Exercise 8. Let X denote the two-element set $\{0, 1\}$; let $\mathscr{B}$ be the set of *countable* subsets of X^ω. Show that X^ω and $\mathscr{B}$ have the same cardinality.

Proof. Let

$$\begin{aligned} \phi : \mathscr{B} &\to X^\omega \\ B &\mapsto \begin{cases} (0, 0, \dots) & \text{if } B = \varnothing \\ \beta & \text{for some } \beta \in B \text{ otherwise} \end{cases} \end{aligned}$$

with the added rule that if B_1 and B_2 are elements of $\mathscr{B}$ such that $B_1 = B_2$, then we choose the same element in B_1 and B_2 to be the value of $\phi(B_1) = \phi(B_2)$. This rule is necessary to make ϕ a function, and is possible with the axiom of choice. For all $x \in X^\omega$, we have $\phi(\{x\}) = x$, so ϕ is surjective.

For all nonempty $B = \{b_1, b_2, \dots\} \in \mathscr{B}$, we note $b_p = (b_{p,1}, b_{p,2}, \dots)$ for all $p \in \mathbb{Z}_+$. Since $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable, let $\sigma : \mathbb{Z}_+ \to \mathbb{Z}_+ \times \mathbb{Z}_+$ be a bijection, and let $\psi : X^\omega \to \mathscr{B}$ be defined by:

- $\psi\big((0, 0, \dots)\big) = \varnothing$
- Otherwise, let $x = (x_1, x_2, \dots) \in X^\omega$ and S be the subset of $\mathbb{Z}_+$ of all integers p such that, for $a \geq p$, any b, and $\sigma(i) = (a, b)$, $x_i = 0$. If S is nonempty, it has a smallest element n. Let then $\psi(x) = \{b_1, b_2, \dots, b_n\}$ with $b_{p,q} = x_i$ for $(p, q) = \sigma(i)$. Since σ is bijective, the $b_{p,q}$ are defined for all q and all $p \leq n$, so each b_p is an element of X^ω as expected.
- Otherwise, if S is empty, define $b_{p,q} = x_i$ for $\sigma(i) = (p, q)$ and let $\psi(x) = \{b_1, b_2, \dots\}$. There are countably many x_i, and σ is bijective, so there are countably many b_i. As a consequence, $\psi(x)$ is an element of $\mathscr{B}$ as expected.

The function ψ is surjective. Let $B \in \mathscr{B}$; if $B = \varnothing$, then $B = \psi\big((0, 0, \dots)\big)$. If $B = \{b_1, b_2, \dots, b_n\}$, let $x = (x_1, x_2, \dots)$ with $x_i = b_{p,q}$ for $p \leq n$ and $\sigma(i) = (p, q)$ and $x_i = 0$ otherwise. Then $\psi(x) = \{b_1, b_2, \dots, b_n\}$. Finally, if $B = \{b_1, b_2, \dots\}$ is countably infinite, let $x = (x_1, x_2, \dots)$ with $x_i = b_{p,q}$ for $\sigma(i) = (p, q)$. For such an x we have $\psi(x) = B$.

Since the exists a surjection from X^ω to $\mathscr{B}$, there exists an injection from $\mathscr{B}$ to X^ω. And since there exists a surjection from $\mathscr{B}$ to X^ω, there exists an injection from X^ω to $\mathscr{B}$. Finally, since there exist injections both from X^ω to $\mathscr{B}$ and from $\mathscr{B}$ to X^ω, there exists a bijection from $\mathscr{B}$ to X^ω, so that $\mathscr{B}$ and X^ω have the same cardinality (and in particular, $\mathscr{B}$ is uncountable).

$\square$