Exercise 6. Let S_Ω be the minimal uncountable well-ordered set.

(a) Show that S_Ω has no largest element.

(b) Show that for every $\alpha \in S_\Omega$, the subset $\{x|\alpha < x\}$ is uncountable.

(c) Let X_0 be the subset of S_Ω consisting of all elements x such that x has no immediate predecessor. Show that X_0 is uncountable.

Proof.

(a) Suppose S_Ω has a largest element M. Then the section S_M is countable, and the set $\bar{S}_M = S_M \cup \{M\}$ is also countable. For all $x \in S_\Omega$, we have $x \leq M$, so that $S_\Omega \subset \bar{S}_M$ is countable. This is a contradiction, so S_Ω does not have a largest element.

(b) Let $\alpha \in S_\Omega$, and $\bar{S}_\alpha = S_\alpha \cup \{\alpha\}$. Let $x \in S_\Omega$, x is comparable with α, so that exactly one of the following conditions is true: $x = \alpha$, or $x < \alpha$, or $\alpha < x$. From this we deduce that $S_\Omega = \bar{S}_\alpha \cup \{x|\alpha < x\}$. As the finite union of countable sets, $\bar{S}_\alpha$ is countable. Since S_Ω is uncountable, $\{x|\alpha < x\}$ must also be uncountable.

(c) Suppose that X_0 is countable, then it has an upper bound m in S_Ω. Since S_Ω does not have a largest element, m has an immediate successor α_1. By induction over $n \in \mathbb{Z}_+$, we define a function

$$\begin{aligned} \alpha : \mathbb{Z}_+ &\to S_\Omega \\ n &\mapsto \begin{cases} \alpha_1 & \text{if } n = 1 \\ \text{the immediate successor of } \alpha_{n-1} & \text{otherwise} \end{cases} \end{aligned}$$

Note that, since S_Ω does not have a largest element, the immediate successor of α_n exists for all $n \in \mathbb{Z}_+$, so that the rule above defines a unique function $\alpha : \mathbb{Z}_+ \to S_\Omega$. Let $A = \{\alpha_n, n \in \mathbb{Z}_+\}$, α is a surjection from $\mathbb{Z}_+$ onto A, so A is a countable subset of S_Ω. Therefore A has an upper bound in S_Ω, and since S_Ω is a well-ordered set, A has a least upper bound $u \in S_\Omega$ which verifies

$$m < \alpha_1 < \alpha_2 < \cdots \leq u \tag{1}$$

Suppose that there exists $p \in \mathbb{Z}_+$ such that $\alpha_p = u$. Then as the immediate successor of α_p, α_{p+1} must verify $\alpha_{p+1} > u$, which contradicts the fact that u is an upper bound for A. Therefore

$$\forall n \in \mathbb{Z}_+, \quad \alpha_n < u. \tag{2}$$

Suppose that u has an immediate predecessor v.

If there exists $r \in \mathbb{Z}_+$ such that $v < \alpha_r$, then from (2), we deduce that $\alpha_r < u$, which contradicts the fact that v is the immediate predecessor of u. Therefore

$$\forall n \in \mathbb{Z}_+, \quad \alpha_n \leq v$$

and this last proposition contradicts the fact that u is the least upper bound of A.

From the above we deduce that u does not have an immediate predecessor, which implies that $u \in X_0$, and therefore $u \leq m$. This contradicts (1), so X_0 does not have an upper bound in S_Ω and is therefore an uncountable subset of S_Ω.

$\square$