Exercise 1. Let $(X, \mathscr{A}, \mu)$ be a measure space, and let A and $A_1, A_2, \ldots$ belong to $\mathscr{A}$. Show that

(a) $\{\chi_{A_n}\}$ converges to 0 in measure if and only if $\lim_n \mu(A_n) = 0$.

(b) $\{\chi_{A_n}\}$ converges to 0 almost everywhere if and only if $\mu(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k) = 0$

(c) $\{\chi_{A_n}\}$ converges to χ_A almost everywhere if and only if the three sets A, $\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$ and $\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k$ differ only by μ-null sets (Hint: see exercise 2.1.1)

For reference, exercise 2.1.1 state that for (A_k) a sequence of subsets of X, and $B = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k$ and $C = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$, we have $\liminf_k \chi_{A_k} = B$ and $\limsup_k \chi_{A_k} = C$.

Proof.

(a) Let $\epsilon > 0$, we have $\{x \in X \mid |\chi_{A_n}(x)| > \epsilon\} = \{x \in X \mid \chi_{A_n}(x) = 1\} = A_n$, from which we deduce $\mu\big(\{x \in X \mid |\chi_{A_n}(x)| > \epsilon\}\big) = \mu(A_n)$. So $\{\chi_{A_n}\}$ converges in measure to 0 is equivalent to $\mu(A_n)$ converges to 0.

(b) Let $\epsilon > 0$ and $n \in \mathbb{N}$. As before, we note that $A_n = \{x \in X \mid |\chi_{A_n}(x)| > \epsilon\}$. Define $B_n = \cup_{k=n}^{\infty} A_k$ and $B = \cap_{n=1}^{\infty} B_n$, and remark that $\{x \in X \mid \{\chi_{A_n}(x)\}$ does not converge to 0 $\}$ is a subset of B. Therefore if $\mu(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k) = 0$, then $\{\chi_{A_n}\}$ converges to 0 almost everywhere.

Conversely, suppose that $\{\chi_{A_n}\}$ converges to 0 almost everywhere, and choose a μ-null set $N \subset \mathscr{A}$ such that $\{x \in X \mid \{\chi_{A_n}\}$ does not converge to 0 $\} \subset N$. Define $D_n = A_n \cap N^C$; the series (χ_{D_n}) converges simply to 0 everywhere. Let $D = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} D_k$ and $x \in D$. Then for all n, $\exists k \geq n, \quad \chi_{D_n}(x) = 1$, which contradicts the simple convergence of (χ_{D_n}) to 0 on N^C. It follows that $D = \varnothing$, and thus $\mu(D) = 0$.

Next, define $S = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k$. We have:

$$
\begin{aligned}
\mu(S) &= \mu(S \cap N) + \mu(S \cap N^C) \\
\mu(S \cap N) &\leq \mu(N) = 0 \\
\mu(S \cap N^C) &= \mu\left(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} \left(A_k \cap N^C\right)\right) = \mu(D) = 0
\end{aligned}
$$

which gives us $\mu(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k) = 0$

(c)

$\square$