Exercise 8. Find all the σ-algebras on $\mathbb{N}$.

Proof. Let $\mathscr{A}$ be a σ-algebra on $\mathbb{N}$. For all $n \in \mathbb{N}$, let $C_n = \{A \in \mathscr{A} \mid n \in A\}$ and $U_n = \{m \in \mathbb{N} \mid C_m = C_n\}$. The family of sets $\{U_n\}_{n\in\mathbb{N}}$ is a partition of $\mathbb{N}$:

- for all $n \in \mathbb{N}$, $n \in U_n$ so U_n is nonempty, and $\mathbb{N} \subset \cup_{n\in\mathbb{N}} U_n$.
- if $n, m \in \mathbb{N}$ are such that $U = U_n \cap U_m$ is nonempty, then let $p \in U$. Since $p \in U_n$, we have $C_p = C_n$, and since $p \in U_m$, we also have $C_p = C_m$; from which we deduce that $C_n = C_m$. This last equality implies that $U_n = U_m$, so any two sets in $\{U_n\}_{n\in\mathbb{N}}$ are either disjoint or equal.

Let $\mathscr{U} = \{\cup_{n\in A} U_n, A \subset \mathbb{N}\}$. Then $\mathscr{U}$ is a σ-algebra on $\mathbb{N}$:

- $\mathscr{U}$ is stable under countable unions: let $\{A_n\}_{n\in\mathbb{N}}$ be a countable family of elements of $\mathscr{U}$. For all n, there exists a subset B_n of $\mathbb{N}$ such that $A_n = \cup_{k\in B_n} U_k$. As subsets of the countable set $\mathbb{N}$, the B_n are countable, so that

  $$\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} \bigcup_{k\in B_n} U_k = \bigcup_{k\in B_0\cup B_1\cup\ldots} U_k$$

  is a countable union of sets U_k, and therefore an element of $\mathscr{U}$.
- We have $\mathbb{N} = \cup_{n\in\mathbb{N}} \{n\} \subset \cup_{n\in\mathbb{N}} U_n$, and conversely $\cup_{n\in\mathbb{N}} U_n \subset \mathbb{N}$, so $\mathbb{N} \in \mathscr{U}$.
- Let $B \in \mathscr{U}$; there exists a subset A of $\mathbb{N}$ such that $B = \cup_{n\in A} U_n$. Suppose that $x \notin B$, then $x \notin U_n$ for all n in A. But since $\{U_n\}_{n\in\mathbb{N}}$ cover $\mathbb{N}$, we deduce that $x \in \cup_{n\notin A} U_n$. And since $\mathbb{N}$ is countable, A^c is countable, and this union ranges over a countable set.

  Conversely, let $x \in \cup_{n\notin A} U_n$. There exists $n \in A^c$ such that $x \in U_n$. If there exists $m \in A$ such that $x \in U_m$, then $x \in U_m \cap U_n$. The case $m = n$ implies that both $n \in A$ and $n \notin A$, a contradition. And the case $m \neq n$ implies $U_m \cap U_n = \varnothing$ since the U_k are pairwise disjoint; and this contradicts the existence of x. Therefore $x \notin \cup_{k\in A} U_k = B$, and $B^c = \cup_{x\notin A} U_k \in \mathscr{U}$.

Let $A \in \mathscr{A}$, we have $A = \cup_{n\in A} \{n\} \subset \cup_{n\in A} U_n \in \mathscr{U}$, from which we deduce that $\mathscr{A} \subset \mathscr{U}$. Conversely, if $x \in \cup_{n\in A} U_n$, then there exists $m \in A$ such that $x \in U_m$. Thus $C_x = C_m$ and $x \in A$. Combining both results we conclude that $\mathscr{A} = \mathscr{U}$.

$\square$