Exercise 7. Show that the Tukey lemma implies the Hausdorff maximum principle. [*Hint:* if $\prec$ is a strict partial order on A, let $\mathscr{A}$ be the collection of all subsets of A that are simply ordered by $\prec$. Show that $\mathscr{A}$ is of finite type.]

Proof. Let $\mathscr{A}$ be the collection of all subsets of A that are simply ordered by $\prec$, let $B \in \mathscr{A}$, and let C be a finite subset of B. As a subset of a simply ordered set , C is simply ordered, (elements of C are comparable as elements of B) and therefore $C \in \mathscr{A}$. From this we deduce that $\mathscr{A}$ is of finite type, and, from the Tukey lemma, there exists $M \in \mathscr{A}$ such that no element of A properly contains M. The element M is a simply ordered subset of A such that for all $B \in \mathscr{A}$, $M \subset B \Rightarrow M = B$, and therefore M is maximal in the sense of Hausdorff's maximum principle.

$\square$