

Exercise 6. A collection \mathcal{A} of subsets of a set X is said to be of *finite type* provided that a subset B of X belongs to \mathcal{A} if and only if every finite subset of B belongs to \mathcal{A} . Show that the Kuratowski lemma implies the following:

Lemma (Tukey, 1940). *Let \mathcal{A} be a collection of sets. If \mathcal{A} is of finite type, then \mathcal{A} has an element that is properly contained in no other element of \mathcal{A} .*

Proof. Let \mathcal{A} be a collection of sets such that \mathcal{A} is of finite type, and let \mathcal{B} be a subcollection of \mathcal{A} that is simply ordered by proper inclusion. For all $B \in \mathcal{B}$, $B \in \mathcal{A}$, and \mathcal{A} is of finite type, so every finite subset of B is an element of \mathcal{A} .

Let $U = \cup_{B \in \mathcal{B}} B$, and let C be a finite subset of U . If C is empty, then C is a subset of \mathcal{A} . Otherwise, $C = \{c_1, \dots, c_n\}$, and for all $i \in \{1, \dots, n\}$, there exists $B_i \in \mathcal{B}$ such that $c_i \in B_i$. As a subset of the simply ordered set \mathcal{B} , the set $\{B_1, \dots, B_n\}$ is simply ordered. Since it is also finite, it has a largest element B_m . For all $i \in \{1, \dots, n\}$, we have $B_i \subset B_m$. From this we deduce that C is a finite subset of B_m , and hence an element of \mathcal{A} .

Every finite subset of the set U is an element of \mathcal{A} , and \mathcal{A} is of finite type, so $U \in \mathcal{A}$. We can therefore apply Kuratowski's lemma and conclude that there exists an element of \mathcal{A} that is properly contained in no other element of \mathcal{A} .

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