Exercise 1. Find the σ-algebra on $\mathbb{R}$ that is generated by all one-point subsets of $\mathbb{R}$.

Proof. Let $\mathscr{C} = \big\{\{x\}, x \in \mathbb{R}\big\}$, and let $\mathscr{A} = \{A \subset \mathbb{R}, A \text{ of } A^c \text{ is countable}\}$. Then $\mathscr{A}$ is a σ-algebra on $\mathbb{R}$: $\varnothing \in \mathscr{A}$ since it is finite, and by definition, $\mathscr{A}$ is stable by complementation.

Let $A, B \in \mathscr{A}$. From $(A \cup B)^c = A^c \cap B^c$, we deduce that if A^c or B^c is countable, then $A \cup B \in \mathscr{A}$. Otherwise both A and B are countable, and then $A \cup B$ is also countable. Therefore for all $A, B \in \mathscr{A}$, $A \cup B \in \mathscr{A}$.

Let A be a countable set of elements of $\mathscr{A}$. Let $B = \{a \in A, a \text{ is countable}\}$ and $C = \{a \in A, A^c \text{ is countable}\}$. As subsets of the countable set A, both B and C are countable, and we have $A = B \cup C$. The set $\cup_{x \in B} x$ is a countable union of countable sets, and therefore countable; and the set $\cup_{x \in C} x$ is such that

$$\left(\bigcup_{x \in C} x\right)^c = \bigcap_{x \in C} x^c \subset x^c \text{ for all } x \in C$$

Since for all $x \in C$, x^c is countable, $\cap_{x \in C} x^c$ is a subset of a countable set, and therefore countable. From this we deduce that

$$\bigcup_{x \in A} x = \left(\bigcup_{x \in B} x\right) \cup \left(\bigcup_{x \in C} x\right)$$

is an element of $\mathscr{A}$ as the union of two elements of $\mathscr{A}$.

Every element of $\mathscr{C}$ is finite and therefore an element of $\mathscr{A}$; thus we have $\mathscr{C} \subset \mathscr{A}$. Since $\mathscr{A}$ is a σ-algebra containing $\mathscr{C}$, we deduce that $\sigma(\mathscr{C}) \subset \mathscr{A}$.

Conversely, let A be a subset of $\mathscr{A}$. We have $A = \cup_{x \in A} \{x\}$, from which we deduce that if A is countable, then $A \in \sigma(\mathscr{C})$ as the countable union of elements of $\mathscr{C}$. Otherwise, $A^c = \cup_{x \in A^c} \{x\}$ is countable, and $A = \cap_{x \in A^c} \{x\}^c$ is the countable intersection of elements of $\sigma(\mathscr{C})$, since $\{x\} \in \mathscr{C}$ for all $x \in \mathbb{R}$ and $\sigma(\mathscr{C})$ is stable by complementation. Therefore $A \in \sigma(\mathscr{C})$.

From the above, we deduce that $\sigma(\mathscr{C}) = \mathscr{A}$.

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