Exercise 13. Prove the following:
Theorem: if an ordered set A has the least upper bound property, then it has the greatest lower bound property.

Proof. Let A be an ordered set. If $A = \varnothing$, then there is no nonempty subset of A, so the theorem is vacuously true. Suppose then that $A \neq \varnothing$, and let C be a nonempty subset of A that is bounded below by $c \in A$. Let B be the set of lower bounds of C; we have $c \in B$ so B is nonempty. Since C is nonempty, we can take $c_0 \in C$, and we have $\forall y \in B, \quad y \leq c_0$, so that B is bounded above. Since A has the least upper bound property, B has a lowest upper bound u.

By definition, u is the smallest element of the set of upper bounds of B in A. If there exists $c_1 \in C$ such that $c_1 < u$, then let $b \in B$. By definition of B, we have $b \leq c_1$, so that c_1 is an upper bound of B that is smaller than u. This is a contradiction, so $\forall c \in C, \quad u \leq c$, and u is a lower bound of C.

If $v \in A$ is a lower bound of C, then by definition of B, $v \in B$, so that $v \leq u$. From this we deduce that u is the greatest lower bound of C in A, so that A has the greatest lower bound property. $\square$