Exercise 9. Check that the dictionary order is an order relation.

Proof.

comparability Let $(x_0, y_0), (x_1, y_1) \in \mathbb{R}^2$ such that $(x_0, y_0) \neq (x_1, y_1)$. Then either $x_0 \neq x_1$ or $y_0 \neq y_1$. By comparability of x_0 with x_1 and y_0 with y_1 for the usual order relation $<$ on $\mathbb{R}$, we deduce the following cases:

- if $x_0 < x_1$, then we have $(x_0, y_0) < (x_1, y_1)$
- if $x_1 < x_0$, then we have $(x_1, y_1) < (x_0, y_0)$
- if $x_0 = x_1$, then we have either $y_0 < y_1$ or $y_1 < y_0$, from which we deduce either $(x_0, y_0) < (x_1, y_1)$ or $(x_1, y_1) < (x_0, y_0)$

non-reflexivity Let $(x_0, y_0) \in \mathbb{R}$. By non-reflexivity of the usual order relation $<$ on $\mathbb{R}$, we deduce that we can have neither $x_0 < x_0$ nor $y_0 < y_0$. From this and the definition of the dictionary order relation, we deduce that we cannot have $(x_0, y_0) < (x_0, y_0)$.

transitivity Let $(x_0, y_0), (x_1, y_1), (x_2, y_2) \in \mathbb{R}$ such that $(x_0, y_0) < (x_1, y_1)$ and $(x_1, y_1) < (x_2, y_2)$. Suppose $x_0 < x_1$. Then from $(x_1, y_1) < (x_2, y_2)$ we deduce that $(x_0, y_0) < (x_2, y_2)$. Otherwise $x_0 = x_1$ and $y_1 < y_2$. From $(x_1, y_1) < (x_2, y_2)$ we deduce that:

- either $x_1 < x_2$, so that $(x_0, y_0) < (x_2, y_2)$
- or $x_0 = x_1 = x_2$, in which case we deduce from $(x_0, y_0) < (x_1, y_1)$ that $y_0 < y_1 < y_2$, which gives us $(x_0, y_0) < (x_2, y_2)$

From the above we conclude that the dictionary order is an order relation on $\mathbb{R}^2$.

□