Exercise 11. Show that an element in an ordered set has at most one immediate successor and at most one immediate predecessor. Show that a subset of an ordered set has at most one smallest element and at most one largest element.

Proof. Let A be an ordered set. If $A = \varnothing$, the results above are vacuously true, so we suppose A is not empty.

Let $x \in A$ and suppose b_0 and b_1 are immediate successors of x. If $b_0 \neq b_1$, then either $b_0 < b_1$ or $b_1 < b_0$. These cases are symmetric so we can assume for example that $b_0 < b_1$. Then $b_0 \in \{y \in A \mid x < y < b_1\}$, so that b_1 cannot be the immediate successor of x. From the above we deduce that $b_0 = b_1$, so that the immediate successor is unique if it exists.

A similar proof shows that the immediate predecessor of an element is unique if it exists.

Let A be an ordered set, $B \subset A$ and s_0 and s_1 two smallest elements of B in A. Then we have $s_0 \in B$ and $s_1 \in B$. Since s_0 (resp. s_1) is the smallest element of B in A, we must have $s_0 \leq s_1$ (resp. $s_1 \leq s_0$). If $s_0 < s_1$, then we cannot have $s_1 \leq s_0$; and similarly, $s_1 < s_0$ is also impossible. By comparability of the order relation on A, if $s_0 \neq s_1$ then one of $s_0 < s_1$ or $s_1 < s_0$ must be true, though. We deduce that we cannot have $s_0 \neq s_1$ either, and therefore the smallest element of B must be unique.

A similar proof shows that the largest element of a subset of an ordered set is unique.

$\square$