Exercise 7. Let J be a well-ordered set. A subset J_0 of J is said to be *inductive* if for every $\alpha \in J$,

$$(S_\alpha \subset J_0) \implies \alpha \in J_0$$

Theorem. *(The principle of transfinite induction). If J is a well-ordered set and J_0 is an inductive subset of J, then $J_0 = J$.*

Proof. Given an element $x \in J$, we denote by $x+1$ the immediate successor of x, if it exists, and by $x-1$ the immediate predecessor of x, if it exists. Let J_0 be an inductive subset of J.

If $J = \varnothing$, then the only subset of J is $J_0 = J = \varnothing$, which is inductive. Suppose then that J is nonempty. Since J is well-ordered, it has a smallest element m.

Let $A = \{x \in J \mid S_x \subset J_0\}$. From the inductivity of J_0, we deduce that $A \subset J_0$. Since $S_m = \varnothing \subset J_0$, we have $m \in A$ so A is nonempty. Let $x \in A$; for all $y < x$ we have $S_y \subset S_x \subset J_0$, and by inductivity of J_0, we conclude that $y \in A$, so

$$\forall x \in A, \quad S_x \subset A \tag{1}$$

Let $x, y \in J$ such that $y < x$. From $S_y \subset S_x$, we deduce that $\cup_{y<x} S_y \subset S_x$. Conversely, let $z \in S_x$. Suppose that x does not have an immediate predecessor; then there exists $a \in J$ such that $z < a < x$, so that $z \in S_a \subset \cup_{y<x} S_y$. Therefore

$$\forall x \in A, \quad S_x = \begin{cases} S_{x-1} \cup \{x-1\} & \text{if } x-1 \text{ exists} \\ \bigcup_{y<x} S_y & \text{otherwise} \end{cases} \tag{2}$$

Suppose that $J - A \neq \varnothing$ and let $M \in J - A$. If there exists $y \in A$ such that $y > M$, then (1) implies that $S_y \subset A$, so that $M \in A$, which is a contradiction. Therefore for all $y \in A$, we have $y < M$. Let then u be the least upper bound of A in J.

- If u has an immediate predecessor, then $u - 1 \in A$, for otherwise, we have $y < u - 1$ for all y in A which contradicts the fact that u is the least upper bound of A.

  Since $u - 1 \in A$, then from (2) we deduce that $S_u \subset J_0$, which implies that $u \in A$.

- Otherwise, u does not have an immediate predecessor. If $x \in S_u$, then there exists $y \in A$ such that $x < y < u$ (otherwise, u would not be the least upper bound of A in J), so that (2) gives us $S_u = \cup_{x \in A} S_x$. From this we deduce that $S_u \subset J_0$, and $u \in A$.

Since $u \in A$, then u is not the largest element of J (otherwise $J - A$ would be empty). Therefore u has an immediate successor, for which $S_{u+1} = S_u \cup \{u\} \subset J_0$, so that $u + 1 \in A$, contrary to the hypothesis that u is an upper bound for A.

Therefore $J = A$, and since $A \subset J_0$, we conclude that $J_0 = J$.

$\square$