Exercise 3. Suppose that A is a set and $\{f_n\}_{n\in\mathbb{Z}_+}$ is a given indexed family of injective functions

$$f_n : \{1, \dots, n\} \to A$$

Show that A is infinite. Can you define an injective function $f : \mathbb{Z}_+ \to A$ without using the choice axiom?

Proof. For all $n \in \mathbb{Z}_+$, note $F_n = f_n(S_{n+1})$. Define f recursively by

$$\begin{aligned} f(1) &= f_1(1) \\ f(n+1) &= f_{n+1}(j) \\ &\qquad \text{with } j = \min\left\{ i \in S_{n+2} \mid f_{n+1}(i) \notin \{f(1), \dots, f(n)\} \right\} \end{aligned}$$

Let us verify that this defines an injective function.

For all $n \in \mathbb{Z}_+$, we note $G_n = \{f(1), \dots, f(n)\} = f|S_{n+1}(S_{n+1})$. Let $\mathscr{A}$ be the set of positive integers such that $f|S_{n+1}$ is injective. We have $1 \in \mathscr{A}$ since $f|\{1\} = f_1$ is injective. Suppose that $n \in \mathscr{A}$; the definition of f gives $f(n+1) \in F_{n+1}$. Since f_{n+1} is injective, the set F_{n+1} has $n+1$ elements. From the inductive hypothesis, the set G_n has n elements. Therefore there exists some $j \in S_{n+2}$ such that $f_{n+1}(j) \notin G_n$. The set $B = \{i \in S_{n+2} \mid f_{n+1}(i) \notin G_n\}$ is nonempty, and, as a subset of the well-ordered set $\mathbb{Z}_+$, has a smallest element $j_0 = f(n+1)$. From the above we have $f(n+1) \notin \{f(1), \dots, f(n)\}$, so that $f|S_{n+2}$ is injective, and $n+1 \in \mathscr{A}$. We deduce that $\mathscr{A}$ is inductive, and therefore $\mathscr{A} = \mathbb{Z}_+$.

Let $i, j \in \mathbb{Z}_+$ such that $i \neq j$. By switching the roles of i and j if needed, we can suppose that $i < j$. The function $f|\{1, \dots, j\}$ is injective, so that $f(i) \neq f(j)$, and therefore f is injective.

Since there exists an injection $f : \mathbb{Z}_+ \to A$, the set A is infinite. Moreover, the definition of f does not use the axiom of choice.

$\square$