Exercise 4. Show that if $\mathscr{A}$ is an algebra of sets, and if $\cup_n A_n$ belongs to $\mathscr{A}$ whenever $\{A_n\}$ is a sequence of disjoint sets in $\mathscr{A}$, then $\mathscr{A}$ is a σ-algebra.

Proof. Since $\mathscr{A}$ is an algebra, it contains $\varnothing$ and is stable by complementation. To show that $\mathscr{A}$ is a σ-algebra, it is enough to show that it is stable by countable unions. Let $\{A_n\}_{n\in\mathbb{N}}$ be a family of sets of $\mathscr{A}$. Define a family $\{B_n\}_{n\in\mathbb{N}}$ of sets of $\mathscr{A}$ by

$$\begin{aligned} B_0 &= A_0 \\ B_{n+1} &= A_{n+1} - \cup_{k=0}^{n} A_k \end{aligned}$$

Then $\{B_n\}_{n\in\mathbb{N}}$ is a family of disjoint sets of $\mathscr{A}$, so $\cup_{n\in\mathbb{N}} B_n \in \mathscr{A}$. However, $\cup_{n\in\mathbb{N}} B_n = \cup_{n\in\mathbb{N}} A_n$, so $\mathscr{A}$ is stable by countable unions, and is therefore a σ-algebra.

□