Mathematics HL

Internal Assessment

How does the arc length of a curve f(x) connecting A and B affect its time of descent?

An exploration into the Brachistochrone

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Word count: 4781

Page count: 17

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The Brachistochrone problem, posed as a challenge by Johann Bernoulli in the scientific Journal *Acta Eruditorum* in 1696, is one that has been largely solved by the most brilliant minds of Mathematical History. Some names that can be noted are Bernoulli himself, along with Newton, Leibniz et al (J J O'Connor and E F Robertson, 2002) The problem reads as such:

"Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time."

The problem, whilst simple at first glance, would begin to break down once readers realise its nature as an optimisation problem where the solution asks for a most optimal arc or function instead of a value. The curve of quickest descent, titled the brachistochrone, has long since been discovered; its shape, a reflection along the x axis of a cycloid – a curve traced out by a point as a circle rolls along a horizontal plane. Upon doing more research, I found out that such a curve would have to both maximise the speed at which a particle travels along the curve as well as minimizing the distance (length of the curve) in which the particle would have to travel. This bit of intuition made me wonder to what extent the length of the curve affects the time in which a particle would take to descend from point A to point B. This problem interested me as because the vector between A and B would have to always be the same as a control, the length of the curve would not only affect the distance a particle would have the travel, but also the speed at which it does so. A longer curve means it would have to displace more vertically, which means more acceleration due to gravity. I would like, throughout this exploration, to investigate this research question, making use of different curves such as the arc of a circle or a quadratic, or a reciprocal fitted along two points A and B which will not be changed. I will then mathematically model the curves and calculate the theoretical time it would take a particle to descend down them. I will then use the data gathered (time of descent) as well as the Arc length of each of the function, including the brachistochrone itself to conclude whether or not the hypothesis is indeed correct. The hypothesis is that the closer a curve is to the length of a brachistochrone, the less time it would take for a particle to descend the curve.

Defining the problem

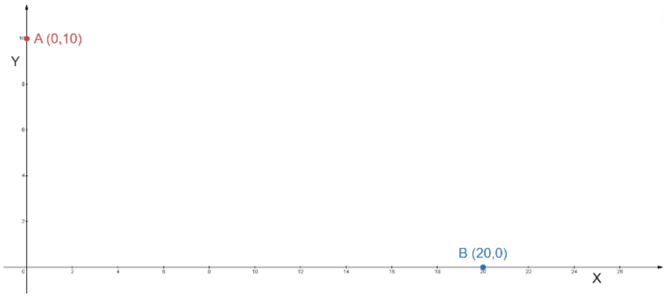


Figure 1: Axis showing the two points of descent A and B

To begin the investigation, I first defined the control variables. The points A(x_1, $f(x_1)$) and B(x_2, $f(x_2)$) will be kept constant throughout the investigation, where $\Delta x = 10$ and $\Delta f(x) = 20$ as shown in *figure 1*. Furthermore, as stated in the original problem posed by Bernoulli, there will be a constant force of gravity acting vertically downwards on the particle as it descends the curve, accelerating it. For the sake of simplicity, the value of g will be approximated to $g = 10ms^{-1}$. Furthermore, air resistance as well as all other forces of drag will be assumed negligible and will not be considered in the model.

Moreover, we will limit the exploration to all curves $y = f(x)$ during the interval $[0,20]$ that are both continuous and differentiable. As in the first place, it would be meaningless to consider curves that are discontinuous; a particle would not, in all cases, be able to travel the curve, more so fall down the curve. And a discontinuous curve will not always have a completely measurable length.

Breaking down the problem

The essential components to this optimization problem could be broken down into the components of kinematics: time, speed, distance, and acceleration. The intuition behind the time for descent can be given as:

$$time = \frac{distance}{speed}$$

The time it would take for a particle to descend a given curve $y = f(x)$ would be the length of the curve, divided by the speed at which the particle travels. However, the particle is travelling at a constant acceleration vertically downwards due to gravity throughout the time of descent. Thus, we need to derive a function *v,* where *v* is the velocity of the particle at any given point $(\mathrm{x}, \mathrm{f(x)})$ as well as a function L[f(x)] to determine the arclength of each function between the interval [0,20]. Firstly, we can introduce the law of conservation of energy; the total energy of interacting parts of a closed system remains constant throughout its operation. At the top of the curve, as the ball is not moving, its energy is fully gravitational and thus can be given as:

$$E_{GP} = E_{total} = mg\Delta f(x)$$

Where E_{GP} = gravitational potential energy, m = mass of the particle, g = acceleration due to gravity,

E_{total} = total energy in the system.

Furthermore, due to the law of conservation of energy, we know that any decrease in gravitational potential energy from the top of the curve translates to an increase in Kinetic Energy, which is given as:

$$E_{KE} = \frac{1}{2}mv^2$$

Thus, as the curve f(x) will always start at the point A(0,10), I derived *v* by equating any change in gravitational potential energy to an increase in kinetic energy, and as the mass terms in both equations cancel out:

$$g\big(10 - f(x)\big) = \frac{1}{2}v^2$$

$$v = \sqrt{2g\big(10 - f(x)\big)}$$

Next, to derive the length L[f(x)] of the curve y = f(x), following the instructions on Paul Dawkins (2020) website which I researched. we first divide up the curve f(x) into *n* subintervals with equal width Δx labelled from P_0 to P_n. This would allow us to estimate the length of the curve of f(x) as:

$$L \approx \sum_{i=0}^{n} |P_{i-1}P_i|$$

Notice that the accuracy of the estimation increases as $n \to \infty$. Thus, the Length of the curve can be represented as:

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i|$$

By using the Pythagorean theorem, the length of the vector $|P_{i-1}P_i|$ can be rewritten as:

$$|P_{i-1}P_i| = \sqrt{[x_i - x_{i-1}]^2 + [f(x_i) - f(x_{i-1})]^2}$$

The *mean value theorem* (Paul Dawkins, 2021) states that along a continuous interval $[x_{i-1}, x_i]$, there is a value c such that $f'(c) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$, And therefore: $f(x_i) - f(x_{i-1}) = (x_i - x_{i-1}) \times f'(c)$. Reading this over, this does not come as a surprise to me. Intuitively, I know that if there was a curve that connected two points together, no matter how much the gradient of the curve changes between the interval of the two points, to be able to reach the endpoint, assuming its gradient function is also continuous, the gradient of the curve would have to eventually be equal to the gradient between the two points ($\frac{\Delta y}{\Delta x}$). So, the length L[f(x)] of the curve f(x) could be written as:

$$L[f(x)] = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (x_i - x_{i-1})^2 \times f'(c)^2}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2(1 + f'(c)^2)}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left((x_i - x_{i-1}) \times \sqrt{1 + f'(c)^2}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\Delta x\sqrt{1 + f'(c)^2}\right)$$

By using the *definition of the definite integral* (Paul Dawkins, 2022). Which, according to the source states that during the continuous interval of [a,b] of a function, if we divide the function into *n* subintervals, as we have before, and from each subinterval choose a point c. The definite integral of the curve between a and b can be derived as:

$$\int_a^b f(x).dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x)\Delta x$$

Therefore, doing this in reverse, we can reduce the formula of the arc length of the curve between the interval [0,20] to:

$$L[f(x)] = \int_0^{20} \sqrt{1 + f'(x)^2}\,.dx$$

From the intuition $time = \frac{distance}{speed}$, as well as the knowledge of the starting point A(0,10) and ending point B(20,0) I inferred using L[f(x)] and v that the time [t] it would take for a particle to descend down a curve f(x) would be the sum of every single value (in other words the definite integral) of arc length travelled at a given point divided by the respective speed travelled at that point:

$$T[f(x)] = \int_0^{20} \frac{\sqrt{1 + f'(x)^2}}{\sqrt{2g\big(10 - f(x)\big)}}.dx$$

$$T[f(x)] = \int_0^{20} \sqrt{\frac{1 + f'(x)^2}{2g\big(10 - f(x)\big)}}.dx$$

Graph-Fitting

The investigation will involve graph fitting different types of graphs with different lengths and substituting f(x) into the T[f(x)] function. I would like to use six different functions, a quadratic with concave up $[f(x) = ax^2 + bx + c]$, a reciprocal function $\left[f(x) = \frac{ax+b}{cx+d}\right]$, the arc of a circle$[(x-a)^2 + (y-b)^2 = r^2]$, a negative exponential function $[f(x) = n^{-a(x+b)}$+c], a straight-line function $[f(x) = ax + b]$ and finally the Brachistochrone, the curve of quickest decent (J J O'Connor and E F Robertson, 2002). Then, will compare the arc length between each of the functions and the Brachistochrone and their respective time of descent. Firstly, I found the functions f(x) of each type that goes through both points A(0,10) and B(20,0). I did this by inputting all the functions inside Desmos and adjusting the coefficients until the graph passes through the two points

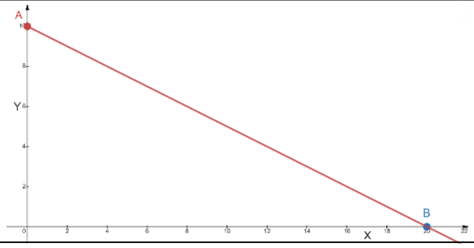
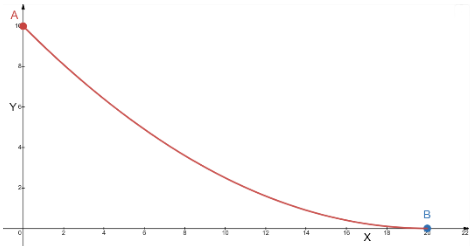
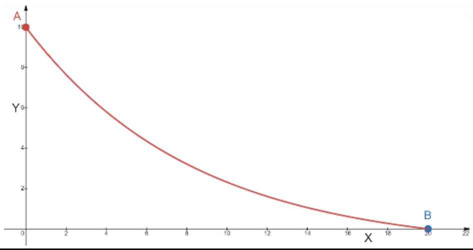
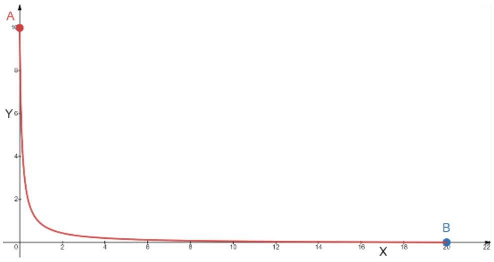
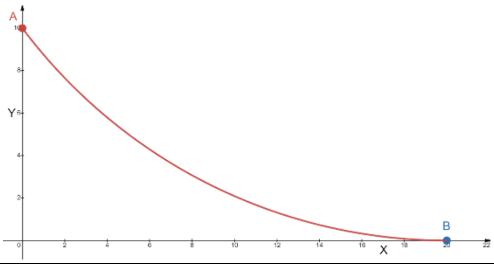
Function number	Function	Function type	Graph
1	$f(x) = -\frac{1}{2}x + 10$	Straight line	
2	$f(x) = \frac{1}{40}x(x-40) + 10$	Quadratic	
3	$f(x) = \frac{11^{-\frac{log11}{log11-1}\left(\frac{1}{20}x-1\right)}}{10} - 1$	Negative exponent	
4	$f(x) = \frac{\frac{-1}{20}x + 1}{x + \frac{1}{10}}$	Reciprocal	
5	$(x-20)^2 + (y-25)^2 = 625$	Arc of a circle	

Figure 2: Table of different functions curve fitted and graphed on Desmos

It was a bit tricky to curve-fit the function of a negative exponent as such a function usually has a horizontal asymptote at y = 0. So, I had to get a little creative. I realised that I could just shift the points A and B up by one, or really any value so long as they are shifted by the same vector, fit the shape of the graph to those points and shift the graph back down to the x axis, which is why the function of the negative exponent ends with negative one

The Brachistochrone [function number 6] and parametric equations

The Brachistochrone curve, literally translating to "shortest" time from Ancient Greek. Is a vertical reflection of a cycloid — the curve traced out by a point on a circle rolling along a plane. As it is a curve that is traced out over an interval of time [t], we can define two functions x(t) and y(t) that denote the cartesian coordinates of the point as a function of time. With these two equations, we can then utilize parametric equations to graph the curve. Therefore, I thought that I should begin by trying to define the parametric equations of a cycloid. I first began drawing the actual circle as well as a few guidelines. *Figure 3* shows the position of a circle After a rotation of T rad. I realised here that because the point A starts at the origin, the horizontal displacement of the center of the circle must always be equal to the arc length subtended by the angle of rotation. This is shown in *Figure 3*. The x coordinate of the point A along the circle is then the displacement of the circle from the origin subtracted by $Rsin(t)$. Therefore, the x position x(t) of the point at a given time [t] can be written as $x(t) = R[t - \sin(t)]$. *Figure 4* shows the same circle after the same rotation of T rad. Here, as the centre of the circle does not ever move up or down, we can infer that the y coordinate y(t) of the point A at a given time [t] is the length of the radius of the circle R subtracted by Rcos(t). Which could be written as $y(t) = R[1 - \cos(t)]$. And so, these are the parametric equations that govern the cycloid. To be able to graph a brachistochrone, we need to transform the graph through a vertical reflection and then a transformation with vector$\begin{pmatrix} 0 \\ 10 \end{pmatrix}$. During which, the sign of the y coordinate of a given point is reversed; the x coordinates are not changed. The parametrics of a brachistochrone can then be concluded to be:

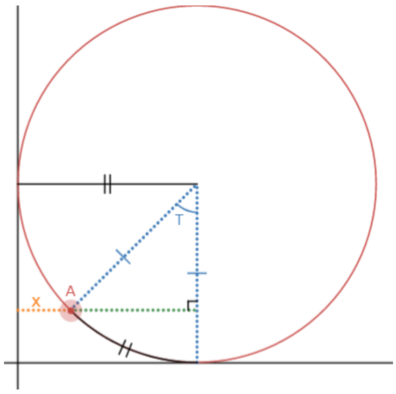


Figure 3: A graph I drew in Desmos showing the rotation after T rad of point A on a circle

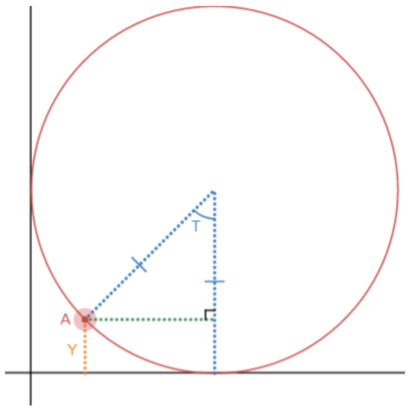


Figure 4: The same circle as Figure 3 but with different lines showing the parametrises of the y position of point A

$$x(t) = R[t - \sin(t)]$$

$$y(t) = -R[1 - \cos(t)] + 10$$

Curve-fitting a brachistochrone along two points

This is where I became quite stuck in my investigation. In the parametrics of the brachistochrone, the only value you can really change is the radius R of the circle as well as the starting position of the circles rotation. It was easy enough to figure out that the starting position of the brachistochrone would need to be shifted vertically upwards by 10 as the coordinates of the start point of the curve is at A(0,10). However graphing a brachistochrone with an arbitrary radius of 2π gave me the following graph (*Figure 5).* I thought I could then take the x intercept of the brachistochrone and enlarging R by a scale facter equal to the distance between it and B(20,0). However this did not work as I soon realised that changing the radius not only affected the x coordinates but also the y coordinates, and therefore such a brachistochrone would be too small to even reach the x axis. Through research (Paul Kunkel 2012), I found that to be able to curve fit a brachistochrone between two points, we need to draw an arbitrary brachistochrone with Radius R and then a linear line connecting the points A and B (the points we are trying to curve-fit it to). Then, we need to find the point C where the line AB intersects the brachistochrone. Using this, we can then then multiply the Radius by a scale factor of $\frac{AB}{AC}$. So, using the parametric equations of a Brachistochrone with a arbitrary radius 2π, as well as the linear function [function 1] I had fitted before to points A and B.

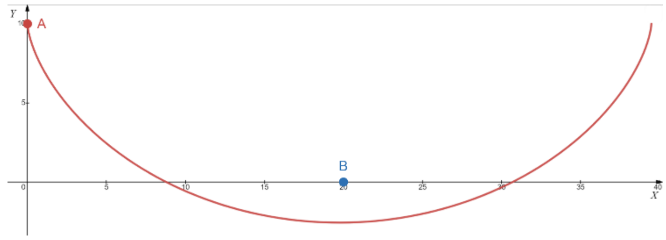


Figure 5: A Brachistochrone I drew in Desmos using the parametrises derived, with an arbitrary Radius = $R = 2\pi$

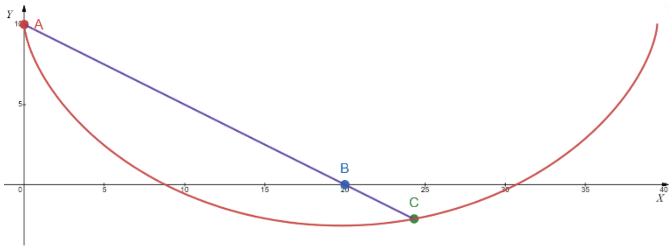


Figure 6: The same Brachistochrone with Point C and a line connecting points A, B and C

I tried first to equate the x and y coordinates by substituting the parametric functions of x and y into the linear function to determine at what time [t] does the brachistochrone intercept with the linear function.

$$-2\pi[1-\cos(t)]+10 = -\frac{1}{2}(2\pi[t-\sin(t)])+10$$

$$1-\cos(t) = \frac{1}{2}([t-\sin(t)])$$

$$t-2 = \sin(t) - 2\cos(t)$$

I noticed that the radius [R] on both sides as well as the signs of the equations and the y-intercepts cancelled each other out, so therefore, it probably would not have mattered what abitrary value of R I picked as well as whether or not I was fitting the curve of a cycloid or a Brachistochrone. However, this is where I really became stuck as no matter what I tried I could not figure out how to solve the equation for T. I had never seen a question that involved finding out the value of a variable that was both inside and outside trigonometric functions. I could not find any solutions through research either.

However, I eventually came to the conclusion to graph two separate functions $y = x - 2$ and $y = \sin(x) - 2\cos(x)$ in Desmos and computing the point interception. Doing this would not give me the exact value, however it would provide me with an accurate estimation. Moreover, in a Physics lab, physical measurements (even those measured by electronic devices) usually don't go into a highler precision level than 3 or 4 significant figures, so I thought that within the context of this experiment being closely tied to the real world, it would not affect the results too much. Doing so provided me with the final Time [T] ≈ 3.5084. I then calculated the length of the lines AB and AC:

$$|AB| = \sqrt{10^2 + 20^2} = 10\sqrt{5}$$

$$|AC| = \sqrt{4\pi^2[3.0688 - \sin(3.0688)]^2 + 4\pi^2[1 - \cos(3.0688)]^2} \approx 27.165$$

And so the final value of radius R that is needed to fit the curve onto the two points is $R \approx 2\pi\frac{10\sqrt{5}}{27.165}$

The final parametric equations that govern the Brachistochrone in this investigation can then be written as:

$$x(t) \approx 2\pi\frac{10\sqrt{5}}{27.165}[t - \sin(t)]$$

$$y(t) \approx -2\pi\frac{10\sqrt{5}}{27.165}[1 - \cos(t)] + 10$$

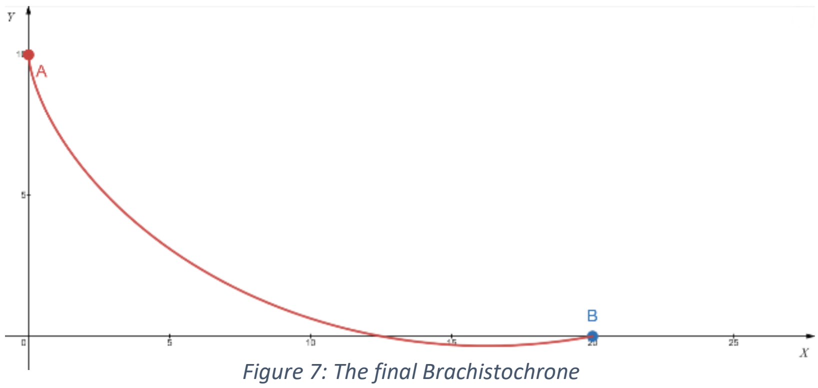


Figure 7: The final Brachistochrone

Time of descent

With all six of the equations of different curves of descent, I then tried to calculate each of their respective time of descent by substituing the functions into the formula for time of descent I had derived previously. I started with the straight-line function as I thought it would be the fastest and easiest to do:

$$T[f(x)] = \int_0^{20} \sqrt{\frac{1+f'(x)^2}{2g\big(10-f(x)\big)}}.dx$$

$$T[f(x)] = \int_0^{20} \sqrt{\frac{1+\left(\frac{d}{dx}\left((-\frac{1}{2}x+10)\right)\right)^2}{2g\left(10+\frac{1}{2}x-10\right)}}.dx$$

$$T[f(x)] = \int_0^{20} \sqrt{\frac{1+\frac{1}{4}}{2g\left(\frac{1}{2}x\right)}}.dx$$

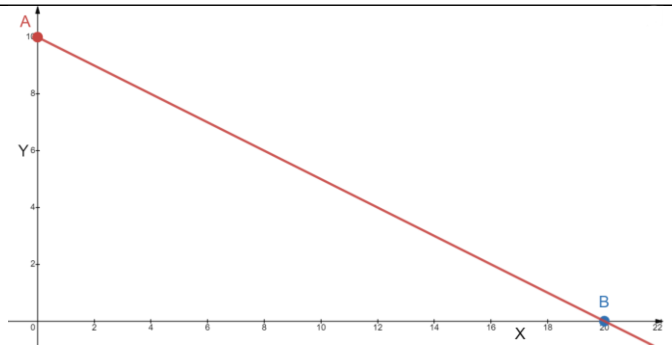
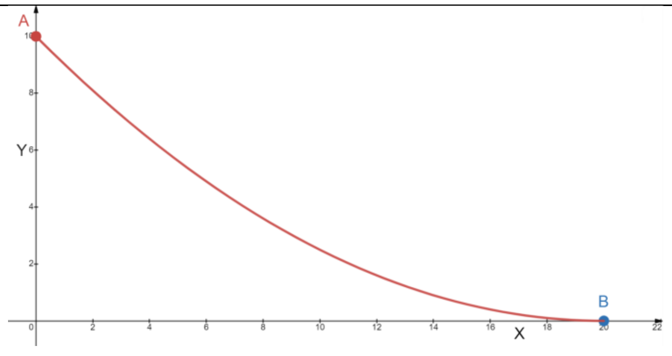
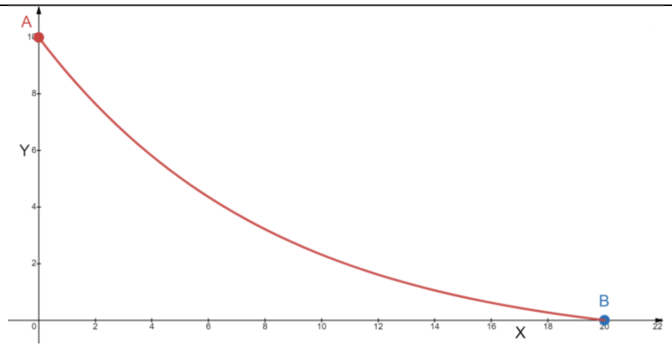
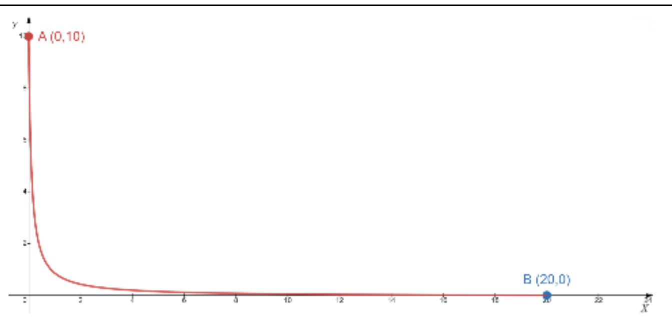
$$T[f(x)] = \int_0^{20} \sqrt{\frac{1}{8}\times\frac{1}{x}}.dx$$

$$T[f(x)] = \frac{1}{\sqrt{8}}\int_0^{20} \frac{1}{\sqrt{x}}.dx$$

$$T[f(x)] = \frac{1}{\sqrt{8}}\left(2\sqrt{20}-2\sqrt{0}\right)$$

$$T[f(x)] = \sqrt{10}$$

Because this process was quite repetitive and had a lot of room for human error, especially in the substituting and integration of the equation, I decided to complete as many of the remaining equations as I could using the Wolfram Alpha Online Integral Calculator as it would decrease the likelihood of human error during calculation. It would also speed up the process of analysis, which, done by hand, would have taken a very long time. For the sake of consistency and reliability, I will provide the values to the same precision as the final time value calculated before throughout the investigation:

Function type	Graph	Time of descent
Straight line		3.1623s
Quadratic		2.9925s
Negative exponent		2.64051s
Reciprocal		2.7261s

Integration and implicit differentiation of the arc of a circle

For the arc circle. I realised that to be able to complete the calculation of the time of descent of the circle, I needed to isolate the y in the equation of the arc-circle so that I can substitute it in the form y = f(x) into T[f(x)]. This would be the simplest and most straightforward way to compute the calculation. Therefore, before I was able to substitute the equation into Wolfram Alpha:

$$(x-20)^2 + (y-25)^2 = 625$$

$$y = 25 - \sqrt{625 - (x-20)^2}$$

Here, the sign of the square root is negative as the interval [0,20] of the curve where the theoretical particle would descend is below the centre of the circle. Then, I proceeded to compute the derivative of the equation using Wolfram Alpha, this is needed in the T[f(x)] formula:

$$\frac{dy}{dx} = \frac{x-20}{\sqrt{-x^2+40x+225}}$$

I then substituted both equations into T[f(x)], this final step was also done using Wolfram Alpha:

$$T[f(x)] = \int_0^{20} \sqrt{\frac{1+\left(\frac{x-20}{\sqrt{-x^2+40x+225}}\right)^2}{20\sqrt{625-(x-20)^2}-300}}\,.\,dx$$

$$\approx 2.6241s$$

Integration of the Brachistochrone

This title is a bit misleading as the final steps I took in solving this problem did not include integration at all. I did at first thought to repeat the same steps I took in solving for the time of descent for the previous functions, using the chain rule: $\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$ to compute the derivative of the parametric and then trying to convert the equation from parametric form into cartesian form in order to substitute into the integral. However, this was very difficult and even after doing some research on the cartesian form of cycloids/Brachistochrone, I could not find a form of the equation that could be substituted into T[f(x)]. However, I then found a website (proofwiki, 2020) which had proven that the time of travel down a Brachistochrone could be computed using the formula:

$$T = \pi\sqrt{\frac{r}{g}}$$

Where r = the radius of the circle in which the Brachistochrone is constructed from, and g is the acceleration due to gravity. However, this formula can only be used to compute the time of descent down from the top of the brachistochrone to the bottom, at the midpoint between the beginning and the end, Where the circle has turned a rotation of $\pi\ rad$. My brachistochrone does not end there and continues as the point of the curve moves up to meet the x axis at [0,20]. I realised here that this equation could be rewritten in a more general form:

$$T = \theta\sqrt{\frac{r}{g}}$$

Where θ = the angle of rotation of the circle, in my case, this is equal to the value of t in my parametric equations, **from now on T will be written as $\boldsymbol{\theta}$ to reduce complications of having two Ts in the formula that mean different things**. Firstly, we know from *Curve-fitting a brachistochrone along two points,* that at the point B[0,20], the circle has completed a rotation $\approx 3.5084rad$. Therefore, substituting this value into the general formula, we get a final time value of $T \approx 1.78391$ which is indeed the shortest time of descent compared to all the other functions.

Arc length of curves.

Finally, using the Arc length function L[f(x)], I computed the respective Arc lengths of the different functions. This was in general quite simple as all I had to do was substitute their functions into the function of arc length, this same method was also used for the circle as I had already found the derivative of the equation in *Integration and implicit differentiation of the arc of a circle*.

$$L[f(x)] = \int_0^{20} \sqrt{1 + f'(x)^2}\,.dx$$

However, the Arc length of the Brachistochrone was calculated using a different function, this was found using research: The arc length of a cycloid taken from proofwiki (2022):

$$L = \int_0^{2\pi} 2r \sin\left(\frac{\theta}{2}\right) d\theta$$

Where θ = angle of rotation of the circle. It did not really matter if it were a function for the brachistochrone or the cycloid as of course if both maintained the same radius, they would have the same arc length. 2π was substituted for my value of $3.5084rad$ as I am not trying to calculate the arc length of the entire brachistochrone. Using this function, the Arc length of the Brachistochrone used in this exploration was:

$$\text{L} \approx 24.461$$

Final Data

Curve	Equation	Arc length/m	Time of descent/s
Straight line	$f(x) = -\frac{1}{2}x + 10$	22.361	3.1623
Quadratic	$f(x) = \frac{1}{40}x(x-40) + 10$	22.956	2.9925
Negative exponent	$f(x) = \frac{11^{-\frac{log11}{log11-1}\left(\frac{1}{20}x-1\right)}}{10} - 1$	23.071	2.63051
Reciprocal	$f(x) = \frac{\frac{-1}{20}x + 1}{x + \frac{1}{10}}$	28.451	2.7261
Arc of a circle	$(x-20)^2 + (y-25)^2 = 625$	23.182	2.6241
Brachistochrone	$x(t) \approx 2\pi\frac{10\sqrt{5}}{27.165}[t - \sin(t)]$ $y(t) \approx -2\pi\frac{10\sqrt{5}}{27.165}[1 - \cos(t)] + 10$	24.461	1.78391

Figure 8: Table of all curves with their corresponding arc lengths and time of descent

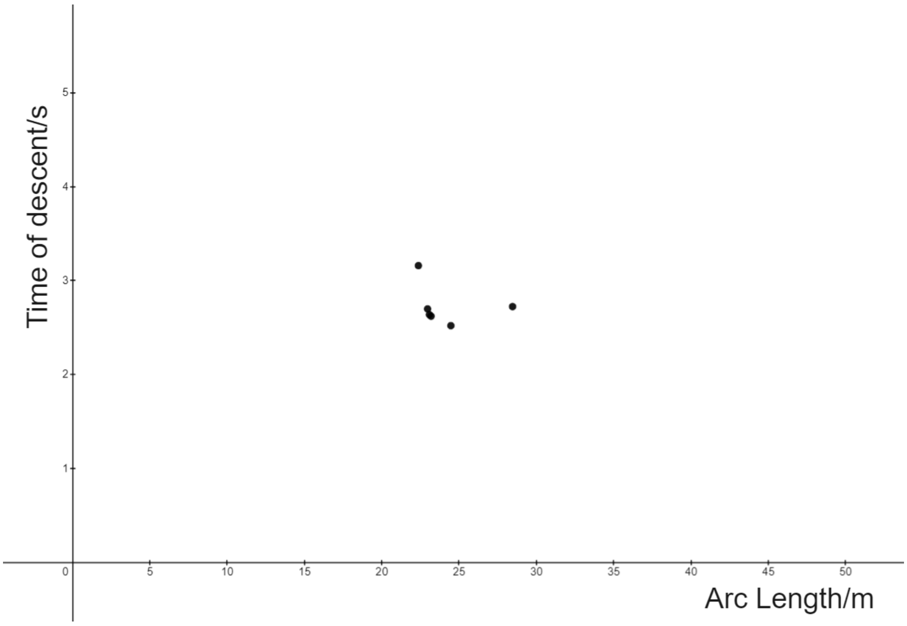


Figure 9: plotted graph of relationship between Arc Length and Time of descent

Finally, I graphed these different values using Desmos again as it has an inbuilt function to construct regression analysis. That would allow me to approximate the curve of best-fit for these sets of points: This is where I first noticed that the relationship shown by the graph here very closely resembles a quadratic with concave up. Its apex being near the point calculated from the arc length and time of descent of the brachistochrone. Using Desmos's inbuilt automatic regression algorithm, I was able to conclude that this was indeed correct, drawing a correlation curve:

$$f(x) = 0.144247x^2 - 7.4121x + 96.8406$$

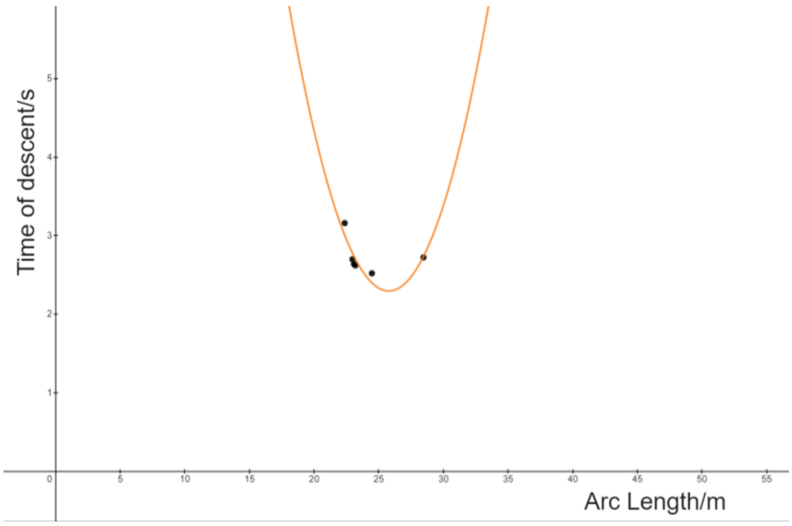


Figure 10: plotted graph with correlation curve

This curve of best fit had a correlation coefficient (R^2 coefficient) of 0.9838. The (R^2) is a statistical function used by Desmos, and many other graphing programs that allow it to quantitively compute the proportion of variation (how much percent the points differ) between recorded data and the line of regression (line of best fit) (corporatefinanceinstitute.com), with a R^2 of 1 being completely identical and 0 meaning having completely no correlation to one another.

Within this context, we can see that an R^2 coefficient of 0.9838 shows a very strong correlation between the Arc Length and the Time of Descent. Intuitively, this made sense to me, as looking at the simplified equation of time in terms of distance and speed ($time = \frac{distance}{speed}$) to minimise time is to minimise distance and maximise speed of travel. A curve that was long would have been able to accelerate a lot during its descent path but would also have had to travel a longer distance, whereas the curve with the shortest path (straight line) would not have had enough vertical displacement to accelerate very much. I wanted to test out this conjecture, so I first reused the formula of arc length for each of the curves, this time instead of integrating between 0 and 20, I substituted in the numbers (2,4,6,8,10,12,14,16,18) to plot out a graph of time of descent against horizontal displacement for each of the curves. To compute the respective graph for the brachistochrone, I also had to calculate the respective angle of rotation of the circle when the particle has been horizontally displaced at those different points as the equation for time of descent down the cycloid has θ as an input, not x. This was done by plotting the graph of $y = 2\pi \frac{10\sqrt{5}}{27.165}[x - \sin(x)]$ (the function which describes the x parametric of the brachistochrone) and then using desmos to compute the point of intersection between this graph and y = (2,4,6,….,18). I then plotted all of the points and for each, computed using desmos a line of regression. Below is the compound graph of all the curves. Because it was quite confusing to try and draw comparisons and differences between the graphs, I split it into 6 individual Graphs.

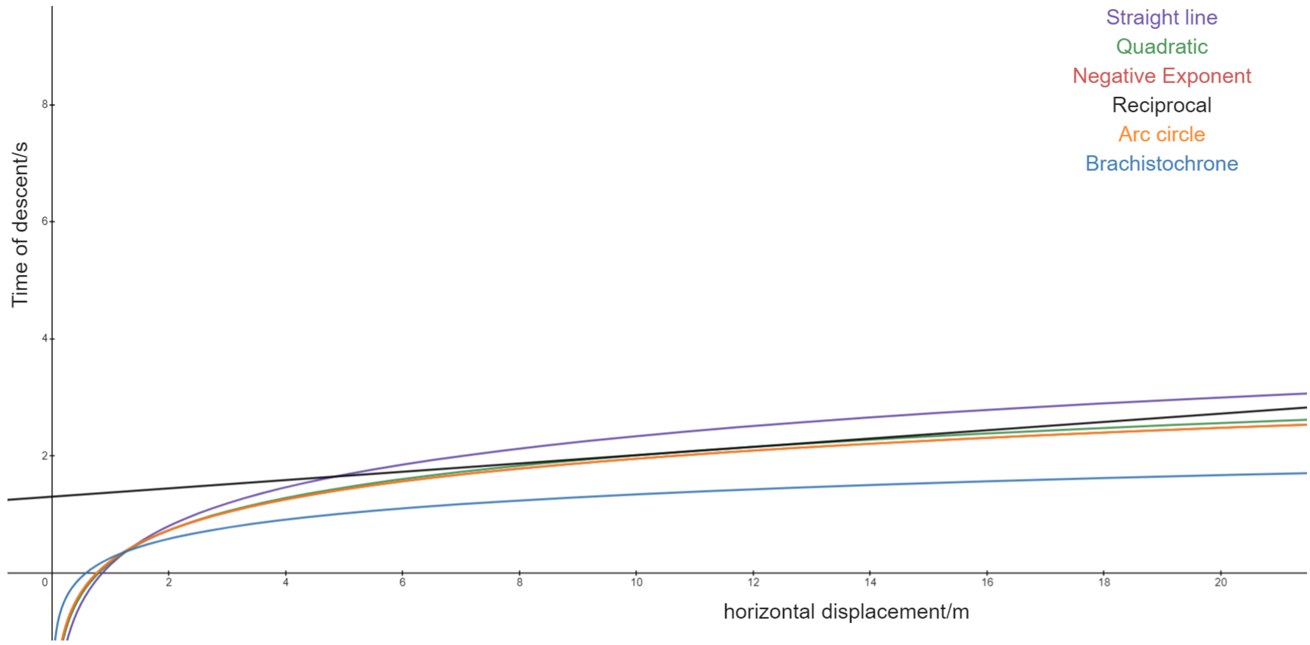


Figure 11: comparison of Arc Length vs horizontal displacement for all curves

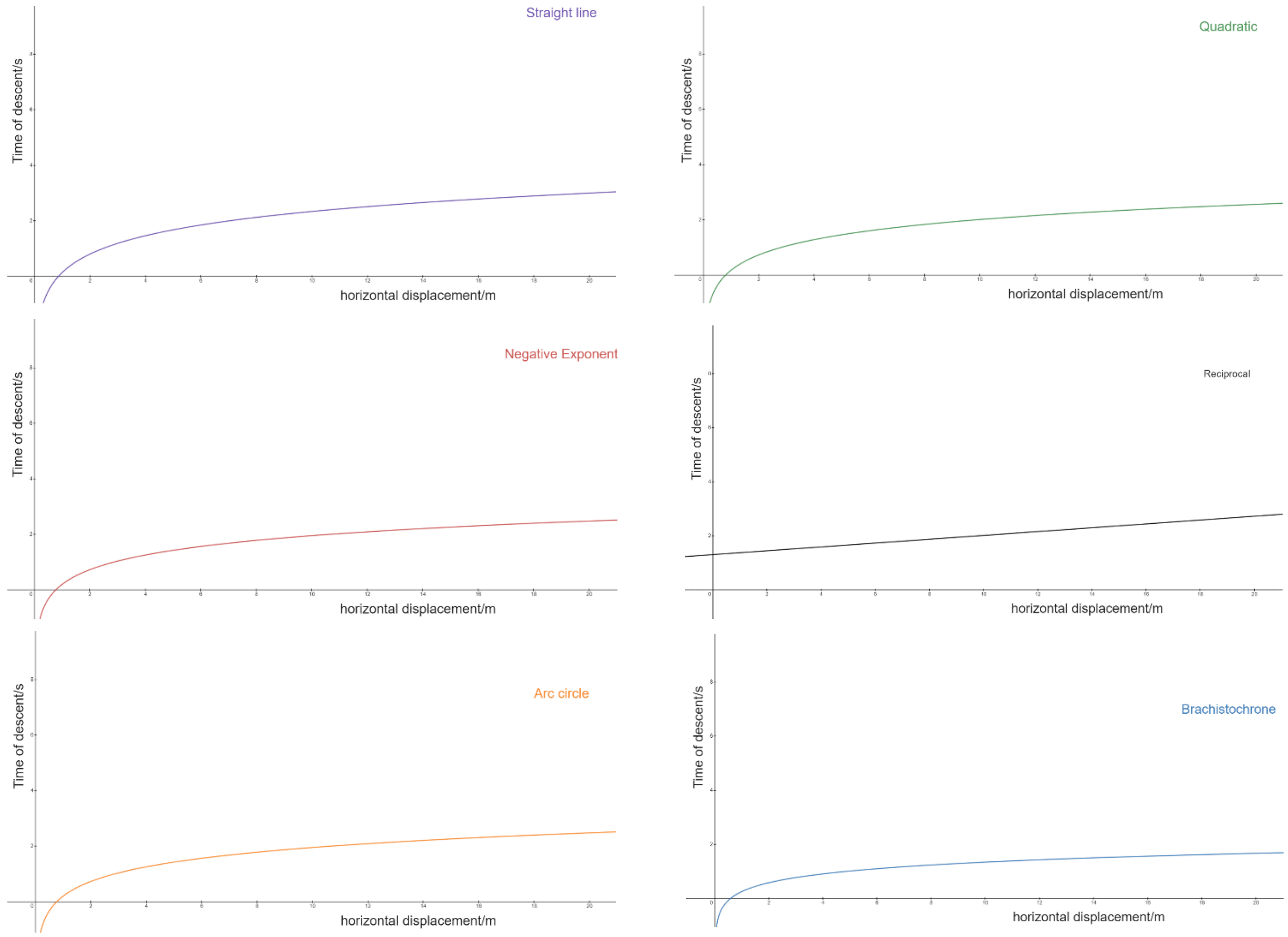


Figure 12-16: Respective graphs of Time of descent vs horizontal displacement for each curve

Although the graphs are quite difficult to comprehend, these graphs generally followed the same regression curve type (a logarithmic curve) and did not differ too much from each other. A notable outlier would be the curve of the Reciprocal function. It, as can be predicted, spent a lot of its time descending at the beginning of the curve (after 4m of horizontal displacement, 1.59s has already passed in comparison to the brachistochrone, which crossed the 4m line at 0.89s). It also had a straight line as a regression line that had a computed R^2 value of 1, meaning a perfect linear relationship between Time of descent and horizontal displacement. Furthermore, the Negative exponent, Quadratic and the Arc circle curves surprisingly followed a very similar path in terms of time taken to travel a set horizontal displacement, on *Figure 11,* They overlap each other for most of the displacement in the interval [0,20]. On the other hand, the Brachistochrone stayed uncontested for the entire duration of the descent path, showing clearly how much more optimised it is than the other curves.

During my exploration on the topic chosen, I have successfully concluded the relationship between the arc length of a function of descent and its time of descent between points A and B. This was done by deriving functions L[f(x)] & T[f(x)] which outputs a L and T value given a function f(x) that connects points A and B. Using these functions, I then mathematically modelled different curves between these points, including the brachistochrone and computed their respective arc lengths and time of descent. This was then graphed in logger pro with a curve of best-fit concluding that the correlation between arc-length and time of descent was a quadratic relationship with its apex being around the arc length of the brachistochrone, which validates my hypothesis. Intuitively this does make sense, as any curve that was too short would not have enough length to curve downwards, meaning it would not have enough velocity throughout to descend quickly, however a curve that was too long would have to spend more time traversing a longer distance to reach its end. For example, the graph of the reciprocal with by far the longest arc length (28.451m) also had the second highest time of descent, only after the equation of the straight line. This because the particle had to descend through such a long path that it invalidates the near vertical drop at the beginning of the slope where the particle accelerated the most. On the other side of the spectrum, the slope of the Straight line had the shortest arc length out of all the slopes (22.361m) but had the longest time of descent (3.1623s) This could be explained by the conjecture that such a slope did not have the necessary vertical displacement early on into the descent to accelerate the particle as much as other slopes. However, the final quadratic model of Arc length vs Time of Descent has a limitation. It had a y-intercept at $y = 96.8406$. Realistically, with an arc length of zero, a curve would have a time of descent of either zero or infinity (the particle could have descended it instantaneously or never descended depending on your perspective.). Furthermore, the regression curves on *figures 12-16* did not always accurately model all the data available. The logarithmic functions had a limitation of having a vertical asymptote at x = 0, meaning that at the point where the horizontal displacement was 0, the time of descent was not defined (although this could have been linked to the 0/infinity paradox stated above).

In conclusion, this exploration displays the importance of optimisation within the fields of architecture, engineering, and design. As in a lot of cases, there are limitation requirements on resources (in this case, it would be the time of descent and length of the curve) that must be met or at least minimised and the difference between an optimal and suboptimal curve could be significantly large. Furthermore, the Brachistochrone of course has a plethora of applications within the field of engineering and design; the design of roofs to maximise precipitation offload, or the design of roller coasters that wish to accelerate the car in the shortest amount of time for example.

Aside from the fun I had working my way through the many different problems that came up during my exploration, I learnt a lot of different things, both about calculus and its uses for optimisation as well as the brachistochrone itself. The arc length function, for example was a completely new field of calculus to me, and the way in which it was derived opened my eyes to the power of infinite and infinitesimal limits. All in all, it was very much enjoyable to complete the exploration as well as learn many new aspects of mathematics.

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