Exercise 1. *Let $(X, \mathscr{A})$ be a measurable space. Use Proposition 2.6.1 and Example 2.1.2(a) to give another proof that if $f, g : X \to \mathbb{R}$ are measurable, then $f+g$ and fg are measurable. (Hint: Consider the function $H : X \to \mathbb{R}^2$ defined by $H(x) = (f(x), g(x))$.)*

Proof. As a reminder, Proposition 2.6.1 shows that $f \circ g$ is measurable when both f and g are measurable. Example 2.1.2(a) shows that f non-decreasing on an interval I is Borel-measurable.

Example 2.6.5 noted that $f(x) = \big(f_1(x), f_2(x), \cdots, f_n(x)\big)$ is measurable on $\big(\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n)\big)$ if and only if $\forall i, f_i$ is measurable on $\big(\mathbb{R}, \mathscr{B}(\mathbb{R})\big)$. From this we deduce that the function H is measurable.

Next, we note that the functions $\mathbb{R}^2 \to \mathbb{R}$ defined by $\phi : (x, y) \mapsto x + y$ and $\psi : (x, y) \mapsto xy$ are continuous on $\mathbb{R}^2$, and therefore measurable on $\big(\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2)\big)$. From proposition 2.6.1, we deduce that the composites $\phi \circ H$ and $\psi \circ H$ are Borel-measurable $\square$

Exercise 2. *Show that if f is a measurable complex-valued function on $(X, \mathscr{A})$, then $|f|$ is also measurable.*

Proof. The functions $\mathbb{C} \to \mathbb{R}^2, z \mapsto (\Re z, \Im z)$ and $\mathbb{R}^2 \to \mathbb{R}_+, (x, y) \mapsto \sqrt{x^2 + y^2}$ are composable and continuous on their respective domains, so their composite $\mathbb{C} \to \mathbb{R}_+, z \mapsto |z|$ is also continuous and thus Borel-measurable $\square$

Exercise 3. *Let $(X, \mathscr{A})$ be a measurable space, and let $f, g : X \to \mathbb{C}$ be measurable. Show that if g does not vanish, then f/g is measurable.*

Proof. We note that:

$$\forall z \in \mathbb{C} \mid g(z) \neq 0, \quad \frac{f(z)}{g(z)} = \frac{f(z)\overline{g(z)}}{|g(z)|^2}$$

The functions $\mathbb{R}_+^* \to \mathbb{R}_+^*, x \mapsto 1/x$, $\mathbb{C} \to \mathbb{R}_+, z \mapsto |z|^2$, and $\mathbb{C} \to \mathbb{C}, z \mapsto \overline{z}$ are all continuous. Whenever $g(z)$ is not zero, the expression given in the equation above is the product of a real-valued function $\phi(z) = 1/|g(z)|^2$ and complex-valued function $\psi(z) = f(z)\overline{g(z)}$. As the composition of Borel-measurable functions wherever g does not vanish, ϕ is Borel-measurable. The function ψ is continuous as the composition of continuous fonctions, and is therefore Borel-measurable. We deduce that the product $\phi(z)\psi(z) = f(z)/g(z)$ is Borel-measurable wherever both ϕ and ψ are Borel-measurable, e.g. wherever g is not zero $\square$

Exercise 4. *Not done.*

Exercise 5. *Let X and Y be sets, and let f be a function from X to Y. Show that*

1. *if $\mathscr{A}$ is a σ-algebra on X, then $\{B \subset Y : f^{-1}(B) \in \mathscr{A}\}$ is a σ-algebra on Y*

2. *if $\mathscr{B}$ is a σ-algebra on Y, then $\{f^{-1}(B) : B \in \mathscr{B}\}$ is a σ-algebra on X, and*

3. *if $\mathscr{C}$ is a collection of subsets of Y, then*

$$\sigma\big(\{f^{-1}(C) : C \in \mathscr{C}\}\big) = \{f^{-1}(B) : B \in \sigma(\mathscr{C})\}$$

Proof.

1. Let us note $\mathscr{B} = \{B \subset Y : f^{-1}(B) \in \mathscr{A}\}$. $Y \subset Y$ and $f^{-1}(Y) = \{x \in X \mid f(x) \in Y\} = X \in \mathscr{A}$, so $Y \in \mathscr{B}$. Next, we note that for all $B \subset Y$, we have $f^{-1}(B)^c = f^{-1}(B^c)$, so $\mathscr{B}$ is stable by complementation. Finally, for any family of sets B, $f^{-1}\left(\bigcup B\right) = \bigcup f^{-1}(B)$, which gives us the stability of $\mathscr{B}$ for unions (we only need countable unions). Combining the preceding results, we conclude that $\mathscr{B}$ is a σ-algebra on Y.

2. Let us note $\mathscr{A} = \{f^{-1}(B) : B \in \mathscr{B}\}$. As noted above, $f^{-1}(Y) = X$, so $X \in \mathscr{A}$. The same arguments on complementation and stability by union from the previous point apply here too, completing the proof that $\mathscr{A}$ is a σ-algebra on X.

3. For every σ-algebra $\mathscr{B}$ on Y that contains $\mathscr{C}$, $g(\mathscr{B}) = \{f^{-1}(B) : B \in \mathscr{B}\}$ is a σ-algebra on X. The family $\mathscr{F}$ of such σ-algebras respects inclusion: if for $\mathscr{B} \in \mathscr{F}$ and $\mathscr{D} \in \mathscr{F}$, we have $\mathscr{B} \subset \mathscr{D}$, then $g(\mathscr{B}) \subset g(\mathscr{D})$ is also true. We conclude the following:

$$\begin{aligned} g\left(\bigcap_{F \in \mathscr{F}} F\right) &= \bigcap_{F \in \mathscr{F}} g(F) && \textit{with} \\ g\left(\bigcap_{F \in \mathscr{F}} F\right) &= g\big(\sigma(\mathscr{C})\big) = \{f^{-1}(B) : B \in \sigma(\mathscr{C})\} && \textit{and} \\ \bigcap_{F \in \mathscr{F}} g(F) &= \sigma\big(\{f^{-1}(B) : B \in \mathscr{C}\}\big) \end{aligned}$$

$\square$

Exercise 6. *Let μ be a nonzero finite Borel measure on $\mathbb{R}$, and let $F : \mathbb{R} \to \mathbb{R}$ be the function defined by $F(x) = \mu\big((-\infty, x]\big)$. Define a function g on the interval $\big(0, \lim_{x \to +\infty} F(x)\big)$ by*

$$g(x) = \inf\{t \in \mathbb{R} : F(t) \geq x\}.$$

1. *Show that g is nondecreasing, finite valued, and Borel measurable.*

2. *Show that $\mu = \lambda g^{-1}$. (Hint: start by showing that $\mu(B) = \lambda\big(g^{-1}(B)\big)$ when B has the form $(-\infty, b]$.)*

Proof.

1. μ being a nonzero finite measure, $0 \leq F(x) \leq \mu(\mathbb{R})$, $\lim_{x \to -\infty} F(x) = 0$ and $\lim_{x \to +\infty} F(x) = \mu(\mathbb{R})$. Further, F is non-decreasing, non-negative, right-continuous (Proposition 1.3.9).

   Let x and y in $\big(0, \lim_{x \to +\infty} F(x)\big)$ such that $x < y$. For all t such that $F(t) \geq y$, we also have $F(t) \geq x$, and therefore $\{t \in \mathbb{R} : F(t) \geq y\} \subset \{t \in \mathbb{R} : F(t) \geq x\}$, from which we deduce that $g(y) \geq g(x)$.

   g being non-decreasing on the interval $\big(0, \lim_{x \to +\infty} F(x)\big)$ is therefore Borel-measurable on that interval.

   Let $x \in \mathbb{R}$ such that $0 < x < \lim_{+\infty} F$. Since μ is nonzero, such an x exists, and thus there also exists $t_0 \in \mathbb{R}$ such that $F(t_0) \geq x$. This shows that $\{t \in \mathbb{R} : F(t) \geq x\} \neq \varnothing$. Since $\lim_{-\infty} F = 0$ and $x > 0$, there exists $t_1 \in \mathbb{R}$ such that $0 \leq F(t_1) < x$. F being non-decreasing, we have $t_1 < t_0$, which shows that $\{t \in \mathbb{R} : F(t) \geq x\}$ has a lower bound in $\mathbb{R}$, e.g. g has finite values.

2. Let $b \in \mathbb{R}$ and $B = (-\infty, b]$. We have

$$\begin{aligned}
\lambda\left(g^{-1}(B)\right) &= \int_{\mathbb{R}} \chi_{g^{-1}(B)} \,\mathrm{d}\lambda = \int_0^{\lim_{+\infty} F} \chi_B \circ g \,\mathrm{d}\lambda = \int_{\mathbb{R}} \chi_B \,\mathrm{d}\big(\lambda g^{-1}\big) \\
&= \lambda g^{-1}(B) \\
\mu(B) &= \int_{-\infty}^{b} \mathrm{d}\mu = \int_0^{F(b)} g \,\mathrm{d}\lambda = \int_0^{\lim_{+\infty} F} \chi_B \circ g \,\mathrm{d}\lambda
\end{aligned}$$

   so $\mu(B) = \lambda g^{-1}(B)$. Since the Borel σ-algebra on $\mathbb{R}$ is generated by sets of the form $(-\infty, b]$ for $b \in \mathbb{R}$ and the measures μ and λg^{-1} agree on such sets, we conclude that they agree on the whole of the Borel σ-algebra.

$\square$

Exercise 7. *Show that a convex subset of $\mathbb{R}^2$ need not be a Borel set. (Hint: consider an open ball, together with part of its boundary).*

Proof. Let B be a non-Borel set in $[0,1]$ and $f : [0,1] \to \mathscr{C}, x \mapsto \big(cos(2\pi x), sin(2\pi x)\big)$ with $\mathscr{C} = \{(x,y) \mid x^2 + y^2 = 1\}$ the unit circle for the Euclidean norm.

f is continuous, hence Borel-measurable. f is also injective, which implies that $f^{-1}\big(f(B)\big) = B$; hence $f(B)$ is not a Borel set.

Let $\mathscr{D} = \{(x,y) \mid x^2 + y^2 < 1\}$ be the open unit ball of $\mathbb{R}^2$. $\mathscr{D}$ is convex. Since $f(B) \subset \mathscr{C}$, line segments from points in $f(B)$ to any other point in $f(B)$ or $\mathscr{D}$ stay in $\mathscr{D}$ (except for the end points which can be in $f(B)$). Therefore $\mathscr{D} \cup f(B)$ is a convex set.

However, $\mathscr{D} \cup f(B)$ is not a Borel set, since $\mathscr{D}$ is a Borel set but $f(B)$ is not $\square$