Exercise 4.

(a) If $\{\mathscr{T}_\alpha\}$ is a family of topologies on X, show that $\cap\mathscr{T}_\alpha$ is a topology on X. Is $\cup\mathscr{T}_\alpha$ a topology on X?

(b) Let $\mathscr{T}_\alpha$ be a family of topologies on X. Show that there is a unique smallest topology on X containing all the collections $\mathscr{T}_\alpha$, and a unique largest topology contained in all $\mathscr{T}_\alpha$.

(c) If $X = \{a, b, c\}$, let

$$\mathscr{T}_1 = \big\{\varnothing, X, \{a\}, \{a, b\}\big\} \qquad \text{and} \qquad \mathscr{T}_2 = \{\varnothing, X, \{a\}, \{b, c\}\}$$

Find the smallest topology containing $\mathscr{T}_1$ and $\mathscr{T}_2$, and the largest topology contained in $\mathscr{T}_1$ and $\mathscr{T}_2$.

Proof.

(a) Since $\mathscr{T}_\alpha$ is a topology for all α, it contains both X and $\varnothing$. These two sets are thus elements of $\cap\mathscr{T}_\alpha$.

Let $(U_\beta)_{\beta\in J}$ be a family of elements of $\cap\mathscr{T}_\alpha$. For all $\mathscr{T}_\alpha$ and all $\beta \in J$, $U_\beta \in \mathscr{T}_\alpha$, which is a topology; therefore $\cup_{\beta\in J} U_\beta \in \mathscr{T}_\alpha$. This implies that $\cup_{\beta\in J} U_\beta \in \cap\mathscr{T}_\alpha$.

Similarly, for all $\mathscr{T}_\alpha$ and all finite collection $\{U_1, \ldots, U_n\}$ of elements of $\mathscr{T}_\alpha$, $U_1 \cap \cdots \cap U_n$ is an element of $\mathscr{T}_\alpha$, and therefore an element of $\cap\mathscr{T}_\alpha$.

From the above we conclude that $\cap\mathscr{T}_\alpha$ is a topology.

Let $X = \{a, b\}$, $T_1 = \big\{\varnothing, X, \{a\}\big\}$ and $T_2 = \{\varnothing, X, \{b\}\}$. Then $\{a\}$ and $\{b\}$ are elements of $T_1 \cup T_2$, but $\{a, b\}$ is not. Therefore $\cup\mathscr{T}_\alpha$ is not in general a topology.

(b) Let $T = \cap\mathscr{T}_\alpha$; it is a topology contained in all $\mathscr{T}_\alpha$. Let T' be a topology contained in all $\mathscr{T}_\alpha$, and let $U \in T'$. For all α, $U \in \mathscr{T}_\alpha$, so that $T' \subset T$, and T is then the largest topology contained in all $\mathscr{T}_\alpha$.

Let $S = \cup\mathscr{T}_\alpha$, we have $\cup_{v\in S}\, v = X$, since X is an element of $\mathscr{T}_\alpha$ for all α. Let T be the topology generated by the subbasis S; T contains all elements of S, and therefore contains $\mathscr{T}_\alpha$ for all α. Conversely, let T' be a topology containing $\mathscr{T}_\alpha$ for all α. Then T' contains S, and therefore all unions of finite intersections of elements of S; from this we deduce that T' contains T, and T is the smallest topology containing all $\mathscr{T}_\alpha$.

(c) Applying the above, and noting $\tau(A)$ for the topology generated by a subbasis A, we find that the largest topology containing $\mathscr{T}_1$ and $\mathscr{T}_2$ is

$$\mathscr{T}_1 \cap \mathscr{T}_2 = \big\{\varnothing, X, \{a\}\big\}$$

For $\tau(\mathscr{T}_1 \cup \mathscr{T}_2)$, the finite intersections of elements of $\mathscr{T}_1$ and $\mathscr{T}_2$ are

$$\varnothing, X, \{a\}, \{a, b\}, \{b\}, \{b, c\}$$

and their unions are

$$\tau(\mathscr{T}_1 \cup \mathscr{T}_2) = \big\{\varnothing, X, \{a\}, \{a, b\}, \{b\}, \{b, c\}\big\}$$

$\square$