Exercise 6. Let V be a normed linear space. Show that the dual V^* of V is complete under the norm $\| \cdot \|$ defined above. (Hint: Let $\{F_n\}$ be a Cauchy sequence in V^*. Show that for each v in V the sequence $\{F_n(v)\}$ is a Cauchy sequence in $\mathbb{R}$ (or in $\mathbb{C}$) and so is convergent. Then show that the formula $F(v) = \lim_n F_n(v)$ defines a bounded linear functional on V and that $\lim_n \|F_n - F\| = 0$.)

Proof. Let K ($K = \mathbb{R}$ or $K = \mathbb{C}$) be the field of scalars for V. Let $\varepsilon > 0$; since the sequence $\{F_n\}$ is a Cauchy sequence, we have

$$\exists N \in \mathbb{N}, \quad \forall n, m \in \mathbb{N}, \quad n \geq N \wedge m \geq N \implies \|F_n - F_m\| < \varepsilon$$

where $\| \cdot \|$ is the norm of linear functionals on V. Let $v \in V$, we have

$$|F_n(v) - F_m(v)| = |(F_n - F_m)(v)| \leq \|F_n - F_m\| \cdot \|v\| < \|v\| \cdot \varepsilon \tag{1}$$

for all $n, m \in \mathbb{N}$ such that $n \geq N$ and $m \geq N$. From this we deduce that the sequence $\{F_n(v)\}$ is a Cauchy sequence of elements of K. Since K is complete, $\{F_n(v)\}$ converges to some scalar in K. Let

$$\begin{aligned} F : V &\to K \\ v &\mapsto \lim_n F_n(v) \end{aligned}$$

For all $u, v \in V$ and all $\alpha \in K$ we have

$$\begin{aligned} F(u + v) &= \lim_n F_n(u + v) = \lim_n \big(F_n(u) + F_n(v)\big) \\ &= \lim_n F_n(u) + \lim_n F_n(v) = F(u) + F(v) \\ F(\alpha \cdot u) &= \lim_n F_n(\alpha \cdot u) = \lim_n \big(\alpha \cdot F_n(u)\big) \\ &= \alpha \cdot \lim_n F_n(u) = \alpha \cdot F(u) \end{aligned}$$

so that F is linear.

Let $B_1 = \{v \in V \mid \|v\| \leq 1\}$ and $v \in B_1$. For all nonzero $v \in V$ and all $\varphi \in V^*$, we have $\varphi(\frac{v}{\|v\|}) = \frac{1}{\|v\|}\varphi(v)$, so that

$$\begin{aligned} \|\varphi\| &= \inf\{M \in \mathbb{R} \mid \forall v \in V, |\varphi(v)| \leq M \cdot \|v\|\} \\ &= \inf\{M \in \mathbb{R} \mid \forall v \in B_1, |\varphi(v)| \leq M\} \end{aligned}$$

The convergence of $\{F_n(v)\}$ to $F(v)$ implies the existence of $n_0 \in \mathbb{N}$ such that $|F_{n_0}(v) - F(v)| < 1$. From this we get

$$|F(v)| \leq |F(v) - F_{n_0}(v)| + |F_{n_0}(v)| \leq 1 + \|F_{n_0}\| \cdot \|v\| \leq 1 + \|F_{n_0}\|$$

so that F is bounded, and therefore continuous.

Let $v \in B_1$ and $n \in \mathbb{N}$ such that $n \geq N$. For all $m \in \mathbb{N}$ such that $m \geq M$, we have by (1)

$$\begin{aligned} |F_n(v) - F(v)| &\leq |F_n(v) - F_m(v)| + |F_m(v) - F(v)| \\ &\leq \varepsilon \cdot \|v\| + \varepsilon \leq 2\varepsilon \end{aligned}$$

where M depends only on v. Therefore

$$\forall v \in B_1, \quad \forall n \in \mathbb{N}, \quad |(F_n - F)(v)| \leq 2\varepsilon$$

which implies $\|F_n - F\| \leq 2\varepsilon$. Thus $\lim_n \|F_n - F\| = 0$, and V^* is complete for the norm of linear functionals.

$\square$