Exercise 5. Let S and S' be the following subsets of the plane:

$$S = \{(x,y) \mid y = x+1 \text{ and } 0 < x < 2\}$$
$$S' = \{(x,y) \mid y - x \text{ is an integer}\}$$

(a) Show that S' is an equivalence relation on the real line and $S' \supset S$. Describe the equivalence classes of S'.

(b) Show that given any collection of equivalence relations on a set A, their intersection is an equivalence relation on A.

(c) Describe the equivalence relation T on the real line that is the intersection of all equivalence relations on the real line that contain S. Describe the equivalence classes of T.

Proof.

(a) **reflexivity** $\forall x \in \mathbb{R}, \quad x - x = 0 \in \mathbb{Z}$, so $\forall x \in \mathbb{R}, \quad xS'x$

symmetry Let $x, y \in \mathbb{R}$ such that $xS'y$. Then $y - x = k \in \mathbb{Z}$, so $x - y = -k \in \mathbb{Z}$. Hence $yS'x$.

transitivity Let $x, y, z \in \mathbb{R}$ such that $xS'y$ and $yS'z$. Then there exist $k, k' \in \mathbb{Z}$ such that $y - x = k$ and $z - y = k'$. Adding these together we get $z - x = k + k' \in \mathbb{Z}$, so that $xS'z$.

S' being reflexive, symmetric and transitive on $\mathbb{R}$ is an equivalence relation on $\mathbb{R}$. Let $x \in \mathbb{R}$, and note $\dot{x}$ the equivalence class of x for S'. Let $x + \mathbb{Z} = \{x + k \text{ for all } k \in \mathbb{Z}\}$. Every $y \in x + \mathbb{Z}$ verifies $yS'x$, so $x + \mathbb{Z} \subset \dot{x}$. Conversely, let $y \in \dot{x}$, there exists $k \in \mathbb{Z}$ such that $y - x = k$, so that $y = x + k \in x + \mathbb{Z}$, so $\dot{x} \subset x + \mathbb{Z}$. From both, we conclude that $\forall x \in \mathbb{R}, \quad \dot{x} = x + \mathbb{Z}$.

(b) Let I be a set and $(R)_I$ be a family of equivalence relations on A, indexed by I. For each $i \in I$, $R_i \subset A \times A$, so $\cap_{i \in I} R_i \subset A \times A$, and therefore $R = \cap_{i \in I} R_i$ is a relation on A.

reflexivity $\forall x \in A, \quad \forall i \in I, \quad xR_ix$, from which we conclude $\forall x \in A, xRx$. In particular, R is not empty.

symmetry Let $x, y \in A$ such that xRy. Then $\forall i \in I, \quad xR_iy$, from which we conclude $\forall i \in I, yR_ix$ by symmetry of all R_i. This last proposition implies that yRx, so that R is symmetric.

transitivity Let $x, y, z \in A$ such that xRy and yRz. Then $\forall i \in I, \quad xR_iy$ and yR_iz, so that $\forall i \in I, xR_iz$ by transitivity of all R_i. The last proposition implies that xRz, so that R is transitive.

From the above, R is an equivalence relation on A.

(c) Let R be an equivalence relation on $\mathbb{R}$ that contains S. For all $x \in (0,2)$, we have $xR(x+1)$, and by symmetry of R, $(x+1)Rx$. Also, for all $x \in (0,1)$, $x+1 \in (1,2)$, so $(x+1)S(x+2)$ and therefore we must have $(x+1)R(x+2)$. This implies by transitivity that $xR(x+2)$. For all $x \in (1,2)$, $x-1 \in (0,1)$ so $x-1$ was already accounted for in the previous cases. The case $x=1$ gives only $1S2$, from which we deduce $1R2$ and $2R1$. Finally, R being defined on all $\mathbb{R}$ must be reflexive and thus $\forall x \in (-\infty,0] \cup [3,+\infty)$ we have xRx.

All the conditions above are necessary to any equivalence relation containing S; since R defined in this way is an equivalence relation, we deduce that $T=R$.

The equivalence classes for T are:

$$\begin{aligned}
&\overline{x} = \{x\} \text{ if } x \leq 0 \text{ or } x \geq 3 \\
&\overline{x} = \{(x,x),(x,x+1),(x,x+2), \\
&\quad (x+1,x),(x+1,x+1),(x+1,x+2), \\
&\quad (x+2,x),(x+2,x+1),(x+2,x+2)\} \text{ for } x \in (0,1) \\
&\overline{x} = \overline{x-1} \text{ if } x \in (1,2) \\
&\overline{x} = \overline{x-2} \text{ if } x \in (2,3)
\end{aligned}$$

$\square$