Exercise 8. A typical use of Zorn's lemma in algebra is the proof that every vector space has a basis. Recall that is A is a subset of the vector space V, we say a vector belongs to the *span* of A if it equals a finite linear combination of elements of A. The set A is *independent* if the only finite linear combination of elements of A that equals the zero vector is the trivial one having all coefficients zero. If A is independent and if every vector in V belongs to the span of A, then A is a *basis* for V.

(a) If A is independent and $v \in V$ does not belong to the span of A, show that $A \cup \{v\}$ is independent.

(b) Show the collection of all independents sets in V has a maximal element.

(c) Show that V has a basis.

Proof.

If the only vector in V is the zero vector, then V does not have a basis. Therefore we suppose here that V has at least one nonzero vector.

(a) Suppose that A is an independent set of vectors of V. Suppose that $v \in V$ is not in the span of A. and suppose that for some linear combination c of vectors of A and some scalar α, we have $\alpha \cdot v + c = 0$. If $\alpha = 0$, then $c = 0$, and since A is independent, every coefficient in c is also 0; from this we deduce that $A \cup \{v\}$ is independent. Otherwise, the inverse α^{-1} exists and we have $v = -\alpha^{-1} \cdot c$, a linear combination of vectors of A, so that v is in the span of A, contrary to the hypothesis. Therefore $A \cup \{v\}$ is independent.

(b) Let $\mathscr{A}$ be the set of all independent sets of vectors of V. Proper inclusion is a strict partial order on $\mathscr{A}$, so from the maximum principle, there exists a subset M of $\mathscr{A}$ that is simply ordered and maximal. Let $U = \cup_{m \in M} m$, and let v be a linear combination of vectors $v_1, \ldots, v_n$ in U such that $v = 0$. For all i, there exists $V_i \in M$ such that $v_i \in V_i$. The finite set $\{V_1, \ldots, V_n\}$, as a subset of the linearly ordered set M, is linearly ordered; therefore it has a maximal element V_m. For all $i \in \{1, \ldots, n\}$, we have $v_i \in V_m$. As an element of $\mathscr{A}$, V_m is independent. From this we conclude that all coefficients in the linear combination v are zero, and therefore U is independent.

For all $m \in M$, we have $m \subset U$. Suppose that there exists $W \in \mathscr{A}$ such that $U \subset W$. Then for all $m \in M$, we also have $m \subset W$, so that $M \cup W$ is simply ordered by proper inclusion and strictly contains M, contrary to the fact that M is maximal for proper inclusion. Therefore U is a maximal element of $\mathscr{A}$.

(c) Let U be the element defined in the above point, and let $v \in V$. Suppose that v is not in the span of U; then the set $U \cup \{v\}$ is independent. For all $B \subset U$, we have $B \subset U \cup \{v\}$, so $U \cup \{v\}$ is simply ordered, and properly contains U, contrary to the fact that U is a maximal element of $\mathscr{A}$. Therefore v is in the span of U. Since U is independent, we deduce that U is a basis for V.

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