Exercise 5. (a) Use the choice axiom to show that if $f : A \to B$ is surjective, then f has a right inverse $h : B \to A$.

(b) Show that if $f : A \to B$ is injective and A is not empty, then f has a left inverse. Is the axiom of choice needed?

Proof.

(a) Let $f : A \to B$ be a surjective function. Since f is a function, B is nonempty. For all $b \in B$, $f^{-1}(\{b\})$ is nonempty. Let $\mathscr{B} = \big\{f^{-1}(\{b\}), b \in B\big\}$; there exists a choice function $c : \mathscr{B} \to \cup_{D\in\mathscr{B}} D$ such that for all $D \in \mathscr{B}$, $c(D) \in D$. Let

$$\begin{aligned} h : B &\to A \\ b &\mapsto c\left(f^{-1}\left(\{b\}\right)\right) \end{aligned}$$

Let $x \in B$, we have $f \circ f^{-1}(\{x\}) = \{x\}$ since f is surjective. Since $h(x) \in f^{-1}(\{x\})$, we have $f \circ h(x) \in f \circ f^{-1}(\{x\}) = \{x\}$, from which we deduce that $f \circ h(x) = x$. The function f has a right inverse h.

(b) Let $f : A \to B$ be an injective function with $A \neq \varnothing$. f is surjective from A to $f(A)$, and therefore $g : A \to f(A)$, $x \mapsto f(x)$ is a bijection. Let $x \in A$ and define

$$\begin{aligned} h : B &\to A \\ y &\mapsto \begin{cases} g^{-1}(y) & \text{if } y \in f(A) \\ x & \text{otherwise} \end{cases} \end{aligned}$$

The function h is a left inverse of f: let $z \in A$, we have

$$\begin{aligned} h \circ f(z) &= g^{-1}\big(f(z)\big) && \text{since } f(z) \in f(A) \\ &= z && \text{by definition of } g \end{aligned}$$

Note that defining g does not require the axiom of choice. But to be able to define $k \circ f$ for some function k, we need the range of f to be equal to the domain of k. So to define h, we need to use the axiom of choice to pick a value for $x \in A$, although that value is not used to compute the value of $h \circ f$ at some $z \in A$. This is consistent with the fact that a left inverse is not unique.

$\square$