Exercise 5. In general, let us denote the *identity function* for a set C by i_C. That is, define $i_C : C \to C$ to be the function given by the rule $i_C(x) = x$ for all $x \in C$. Given $f : A \to B$, we say that a function $g : B \to A$ is a *left inverse* for f if $g \circ f = i_A$; and we say that $h : B \to A$ is a *right inverse* for f if $f \circ h = i_B$.

(a) Show that if f has a left inverse, f is injective; and if f has a right inverse, f is surjective

(b) Give an example of a function that has a left inverse but no right inverse

(c) Give an example of a function that has a right inverse but no left inverse

(d) Can a function have more than one left inverse? More than one right inverse?

(e) Show that if f has both a left inverse g and a right inverse h, then f is bijective and $g = h = f^{-1}$

Proof.

(a) If f has a left inverse g, then $g \circ f = i_A$. The function i_A being injective, we deduce that f is injective. Similarly, if f has a right inverse h, then $f \circ h = i_B$. The function i_B being surjective, we deduce that f is surjective.

(b) Let $A = \{1, 2\}$, $B = \{a, b, c\}$, $f(1) = a$, $f(2) = b$. The function $g : B \to A$ defined by $g(a) = 1$, $g(b) = 2$, $g(c) = 1$ is a left inverse for f. However, f not being surjective implies that it does not have a right inverse.

(c) Consider A, B, f, g as in the exemple defined above. Then f is a right inverse for g. Similarly, g not being injective implies it does not have a left inverse.

(d) Left and right inverses are not unique. Still with the definitions from the same example, let $h : B \to A$ defined by $h(a) = 1$, $h(b) = 2$, $h(c) = 2$. Then both g and h are left inverses for f. Define also $k : A \to B$ by $k(1) = a$, $k(2) = c$. Then both f and k are left inverses for g.

(e) Let f, g and h be such that $g \circ f = i_A$ and $f \circ h = i_B$. Let $y \in B$; we have $y = f \circ h(y)$, so $g(y) = (g \circ f) \circ h(y) = h(y)$, so g and h are equal. Since $f \circ h = i_B$ is surjective, then f is surjective. Since $g \circ f = i_A$ is injective, then f is injective. Since f is both surjective and injective, it is bijective. From $f \circ h = i_B = f \circ f^{-1}$, we deduce that $f^{-1} = h$. And from $h = g$, we deduce $g = h = f^{-1}$ $\square$