Exercise 2.

(a) Show that if $n > 1$ there is a bijective correspondence of

$$A_1 \times A_2 \times \cdots \times A_n \qquad \text{with} \qquad (A_1 \times A_2 \times \cdots \times A_{n-1}) \times A_n$$

(b) Given the indexed family $\{A_1, A_2, \dots\}$, let $B_i = A_{2i-1} \times A_{2i}$ for each positive integer i. Show that there is a bijective correspondence of $A_1 \times A_2 \times \cdots$ with $B_1 \times B_2 \times \cdots$.

Proof.

(a) Let $n > 1$. If for some $i \leq n$, $A_i = \varnothing$, then $A_1 \times \cdots \times A_n = \varnothing$. If $i < n$, then $A_1 \times \cdots \times A_{n-1}$ is also empty, which implies that $(A_1 \times \cdots \times A_{n-1}) \times A_n$ is empty. If $i = n$, then $(A_1 \times \cdots \times A_{n-1}) \times A_n = (A_1 \times \cdots \times A_{n-1}) \times \varnothing = \varnothing$.

Let us then suppose that $\forall i,\ 1 \leq i \leq n,\ A_i \neq \varnothing$. Let

$$\begin{aligned} p_1 : A_1 \times \ldots \times A_{n-1} \times A_n &\to A_1 \times \cdots \times A_{n-1} \\ (x_1, \ldots, x_{n-1}, x_n) &\mapsto (x_1, \ldots, x_{n-1}) \\ p_2 : A_1 \times \ldots \times A_{n-1} \times A_n &\to A_n \\ (x_1, \ldots, x_{n-1}, x_n) &\mapsto x_n \end{aligned}$$

and define

$$\begin{aligned} q : A_1 \times \ldots \times A_n &\to (A_1 \times \ldots \times A_{n-1}) \times A_n \\ x &\mapsto \big(p_1(x), p_2(x)\big) \end{aligned}$$

Let $x, y \in A_1 \times \ldots \times A_n$. If

$$\begin{aligned} q(x) = q(y) &\quad \text{then} \\ \big(p_1(x), p_2(x)\big) &= \big(p_1(y), p_2(y)\big) \end{aligned}$$

so $p_1(x) = p_1(y)$ and $p_2(x) = p_2(y)$ by definition of the cartesian product of 2 sets. From $p_1(x) = p_1(y)$, we deduce that the first $n - 1$ coordinates of x and y are equal, and from $p_2(x) = p_2(y)$ we deduce that their n-th coordinates are also equal; so $x = y$ and q is injective.

Let

$$x = \big((x_1, \ldots, x_{n-1}), x_n\big) \in (A_1 \times \ldots \times A_{n-1}) \times A_n$$

Take $y = (x_1, \ldots, x_n)$; we have

$$\begin{aligned} p_1(y) &= (x_1, \ldots, x_{n-1}) \quad \text{and} \\ p_2(y) &= x_n \end{aligned}$$

so that $q(y) = x$ and q is surjective. Therefore $A_1 \times \ldots \times A_{n-1} \times A_n$ and $(A_1 \times \ldots \times A_{n-1}) \times A_n$ are in bijective correspondence to each other. This result justifies reasoning by induction on the number of sets in a cartesian product with index set a subset of $\mathbb{Z}_+$.

(b) For A and B nonempty sets, define

$$\begin{aligned} \pi_1 : A \times B &\to A \\ (x,y) &\mapsto x \end{aligned} \qquad\qquad \begin{aligned} \pi_2 : A \times B &\to B \\ (x,y) &\mapsto y \end{aligned}$$

We will use such functions for $A = A_{2i-1}$ and $B = A_{2i}$, for all $i \in \mathbb{Z}_+$. Since the domain and range are different for each i, we get two families of distinct functions $(\pi_{1,i})$ and $(\pi_{2,i})$ for $i \in \mathbb{Z}_+$. In order to keep notations simple in what follows, we will abusively write π_1 (respectively π_2) when we actually mean $\pi_{1,i}$ (respectively $\pi_{2,i}$) for some i. Let

$$\begin{aligned} \phi : B_1 \times B_2 \times \cdots &\to A_1 \times A_2 \times \cdots \\ (x_1, x_2, \dots) &\mapsto \big(\pi_1(x_1), \pi_2(x_1), \pi_1(x_2), \pi_2(x_2), \dots\big) \end{aligned}$$

and

$$\begin{aligned} \psi : A_1 \times A_2 \times \cdots &\to B_1 \times B_2 \times \cdots \\ (x_1, x_2, \dots) &\mapsto (y_1, y_2, \dots) \quad \text{where} \quad y_i = (x_{2i-1}, x_{2i}) \end{aligned}$$

For all $i \in \mathbb{Z}_+$, $y_i = (x_{2i-1}, x_{2i}) = \big(\pi_1(y_i), \pi_2(y_i)\big)$), so

$$\begin{aligned} \phi \circ \psi = i_A \quad &\text{with} \quad A = A_1 \times A_2 \times \cdots \\ \psi \circ \phi = i_B \quad &\text{with} \quad B = B_1 \times B_2 \times \cdots \end{aligned}$$

So ϕ and ψ are bijective and inverse of each other. From this we deduce that $A_1 \times A_2 \times \cdots$ and $(A_1 \times A_2) \times (A_3 \times A_4) \times \cdots$ are in bijective relation to each other.

$\square$