Exercise 8. (This exercise depends on the Hahn-Banach theorem, which is stated without proof in appendix E.) Let V be the subspace of ℓ^∞ consisting of those sequences $\{x_n\}$ for which $\lim_n x_n$ exists, and let $F_0 \colon V \to \mathbb{R}$ be defined by $F_0(\{x_n\}) = \lim_n x_n$.

(a) Show that F_0 is a bounded linear functional on V and that $\|F_0\| = 1$.

(b) Let F be a linear functional on ℓ^∞ that satisfies $\|F\| = 1$ and agrees with F_0 on V (see Theorem E.7). Show that if $\{x_n\}$ is a nonnegative element of ℓ^∞ (that is, if $\{x_n\}$ belongs to ℓ^∞ and satisfies $x_n \geq 0$ for each n), then $F(\{x_n\}) \geq 0$. (Hint: consider the sequence $\{x'_n\}$ defined by $x'_n = x_n - c$, where c is a suitably chosen constant.)

(c) For each subset A of $\mathbb{N}$ let $\{\chi_{A,n}\}_{n=1}^\infty$ be the sequence defined by

$$\chi_{A,n} = \begin{cases} 1 & \text{if } n \in A \\ 0 & \text{if } n \notin A \end{cases}$$

Show that the function $\mu \colon \mathscr{P} \to \mathbb{R}$ defined by $\mu(A) = F(\{X_{A,n}\})$ is a finitely additive measure, but is not countably additive.

For reference, theorem E.7 is:

Theorem (Hahn-Banach)**.** *Let E be a normed real or complex linear space, let F be a linear subspace of E, and let φ_0 be a continuous linear functional on F. Then there is a continuous linear funcional φ on E such that $\|\varphi\| = \|\varphi_0\|$ and such that φ_0 is the restriction of φ to F. In other words, φ_0 can be extended to a continuous linear functional on all of E without increasing its norm.*

Proof.

(a) The linearity of F_0 is an immediate consequence of the linearity of limits. Moreover, for all $\{x_n\} \in V$, $\{|x_n|\}$ converges since $\{x_n\}$ converges, and

$$\begin{aligned} \forall n \in \mathbb{N}, \quad x_n &\leq |x_n| \\ \lim_{n\to\infty} x_n &\leq \lim_{n\to\infty} |x_n| \\ \left|\lim_{n\to\infty} x_n\right| &\leq \lim_{n\to\infty} |x_n| \\ |F_0(\{x_n\})| &\leq \sup\{|x_n|, n \in \mathbb{N}\} = \|x_n\|_\infty \end{aligned}$$

so that F_0 is bounded and $\|F_0\| \leq 1$. Since we also have

$$|F_0(\{1, 1, \dots\})| = 1 = \|\{1, 1, \dots\}\|_\infty$$

we deduce that $\|F_0\| = 1$.

(b) First suppose that for all $n \in \mathbb{N}$, $0 \leq x_n \leq 1$, $\inf_{n\in\mathbb{N}} = 0$ and $\|x_n\|_\infty = 1$. Then we have

$$\left\| x_n - \frac{1}{2} \right\|_\infty = \frac{1}{2}$$

From the linearity of F and the fact that $\|F\| = 1$, we deduce

$$\begin{aligned}
\left\| x_n - \frac{1}{2} \right\|_\infty &\geq \left| F\left(\left\{ x_n - \frac{1}{2} \right\} \right) \right| \\
\frac{1}{2} &\geq \left| F\left(\{x_n\}\right) - F\left(\left\{ \frac{1}{2} \right\}_{n\in\mathbb{N}} \right) \right| \\
\frac{1}{2} &\geq \left| F\left(\{x_n\}\right) - \frac{1}{2} \right| \\
-\frac{1}{2} &\leq F\left(\{x_n\}\right) - \frac{1}{2} \leq \frac{1}{2}
\end{aligned}$$

which in turn gives us

$$F\left(\{x_n\}\right) \geq 0 \tag{1}$$

Suppose now that $\{x_n\} \in \ell^\infty$ is such that $0 \leq x_n$ for all $n \in \mathbb{N}$, and let $c = \inf_{n\in\mathbb{N}}\{x_n\} \geq 0$. If $\|x_n - c\|_\infty = 0$, then for all $n \in \mathbb{N}$, $x_n = c$, in which case $F(\{x_n\}) = F(\{c\}_{n\in\mathbb{N}}) = c \geq 0$. Otherwise, $\|x_n - c\|_\infty > 0$, and the sequence defined by

$$\forall n \in \mathbb{N}, \quad y_n = \frac{x_n - c}{\|x_n - c\|_\infty}$$

satisfies $y_n \geq 0$ for all $n \in \mathbb{N}$, $\inf_{n\in\mathbb{N}}\{y_n\} = 0$ and $\|y_n\|_\infty = 1$. We can then apply (1) to deduce that

$$\begin{aligned}
F(\{y_n\}) &\geq 0 \\
F\left(\frac{x_n - c}{\|x_n - c\|_\infty} \right) &\geq 0 \\
\frac{1}{\|x_n - c\|_\infty} \left(F\left(\{x_n\}\right) - c \right) &\geq 0 \\
F\left(\{x_n\}\right) &\geq c \geq 0
\end{aligned}$$

(c) $\mathscr{P}(\mathbb{N})$ is a σ-algebra on $\mathbb{N}$; for all $A \subset \mathbb{N}$, the sequence $\{\chi_{A,n}\}$ is nonnegative and bounded. From the preceding points, we deduce that

$$\begin{aligned}
\mu(A) &= F(\{\chi_{A,n}\}) \geq 0 \\
\mu(\varnothing) &= F(\{0\}_{n\in\mathbb{N}}) = 0
\end{aligned}$$

Let A, B be disjoint subsets of $\mathbb{N}$. We have $\chi_{A\cup B,n} = \chi_{A,n} + \chi_{B,n}$, and, from the lineairy of F,

$$\begin{aligned}
\mu(A \cup B) &= F(\{\chi_{A\cup B,n}\}) = F(\{\chi_{A,n}\} + \{\chi_{B,n}\}) \\
&= F(\{\chi_{A,n}\}) + F(\{\chi_{B,n}\}) = \mu(A) + \mu(B)
\end{aligned}$$

so that μ is a finitely additive measure.

Let $A = \mathbb{N}$ and $A_n = \{n\}$ for all $n \in \mathbb{N}$. The A_n are pairwise disjoint and $A = \cup_{n\in\mathbb{N}} A_n$. However,

$$\begin{aligned} \mu(A) = \mu(\mathbb{N}) &= F(\{1\}_{n\in\mathbb{N}}) = 1 \\ \forall n \in \mathbb{N}, \quad \mu(A_n) &= F(\{0, 0, \dots, n, 0, \dots\}) = 0 \end{aligned}$$

so that $\mu(A) \neq \sum_{n=0}^{\infty} \mu(A_n)$, and μ is not countably additive.

$\square$