Exercise 6. Suppose that for each subset A of $\mathbb{R}^2$ and each real number x we denote the set $\{y \in \mathbb{R} \mid (x,y) \in A\}$ by A_x. Let $\mathscr{A}$ consist of those subsets A of $\mathbb{R}^2$ that satisfy $A_x \in \mathscr{B}(\mathbb{R})$ for each x in $\mathbb{R}$, and define $\mu : \mathscr{A} \to [0, +\infty]$ by

$$\mu(A) = \begin{cases} \sum_x \lambda(A_x) & \text{if } A_x \neq \varnothing \text{ for only countably many } x \\ +\infty & \text{otherwise} \end{cases}$$

(a) Show that $\mathscr{A}$ is a σ-algebra on $\mathbb{R}^2$ and that μ is a measure on $(\mathbb{R}^2, \mathscr{A})$.

(b) Show that $\{(x,y) \in \mathbb{R}^2 \mid y = 0\}$ is locally μ-null but not μ-null.

Proof. (a) First, note that $\varnothing \in \mathscr{A}$ since the property defining elements of $\mathscr{A}$ is vacuously true for $\varnothing$, and that $\mathbb{R}^2 \in \mathscr{A}$ since the condition defining $\mathbb{R}^2_x$ holds by definition.

Next, take $A_1, A_2, \ldots$ a countable family of elements of $\mathscr{A}$, and consider $A = \cap_{i \in \mathbb{N}} A_i$. For each $x \in A$, and each $i \in \mathbb{N}$, $A_{i,x} \in \mathscr{B}(\mathbb{R})$, so that $A_x = \cap_{i \in \mathbb{N}} A_{i,x} \in \mathscr{B}(\mathbb{R})$ and thus $A \in \mathscr{A}$. Therefore $\mathscr{A}$ is stable by countable intersection.

Let $A \in \mathscr{A}$ and $B = \mathbb{R}^2 - A$. Let $x \in \mathbb{R}$. For all $y \in \mathbb{R}$, either $(x,y) \in A$, and then $y \in A_x$, or $(x,y) \in B$, and then $y \in B_x$. From this we deduce that $B_x = \mathbb{R} - A_x \in \mathscr{B}(\mathbb{R})$. This last point is the definition of $B \in \mathscr{A}$, so $\mathscr{A}$ is stable by complement.

From the above points we conclude that $\mathscr{A}$ is a σ-algebra on $\mathbb{R}^2$.

Let $A = \varnothing$. For all $x \in \mathbb{R}$, we have $\lambda(A_x) = \lambda(\varnothing) = 0$ so that $\mu(A) = 0$. Also, $\forall B \in \mathscr{B}(\mathbb{R})$, $\lambda(B) \geq 0$ so that $\mu(A) \geq 0$ for all $A \in \mathscr{A}$. Let $A \in \mathscr{A}$, we have $A = \cup_{x \in \mathbb{R}} \{x\} \times A_x$. Let $A_1, A_2, \ldots$ be a countable sequence of disjoint sets in $\mathscr{A}$, and let, for all $x \in \mathbb{R}$, $C_x = \cup_{i \in \mathbb{N}} A_{i,x} \in \mathscr{B}(\mathbb{R})$. If for $x \in \mathbb{R}$ and $i, j \in \mathbb{N}$ distinct, there exists $z \in A_{i,x} \cap A_{j,x}$, then $(x,z) \in A_i \cap A_j$, which is impossible since the A_i are disjoint. Therefore $\lambda(C_x) = \sum_{i \in \mathbb{N}} A_{i,x}$. We also have

$$\begin{aligned} \mu\left(\bigcup_{i \in \mathbb{N}} A_i\right) &= \mu\left(\bigcup_{i \in \mathbb{N}} \bigcup_{x \in \mathbb{R}} \{x\} \times A_{i,x}\right) \\ &= \mu\left(\bigcup_{x \in \mathbb{R}} \{x\} \times C_x\right) \\ &= \begin{cases} \sum_{x \in \mathbb{R}} \lambda\left(C_x\right) & \text{if } C_x \neq \varnothing \text{ for countably many } C_x \\ +\infty & \text{otherwise} \end{cases} \end{aligned} \tag{1}$$

For countably many nonempty C_x, we get

$$\sum_{x\in\mathbb{R}} \lambda(C_x) = \sum_{x\in\mathbb{R}} \sum_{i\in\mathbb{N}} \lambda(A_{i,x}) = \sum_{i\in\mathbb{N}} \sum_{x\in\mathbb{R}} \lambda(A_{i,x}) = \sum_{i\in\mathbb{N}} \mu\left(A_i\right)$$

If countably many $A_{i,x}$ are nonempty, then countably many C_x are nonempty, so if uncountably many C_x are nonempty, we get from (1) $\mu(\cup_{i\in\mathbb{N}} A_i) = +\infty$. In that case, also, there is at least one A_j such that there are uncountably many nonempty $A_{j,x}$. This gives us $\mu(A_j) = +\infty$, from which we deduce that $\sum_{i\in\mathbb{N}} \mu(A_i) = +\infty$. We can then conclude that

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mu(A_i)$$

and that μ is a measure on $(\mathbb{R}^2, \mathscr{A})$.

(b) Let $N = \{(x,y) \in \mathbb{R}^2 \mid y = 0\} = \mathbb{R} \times \{0\}$, and let $A \in \mathscr{A}$ such that $\mu(A)$ is finite. If $A = \varnothing$, then $\mu(N \cap A) = \mu(\varnothing) = 0$. Otherwise, for all $x \in A$, we have $N \cap \{x\} \times A_x \subset \{(x,0)\} \in \mathscr{A}$. Since $A_x \neq \varnothing$ for only countably many values of x, we have $\mu(N \cap A) = \sum_x \lambda(N \cap A_x) = 0$, from which we conclude that N is a locally μ-null set.

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