Exercise 2. Show that $\mathscr{B}(\mathbb{R})$ is generated by the collection of intervals $(-\infty, b]$ for which the endpoint b is a rational number.

Proof. Let $\mathscr{A} = \{(-\infty, r], r \in \mathbb{Q}\}$ and $\mathscr{S} = \{(-\infty, x], x \in \mathbb{R}\}$. We have $\mathscr{A} \subset \mathscr{S}$, so that $\sigma(\mathscr{A}) \subset \sigma(\mathscr{S}) = \mathscr{B}(\mathbb{R})$.

Conversely, let $x \in \mathbb{R}$. For all $n \in \mathbb{N}$, let $r_n = x + 1/(n+1)$ and $I_n = (-\infty, r_n]$. Then $(-\infty, x] = \cap_{n \in \mathbb{N}} I_n$ is the countable intersection of elements of $\mathscr{A}$ and therefore an element of $\sigma(\mathscr{A})$. From this we deduce that $\sigma(\mathscr{S}) \subset \sigma(\mathscr{A})$.

Therefore $\mathscr{B}(\mathbb{R}) = \sigma(\mathscr{A})$.

$\square$