Exercise 10. Let l^2 be the set of all infinite sequences $\{x_n\}$ of real numbers for which $\sum_n x_n^2 < +\infty$.

(a) Show that l^2 is a vector space over $\mathbb{R}$. (Hint: show that $(x+y)^2 \leq 2x^2 + 2y^2$ holds for all reals x and y.)

(b) Show that the formula $(\{x_n\},\{y_n\}) = \sum_n x_n y_n$ defines an inner product on l^2 and hence (see part (b) of Exercise 7) that the formula $\|\{x_n\}\| = \left(\sum_n x_n^2\right)^{1/2}$ defines a norm on l^2. (The issue is the convergence of $\sum_n x_n y_n$.)

(c) Show that l^2 is complete under the norm defined in part (b) of this exercise.

Proof.

(a) The null sequence makes a convergent series of squares, so $\{0,0,\dots\} \in l^2$. Let $x, y \in \mathbb{R}$; we have:

$$(x+y)^2 = x^2 + 2xy + y^2 = 2x^2 + 2y^2 - (x-y)^2 \geq 0$$

and since $(x-y)^2 \geq 0$, we deduce that $(x+y)^2 \leq 2x^2 + 2y^2$. Let $\{x_n\},\{y_n\} \in l^2$; we have, for all $n \in \mathbb{N}$, $0 \leq (x_n+y_n)^2 \leq 2x_n^2 + 2y_n^2$, and since the series of terms x_n^2 and y_n^2 converge, the series of term $(x_n+y_n)^2$ converges too, so that $\{x_n+y_n\} \in l^2$. Similarly, for $\alpha, \beta \in \mathbb{R}$, $0 \leq (\alpha x_n + \beta y_n)^2 \leq 2\alpha^2 x_n^2 + 2\beta y_n^2$, so the same reasoning shows that $\{\alpha x_n + \beta y_n\} \in l^2$. From the above, we conclude that l^2 is a vector space on $\mathbb{R}$.

(b) Let $\{x_n\} \in l^2$, the series $(\{x_n\},\{x_n\}) = \sum_n x_n^2$ converges, and is a series of nonnegative terms, so $(\{x_n\},\{x_n\}) \geq 0$ and is zero if and only if $x_n = 0$ for all $n \in \mathbb{N}$.

Let $\{x_n\},\{y_n\},\{z_n\} \in l^2$ and $\alpha, \beta \in \mathbb{R}$; $(\{\alpha x_n + \beta y_n\},\{z_n\})$ is by definition the series of term $(\alpha x_n + \beta y_n) z_n = \alpha x_n z_n + \beta y_n z_n$. Since for all $x, y \in \mathbb{R}$ we have $4xy = (x+y)^2 - (x-y)^2$ and $(x+y)^2 \leq 2x^2 + 2y^2$, we deduce that, for all $n \in \mathbb{N}$:

$$\begin{aligned}
|\alpha x_n z_n + \beta y_n z_n| &= \left| \frac{\alpha}{4}(x_n+z_n)^2 - \frac{\alpha}{4}(x_n-z_n)^2 \right. \\
&\qquad \left. + \frac{\beta}{4}(y_n+z_n)^2 - \frac{\beta}{4}(y_n-z_n)^2 \right| \\
&\leq \frac{|\alpha|}{2}(x_n^2+z_n^2) + \frac{|\alpha|}{2}(x_n^2+z_n^2) \\
&\quad + \frac{|\beta|}{2}(y_n^2+z_n^2) + \frac{|\beta|}{2}(y_n^2+z_n^2) \\
&\leq |\alpha| x_n^2 + |\beta| y_n^2 + (|\alpha|+|\beta|)\, z_n^2
\end{aligned}$$

The convergence of the series $\sum_n x_n^2$, $\sum_n y_n^2$ and $\sum_n z_n^2$ implies the absolute convergence of the series $\sum_n \alpha x_n z_n + \beta y_n z_n$, hence the convergence of $(\{\alpha x_n + \beta y_n\}, \{z_n\})$.

Last, the commutativity of multiplication of real numbers implies that $(\{x_n\}, \{y_n\}) = (\{y_n\}, \{x_n\})$, so the defined formula is an inner product on l^2.

From Exercise 7, we conclude that $\|\{x_n\}\| = \left(\sum_n x_n^2\right)^{1/2}$ is a norm on l^2.

(c) Let $(X_n)_{n\in\mathbb{N}}$ be a Cauchy sequence of elements of l^2; for all $n \in \mathbb{N}$, we note $X_n = (x_{n,0}, x_{n,1}, \dots)$ and $\|X_n\| = \lambda_n \geq 0$.

We have $|\lambda_n - \lambda_m| = |\|X_n\| - \|X_m\|| \leq \|X_n - X_m\|$, so $\{\lambda_n\}$ is a Cauchy sequence of real numbers. Since $\mathbb{R}$ is complete, $\{\lambda_n\}$ converges to a real number $\lambda \geq 0$.

Let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that for all n, m greater than N,

$$\|X_n - X_m\|^2 = \sum_{k=0}^{+\infty} (x_{n,k} - x_{m,k})^2 < \varepsilon^2 \tag{1}$$

For all $k \in \mathbb{N}$, we have:

$$(x_{n,k} - x_{m,k})^2 \leq \|X_n - X_m\|^2 < \varepsilon^2$$

for all n, m greater than N, so the sequence $(x_{i,k})_{i\in\mathbb{N}}$ is a Cauchy sequence of real numbers and therefore converges to $\alpha_k \in \mathbb{R}$.

From this and (1), we deduce that

$$0 \leq \sum_{k=0}^{+\infty}(x_{n,k} - \alpha_k)^2 \leq \sum_{k=0}^{+\infty}(x_{n,k} - x_{m,k})^2 < \varepsilon^2$$

as soon as n, m are greater than N. So $\sum_k \alpha_k^2$ converges and

$$\sum_{k=0}^{+\infty} \alpha_k^2 = \lambda^2$$

Therefore the sequence $\{X_n\}$ converges to the sequence $\{\alpha_0, \alpha_1, \dots\}$ for the norm $\|\cdot\|$, and l^2 is complete for this norm.

$\square$