Exercise 6. A collection $\mathscr{A}$ of subsets of a set X is said to be of *finite type* provided that a subset B of X belongs to $\mathscr{A}$ if and only if every finite subset of B belongs to $\mathscr{A}$. Show that the Kuratowski lemma implies the following:

Lemma (Tukey, 1940). *Let $\mathscr{A}$ be a collection of sets. If $\mathscr{A}$ is of finite type, then $\mathscr{A}$ has an element that is properly contained in no other element of $\mathscr{A}$.*

Proof. Let $\mathscr{A}$ be a collection of sets of finite type, and let $\mathscr{B}$ be a subcollection of $\mathscr{A}$ that is simply ordered by proper inclusion. For all $B \in \mathscr{B}$, $B \in \mathscr{A}$, and $\mathscr{A}$ is of finite type, so every finite subset of B is an element of $\mathscr{A}$.

Let $U = \cup_{b \in \mathscr{B}} b$, and let C be a finite subset of U. If C is empty, then C is a subset of $\mathscr{A}$. Otherwise, $C = \{c_1, \dots, c_n\}$, and for all $i \in \{1, \dots, n\}$, there exists $B_i \in \mathscr{B}$ such that $c_i \in B_i$. As a subset of the simply ordered set $\mathscr{B}$, the set $\{B_1, \dots, B_n\}$ is simply ordered. Since it is also finite, it has a largest element B_m. For all $i \in \{1, \dots, n\}$, we have $B_i \subset B_m$. From this we deduce that C is a finite subset of B_m, and hence an element of $\mathscr{A}$.

Every finite subset of the set U is an element of $\mathscr{A}$, and $\mathscr{A}$ is of finite type, so $U \in \mathscr{A}$. We can therefore apply Kuratowski's lemma and conclude that there exists and element of $\mathscr{A}$ that is properly contained in no other element of $\mathscr{A}$.

$\square$