Exercise 6. We say that two sets A and B *have the same cardinality* if there is a bijection of A with B.

(a) Show that if $B \subset A$ and there is an injection

$$f : A \to B$$

the A and B have the same cardinality. [*Hint:* Define $A_1 = A$, $B_1 = B$, and for $n > 1$, $A_n = f(A_{n-1})$ and $B_n = f(B_{n-1})$. (Recursive definition again!) Note that $A_1 \supset B_1 \supset A_2 \supset B_2 \supset A_3 \supset \cdots$. Define a bijection $h : A \to B$ by the rule

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_n - B_n \text{ for some } n \\ x & \text{otherwise.]} \end{cases}$$

(b) *Theorem (Schröder-Bernstein theorem). If there are injections $f : A \to C$ and $g : C \to A$, then A and C have the same cardinality.*

Proof.

(a) Let A_i, B_i be defined as above for all $i \in \mathbb{Z}_+$, and let $\mathscr{A}$ be the subset of $\mathbb{Z}_+$ such that for all $n \in \mathscr{A}$, $A_n \supset B_n \supset A_{n+1}$. Then $1 \in \mathscr{A}$: we already have $B_1 = B \subset A = A_1$; from the definition of f, we also have $A_2 = f(A_1) = f(A) \subset B = B_1$.

Suppose now that $S_{n+1} \subset \mathscr{A}$ for some n. Since $B_n \subset A_n$, we have $B_{n+1} = f(B_n) \subset f(A_n) = A_{n+1}$. Similarly, $A_{n+1} \subset B_n$, so $A_{n+2} = f(A_{n+1}) \subset f(B_n) = B_{n+1}$. From this we deduce that $S_{n+2} \subset \mathscr{A}$, so that $\mathscr{A}$ is inductive, and finally $\mathscr{A} = \mathbb{Z}_+$.

Since $f : A \to B$ exists, we have $B \neq \varnothing$. And since $B \subset A$, we also have $A \neq \varnothing$. If A has only one element, then $B = A$, and $h = i_A$, which is bijective.

Injectivity of h

Suppose then that there are $x, y \in A$ such that $x \neq y$. If we have $h(x) = x$ and $h(y) = y$, then $h(x) \neq h(y)$. Similarly, if we have $h(x) = f(x)$ and $h(y) = f(y)$, then $h(x) \neq h(y)$ since f is injective.

Otherwise, we can suppose that $h(x) = f(x)$ and $h(y) = y$, possibly switching the roles of x and y. Then there exists n such that $x \in A_n - B_n$, but for all $n \in \mathbb{Z}_+$, $y \notin A_n - B_n$. Therefore $y \in \cup_{n \in \mathbb{Z}_+} B_n - A_{n+1}$, but $f(x) \in f(A_n) = A_{n+1}$, so that $h(x) \neq h(y)$. From the above, we deduce that h is injective.

Surjectivity of h

Let $y \in B$. If $y \in \mathscr{B} = \cup_{n\in\mathbb{Z}_+} B_n - A_{n+1}$, then $h(y) = y$.

Otherwise, there exists n such that $y \in A_n - B_n$. Since we also have $y \in B$, then $n \geq 2$ and $y \in f(A)$. From this we deduce that $f^{-1}(\{y\}) \neq \varnothing$.

If $f^{-1}(\{y\}) \cap \mathscr{B} \neq \varnothing$, then take x in it. There exists $m \in \mathbb{Z}_+$ such that $x \in B_m - A_{m+1}$, so that $f(x) = y \in f(B_m - A_{m+1})$. Since f is injective, we have $f(B_m - A_{m+1}) = f(B_m) - f(A_{m+1}) = B_{m+1} - A_{m+2}$.

From $y \in A_n$ and $y \in B_{m+1}$, we deduce that $m + 1 < n$ and $m + 2 \leq n$. We must therefore have $y \notin A_{m+2} \supset A_n$. This contradicts $y \in A_n$, so $f^{-1}(\{y\}) \cap \mathscr{B}$ must be empty.

So there exists $p \in \mathbb{Z}_+$ such that $x \in A_p - B_p$; in this case, we have $h(x) = f(x) = y$.

From the above we deduce that h is bijective, so that A and B have the same cardinality.

(b) Let $f : A \to C$ and $g : C \to A$ be injections and $B = g(C) \subset A$. The function $g \circ f : A \to B$ is injective, so we are in the hypotheses of (a). There exists a bijection $h : A \to g(C)$. Furthermore, g is surjective from C to $g(C)$, so $k : C \to B$, $x \mapsto g(x)$ is bijective. Therefore the function $h^{-1} \circ k : C \to A$ is a bijection, and A and C have the same cardinality.

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