Exercise 7. Prove

Theorem 8.4 (Principle of recursive definition). *Let A be a set; let a_0 be an element of A. Suppose ρ is a function that assigns, to each function f mapping a nonempty section of the positive integers into A, an element of A. Then there exists a unique function*

$$h : \mathbb{Z}_+ \to A$$

such that

$$\begin{aligned} h(1) &= a_0 \\ h(i) &= \rho(h|\{1, \ldots, i-1\}) \qquad\qquad \text{for } i > 1. \end{aligned} \tag{1}$$

Let us follow the schema that was used earlier in the section and prove the following results:

Lemma 1. *Given $n \in \mathbb{Z}_+$, there exists a function $h : \{1, \ldots, n\} \to A$ that verifies (1) for all i in its domain.*

Proof. Let B be the subset of $\mathbb{Z}_+$ such that the lemma is true. If $n = 1$, then (1) gives $h(1) = 1$, which completely defines h, so $1 \in B$. Suppose now that the lemma is true for some $n \geq 1$, and let $h : \{1, \ldots, n\} \to A$ be defined by (1). Let

$$\begin{aligned} h' : \{1, \ldots, n+1\} &\to A \\ x &\mapsto \begin{cases} h(x) & \text{if } x \leq n \\ \rho(h) & \text{otherwise} \end{cases} \end{aligned}$$

The induction hypothesis allows us to conclude that h' is well-defined for $x \in \{1, \ldots, n\}$. Since h is defined from $\{1, \ldots, n\}$ to A, it is in the domain of ρ, so $\rho(h)$ exists and is an element of A. Therefore $h' : \{1, \ldots, n+1\} \to A$ is well-defined by the above formula. Furthermore, we have $h'(1) = h(1) = a_0$, and for $2 \leq i \leq n$,

$$h'(i) = h(i) = \rho(h|\{1, \ldots, i-1\}) = \rho(h'|\{1, \ldots, i-1\})$$

since $h = h'$ for $i \leq n$. Last, $h'(n+1) = \rho(h) = \rho(h'|\{1, \ldots, n\})$. From this we deduce that h' verifies (1), so B is inductive, and therefore $B = \mathbb{Z}_+$. $\square$

Lemma 2. *Given $f : \{1, \ldots, n\} \to A$ and $g : \{1, \ldots, m\} \to A$ two functions that verify (1) on their domains, $f(i) = g(i)$ for all $i \in \{1, \ldots, n\} \cap \{1, \ldots, m\}$.*

Proof. Suppose that there exists $i \in \{1, \ldots, n\} \cap \{1, \ldots, m\}$ such that $f(i) \neq g(i)$. Then let i_0 be the smallest such i. Note that (1) gives $f(1) = g(1) = a_0$

so $i_0 > 1$. For all $1 \leq i < i_0$, we have $f(i) = g(i)$. Since f and g verify (1), we have

$$f(i_0) = \rho(f|\{1, \ldots, i_0 - 1\}) = \rho(g|\{1, \ldots, i_0 - 1\}) = g(i_0)$$

This contradicts our hypothesis that $f(i_0) \neq g(i_0)$, so that i_0 does not exist, and therefore for all $i \in \{1, \ldots, n\} \cap \{1, \ldots, m\}$, $f(i) = g(i)$ $\square$

Let us now show that there exists a unique function $h : \mathbb{Z}_+ \to A$ that satisfies (1).

Proof. From lemma 1, for all $n \in \mathbb{Z}_+$ there is a function $f_n : \{1, \ldots, n\} \to A$ which verifies (1), and from lemma 2, this function is unique. Let us consider $R \subset \mathbb{Z}_+ \times A$ the union of the rules of f_n for all n. Let $i \in \mathbb{Z}_+$, for all $n \geq i$ and $m \geq i$, lemma 2 gives us $f_n(i) = f_m(i)$, so that there is a unique tuple $\big(i, f_n(i)\big)$ in R with i as its first coordinate. From this we deduce that R is the rule of a function $h : \mathbb{Z}_+ \to A$.

Since for all $n \in \mathbb{Z}_+$, $f_n(1) = a_0$, we deduce that $h(1) = a_0$. Then, for all $n > 1$, we have

$$\begin{aligned}
h(n+1) &= f_{n+1}(n+1) && \text{by definition of } R \\
&= \rho(f_{n+1}|\{1, \ldots, n\}) && \text{since } f_{n+1} \text{ verifies (1)} \\
&= \rho(h|\{1, \ldots, n\}) && \text{by definition of } R
\end{aligned}$$

so h verifies (1).

Last, suppose that there exist h and h' that satisfy (1) and are such that $h \neq h'$. Let i_0 be the smallest element of $\mathbb{Z}_+$ such that $h(i_0) \neq h'(i_0)$. We have $h(1) = h'(1) = a_0$ so that $i_0 > 1$. Also,

$$h'(i_0) = \rho(h'|\{1, \ldots, i_0 - 1\}) = \rho(h|\{1, \ldots, i_0 - 1\}) = h(i_0)$$

since h' and h are equal for all $i < i_0$. The above equation contradicts our hypothesis on the existence of i_0, so $h = h'$.

$\square$