Exercise 3. Let X be the two-elements set $\{0, 1\}$. Find a bijective correspondence between X^ω and a proper subset of itself.

Proof. Let A be the subset of X^ω of all ω-tuples of elements of X that start with 0. Then A is a strict subset of X^ω. Let

$$f : X^\omega \to A$$
$$(x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots)$$

Let $y' = (0, y_1, y_2, \dots) \in A$, we have $y' = f(y_1, y_2, \dots)$ so that f is surjective. Now let $z = (z_1, z_2, \dots)$ and $y = (y_1, y_2, \dots)$ be elements of X^ω, and suppose that $f(y) = f(z)$. Then $(0, y_1, y_2, \dots) = (0, z_1, z_2, \dots)$, so that $\forall i \in \mathbb{Z}_+$, $y_i = z_i$ and finally $y = z$. Therefore f is also injective, and is thus a bijection between X^ω and its proper subset A. This shows that X^ω is infinite.

□