Exercise 8. Let $(\cdot, \cdot)$ be an inner product on the real vector space V, and let $\|\cdot\|$ be the associated norm. Show that

(a) $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

(b) $\|x+y\|^2 - \|x-y\|^2 = 4(x,y)$

hold for all x, y in V. The identity in part (a) is called the *parallelogram law*

Proof.

Let $x, y \in V$, we have:

$$\begin{aligned}\|x+y\|^2 &= (x+y, x+y) = (x,x) + 2(x,y) + (y,y) \\ \|x-y\|^2 &= (x-y, x-y) = (x,x) - 2(x,y) + (y,y)\end{aligned}$$

By adding these two equalities, we get:

$$\|x+y\|^2 + \|x-y\|^2 = 2(x,x) + 2(y,y) = 2\|x\|^2 + 2\|y\|^2$$

and by substracting the second from the first:

$$\|x+y\|^2 - \|x-y\|^2 = 4(x,y)$$

□