Exercise 8. Verify the following version of the principle of recursive definition: let A be a set. Let ρ be a function assigning, to every function f mapping a section S_n of $\mathbb{Z}_+$ into A, an element $\rho(f)$ of A. Then there exists a unique function $h : \mathbb{Z}_+ \to A$ such that $h(n) = \rho(h|S_n)$ for each $n \in \mathbb{Z}_+$.

Proof. Let ρ' be a function assigning, to every function f mapping a nonempty section S_{n+1} of $\mathbb{Z}_+$ into A, an element $\rho'(f) = \rho(f)$, and let $a_0 = \rho(\varnothing) \in A$. Theorem 8.4 allows us to conclude that there exists a unique function $h : \mathbb{Z}_+ \to A$ such that:

$$\begin{aligned} h(1) &= a_0 = \rho(\varnothing) \\ h(n) &= \rho'(h|S_n) = \rho(h|S_n) \qquad \text{for } n > 1 \end{aligned}$$

which is the expected result. $\square$