Exercise 4. The *Fibonacci numbers* of number theory are defined recursively by the formula

$$\begin{aligned} &\lambda_1 = \lambda_2 = 1 \\ &\lambda_n = \lambda_{n-1} + \lambda_{n-2} && \text{for } n > 2. \end{aligned}$$

Define them rigorously by use of Theorem 8.4.

Proof. Let $A = \mathbb{Z}_+$ and $\mathscr{A}$ be the set of functions mapping a nonempty section of the positive integers into $\mathbb{Z}_+$. Define

$$\begin{aligned} \rho : \mathscr{A} &\to \mathbb{Z}_+ \\ f &\mapsto \begin{cases} 1 & \text{if the domain of f is } \{1\} \\ f(n) + f(n-1) & \text{otherwise} \end{cases} \end{aligned}$$

In the last case, the domain of f is a section $\{1, \ldots, n\}$ of the positive integers with $n > 1$. Let $a_0 = 1$, and apply Theorem 8.4 with these values of a_0 and ρ to deduce the existence of a function h such that:

$$\begin{aligned} h(1) &= a_0 = 1 \\ h(i) &= \rho(h|\{1, \ldots, i-1\}) && \text{if } i > 1 \end{aligned}$$

For $i = 2$, we get

$$h(2) = \rho(h|\{1\}) = 1$$

and for $i > 2$, we get

$$h(i) = \rho(h|\{1, \ldots, i-1\}) = h(i-1) + h(i-2)$$

so that $h(i)$ is the i-th Fibonacci number.

□