Exercise 6. Find an infinite collection of subsets of $\mathbb{R}$ that contains $\mathbb{R}$, is closed under the formation of countable unions, and is closed under the formation of countable intersections, but is not a σ-algebra.

Proof. Let $\mathscr{A} = \{\mathbb{R}\} \cup \{[-n, n], n \in \mathbb{N}\}$, with $\varnothing = [-0, 0]$, and let $\{A_n\}_{n\in\mathbb{N}}$ be a family of elements of $\mathscr{A}$. For all $n \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $A_n = [-k, k]$. Let K be the set of all such integers k.

If K is bounded above, then it has a largest element M since $\mathbb{N}$ is well-ordered. Then for all $n \in \mathbb{N}$, $A_n \subset [-M, M]$, and there exists $m \in \mathbb{N}$ such that $A_m = [-M, M]$. From this we deduce that $\cup_{n\in\mathbb{N}} A_k = [-M, M] \in \mathscr{A}$. Otherwise, let $x \in \mathbb{R}$. Since K is not bounded above, there exists $M, m \in \mathbb{N}$ such that $x \in A_m = [-M, M]$. From this we deduce that $\cup_{n\in\mathbb{N}} A_n = \mathbb{R} \in \mathscr{A}$.

Since K is nonempty, it has a smallest element P since $\mathbb{N}$ is well-ordered, and there is some $p \in \mathbb{N}$ such that $A_p = [-p, p]$. From this we deduce that $A_p = \cap_{n\in\mathbb{N}} A_n \in \mathscr{A}$.

The set $\mathscr{A}$ is stable by countable union and countable intersection, and contains $\mathbb{R}$. However it is not stable by complementation: for example $[-1, 1] \in \mathscr{A}$, but $[-1, 1]^c = (-\infty, 1) \cup (1, +\infty)$ is not a closed interval with integer boundaries. Therefore $\mathscr{A}$ is not a σ-algebra.

$\square$