**Exercise 10.**

**Theorem.** *Let $J$ and $C$ be well-ordered sets; assume that there is no surjective function mapping a section of $J$ onto $C$. Then there exists a unique function $h : J \to C$ satisfying the equation:*

$$h(x) = \text{smallest of } [C - h(S_x)] \tag{1}$$

*for each $x \in J$, where $S_x$ is the section of $J$ by $x$.*
Proof.

(a) *If $h$ and $k$ map sections of $J$, or all of $J$, into $C$ and satisfy (1) for all $x$ in their respective domains, show that $h(x) = k(x)$ for all $x$ in both domains.*

(b) *If there exists a function $h : S_\alpha \to C$ satisfying (1), show that there is a function $k : S_\alpha \cup \{\alpha\} \to C$ satisfying (1).*

(c) *If $K \subset J$ and for all $\alpha \in K$ there exists a function $h_\alpha : S_\alpha \to C$ satisfying (1), show that there exists a function*

$$k : \bigcup_{\alpha \in K} S_\alpha \to C$$

*satisfying (1).*

(d) *Show by transfinite induction that for every $\beta \in J$, there exists a function $h_\beta : S_\beta \to C$ satisfying (1). [*Hint: *If $\beta$ has an immediate predecessor $\alpha$, then $S_\beta = S_\alpha \cup \{\alpha\}$. If not, $S_\beta$ is the union of all $S_\alpha$ with $\alpha < \beta$.]*

(e) *Prove the theorem.*

*Proof.*

If $C = \varnothing$, there is no function from $J$ to $C$, so we assume that $C$ is nonempty.

(a) Let $\mathscr{D}_h$ and $\mathscr{D}_k$ be the respective domains of $h$ and $k$, and suppose there exists and element $x \in \mathscr{D}_h \cap \mathscr{D}_k$ such that $h(x) \neq k(x)$. Since $J$ is well-ordered, there exists a smallest such element $m$. For all $x < m$, we have $h(x) = k(x)$, so that $h(S_m) = k(S_m)$. Since there is no surjection of $J$ onto $C$, the set $C - h(S_m)$ is nonempty, and, as a subset of the well-ordered set $C$, has a smallest element $x_0$, which is also the smallest element of $C - k(S_m)$. From (1), we deduce that $h(m) = x_0 = k(m)$, which contradicts the existence of $m$.

Therefore, for all $x \in \mathscr{D}_h \cap \mathscr{D}_k$ we have $h(x) = k(x)$.

(b) Since there is no surjection from $J$ onto $C$ and $S_\alpha \subset J$, $h$ is not surjective and therefore $C - h(S_\alpha)$ is a nonempty subset of the well-ordered set $C$. From this we deduce that the smallest element $m$ of $C - h(S_\alpha)$ exists. Let

$$\begin{aligned} k : S_\alpha \cup \{\alpha\} &\to C \\ x &\mapsto \begin{cases} h(x) & \text{if } x \in S_\alpha \\ m & \text{otherwise} \end{cases} \end{aligned}$$

For all $x \in S_\alpha$, $h(x) = k(x)$, so $C - k(S_\alpha) = C - h(S_\alpha)$ and $k$ satisfies (1) on $S_\alpha$. Since $m = k(\alpha)$ is the smallest element of $C - k(S_\alpha)$, we conclude that $k$ satisfies (1) on $S_\alpha \cup \{\alpha\}$.

(c) Let $R \subset \cup_{\alpha \in K} S_\alpha \times C$ consist of all the tuples $(x, h_\alpha(x))$ for $\alpha \in K$ and $x \in S_\alpha$. Then $R$ is the rule of a function: let $x \in \cup_{\alpha \in K} S_\alpha$, there exists $\beta \in K$ such that $x \in S_\beta$, and for all $\gamma \in K$ such that $\beta < \gamma$, we deduce from point (a) that $h_\beta = h_\gamma$ on $S_\beta$. Therefore there is a unique element $(a, b) \in R$ such that $a = x$. Let

$$\begin{aligned} k : \bigcup_{\alpha \in K} S_\alpha &\to C \\ x &\mapsto h_\alpha(x) \quad \text{for } x \in S_\alpha \end{aligned}$$

The function $k$ satisfies (1): let $x \in \cup_{\alpha \in K} S_\alpha$, there exists $\alpha \in K$ such that $x \in S_\alpha$. The function $h_\alpha$ satisfies (1) on its domain $S_\alpha$, so that

$$k(x) = h_\alpha(x) = \min\left(C - h_\alpha\left(S_x\right)\right) = \min\left(C - k\left(S_x\right)\right)$$

The last equality comes from the fact that $x \in S_\alpha$ implies $x < \alpha$, so that for all $y \in S_x$ we have $k(y) = h_x(y) = h_\alpha(y)$.

(d) Let $J_0$ be the subset of $J$ such that for all $\beta \in J_0$, there exists a function $h_\beta : S_\beta \to C$ which satisfies (1). Suppose that $J - J_0$ is nonempty and let $\beta$ be its smallest element.

If $\beta$ has an immediate predecessor $\alpha$, then $\alpha \in J_0$. There exists a function $h_\alpha : S_\alpha \to C$ satisfying (1), and from point (b), we deduce that there exists a function $h_\beta$ from $S_\beta = S_\alpha \cup \{\alpha\}$ to $C$ satisfying (1), which is a contradiction with the definition of $\beta$.

Therefore $\beta$ does not have an immediate predecessor. For all $x < \beta$, we have $S_x \subset J_0$, and since $S_\beta = \cup_{x<\beta} S_x$, we deduce from (c) that $\beta \in J_0$, again a contradiction.

Therefore $\beta$ does not exist, and for all $\alpha \in J$, there exists $h_\alpha : S_\alpha \to C$ which satisfies (1).

(e) Suppose that $J$ has a largest element $M$, then $J = S_M \cup \{M\}$. From (d), we deduce the existence of a function $h_M : S_M \to C$ that satisfies (1). Then from (b), we deduce the existence of a function $h : J \to C$ which satisfies (1).

Suppose otherwise that $J$ does not have a largest element. For all $\alpha \in J$, we deduce from (d) the existence of a function $h_\alpha : S_\alpha \to C$ satisfying (1). Let $x \in J$; since $J$ does not have a largest element, there exists $y \in J$ such that $x < y$. From this we deduce that $J = \cup_{x \in J} S_x$.

Let $R$ the subset of $\cup_{x \in J} S_x \times C$ consisting of all the tuples $(y, h_x(y))$ for $x \in J$ and $y \in S_x$, and let $\alpha \in J$. There exists $\beta \in J$ such that $\alpha < \beta$, so $\alpha \in S_\beta$; from (d) we deduce the existence of $h_\alpha : S_\alpha \to C$ and $h_\beta : S_\beta \to C$ satisfying (1). Then from (a), we deduce that for all $x \in S_\alpha$, $h_\alpha(x) = h_\beta(x)$, so that there is only one tuple $(a, b) \in R$ such that $a = x$. Let $h : J \to C$ be the function with rule $R$. Then $h$ satisfies (1): let $x \in J$, there exists $\alpha > x$. We have:

$$\begin{aligned}
h(x) &= h_\alpha(x) \\
&= \min\big(C - h_\alpha(S_x)\big) && \text{since } h_\alpha \text{ satisfies (1)} \\
&= \min\big(C - h(S_x)\big) && \text{since } h_\alpha = h \text{ on } S_x
\end{aligned}$$

Suppose that there are 2 functions $h$ and $k$ from $J$ to $C$ that satisfy (1), and suppose that there exists $x \in J$ such that $h(x) = k(x)$. Let $\alpha \in J$ be the smallest such element; we have $h(y) = k(y)$ for all $y \in S_x$. From (1) we deduce that $h(x) = k(x)$, which is a contradiction with the existence of $x$. Therefore such an $x$ does not exist and $h$ is unique.

$\square$