**Exercise 1.** Use Proposition 3.4.3 to show that if $1 \leq p < +\infty$ then $L^p([a,b])$ is separable.

**Proposition 3.4.3.** *Suppose that $[a,b]$ is a closed bounded interval and that $p$ satisfies $1 \leq p < +\infty$. Then the subspace of $L^p([a,b])$ determined by the step functions on $[a,b]$ is dense in $L^p([a,b])$.*

*Proof.* Given Proposition 3.4.3, it is enough to show that the subspace of $L^p([a,b])$ determined by the step functions on $[a,b]$ has a countable dense subspace.

Furthermore, complex-valued functions can be decomposed into their real and imaginary parts and the following result applied in turn to each of them; so we are only considering real-valued functions.

Let $\alpha, \beta \in [a,b]$ such that $\alpha < \beta$ and let $\varepsilon > 0$. There exist $q, r \in \mathbb{Q}$ such that $0 \leq q - \alpha \leq \varepsilon$ and $0 \leq \beta - r \leq \varepsilon$. From this we deduce that $\|\chi_{[\alpha,\beta]} - \chi_{[q,r]}\|_p = (q-\alpha)^{1/p} + (\beta - r)^{1/p} \leq 2\varepsilon^{1/p}$, so that $\chi_{[\alpha,\beta]}$ can be approximated by the characteristic function of a bounded interval with rational endpoints.

Next, let $f = \sum_{i=0}^{n} f_i \chi_{A_i}$ with $A_i = [a_i, a_{i+1}]$ be a step function on $[a,b]$. Let $g_i$ be rational numbers such that $|f_i - g_i| \leq \varepsilon$ for all $i$, and let $Q_i = [q_i, q_{i+1}]$ be a subinterval of $(a_i, a_{i+1})$ with rational endpoints such that $\lambda(A_i - Q_i) \leq \varepsilon$. Define a step function $h$ on $[a,b]$ such that

$$h(x) = \begin{cases} g_i & \text{if } q_i \leq x < q_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned} \|f - h\|_p^p &= \int_a^b \left| \sum_{i=0}^{n} f_i \chi_{A_i} - \sum_{i=0}^{n} g_i \chi_{Q_i} \right|^p \mathrm{d}\lambda \\ &= \sum_{i=0}^{n} |f_i - g_i|^p \lambda(Q_i) + \sum_{i=0}^{n} |f_i|^p \lambda(A_i - Q_i) \\ &\leq \varepsilon^p \cdot (b-a) + \varepsilon \cdot (n+1) \cdot \max_i \{|f_i|^p\} \end{aligned}$$

Since $n$ and the $f_i$ only depend on $f$, we conclude that each step function $f$ on $[a,b]$ can be approximated by a step function on $[a,b]$ with rational values and a subdivision $a = q_0 < q_1 < \cdots < q_{n-1} < q_n = b$ with $q_i$ rational for $0 < i < n$.

The set $R = \{(q,r) \in \mathbb{Q}^2 \mid [q,r] \in [a,b]\}$ is countable, as it is a subset of $\mathbb{Q}^2$ which is countable. For a given $n \in \mathbb{N}$, there is an injection of the set of step functions taking $n$ rational values on a subdivision $\{q_i\}$ as above into $T_n = (\mathbb{Q} \times R)^n$, which is countable. Finally, there is an injection from the set $H$ of step functions defined as $h$ above into the set $\cup_{n \in \mathbb{N}} T_n$, which is countable as the countable union of countable sets.

The set $H$ is stable by pointwise addition, multiplication by a rational constant, and contains the function 0, so it is a $\mathbb{Q}$-vector space, and a subspace of the space of step functions on $[a, b]$. Thus $H$ determines a countable dense subspace of $L^p([a, b])$.

$\square$