Exercise 4. Let $m, n \in \mathbb{Z}_+$. Let $X \neq \varnothing$.

(a) If $m \leq n$, find an injective map $f : X^m \to X^n$.

(b) Find a bijective map $g : X^m \times X^n \to X^{m+n}$.

(c) Find an injective map $h : X^n \to X^\omega$.

(d) Find a bijective map $k : X^n \times X^\omega \to X^\omega$.

(e) Find a bijective map $l : X^\omega \times X^\omega \to X^\omega$.

(f) If $A \subset B$, find an ijective map $m : (A^\omega)^n \to B^\omega$.

Proof.

(a) Since $X \neq \varnothing$, let $x_0 \in X$, and define

$$\begin{aligned} f : X^m &\to X^n \\ (x_1, x_2, \ldots, x_m) &\mapsto (y_1, y_2, \ldots, y_n) \end{aligned}$$

with

$$\forall i \in \{1, 2, \ldots, n\}, \begin{cases} y_i = x_i & \text{if} \quad 1 \leq i \leq m \\ y_i = x_0 & \text{if} \quad m+1 \leq i \leq n \end{cases}$$

Let

$$\begin{aligned} (y_1, y_2, \ldots, y_n) &= f(x_1, x_2, \ldots x_m) \quad \text{and} \\ (y'_1, y'_2, \ldots, y'_n) &= f(x'_1, x'_2, \ldots, x'_m) \end{aligned}$$

and suppose that $(y_1, y_2, \ldots, y_n) = (y'_1, y'_2, \ldots, y'_n)$. Then by definition of f, $y_i = x_0 = y'_i$ for all i such that $m+1 \leq i \leq n$, and $y_i = x_i = y'_i = x'_i$ for all i such that $1 \leq i \leq m$. From this we deduce that $\forall i,\ 1 \leq i \leq m \implies x_i = x'_i$, so that f is injective.

(b) Define

$$\begin{aligned} g : X^m \times X^n &\to X^{m+n} \\ (x, y) &\mapsto (z_1, z_2, \ldots, z_{m+n}) \end{aligned}$$

with

$$\forall i \in \{1, 2, \ldots, m+n\}, \begin{cases} z_i = x_i & \text{if} \quad 1 \leq i \leq m \\ z_i = y_i & \text{if} \quad m+1 \leq i \leq m+n \end{cases}$$

Let

$$\begin{aligned} z = (z_1, z_2, \ldots, z_{m+n}) &= g((x_1, x_2, \ldots, x_m), (y_1, y_2, \ldots, y_n)) \\ z' = (z'_1, z'_2, \ldots, z'_{m+n}) &= g((x'_1, x'_2, \ldots, x'_m), (y'_1, y'_2, \ldots, y'_n)) \end{aligned}$$

and suppose that $z = z'$. Then for all i such that $1 \le i \le m$, $z_i = x_i = z'_i = x'_i$, and for all i such that $m+1 \le i \le m+n$, $z_i = y_i = z'_i = y'_i$, from which we deduce that g is injective. Let now $z = (z_1, z_2, \dots, z_{m+n}) \in X^{m+n}$, and define

$$\begin{aligned} x &= (z_1, z_2, \dots, z_m) \\ y &= (z_{m+1}, z_{m+2}, \dots, z_{m+n}) \end{aligned}$$

Then $(x, y) \in X^m \times X^n$, and $g(x, y) = z$, so that g is surjective.

(c) Let $x_0 \in X$ and define

$$\begin{aligned} h : X^n &\to X^\omega \\ (x_1, x_2, \dots, x_n) &\mapsto (z_1, z_2, \dots) \end{aligned}$$

such that

$$\begin{aligned} z_i &= x_i \quad \text{if} \quad 1 \le i \le n \\ z_i &= x_0 \quad \text{if} \quad i > n \end{aligned}$$

Let

$$\begin{aligned} z &= (z_1, z_2, \dots) = h(x_1, x_2, \dots, x_n) \\ z' &= (z'_1, z'_2, \dots) = h(x'_1, x'_2, \dots, x'_n) \end{aligned}$$

and suppose that $z = z'$. Then we have

$$\forall i \in \mathbb{Z}_+, \quad \begin{cases} z_i = z'_i = x_0 & \text{if} \quad i > n \\ z_i = x_i = z'_i = x'_i & \text{if} \quad 1 \le i \le n \end{cases}$$

so that h is injective.

(d) Let

$$\begin{aligned} k : X^n \times X^\omega &\to X^\omega \\ (x, y) &\mapsto z \end{aligned}$$

such that

$$\forall i \in \mathbb{Z}_+, \quad \begin{cases} z_i = x_i & \text{if} \quad 1 \le i \le n \\ z_i = y_i & \text{if} \quad i > n \end{cases}$$

Let

$$\begin{aligned} z &= (z_1, z_2, \dots) & x &= (x_1, x_2, \dots, x_n) & y &= (y_1, y_2, \dots) \\ z' &= (z'_1, z'_2, \dots) & x' &= (x'_1, x'_2, \dots, x'_n) & y' &= (y'_1, y'_2, \dots) \end{aligned}$$

and suppose that $z = k(x, y) = z' = k(x', y')$. Then,

$$\forall i \in \mathbb{Z}_+, \begin{cases} z_i = z'_i = y_i = y'_i & \text{if} \quad i > n \\ z_i = z'_i = x_i = x'_i & \text{if} \quad 1 \leq i \leq n \end{cases}$$

so that k is injective. Next, let $z = (z_1, z_2, \dots) \in X^\omega$ and take $x \in X^n$ and $y \in X^\omega$ such that

$$\forall i \in \mathbb{Z}_+, \begin{cases} x_i = z_i & \text{if} \quad 1 \leq i \leq n \\ y_i = z_i & \text{if} \quad i > n \end{cases}$$

Then $k(x, y) = z$, so that k is surjective.

(e) Let

$$\begin{aligned} l : X^\omega \times X^\omega &\to X^\omega \\ (x, y) &\mapsto z \end{aligned}$$

such that

$$\forall i \in \mathbb{Z}_+, \begin{cases} z_i = x_i & \text{if } i \text{ is even} \\ z_i = y_i & \text{if } i \text{ is odd} \end{cases}$$

Let

$$\begin{aligned} z &= (z_1, z_2, \dots) & x &= (x_1, x_2, \dots) & y &= (y_1, y_2, \dots) \\ z' &= (z'_1, z'_2, \dots) & x' &= (x'_1, x'_2, \dots) & y' &= (y'_1, y'_2, \dots) \end{aligned}$$

and suppose that $z = l(x, y) = z' = l(x', y')$. Then,

$$\forall i \in \mathbb{Z}, \begin{cases} z_{2i} = x_i = z'_{2i} = x'_i \\ z_{2i+1} = y_i = z'_{2i+1} = y'_i \end{cases}$$

so that for all i, $x_i = x'_i$ and $y_i = y'_i$, and l is injective. Next, let $z \in X^\omega$, and define $x \in X^\omega$ and $y \in X^\omega$ by

$$\forall i \in \mathbb{Z}_+,\ x_i = z_{2i} \text{ and } y_i = z_{2i+1}$$

Then x and y are elements of X^ω and $l(x, y) = z$, so that l is surjective.

(f) To define such a function, we need $B^\omega \neq \varnothing$, which implies $B \neq \varnothing$. Let

$$\begin{aligned} m : (A^\omega)^n &\to B^\omega \\ (a_1, a_2, \dots, a_n) &\mapsto (a_{1,1}, a_{1,2}, \dots, a_{1,n}, a_{2,1}, \dots) \end{aligned}$$

noting $a_i = (a_{1,i}, a_{2,i}, \dots)$. Suppose that $a, a' \in (A^n)^\omega$ are such that $a \neq a'$. Then there exists $r \in \{1, 2, \dots, n\}$ such that $a_r \neq a'_r$, which in turns implies that there exists $q \in \mathbb{Z}_+$ such that $a_{q,r} \neq a'_{q,r}$. The terms $a_{q,r}$ and $a'_{q,r}$ appear in $m(a)$ and $m(a')$ at the same index i, so $m(a) \neq m(a')$ and m is therefore injective.

$\square$