

Exercise 6. Show that the hypothesis of σ -finiteness cannot be omitted from Proposition 3.4.5. (*Hint:* Consider counting measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.)

Proof. Let μ be the counting measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. The countable set $\mathcal{A} = \{(-\infty, r], r \in \mathbb{Q}\}$ satisfies $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R})$ (for example from exercise 1.1.2), so $\mathcal{B}(\mathbb{R})$ is countably generated by \mathcal{A} . Let $\{X_n\}_{n \in \mathbb{N}}$ be a family of Borel subsets of \mathbb{R} such that $\mu(X_i)$ is finite for all i . Since μ is the counting measure, X_i is also finite for all i . As a countable union of finite sets, $\cup_{n \in \mathbb{N}} X_n$ is countable; since \mathbb{R} is uncountable, $\mathbb{R} \neq \cup_{n \in \mathbb{N}} X_n$, and therefore μ is not σ -finite.

Let $\varepsilon > 0$ and suppose that there exists a family of functions $\{\varphi_n\}_{n \in \mathbb{N}}$ in $\mathcal{L}^p(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ such that:

$$\forall f \in \mathcal{L}^p(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu), \quad \exists n \in \mathbb{N}, \quad \|f - \varphi_n\|_p < \varepsilon$$

Let $\{f_x\}_{x \in \mathbb{R}}$ be the family of functions of $\mathcal{L}^p(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ defined by:

$$\forall y \in \mathbb{R}, \quad f(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

Let $x, y \in \mathbb{R}$ such that $x \neq y$; we have $\|f_x - f_y\|_p = 2^{1/p} \geq 1$. If $m, n \in \mathbb{N}$ are such that

$$\begin{aligned} \|f_x - \varphi_n\|_p &< \varepsilon \\ \|f_y - \varphi_m\|_p &< \varepsilon \end{aligned}$$

then the triangular inequality gives us

$$\|\varphi_n - \varphi_m\|_p \geq \|f_x - f_y\| - (\|f_x - \varphi_n\|_p + \|f_y - \varphi_m\|_p) > 1/3$$

as soon as $\varepsilon < 1/3$. From this we deduce that the function

$$\begin{aligned} \psi : \{f_x\}_{x \in \mathbb{R}} &\rightarrow \{\varphi_n\}_{n \in \mathbb{N}} \\ f_y &\mapsto \varphi_m \quad \text{such that } \|f_y - \varphi_m\|_p < \varepsilon \end{aligned}$$

is injective, and therefore $\{f_x\}_{x \in \mathbb{R}}$ must be countable, which is absurd. Therefore, the family $\{\varphi_n\}_{n \in \mathbb{N}}$ does not exist and $L^p(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ is not separable.

□