Exercise 3. Let J and E be well-ordered sets; suppose there is an order-preserving map $k : J \to E$. Using Exercises 1 and 2, show that J has the order type of E or a section of E. [*Hint:* choose $e_0 \in E$. Define $h : J \to E$ by the recursion formula

$$h(\alpha) = \text{smallest } \big(E - h(S_\alpha)\big) \quad \text{if } h(S_\alpha) \neq E,$$

and $h(\alpha) = e_0$ otherwise. Show that $h(\alpha) \leq k(\alpha)$ for all α; conclude that $h(S_\alpha) \neq E$ for all α.]

Proof. The existence of k implies that E is nonempty. If J is empty, then there is no bijection between J and E or a section of E. Therefore we suppose that J is nonempty. Suppose that $k : J \to E$ is an order-preserving map, and let $e_0 \in E$.

Let $\mathscr{F}$ be the set of all functions mapping sections of J into E. For all $f \in \mathscr{F}$, let S_α for some α in J, be the domain of f. Let

$$\begin{aligned} \rho : \mathscr{F} &\to E \\ f &\mapsto \begin{cases} \min\big(E - f(S_\alpha)\big) & \text{if } f(S_\alpha) \neq E \\ e_0 & \text{otherwise} \end{cases} \end{aligned}$$

From exercise 1 (general principle of recursive definition), the function ρ above defines a unique function $h : J \to E$ such that

$$\forall \alpha \in J, \quad h(\alpha) = \rho(h|S_\alpha) = \begin{cases} \min\big(E - h(S_\alpha)\big) & \text{if } h(S_\alpha) \neq E \\ e_0 & \text{otherwise} \end{cases}$$

Let J_0 be the set of elements α of J such that $h(\alpha) \leq k(\alpha)$. Suppose that $S_\beta \subset J_0$ for some $\beta \in J$. For all $\alpha \in S_\beta$, we have $h(\alpha) \leq k(\alpha) < k(\beta)$, so $k(\beta)$ is an upper bound for $h(S_\beta)$, and therefore an element of $E - h(S_\beta)$. Since $h(\beta)$ is the smallest element of $E - h(S_\beta)$, we deduce that $h(\beta) \leq k(\beta)$. Therefore $\beta \in J_0$, so J_0 is an inductive subset of J; thus $J_0 = J$, and

$$\forall \alpha \in J, \quad h(\alpha) \leq k(\alpha)$$

Suppose that there exists $\alpha \in J$ such that $h(S_\alpha) = E$. Since $k(\alpha) \in E$, we deduce that there exists $\beta \in S_\alpha$ such that $h(\beta) = k(\alpha)$. From this, we get

$$k(\alpha) = h(\beta) \leq k(\beta) < k(\alpha)$$

This is a contradiction, so for all $\alpha \in J$, $h(S_\alpha) \neq E$.

From the above we deduce that h is defined by:

$$\forall \alpha \in J, \quad h(\alpha) = \min\big(E - h(S_\alpha)\big)$$

We are therefore in the hypotheses of exercise 2, and conclude that h is order-preserving and its image is E or a section of E. As an order-preserving map, h is injective, and therefore a bijection from J to $h(J)$. Thus J has the order type of either E or a section of E.

$\square$