Exercise 11. Let A and B be two sets. Using the well-ordering theorem, prove that either they have the same cardinality, or one has cardinality greater than the other. [*Hint:* if there is no surjection $f : A \to B$, apply the preceding exercise.]

Proof. If there is a bijection from A to B, then they have the same cardinality. Suppose therefore that there is no such bijection.

From the well-ordering theorem, there exist order relations such that A and B are well-ordered. Let us consider A and B as well-ordered sets.

If there is a surjection from a section of A onto B, then using the axiom of choice we can define an injection of B into A. If there also exists an injection of A into B, then there exists a bijection between A and B, which is contrary to our hypothesis. Therefore there is no injection of A into B, and A has greater cardinality than B.

Suppose there is no surjection from a section of A onto B. Using the previous exercise, we can define a function $h : A \to B$ such that

$$\forall x \in A, \quad h(x) = \min(B - h(S_x)) \tag{1}$$

Let J be the subset of A such that $h|J$ is injective, and suppose that there exists $x \in A - J$. Since A is well-ordered, there is a smallest such element β.

From (1) we deduce that $h(\beta) \notin h(S_\beta)$, so that $\beta \in J$, which is a contradiction. Therefore $J = A$ and h is injective.

If there also exists an injection from B into A, then there exists a bijection between A and B, which is contrary to our hypothesis. Therefore there is no such injection, and B has greater cardinality than A.

$\square$