Exercise 5. Let μ be a measure on $(X, \mathscr{A})$, and et $f, f_1, f_2, \ldots$ and $g, g_1, g_2, \ldots$ be real-valued $\mathscr{A}$-measurable functions on X.

(a) Show that if μ is finite, if $\{f_n\}$ converges to f in measure, and if $\{g_n\}$ converges to g in measure, then $\{f_n g_n\}$ converges to fg in measure.

(b) Can the assumption that μ is finite be omitted in part (a)?

Proof.

(a) First, note that $\forall x \in \mathbb{R}$,

$$\begin{aligned} f_n(x)g_n(x) - f(x)g(x) = \big(f_n(x) - f(x)\big)\big(g_n(x) - g(x)\big) \\ + f(x)\big(g_n(x) - g(x)\big) + g(x)\big(f_n(x) - f(x)\big) \end{aligned}$$

Let $\alpha > 0$. For all positive reals $0 < \alpha_1 < \alpha_2 < \alpha_3$ such that $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, we have $\alpha_3 \geq \alpha/3$ (otherwise their sum would be less than α). From this we deduce that:

$$\begin{aligned} &\big\{x \in X \mid |f_n(x)g_n(x) - f(x)g(x)| > 3\alpha\big\} \subset \\ &\qquad \big\{x \in X \mid |f_n(x) - f(x)||g_n(x) - g(x)| > \alpha\big\} \\ &\qquad \cup \big\{x \in X \mid |f(x)||g_n(x) - g(x)| > \alpha\big\} \\ &\qquad \cup \big\{x \in X \mid |g(x)||f_n(x) - f(x)| > \alpha\big\} \end{aligned} \tag{1}$$

Now let $n \in \mathbb{N}$. Define $F_n = \{x \in X \mid |f(x)| > n\}$ and $G_n = \{x \in X \mid |g(x)| > n\}$. Since μ is finite, $\mu(F_n)$ is finite, and we also have $F_{n+1} \subset F_n$. Since f is real-valued, we deduce:

$$\mu\left(\bigcap_{n=0}^{\infty} F_n\right) = \lim_{n \to \infty} \mu(F_n) = 0$$

The same holds for g and G_n.

Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ such that $\mu(F_n) < \epsilon$ and $\mu(G_n) < \epsilon$ for all $n \geq N$.

Note that

$$\begin{aligned} \big\{x \in X \mid |f(x)| \leq N\big\} \cap \big\{x \in X \mid |g_n(x) - g(x)| \leq \alpha/N\big\} \\ \subset \big\{x \in X \mid |f(x)||g_n(x) - g(x)| \leq \alpha\big\} \end{aligned}$$

Taking the complement in X, we get

$$\begin{aligned} &\big\{x \in X \mid |f(x)||g_n(x) - g(x)| > \alpha\big\} \\ &\qquad \subset F_n \cup \big\{x \in X \mid |g_n(x) - g(x)| > \alpha/N\big\} \end{aligned}$$

which implies

$$\begin{aligned}
&\mu\left(\left\{x \in X \mid |f(x)||g_n(x) - g(x)| > \alpha\right\}\right) \\
&\qquad\qquad \leq \mu(F_n) + \mu\left(\left\{x \in X \mid |g_n(x) - g(x)| > \alpha/N\right\}\right)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\mu\left(\left\{x \in X \mid |g(x)||f_n(x) - f(x)| > \alpha\right\}\right) \\
&\qquad\qquad \leq \mu(G_n) + \mu\left(\left\{x \in X \mid |f_n(x) - f(x)| > \alpha/N\right\}\right)
\end{aligned}$$

Finally, we also have

$$\begin{aligned}
&\left\{x \in X \mid |f_n(x) - f(x)| \leq \sqrt{\alpha}\right\} \cap \left\{x \in X \mid |g_n(x) - g(x)| \leq \sqrt{\alpha}\right\} \\
&\qquad\qquad \subset \left\{x \in X \mid |f_n(x) - f(x)||g_n(x) - g(x)| \leq \alpha\right\}
\end{aligned}$$

which implies

$$\begin{aligned}
&\mu\left(\left\{x \in X \mid |f_n(x) - f(x)||g_n(x) - g(x)| > \alpha\right\}\right) \\
&\qquad \leq \mu\left(\left\{x \in X \mid |f_n(x) - f(x)| > \sqrt{\alpha}\right\}\right) \\
&\qquad\qquad + \mu\left(\left\{x \in X \mid |g_n(x) - g(x)| > \sqrt{\alpha}\right\}\right)
\end{aligned}$$

Summarizing the results above, we get:

$$\begin{aligned}
&\mu\left(\left\{x \in X \mid |f_n(x)g_n(x) - f(x)g(x)| > 3\alpha\right\}\right) \\
&\quad \leq \mu\left(\left\{x \in X \mid |f_n(x) - f(x)| > \sqrt{\alpha}\right\}\right) \\
&\qquad + \mu\left(\left\{x \in X \mid |g_n(x) - g(x)| > \sqrt{\alpha}\right\}\right) \\
&\qquad + \mu(F_n) + \mu\left(\left\{x \in X \mid |g_n(x) - g(x)| > \alpha/N\right\}\right) \\
&\qquad + \mu(G_n) + \mu\left(\left\{x \in X \mid |f_n(x) - f(x)| > \alpha/N\right\}\right)
\end{aligned}$$

The convergence of $\{f_n\}$ (resp. $\{g_n\}$) to f (resp. g) in measure allows us to choose $M \in \mathbb{N}$ greater than N and such that the 4 measures above involving $|f_n - f|$ and $|g_n - g|$ are simultaneously less than ϵ when $n \geq M$. From this we deduce that $\{f_n g_n\}$ converges to fg in measure.

(b) The hypothesis that μ is finite is necessary. Consider for example for $x > 0$, $f_n(x) = \exp(x)$ and $g_n(x) = \frac{1}{x} + \frac{1}{n}$ for $n \geq 1$. We have $|g_n(x) - \frac{1}{x}| = \frac{1}{n}$; then $\{g_n\}$ converges uniformly to $x \mapsto \frac{1}{x}$, and therefore converges also in measure.

$$\begin{aligned}
f_n(x)g_n(x) - f(x)g(x) &= \exp(x)\left(\frac{1}{x} + \frac{1}{n}\right) - \frac{\exp(x)}{x} \\
&= \frac{1}{n}\exp(x)
\end{aligned}$$

Let $\epsilon > 0$, $\{x > 0 \mid \frac{1}{n}\exp(x) > \epsilon\} = [\ln(n\epsilon), +\infty)$, whose Lebesgue measure is always $+\infty$, so $\{f_n g_n\}$ does not converge in measure to fg. $\square$