Exercise 4. Let $f : A \to B$ and $g : B \to C$.

(a) If $C_0 \subset C$, show that $(g \circ f)^{-1}(C_0) = f^{-1} \circ g^{-1}(C_0)$

(b) If f and g are injective, show that $g \circ f$ is injective

(c) If $g \circ f$ is injective, what can you say about the injectivity of f and g?

(d) If f and g are surjective, show that $g \circ f$ is surjective

(e) If $g \circ f$ is surjective, what can you say about the surjectivity of f and g?

Proof.

(a) Let $x \in (g \circ f)^{-1}(C_0)$. Then $g \circ f(x) \in C_0$, from which we deduce that $f(x) \in g^{-1}(C_0)$, and then that $x \in f^{-1} \circ g^{-1}(C_0)$. Thus we have $(g \circ f)^{-1}(C_0) \subset f^{-1} \circ g^{-1}(C_0)$.

Conversely, let $x \in f^{-1} \circ g^{-1}(C_0) = \left\{x \in A \mid f(x) \in g^{-1}(C_0)\right\} = f^{-1}\left(g^{-1}(C_0)\right)$. Then we have $f(x) \in g^{-1}(C_0) = \{y \in B \mid g(y) \in C_0\}$, so that $g \circ f(x) \in C_0$, from which we deduce that $f^{-1} \circ g^{-1}(C_0) = (g \circ f)^{-1}(C_0)$.

(b) Let x and y in A such that $g \circ f(x) = g \circ f(y)$. By injectivity of g, we deduce that $f(x) = f(y)$, and by injectivity of f, that $x = y$. Thus $g \circ f$ is injective.

(c) Let x and y in A such that $f(x) = f(y)$. g being a function, we deduce $g \circ f(x) = g \circ f(y)$. By injectivity of $g \circ f$, we deduce that $x = y$, hence that f is injective. Note that g is not necessarily injective: take $A = \{1, 2\}$, $B = \{3, 4, 5\}$, $C = \{6, 7\}$, $f(1) = 3$, $f(2) = 4$, $g(3) = 6$, $g(4) = 7$, $g(5) = 7$. Then $g \circ f$ is injective but g itself is not.

(d) Let $z \in C$. By surjectivity of g, there exists $y \in B$ such that $z = g(y)$. By surjectivity of f, there exists $x \in A$ such that $y = f(x)$. Composing both, we get $z = g \circ f(x)$, so $g \circ f$ is surjective.

(e) Let $z \in C$. By surjectivity of $g \circ f$, there exists $x \in A$ such that $z = g \circ f(x) = g\left(f(x)\right)$. So $f(x)$ is an antecedent of z by g, from which we conclude that g is surjective. f does not need to be surjective, however: take $A = \{1, 2\}$, $B = \{3, 4, 5\}$, $C = \{6, 7\}$, $f(1) = 3$, $f(2) = 4$, $g(3) = 6$, $g(4) = 7$, $g(5) = 7$. Then $g \circ f$ is injective, but f is not $\square$