Non-split alternating links bound unique minimal genus Seifert surfaces up to isotopy in the 4-ball

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Joint work with Seungwon Kim and Maggie Miller

July 27, 2024

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3 Main Theorem

2019

MIMIC

Sungkyunkwan Universtiy

Abstract

수학 학술 동아리 MIMIC에서 신입 회원을 모집함.

Definition 1. MIMIC is Math Interchange Club.

Theorem 1. 다음 조건을 만족하는 사람은 MIMIC에 지원할 수 있다.

(1) 자유롭고 학술적인 분위기에서 수학을 공부하고 싶은 사람.
(2) 함께 스터디 할 사람이 필요한 사람.
(3) 수학공부에 어려움을 느끼는 사람.

Proof. It suffices to prove in the following steps.

Step 1 매주 화요일 저녁 7시에 정기 세미나.
Step 2 동아리 내 튜터링 및 스터디 자체 운영.

□

Remark.

(1) 개강총회 : 9월 3일 화요일 저녁 6시 31316 (수학과 전공 강의실)
(2) 회장 이호빈 010-9137-2087
(3) 총무 박승열 010-2770-4915

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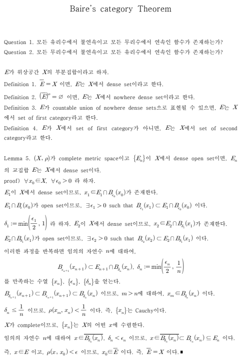
Once upon a time



Baire's category Theorem

Question 1. 모든 유리수에서 불연속이고 모든 무리수에서 연속인 함수가 존재하는가?
Question 2. 모든 무리수에서 불연속이고 모든 유리수에서 연속인 함수가 존재하는가?

E가 위상공간 X의 부분집합이라고 하자.

Definition 1. $\overline{E}=X$ 이면, E는 X에서 dense set이라고 한다.

Definition 2. $(\overline{E})^o=\varnothing$ 이면, E는 X에서 nowhere dense set이라고 한다.

Definition 3. E가 countable union of nowhere dense sets으로 표현될 수 있으면, E는 X에서 set of first category라고 한다.

Definition 4. E가 X에서 set of first category가 아니면, E는 X에서 set of second category라고 한다.

Lemma 5. (X,ρ)가 complete metric space이고 $\{E_n\}$이 X에서 dense open set이면, E_n의 교집합 E는 X에서 dense set이다.

proof) $\forall x_0 \in X$, $\forall \epsilon_0>0$ 라 하자.

E_1이 X에서 dense set이므로, $x_1 \in E_1 \cap B_{\epsilon_0}(x_0)$가 존재한다.

$E_1 \cap B_\epsilon(x_0)$가 open set이므로, $\exists \epsilon_1>0$ such that $B_{\epsilon_1}(x_1) \subset E_1 \cap B_{\epsilon_0}(x_0)$ 이다.

$\delta_1 := \min\left(\frac{\epsilon_1}{2}, 1\right)$ 라 하자. E_2이 X에서 dense set이므로, $x_2 \in E_2 \cap B_{\delta_1}(x_1)$가 존재한다.

$E_2 \cap B_{\delta_1}(x_1)$가 open set이므로, $\exists \epsilon_2>0$ such that $B_{\epsilon_2}(x_2) \subset E_2 \cap B_{\delta_1}(x_1)$ 이다.

이러한 과정을 반복하면 임의의 자연수 n에 대하여,

$$B_{\epsilon_{n+1}}(x_{n+1}) \subset E_{n+1} \cap B_{\delta_n}(x_n),\ \delta_n := \min\left(\frac{\epsilon_n}{2}, \frac{1}{n}\right)$$

를 만족하는 수열 $\{x_n\}$, $\{\epsilon_n\}$, $\{\delta_n\}$을 얻는다.

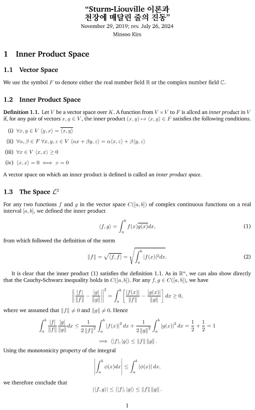
$B_{\delta_{n+1}}(x_{n+1}) \subset B_{\epsilon_{n+1}}(x_{n+1}) \subset B_{\delta_n}(x_n)$ 이므로, $m>n$에 대하여, $x_m \in B_{\delta_n}(x_n)$ 이다.

$\delta_n \le \frac{1}{n}$ 이므로, $\rho(x_m, x_n) < \frac{1}{n}$ 이다. 즉, $\{x_n\}$는 Cauchy이다.

X가 complete이므로, $\{x_n\}$는 X의 어떤 x에 수렴한다.

임의의 자연수 n에 대하여 $x \in \overline{B_{\delta_n}(x_n)}$, $\delta_n < \epsilon_n$ 이므로, $x \in \overline{B_{\delta_n}(x_n)} \subset B_{\epsilon_n}(x_n) \subseteq E_n$ 이다.

즉, $x \in E$ 이고, $\rho(x, x_0) < \epsilon$ 이므로, $x_0 \in \overline{E}$ 이다. 즉, $\overline{E}=X$ 이다.■



"Sturm-Liouville 이론과
천장에 매달린 줄의 진동"

November 29, 2019; rev. July 26, 2024

Minsoo Kim

1 Inner Product Space

1.1 Vector Space

We use the symbol F to denote either the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$.

1.2 Inner Product Space

Definition 1.1. Let V be a vector space over K. A function from $V\times V$ to F is allced an *inner product* in V if, for any pair of vectors $x, y \in V$, the inner product $(x,y) \mapsto \langle x,y\rangle \in F$ satisfies the following conditions.

(i) $\forall x, y \in V\ \langle y,x\rangle = \overline{\langle x,y\rangle}$

(ii) $\forall \alpha, \beta \in F\ \forall x,y,z \in V\ \langle \alpha x+\beta y, z\rangle = \alpha\langle x,z\rangle + \beta\langle y,z\rangle$

(iii) $\forall x \in V\ \langle x,x\rangle \ge 0$

(iv) $\langle x,x\rangle = 0 \iff x=0$

A vector space on which an inner product is defined is called an *inner product space*.

1.3 The Space $\mathcal{L}^2$

For any two functions f and g in the vector space $C([a,b])$ of complex continuous functions on a real interval $[a,b]$, we defined the inner product

$$\langle f,g\rangle = \int_a^b f(x)\overline{g(x)}dx, \tag{1}$$

from which followed the definition of the norm

$$\|f\| = \sqrt{\langle f,f\rangle} = \sqrt{\int_a^b |f(x)|^2 dx}. \tag{2}$$

It is clear that the inner product (1) satisfies the definition 1.1. As in $\mathbb{R}^n$, we can also show directly that the Cauchy-Schwarz inequality holds in $C([a,b])$. For any $f,g \in C([a,b])$, we have

$$\left\| \frac{|f|}{\|f\|} - \frac{|g|}{\|g\|} \right\|^2 = \int_a^b \left[\frac{|f(x)|}{\|f\|} - \frac{|g(x)|}{\|g\|} \right] dx \ge 0,$$

where we assumed that $\|f\| \ne 0$ and $\|g\| \ne 0$. Hence

$$\int_a^b \frac{|f|}{\|f\|}\frac{|g|}{\|g\|} dx \le \frac{1}{2\|f\|^2}\int_a^b |f(x)|^2\,dx + \frac{1}{2\|g\|^2}\int_a^b |g(x)|^2\,dx = \frac{1}{2}+\frac{1}{2} = 1$$
$$\implies \langle |f|, |g|\rangle \le \|f\|\,\|g\|.$$

Using the monotonicity property of the integral

$$\left|\int_a^b \phi(x)dx\right| \le \int_a^b |\phi(x)|\,dx,$$

we therefore conclude that

$$|\langle f,g\rangle| \le \langle |f|, |g|\rangle \le \|f\|\,\|g\|.$$

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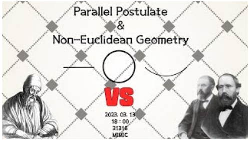
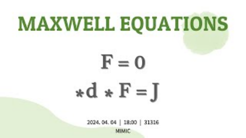
(a) Freshman can do (b) Can do if you know L.A.

2023 ∼ 2024





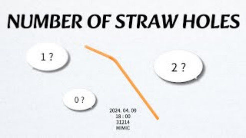


Figure: Ajou University

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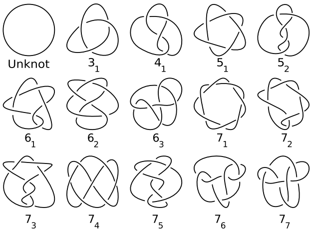
Knots & Links

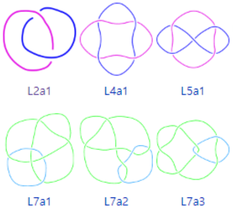
Links

A link L of m components is a subspace of $S^3 = R^3 \cup \{\infty\}$, that consists of m disjoint, smooth, simple closed curves.
A link of one component is a knot.

In this talk, all the links are oriented.



(a) Knot Table



(b) Link Table

Link Diagram

Also the link diagram is the image of a projection $\pi : S^3 \to R^2$ with under and over information in each crossing.

Reduced diagram

If the diagram has minimal crossing, it is a reduced diagram.

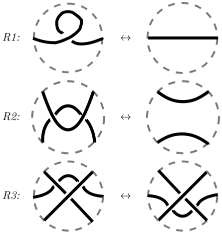


Figure: Reducing

Alternating Link

Alternating diagram

A link diagram is alternating if the crossings alternate under, over, under, over, as one travels along each component of the link.

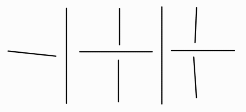


Figure: Alternating Diagram

Alternating Link

A link is alternating if it has an alternating diagram.

Non-split Link

Definition

A link diagram D in S^2 is a split diagram if there is a simple closed curve in $S^2 - D$ separating S^2 into two discs each containing part of D. A link is a split link, if it has a split diagram.

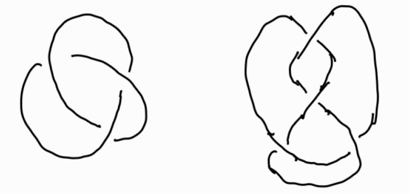


Figure: Split diagram

Menasco's result

There is a relation between non-split link and alternating link.

Theorem[Men84]

An alternating link, will be non-split if and only if its alternating diagram is non-split.

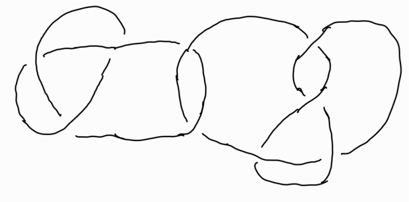
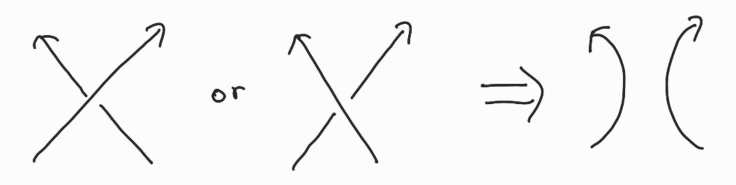


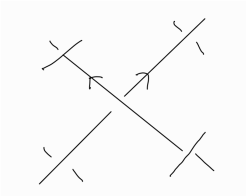
Figure: Non-split diagram

Smoothing



After smoothing

Smoothing does not change the alternatingness, non-splittness. If the link diagram D is alternating, non-split and reduced, the smoothed diagram D' is alternating and non-split.



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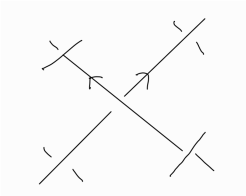
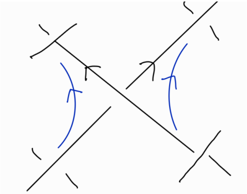

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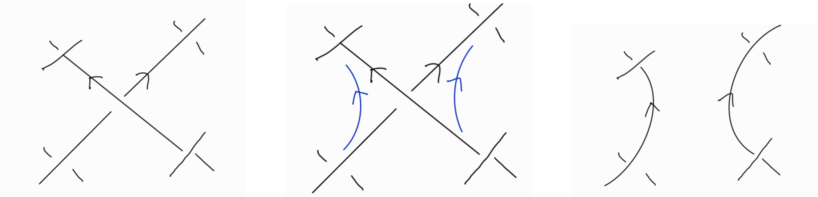

Figure: Smoothing does not change the alternatingness.

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Seifert Surface

Definition

A Seifert surface S for a knot or link $L \subset S^3$ is a smooth, compact manifold with $\partial S = L$. Also, it should not have closed manifold component.

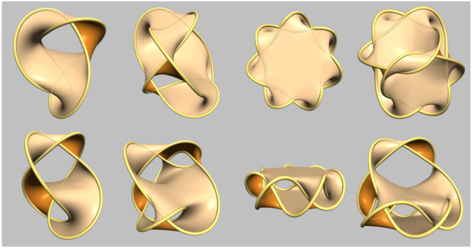


Figure: Seifert surfaces of some links

Seifert Algorithm

Every links have a Seifert surface through the Seifert algorithm.

Seifert Algorithm

1. For all crossings, smoothing.
2. For each components, attaching discs.
3. Attach the twisted band in the position of the crossing.

Figure 8 example

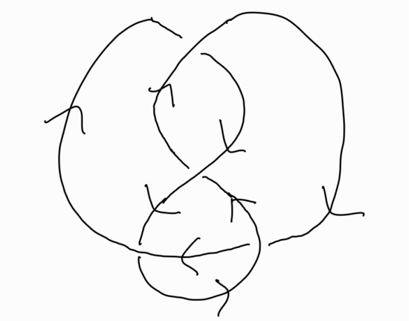
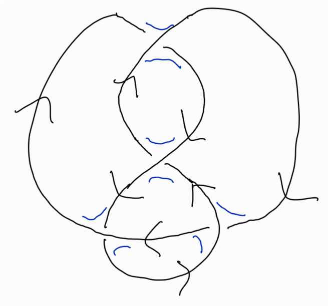
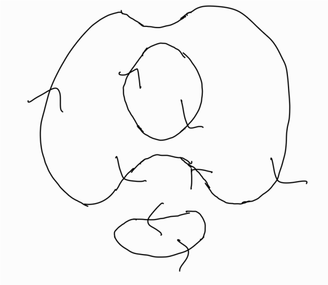


Figure: figure 8 knot with orientation

Smoothing



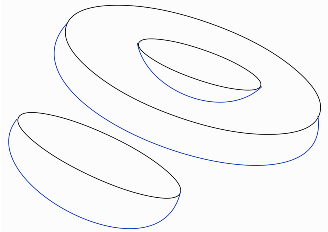
(a) Smoothing



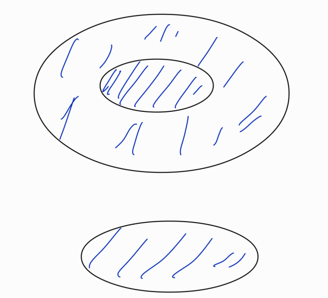
(b) After smoothing

Figure: Smoothing Algorithm

Attaching discs



(a) Attaching discs in a side view



(b) Attaching discs in projections

Figure: Attaching discs

Twisted Bands

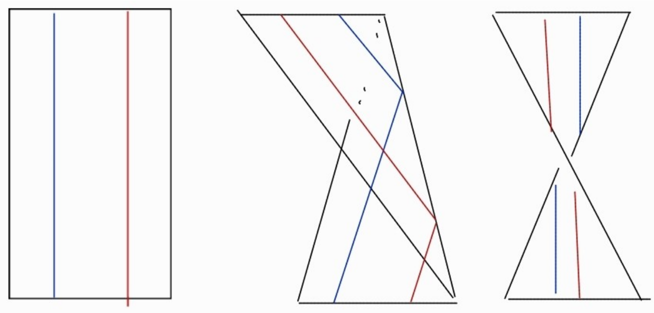


Figure: Twisted band

Attaching Twisted Bands

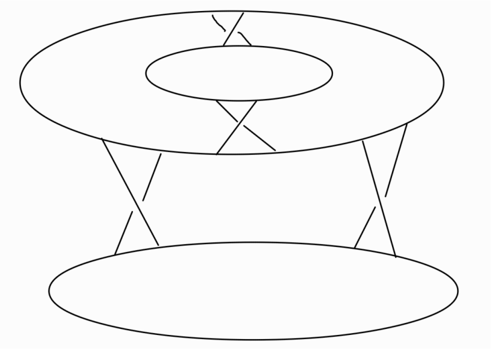


Figure: Attatching Twisted Band

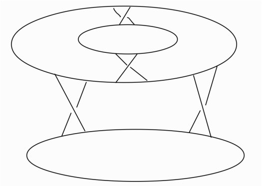
Genus of the Seifert Surface

Euler characteristic

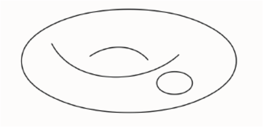
We can use the Euler characteristic argument to check the genus.

$$\chi(S) = \#\text{discs} - \#\text{twisted bands} = 2 - 2g(S) - n$$

where n is the number of the link components.



(a) Seifert surface of figure 8 knot with embedding.



(b) Same Seifert surface of figure 8 knot without embedding.

Does not need to minimal Genus

Remark

Definition does not need to have minimal genus.

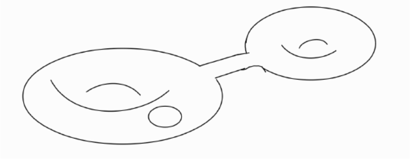


Figure: Do not need to minimal genus

Genus of Knot

Genus of knot

The *genus* $g(S)$ of a link L is defined by

$$g(L) = \min\{g(S) : F \text{ is a Seifert surface for } L\}.$$

Also that F is a minimal Seifert surface.

Minimal Seifert Surface

Theorem

Let L be an alternating knot. If S is a Seifert surface for L obtained by applying Seifert's algorithm to an alternating diagram of L, then S is a minimal Seifert surface for L.

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Motivation

Link

If we push links in S^3 to B^4, all links are equivalent to the unlinks.

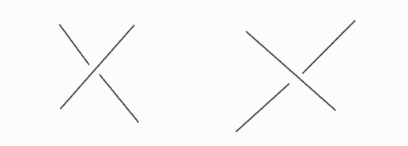

Figure: Two crossings in 4-dimension are equivalent.

Motivation

Link

If we push links in S^3 to B^4, all links are equivalent to the unlinks.

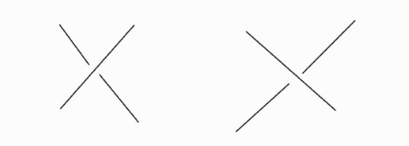

Figure: Two crossings in 4-dimension are equivalent.

Surface

How about the case for the surfaces in B^4, in particular fixing boundary in S^3?

Livingston's work and questions

In [Liv82], Livingston asked about the case when push the non-isotopic surface in S^3 to B^4, rel to boundary.

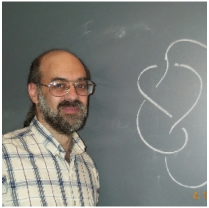

Figure: Charles Livingston

Answer

Trotter's example

There is an example in [Tro75] are not isotopic in S^3 but isotopic in B^4.

False

In [AFMW24, HKM$^+$23], they found the examples not even topologically isotopic.

How many classes?

How many classes that topologically or smoothly isotopic?

Construction in S^3

There is a sequence of minimal genus Seifert surfaces $\Sigma_1 = S_1, S_2, \ldots, S_n = \Sigma_2$ for L such that the interiors of S_i, S_{i+1} are disjoint for each i. [ST88, Kak92]

Euler Characteristic Argument

Let $F = \Sigma_1 \cup_L \Sigma_2$. Since Σ_1 and Σ_2 are minimal Seifert surfaces, i.e. connected which has maximal Euler characteristic, F has a maximal Euler characteristic

$$g(F) = 1 - \chi(\Sigma_i).$$

If $g(F) = 0$, $\chi(\Sigma_i) = 1$ that Σ_i is a disk that L is an unlink by the Schoenflies theorem. Let using the induction argument in $g(F)$ with the base case 0.

Crossing Tube Lemma [Kin18]

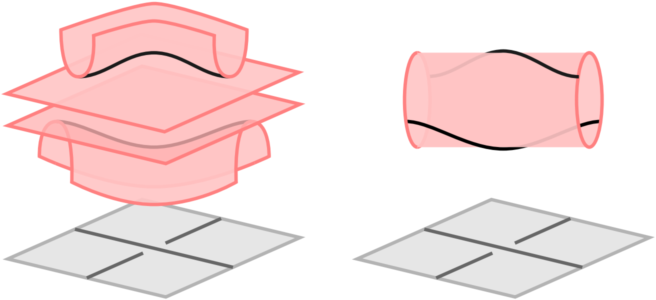

Figure: Crossing Tube Lemma

Band attaching

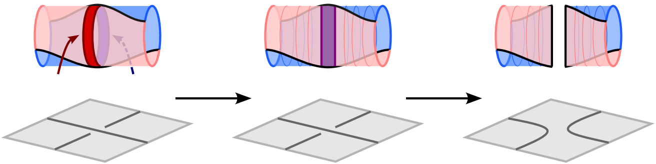

Figure: Band Attaching

After Surgery

In point of the link, the link L' is still non-split, alternating link. So, we can do this process inductively in all crossings until $g(F) = 0$.

Main Theorem

So, let's think in reverse way. Starting from the unlink, we can make the link L, and following Σ_1 and Σ_2.

Main Theorem

Any two minimal genus Seifert surfaces for a non-split, alternating link are smoothly isotopic rel. boundary in B^4.

In summary

In S^3

For a non-split, alternating link, there are non-isotopic minimal Seifert surfaces in S^3.

In B^4

However, when push those surfaces into B^4, they are isotopic.

Future Open Questions

How about the other link classes?
Is there any lower bound or upper bound for the classes?

In summary

In S^3

For a non-split, alternating link, there are non-isotopic minimal Seifert surfaces in S^3.

In B^4

However, when push those surfaces into B^4, they are isotopic.

Future Open Questions

How about the other link classes?
Is there any lower bound or upper bound for the classes?

Thank you for listening!

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