Exercise 2. Show that $L^{\infty}([a,b])$ is not separable. (Hint: consider the elements of $L^{\infty}([a,b])$) determined by the characteristic functions of the sets [a,c] for a < c < b.

Proof. Let X be the set of characteristic functions of the sets [a, c] for a < c < b, and let $c_1, c_2 \in [a, b]$ such that $c_1 < c_2$. Then

$$\|\chi_{[a,c_1]} - \chi_{[a,c_2]}\|_{\infty} = \|\chi_{(c_1,c_2)}\|_{\infty} = 1$$

Suppose that $(\varphi_n)_{n\in\mathbb{N}}$ is a countable dense family of functions in $\mathscr{L}^{\infty}([a,b])$, and let $\varepsilon > 0$. For all $n, m \in \mathbb{N}$ such that

$$\left\|\chi_{[a,c_1]} - \varphi_n\right\|_{\infty} < \frac{\varepsilon}{3}$$
$$\left\|\chi_{[a,c_2]} - \varphi_m\right\|_{\infty} < \frac{\varepsilon}{3}$$

we have

$$\|\varphi_{n} - \varphi_{m}\|_{\infty} = \|(\varphi_{n} - \chi_{[a,c_{1}]}) + (\chi_{[a,c_{1}]} - \chi_{[a,c_{2}]}) + (\chi_{[a,c_{2}]} - \varphi_{m})\|_{\infty}$$

$$\geq \|\chi_{(c_{1},c_{2})}\|_{\infty} - (\|\chi_{[a,c_{1}]} - \varphi_{n}\|_{\infty} + \|\chi_{[a,c_{2}]} - \varphi_{m}\|_{\infty})$$

$$\geq 1 - \frac{2}{3}\varepsilon$$
(1)

which is positive for ε sufficiently small. Choose such an ε , and, for all $\chi_{[a,c]} \in X$, choose one φ_n such that $\|\chi_{[a,c]} - \phi_n\|_{\infty} < \varepsilon/3$, and call it φ_c . The function

$$\psi: X \to \{\varphi_n, n \in \mathbb{N}\}$$
$$\chi_{[a,c]} \mapsto \varphi_c$$

is injective: if $c \neq d$, then from (1) we get $\|\varphi_c - \varphi_d\|_{\infty} > 0$, so that $\varphi_c \neq \varphi_d$. Since X is uncountable, we deduce that $\{\varphi_n, n \in \mathbb{N}\}$ is also uncountable, which is a contradiction. Therefore the family $(\varphi_n)_{n \in \mathbb{N}}$ does not exist, and $L^{\infty}([a,b])$ is not separable.