the entire population. For a sample, the symbol for mean is $\bar{x}$ and the symbol for standard deviation is s. The mean and standard deviation of a sample may or may not be identical to that of the entire population due to a sample only being a subset of the population. However, if the sample is random and large enough, statistically significant values can be attained. Samples are generally used when the population is too large to justify including every element or when acquiring data for the entire population is impossible.

Inferential Statistics

Inferential statistics is the branch of statistics that uses samples to make predictions about an entire population. This type of statistic is often seen in political polls, where a sample of the population is questioned about a particular topic or politician to gain an understanding of the attitudes of the entire population of the country. Often, exit polls are conducted on election days using this method. Inferential statistics can have a large margin of error if you do not have a valid sample.

Sampling Distribution

Statistical values calculated from various samples of the same size make up the **sampling distribution.** For example, if several samples of identical size are randomly selected from a large population and then the mean of each sample is calculated, the distribution of values of the means would be a sampling distribution.

The **sampling distribution of the mean** is the distribution of the sample mean, $\bar{x}$, derived from random samples of a given size. It has three important characteristics. First, the mean of the sampling distribution of the mean is equal to the mean of the population that was sampled. Second, assuming the standard deviation is non-zero, the standard deviation of the sampling distribution of the mean equals the standard deviation of the sampled population divided by the square root of the sample size. This is sometimes called the standard error. Finally, as the sample size gets larger, the sampling distribution of the mean gets closer to a normal distribution via the central limit theorem.

Survey Study

A **survey study** is a method of gathering information from a small group in an attempt to gain enough information to make accurate general assumptions about the population. Once a survey study is completed, the results are then put into a summary report.

Survey studies are generally in the format of surveys, interviews, or questionnaires as part of an effort to find opinions of a particular group or to find facts about a group.

It is important to note that the findings from a survey study are only as accurate as the sample chosen from the population.

Correlational Studies

Correlational studies seek to determine how much one variable is affected by changes in a second variable. For example, correlational studies may look for a relationship between the amount of time a student spends studying for a test and the grade that student earned on the test or between student scores on college admissions tests and student grades in college.

It is important to note that correlational studies cannot show a cause and effect, but rather can show only that two variables are or are not potentially correlated.

Experimental Studies

Experimental studies take correlational studies one step farther, in that they attempt to prove or disprove a cause-and-effect relationship. These studies are performed by conducting a series of experiments to test the hypothesis. For a study to be scientifically accurate, it must have both an experimental group that receives the specified treatment and a control group that does not get the treatment. This is the type of study pharmaceutical companies do as part of drug trials for new medications. Experimental studies are only valid when the proper scientific method has been followed. In other words, the experiment must be well-planned and executed without bias in the testing process, all subjects must be selected at random, and the process of determining which subject is in which of the two groups must also be completely random.

OBSERVATIONAL STUDIES

Observational studies are the opposite of experimental studies. In observational studies, the tester cannot change or in any way control all of the variables in the test. For example, a study to determine which gender does better in math classes in school is strictly observational. You cannot change a person's gender, and you cannot change the subject being studied. The big downfall of observational studies is that you have no way of proving a cause-and-effect relationship because you cannot control outside influences. Events outside of school can influence a student's performance in school, and observational studies cannot take that into consideration.

RANDOM SAMPLES

For most studies, a **random sample** is necessary to produce valid results. Random samples should not have any particular influence to cause sampled subjects to behave one way or another. The goal is for the random sample to be a **representative sample**, or a sample whose characteristics give an accurate picture of the characteristics of the entire population. To accomplish this, you must make sure you have a proper **sample size**, or an appropriate number of elements in the sample.

BIASES

In statistical studies, biases must be avoided. **Bias** is an error that causes the study to favor one set of results over another. For example, if a survey to determine how the country views the president's job performance only speaks to registered voters in the president's party, the results will be skewed because a disproportionately large number of responders would tend to show approval, while a disproportionately large number of people in the opposite party would tend to express disapproval. **Extraneous variables** are, as the name implies, outside influences that can affect the outcome of a study. They are not always avoidable but could trigger bias in the result.

DATA ANALYSIS

DISPERSION

A **measure of dispersion** is a single value that helps to "interpret" the measure of central tendency by providing more information about how the data values in the set are distributed about the measure of central tendency. The measure of dispersion helps to eliminate or reduce the disadvantages of using the mean, median, or mode as a single measure of central tendency, and give a more accurate picture of the dataset as a whole. To have a measure of dispersion, you must know or calculate the range, standard deviation, or variance of the data set.

RANGE

The **range** of a set of data is the difference between the greatest and lowest values of the data in the set. To calculate the range, you must first make sure the units for all data values are the same, and then identify the greatest and lowest values. If there are multiple data values that are equal for the highest or lowest, just use one of the values in the formula. Write the answer with the same units as the data values you used to do the calculations.

Review Video: Statistical Range
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Sample Standard Deviation

Standard deviation is a measure of dispersion that compares all the data values in the set to the mean of the set to give a more accurate picture. To find the **standard deviation of a sample**, use the formula

$$s = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}}$$

Note that s is the standard deviation of a sample, x_i represents the individual values in the data set, $\bar{x}$ is the mean of the data values in the set, and n is the number of data values in the set. The higher the value of the standard deviation is, the greater the variance of the data values from the mean. The units associated with the standard deviation are the same as the units of the data values.

Review Video: Standard Deviation
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Sample Variance

The **variance of a sample** is the square of the sample standard deviation (denoted s^2). While the mean of a set of data gives the average of the set and gives information about where a specific data value lies in relation to the average, the variance of the sample gives information about the degree to which the data values are spread out and tells you how close an individual value is to the average compared to the other values. The units associated with variance are the same as the units of the data values squared.

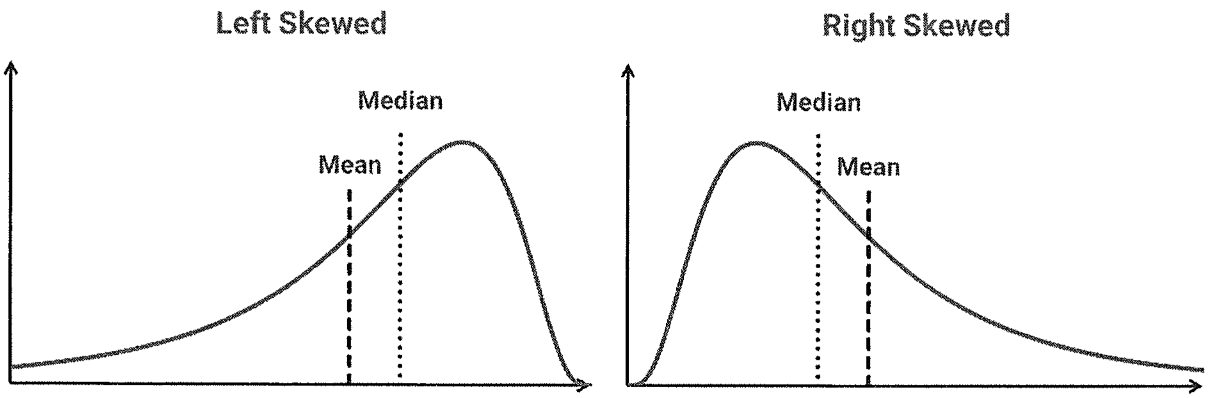
Percentile

Percentiles and quartiles are other methods of describing data within a set. **Percentiles** tell what percentage of the data in the set fall below a specific point. For example, achievement test scores are often given in percentiles. A score at the 80th percentile is one which is equal to or higher than 80 percent of the scores in the set. In other words, 80 percent of the scores were lower than that score.

Quartiles are percentile groups that make up quarter sections of the data set. The first quartile is the 25th percentile. The second quartile is the 50th percentile; this is also the median of the dataset. The third quartile is the 75th percentile.

Skewness

Skewness is a way to describe the symmetry or asymmetry of the distribution of values in a dataset. If the distribution of values is symmetrical, there is no skew. In general the closer the mean of a data set is to the median of the data set, the less skew there is. Generally, if the mean is to the right of the median, the data set is *positively skewed*, or right-skewed, and if the mean is to the left of the median, the data set is *negatively skewed*, or left-skewed. However, this rule of thumb is not infallible. When the data values are graphed on a curve, a set with no skew will be a perfect bell curve.



To estimate skew, use the formula:

$$\text{skew} = \frac{\sqrt{n(n-1)}}{n-2}\left(\frac{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^3}{\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right)^{\frac{3}{2}}}\right)$$

Note that n is the datapoints in the set, x_i is the i^{th} value in the set, and $\bar{x}$ is the mean of the set.

Review Video: Skew
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Unimodal vs. Bimodal

If a distribution has a single peak, it would be considered **unimodal**. If it has two discernible peaks it would be considered **bimodal**. Bimodal distributions may be an indication that the set of data being considered is actually the combination of two sets of data with significant differences. A **uniform distribution** is a distribution in which there is *no distinct peak or variation* in the data. No values or ranges are particularly more common than any other values or ranges.

Outlier

An outlier is an extremely high or extremely low value in the data set. It may be the result of measurement error, in which case, the outlier is not a valid member of the data set. However, it may also be a valid member of the distribution. Unless a measurement error is identified, the experimenter cannot know for certain if an outlier is or is not a member of the distribution. There are arbitrary methods that can be employed to designate an extreme value as an outlier. One method designates an outlier (or possible outlier) to be any value less than $Q_1 - 1.5(IQR)$ or any value greater than $Q_3 + 1.5(IQR)$.

Data Analysis

Simple Regression

In statistics, **simple regression** is using an equation to represent a relation between independent and dependent variables. The independent variable is also referred to as the explanatory variable or the predictor and is generally represented by the variable x in the equation. The dependent variable, usually represented by the variable y, is also referred to as the response variable. The equation may be any type of function – linear, quadratic, exponential, etc. The best way to handle this task is to use the regression feature of your graphing calculator. This will easily give you the curve of best fit and provide you with the coefficients and other information you need to derive an equation.

LINE OF BEST FIT

In a scatter plot, the **line of best fit** is the line that best shows the trends of the data. The line of best fit is given by the equation $\hat{y} = ax + b$, where a and b are the regression coefficients. The regression coefficient a is also the slope of the line of best fit, and b is also the y-coordinate of the point at which the line of best fit crosses the y-axis. Not every point on the scatter plot will be on the line of best fit. The differences between the y-values of the points in the scatter plot and the corresponding y-values according to the equation of the line of best fit are the residuals. The line of best fit is also called the least-squares regression line because it is also the line that has the lowest sum of the squares of the residuals.

CORRELATION COEFFICIENT

The **correlation coefficient** is the numerical value that indicates how strong the relationship is between the two variables of a linear regression equation. A correlation coefficient of −1 is a perfect negative correlation. A correlation coefficient of +1 is a perfect positive correlation. Correlation coefficients close to −1 or +1 are very strong correlations. A correlation coefficient equal to zero indicates there is no correlation between the two variables. This test is a good indicator of whether or not the equation for the line of best fit is accurate. The formula for the correlation coefficient is

$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

where r is the correlation coefficient, n is the number of data values in the set, (x_i, y_i) is a point in the set, and $\bar{x}$ and $\bar{y}$ are the means.

Z-SCORE

A **z-score** is an indication of how many standard deviations a given value falls from the sample mean. To calculate a z-score, use the formula:

$$\frac{x - \bar{x}}{\sigma}$$

In this formula x is the data value, $\bar{x}$ is the mean of the sample data, and σ is the standard deviation of the population. If the z-score is positive, the data value lies above the mean. If the z-score is negative, the data value falls below the mean. These scores are useful in interpreting data such as standardized test scores, where every piece of data in the set has been counted, rather than just a small random sample. In cases where standard deviations are calculated from a random sample of the set, the z-scores will not be as accurate.

CENTRAL LIMIT THEOREM

According to the **central limit theorem**, regardless of what the original distribution of a sample is, the distribution of the means tends to get closer and closer to a normal distribution as the sample size gets larger and larger (this is necessary because the sample is becoming more all-encompassing of the elements of the population). As the sample size gets larger, the distribution of the sample mean will approach a normal distribution with a mean of the population mean and a variance of the population variance divided by the sample size.

MEASURES OF CENTRAL TENDENCY

A **measure of central tendency** is a statistical value that gives a reasonable estimate for the center of a group of data. There are several different ways of describing the measure of central tendency. Each one has a unique way it is calculated, and each one gives a slightly different perspective on the data set. Whenever you give a measure of central tendency, always make sure the units are the same. If the data has different units, such as hours, minutes, and seconds, convert all the data to the same unit, and use the same unit in the measure of central tendency. If no units are given in the data, do not give units for the measure of central tendency.

MEAN

The **statistical mean** of a group of data is the same as the arithmetic average of that group. To find the mean of a set of data, first convert each value to the same units, if necessary. Then find the sum of all the values, and count the total number of data values, making sure you take into consideration each individual value. If a value appears more than once, count it more than once. Divide the sum of the values by the total number of values and apply the units, if any. Note that the mean does not have to be one of the data values in the set, and may not divide evenly.

$$\text{mean} = \frac{\text{sum of the data values}}{\text{quantity of data values}}$$

For instance, the mean of the data set {88, 72, 61, 90, 97, 68, 88, 79, 86, 93, 97, 71, 80, 84, 89} would be the sum of the fifteen numbers divided by 15:

$$\frac{88+72+61+90+97+68+88+79+86+93+97+71+80+84+89}{15} = \frac{1242}{15}$$
$$= 82.8$$

While the mean is relatively easy to calculate and averages are understood by most people, the mean can be very misleading if it is used as the sole measure of central tendency. If the data set has outliers (data values that are unusually high or unusually low compared to the rest of the data values), the mean can be very distorted, especially if the data set has a small number of values. If unusually high values are countered with unusually low values, the mean is not affected as much. For example, if five of twenty students in a class get a 100 on a test, but the other 15 students have an average of 60 on the same test, the class average would appear as 70. Whenever the mean is skewed by outliers, it is always a good idea to include the median as an alternate measure of central tendency.

A **weighted mean**, or weighted average, is a mean that uses "weighted" values. The formula is weighted mean $= \frac{w_1x_1+w_2x_2+w_3x_3...+w_nx_n}{w_1+w_2+w_3+\cdots+w_n}$. Weighted values, such as $w_1, w_2, w_3, \ldots w_n$ are assigned to each member of the set $x_1, x_2, x_3, \ldots x_n$. When calculating the weighted mean, make sure a weight value for each member of the set is used.

Review Video: All About Averages
Visit mometrix.com/academy and enter code: 176521

MEDIAN

The **statistical median** is the value in the middle of the set of data. To find the median, list all data values in order from smallest to largest or from largest to smallest. Any value that is repeated in the set must be listed the number of times it appears. If there are an odd number of data values, the median is the value in the middle of the list. If there is an even number of data values, the median is the arithmetic mean of the two middle values.

For example, the median of the data set {88, 72, 61, 90, 97, 68, 88, 79, 86, 93, 97, 71, 80, 84, 88} is 86 since the ordered set is {61, 68, 71, 72, 79, 80, 84, **86**, 88, 88, 88, 90, 93, 97, 97}.

The big disadvantage of using the median as a measure of central tendency is that is relies solely on a value's relative size as compared to the other values in the set. When the individual values in a set of data are evenly dispersed, the median can be an accurate tool. However, if there is a group of rather large values or a group of rather small values that are not offset by a different group of values, the information that can be inferred from the median may not be accurate because the distribution of values is skewed.

Mode

The **statistical mode** is the data value that occurs the greatest number of times in the data set. It is possible to have exactly one mode, more than one mode, or no mode. To find the mode of a set of data, arrange the data like you do to find the median (all values in order, listing all multiples of data values). Count the number of times each value appears in the data set. If all values appear an equal number of times, there is no mode. If one value appears more than any other value, that value is the mode. If two or more values appear the same number of times, but there are other values that appear fewer times and no values that appear more times, all of those values are the modes.

For example, the mode of the data set {**88**, 72, 61, 90, 97, 68, **88**, 79, 86, 93, 97, 71, 80, 84, **88**} is 88.

The main disadvantage of the mode is that the values of the other data in the set have no bearing on the mode. The mode may be the largest value, the smallest value, or a value anywhere in between in the set. The mode only tells which value or values, if any, occurred the greatest number of times. It does not give any suggestions about the remaining values in the set.

Review Video: Mean, Median, and Mode
Visit mometrix.com/academy and enter code: 286207

Standard Deviation

The **standard deviation** (σ) of a data set measures variation, or how spread out the values are. To calculate it, first find the mean of the data set. Then find the difference of each value and the mean and square the differences. Find the **variance** (σ^2)by adding each of these squared differences together and dividing by one less than the number of values. Finally, take the root of the variance and you have the standard deviation.

Review Video: Standard Deviation
Visit mometrix.com/academy and enter code: 419469

Example

For the following data set (4, 8, 2, 7, 11, 13, 6, 5, 7), we first find the mean:

$$\frac{4+8+2+7+11+13+6+5+7}{9}=7$$

Then we find the difference of each value and the mean, square and add them, and divide by one less than the number of data points:

$$\begin{aligned}\sigma^2 &= \frac{(7-4)^2 + (7-8)^2 + (7-2)^2 + (7-7)^2 + (7-11)^2 + (7-13)^2 + (7-6)^2 + (7-5)^2 + (7-7)^2}{9-1} \\ &= \frac{9+1+25+0+16+36+1+4+0}{8} \\ &= \frac{92}{8} \\ &= 11.5\end{aligned}$$

So the variance is 11.5, and we take the root to find the standard deviation: $\sigma = \sqrt{11.5} \approx 3.39$.

DISTRIBUTIONS

NORMAL DISTRIBUTIONS

If a distribution is **normal**, this means that the variables are mostly symmetrical about the mean. Approximately half of the values are below the mean and half are above. Most values are clustered closely about the mean, with approximately 68% within one standard deviation, 95% within two standard deviations, and 99.7% within three standard deviations. Anything beyond this is an **outlier**.

EXAMPLE

If the height of freshman women at a university is normally distributed with a mean of 66" and a standard deviation of 1.5", we can calculate the percentage of women in various height brackets. Since the standard deviation is 1.5" and 68% of values in a normal distribution fall within one distribution, we can say that approximately 68% of the freshman women are between 64.5" and 67.5", 95% are between 63" and 69", and 99.7% within 61.5" and 70.5". Anyone shorter than 61.5" or taller than 70.5" is an outlier.

UNIFORM DISTRIBUTIONS

In a **uniform distribution**, every outcome is equally likely. For instance, in flipping a fair coin or rolling a die, each outcome is as likely as any of the others. A random generator is another example, since each number within the parameters has an equal chance of being selected.

EXAMPLE

Suppose a collection of songs has a range of anywhere from 4 to 11 notes, inclusive, and they are distributed uniformly. We could find the probability that a given song has six or fewer notes. There are eight possibilities (from 4 to 11) and three of them have six or fewer notes (4, 5, 6). So the probability is $\frac{3}{8}$ or 0.375.

BINOMIAL DISTRIBUTIONS

A binomial distribution has two possible outcomes, like flipping a coin or a true/false quiz. It is typically used on a series of trials (like flipping a coin 50 times). None of the trials impact the others, so the probability of success or failure is the same on each, but the number of trials can affect overall probability. For example, flipping a coin 20 times gives a much higher probability of getting at least one heads than flipping it once, even though the probability for each individual flip is the same. The formula for **binomial distribution** is $P_x = \binom{n}{x} p^x q^{n-x}$, where P is the probability, n is the number of trials, x is the number of times of the desired outcome, p is the probability of success on any trial, and q is the probability of failure on any trial.

EXAMPLE

Suppose a jar has 20 marbles, 10 red and 10 blue. If Lea draws a marble 10 times, replacing it after each draw, there is a 50% chance on each draw of getting red, and a 50% chance of getting blue, so this is a binomial distribution. The probability of getting three reds can be calculated:

$$P_3 = \binom{10}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^7 = \frac{10!}{3!\,7!}(0.5)^3(0.5)^7 = \frac{10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 1}{7\times 6\times 5\times 4\times 3\times 2\times 1\times 3\times 2\times 1}\left(\frac{1}{2}\right)^{10} = \frac{15}{128}$$

DISPLAYING INFORMATION

FREQUENCY TABLES

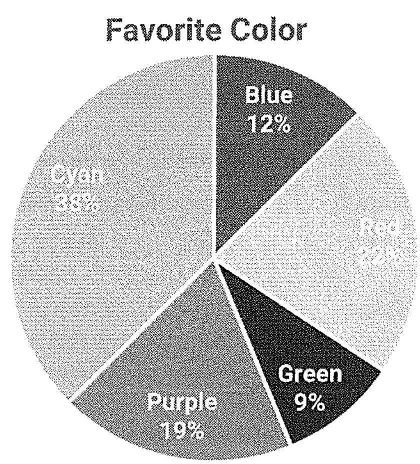
Frequency tables show how frequently each unique value appears in a set. A **relative frequency table** is one that shows the proportions of each unique value compared to the entire set. Relative frequencies are given as percentages; however, the total percent for a relative frequency table will not necessarily equal 100 percent due to rounding. An example of a frequency table with relative frequencies is below.

Favorite Color	Frequency	Relative Frequency
Blue	4	13%
Red	7	22%
Green	3	9%
Purple	6	19%
Cyan	12	38%

Review Video: Data Interpretation of Graphs
Visit mometrix.com/academy and enter code: 200439

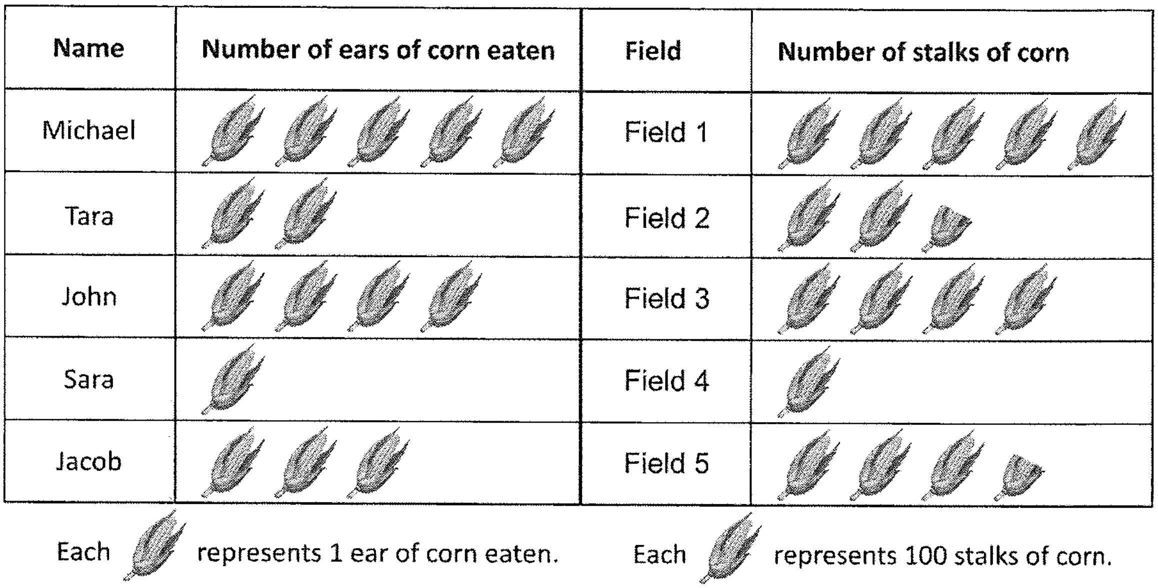
CIRCLE GRAPHS

Circle graphs, also known as *pie charts*, provide a visual depiction of the relationship of each type of data compared to the whole set of data. The circle graph is divided into sections by drawing radii to create central angles whose percentage of the circle is equal to the individual data's percentage of the whole set. Each 1% of data is equal to 3.6° in the circle graph. Therefore, data represented by a 90° section of the circle graph makes up 25% of the whole. When complete, a circle graph often looks like a pie cut into uneven wedges. The pie chart below shows the data from the frequency table referenced earlier where people were asked their favorite color.



Pictographs

A **pictograph** is a graph, generally in the horizontal orientation, that uses pictures or symbols to represent the data. Each pictograph must have a key that defines the picture or symbol and gives the quantity each picture or symbol represents. Pictures or symbols on a pictograph are not always shown as whole elements. In this case, the fraction of the picture or symbol shown represents the same fraction of the quantity a whole picture or symbol stands for. For example, a row with $3\frac{1}{2}$ ears of corn, where each ear of corn represents 100 stalks of corn in a field, would equal $3\frac{1}{2} \times 100 = 350$ stalks of corn in the field.



Review Video: Pictographs
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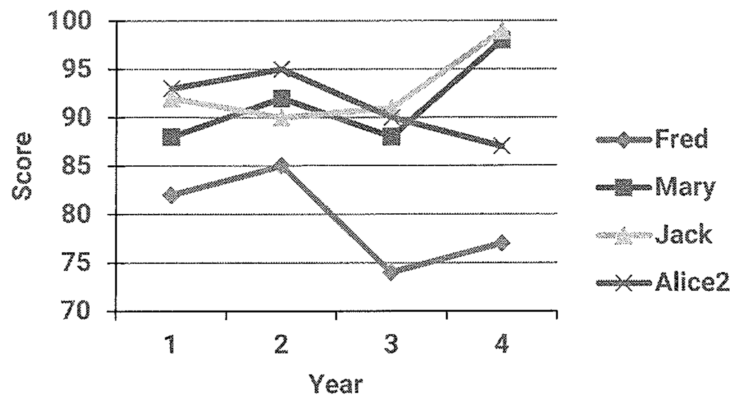
Line Graphs

Line graphs have one or more lines of varying styles (solid or broken) to show the different values for a set of data. The individual data are represented as ordered pairs, much like on a Cartesian plane. In this case, the x- and y-axes are defined in terms of their units, such as dollars or time. The individual plotted points are joined by line segments to show whether the value of the data is increasing (line sloping upward), decreasing (line sloping downward), or staying the same (horizontal line). Multiple sets of data can be graphed on the same line graph to give an easy visual comparison. An example of this would be graphing achievement test scores for different groups of students over the same time period to see which group had the greatest increase or decrease in performance from year to year (as shown below).



Review Video: How to Create a Line Graph
Visit mometrix.com/academy and enter code: 480147

LINE PLOTS

A **line plot**, also known as a *dot plot*, has plotted points that are not connected by line segments. In this graph, the horizontal axis lists the different possible values for the data, and the vertical axis lists the number of times the individual value occurs. A single dot is graphed for each value to show the number of times it occurs. This graph is more closely related to a bar graph than a line graph. Do not connect the dots in a line plot or it will misrepresent the data.

Review Video: Line Plot
Visit mometrix.com/academy and enter code: 754610

STEM AND LEAF PLOTS

A **stem and leaf plot** is useful for depicting groups of data that fall into a range of values. Each piece of data is separated into two parts: the first, or left, part is called the stem; the second, or right, part is called the leaf. Each stem is listed in a column from smallest to largest. Each leaf that has the common stem is listed in that stem's row from smallest to largest. For example, in a set of two-digit numbers, the digit in the tens place is the stem, and the digit in the ones place is the leaf. With a stem and leaf plot, you can easily see which subset of numbers (10s, 20s, 30s, etc.) is the largest. This information is also readily available by looking at a histogram, but a stem and leaf plot also allows you to look closer and see exactly which values fall in that range. Using a sample set of test scores (82, 88, 92, 93, 85, 90, 92, 95, 74, 88, 90, 91, 78, 87, 98, 99), we can assemble a stem and leaf plot like the one below.

Test Scores

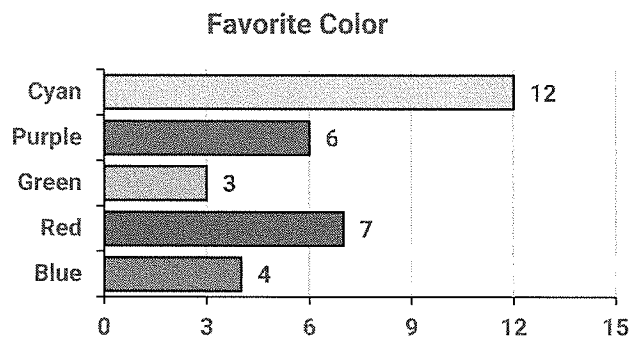
7	4	8							
8	2	5	7	8	8				
9	0	0	1	2	2	3	5	8	9

Review Video: Stem and Leaf Plots
Visit mometrix.com/academy and enter code: 302339

BAR GRAPHS

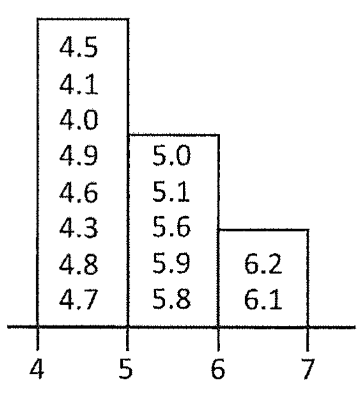
A **bar graph** is one of the few graphs that can be drawn correctly in two different configurations – both horizontally and vertically. A bar graph is similar to a line plot in the way the data is organized on the graph. Both axes must have their categories defined for the graph to be useful. Rather than placing a single dot to mark the point of the data's value, a bar, or thick line, is drawn from zero to the exact value of the data, whether it is a number, percentage, or other numerical value. Longer bar lengths correspond to greater data values. To read a bar graph, read the labels for the axes to find the units being reported. Then, look where the bars end in relation to the scale given on the corresponding axis and determine the associated value.

The bar chart below represents the responses from our favorite-color survey.



HISTOGRAMS

At first glance, a **histogram** looks like a vertical bar graph. The difference is that a bar graph has a separate bar for each piece of data and a histogram has one continuous bar for each *range* of data. For example, a histogram may have one bar for the range 0–9, one bar for 10–19, etc. While a bar graph has numerical values on one axis, a histogram has numerical values on both axes. Each range is of equal size, and they are ordered left to right from lowest to highest. The height of each column on a histogram represents the number of data values within that range. Like a stem and leaf plot, a histogram makes it easy to glance at the graph and quickly determine which range has the greatest quantity of values. A simple example of a histogram is below.

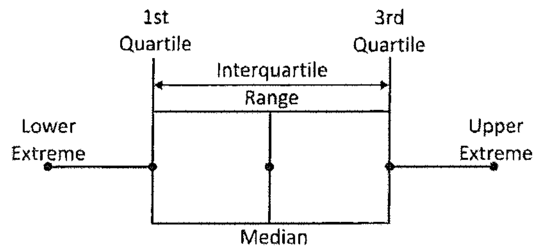


5-NUMBER SUMMARY

The **5-number summary** of a set of data gives a very informative picture of the set. The five numbers in the summary include the minimum value, maximum value, and the three quartiles. This information gives the reader the range and median of the set, as well as an indication of how the data is spread about the median.

Box and Whisker Plots

A **box-and-whiskers plot** is a graphical representation of the 5-number summary. To draw a box-and-whiskers plot, plot the points of the 5-number summary on a number line. Draw a box whose ends are through the points for the first and third quartiles. Draw a vertical line in the box through the median to divide the box in half. Draw a line segment from the first quartile point to the minimum value, and from the third quartile point to the maximum value.



Review Video: Box and Whisker Plots
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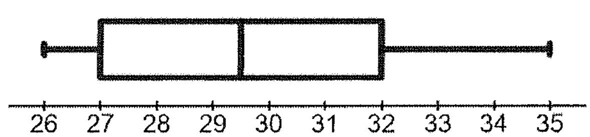
Example

Given the following data (32, 28, 29, 26, 35, 27, 30, 31, 27, 32), we first sort it into numerical order: 26, 27, 27, 28, 29, 30, 31, 32, 32, 35. We can then find the median. Since there are ten values, we take the average of the 5^{th} and 6^{th} values to get 29.5. We find the lower quartile by taking the median of the data smaller than the median. Since there are five values, we take the 3^{rd} value, which is 27. We find the upper quartile by taking the median of the data larger than the overall median, which is 32. Finally, we note our minimum and maximum, which are simply the smallest and largest values in the set: 26 and 35, respectively. Now we can create our box plot:



This plot is fairly "long" on the right whisker, showing one or more unusually high values (but not quite outliers). The other quartiles are similar in length, showing a fairly even distribution of data.

Interquartile Range

The **interquartile range, or IQR**, is the difference between the upper and lower quartiles. It measures how the data is dispersed: a high IQR means that the data is more spread out, while a low IQR means that the data is clustered more tightly around the median. To find the IQR, subtract the lower quartile value (Q_1) from the upper quartile value (Q_3).

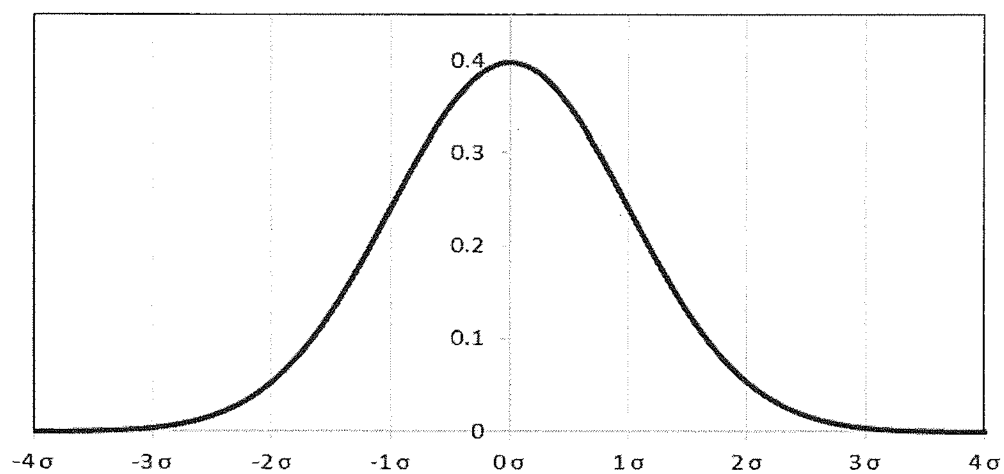
Example

To find the upper and lower quartiles, we first find the median and then take the median of all values above it and all values below it. In the following data set (16, 18, 13, 24, 16, 51, 32, 21, 27, 39), we first rearrange the values in numerical order: 13, 16, 16, 18, 21, 24, 27, 32, 39, 51. There are

10 values, so the median is the average of the 5th and 6th: $\frac{21+24}{2} = \frac{45}{2} = 22.5$. We do not actually need this value to find the upper and lower quartiles. We look at the set of numbers below the median: 13, 16, 16, 18, 21. There are five values, so the 3rd is the median (16), or the value of the lower quartile (Q_1). Then we look at the numbers above the median: 24, 27, 32, 39, 51. Again there are five values, so the 3rd is the median (32), or the value of the upper quartile (Q_3). We find the IQR by subtracting Q_1 from Q_3: $32 - 16 = 16$.

68-95-99.7 Rule

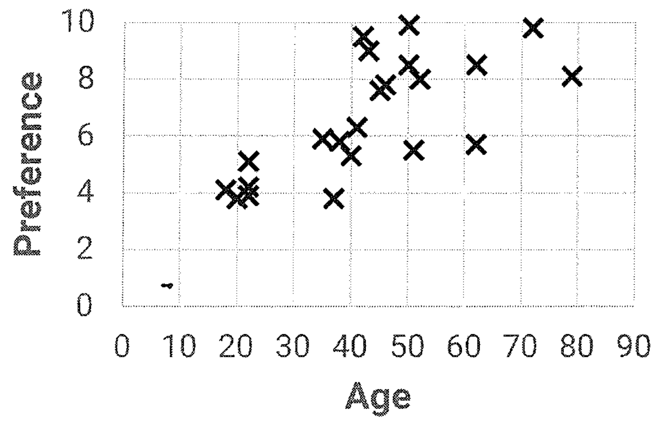
The **68–95–99.7 rule** describes how a normal distribution of data should appear when compared to the mean. This is also a description of a normal bell curve. According to this rule, 68 percent of the data values in a normally distributed set should fall within one standard deviation of the mean (34 percent above and 34 percent below the mean), 95 percent of the data values should fall within two standard deviations of the mean (47.5 percent above and 47.5 percent below the mean), and 99.7 percent of the data values should fall within three standard deviations of the mean, again, equally distributed on either side of the mean. This means that only 0.3 percent of all data values should fall more than three standard deviations from the mean. On the graph below, the normal curve is centered on the y-axis. The x-axis labels are how many standard deviations away from the center you are. Therefore, it is easy to see how the 68-95-99.7 rule can apply.



Scatter Plots

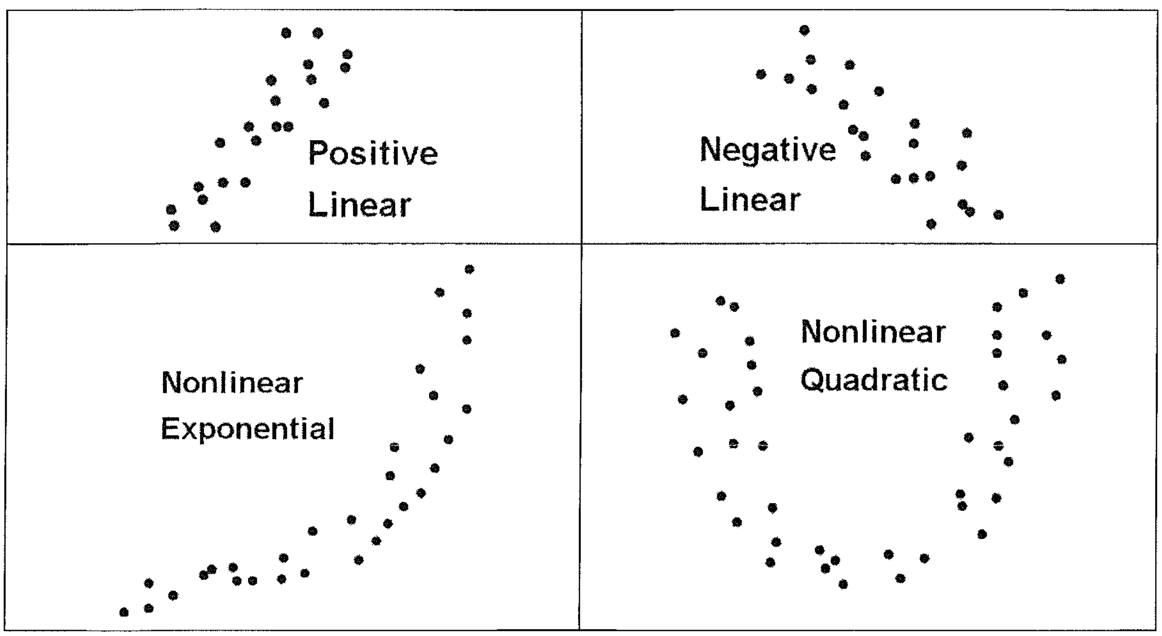
Bivariate Data

Bivariate data is simply data from two different variables. (The prefix *bi-* means *two.*) In a *scatter plot,* each value in the set of data is plotted on a grid similar to a Cartesian plane, where each axis represents one of the two variables. By looking at the pattern formed by the points on the grid, you can often determine whether or not there is a relationship between the two variables, and what that relationship is, if it exists. The variables may be directly proportionate, inversely proportionate, or show no proportion at all. It may also be possible to determine if the data is linear, and if so, to find an equation to relate the two variables. The following scatter plot shows the relationship between preference for brand "A" and the age of the consumers surveyed.



SCATTER PLOTS

Scatter plots are also useful in determining the type of function represented by the data and finding the simple regression. Linear scatter plots may be positive or negative. Nonlinear scatter plots are generally exponential or quadratic. Below are some common types of scatter plots:



Review Video: What is a Scatter Plot?
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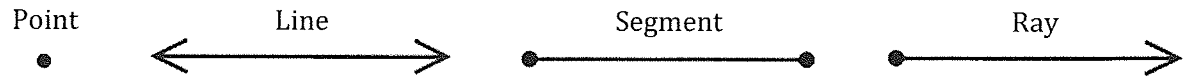
Geometry and Trigonometry

POINTS, LINES, AND PLANES

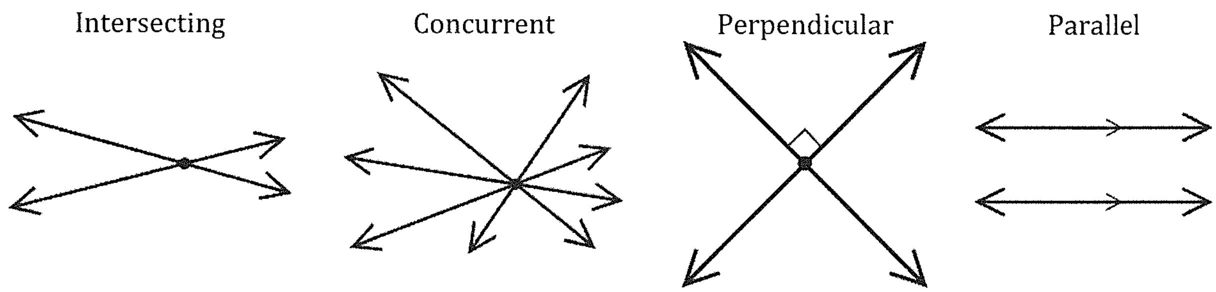
POINTS AND LINES

A **point** is a fixed location in space, has no size or dimensions, and is commonly represented by a dot. A **line** is a set of points that extends infinitely in two opposite directions. It has length, but no width or depth. A line can be defined by any two distinct points that it contains. A **line segment** is a portion of a line that has definite endpoints. A **ray** is a portion of a line that extends from a single point on that line in one direction along the line. It has a definite beginning, but no ending.



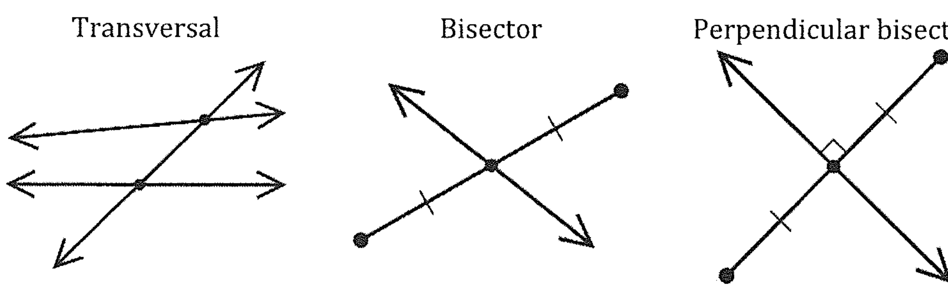
INTERACTIONS BETWEEN LINES

Intersecting lines are lines that have exactly one point in common. **Concurrent lines** are multiple lines that intersect at a single point. **Perpendicular lines** are lines that intersect at right angles. They are represented by the symbol ⊥. The shortest distance from a line to a point not on the line is a perpendicular segment from the point to the line. **Parallel lines** are lines in the same plane that have no points in common and never meet. It is possible for lines to be in different planes, have no points in common, and never meet, but they are not parallel because they are in different planes.



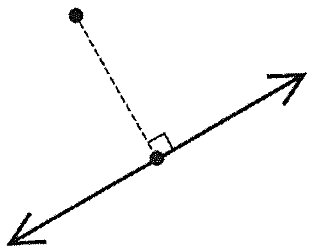
Review Video: Parallel and Perpendicular Lines
Visit mometrix.com/academy and enter code: 815923

A **transversal** is a line that intersects at least two other lines, which may or may not be parallel to one another. A transversal that intersects parallel lines is a common occurrence in geometry. A **bisector** is a line or line segment that divides another line segment into two equal lengths. A **perpendicular bisector** of a line segment is composed of points that are equidistant from the endpoints of the segment it is dividing.

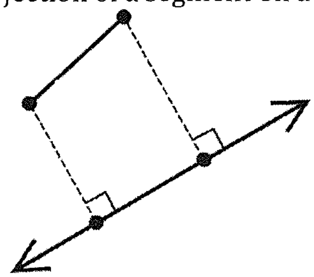


The **projection of a point on a line** is the point at which a perpendicular line drawn from the given point to the given line intersects the line. This is also the shortest distance from the given point to the line. The **projection of a segment on a line** is a segment whose endpoints are the points formed when perpendicular lines are drawn from the endpoints of the given segment to the given line. This is similar to the length a diagonal line appears to be when viewed from above.

Projection of a point on a line

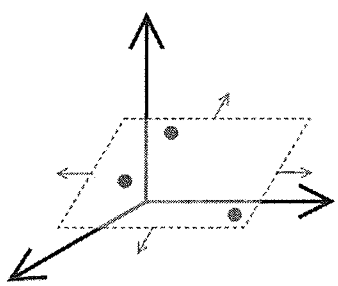


Projection of a segment on a line



PLANES

A **plane** is a two-dimensional flat surface defined by three non-collinear points. A plane extends an infinite distance in all directions in those two dimensions. It contains an infinite number of points, parallel lines and segments, intersecting lines and segments, as well as parallel or intersecting rays. A plane will never contain a three-dimensional figure or skew lines, which are lines that don't intersect and are not parallel. Two given planes are either parallel or they intersect at a line. A plane may intersect a circular conic surface to form **conic sections**, such as a parabola, hyperbola, circle or ellipse.



Review Video: Lines and Planes
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ANGLES

ANGLES AND VERTICES

An **angle** is formed when two lines or line segments meet at a common point. It may be a common starting point for a pair of segments or rays, or it may be the intersection of lines. Angles are represented by the symbol ∠.

The **vertex** is the point at which two segments or rays meet to form an angle. If the angle is formed by intersecting rays, lines, and/or line segments, the vertex is the point at which four angles are formed. The pairs of angles opposite one another are called vertical angles, and their measures are equal.

- An **acute** angle is an angle with a degree measure less than 90°.
- A **right** angle is an angle with a degree measure of exactly 90°.
- An **obtuse** angle is an angle with a degree measure greater than 90° but less than 180°.

- A **straight angle** is an angle with a degree measure of exactly 180°. This is also a semicircle.
- A **reflex angle** is an angle with a degree measure greater than 180° but less than 360°.
- A **full angle** is an angle with a degree measure of exactly 360°. This is also a circle.

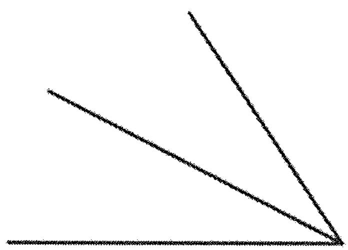
Review Video: Angles
Visit mometrix.com/academy and enter code: 264624

Relationships Between Angles

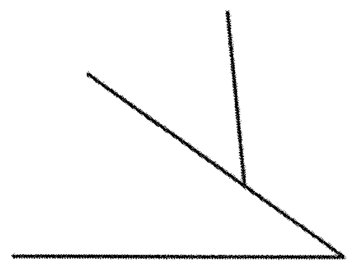
Two angles whose sum is exactly 90° are said to be **complementary**. The two angles may or may not be adjacent. In a right triangle, the two acute angles are complementary.

Two angles whose sum is exactly 180° are said to be **supplementary**. The two angles may or may not be adjacent. Two intersecting lines always form two pairs of supplementary angles. Adjacent supplementary angles will always form a straight line.

Two angles that have the same vertex and share a side are said to be **adjacent**. Vertical angles are not adjacent because they share a vertex but no common side.



Adjacent
Share vertex and side



Not adjacent
Share part of a side, but not vertex

When two parallel lines are cut by a transversal, the angles that are between the two parallel lines are **interior angles**. In the diagram below, angles 3, 4, 5, and 6 are interior angles.

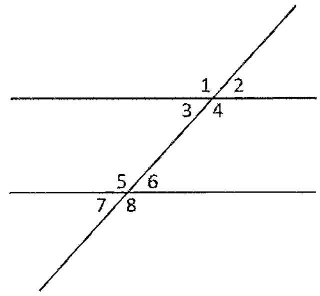
When two parallel lines are cut by a transversal, the angles that are outside the parallel lines are **exterior angles**. In the diagram below, angles 1, 2, 7, and 8 are exterior angles.

When two parallel lines are cut by a transversal, the angles that are in the same position relative to the transversal and a parallel line are **corresponding angles**. The diagram below has four pairs of corresponding angles: angles 1 and 5, angles 2 and 6, angles 3 and 7, and angles 4 and 8. Corresponding angles formed by parallel lines are congruent.

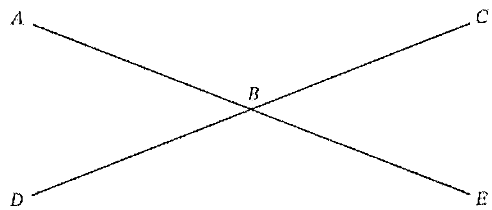
When two parallel lines are cut by a transversal, the two interior angles that are on opposite sides of the transversal are called **alternate interior angles**. In the diagram below, there are two pairs of alternate interior angles: angles 3 and 6, and angles 4 and 5. Alternate interior angles formed by parallel lines are congruent.

When two parallel lines are cut by a transversal, the two exterior angles that are on opposite sides of the transversal are called **alternate exterior angles**.

In the diagram below, there are two pairs of alternate exterior angles: angles 1 and 8, and angles 2 and 7. Alternate exterior angles formed by parallel lines are congruent.



When two lines intersect, four angles are formed. The non-adjacent angles at this vertex are called vertical angles. Vertical angles are congruent. In the diagram, $\angle ABD \cong \angle CBE$ and $\angle ABC \cong \angle DBE$. The other pairs of angles, $(\angle ABC, \angle CBE)$ and $(\angle ABD, \angle DBE)$, are supplementary, meaning the pairs sum to 180°.



POLYGONS

A **polygon** is a closed, two-dimensional figure with three or more straight line segments called **sides**. The point at which two sides of a polygon intersect is called the **vertex**. In a polygon, the number of sides is always equal to the number of vertices. A polygon with all sides congruent and all angles equal is called a **regular polygon**. Common polygons are:

Triangle = 3 sides
Quadrilateral = 4 sides
Pentagon = 5 sides
Hexagon = 6 sides
Heptagon = 7 sides
Octagon = 8 sides
Nonagon = 9 sides
Decagon = 10 sides
Dodecagon = 12 sides

More generally, an n-gon is a polygon that has n angles and n sides.

Review Video: Intro to Polygons
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The sum of the interior angles of an n-sided polygon is $(n - 2) \times 180°$. For example, in a triangle $n = 3$. So the sum of the interior angles is $(3 - 2) \times 180° = 180°$. In a quadrilateral, $n = 4$, and the sum of the angles is $(4 - 2) \times 180° = 360°$.

Review Video: Sum of Interior Angles
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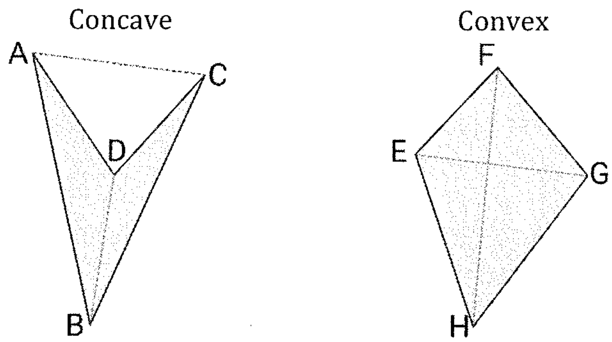
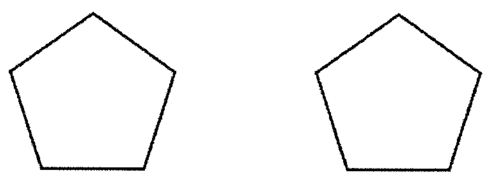
CONVEX AND CONCAVE POLYGONS

A **convex polygon** is a polygon whose diagonals all lie within the interior of the polygon. A **concave polygon** is a polygon with a least one diagonal that is outside the polygon. In the diagram below, quadrilateral $ABCD$ is concave because diagonal $\overline{AC}$ lies outside the polygon and quadrilateral $EFGH$ is convex because both diagonals lie inside the polygon.



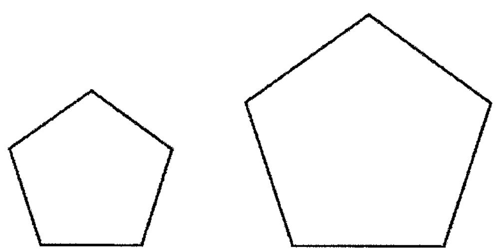
CONGRUENCE AND SIMILARITY

Congruent figures are geometric figures that have the same size and shape. All corresponding angles are equal, and all corresponding sides are equal. Congruence is indicated by the symbol $\cong$.



Congruent polygons

Similar figures are geometric figures that have the same shape, but do not necessarily have the same size. All corresponding angles are equal, and all corresponding sides are proportional, but they do not have to be equal. It is indicated by the symbol $\sim$.



Similar polygons

Note that all congruent figures are also similar, but not all similar figures are congruent.

Review Video: Congruent Shapes
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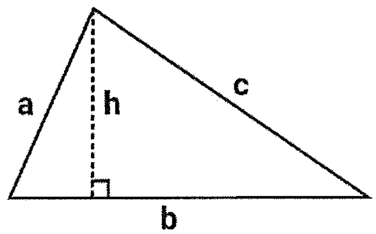
TRIANGLES

A triangle is a three-sided figure with the sum of its interior angles being 180°. The **perimeter of any triangle** is found by summing the three side lengths; $P = a + b + c$. For an equilateral triangle, this is the same as $P = 3a$, where a is any side length, since all three sides are the same length.



Review Video: Proof that a Triangle is 180 Degrees
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Review Video: Area and Perimeter of a Triangle
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The **area of any triangle** can be found by taking half the product of one side length referred to as the base, often given the variable b and the perpendicular distance from that side to the opposite vertex called the altitude or height and given the variable h. In equation form that is $A = \frac{1}{2}bh$. Another formula that works for any triangle is $A = \sqrt{s(s-a)(s-b)(s-c)}$, where s is the semiperimeter: $\frac{a+b+c}{2}$, and a, b, and c are the lengths of the three sides. Special cases include isosceles triangles, $A = \frac{1}{2}b\sqrt{a^2 - \frac{b^2}{4}}$, where b is the unique side and a is the length of one of the two congruent sides, and equilateral triangles, $A = \frac{\sqrt{3}}{4}a^2$, where a is the length of a side.

Review Video: Area of Any Triangle
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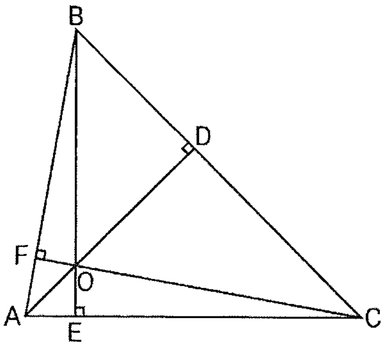
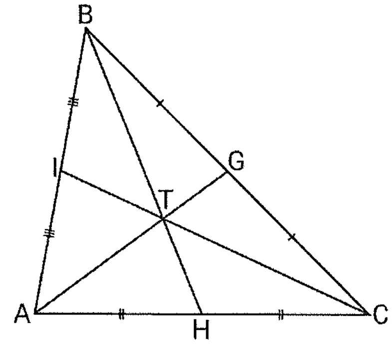
PARTS OF A TRIANGLE

An **altitude** of a triangle is a line segment drawn from one vertex perpendicular to the opposite side. In the diagram that follows, $\overline{BE}$, $\overline{AD}$, and $\overline{CF}$ are altitudes. The length of an altitude is also called the height of the triangle. The three altitudes in a triangle are always concurrent. The point of concurrency of the altitudes of a triangle, O, is called the **orthocenter**. Note that in an obtuse triangle, the orthocenter will be outside the triangle, and in a right triangle, the orthocenter is the vertex of the right angle.

A **median** of a triangle is a line segment drawn from one vertex to the midpoint of the opposite side. In the diagram that follows, $\overline{BH}$, $\overline{AG}$, and $\overline{CI}$ are medians. This is not the same as the altitude, except the altitude to the base of an isosceles triangle and all three altitudes of an equilateral triangle. The point of concurrency of the medians of a triangle, T, is called the **centroid**. This is the same point as the orthocenter only in an equilateral triangle. Unlike the orthocenter, the centroid is always inside the triangle. The centroid can also be considered the exact center of the triangle. Any shape triangle can be perfectly balanced on a tip placed at the centroid. The centroid is also the point that is two-thirds the distance from the vertex to the opposite side.





Review Video: Centroid, Incenter, Circumcenter, and Orthocenter
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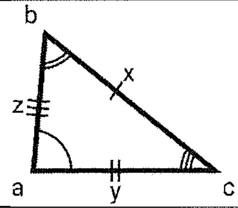
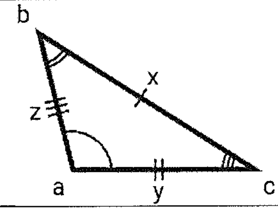
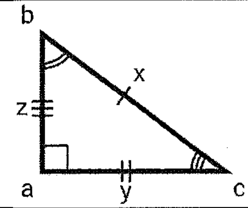
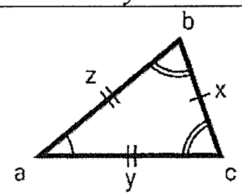
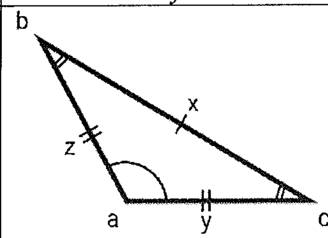
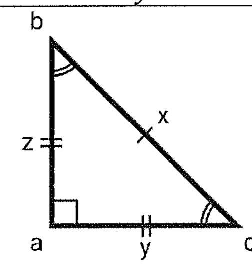
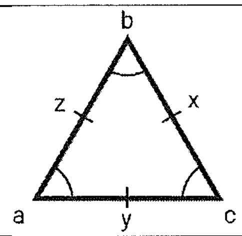
TRIANGLE PROPERTIES

CLASSIFICATIONS OF TRIANGLES

A **scalene triangle** is a triangle with no congruent sides. A scalene triangle will also have three angles of different measures. The angle with the largest measure is opposite the longest side, and the angle with the smallest measure is opposite the shortest side. An **acute triangle** is a triangle whose three angles are all less than 90°. If two of the angles are equal, the acute triangle is also an **isosceles triangle**. An isosceles triangle will also have two congruent angles opposite the two congruent sides. If the three angles are all equal, the acute triangle is also an **equilateral triangle**. An equilateral triangle will also have three congruent angles, each 60°. All equilateral triangles are also acute triangles. An **obtuse triangle** is a triangle with exactly one angle greater than 90°. The other two angles may or may not be equal. If the two remaining angles are equal, the obtuse triangle is also an isosceles triangle. A **right triangle** is a triangle with exactly one angle equal to 90°. All right triangles follow the Pythagorean theorem. A right triangle can never be acute or obtuse.

The table below illustrates how each descriptor places a different restriction on the triangle:

Angles / Sides	**Acute:** **All angles < 90°**	**Obtuse:** **One angle > 90°**	**Right:** **One angle = 90°**
Scalene: **No equal side lengths**			
	$90° > \angle a > \angle b > \angle c$ $x > y > z$	$\angle a > 90° > \angle b > \angle c$ $x > y > z$	$90° = \angle a > \angle b > \angle c$ $x > y > z$
Isosceles: **Two equal side lengths**			
	$90° > \angle a, \angle b, or\ \angle c$ $\angle b = \angle c, \quad y = z$	$\angle a > 90° > \angle b = \angle c$ $x > y = z$	$\angle a = 90°$ $\angle b = \angle c = 45°$ $x > y = z$
Equilateral: **Three equal side lengths**			
	$60° = \angle a = \angle b = \angle c$ $x = y = z$		

Review Video: Introduction to Types of Triangles
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General Rules for Triangles

The **triangle inequality theorem** states that the sum of the measures of any two sides of a triangle is always greater than the measure of the third side. If the sum of the measures of two sides were equal to the third side, a triangle would be impossible because the two sides would lie flat across the third side and there would be no vertex. If the sum of the measures of two of the sides was less than the third side, a closed figure would be impossible because the two shortest sides would never meet. In other words, for a triangle with sides lengths A, B, and C: $A + B > C$, $B + C > A$, and $A + C > B$.

The sum of the measures of the interior angles of a triangle is always 180°. Therefore, a triangle can never have more than one angle greater than or equal to 90°.

In any triangle, the angles opposite congruent sides are congruent, and the sides opposite congruent angles are congruent. The largest angle is always opposite the longest side, and the smallest angle is always opposite the shortest side.

The line segment that joins the midpoints of any two sides of a triangle is always parallel to the third side and exactly half the length of the third side.

Review Video: General Rules (Triangle Inequality Theorem)
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Similarity and Congruence Rules

Similar triangles are triangles whose corresponding angles are equal and whose corresponding sides are proportional. Represented by AAA. Similar triangles whose corresponding sides are congruent are also congruent triangles.

Triangles can be shown to be **congruent** in 5 ways:

- **SSS**: Three sides of one triangle are congruent to the three corresponding sides of the second triangle.
- **SAS**: Two sides and the included angle (the angle formed by those two sides) of one triangle are congruent to the corresponding two sides and included angle of the second triangle.
- **ASA**: Two angles and the included side (the side that joins the two angles) of one triangle are congruent to the corresponding two angles and included side of the second triangle.
- **AAS**: Two angles and a non-included side of one triangle are congruent to the corresponding two angles and non-included side of the second triangle.
- **HL**: The hypotenuse and leg of one right triangle are congruent to the corresponding hypotenuse and leg of the second right triangle.

Review Video: Similar Triangles
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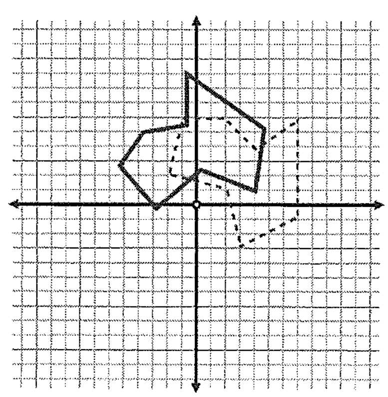
Transformations

Rotation

A **rotation** is a transformation that turns a figure around a point called the **center of rotation**, which can lie anywhere in the plane. If a line is drawn from a point on a figure to the center of rotation, and another line is drawn from the center to the rotated image of that point, the angle between the two lines is the **angle of rotation**. The vertex of the angle of rotation is the center of rotation.



Review Video: Rotation
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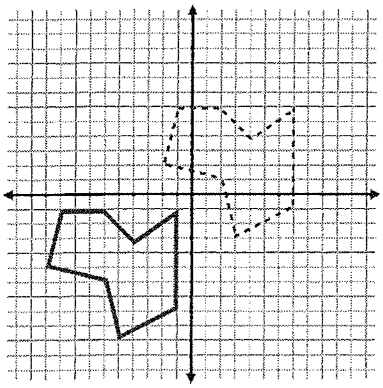
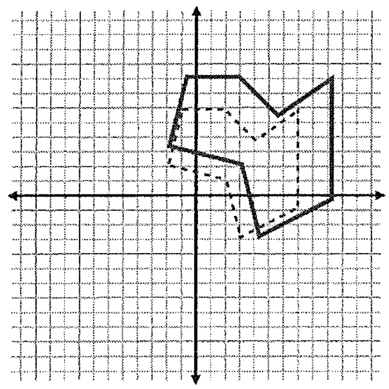
Translation and Dilation

A **translation** is a transformation which slides a figure from one position in the plane to another position in the plane. The original figure and the translated figure have the same size, shape, and orientation. A **dilation** is a transformation which proportionally stretches or shrinks a figure by a **scale factor**. The dilated image is the same shape and orientation as the original image but a different size. A polygon and its dilated image are similar.

Translation



Dilation



Review Video: Translation
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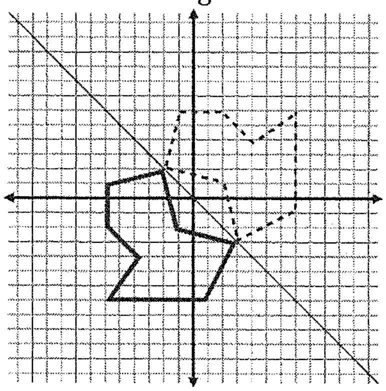
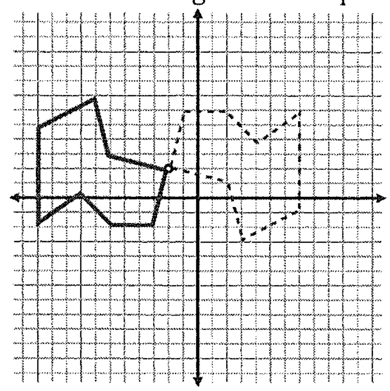
Review Video: Dilation
Visit mometrix.com/academy and enter code: 471630

A **reflection of a figure over a line** (a "flip") creates a congruent image that is the same distance from the line as the original figure but on the opposite side. The **line of reflection** is the perpendicular bisector of any line segment drawn from a point on the original figure to its reflected image (unless the point and its reflected image happen to be the same point, which happens when a figure is reflected over one of its own sides). A **reflection of a figure over a point** (an inversion) in two dimensions is the same as the rotation of the figure 180° about that point. The image of the figure is congruent to the original figure. The **point of reflection** is the midpoint of a line segment which connects a point in the figure to its image (unless the point and its reflected image happen to be the same point, which happens when a figure is reflected in one of its own points).

Reflection of a figure over a line



Reflection of a figure over a point



Review Video: Reflection
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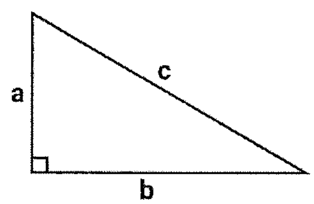
PYTHAGOREAN THEOREM

The side of a triangle opposite the right angle is called the **hypotenuse**. The other two sides are called the legs. The Pythagorean theorem states a relationship among the legs and hypotenuse of a right triangle: $(a^2 + b^2 = c^2)$, where a and b are the lengths of the legs of a right triangle, and c is the length of the hypotenuse. Note that this formula will only work with right triangles.



Review Video: Pythagorean Theorem
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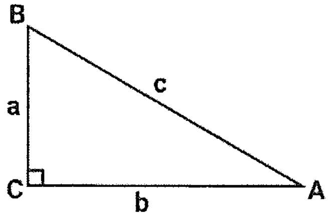
TRIGONOMETRIC FORMULAS

In the diagram below, angle C is the right angle, and side c is the hypotenuse. Side a is the side opposite to angle A and side b is the side opposite to angle B. Using ratios of side lengths as a means to calculate the sine, cosine, and tangent of an acute angle only works for right triangles.



$$\sin A = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{c} \qquad \csc A = \frac{1}{\sin A} = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{c}{a}$$

$$\cos A = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{c} \qquad \sec A = \frac{1}{\cos A} = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{c}{b}$$

$$\tan A = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{b} \qquad \cot A = \frac{1}{\tan A} = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{b}{a}$$

Laws of Sines and Cosines

The **law of sines** states that $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$, where A, B, and C are the angles of a triangle, and a, b, and c are the sides opposite their respective angles. This formula will work with all triangles, not just right triangles.

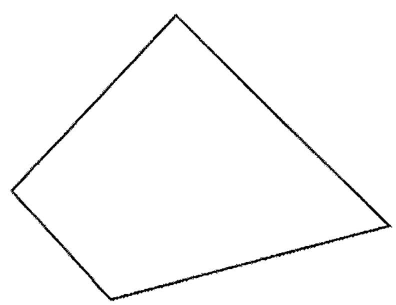
The **law of cosines** is given by the formula $c^2 = a^2 + b^2 - 2ab(\cos C)$, where a, b, and c are the sides of a triangle, and C is the angle opposite side c. This is a generalized form of the Pythagorean theorem that can be used on any triangle.

Review Video: Upper Level Trig: Law of Sines
Visit mometrix.com/academy and enter code: 206844

Review Video: Upper Level Trig: Law of Cosines
Visit mometrix.com/academy and enter code: 158911

Quadrilaterals

A **quadrilateral** is a closed two-dimensional geometric figure that has four straight sides. The sum of the interior angles of any quadrilateral is 360°.



Review Video: Diagonals of Parallelograms, Rectangles, and Rhombi
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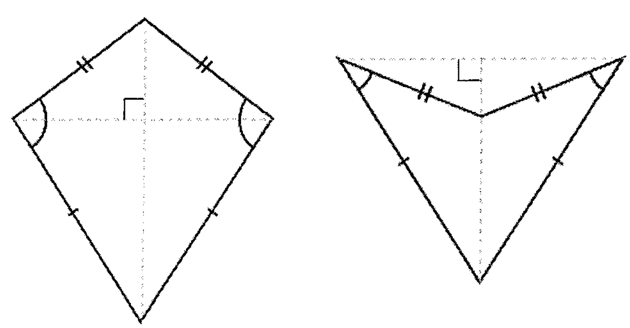
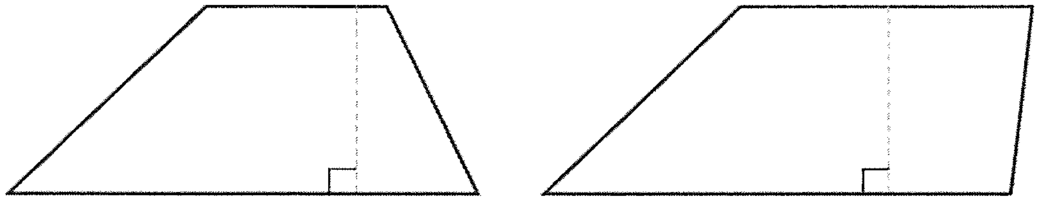
Kite

A **kite** is a quadrilateral with two pairs of adjacent sides that are congruent. A result of this is perpendicular diagonals. A kite can be concave or convex and has one line of symmetry.



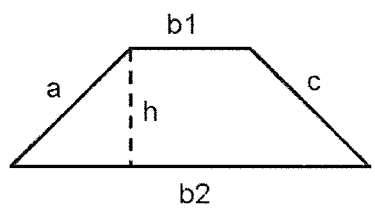
TRAPEZOID

Trapezoid: A trapezoid is defined as a quadrilateral that has at least one pair of parallel sides. There are no rules for the second pair of sides. So, there are no rules for the diagonals and no lines of symmetry for a trapezoid.



The **area of a trapezoid** is found by the formula $A = \frac{1}{2}h(b_1 + b_2)$, where h is the height (segment joining and perpendicular to the parallel bases), and b_1 and b_2 are the two parallel sides (bases). Do not use one of the other two sides as the height unless that side is also perpendicular to the parallel bases.

The **perimeter of a trapezoid** is found by the formula $P = a + b_1 + c + b_2$, where a, b_1, c, and b_2 are the four sides of the trapezoid.



Review Video: Area and Perimeter of a Trapezoid
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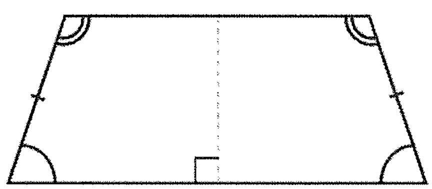
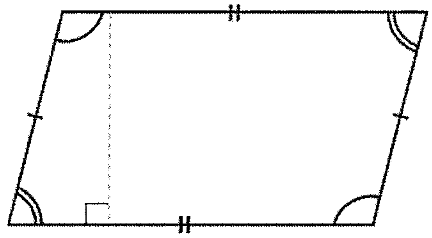
Isosceles trapezoid: A trapezoid with equal base angles. This gives rise to other properties including: the two nonparallel sides have the same length, the two non-base angles are also equal, and there is one line of symmetry through the midpoints of the parallel sides.



PARALLELOGRAM

A **parallelogram** is a quadrilateral that has two pairs of opposite parallel sides. As such it is a special type of trapezoid. The sides that are parallel are also congruent. The opposite interior angles are always congruent, and the consecutive interior angles are supplementary. The diagonals of a parallelogram divide each other. Each diagonal divides the parallelogram into two congruent triangles. A parallelogram has no line of symmetry, but does have 180-degree rotational symmetry about the midpoint.



The **area of a parallelogram** is found by the formula $A = bh$, where b is the length of the base, and h is the height. Note that the base and height correspond to the length and width in a rectangle, so this formula would apply to rectangles as well. Do not confuse the height of a parallelogram with the length of the second side. The two are only the same measure in the case of a rectangle.

The **perimeter of a parallelogram** is found by the formula $P = 2a + 2b$ or $P = 2(a + b)$, where a and b are the lengths of the two sides.

Review Video: How to Find the Area and Perimeter of a Parallelogram
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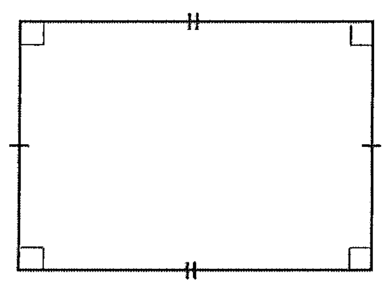
RECTANGLE

A **rectangle** is a quadrilateral with four right angles. All rectangles are parallelograms and trapezoids, but not all parallelograms or trapezoids are rectangles. The diagonals of a rectangle are congruent. Rectangles have two lines of symmetry (through each pair of opposing midpoints) and 180-degree rotational symmetry about the midpoint.

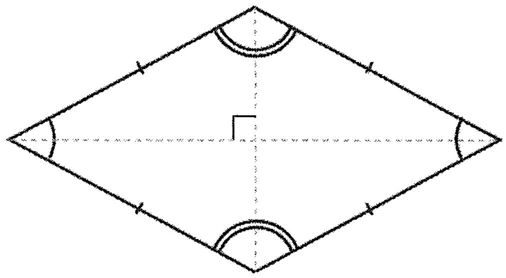


The **area of a rectangle** is found by the formula $A = lw$, where A is the area of the rectangle, l is the length (usually considered to be the longer side) and w is the width (usually considered to be the shorter side). The numbers for l and w are interchangeable.

The **perimeter of a rectangle** is found by the formula $P = 2l + 2w$ or $P = 2(l + w)$, where l is the length, and w is the width. It may be easier to add the length and width first and then double the result, as in the second formula.

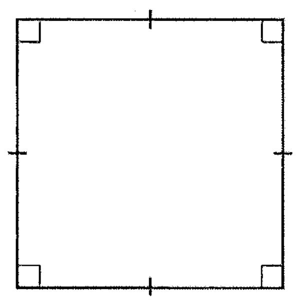
RHOMBUS

A **rhombus** is a quadrilateral with four congruent sides. All rhombuses are parallelograms and kites; thus, they inherit all the properties of both types of quadrilaterals. The diagonals of a rhombus are perpendicular to each other. Rhombi have two lines of symmetry (along each of the diagonals) and 180° rotational symmetry. The **area of a rhombus** is half the product of the diagonals: $A = \frac{d_1 d_2}{2}$ and the perimeter of a rhombus is: $P = 2\sqrt{(d_1)^2 + (d_2)^2}$.



SQUARE

A **square** is a quadrilateral with four right angles and four congruent sides. Squares satisfy the criteria of all other types of quadrilaterals. The diagonals of a square are congruent and perpendicular to each other. Squares have four lines of symmetry (through each pair of opposing midpoints and along each of the diagonals) as well as 90° rotational symmetry about the midpoint.



The **area of a square** is found by using the formula $A = s^2$, where s is the length of one side. The **perimeter of a square** is found by using the formula $P = 4s$, where s is the length of one side. Because all four sides are equal in a square, it is faster to multiply the length of one side by 4 than to add the same number four times. You could use the formulas for rectangles and get the same answer.

Review Video: Area and Perimeter of Rectangles and Squares
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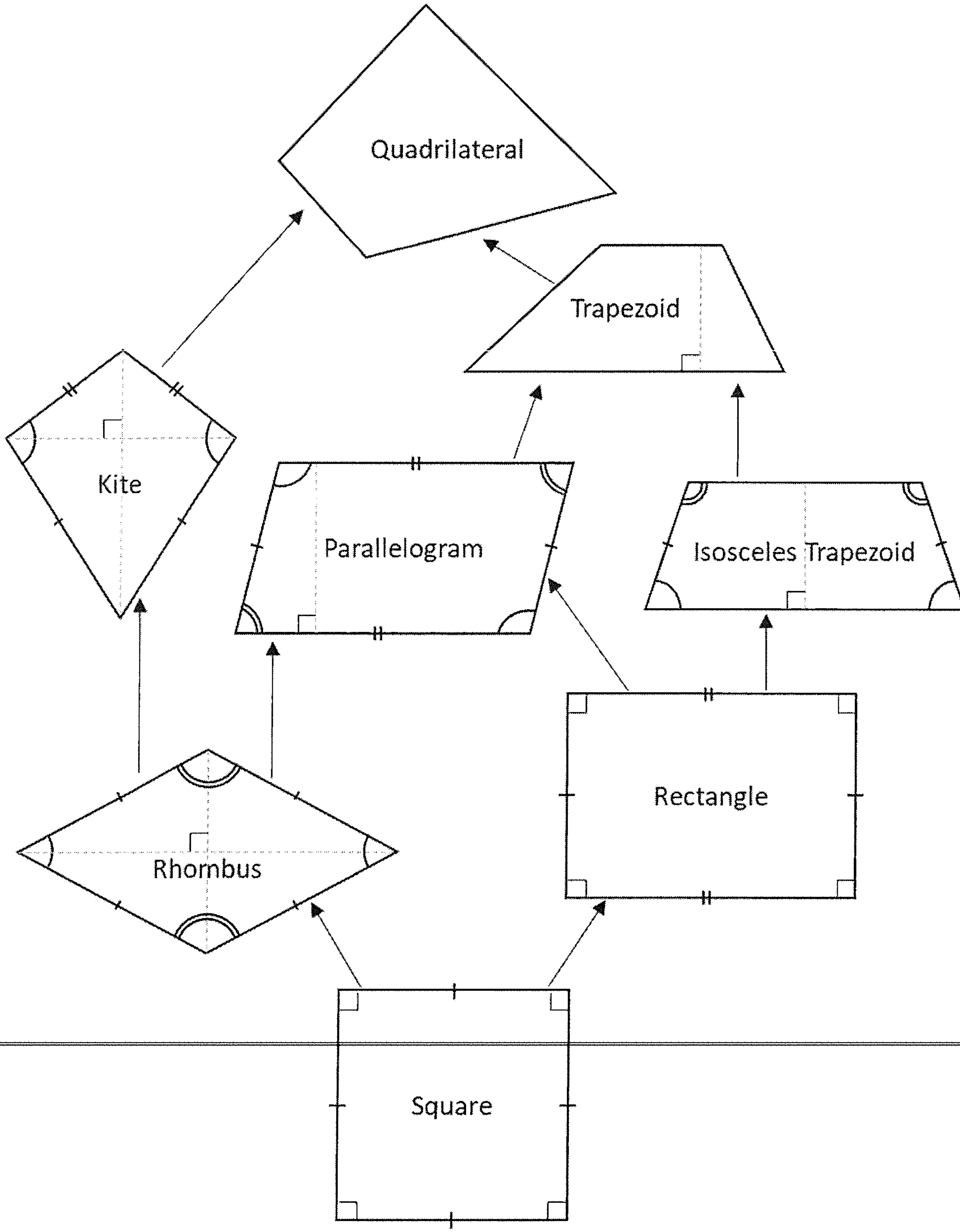
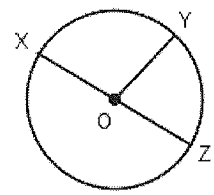
HIERARCHY OF QUADRILATERALS

The hierarchy of quadrilaterals is as follows:



CIRCLES

The **center** of a circle is the single point from which every point on the circle is **equidistant**. The **radius** is a line segment that joins the center of the circle and any one point on the circle. All radii of a circle are equal. Circles that have the same center but not the same length of radii are **concentric**. The **diameter** is a line segment that passes through the center of the circle and has both endpoints on the circle. The length of the diameter is exactly twice the length of the radius. Point O in the diagram below is the center of the circle, segments $\overline{OX}$, $\overline{OY}$, and $\overline{OZ}$ are radii; and segment $\overline{XZ}$ is a diameter.



Review Video: Points of a Circle
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Review Video: The Diameter, Radius, and Circumference of Circles
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The **area of a circle** is found by the formula $A = \pi r^2$, where r is the length of the radius. If the diameter of the circle is given, remember to divide it in half to get the length of the radius before proceeding.

The **circumference** of a circle is found by the formula $C = 2\pi r$, where r is the radius. Again, remember to convert the diameter if you are given that measure rather than the radius.

Review Video: Area and Circumference of a Circle
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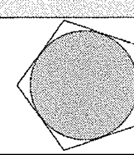
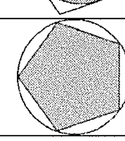
INSCRIBED AND CIRCUMSCRIBED FIGURES

These terms can both be used to describe a given arrangement of figures, depending on perspective. If each of the vertices of figure A lie on figure B, then it can be said that figure A is **inscribed** in figure B, but it can also be said that figure B is **circumscribed** about figure A. The following table and examples help to illustrate the concept. Note that the figures cannot both be circles, as they would be completely overlapping and neither would be inscribed or circumscribed.

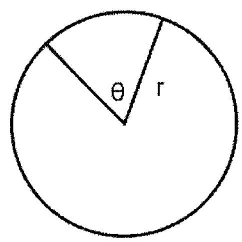
Given	Description	Equivalent Description	Figures
Each of the sides of a pentagon is tangent to a circle	The circle is inscribed in the pentagon	The pentagon is circumscribed about the circle	
Each of the vertices of a pentagon lie on a circle	The pentagon is inscribed in the circle	The circle is circumscribed about the pentagon	

CIRCLE PROPERTIES

ARCS

An **arc** is a portion of a circle. Specifically, an arc is the set of points between and including two points on a circle. An arc does not contain any points inside the circle. When a segment is drawn from the endpoints of an arc to the center of the circle, a sector is formed. A **minor arc** is an arc that has a measure less than 180°. A **major arc** is an arc that has a measure of at least 180°. Every minor arc has a corresponding major arc that can be found by subtracting the measure of the minor arc from 360°. A **semicircle** is an arc whose endpoints are the endpoints of the diameter of a circle. A semicircle is exactly half of a circle.

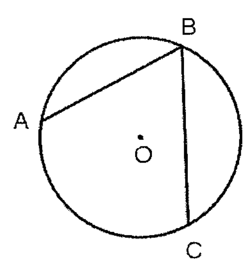
Arc length is the length of that portion of the circumference between two points on the circle. The formula for arc length is $s = \frac{\pi r \theta}{180°}$, where s is the arc length, r is the length of the radius, and θ is the angular measure of the arc in degrees, or $s = r\theta$, where θ is the angular measure of the arc in radians (2π radians $= 360$ degrees).



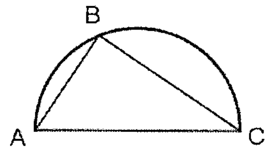
ANGLES OF CIRCLES

A **central angle** is an angle whose vertex is the center of a circle and whose legs intercept an arc of the circle. The measure of a central angle is equal to the measure of the minor arc it intercepts.

An **inscribed angle** is an angle whose vertex lies on a circle and whose legs contain chords of that circle. The portion of the circle intercepted by the legs of the angle is called the intercepted arc. The measure of the intercepted arc is exactly twice the measure of the inscribed angle. In the following diagram, angle ABC is an inscribed angle. $\widehat{AC} = 2(\text{m}\angle ABC)$.



Any angle inscribed in a semicircle is a right angle. The intercepted arc is 180°, making the inscribed angle half that, or 90°. In the diagram below, angle ABC is inscribed in semicircle ABC, making angle ABC equal to 90°.



Review Video: Arcs and Angles of Circles
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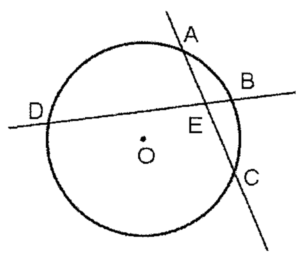
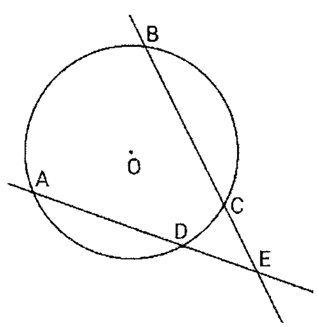
Secants, Chords, and Tangents

A **secant** is a line that intersects a circle in two points. The segment of a secant line that is contained within the circle is called a **chord**. Two secants may intersect inside the circle, on the circle, or outside the circle. When the two secants intersect on the circle, an inscribed angle is formed. When two secants intersect inside a circle, the measure of each of two vertical angles is equal to half the sum of the two intercepted arcs. Consider the following diagram where $m\angle AEB = \frac{1}{2}(\widehat{AB} + \widehat{CD})$ and $m\angle BEC = \frac{1}{2}(\widehat{BC} + \widehat{AD})$.

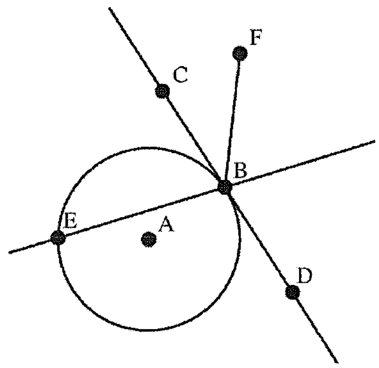


When two secants intersect outside a circle, the measure of the angle formed is equal to half the difference of the two arcs that lie between the two secants. In the diagram below, $m\angle AEB = \frac{1}{2}(\widehat{AB} - \widehat{CD})$.



A **tangent** is a line in the same plane as a circle that touches the circle in exactly one point. The point at which a tangent touches a circle is called the **point of tangency**. While a line segment can be tangent to a circle as part of a line that is tangent, it is improper to say a tangent can be simply a line segment that touches the circle in exactly one point.

In the diagram below, $\overleftrightarrow{EB}$ is a secant and contains chord $\overline{EB}$, and $\overleftrightarrow{CD}$ is tangent to circle A. Notice that $\overline{FB}$ is not tangent to the circle. $\overline{FB}$ is a line segment that touches the circle in exactly one point, but if the segment were extended, it would touch the circle in a second point. In the diagram below, point B is the point of tangency.



Review Video: Secants, Chords, and Tangents
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Review Video: Tangent Lines of a Circle
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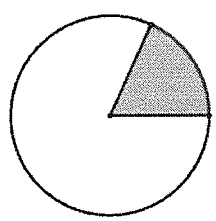
Sectors

A **sector** is the portion of a circle formed by two radii and their intercepted arc. While the arc length is exclusively the points that are also on the circumference of the circle, the sector is the entire area bounded by the arc and the two radii.



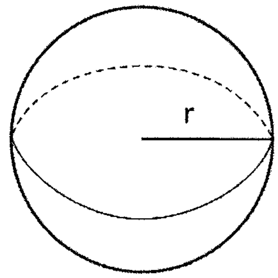
The **area of a sector** of a circle is found by the formula, $A = \frac{\theta r^2}{2}$, where A is the area, θ is the measure of the central angle in radians, and r is the radius. To find the area with the central angle in degrees, use the formula, $A = \frac{\theta \pi r^2}{360}$, where θ is the measure of the central angle and r is the radius.

3D Shapes

Solids

The **surface area of a solid object** is the area of all sides or exterior surfaces. For objects such as prisms and pyramids, a further distinction is made between base surface area (B) and lateral surface area (LA). For a prism, the total surface area (SA) is $SA = LA + 2B$. For a pyramid or cone, the total surface area is $SA = LA + B$.

The **surface area of a sphere** can be found by the formula $A = 4\pi r^2$, where r is the radius. The volume is given by the formula $V = \frac{4}{3}\pi r^3$, where r is the radius. Both quantities are generally given in terms of π.



Review Video: Volume and Surface Area of a Sphere
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Review Video: How to Calculate the Volume of 3D Objects
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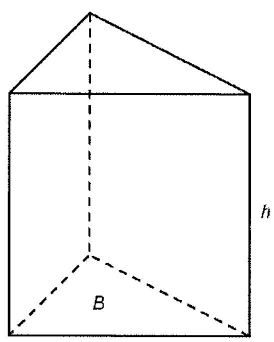
The **volume of any prism** is found by the formula $V = Bh$, where B is the area of the base, and h is the height (perpendicular distance between the bases). The surface area of any prism is the sum of the areas of both bases and all sides. It can be calculated as $SA = 2B + Ph$, where P is the perimeter of the base.



Review Video: Volume and Surface Area of a Prism
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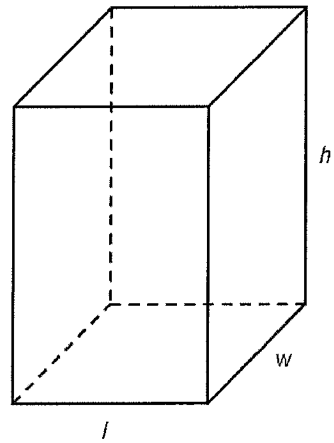
For a **rectangular prism**, the volume can be found by the formula $V = lwh$, where V is the volume, l is the length, w is the width, and h is the height. The surface area can be calculated as $SA = 2lw + 2hl + 2wh$ or $SA = 2(lw + hl + wh)$.



Review Video: Volume and Surface Area of a Rectangular Prism
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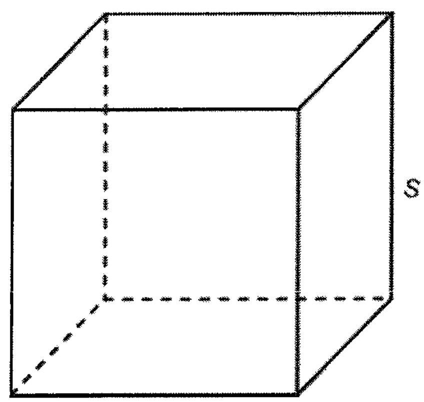
The **volume of a cube** can be found by the formula $V = s^3$, where s is the length of a side. The surface area of a cube is calculated as $SA = 6s^2$, where SA is the total surface area and s is the length of a side. These formulas are the same as the ones used for the volume and surface area of a rectangular prism, but simplified since all three quantities (length, width, and height) are the same.



Review Video: Volume and Surface Area of a Cube
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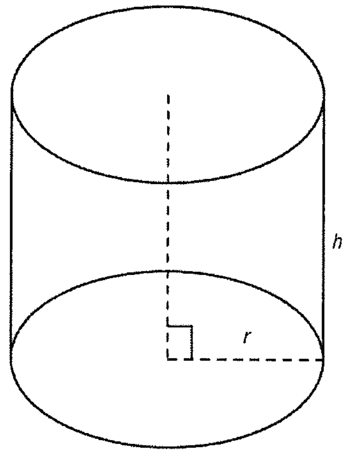
The **volume of a cylinder** can be calculated by the formula $V = \pi r^2 h$, where r is the radius, and h is the height. The surface area of a cylinder can be found by the formula $SA = 2\pi r^2 + 2\pi rh$. The first term is the base area multiplied by two, and the second term is the perimeter of the base multiplied by the height.



Review Video: Finding the Volume and Surface Area of a Right Circular Cylinder
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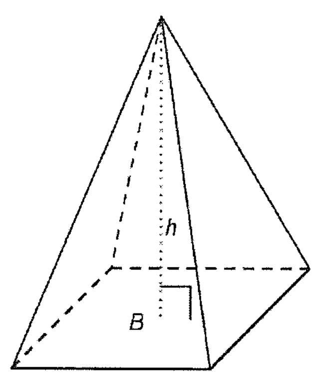
The **volume of a pyramid** is found by the formula $V = \frac{1}{3}Bh$, where B is the area of the base, and h is the height (perpendicular distance from the vertex to the base). Notice this formula is the same as $\frac{1}{3}$ times the volume of a prism. Like a prism, the base of a pyramid can be any shape.

Finding the **surface area of a pyramid** is not as simple as the other shapes we've looked at thus far. If the pyramid is a right pyramid, meaning the base is a regular polygon and the vertex is directly over the center of that polygon, the surface area can be calculated as $SA = B + \frac{1}{2}Ph_s$, where P is the perimeter of the base, and h_s is the slant height (distance from the vertex to the midpoint of one side of the base). If the pyramid is irregular, the area of each triangle side must be calculated individually and then summed, along with the base.



Review Video: Finding the Volume and Surface Area of a Pyramid
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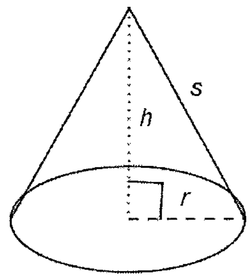
The **volume of a cone** is found by the formula $V = \frac{1}{3}\pi r^2 h$, where r is the radius, and h is the height. Notice this is the same as $\frac{1}{3}$ times the volume of a cylinder. The surface area can be calculated as $SA = \pi r^2 + \pi rs$, where s is the slant height. The slant height can be calculated using the Pythagorean theorem to be $\sqrt{r^2 + h^2}$, so the surface area formula can also be written as $SA = \pi r^2 + \pi r\sqrt{r^2 + h^2}$.



Review Video: Volume and Surface Area of a Right Circular Cone
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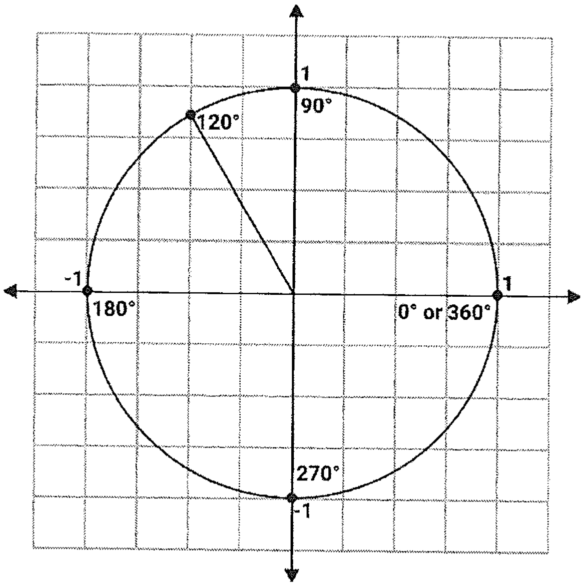
THE UNIT CIRCLE

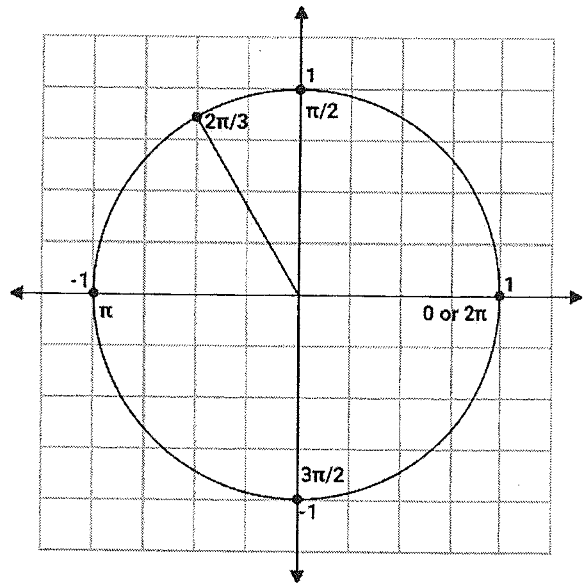
DEGREES, RADIANS, AND THE UNIT CIRCLE

It is important to understand the deep connection between trigonometry and circles. Specifically, the two main units, **degrees** (°) and **radians** (rad), that are used to measure angles are related this way: 360° in one full circle and 2π radians in one full circle: ($360° = 2\pi$ rad). The conversion factor relating the two is often stated as $\frac{180°}{\pi}$. For example, to convert $\frac{3\pi}{2}$ radians to degrees, multiply by the conversion factor: $\frac{3\pi}{2} \times \frac{180°}{\pi} = 270°$. As another example, to convert 60° to radians, divide by the conversion factor or multiply by the reciprocal: $60° \times \frac{\pi}{180°} = \frac{\pi}{3}$ radians.

Recall that the standard equation for a circle is $(x - h)^2 + (y - k)^2 = r^2$. A **unit circle** is a circle with a radius of 1 ($r = 1$) that has its center at the origin ($h = 0, k = 0$). Thus, the equation for the unit circle simplifies from the standard equation down to $x^2 + y^2 = 1$.

Standard position is the position of an angle of measure θ whose vertex is at the origin, the initial side crosses the unit circle at the point $(1, 0)$, and the terminal side crosses the unit circle at some other point (a, b). In the standard position, $\sin\theta = b$, $\cos\theta = a$, and $\tan\theta = \frac{b}{a}$.





Review Video: Upper Level Trig: Unit Circle
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TABLE OF COMMONLY ENCOUNTERED ANGLES

$0° = 0$ radians, $30° = \frac{\pi}{6}$ radians, $45° = \frac{\pi}{4}$ radians, $60° = \frac{\pi}{3}$ radians, and $90° = \frac{\pi}{2}$ radians

$\sin 0° = 0$	$\cos 0° = 1$	$\tan 0° = 0$
$\sin 30° = \frac{1}{2}$	$\cos 30° = \frac{\sqrt{3}}{2}$	$\tan 30° = \frac{\sqrt{3}}{3}$
$\sin 45° = \frac{\sqrt{2}}{2}$	$\cos 45° = \frac{\sqrt{2}}{2}$	$\tan 45° = 1$
$\sin 60° = \frac{\sqrt{3}}{2}$	$\cos 60° = \frac{1}{2}$	$\tan 60° = \sqrt{3}$
$\sin 90° = 1$	$\cos 90° = 0$	$\tan 90° =$ undefined
$\csc 0° =$ undefined	$\sec 0° = 1$	$\cot 0° =$ undefined
$\csc 30° = 2$	$\sec 30° = \frac{2\sqrt{3}}{3}$	$\cot 30° = \sqrt{3}$
$\csc 45° = \sqrt{2}$	$\sec 45° = \sqrt{2}$	$\cot 45° = 1$
$\csc 60° = \frac{2\sqrt{3}}{3}$	$\sec 60° = 2$	$\cot 60° = \frac{\sqrt{3}}{3}$
$\csc 90° = 1$	$\sec 90° =$ undefined	$\cot 90° = 0$

The values in the upper half of this table are values you should have memorized or be able to find quickly and those in the lower half can easily be determined as the reciprocal of the corresponding function.