Exercise 8. Show that in Lemma 3.4.7 condition (b) cannot be replaced with the assumption that μ is σ-finite. (*Hint:* Let $\mathscr{A}_0$ be the algebra on $\mathbb{R}$ defined in example 1.1.1(g) let $\{r_n\}$ be an enumeration of $\mathbb{Q}$, and let μ be the Borel measure on $\mathbb{R}$ defined by $\mu = \sum_n \delta_{r_n}$.)

Lemma 3.4.7. *Let $(X, \mathscr{A}, \mu)$ be a measure space. Suppose that $\mathscr{A}_0$ is an algebra of subsets of X such that*

(a) $\sigma(\mathscr{A}_0) = \mathscr{A}$, and

(b) X is the union of a sequence of sets that belong to $\mathscr{A}_0$ and have finite measure under μ.

Then for each positive ε and each set A that belongs to $\mathscr{A}$ and satisfies $\mu(A) < +\infty$ there is a set A_0 that belongs to $\mathscr{A}_0$ and satisfies $\mu(A \triangle A_0) < \varepsilon$.

Example 1.1.1. *(g) Let $\mathscr{A}$ be the collection of all subsets of $\mathbb{R}$ that are unions of finitely many intervals of the form $(a, b]$, $(a, +\infty)$ or $(-\infty, b]$. It is easy to check that each set that belongs to $\mathscr{A}$ is the union of a finite disjoint collection of intervals of the types listed above, and then to check that $\mathscr{A}$ is an algebra on $\mathbb{R}$ (the empty set belongs to $\mathscr{A}$, since it is the union of the empty, and hence finite, collection of intervals). The algebra $\mathscr{A}$ is not a σ-algebra; for example the bounded open subintervals of $\mathbb{R}$ are unions of sequences of sets of $\mathscr{A}$ but do not themselves belong to $\mathscr{A}$.*

Proof. For all $b \in \mathbb{R}$, $(-\infty, b] \in \mathscr{A}$, so $\mathscr{B}(\mathbb{R}) \subset \sigma(\mathscr{A})$. Moreover, $\mathbb{R} = \cup_{r \in \mathbb{Q}} \{r\} \cup (\mathbb{R} - \mathbb{Q})$ is a countable union of sets of $\sigma(\mathscr{A})$ with finite measure: $\mu(\{r\}) = 1$ for all $r \in \mathbb{Q}$, and $\mu(\mathbb{R} - \mathbb{Q}) = 0$. So μ is σ-finite on $\sigma(\mathscr{A})$.

Every nonempty $A \in \mathscr{A}$ is the finite union of intervals mentioned in the example, and therefore contains a nonempty open interval. So there are infinitely many rational numbers in A, and $\mu(A)$ is infinite. From this we deduce that $\mathbb{R}$ is not the union of sets of $\mathscr{A}$ with finite measure, and so hypothesis (b) of the lemma does not hold.

Let $B = \{\sqrt{2}\} \in \sigma(\mathscr{A})$, and $A \in \mathscr{A}$ such that A is nonempty. Then A contains at least one rational number r, and since $r \notin B$, we have $r \in A \triangle B$. Therefore $\mu(A \triangle B) \geq 1$, but $\mu(\{B\}) = 0$, so the conclusion of the lemma does not hold if X is only σ-finite.

$\square$