Exercise 6. Show that the topologies of $\mathbb{R}_\ell$ and $\mathbb{R}_K$ are not comparable.

Proof. Let $x = 0$ and $B = (-1, 1) - K$; suppose that there exists $B' = [a, b)$ containing x such that $B' \subset B$. If there exists $n \in \mathbb{Z}_+$ such that $b > 1/n$, then $1/(n + 1)$ is in B' but not in B, which contradicts $B' \subset B$. Therefore $b < 1/n$ for all n; but this implies that $b \leq 0$, so that $0 \notin B'$, contrary to our hypothesis. Thus the topology of $\mathbb{R}_\ell$ is not finer than that of $\mathbb{R}_K$.

Consider now $B' = [0, 1)$ and suppose that there exists $B \in \mathbb{R}_K$ containing x and included in B'. The set B has a greatest lower bound a. Since $0 \in B$ we have $a \leq 0$, and since $B \subset B'$ we have $a \geq 0$. Therefore $B = (0, b)$ or $(0, b) - K$ for some real $b \geq 0$; but this implies that $0 \notin B$, contrary to hypothesis. Therefore the topology of $\mathbb{R}_K$ is not finer than that of $\mathbb{R}_\ell$.

From the above we conclude that these two topologies are not comparable.

$\square$