Exercise 10. (a) Show that the map $f : (-1,1) \to \mathbb{R}, x \mapsto x/(1-x^2)$ is order-preserving.

(b) Show that the equation $g(y) = 2y \,/\, \left(1 + \left(1+4y^2\right)^{1/2}\right)$ defines a function $g : \mathbb{R} \to (-1,1)$ that is both a left and a right inverse for f.

Proof.

(a) The function f is strictly increasing in the interval $(-1,1)$ and thus order-preserving. Remark that $f(x) = 1/2 \tan\theta$ for $x = \tan(\theta/2)$ and $\theta \in (-\pi/2, \pi/2)$, so that f is the composition of strictly increasing functions, and thus strictly increasing.

(b) As a continuous and strictly increasing function on its domain $(-1,1)$, f is bijective and has thus a unique inverse (which is both a left and a right inverse). It is thus enough to verify that $f \circ g(y) = y$ for all y in $\mathbb{R}$. Letting $t = \sqrt{1+4y^2}$, we get

$$\begin{aligned}
1 - g^2(y) &= 1 - \frac{t^2-1}{(1+t)^2} \\
&= 1 - \frac{t-1}{t+1} = \frac{2}{t+1} \\
\frac{g(y)}{1-g^2(y)} &= \frac{t+1}{2}\sqrt{\frac{t-1}{t+1}} \\
&= \sqrt{\frac{t^2-1}{4}} = y
\end{aligned}$$

for $y \geq 0$. Since $f(-x) = -f(x)$, $g(-y) = -g(y)$, and $y \geq 0 \iff g(y) \geq 0$, we apply the above to $-y$ when $y < 0$ to get

$$\begin{aligned}
f \circ g(-y) &= f\left(-g(y)\right) = -f \circ g(y) \\
&= -y
\end{aligned}$$

which gives the expected result.

$\square$