Exercise 7. Let V be an inner product space, and for each y in V define $F_y : V \to \mathbb{R}$ by $F_y(x) = (x, y)$.

(a) Show that F_y belongs to V^* and satisfies $\|F_y\| = \|y\|$. (Hint: use the Cauchy-Schwarz inequality; see exercise 3.2.7. To check that $\|F_y\|$ is equal to (rather than less than) $\|y\|$, consider $F_y(y)$.)

(b) Show that if $y \neq y'$, then $F_y \neq F_{y'}$.

(c) Show that if the inner product space V is a Hilbert space and if F belongs to V^*, then there is an element y of V such that $F = F_y$. (Hint: let $y = 0$ if $F = 0$. Otherwise choose a nonzero element v of V such that $(u, v) = 0$ holds whenever $F(u) = 0$ (see exercise 3.2.12), and check that a suitable multiple of v works.)

Proof.

(a) Let K be $\mathbb{R}$ or $\mathbb{C}$, the field on which V is defined, and let $y \in V$. For all $x, z \in V$ and $\alpha \in K$, we have

$$F_y(x + z) = (y, x + z) = (y, x) + (y, z) = F_y(x) + F_y(z)$$
$$F_y(\alpha \cdot x) = (y, \alpha \cdot x) = \alpha \cdot (y, x) = \alpha \cdot F_y(x)$$

so that F_y is linear; since the inner product is defined from $V \times V$ to K, $F_y(x) \in K$ for all $x \in V$. Using Cauchy-Schwarz inequality, we get

$$|F_y(x)| = |(y, x)| \leq \|y\| \cdot \|x\|$$

from which we deduce that F_y is bounded. Therefore $F_y \in V^*$, and $\|F_y\| \leq \|y\|$. Moreover, we have

$$|F_y(y)| = |(y, y)| = \|y\|^2$$

so there exists an element $x \in V$ such that $|F_y(x)| = \|y\| \cdot \|x\|$. From this we deduce that $\|F_y\| \geq \|y\|$, and, combining both inequalities, that $\|F_y\| = \|y\|$.

(b) Let y, y' be distinct elements of V. We have

$$\begin{aligned} |F_y(y - y') - F_{y'}(y - y')| &= |(y, y - y') - (y', y - y')| \\ &= |(y - y', y - y')| = \|y - y'\|^2 > 0 \end{aligned}$$

F_y and $F_{y'}$ take different values on the vector $y - y'$, and are therefore different.

(c) If $F = 0$, then $F = F_0$. Otherwise, there exists $w \in V$ such that $F(w) \neq 0$.

The set $k_F = \{u \in V \mid F(u) = 0\}$ is nonempty (it contains the null vector), and, by linearity of F, is a K-vector space. As a Hilbert space, V is complete, and hence closed; let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of elements of k_F that converges to some $u \in V$. We have

$$\begin{aligned} |F(u)| &\le |F(u) - F(u_n)| + |F(u_n)| \\ &\le \|F\| \cdot \|u - u_n\| \end{aligned}$$

since $F(u_n) = 0$ and F is continuous. From this we deduce that for all $\varepsilon > 0$, $|F(u)| \le \varepsilon$, so that $F(u) = 0$ and $u \in k_F$. k_F is then a closed subspace of V.

From exercise 3.2.12, we deduce the existence of a vector $h \in k_F$ satisfying $\|w - h\| = \inf\{\|w - x\|, x \in k_F\}$, and such that $w - h$ is orthogonal to k_F. Note that $F(w-h) = F(w) - F(h) = F(w)$, so that $w - h \notin k_F$.

Let $v = \frac{w-h}{F(w-h)}$, for all $x \in V$, we have

$$F(x) = F(x) \cdot F(v) = F\big(F(x) \cdot v\big)$$

so that $x - F(x) \cdot v = z \in k_F$. Moreover,

$$\begin{aligned} (x, v) &= \big(F(x) \cdot v + z, v\big) = F(x) \cdot (v, v) + (z, v) \\ &= F(x) \cdot \|v\|^2 \end{aligned}$$

since v is orthogonal to k_F. From this we deduce that

$$F = F_y \quad \text{with } y = \frac{v}{\|v\|^2}$$

$\square$