Exercise 7. Let V be a vector space over $\mathbb{R}$. A function $(\cdot,\cdot) : V \times V \to \mathbb{R}$ is an *inner product* on V if

(i) $(x, x) \geq 0$

(ii) $(x, x) = 0$ if and only if $x = 0$

(iii) $(x, y) = (y, x)$

(iv) $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$

hold for all x, y, z in V and all α, β in $\mathbb{R}$. An *inner product space* is a vector space, together with an inner product on it. The norm $\|\cdot\|$ associated with the inner product $(\cdot,\cdot)$ is defined by $\|x\| = \sqrt{(x, x)}$.

(a) Prove that an inner product satisfies the *Cauchy-Schwartz inequality:* if $x, y \in V$, then $|(x, y)| \leq \|x\|\|y\|$. (Hint: define a function $p : \mathbb{R} \to \mathbb{R}$ by $p(t) = \|x\|^2 + 2t(x, y) + t^2\|y\|^2$ and note that $p(t) = \|x + ty\|^2 \geq 0$ holds for each real t; then recall that a quadratic polynomial at^2+bt+c is nonnegative for each t if and only if $b^2 - 4ac \leq 0$).

(b) Verify that the norm associated with $(\cdot,\cdot)$ is indeed a norm. (Hint: use the Cauchy-Schwartz inequality when checking the triangle inequality).

Proof.

(a) Fix $x, y \in V$ and consider $p(t) = \|x + ty\|^2$ for all $t \in \mathbb{R}$. We have:

$$\begin{aligned} p(t) &= (x + ty, x + ty) = (x, x) + (x, ty) + (ty, x) + (y, y) \\ &= \|x\|^2 + 2t(x, y) + t^2\|y\|^2 \end{aligned}$$

by the rules of the inner product, and $p(t) \geq 0$ since $(z, z) \geq 0$ for all $z \in V$. $p(t)$ can only be positive if, as a polynomial in $t \in \mathbb{R}$ with real coefficients, it does not have any real roots, which in turns necessitates that its discriminant $4(x, y)^2 - 4\|x\|^2\|y\|^2$ be negative. This gives us $(x, y)^2 \leq \|x\|^2\|y\|^2$, which in turn implies that $|(x, y)| \leq \|x\|\|y\|$.

(b) Since $(x, x) \geq 0$ for all $x \in V$ and $(x, x) = 0$ if and only $x = 0$, $\|x\| = \sqrt{(x, x)} \geq 0$ and is zero if and only if $x = 0$. For all $\alpha \in \mathbb{R}$ and $x \in V$, $\|\alpha x\|^2 = (\alpha x, \alpha x) = \alpha(x, \alpha x) = \alpha(\alpha x, x) = \alpha^2(x, x) \geq 0$, so that $\|\alpha x\| = |\alpha|\|x\|$. Finally, for $x, y \in V$, we have:

$$\begin{aligned} &\|x + y\|^2 = \|x\|^2 + 2(x, y) + \|y\|^2 \\ &\|x + y\|^2 \leq \|x\|^2 + 2|(x, y)| + \|y^2\| \\ &\|x + y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \quad \text{by Cauchy-Schwartz} \\ &\|x + y\|^2 \leq (\|x\| + \|y\|)^2 \end{aligned}$$

and since both sides of the last inequality are the squares of positive reals, we conclude that $\|x+y\| \leq \|x\| + \|y\|$, and the norm associated with the inner product is a norm.

$\square$