Solution to Homework 2

Task 1. (10 points)

Given matrix $M = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 2 & 2 & 3 \\ 0 & 1 & 9 & 10 \end{bmatrix}$

a. Write down MATLAB code including the elementary row operations that put the matrix in reduced row echelon form.

```
M=[2 3 4 5;1 2 2 3;0 1 9 10];

M(1,:) = M(1,:) - M(2,:)
```

```
M = 3×4
     1     1     2     2
     1     2     2     3
     0     1     9    10
```

```
M(2,:) = M(2,:) - M(1,:)
```

```
M = 3×4
     1     1     2     2
     0     1     0     1
     0     1     9    10
```

```
M(3,:) = M(3,:) - M(2,:)
```

```
M = 3×4
     1     1     2     2
     0     1     0     1
     0     0     9     9
```

```
M(3,:) = 1/9*M(3,:)
```

```
M = 3×4
     1     1     2     2
     0     1     0     1
     0     0     1     1
```

```
M(1,:) = M(1,:) - M(2,:)
```

```
M = 3×4
     1     0     2     1
     0     1     0     1
     0     0     1     1
```

```
M(1,:) = M(1,:) - 2*M(3,:)
```

```
M = 3×4
     1     0     0    -1
     0     1     0     1
     0     0     1     1
```

We check the solution using *rref()*

```
M=[2 3 4 5;1 2 2 3;0 1 9 10];
M_red = rref(M)
```

```
M_red = 3×4
     1     0     0    -1
     0     1     0     1
     0     0     1     1
```

b. What is the rank of the matrix? How can you read the information from the reduced row echelon form of matrix ?

The rank of matrix M is the number of non-zero rows in M_{red} (M in reduched row echolon form). Hence, the rank of $M$is 3. We can check that result using function *rank().*

```
rank(M)
```

```
ans = 3
```

c. Use the reduced row echelon form of matrix to solve = 0. Write down the general solution vector! How many variables are free? What is the number of constraints?

The pivot variables can be expressed in terms of the remaining variables:

$x_1 = x_4$

$x_2 = x_3 = \ -x_4$

The system has one free variable x_4 and three constraints (number of constraints is equal to the number of pivots).

The general solution vector is $(x_4, -x_4, -x_4, x_4)$.

In matrix form $x = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} x_4$. Also compare with output of *null(M,'r')* in Matlab.

```
null(M,'r')
```

```
ans = 4×1
     1
    -1
    -1
     1
```

It returns all linearly independent vectors that span the null space.

d. Does system = 0 have a unique solution?

No, since there is a solution other then the trivial solution.

From the lecture we know that if

- the system has a free variable there must be a non-trivial solution (fulfilled)

- every variable is a pivot variable the system has a unique solution (not fullfilled)

e. Check if vectors $s_1 = [2 \quad -1 \quad -1 \quad 1]^T$ and $s_2 = [2 \quad -2 \quad -2 \quad 2]^T$ are in the null space of . Are vectors s_1 and s_2 linearly dependent?

From the general solution vector obrained above $(x_4, -x_4, -x_4, x_4)$ we can see that s_1 is no solution to the homogenous system, hence, is not in the null space, but s_2 is in the null space.

We cannot find a scalar α that transforms s_1 into s_2, $s_1 \neq \alpha{\cdot}s_2$, hence the two vetors are not linearly dependent.

Task 2. (5 points)

Given subspace W, spanned by the vectors $V = \{[1\,2\,3]^T, [0\,2\,1]^T, [1\,1\,1]^T\}$

a. Is vector $b = [4\,3\,1]^T$ in subspace W?

If b was in W, it could be obtained by a linear combination of the vectors in V.

To check this, we can append the vector b to V from the right side and transform the resutling matrix into the reduced row echelon form (by hand or using Matlab):

```
b = [4;3;1];
V = [1 2 3; 0 2 1; 1 1 1]';
% reduced row echelon form
disp("Reduced row echelon form of [V|b]:")
```

```
Reduced row echelon form of [V|b]:
```

```
disp(rref([V b]))
```

```
    1.0000         0         0   -1.6667
         0    1.0000         0    0.3333
         0         0    1.0000    5.6667
```

```
S=rref([V b]);
```

From the reduced row echelon form, we can see that the system is consistent (has a solution).

Hence, we can obtain b as a linear combination of the vectors in V.

Confirm that $\alpha_1 \cdot V_{.1} + \alpha_2 \cdot V_{.2} + \alpha_3 \cdot V_{.3} = b$:

```
S(1,end)*V(:,1)+S(2,end)*V(:,2)+S(3,end)*V(:,3)
```

```
ans = 3×1
     4
     3
     1
```

b. Is V a basis for subspace W?

V is a basis if all vectors in V are linearly independent and V spans W.

From the reduced row echelon form we can easily see that all vectors in V are linearly independent, since there is no non-trivial solution to the homogenous system $\mathrm{V}\mathrm{x} = 0$.

V is a standard basis, since V_{ref} is an identity matrix.

```
rref(V)
```

```
ans = 3×3
     1     0     0
     0     1     0
     0     0     1
```

c. Form a matrix A, with its columns corresponding to the vectors in , and then use appropriate MATLAB functions to confirm that the equation dim(col(A))+dim(N(A))=number of columns in A holds for this matrix. (hint: dim(col(A)) is equal to the number of pivots in the reduced echelon form of the matrix)

```
A = V;
% number of columns in A ==> four columns
disp("Matrix A:")
```

```
Matrix A:
```

```
disp(A)
```

```
     1     0     1
     2     2     1
     3     1     1
```

```
disp("A in reduced row echelon form:")
```

```
A in reduced row echelon form:
```

```
disp(rref(A))
```

```
     1     0     0
     0     1     0
     0     0     1
```

```
% three pivots
dim_col_A = rank(A); % column rank

disp("Null space of A:")
```

```
Null space of A:
```

```
disp(null(A))
dim_N_A = size(null(A),2);

% 3+0 = 3 -> the equation holds
dim_col_A + dim_N_A == size(A,2)
```

```
ans = logical
   1
```