MERTON'S PORTFOLIO PROBLEM

1. Code

The PDE is given as

$$\frac{\partial V}{\partial t} - \frac{(\mu - r)^2}{2\sigma^2} \cdot \frac{\left(\frac{\partial V}{\partial x}\right)^2}{\frac{\partial^2 V}{\partial x^2}} + \frac{\partial V}{\partial x} \cdot r \cdot x + \frac{\gamma}{1-\gamma} \cdot \left(\frac{\partial V}{\partial x}\right)^{\frac{\gamma-1}{\gamma}} = \rho V. \tag{1.1}$$

The boundary condition is

$$V(T, x) = \epsilon^\gamma \cdot \frac{x^{1-\gamma}}{1-\gamma}. \tag{1.2}$$

The domain for (x, t) is $[0, \infty) \times [0, T]$.

1.1. Solving the PDE with ansatz.

Let $V(t, x) = f(t)^\gamma \cdot \frac{x^{1-\gamma}}{1-\gamma}$. Then,

$$\begin{aligned} \partial_t V(t, x) &= \gamma f(t)^{\gamma-1} f'(t) \cdot \frac{x^{1-\gamma}}{1-\gamma} \\ \partial_x V(t, x) &= f(t)^\gamma \cdot x^{-\gamma} \\ \partial_{xx} V(t, x) &= -f(t)^\gamma \cdot \gamma \cdot x^{-\gamma-1}. \end{aligned} \tag{1.3}$$

Inserting these computations, we find

$$f'(t) = \nu \cdot f(t) - 1 \tag{1.4}$$

where

$$\nu = \frac{\rho - (1-\gamma) \cdot \left(\frac{(\mu-r)^2}{2\sigma^2\gamma} + r\right)}{\gamma} \tag{1.5}$$

with boundary condition $f(T) = \epsilon$. The solution to this ODE is

$$f(t) = \begin{cases} \frac{1 + (\nu\epsilon - 1) \cdot e^{-\nu(T-t)}}{\nu} & \text{for } \nu \neq 0 \\ T - t + \epsilon & \text{for } \nu = 0. \end{cases} \tag{1.6}$$