Exercise 7. Let $(X, \mathscr{A}, \mu)$ be a finite measure space, and let f be an $\mathscr{A}$-measurable real- or complex-valued function on X.

(a) Show that f belongs to $\mathscr{L}^\infty(X, \mathscr{A}, \mu)$ if and only if:

  (i) f belongs to $\mathscr{L}^p(X, \mathscr{A}, \mu)$ for each p in $[1, +\infty)$ and

  (ii) $\sup\{\|f\|_p : 1 \le p < +\infty\}$ is finite.

(b) Show that if these conditions hold, then $\|f\|_\infty = \lim_{p \to +\infty} \|f\|_p$.

Proof. A function $f : X \to \mathbb{R}$ can verify $\sup\{\|f\|_p, 1 \le p < +\infty\} < +\infty$ without being bounded: for example, let $X = \mathbb{R}$, $\mathscr{A} = \mathscr{B}(\mathbb{R})$, μ be the point mass $\mu(A) = 1 \iff 0 \in A$, and

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \begin{cases} n & \text{if } x = 1/n \text{ for } n \in \mathbb{N}_+ \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Then f is measurable, and $\int_{\mathbb{R}} |f|^p \, \mathrm{d}\mu = 0$ for all $p \in [1, +\infty)$. However, $f(1/n) = n$ for all $n \in \mathbb{N}$ so f is not bounded.

Therefore, for this exercise, we take $\mathscr{L}^\infty(X, \mathscr{A}, \mu, \mathbb{R})$ to be the $\mathbb{R}$-vector space of $\mathscr{A}$-measurable functions $X \to \mathbb{R}$ that are essentially bounded; and, replacing $\mathbb{R}$ with $\mathbb{C}$ in the previous sentence, we define $\mathscr{L}^\infty(X, \mathscr{A}, \mu, \mathbb{C})$.

(a) Let $f \in \mathscr{L}^\infty(X, \mathscr{A}, \mu)$. For all $p \in [1, +\infty)$, f^p is $\mathscr{A}$-measurable as the composition of $\mathscr{A}$-measurable functions on their respective domains; and f^p is essentially bounded. Let N be a locally μ-null set where f^p is not bounded. Since $\mu(X)$ is finite, we have $\mu(N \cap X) = \mu(N) = 0$. From this we deduce that

$$\int_X |f|^p \, \mathrm{d}\mu = \int_{X-N} |f|^p \, \mathrm{d}\mu$$

and, since f^p is bounded on $X - N$, this integral is finite. Therefore $f \in \mathscr{L}^p(X, \mathscr{A}, \mu)$ for all $1 \le p < +\infty$.

The set $M = \{x \in X \mid |f(x)| > \|f\|_\infty\}$ is locally μ-null since $f \in \mathscr{L}^\infty(X, \mathscr{A}, \mu)$. Since $\mu(X)$ is finite, $\mu(M) = 0$, and

$$\int_X |f|^p \, \mathrm{d}\mu = \int_{X-M} |f|^p \, \mathrm{d}\mu \le \|f\|_\infty^p \cdot \mu(X) \qquad \text{so that}$$

$$\|f\|_p \le \|f\|_\infty \cdot \mu(X)^{1/p}$$

- If $\mu(X) = 1$, then we get $\|f\|_p \le \|f\|_\infty$
- If $\mu(X) > 1$, the function $p \mapsto \mu(X)^{1/p}$ is nonnegative and non-increasing; we get $\|f\|_p \le \|f\|_\infty \cdot \mu(X)$

- If $\mu(X) < 1$, the function $p \mapsto \mu(X)^{1/p}$ is nonnegative and nondecreasing; we get $\|f\|_p \leq \|f\|_\infty$

Therefore, $\mathrm{P} = \{\|f\|_p, 1 \leq p < \infty\}$ is a nonempty subset of $\mathbb{R}$ that is bounded above, so $\sup P$ exists and is finite.

Conversely, suppose that f verifies (i) and (ii), and suppose that f is not essentially bounded. Let $M > \sup P$, there exists $N = \{x \in X \mid |f| > M\}$ such that $\mu(N) > 0$. Then

$$\int_X |f|^p \, \mathrm{d}\mu \geq M^p \cdot \mu(N)$$
$$\|f\|_p > \sup P \cdot \mu(N)^{1/p}$$
$$\sup P > \sup P \cdot \mu(N)^{1/p}$$

This last inequality holds for any value of $p \in [1, +\infty)$. Using a disjunction of the cases $\mu(N) < 1$, $\mu(N) = 1$, and $\mu(N) > 1$, we always arrive at $\sup P > \sup P$, which is impossible. Therefore f is essentially bounded.

(b) Let $f \in \mathscr{L}^\infty(X, \mathscr{A}, \mu)$. Since $\mu(X)$ is finite, f is bounded almost everywhere; let N be a μ-null set where f is not bounded. There exists $\{f_n\}_{n\in\mathbb{N}}$ a nondecreasing sequence of simple functions converging simply to f on $X - N$. For all n, f_n is bounded. Let us note

$$f_n(x) = \sum_{i=0}^{k} a_i \chi_{A_i}(x) \text{ with } |a_1| > |a_2| > \cdots > |a_k|$$

Then, for all $p \in [1, +\infty)$, we deduce from (a) that f_n^p is integrable, and, since the A_i are pairwise disjoint, we get

$$\int_X |f_n|^p \, \mathrm{d}\mu = \int_X \sum_{i=0}^{k} |a_i|^p \chi_{A_i} \, \mathrm{d}\mu$$

From this we deduce

$$\int_X |a_1|^p \chi_{A_1} \, \mathrm{d}\mu \leq \int_X |f_n|^p \, \mathrm{d}\mu \leq \int_X |a_1|^p \chi_X \, \mathrm{d}\mu$$
$$|a_1|^p \mu(A_1) \leq \int_X |f_n|^p \, \mathrm{d}\mu \leq |a_1|^p \mu(X)$$

Since for all $x > 0$ $\lim_{p\to+\infty} x^{1/p} = 1$, we deduce that

$$\lim_{p\to+\infty} \|f_n\|_p = |a_1| = \|f_n\|_\infty \tag{1}$$

The sequence $\{\|f_n\|_\infty\}_{n\in\mathbb{N}}$ of real numbers is nondecreasing and bounded above by $\|f\|_\infty$, and therefore converges to λ such that $0 \leq \lambda \leq \|f\|_\infty$.

Let $\varepsilon > 0$, and suppose that $\lambda < \|f\|_\infty - \varepsilon$. Since f is bounded on $X - N$, there exists $x \in X - N$ such that $\left|f(x) - \|f\|_\infty\right| < \varepsilon/3$. By convergence of $\{f_n(x)\}$ to $f(x)$, there exists $M \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon/3$ for all $n > M$. Then, for $n > M$, we have $\left|f_n(x) - \|f\|_\infty\right| < 2\varepsilon/3$. Taking the limit as n reaches $+\infty$, we get $\varepsilon < \left|\lambda - \|f\|_\infty\right| \leq 2\varepsilon/3$, which is a contradiction. Therefore

$$\lim_{n \to +\infty} \|f_n\|_\infty = \|f\|_\infty \tag{2}$$

Last, we have $\left|\|f_n\|_p - \|f\|_p\right| \leq \|f_n - f\|_p$. Since $f_n \leq f$ and f is bounded, the dominated convergence theorem allows us to conclude that

$$\lim_{n \to +\infty} \|f_n - f\|_p = 0 \tag{3}$$

For all n and all p, the following inequality stands:

$$\begin{aligned} \left|\|f\|_p - \|f\|_\infty\right| \leq & \left|\|f\|_p - \|f_n\|_p\right| \\ & + \left|\|f_n\|_p - \|f_n\|_\infty\right| \\ & + \left|\|f_n\|_\infty - \|f\|_\infty\right| \end{aligned}$$

Let $\varepsilon > 0$. From (2), there exists $N_1 \in \mathbb{N}$ such that $\left|\|f_{N_1}\|_\infty - \|f\|_\infty\right| < \varepsilon$. From (1), there exists $P_1 \in \mathbb{N}$ such that $\left|\|f_{N_1}\|_p - \|f_{N_1}\|_\infty\right| < \varepsilon$ for all $p \geq P_1$. From (3), there exists $P_2 \in \mathbb{N}$ such that $\left|\|f\|_p - \|f_{N_1}\|_p\right| < \varepsilon$ for all $p \geq P_2$. Therefore, for all $p \geq \max(P_1, P_2)$, we have

$$\left|\|f\|_p - \|f\|_\infty\right| < 3\varepsilon$$

so that $\lim_{p \to \infty} \|f\|_p = \|f\|_\infty$

$\square$