Exercise 8. Show that $\mathscr{P}(\mathbb{Z}_+)$ and $\mathbb{R}$ have the same cardinality. [*Hint:* you may usr the fact that every real number has a decimal expansion, which is unique if expansions that end in an infinite string of 9s are forbidden.]

Proof.

Injection of $\mathbb{R}$ into $\mathscr{P}(\mathbb{Z}_+)$

For all $x \in [0,1]$, we admit that x has a unique proper base 2 development noted $(0, x_1 x_2 \dots)_2$, with $x_i \in \{0,1\}$ for all $i \in \mathbb{Z}_+$, and such that for all $i \in \mathbb{Z}_+$ there exists $j > i$ such that $x_j = 0$. Let

$$\begin{aligned} \phi : (0,1) &\to \mathscr{P}(\mathbb{Z}_+) \\ x = (0, x_1, x_2 \dots)_2 &\mapsto \{i \in \mathbb{Z}_+ \mid x_i = 1\} \end{aligned}$$

The unicity of the proper base 2 development of x implies that if x, y are elements of $(0,1)$ such that $x \neq y$, then there exists $j \in \mathbb{Z}_+$ such that $x_j \neq y_j$; so that $j \notin \phi(x) \cap \phi(y)$, from which we deduce that ϕ is injective.

Note that ϕ is not surjective: suppose that there exists $x = (0, x_1, x_2 \dots)_2$ such that $\phi(x) = \mathbb{Z}_+$. If $x_i = 0$ for some $i \in \mathbb{Z}_+$, then $i \notin \phi(x) = \mathbb{Z}_+$, which is a contradiction. Therefore $x_i = 1$ for all i, so that $(0, x_1 x_2 \dots)_2$ is not a proper base 2 development.

Let

$$\begin{aligned} \theta : \mathbb{R} &\to (0,1) \\ x &\mapsto \frac{1}{2} + \frac{1}{\pi} \arctan(x) \end{aligned}$$

θ is a bijection, and thus $\phi \circ \theta$ is an injection of $\mathbb{R}$ into $\mathscr{P}(\mathbb{Z}_+)$.

Injection of $\mathscr{P}(\mathbb{Z}_+)$ into $\mathbb{R}$

Let X be a nonempty subset of $\mathbb{Z}_+$. For all $n \in \mathbb{Z}_+$, let

$$S_n(X) = \sum_{i \in X \cap \{1, \dots, n\}} 2^{-i}$$

We have

$$0 \leq S_n(X) \leq S_{n+1}(X) \leq \sum_{i \in \{1, \dots, n+1\}} 2^{-i} \leq 1$$

so that the sequence $S_n(X)$ converges to some real number $S(X)$ in $[0,1]$. For all $n \in \mathbb{Z}_+$, let $Q_n = \{i \in \mathbb{Z}_+ \mid i \geq n\}$, and

$$\begin{aligned} \psi : \mathscr{P}(\mathbb{Z}_+) &\to \mathbb{R} \\ X &\mapsto \begin{cases} 0 & \text{if } X = \varnothing \\ S(X) & \text{if } Q_n \not\subset X \text{ for all } n \\ 1 + S(X) & \text{otherwise} \end{cases} \end{aligned}$$

Let X, Y be subsets of $\mathbb{Z}_+$ such that $X \neq Y$. Switching the roles of X and Y if necessary, we can suppose that $X - Y \neq \varnothing$. Let $i \in X - Y$; if $Y = \varnothing$, then $\psi(Y) = 0 < 2^{-i} \leq \psi(X)$, so $\psi(X) \neq \psi(Y)$. Otherwise, if there is some n such that $Q_n \subset Y$, but for all n, $Q_n \not\subset X$, then $\psi(X) \leq 1 < 1 + \sum_{i \in Q_n} 2^{-i} \leq \psi(Y)$, so that $\psi(X) \neq \psi(Y)$. The same reasoning applies if X contains some Q_n but Y does not contain any.

Otherwise, suppose that neither X nor Y contains any Q_n, and let $m = \min(X - Y)$. Switching the roles of X and Y if necessary, we can suppose that $x_m = 1$. For all $i < m$, we have $x_i = y_i$, so that

$$\begin{aligned} 0 \leq \psi(Y) &< \sum_{i \in \{1,\dots,m-1\} \cap Y} 2^{-i} + \sum_{i > m} 2^{-i} && \text{since } Q_{m+1} \not\subset Y \\ &= \sum_{i \in \{1,\dots,m-1\} \cap X} + 2^{-m} && \text{since } x_i = y_i \text{ for } i < m \\ &\leq \psi(X) \end{aligned}$$

From this we deduce that $\psi(X) \neq \psi(Y)$.

The last case to check is when there is some positive n such that $Q_n \subset X$ and some p such that $Q_p \subset Y$. Again let $m = \min(X - Y)$, and suppose that $x_m = 1$, switching the roles of X and Y if necessary. Then

$$\begin{aligned} \psi(Y) &\leq 1 + \sum_{i \in \{1,\dots,m-1\} \cap Y} 2^{-i} + \sum_{i > m} 2^{-i} \\ &< 1 + \sum_{i \in \{1,\dots,m-1\} \cap X} 2^{-i} + 2^{-m} \\ &\leq \psi(X) \end{aligned}$$

and we have $\psi(X) \neq \psi(Y)$.

From the above, we deduce that ψ is injective.

Since there exists an injection of $\mathbb{R}$ into $\mathscr{P}(\mathbb{Z}_+)$, and an injection of $\mathscr{P}(\mathbb{Z}_+)$ into $\mathbb{R}$, there exists a bijection from $\mathscr{P}(\mathbb{Z}_+)$ to $\mathbb{R}$, so that they have the same cardinality.

$\square$