Exercise 9.

(a) Show that every nonempty subset of $\mathbb{Z}$ that is bounded above has a largest element.

(b) If $x \notin \mathbb{Z}$, show there is exacty one $n \in \mathbb{Z}$ such that $n < x < n + 1$.

(c) If $x - y > 1$, show there is at least one $n \in \mathbb{Z}$ such that $y < n < x$.

(d) If $y < x$, show there is a rational number z such that $y < z < x$.

Proof.

(a) Let A be a nonempty subset of $\mathbb{Z}$ that is bounded above. There exists $b \in \mathbb{R}$ such that $\forall x \in A, \quad x < b$. Since $\mathbb{Z}$ is unbounded, there exists $m \in \mathbb{Z}$ such that $b < m$, so the subset B of $\mathbb{Z}$ containing all integer upper bounds of A is nonempty. Therefore B has a smallest element u. If $u \notin A$, then the largest element in A is at most $u - 1$ since $u \in \mathbb{Z}$, so that $u - 1 < u$ is an upper bound for A, which contradicts the fact that u is the smallest such upper bound; so u is the largest element of A.

(b) Let $x \in \mathbb{R} - \mathbb{Z}$. Since $\mathbb{Z}$ is unbounded, there exists $m \in \mathbb{Z}$ such that $m > x$. The subset B of $\mathbb{Z}$ containing all integer upper bounds of $\{x\}$ is nonempty, and thus has a smallest element u. Then $u > x$ since x is not an integer, and $u - 1 < x$ since u is the smallest element of B.

(c) Let $x, y \in \mathbb{R}$ such that $x - y > 1$, and let $X = \{z \in \mathbb{Z} \mid z < x\}$. Note that since $\mathbb{Z}$ is unbounded, there is an element $m \in \mathbb{Z}$ such that $m > -x$. Then we have $-m < x$, so that X is not empty. Since X is also bounded above, it has a largest element u, and we have $u < x$. If $u < y$, then $x - u > x - y > 1$, so that $x > u + 1$, which contradicts the fact that u is the largest element of X. So we have $y < u < x$.

(d) Let $x, y \in \mathbb{R}$ such that $y < x$. The real number $1/(x - y)$ exists, and since $\mathbb{Z}_+$ is unbounded, $A = \{n \in \mathbb{Z}_+ \mid n > 1/(x - y)\}$ is nonempty. Let u be the smallest element in A; since $x - y > 0$, we have $u > 0$. From the definition of u we deduce that $ux - uy > 1$, so that there exists an integer d such that $uy < d < ux$. This implies that the rational number d/u verifies $y < d/u < x$.

$\square$