Exercise 6. *Let μ be a nonzero finite Borel measure on $\mathbb{R}$, and let $F : \mathbb{R} \to \mathbb{R}$ be the function defined by $F(x) = \mu((-\infty, x])$. Define a function g on the interval $(0, \lim_{x\to+\infty} F(x))$ by*

$$g(x) = \inf \{t \in \mathbb{R} : F(t) \geq x\}.$$

1. *Show that g is nondecreasing, finite valued, and Borel measurable.*
2. *Show that $\mu = \lambda g^{-1}$. (Hint: start by showing that $\mu(B) = \lambda\left(g^{-1}(B)\right)$ when B has the form $(-\infty, b]$.)*

Proof.

1. g is non-decreasing: let $0 < x < y < \lim_{+\infty} F$, we have for all t in $\mathbb{R}$ $F(t) \geq y \implies F(t) \geq x$, so $\{t \in \mathbb{R} \mid F(t) \geq y\} \subset \{t \in \mathbb{R} \mid F(t) \geq x\}$. As $g(x)$ and $g(y)$ are the infimums of these sets, $g(x) \leq g(y)$.

   g is finite-valued: F is non-negative, non-decreasing, right-continuous, and $\lim_{-\infty} F = 0$ (Proposition 1.3.9). Further, μ being finite, F is bounded and $\lim_{+\infty} F = \mu(\mathbb{R})$. Let x be in $(0, \mu(\mathbb{R}))$, there exists t_0 in $\mathbb{R}$ such that $F(t_0) \geq x$ (since $\mu(\mathbb{R}) = \lim_{+\infty} F$ and μ is non-zero), so $\{t \in \mathbb{R} \mid F(t) \geq x\}$ is non empty. Also, there exists t_1 in $\mathbb{R}$ such that $0 \leq F(t_1) \leq x$ (since $0 = \lim_{-\infty} F$), so $\{t \in \mathbb{R} \mid F(t) \geq x\}$ has a lower bound in $\mathbb{R}$, and therefore an infimum in $\mathbb{R}$, so $g(x)$ is finite for all x in $(0, \mu(\mathbb{R}))$.

   g is Borel-measurable: as g is non-decreasing on the interval $(0, \mu(\mathbb{R}))$, it is Borel-measurable on this interval.

2. Let (F_n) be a non-decreasing series of non-negative, non-decreasing, right-continuous simple functions such that for all $x \in \mathbb{R}$, $\lim_{n\to\infty} F_n(x) = F(x)$. For each n, define on $(0, \lim_{+\infty} F_n)$ a function g_n by $g_n(x) = \inf\{t \in \mathbb{R} \mid F_n(t) \geq x\}$. Each F_n has the same properties as F, which implies that each g_n has the same properties as g (listed in the preceding point).

   Further, since the F_n are right-continuous and non-decreasing, for each $x \in \mathbb{R}$, $\{t \in \mathbb{R} \mid F_n(t) = x\}$ is an interval. This allows us to write $F_n(x) = \sum_{i=0}^{K_n} F_n(a_i)\chi_{[a_i, a_{i+1})}$.

   From the above, we deduce that on each interval $[F(a_i), F(a_{i+1}))$, g_n is constant, equal to a_i.

   Let $b \in \mathbb{R}$, we write, for the biggest i such that $F_n(a_i) \leq F_n(b)$:

$$\begin{aligned} F_n(b) =& \big(F_n(b) - F_n(a_i)\big) + \big(F_n(a_i) - F_n(a_{i-1})\big) \\ &+ \cdots + \big(F_n(a_1) - F_n(a_0)\big) + \big(F_n(a_0) - 0\big) \\ F_n(b) =& \big(F_n(b) - F_n(a_i)\big) + \lambda\big(\{x \in \mathbb{R} \mid g_n(x) = a_i\}\big) \\ &+ \cdots + \lambda\big(\{x \in \mathbb{R} \mid g_n(x) = a_0\}\big) + (F(a_0) - 0) \end{aligned}$$

Let $\epsilon > 0$. Since F_n is simple, there exists k such that $0 \le k \le K_n$ and $F_n(b) = F_n(a_k)$. And since $lim_{-\infty}F = 0$, we can choose n such that $0 \le F_n(a_0) < \epsilon$. By convergence of $\big(F_n(b)\big)$ to $F(b)$, we can, by increasing n as necessary, also have $\big|F_n(b) - F(b)\big| < \epsilon$. Combining the above, we get

$$\left| F(b) - \sum_{i=0}^{k} \lambda\big(\{x \in \mathbb{R} \mid g_n(x) = a_i\}\big) \right| \le 2\epsilon$$

Noting that g_n only takes the discrete values a_i, the above inequality can be rewritten as

$$\left|F(b) - \lambda\left(g_n^{-1}\left(-\infty, b\right]\right)\right| \le 2\epsilon$$

Since the g_n are non-decreasing, for each $n \in \mathbb{N}$ and each $x \in (0, lim_{+\infty}F_n)$, $g_{n+1}(x) \le b \implies g_n(x) \le b$, from which we get $g_{n+1}^{-1}(-\infty, b] \subset g_n^{-1}(-\infty, b] \subset g_0^{-1}(-\infty, b] \subset (0, lim_{+\infty}F)$, and this limit is a finite real number. From the above, we conclude that

$$\lim_{n \to +\infty} \lambda\big(g_n^{-1}(-\infty, b]\big) = \lambda\left(\bigcap_{n=0}^{+\infty} g_n^{-1}(-\infty, b]\right)$$

With our previous notation $F_n(b) = F_n(a_k)$, the simplicity of F_n implies that $a_k = g_n\big(F_n(b)\big)$. Note that for n fixed in $\mathbb{N}$, $(g_k \circ F_n(b))_{k \ge n}$ is a non-decreasing sequence converging to $g\big(F_n(b)\big)$ (since $F_n(b)$ is in the domain of every g_n and the $g_n(x)$ are non-decreasing, simply converging to $g(x)$). $\big(g \circ F_n(b)\big)_{n \in \mathbb{N}}$ is also a non-decreasing sequence, converging to $g\big(F(b-)\big)$.

Note that $g(F(b-)) \le b$, since $F(b-) \le F(b)$. If F is left-continuous at b, then $g(F(b-)) = g(F(b)) = b$. Otherwise, let us take x_0 such that $F(b-) < x_0 < F(b)$. By definition of g, we have $g(x_0) = b$. However, g is right-continuous at every point in its domain, so $g(F(b-)) = g(x_0) = b$. This allows us to deduce that

$$\bigcap_{n \in \mathbb{N}} g_n^{-1}(-\infty, b] = g^{-1}(-\infty, b]$$

from which we conclude that $\mu(-\infty, b] = \lambda\big(g^{-1}(-\infty, b]\big)$

Finally, we note that for any Borel set B,

$$\lambda(g^{-1}B) = \int_{\mathbb{R}} \chi_{g^{-1}B} \,\mathrm{d}\lambda = \int_{\mathbb{R}} \chi_B \circ g \,\mathrm{d}\lambda = \int_B \mathrm{d}(\lambda g^{-1}) = \lambda g^{-1}(B)$$

Thus the measures μ and λg^{-1} agree on the open sets $(-\infty, b]$ for any $b \in \mathbb{R}$. Therefore they agree on the σ-algebra generated by these open sets, which is the Borel σ-algebra. μ and λg^{-1} are therefore equal on $\mathscr{B}(\mathbb{R})$.

$\square$