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Convergence Behaviour of Newton-Raphson Method in Node- and Loop-Based Non-linear Magnetic Equivalent Circuit Analysis

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***Abstract*—In this paper, the non-linear magnetic system equations are derived. The Newton-Raphson method is used to solve the system of non-linear equations. A simplified Jacobian is derived for both node- and loop-based magnetic equivalent circuit analysis. The partial derivative term is eliminated in this simplified Jacobian using differential relative permeability. The convergence behaviour of the Newton-Raphson method is studied. The loop analysis exhibits more stable convergence than that of the node analysis while solving non-linear magnetic equivalent circuit using the Newton-Raphson method.**

***Index Terms*—Magnetic equivalent circuit, Newton-Raphson method, Jacobian, non-linear, node, loop, differential relative permeability**

I. Introduction

Magnetic Equivalent Circuit (MEC) modeling is an approximate method used for the analysis of magnetic systems. The geometric description of an electrical machine with coils, core, permanent magnet material, and medium (air) are translated into an electrical circuit model using this method. The solution of this equivalent electrical circuit is used to find out various machine parameters like flux linkage, induced emf, cogging torque, average torque, and instantaneous torque.

In the magnetic equivalent circuit model, the circuit parameter is represented by either the permeance or reluctance. If the magnetic core is saturated, the relative permeability of the core reduces with an increase in the magnetic field density/intensity. Because of this, the permeance or reluctance can not be calculated directly without knowing the field density/intensity. Also, the MEC system becomes non-linear. Hence, iterative method is necessary to solve the MEC system.

Among the several numerical methods, Newton-Raphson method has advantages such as fast convergence and ease in representing the multi-dimensional system [1]. Because of this, the Newton-Raphson method is chosen to solve the non-linear MEC system. In [2], derivation of node- and loop-based MEC system is presented. Therein for node analysis, the relative permeability is expressed as a function of magnetic field density. Due to this, a relative ease Jacobian is not derived for node analysis as compared to loop analysis. Also, the Jacobian in the loop analysis has two terms; one is the reluctance matrix, and the other is partial derivative of the reluctance matrix. Herein a simplified Jacobian is derived for both node and loop analysis.

The relative permeability of the core can be represented as a function of field intensity or field density as illustrated in Fig. 1. The MEC of non-linear magnetic system can be solved using node or loop analysis. For easy calculation, the permeability should be expressed as a function of field intensity in case of node analysis; and in loop analysis, it is to be described as a function of field density [3]. In Fig. 1, two types of relative permeability curves are shown. The apparent relative permeability μ_r is used to calculate permeance or reluctance; and the differential permeability μ_{rd} is used to determine a simplified Jacobian as discussed in the following section.

II. Derivation of non-linear MEC system equations

A. Node analysis

Consider a sample non-linear MEC system shown in Fig. 2. In this figure, $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$, and $\mathcal{P}_5$ are flux tube permeances (function of magnetic field intensity), and $\mathcal{F}_1, \mathcal{F}_2$, and $\mathcal{F}_3$ are node MMF. There are four nodes in the circuit, hence minimum three equations are required to describe the system. At node ①, ② & ③, applying Gauss law gives

$$g_1(\mathcal{F}) = (\mathcal{F}_1 - \mathcal{F}_s)\mathcal{P}_1 + (\mathcal{F}_1 - \mathcal{F}_2)\mathcal{P}_2 = 0 \tag{1}$$

$$g_2(\mathcal{F}) = (\mathcal{F}_2 - \mathcal{F}_1)\mathcal{P}_2 + (\mathcal{F}_2 - \mathcal{F}_3)\mathcal{P}_3 + \mathcal{F}_2\mathcal{P}_4 = 0 \tag{2}$$

$$g_3(\mathcal{F}) = (\mathcal{F}_3 - \mathcal{F}_2)\mathcal{P}_3 + \mathcal{F}_3\mathcal{P}_5 = 0 \tag{3}$$

where $\mathcal{F}_s$ is MMF source, and $g_1(\mathcal{F})$, $g_2(\mathcal{F})$ & $g_3(\mathcal{F})$ are node residual functions.
The above equations can be written in matrix form as

$$\begin{bmatrix} g_1(\mathcal{F}) \\ g_2(\mathcal{F}) \\ g_3(\mathcal{F}) \end{bmatrix} = \begin{bmatrix} \mathcal{P}_1 + \mathcal{P}_2 & -\mathcal{P}_2 & 0 \\ -\mathcal{P}_2 & \mathcal{P}_2 + \mathcal{P}_3 + \mathcal{P}_4 & -\mathcal{P}_3 \\ 0 & -\mathcal{P}_3 & \mathcal{P}_3 + \mathcal{P}_5 \end{bmatrix} \begin{bmatrix} \mathcal{F}_1 \\ \mathcal{F}_2 \\ \mathcal{F}_3 \end{bmatrix} - \begin{bmatrix} \mathcal{F}_s\mathcal{P}_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{4}$$

$$\left[\boldsymbol{\mathcal{G}}_n(\boldsymbol{\mathcal{F}})\right] = \left[\boldsymbol{\mathcal{P}}(\boldsymbol{\mathcal{H}})\right]\left[\boldsymbol{\mathcal{F}}\right] - \left[\boldsymbol{\Phi}_s\right] = \left[0\right]$$

where $\boldsymbol{\mathcal{G}}_n(\boldsymbol{\mathcal{F}})$ is residual function vector, $\boldsymbol{\mathcal{F}}$ is node MMF vector, $\boldsymbol{\mathcal{P}}(\boldsymbol{\mathcal{H}})$ is node permeance matrix, $\boldsymbol{\mathcal{H}}$ is flux-tube field intensity vector, and $\boldsymbol{\Phi}_s$ is flux source (input) vector.

The residual function vector can be expressed in Taylor series (first order approximation) as

$$\left[\boldsymbol{\mathcal{G}}_n(\boldsymbol{\mathcal{F}})\right]_{(k+1)} = \left[\boldsymbol{\mathcal{G}}_n(\boldsymbol{\mathcal{F}})\right]_{(k)} + \left[\boldsymbol{\mathcal{G}'}_n(\boldsymbol{\mathcal{F}})\right]_{(k)} \left(\left[\boldsymbol{\mathcal{F}}\right]_{(k+1)} - \left[\boldsymbol{\mathcal{F}}\right]_{(k)}\right) = 0$$

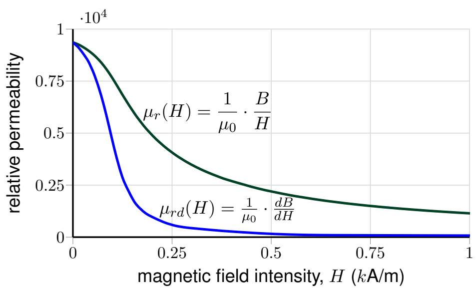
where the subscript k is iteration number.

$$\left[\boldsymbol{\mathcal{G}'}_n(\boldsymbol{\mathcal{F}})\right]_{(k)} \left(\left[\boldsymbol{\mathcal{F}}\right]_{(k+1)} - \left[\boldsymbol{\mathcal{F}}\right]_{(k)}\right) = \left[0 - \boldsymbol{\mathcal{G}}_n(\boldsymbol{\mathcal{F}})\right]_{(k)}$$

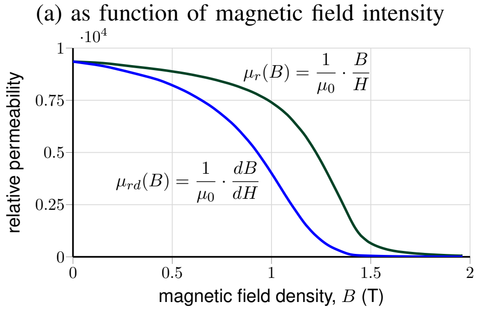
$$\left[\boldsymbol{\mathcal{F}}\right]_{(k+1)} = \left[\boldsymbol{\mathcal{F}}\right]_{(k)} - \frac{\left[\boldsymbol{\mathcal{G}}_n(\boldsymbol{\mathcal{F}})\right]_{(k)}}{\left[\boldsymbol{\mathcal{G}'}_n(\boldsymbol{\mathcal{F}})\right]_{(k)}}$$

where $\boldsymbol{\mathcal{G}'}_n(\boldsymbol{\mathcal{F}})$ is the Jacobian matrix and it is given by

$$\left[\boldsymbol{\mathcal{G}'}_n(\boldsymbol{\mathcal{F}})\right] = \begin{bmatrix} \frac{\partial g_1}{\partial \mathcal{F}_1} & \frac{\partial g_1}{\partial \mathcal{F}_2} & \frac{\partial g_1}{\partial \mathcal{F}_3} \\ \frac{\partial g_2}{\partial \mathcal{F}_1} & \frac{\partial g_2}{\partial \mathcal{F}_2} & \frac{\partial g_2}{\partial \mathcal{F}_3} \\ \frac{\partial g_3}{\partial \mathcal{F}_1} & \frac{\partial g_3}{\partial \mathcal{F}_2} & \frac{\partial g_3}{\partial \mathcal{F}_3} \end{bmatrix} \tag{5}$$



(a) as function of magnetic field intensity



(b) as function of magnetic field density

Fig. 1: Apparent and differential relative permeability of silicon steel core - 35C250

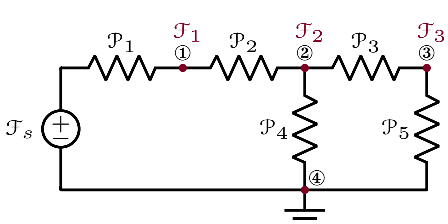


Fig. 2: A sample non-linear MEC - permeance network

<u>Evaluation of $\frac{\partial g_1}{\partial \mathcal{F}_1}$:-</u>

Partial derivative of (1) with respect to $\mathcal{F}_1$ is

$$\frac{\partial g_1}{\partial \mathcal{F}_1} = \mathcal{P}_1 + \mathcal{P}_2 + (\mathcal{F}_1 - \mathcal{F}_s)\frac{\partial \mathcal{P}_1}{\partial \mathcal{F}_1} + (\mathcal{F}_1 - \mathcal{F}_2)\frac{\partial \mathcal{P}_2}{\partial \mathcal{F}_1} \tag{6}$$

Using the chain rule,

$$\frac{\partial \mathcal{P}_1}{\partial \mathcal{F}_1} = \frac{\partial \mathcal{P}_1}{\partial \mu_{r1}} \cdot \frac{\partial \mu_{r1}}{\partial \mathcal{H}_1} \cdot \frac{\partial \mathcal{H}_1}{\partial \mathcal{F}_1} \tag{7}$$

The permeance $\mathcal{P}_1$ is given by

$$\mathcal{P}_1 = \frac{\mu_0 \mu_{r1} A_1}{l_1} \Rightarrow \frac{\partial \mathcal{P}_1}{\partial \mu_{r1}} = \frac{\mu_0 A_1}{l_1} = \frac{\mathcal{P}_1}{\mu_{r1}}$$

The magnetic field intensity $\mathcal{H}_1$ at $\mathcal{P}_1$ is given by

$$\mathcal{H}_1 = \frac{\mathcal{F}_1 - \mathcal{F}_s}{l_1} \Rightarrow \frac{\partial \mathcal{H}_1}{\partial \mathcal{F}_1} = \frac{1}{l_1}$$

substituting the value $\frac{\partial \mathcal{P}_1}{\partial \mu_{r1}}$ and $\frac{\partial \mathcal{H}_1}{\partial \mathcal{F}_1}$ in (7) gives

$$\frac{\partial \mathcal{P}_1}{\partial \mathcal{F}_1} = \frac{\mathcal{P}_1}{\mu_{r1}} \cdot \frac{\partial \mu_{r1}}{\partial \mathcal{H}_1} \cdot \frac{1}{l_1}$$

Hence,

$$(\mathcal{F}_1 - \mathcal{F}_s)\frac{\partial \mathcal{P}_1}{\partial \mathcal{F}_1} = \frac{\mathcal{P}_1}{\mu_{r1}} \cdot \frac{\partial \mu_{r1}}{\partial \mathcal{H}_1} \cdot \frac{1}{l_1}(\mathcal{F}_1 - \mathcal{F}_s) = \frac{\mathcal{P}_1}{\mu_{r1}} \cdot \frac{\partial \mu_{r1}}{\partial \mathcal{H}_1} \cdot \mathcal{H}_1 \tag{8}$$

Using the chain rule,

$$\frac{\partial \mathcal{P}_2}{\partial \mathcal{F}_1} = \frac{\partial \mathcal{P}_2}{\partial \mu_{r2}} \cdot \frac{\partial \mu_{r2}}{\partial \mathcal{H}_2} \cdot \frac{\partial \mathcal{H}_2}{\partial \mathcal{F}_1} \tag{9}$$

The permeance $\mathcal{P}_2$ is given by

$$\mathcal{P}_2 = \frac{\mu_0 \mu_{r2} A_2}{l_2} \Rightarrow \frac{\partial \mathcal{P}_2}{\partial \mu_{r2}} = \frac{\mu_0 A_2}{l_2} = \frac{\mathcal{P}_2}{\mu_{r2}}$$

The magnetic field intensity $\mathcal{H}_2$ at $\mathcal{P}_2$ is given by

$$\mathcal{H}_2 = \frac{\mathcal{F}_1 - \mathcal{F}_2}{l_2} \Rightarrow \frac{\partial \mathcal{H}_2}{\partial \mathcal{F}_1} = \frac{1}{l_2}$$

substituting the value of $\frac{\partial \mathcal{P}_2}{\partial \mu_{r2}}$ and $\frac{\partial \mathcal{H}_2}{\partial \mathcal{F}_1}$ in (9) gives

$$\frac{\partial \mathcal{P}_2}{\partial \mathcal{F}_1} = \frac{\mathcal{P}_2}{\mu_{r2}} \cdot \frac{\partial \mu_{r2}}{\partial \mathcal{H}_2} \cdot \frac{1}{l_2}$$

Hence,

$$(\mathcal{F}_1 - \mathcal{F}_2)\frac{\partial \mathcal{P}_2}{\partial \mathcal{F}_1} = \frac{\mathcal{P}_2}{\mu_{r2}} \cdot \frac{\partial \mu_{r2}}{\partial \mathcal{H}_2} \cdot \frac{1}{l_2}(\mathcal{F}_1 - \mathcal{F}_2) = \frac{\mathcal{P}_2}{\mu_{r2}} \cdot \frac{\partial \mu_{r2}}{\partial \mathcal{H}_2} \cdot \mathcal{H}_2 \tag{10}$$

substituting (8) and (10) in (6) yields

$$\frac{\partial g_1}{\partial \mathcal{F}_1} = \mathcal{P}_1 + \mathcal{P}_2 + \frac{\mathcal{P}_1}{\mu_{r1}} \cdot \frac{\partial \mu_{r1}}{\partial \mathcal{H}_1} \cdot \mathcal{H}_1 + \frac{\mathcal{P}_2}{\mu_{r2}} \cdot \frac{\partial \mu_{r2}}{\partial \mathcal{H}_2} \cdot \mathcal{H}_2$$

rearranging the terms and substituting value of $\mathcal{P}_1$ & $\mathcal{P}_2$ gives

$$\frac{\partial g_1}{\partial \mathcal{F}_1} = \frac{\mu_0 \mu_{r1} A_1}{l_1} + \frac{\mu_0 \mu_{r1} A_1}{l_1} \cdot \frac{1}{\mu_{r1}} \cdot \frac{\partial \mu_{r1}}{\partial \mathcal{H}_1} \cdot \mathcal{H}_1 + \frac{\mu_0 \mu_{r2} A_2}{l_2} + \frac{\mu_0 \mu_{r2} A_2}{l_2} \cdot \frac{1}{\mu_{r2}} \cdot \frac{\partial \mu_{r2}}{\partial \mathcal{H}_2} \cdot \mathcal{H}_2$$

$$\frac{\partial g_1}{\partial \mathcal{F}_1} = \frac{\mu_0 A_1}{l_1}(\mu_{r1} + \frac{\partial \mu_{r1}}{\partial \mathcal{H}_1} \cdot \mathcal{H}_1) + \frac{\mu_0 A_2}{l_2}(\mu_{r2} + \frac{\partial \mu_{r2}}{\partial \mathcal{H}_2} \cdot \mathcal{H}_2)$$

Simplification of $\left(\mu_r + \frac{\partial \mu_r}{\partial \mathcal{H}} \cdot \mathcal{H}\right)$:-

Apparent relative permeability of the core is

$$\mu_r = \frac{\mathcal{B}}{\mu_0 \mathcal{H}} \Rightarrow \mathcal{B} = \mu_r \mu_0 \mathcal{H}$$

Differential relative permeability [4] of the core is given by

$$\mu_{rd} = \frac{1}{\mu_0}\frac{d\mathcal{B}}{d\mathcal{H}} = \frac{1}{\mu_0}\frac{d(\mu_r \mu_0 \mathcal{H})}{d\mathcal{H}} = \mu_r + \frac{d\mu_r}{d\mathcal{H}} \cdot \mathcal{H} = \mu_r + \frac{\partial \mu_r}{\partial \mathcal{H}} \cdot \mathcal{H}$$

Hence

$$\frac{\partial g_1}{\partial \mathcal{F}_1} = \frac{\mu_0 A_1}{l_1}(\mu_{rd1}) + \frac{\mu_0 A_2}{l_2}(\mu_{rd2}) = \mathcal{P}'_1 + \mathcal{P}'_2 \tag{11}$$

where $\mathcal{P}'_1 = \frac{\mu_0 \mu_{rd1} A_1}{l_1}$ is differential permeance. In this equation, differential permeability is used to calculate the permeance $\mathcal{P}'_1$, and so it is named as differential permeance. If the BH characteristic of the material is linear, then $\mu_r = \mu_{rd}$ and $\mathcal{P}' = \mathcal{P}$.

Evaluation of $\frac{\partial g_1}{\partial \mathcal{F}_2}$:-

Partial derivative of (1) with respect to $\mathcal{F}_2$ is

$$\frac{\partial g_1}{\partial \mathcal{F}_2} = -\mathcal{P}_2 + (\mathcal{F}_1 - \mathcal{F}_2)\frac{\partial \mathcal{P}_2}{\partial \mathcal{F}_2}$$

using the chain rule,

$$\frac{\partial \mathcal{P}_2}{\partial \mathcal{F}_2} = \frac{\partial \mathcal{P}_2}{\partial \mu_{r2}} \cdot \frac{\partial \mu_{r2}}{\partial \mathcal{H}_2} \cdot \frac{\partial \mathcal{H}_2}{\partial \mathcal{F}_2} \tag{12}$$

The magnetic field intensity $\mathcal{H}_2$ at $\mathcal{P}_2$ is given by

$$\mathcal{H}_2 = \frac{\mathcal{F}_1 - \mathcal{F}_2}{l_2} \Rightarrow \frac{\partial \mathcal{H}_2}{\partial \mathcal{F}_2} = \frac{-1}{l_2}$$

substituting the value of $\frac{\partial \mathcal{P}_2}{\partial \mu_{r2}}$ and $\frac{\partial \mathcal{H}_2}{\partial \mathcal{F}_2}$ in (12) gives

$$\frac{\partial \mathcal{P}_2}{\partial \mathcal{F}_2} = \frac{\mathcal{P}_2}{\mu_{r2}} \cdot \frac{\partial \mu_{r2}}{\partial \mathcal{H}_2} \cdot \frac{-1}{l_2}$$

Hence,

$$\frac{\partial g_1}{\partial \mathcal{F}_2} = -\mathcal{P}_2 + (\mathcal{F}_1 - \mathcal{F}_2) \cdot \frac{\mathcal{P}_2}{\mu_{r2}} \cdot \frac{\partial \mu_{r2}}{\partial \mathcal{H}_2} \cdot \frac{-1}{l_2}$$

$$\frac{\partial g_1}{\partial \mathcal{F}_2} = -\mathcal{P}_2 - \frac{\mathcal{P}_2}{\mu_{r2}} \cdot \frac{\partial \mu_{r2}}{\partial \mathcal{H}_2} \cdot \mathcal{H}_2 = -\frac{\mu_0 A_2}{l_2}(\mu_{r2} + \frac{\partial \mu_{r2}}{\partial \mathcal{H}_2} \cdot \mathcal{H}_2)$$

$$\frac{\partial g_1}{\partial \mathcal{F}_2} = -\frac{\mu_0 A_2}{l_2}(\mu_{rd2}) = -\mathcal{P}'_2 \tag{13}$$

Evaluation of $\frac{\partial g_1}{\partial \mathcal{F}_3}$:-

Partial derivative of (1) with respect to $\mathcal{F}_3$ is

$$\frac{\partial g_1}{\partial \mathcal{F}_3} = 0 \tag{14}$$

Similarly, remaining elements of the Jacobian matrix can be derived as

$$\frac{\partial g_2}{\partial \mathcal{F}_1} = -\mathcal{P}'_2 \qquad \frac{\partial g_3}{\partial \mathcal{F}_1} = 0$$

$$\frac{\partial g_2}{\partial \mathcal{F}_2} = \mathcal{P}'_2 + \mathcal{P}'_3 + \mathcal{P}'_4 \qquad \frac{\partial g_3}{\partial \mathcal{F}_2} = -\mathcal{P}'_3$$

$$\frac{\partial g_2}{\partial \mathcal{F}_3} = -\mathcal{P}'_3 \qquad \frac{\partial g_3}{\partial \mathcal{F}_3} = \mathcal{P}'_3 + \mathcal{P}'_5$$

The complete Jacobian matrix is

$$\left[\boldsymbol{\mathcal{G}'}_n(\boldsymbol{\mathcal{F}})\right] = \begin{bmatrix} \mathcal{P}'_1 + \mathcal{P}'_2 & -\mathcal{P}'_2 & 0 \\ -\mathcal{P}'_2 & \mathcal{P}'_2 + \mathcal{P}'_3 + \mathcal{P}'_4 & -\mathcal{P}'_3 \\ 0 & -\mathcal{P}'_3 & \mathcal{P}'_3 + \mathcal{P}'_5 \end{bmatrix}$$

The elements of Jacobian matrix $\boldsymbol{\mathcal{G}'}_n$ resemble the corresponding element in the permeance matrix $\boldsymbol{\mathcal{P}}$. Differential permeability is used in the Jacobian matrix, whereas apparent permeability is used in permeance matrix calculation. The Jacobian derived in this paper is simpler than that presented in [2], [5]–[7]. Therein the Jacobian has two terms; one is permeance/reluctance matrix, and the other is partial derivative of permeance/reluctance matrix. Herein the partial derivative term is eliminated with the use of differential permeability. This simplification will reduce computation time.

B. Loop analysis

The same MEC system considered in the above subsection is shown in Fig. 3 with equivalent reluctance network parameters. In this figure, $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4,$ and $\mathcal{R}_5$ are flux tube reluctances (function of magnetic field density), and Φ_1 &Φ_2 are loop flux. There are two meshes in the circuit, hence minimum two equations are required to describe the system. In loop 1 & loop 2, applying Ampere's law gives

$$g_1(\Phi) = (\mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_4)\Phi_1 - \mathcal{R}_4\Phi_2 - \mathcal{F}_s = 0 \tag{15}$$

$$g_2(\Phi) = -\mathcal{R}_4\Phi_1 + (\mathcal{R}_3 + \mathcal{R}_4 + \mathcal{R}_5)\Phi_2 = 0 \tag{16}$$

where $\mathcal{F}_s$ is MMF source, and $g_1(\Phi)$ & $g_2(\Phi)$ are loop residual functions.

The above equations can be written in matrix form as

$$\begin{bmatrix} g_1(\Phi) \\ g_2(\Phi) \end{bmatrix} = \begin{bmatrix} \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_4 & -\mathcal{R}_4 \\ -\mathcal{R}_4 & \mathcal{R}_3 + \mathcal{R}_4 + \mathcal{R}_5 \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix} - \begin{bmatrix} \mathcal{F}_s \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{17}$$

$$\left[\boldsymbol{\mathcal{G}}_l(\boldsymbol{\Phi})\right] = \left[\boldsymbol{\mathcal{R}}(\boldsymbol{\mathcal{B}})\right]\left[\boldsymbol{\Phi}\right] - \left[\boldsymbol{\mathcal{F}}_s\right] = \left[0\right]$$

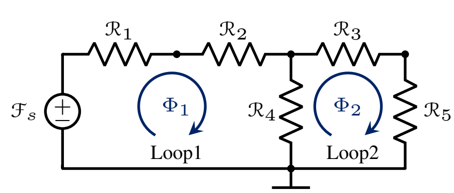


Fig. 3: A sample non-linear MEC - reluctance network

where $\boldsymbol{\mathcal{G}}_l(\boldsymbol{\Phi})$ is residual function vector, $\boldsymbol{\Phi}$ is loop flux vector, $\boldsymbol{\mathcal{R}}(\boldsymbol{\mathcal{B}})$ is loop reluctance matrix, $\boldsymbol{\mathcal{B}}$ is flux-tube field density vector, and $\boldsymbol{\mathcal{F}}_s$ is MMF source (input) vector.

The residual function vector can be expressed in Taylor series (first order approximation) as

$$\left[\boldsymbol{\mathcal{G}}_l(\boldsymbol{\Phi})\right]_{(k+1)} = \left[\boldsymbol{\mathcal{G}}_l(\boldsymbol{\Phi})\right]_{(k)} + \left[\boldsymbol{\mathcal{G}'}_l(\boldsymbol{\Phi})\right]_{(k)} \left(\left[\boldsymbol{\Phi}\right]_{(k+1)} - \left[\boldsymbol{\Phi}\right]_{(k)}\right) = 0 \quad (18)$$

where the subscript k is iteration number.

$$\left[\boldsymbol{\mathcal{G}'}_l(\boldsymbol{\Phi})\right]_{(k)} \left(\left[\boldsymbol{\Phi}\right]_{(k+1)} - \left[\boldsymbol{\Phi}\right]_{(k)}\right) = \left[0 - \boldsymbol{\mathcal{G}}_l(\boldsymbol{\Phi})\right]_{(k)}$$

$$\left[\boldsymbol{\Phi}\right]_{(k+1)} = \left[\boldsymbol{\Phi}\right]_{(k)} - \frac{\left[\boldsymbol{\mathcal{G}}_l(\boldsymbol{\Phi})\right]_{(k)}}{\left[\boldsymbol{\mathcal{G}'}_l(\boldsymbol{\Phi})\right]_{(k)}}$$

where $\boldsymbol{\mathcal{G}'}_l(\boldsymbol{\Phi})$ is the Jacobian matrix and it is given by

$$\left[\boldsymbol{\mathcal{G}'}_l(\boldsymbol{\Phi})\right] = \begin{bmatrix} \frac{\partial g_1}{\partial \Phi_1} & \frac{\partial g_1}{\partial \Phi_2} \\ \frac{\partial g_2}{\partial \Phi_1} & \frac{\partial g_2}{\partial \Phi_2} \end{bmatrix} \quad (19)$$

Evaluation of $\frac{\partial g_1}{\partial \phi_1}$:-

Partial derivative of (15) with respect to Φ_1 is

$$\frac{\partial g_1}{\partial \Phi_1} = (\mathcal{R}_1+\mathcal{R}_2+\mathcal{R}_4)+\left(\frac{\partial \mathcal{R}_1}{\partial \Phi_1} + \frac{\partial \mathcal{R}_2}{\partial \Phi_1} + \frac{\partial \mathcal{R}_4}{\partial \Phi_1}\right)\Phi_1 - \frac{\partial \mathcal{R}_4}{\partial \Phi_1}\Phi_2 \quad (20)$$

Using the chain rule,

$$\frac{\partial \mathcal{R}_1}{\partial \Phi_1} = \frac{\partial \mathcal{R}_1}{\partial \mu_{r1}} \cdot \frac{\partial \mu_{r1}}{\partial \mathcal{B}_1} \cdot \frac{\partial \mathcal{B}_1}{\partial \Phi_1} \quad (21)$$

The reluctance $\mathcal{R}_1$ is given by

$$\mathcal{R}_1 = \frac{l_1}{\mu_0 \mu_{r1} A_1} \Rightarrow \frac{\partial \mathcal{R}_1}{\partial \mu_{r1}} = -\frac{l_1}{\mu_0 \mu_{r1}^2 A_1} = -\frac{\mathcal{R}_1}{\mu_{r1}}$$

The magnetic field density $\mathcal{B}_1$ at $\mathcal{R}_1$ is given by

$$\mathcal{B}_1 = \frac{\Phi_1}{A_1} \Rightarrow \frac{\partial \mathcal{B}_1}{\partial \Phi_1} = \frac{1}{A_1}$$

substituting the value of $\frac{\partial \mathcal{R}_1}{\partial \mu_{r1}}$ and $\frac{\partial \mathcal{B}_1}{\partial \Phi_1}$ in (21) gives

$$\frac{\partial \mathcal{R}_1}{\partial \Phi_1} = -\frac{\mathcal{R}_1}{\mu_{r1}} \cdot \frac{\partial \mu_{r1}}{\partial \mathcal{B}_1} \cdot \frac{1}{A_1}$$

Hence,

$$\mathcal{R}_1 + \frac{\partial \mathcal{R}_1}{\partial \Phi_1}\Phi_1 = \mathcal{R}_1 \left(1 - \frac{\mathcal{B}_1}{\mu_{r1}} \cdot \frac{\partial \mu_{r1}}{\partial \mathcal{B}_1}\right) \quad (22)$$

Using the chain rule,

$$\frac{\partial \mathcal{R}_2}{\partial \Phi_1} = \frac{\partial \mathcal{R}_2}{\partial \mu_{r2}} \cdot \frac{\partial \mu_{r2}}{\partial \mathcal{B}_2} \cdot \frac{\partial \mathcal{B}_2}{\partial \Phi_1} \quad (23)$$

The reluctance $\mathcal{R}_2$ is given by

$$\mathcal{R}_2 = \frac{l_2}{\mu_0 \mu_{r2} A_2} \Rightarrow \frac{\partial \mathcal{R}_2}{\partial \mu_{r2}} = -\frac{l_2}{\mu_0 \mu_{r2}^2 A_2} = -\frac{\mathcal{R}_2}{\mu_{r2}}$$

The magnetic field density $\mathcal{B}_2$ at $\mathcal{R}_2$ is given by

$$\mathcal{B}_2 = \frac{\Phi_1}{A_2} \Rightarrow \frac{\partial \mathcal{B}_2}{\partial \Phi_1} = \frac{1}{A_2}$$

substituting the value of $\frac{\partial \mathcal{R}_2}{\partial \mu_{r2}}$ and $\frac{\partial \mathcal{B}_2}{\partial \Phi_1}$ in (23) gives

$$\frac{\partial \mathcal{R}_2}{\partial \Phi_1} = -\frac{\mathcal{R}_2}{\mu_{r2}} \cdot \frac{\partial \mu_{r2}}{\partial \mathcal{B}_2} \cdot \frac{1}{A_2}$$

Hence,

$$\mathcal{R}_2 + \frac{\partial \mathcal{R}_2}{\partial \Phi_1}\Phi_1 = \mathcal{R}_2 \left(1 - \frac{\mathcal{B}_2}{\mu_{r2}} \cdot \frac{\partial \mu_{r2}}{\partial \mathcal{B}_2}\right) \quad (24)$$

Using the chain rule,

$$\frac{\partial \mathcal{R}_4}{\partial \Phi_1} = \frac{\partial \mathcal{R}_4}{\partial \mu_{r4}} \cdot \frac{\partial \mu_{r4}}{\partial \mathcal{B}_4} \cdot \frac{\partial \mathcal{B}_4}{\partial \Phi_1} \quad (25)$$

The reluctance $\mathcal{R}_4$ is given by

$$\mathcal{R}_4 = \frac{l_4}{\mu_0 \mu_{r4} A_4} \Rightarrow \frac{\partial \mathcal{R}_4}{\partial \mu_{r4}} = -\frac{l_4}{\mu_0 \mu_{r4}^2 A_4} = -\frac{\mathcal{R}_4}{\mu_{r4}}$$

The magnetic field density $\mathcal{B}_4$ at $\mathcal{R}_4$ is given by

$$\mathcal{B}_4 = \frac{(\Phi_1 - \Phi_2)}{A_4} \Rightarrow \frac{\partial \mathcal{B}_4}{\partial \Phi_1} = \frac{1}{A_4}$$

substituting the value of $\frac{\partial \mathcal{R}_4}{\partial \mu_{r4}}$ and $\frac{\partial \mathcal{B}_4}{\partial \Phi_1}$ in (25) gives

$$\frac{\partial \mathcal{R}_4}{\partial \Phi_1} = -\frac{\mathcal{R}_4}{\mu_{r4}} \cdot \frac{\partial \mu_{r4}}{\partial \mathcal{B}_4} \cdot \frac{1}{A_4}$$

Hence,

$$\begin{aligned}\mathcal{R}_4 + \frac{\partial \mathcal{R}_4}{\partial \Phi_1}(\Phi_1 - \Phi_2) &= \mathcal{R}_4 - \frac{\mathcal{R}_4}{\mu_{r4}} \cdot \frac{\partial \mu_{r4}}{\partial \mathcal{B}_4} \cdot \frac{1}{A_4} \cdot (\Phi_1 - \Phi_2) \\ &= \mathcal{R}_4 \left(1 - \frac{\mathcal{B}_4}{\mu_{r4}} \cdot \frac{\partial \mu_{r4}}{\partial \mathcal{B}_4}\right) \quad (26)\end{aligned}$$

substituting (22), (24) and (26) in (20) yields

$$\frac{\partial g_1}{\partial \Phi_1} = \mathcal{R}_1 \left(1 - \frac{\mathcal{B}_1}{\mu_{r1}} \cdot \frac{\partial \mu_{r1}}{\partial \mathcal{B}_1}\right) + \mathcal{R}_2 \left(1 - \frac{\mathcal{B}_2}{\mu_{r2}} \cdot \frac{\partial \mu_{r2}}{\partial \mathcal{B}_2}\right) + \mathcal{R}_4 \left(1 - \frac{\mathcal{B}_4}{\mu_{r4}} \cdot \frac{\partial \mu_{r4}}{\partial \mathcal{B}_4}\right) \quad (27)$$

Simplification of $\left(1 - \frac{\mathcal{B}}{\mu_r} \cdot \frac{\partial \mu_r}{\partial \mathcal{B}}\right)$:-

Apparent relative permeability of the core is

$$\mu_r = \frac{\mathcal{B}}{\mu_0 \mathcal{H}} \Rightarrow \mathcal{H} = \frac{\mathcal{B}}{\mu_0 \mu_r}$$

Differential relative permeability of the core is

$$\mu_{rd} = \frac{1}{\mu_0} \frac{d\mathcal{B}}{d\mathcal{H}}$$

$$\Rightarrow \frac{1}{\mu_{rd}} = \mu_0 \frac{d\mathcal{H}}{d\mathcal{B}} = \mu_0 \frac{d\left(\frac{\mathcal{B}}{\mu_0 \mu_r}\right)}{d\mathcal{B}} = \frac{d\left(\frac{\mathcal{B}}{\mu_r}\right)}{d\mathcal{B}}$$

$$\frac{1}{\mu_{rd}} = \frac{1}{\mu_r} - \frac{\mathcal{B}}{\mu_r^2} \cdot \frac{\partial \mu_r}{\partial \mathcal{B}}$$
$$\Rightarrow \frac{\mu_r}{\mu_{rd}} = 1 - \frac{\mathcal{B}}{\mu_r} \cdot \frac{\partial \mu_r}{\partial \mathcal{B}} \tag{28}$$

substituting (28) in (27) gives

$$\frac{\partial g_1}{\partial \Phi_1} = \mathcal{R}_1 \cdot \frac{\mu_{r1}}{\mu_{rd1}} + \mathcal{R}_2 \cdot \frac{\mu_{r2}}{\mu_{rd2}} + \mathcal{R}_4 \cdot \frac{\mu_{r3}}{\mu_{rd3}} \tag{29}$$

$$\mathcal{R}_1 \cdot \frac{\mu_{r1}}{\mu_{rd1}} = \frac{l_1}{\mu_0 \mu_{r1} A_1} \cdot \frac{\mu_{r1}}{\mu_{rd1}} = \frac{l_1}{\mu_0 \mu_{rd1} A_1} = \mathcal{R}'_1$$

where $\mathcal{R}'_1$ is reluctance of the flux tube 1 calculated using differential permeability. If the BH characteristic of the flux tube material is linear, then $\mu_r = \mu_{rd}$ and $\mathcal{R}' = \mathcal{R}$.
Equation (29) can be written as

$$\frac{\partial g_1}{\partial \Phi_1} = \mathcal{R}'_1 + \mathcal{R}'_2 + \mathcal{R}'_4$$

Evaluation of $\dfrac{\partial g_1}{\partial \phi_2}$:-

Partial derivative of (15) with respect to Φ_2 is

$$\frac{\partial g_1}{\partial \Phi_2} = -\mathcal{R}_4 - \frac{\partial \mathcal{R}_4}{\partial \Phi_2}(\Phi_2 - \Phi_1) \tag{30}$$

Using the chain rule,

$$\frac{\partial \mathcal{R}_4}{\partial \Phi_2} = \frac{\partial \mathcal{R}_4}{\partial \mu_{r4}} \cdot \frac{\partial \mu_{r4}}{\partial \mathcal{B}_4} \cdot \frac{\partial \mathcal{B}_4}{\partial \Phi_2} \tag{31}$$

The magnetic field density $\mathcal{B}_4$ at $\mathcal{R}_4$ is given by

$$\mathcal{B}_4 = \frac{(\Phi_1 - \Phi_2)}{A_4} \Rightarrow \frac{\partial \mathcal{B}_4}{\partial \Phi_2} = -\frac{1}{A_4}$$

substituting the value of $\dfrac{\partial \mathcal{R}_4}{\partial \mu_{r4}}$ and $\dfrac{\partial \mathcal{B}_4}{\partial \Phi_2}$ in (31) gives

$$\frac{\partial \mathcal{R}_4}{\partial \Phi_2} = -\frac{\mathcal{R}_4}{\mu_{r4}} \cdot \frac{\partial \mu_{r4}}{\partial \mathcal{B}_4} \cdot -\frac{1}{A_4}$$

Hence,

$$\frac{\partial g_1}{\partial \Phi_2} = -\mathcal{R}_4 - \frac{\mathcal{R}_4}{\mu_{r4}} \cdot \frac{\partial \mu_{r4}}{\partial \mathcal{B}_4} \cdot \frac{1}{A_4} \cdot (\Phi_2 - \Phi_1)$$
$$= -\mathcal{R}_4 \left(1 - \frac{\mathcal{B}_4}{\mu_{r4}} \cdot \frac{\partial \mu_{r4}}{\partial \mathcal{B}_4}\right) = -\mathcal{R}_4 \frac{\mu_{r4}}{\mu_{rd4}} = -\mathcal{R}'_4$$

Similarly, the remaining elements of Jacobian can be derived as

$$\frac{\partial g_2}{\partial \Phi_1} = -\mathcal{R}'_4$$

$$\frac{\partial g_2}{\partial \Phi_2} = \mathcal{R}'_3 + \mathcal{R}'_4 + \mathcal{R}'_5$$

The complete Jacobian matrix is

$$\left[\boldsymbol{\mathcal{G}'}_l(\boldsymbol{\Phi})\right] = \begin{bmatrix} \mathcal{R}'_1 + \mathcal{R}'_2 + \mathcal{R}'_4 & -\mathcal{R}'_4 \\ -\mathcal{R}'_4 & \mathcal{R}'_3 + \mathcal{R}'_4 + \mathcal{R}'_5 \end{bmatrix}$$

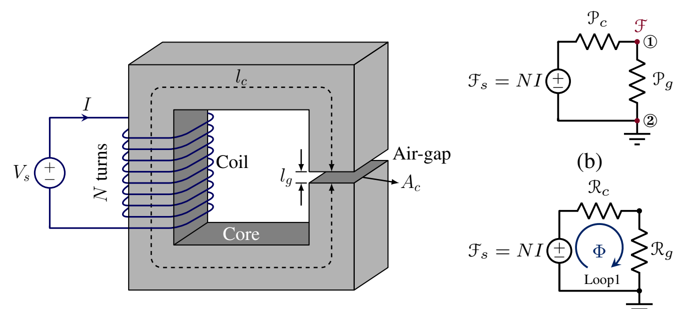
The elements of Jacobian matrix $\boldsymbol{\mathcal{G}'}_l$ resemble the corresponding element in the loop reluctance matrix $\boldsymbol{\mathcal{R}}$.

III. Convergence behaviour of the Newton-Raphson method

Convergence behaviour of the Newton-Raphson method while solving the non-linear MEC system is examined in this section. For simplicity, a sample magnetic system shown in Fig. 4 is considered with the following data. Core length $l_c = 0.17$ m, air-gap length $l_g = 0.003$ m, cross-section area of core $A_c = 7 \times 10^{-4}$ m^2, stacking factor $k_s = 0.97$, current in the coil $I = 1$ A, and number of turns in the coil is $N = 600$. The core is made of silicon steel material 35C250. The coil leakage flux and air-gap fringing flux are neglected in this study. Convergence behaviour of the Newton-Raphson method in both node and loop analysis is shown in Fig. 5, Fig. 6, Fig. 7, and Fig. 8. These figures are prepared using LaTeX animation package to illustrate step by step iteration process. These animations can be played continuously or frame by frame using the control buttons provided below the figures.

A. Node analysis

The permeance network shown in Fig. 4b is solved using the Newton-Raphson method with two different initial conditions. In Fig. 5 and Fig. 6, the variation of residual function with the node ① MMF is shown. At the true solution ($\mathcal{F}^* = 596.39$ A-t), the magnitude of residual function is zero. With the initial condition $\mathcal{F}_{(0)} = 615$ A-t, it is converging in three



(a) (c)

Fig. 4: (a) A sample magnetic system; and its MEC (b) permeance network; (c) reluctance network

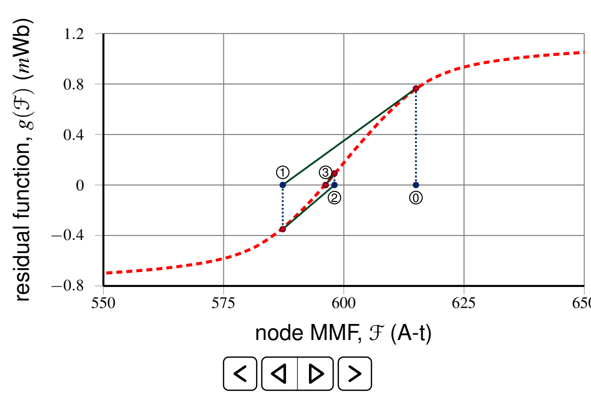


Fig. 5: Node analysis – converging behaviour with $\mathcal{F}_{(0)} = 615$ A-t

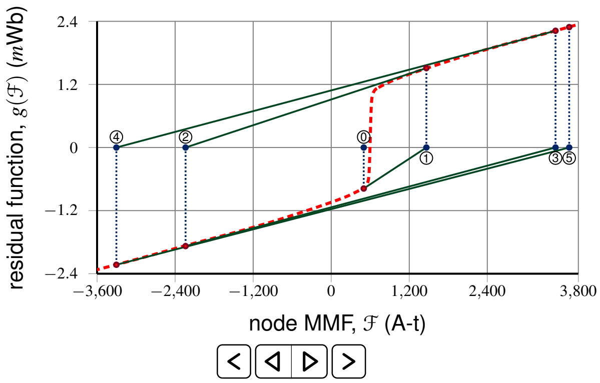


Fig. 6: Node analysis – diverging behaviour with $\mathcal{F}_{(0)} = 500$ A-t

iterations as shown in the Fig. 5. The point marked as ⓪ on x-axis denotes initial guess and remaining points (①, ②, ③) indicates subsequently iterated value of the node MMF. In this case, the initial value and iterated value of the node MMF are situated within the linear region of residual function. The vertical dotted line represents the magnitude of residual function at given node MMF.

The same permeance network is solved with the initial guess $\mathcal{F}_{(0)} = 500$ A-t, and the Newton-Raphson method shows diverging behaviour as illustrated in Fig. 6. In this case, the initial value and subsequently iterated value of the node MMF are located in the saturation region of residual function. From the above cases, it is clear that the initial condition is profoundly affecting the convergence behaviour of the Newton-Raphson method while solving the MEC system using node analysis. Hence, the initial value must be selected with the proper knowledge of the system; otherwise, it may not converge.

B. Loop analysis

The reluctance network shown in Fig. 4c is solved using the Newton-Raphson method with two different initial conditions. In both cases, the initial value of loop flux is located in the saturation region of residual function as shown in Fig. 7 and Fig. 8. The exact solution exists at $\Phi^* = 0.1748$ mWb. In loop analysis, shape of the residual function causes the Newton-Raphson method to converge even though the initial guess lies in the deep saturation region (i.e. $\mathcal{B}_{(0)} = 2.14$ T) as illustrated in Fig. 7 and Fig. 8. Irrespective of the initial guess value, the loop analysis shows stable converging behaviour.

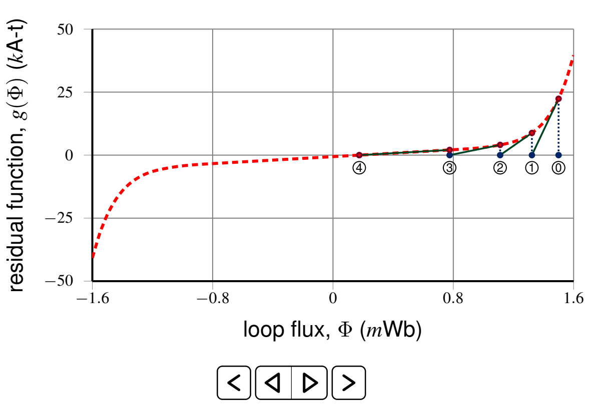


Fig. 7: Loop analysis – converging behaviour with $\Phi_{(0)} = 1.5$ mWb

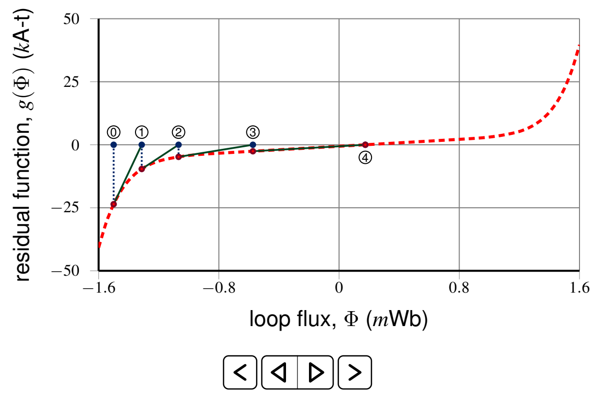


Fig. 8: Loop analysis – converging behaviour with $\Phi_{(0)} = -1.5$ mWb

IV. Conclusion

A simplified Jacobian matrix is determined for both node- and loop-based magnetic equivalent circuit analysis. The elements of Jacobian are similar to that of permeance or reluctance matrix, except it is calculated using differential permeability. Also, the derived Jacobian doesn't have partial derivative term which reduces the computation time. The convergence behaviour of the Newton-Raphson method is investigated. The initial condition is profoundly affecting the convergence of the Newton-Raphson method in the node analysis. Due to the shape of the residual function, loop analysis exhibits more stable convergence than that of node analysis while solving the non-linear MEC network using the Newton-Raphson method. The work presented in this paper can be applied to any magnetic system.

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