Exercise 3. Show that Proposition 3.4.3 would be false if p were allowed to be infinite. (Hint: construct a Borel subset A of $[a,b]$ such that $\|\chi_A - f\|_\infty \geq 1/2$ holds whenever f is a step function.)

Proposition 3.4.3. *Suppose that $[a,b]$ is a closed bounded interval and that p satisfies $1 \leq p < +\infty$. Then the subspace of $L^p([a,b])$ determined by the step functions on $[a,b]$ is dense in $L^p([a,b])$.*

Proof. Let $0 < \alpha < 1$, there exists a Borel subset A of $[a,b]$ such that:

- $\lambda(A) = \alpha \cdot (b-a)$
- A does not contain any nonempty closed interval

Let $f \in \mathscr{L}^\infty([a,b])$ be a step function

$$f = \sum_{i=0}^{n} b_i \chi_{[a_i, a_{i+1}]}$$

with $a = a_0 < \cdots < a_n = b$ real numbers in $[a,b]$ and b_i complex numbers for all i. Suppose that $\|\chi_A - f\|_\infty < \alpha/3$. For all $0 \leq i \leq n$, there exists $x_0 \in [a_i, a_{i+1}]$ such that $x_0 \notin A$, since A does not contain any nonempty closed interval. From this we deduce that $\alpha/3 > \|\chi_A - f\|_\infty \geq |b_i|$ for all i. Both f and χ_A are integrable on $[a,b]$ and we have

$$\begin{aligned}
\int_{[a,b]} \chi_A \,\mathrm{d}\lambda &\leq \int_{[a,b]} |\chi_A - f| \,\mathrm{d}\lambda + \int_{[a,b]} |f| \,\mathrm{d}\lambda \\
\alpha \cdot (b-a) &\leq (b-a)\frac{\alpha}{3} + \sum_{i=0}^{n} |b_i| \cdot (a_{i+1} - a_i) \\
&\leq 2(b-a)\frac{\alpha}{3}
\end{aligned}$$

from which we deduce that $\alpha \leq 2\alpha/3$, which is a contradiction. Therefore no step function can approach χ_A arbitrarily, so that the subspace of $L^\infty([a,b])$ determined by the step functions is not dense in $L^\infty([a,b])$.

Let us show the existence of a set A with the expected properties. Since the Lebesgue measure is invariant by translation, it is enough to show the existence of such a set in the interval $[0, b-a]$. Given a Borel set B and some real number $x > 0$, $x \cdot B$ is also a Borel set, so we can further reduce the interval to $[0,1]$.

Let β be a real such that $0 < \beta < 1$, and let $\sum_k u_k$ be a series of positive real numbers such that $\sum_{k=0}^\infty u_k = 1 - \beta$. Let $(A_n)_{n \in \mathbb{N}}$ be a sequence of closed sets defined recursively by $A_0 = [0,1]$, and, for each n, A_{n+1} be the union of 2^{n+1} closed intervals I_n obtained by removing from each of the 2^n

closed intervals I_n composing A_n an open interval of length $u_n/2^n$ centered on I_n.

For all n, A_n is closed and measurable, and $\lambda(A_{n+1}) = 1 - (u_0 + u_1 + \ldots + u_n)$. Moreover, $A_{n+1} \subset A_n$, and $\lambda(A_0) = 1$ is finite, so

$$\lambda\left(\bigcap_{n=0}^{\infty} A_n\right) = \lim_{n\to\infty} \lambda(A_n) = 1 - \lim_{n\to\infty} \sum_{k=0}^{n} u_k = \beta$$

Let $A = \cap_{n\in\mathbb{N}} A_n$. A is measurable, has measure β, and does not include any nonempty closed interval: suppose that $[a, b] \subset A$, then $[a, b] \subset A_n$ for all n. Since A_n is the union of 2^n disjoint intervals I_n, each of measure $(1 - u_0 - \cdots - u_n)/2^n$, we have

$$0 \leq \lambda([a, b]) \leq \frac{1}{2^n}\left(1 - \sum_{k=0}^{n} u_k\right)$$

And since the series $\sum_k u_k$ of positive reals converges, we deduce that $\lambda([a, b]) = 0$, so that $[a, b] = \varnothing$.

$\square$