Exercise 6. Show that the hypothesis of σ-finiteness cannot be omitted from Proposition 3.4.5. (*Hint:* Consider counting measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.)

Proof. Let μ be the counting measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$. The countable set $\mathscr{A} = \{(-\infty, r], r \in \mathbb{Q}\}$ satisfies $\sigma(\mathscr{A}) = \mathscr{B}(\mathbb{R})$ (for example from exercise 1.1.2), so $\mathscr{B}(\mathbb{R})$ is countably generated by $\mathscr{A}$. Let $\{X_n\}_{n\in\mathbb{N}}$ be a family of Borel subsets of $\mathbb{R}$ such that $\mu(X_i)$ is finite for all i. Since μ is the counting measure, X_i is also finite for all i. As a countable union of finite sets, $\cup_{n\in\mathbb{N}} X_n$ is countable; since $\mathbb{R}$ is uncountable, $\mathbb{R} \neq \cup_{n\in\mathbb{N}} X_n$, and therefore μ is not σ-finite.

Let $\varepsilon > 0$ and suppose that there exists a family of functions $\{\varphi_n\}_{n\in\mathbb{N}}$ in $\mathscr{L}^p(\mathbb{R}, \mathscr{B}(\mathbb{R}), \mu)$ such that:

$$\forall f \in \mathscr{L}^p(\mathbb{R}, \mathscr{B}(\mathbb{R}), \mu), \quad \exists n \in \mathbb{N}, \quad \|f - \varphi_n\|_p < \varepsilon$$

Let $\{\delta_x\}_{x\in\mathbb{R}}$ be the family of functions of $\mathscr{L}^p(\mathbb{R}, \mathscr{B}(\mathbb{R}), \mu)$ defined by:

$$\forall y \in \mathbb{R}, \quad \delta_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

Let $x, y \in \mathbb{R}$ such that $x \neq y$; we have $\|\delta_x - \delta_y\|_p = 2^{1/p} \geq 1$. If $m, n \in \mathbb{N}$ are such that

$$\begin{aligned} \|\delta_x - \varphi_n\|_p &< \varepsilon \\ \|\delta_y - \varphi_m\|_p &< \varepsilon \end{aligned}$$

then the triangular inequality gives us

$$\|\varphi_n - \varphi_m\|_p \geq \|\delta_x - \delta_y\|_p - \big(\|\delta_x - \varphi_n\|_p + \|\delta_y - \varphi_m\|_p\big) > 1/3 \tag{1}$$

as soon as $\varepsilon < 1/3$. For all $x \in \mathbb{R}$, let $D_x = \{n \in \mathbb{N}, \|\delta_x - \varphi_n\|_p < \varepsilon\}$. Using the axiom of choice, we can define a function

$$\begin{aligned} \psi : \{\delta_x\}_{x\in\mathbb{R}} &\to \mathbb{N} \\ \delta_y &\mapsto n \quad \text{such that } n \in D_y \end{aligned}$$

From (1), we deduce that ψ is injective, and therefore $\{\delta_x\}_{x\in\mathbb{R}}$ must be countable, which is absurd. Therefore, the family $\{\varphi_n\}_{n\in\mathbb{N}}$ does not exist and $L^p(\mathbb{R}, \mathscr{B}(\mathbb{R}), \mu)$ is not separable.

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