Exercise 5. Show by an example that the union of a collection of σ-algebras on a set X can fail to be a σ-algebra on X. (Hint: there are examples in which X is a small finite set.)

Proof. Let $X = \{a, b, c\}$, and let

$$\begin{aligned} \mathscr{A}_1 &= \big\{\varnothing, \{a\}, \{b, c\}, X\big\} \\ \mathscr{A}_2 &= \big\{\varnothing, \{b\}, \{a, c\}, X\big\} \end{aligned}$$

Then $\mathscr{A}_1$ and $\mathscr{A}_2$ are σ-algebras on X. Let

$$\begin{aligned} \mathscr{A} &= \mathscr{A}_1 \cup \mathscr{A}_2 \\ &= \big\{\varnothing, \{a\}, \{b\}, \{a, c\}, \{b, c\}, X\big\} \end{aligned}$$

Then $\mathscr{A}$ is not a σ-algebra on X, since $\{a, c\} \cap \{b, c\} = \{c\} \notin \mathscr{A}$.

□