Exercise 7. Let $\mathscr{S}$ be a collection of subsets of the set X. Show that for each A in $\sigma(\mathscr{S})$, there is a countable subfamily $\mathscr{C}_0$ of $\mathscr{S}$ such that $A \in \sigma(\mathscr{C}_0)$. (Hint: let $\mathscr{A}$ be the union of the σ-algebras $\sigma(\mathscr{C})$, where $\mathscr{C}$ ranges over the countable subfamilies of $\mathscr{S}$, and show that $\mathscr{A}$ is a σ-algebra that satisfies $\mathscr{S} \subseteq \mathscr{A} \subseteq \sigma(\mathscr{S})$ and hence is equal to $\sigma(\mathscr{S})$.)

Proof. Using the notations defined above, for all $\sigma(\mathscr{C})$, we have $\varnothing \in \sigma(\mathscr{C})$, so $\varnothing \in \mathscr{A}$. Let $A \in \mathscr{A}$, there exists a countable subset $\mathscr{C}$ of $\mathscr{S}$ such that $A \in \sigma(\mathscr{C})$. Since $\sigma(\mathscr{C})$ is stable by complementation, $A^c \in \sigma(\mathscr{C})$ and therefore $A^c \in \mathscr{A}$ (with $A^c = X - A$).

Let now $\{A_n\}_{n\in\mathbb{N}}$ be a family of subsets of $\mathscr{A}$. For all n, there exists a countable family $\mathscr{C}_n$ of subsets of $\mathscr{S}$, such that $A_n \in \sigma(\mathscr{C}_n)$. The set $A = \cup_{n\in\mathbb{N}} A_n$ is an element of $\mathscr{T} = \cup_{n\in\mathbb{N}} \sigma(\mathscr{C}_n)$, and for all $n \in \mathbb{N}$, $\mathscr{C}_n \subseteq \cup_{n\in\mathbb{N}} \mathscr{C}_n = \mathscr{C}$, so $\sigma(\mathscr{C}_n) \subseteq \sigma(\mathscr{C})$ and finally $\mathscr{T} \subset \sigma(\mathscr{C})$. Since $\mathscr{C}_n$ is countable for all n, we deduce that $\mathscr{C}$ is countable, and therefore that $A \in \mathscr{A}$.

From the above, we conclude that $\mathscr{A}$ is a σ-algebra on X.

For all $A \in \mathscr{A}$, we have $A \in \sigma(\mathscr{C})$ for some $\mathscr{C} \subseteq \mathscr{S}$. From this we deduce that $\sigma(\mathscr{C}) \subseteq \sigma(\mathscr{S})$, so that $\mathscr{A} \subseteq \sigma(\mathscr{S})$. For all $A \in \mathscr{S}$, the σ-algebra $\sigma(A)$ is generated by the countable family $\{A\}$, so $\sigma(A) \subseteq \mathscr{A}$. From this we deduce that $\mathscr{S} \subseteq \mathscr{A}$.

From $\mathscr{S} \subseteq \mathscr{A} \subseteq \sigma(\mathscr{S})$, we deduce that $\mathscr{A}$ is included in the smallest σ-algebra that contains $\mathscr{S}$, and is therefore equal to $\sigma(\mathscr{S})$.

$\square$