Exercise 6. Define a relation on the plane by setting:

$$(x_0, y_0) < (x_1, y_1)$$

if either $y_0 - x_0^2 < y_1 - x_1^2$ or $y_0 - x_0^2 = y_1 - x_1^2$ and $x_0 < x_1$. Show that this is an order relation on the plane, and describe it geometrically.

Proof.

comparability Let (x_0, y_0) and (x_1, y_1) in $\mathbb{R}^2$ such that $(x_0, y_0) \neq (x_1, y_1)$. First suppose that $y_0 - x_0^2 = y_1 - x_1^2$. Then $x_0 \neq x_1$, for otherwise we would have $(x_0, y_0) = (x_1, y_1)$, which is contrary to our hypothesis. From $x_0 \neq x_1$ we deduce that either $x_0 < x_1$ or $x_1 < x_0$. The first case leads to $(x_0, y_0) < (x_1, y_1)$ and the second to $(x_1, y_1) < (x_0, y_0)$, so (x_0, y_0) and (x_1, y_1) are comparable.

Next, suppose that $y_0 - x_0^2 \neq y_1 - x_1^2$. Then either $y_0 - x_0^2 < y_1 - x_1^2$, which gives $(x_0, y_0) < (x_1, y_1)$, or $y_1 - x_1^2 < y_0 - x_0^2$, which gives $(x_1, y_1) < (x_0, y_0)$, so (x_0, y_0) and (x_1, y_1) are comparable.

non-reflexivity Let $(x_0, y_0) \in \mathbb{R}^2$. Then $y_0 - x_0^2 = y_0 - x_0^2$, so to have $(x_0, y_0) < (x_0, y_0)$ we need $x_0 < x_0$, which is impossible. So we never have $(x_0, y_0) < (x_0, y_0)$.

transitivity Let $(x_0, y_0), (x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$, such that $(x_0, y_0) < (x_1, y_1)$ and $(x_1, y_1) < (x_2, y_2)$. Suppose that $y_0 - x_0^2 < y_1 - x_1^2$. From $(x_1, y_1) < (x_2, y_2)$ we deduce that either $y_1 - x_1^2 < y_2 - x_2^2$ which implies by transitivity of the usual order relation on real numbers that $y_0 - x_0^2 < y_2 - x_2^2$, so that $(x_0, y_0) < (x_2, y_2)$. Otherwise, $y_1 - x_1^2 = y_2 - x_2^2$, from which we get by hypothesis $y_0 - x_0^2 < y_2 - x_2^2$, which is $(x_0, y_0) < (x_2, y_2)$.

Suppose now that $y_0 - x_0^2 = y_1 - x_1^2$ and $x_0 < x_1$. If $y_1 - x_1^2 < y_2 - x_2^2$, then $y_0 - x_0^2 < y_2 - x_2^2$ so that $(x_0, y_0) < (x_2, y_2)$. Otherwise, if $y_1 - x_1^2 = y_2 - x_2^2$ and $x_1 < x_2$, we deduce that $y_0 - x_0^2 = y_2 - x_2^2$ and $x_0 < x_1 < x_2$, so that $(x_0, y_0) < (x_2, y_2)$.

Let $c \in \mathbb{R}$, $y - x^2 = c$ is the equation of a parabola with focus $(0, c + 1/4)$ and directrix $y = c - 1/4$. Let (x_0, y_0) and (x_1, y_1) in $\mathbb{R}^2$ and note $y_0 - x_0^2 = c_0$ and $y_1 - x_1^2 = c_1$. Then $(x_0, y_0) < (x_1, y_1)$ if and only if either $c_0 < c_1$, in which case the parabola $y - x^2 = c_1$ is above $y - x^2 = c_0$, or $c_0 = c_1$ and $x_0 < x_1$, in which case both points are on the same parabola and (x_1, y_1) is to the right of (x_0, y_0).

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