Exercise 2. Show that $L^\infty([a,b])$ is not separable. (Hint: consider the elements of $L^\infty([a,b]))$ determined by the characteristic functions of the sets $[a,c]$ for $a<c<b$.

Proof. Let X be the set of characteristic functions of the sets $[a,c]$ for $a < c < b$, and let $c_1, c_2 \in [a,b]$ such that $c_1 < c_2$. Then

$$\left\|\chi_{[a,c_1]} - \chi_{[a,c_2]}\right\|_\infty = \left\|\chi_{(c_1,c_2]}\right\|_\infty = 1$$

Suppose that $(\varphi_n)_{n\in\mathbb{N}}$ is a countable dense family of functions in $\mathscr{L}^\infty([a,b])$, and let $\varepsilon > 0$. For all $n, m \in \mathbb{N}$ such that

$$\left\|\chi_{[a,c_1]} - \varphi_n\right\|_\infty < \frac{\varepsilon}{3}$$
$$\left\|\chi_{[a,c_2]} - \varphi_m\right\|_\infty < \frac{\varepsilon}{3}$$

we have

$$\begin{aligned}
\|\varphi_n - \varphi_m\|_\infty &= \left\|(\varphi_n - \chi_{[a,c_1]}) + \left(\chi_{[a,c_1]} - \chi_{[a,c_2]}\right) + \left(\chi_{[a,c_2]} - \varphi_m\right)\right\|_\infty \\
&\geq \left\|\chi_{(c_1,c_2)}\right\|_\infty - \left(\left\|\chi_{[a,c_1]} - \varphi_n\right\|_\infty + \left\|\chi_{[a,c_2]} - \varphi_m\right\|_\infty\right) \\
&\geq 1 - \frac{2}{3}\varepsilon \qquad (1)
\end{aligned}$$

which is positive for ε sufficiently small. Choose such an ε, and, for all $\chi_{[a,c]} \in X$, choose one φ_n such that $\left\|\chi_{[a,c]} - \phi_n\right\|_\infty < \varepsilon/3$, and call it φ_c. The function

$$\begin{aligned}
\psi : X &\to \{\varphi_n, n \in \mathbb{N}\} \\
\chi_{[a,c]} &\mapsto \varphi_c
\end{aligned}$$

is injective: if $c \neq d$, then from (1) we get $\|\varphi_c - \varphi_d\|_\infty > 0$, so that $\varphi_c \neq \varphi_d$. Since X is uncountable, we deduce that $\{\varphi_n, n \in \mathbb{N}\}$ is also uncountable, which is a contradiction. Therefore the family $(\varphi_n)_{n\in\mathbb{N}}$ does not exist, and $L^\infty([a,b])$ is not separable.

$\square$