Exercise 13. Let V be a Banach space, and let v and $v_1, v_2, \dots$ belong to V. The series $\sum_{k=1}^{+\infty} v_k$ is said to *converge unconditionally* to v if for each positive ε there is a finite subset F_ε of $\mathbb{N}$ such that $\left\|\sum_{k\in F} v_k - v\right\| < \varepsilon$ holds whenever F is a finite subset of $\mathbb{N}$ that includes F_ε.

(a) Show that if $\sum_{k=0}^{+\infty} v_k$ converges absolutely, then it converges unconditionally to some point in V.

(b) Show that the converse of part (a) holds if $V = \mathbb{R}$.

(c) Show that the converse of part (a) is not true in general. (Hint: let V be c_0, l^2, or l^∞.)

Proof.

(a) Suppose that $\sum_k v_k$ converges uniformly and note $v = \sum_{k=0}^{+\infty} v_k$. Let $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\forall n \geq N, \quad m \geq N, \quad \left\| \sum_{k=n}^{m} v_k \right\| < \sum_{k=n}^{m} \|v_k\| < \varepsilon$$

Let $F_\varepsilon = \{0, 1, \dots, N\}$, and let $F_\varepsilon \subset F \subset \mathbb{N}$ for some finite F.

$$\begin{aligned}
\left\| \sum_{k\in F} v_k - v \right\| &= \left\| \left(\sum_{k\in F_\varepsilon} v_k - v \right) + \sum_{k\in F - F_\varepsilon} v_k \right\| \\
&\leq \left\| \sum_{k=0}^{N} v_k - v \right\| + \left\| \sum_{k=N+1}^{+\infty} v_k \right\| \\
&\leq \left\| \sum_{k=0}^{N} v_k - v \right\| + \sum_{k=N+1}^{+\infty} \|v_k\| \\
&\leq 2\varepsilon
\end{aligned}$$

so that if $\sum_k v_k$ converges uniformly, then it converges unconditionally.

(b) Let $\sum_k v_k$ be a series of real numbers that converges unconditionally to $v \in \mathbb{R}$. Let $\varepsilon > 0$ and F_ε a finite subset of $\mathbb{N}$ such that

$$\left| \sum_{k\in F_\varepsilon} v_k - v \right| < \varepsilon$$

Let $n = \max F_\varepsilon$ if $F_\varepsilon \neq \varnothing$ and $n = 1$ otherwise, let $m \geq n$, and let $F^- \subset \{n+1, n+2, \dots, m\}$ such that for all i, $n < i \leq m$, $v_i < 0$

implies $v_i \in F^-$. Then

$$
\begin{aligned}
\sum_{k \in F^-} |v_k| &= \left| \sum_{k \in F^-} v_k \right| \\
&= \left| \sum_{k \in F_\varepsilon \cup F^-} v_k - \sum_{k \in F_\varepsilon} v_k \right| \\
&\leq \left| \sum_{k \in F_\varepsilon \cup F^-} v_k - v \right| + \left| \sum_{k \in F_\varepsilon} v_k - v \right| \\
&\leq 2\varepsilon \qquad (1)
\end{aligned}
$$

Defining similarly F^+ to be the nonnegative terms of $\sum_k v_k$ with k between $n+1$ and m, we get

$$
\sum_{k \in F^+} |v_k| \leq 2\varepsilon \qquad (2)
$$

Combining (1) and (2) we get:

$$
\sum_{n < k \leq m} |v_k| \leq 4\varepsilon
$$

The above inequality is valid for any value of m, as soon as $n \geq \max F_\varepsilon$, so that $\sum_k |v_k|$ is a Cauchy series of real numbers. Since $\mathbb{R}$ is complete, we deduce that $\sum_k |v_k|$ converges in $\mathbb{R}$, so that $\sum_k v_k$ is absolutely convergent.

(c) Let $V = c_0$, the $\mathbb{R}$-vector space of real number sequences $\{x_n\}$ such that $\lim_n x_n = 0$, equipped with the norm $\| \cdot \|_\infty$. Define a sequence $\{v_n\}$ of elements of c_0, noted $v_n = (v_{n,0}, v_{n,0}, \cdots)$ such that

$$
\forall k \in \mathbb{N},\ v_{n,k} = \begin{cases} \frac{1}{n+1} & \text{if } k = n \\ 0 & \text{otherwise} \end{cases}
$$

Then, for all $n \in \mathbb{N}$, $\|v_n\| = 1/(n+1)$, so that $\sum_n v_n$ is not absolutely convergent. Let

$$
v = \left(1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right)
$$

For all $N \in \mathbb{N}$,

$$
\left\| \sum_{k=0}^{N} v_k - v \right\| = \left\| \left(0, 0, \ldots, \frac{1}{N+2}, \frac{1}{N+3}, \ldots\right) \right\| = \frac{1}{N+2}
$$

Let $\varepsilon > 0$ and choose N such that $1/(N+2) < \varepsilon$. Let $F_\varepsilon = \{0, 1, 2, \ldots, N\}$, and let F be any subset of $\mathbb{N}$ that contains F_ε. Then

$$\left\| \sum_{k \in F} v_k - v \right\| \leq \frac{1}{N+2} < \varepsilon$$

so $\sum_k v_k$ converges unconditionally to v.

$\square$