Exercise 4. Let $f : A \to B$ be a surjective function. Let us define a relation on A by setting $a_0 \sim a_1$ if

$$f(a_0) = f(a_1)$$

(a) Show that this is an equivalence relation

(b) Let A^* be the set of equivalence classes. Show there is a bijective correspondence of A^* with B

Proof.

(a) The fact that every $x \in A$ has a unique image by f, and that equality is an equivalence relation, imply that $\sim$ is an equivalence relation on A.

(b) Let $x^* \in A^*$. By definition of $\sim$, for all $y, z \in x^*$, we have $f(y) = f(z)$. So the following definition uniquely associates an image to an element, and is thus a function:

$$\begin{aligned} \phi : A^* &\to B \\ x^* &\mapsto f(x) \end{aligned}$$

The function ϕ is surjective: f being surjective, $\forall y \in B, \quad \exists z \in A, \quad f(x) = y$. Since $\sim$ is an equivalence relation on A, $x \sim x$, from which we deduce that $\phi(x^*) = y$, which is the surjectivity of ϕ.

$\square$