Exercise 4.

(a) A real number a is said to be *algebraic* (over the rationals) if it satisfies some polynomial equation of positive degree

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x_1 + a_0 = 0$$

with rational coefficients a_i. Assuming that each polynomial equation has only finitely many roots, show that the set of algebraic numbers is countable.

(b) A real number is said to be *transcendental* if it is not algebraic. Assuming the reals are uncountable, show that the transcendental numbers are uncountable. (It is a somewhat surprising fact that only two transcendental numbers are familiar to us: e and π. Even proving these numbers transcendental is highly nontrivial.)

Proof.

(a) Let $\mathscr{P}$ be the set of polynomials of the form $X^n + a_{n-1}X^{n-1} + \cdots + a_0$ for some $n \in \mathbb{Z}_+$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{Q}$. Let $Q = \cup_{i \in \mathbb{Z}_+} \mathbb{Q}^n$, and let

$$\begin{aligned} \phi : \mathscr{P} &\to Q \\ X^n + a_{n-1}X^{n-1} + \cdots + a_0 &\mapsto (a_0, a_1, \ldots, a_{n-1}) \end{aligned}$$

Let $S, T \in \mathscr{P}$ such that $\phi(S) = \phi(T)$, and note

$$\begin{aligned} \phi(S) &= s = (s_1, s_2, \ldots, s_n) \\ \phi(T) &= t = (t_1, t_2, \ldots, t_m) \end{aligned}$$

For all $p \in \mathbb{Z}_+$, $\mathbb{Q}^p$ is the set of functions $\{1, \ldots, p\} \to \mathbb{Q}$, so for s and t to be equal, they must have the same domain, and therefore $m = n$. From this and $\phi(S) = \phi(T)$ we deduce that $s_i = t_i$ for all $i \in \{1, \ldots, n\}$, and $S(X) = X^n + s_{n-1}X^{n-1} + \cdots + s_0 = X^n + t_{n-1}X^{n-1} + \cdots + t_0 = T(X)$, so ϕ is injective. Since $\mathbb{Q}$ is countable, the finite product $\mathbb{Q}^n$ is countable for all n, and the countable union Q of countable sets is also countable. From this we deduce that $\mathscr{P}$ is countable.

Let $\mathscr{A} \subset \mathbb{R}$ be the set of algebraic numbers over $\mathbb{Q}$. For all $P \in \mathscr{P}$, let $R_P = \{x \in \mathbb{R} \mid P(x) = 0\}$ be the set of real roots of P. Then $\mathscr{A} = \cup_{P \in \mathscr{P}} R_P$. Since R_P is finite, and $\mathscr{P}$ is countable, $\mathscr{A}$ is the countable union of finite sets, and hence countable.

(b) Let $x \in \mathbb{C}$, $\mathscr{A}$ be the set of algebraic numbers, and $\mathscr{T}$ be the set of transcendental numbers. We have $\mathbb{C} = \mathscr{A} \cup \mathscr{T}$. If $\mathscr{T}$ were countable, then $\mathbb{C}$ would be the union of two countable sets, and therefore countable, which is impossible. Therefore, $\mathscr{T}$ is uncountable.

$\square$