Exercise 5. Which of the following subsets of $\mathbb{R}^\omega$ can be expressed as the cartesian product of subsets of $\mathbb{R}$?

(a) $\{\mathbf{x} \mid x_i \text{ is an integer for all } i\}$

(b) $\{\mathbf{x} \mid x_i \geq i \text{ for all } i\}$

(c) $\{\mathbf{x} \mid x_i \text{ is an integer for all } i \geq 100\}$

(d) $\{\mathbf{x} \mid x_2 = x_3\}$

Proof.

(a) Let $A = \{\mathbf{x} \mid x_i \text{ is an integer for all } i\}$; then for all i, $x_i \in \mathbb{Z}_+$ so that $A \subset \mathbb{Z}_+^\omega$. Conversely, for $x \in \mathbb{Z}_+^\omega$, x_i is an integer by definition of $\mathbb{Z}_+^\omega$, so $A = \mathbb{Z}_+^\omega$.

(b) Let $A = \{\mathbf{x} \mid x_i \geq i \text{ for all } i\}$, and let $B = \prod_{i \in \mathbb{Z}_+} [i, +\infty)$. Then for all i and $y \in [i, +\infty)$, $y \geq i$, so we can take $x_i \in [i, +\infty)$ to satisfy the condition in A. Therefore $B \subset A$. Conversely, for all $\mathbf{x} \in A$ and all $i \in \mathbb{Z}_+$, we have $x_i \geq i$, so that $x_i \in [i, +\infty)$, from which we conclude that $B = A$.

(c) Let $A = \{\mathbf{x} \mid x_i \text{ is an integer for all } i \geq 100\}$, and let $B = \mathbb{R}^{99} \times \mathbb{Z}_+^\omega$. For all $\mathbf{x} \in B$, x_i is an integer if $i \geq 100$, so $B \subset A$. Conversely, for all $\mathbf{x} \in A$, the first 99 coordinates of $\mathbf{x}$ are only required to be real numbers, so that $\mathbf{x} \in B$, and $B = A$.

(d) Let $A = \{\mathbf{x} \mid x_2 = x_3\}$, and suppose there exist $A_1, A_2, \dots$ subsets of $\mathbb{R}$ such that $A = A_1 \times A_2 \times \cdots$. For all $\mathbf{x} \in A$, we have $x_2 = x_3$, so that $A_2 = A_3$. Suppose that A_2 contains at least 2 elements a and b. Then the ω-tuple $(x_1, a, b, x_4, \dots)$ is an element of $A_1 \times A_2 \times \cdots$, but is not an element of A. So A_2 has at most 1 element. Suppose there exists $a \in A_2$. Let $\mathbf{x} = (x_1, a, a, x_4, \dots)$ and $\mathbf{y} = (x_1, a+1, a+1, x_4, \dots)$ for some $x_1, x_4, \dots$ elements of $\mathbb{R}$. Then both $\mathbf{x}$ and $\mathbf{y}$ are elements of A, but $\mathbf{y}$ is not an element of $A_1 \times A_2 \times \cdots$. So A_2 must be empty, in which case (because of the axiom of choice), $A_1 \times A_2 \times \cdots = \varnothing$. Last, note that $(0, 0, 0, \dots) \in A$ so $A \neq \varnothing$. Therefore, $A \neq A_1 \times A_2 \times \cdots$ for any subsets $A_1, A_2, \dots$ of $\mathbb{R}$.

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