Exercise 6. (a) Let $A = \{1, 2, \ldots, n\}$. Show there is a bijection of $\mathscr{P}(A)$ with the cartesian product X^n, where X is the two-elements set $X = \{0, 1\}$.

(b) Show that if A is finite, then $\mathscr{P}(A)$ is finite.

Proof.

(a) Let

$$\begin{aligned} \phi : \mathscr{P}(A) &\to X^n \\ B &\mapsto (x_1, x_2, \ldots, x_n) \end{aligned}$$

such that

$$\forall i \in \{1, 2, \ldots n\},\ x_i = \begin{cases} 1 & \text{if} \quad i \in B \\ 0 & \text{otherwise} \end{cases}$$

Let $B1, B2 \in \mathscr{P}(A)$ such that $B_1 \neq B_2$. The roles of B_1 and B_2 being symmetric in the following, it is sufficient to suppose that there exists $j \in B_1 - B_2$. Then $\phi(B_1)$ and $\phi(B_2)$ have a different j-th coordinate and so $\phi(B_1) \neq \phi(B_2)$ and ϕ is injective.

Let $x = (x_1, x_2, \ldots, x_n) \in X^n$. If $x = (0, 0, \ldots, 0)$, then $\phi(\varnothing) = x$. Otherwise, let $B = \{i \in A \mid x_i = 1\}$. We have $\phi(B) = (y_1, y_2, \ldots, y_n)$, with $y_i = 1$ if and only if $i \in B$, which, in turns, is true if and only if $x_i = 1$, so that $\phi(B) = x$ and ϕ is surjective.

(b) Let A be a finite set. If $A = \varnothing$, then $\mathscr{P}(A) = \varnothing$, so $\mathscr{P}(A)$ is finite. Otherwise, there is a bijection f from $\mathscr{P}(A)$ to X^n for some positive integer n. Since X is finite, the cartesian product X^n is also finite, so there is a bijection g from X^n to some section of positive integers $\{1, 2, \ldots, m\}$. Therefore $g \circ f$ is a bijection from $\mathscr{P}(A)$ to $\{1, 2, \ldots, m\}$, so $\mathscr{P}(A)$ is finite.

□