Exercise 10. Show that every positive number a has exactly one positive square root, as follows:

(a) Show that if $x > 0$ and $0 \le h < 1$, then
$$(x+h)^2 \le x^2 + h(2x+1)$$
$$(x-h)^2 \ge x^2 - h(2x)$$

(b) Let $x > 0$. Show that if $x^2 < a$, then $(x+h)^2 < a$ for some $h > 0$; and if $x^2 > a$, then $(x-h)^2 > a$ for some $h > 0$.

(c) Given $a > 0$, let B be the set of all real numbers x such that $x^2 < a$. Show that B is bounded above and contains at least one positive number. Let $b = \sup B$; show that $b^2 = a$.

(d) Show that if b and c are positive and $b^2 = c^2$, then $b = c$.

Proof.

(a) Let $x > 0$ and $0 \le h < 1$. We have $0 \le h^2 \le h$, so that:
$$(x+h)^2 = x^2 + 2xh + h^2 \le x^2 + h(2x+1)$$
And since $h^2 \ge 0$, we have:
$$(x-h)^2 = x^2 - 2xh + h^2 \ge x^2 - h(2x)$$

(b) Let $x > 0$ such that $x^2 < a$, and let $0 \le h < 1$. Since $2x+1 > 0$, we have:
$$\begin{aligned} &(x+h)^2 \ge a \\ \Longrightarrow\ & x^2 + h(2x+1) \ge a \\ \Longrightarrow\ & h \ge \frac{a - x^2}{2x+1} = m \end{aligned}$$
From $x^2 < a$ we deduce that $m > 0$. Therefore we can choose h such that $0 < h < \sup\{m, 1\}$. By the contrapositive of the implication shown above, we have, for such a choice of h, $(x+h)^2 < a$.

Similarly, from $h \ge 0$, we have:
$$\begin{aligned} &(x-h)^2 \le a \\ \Longrightarrow\ & x^2 - h(2x) \le a \\ \Longrightarrow\ & h \ge \frac{x^2 - a}{2x} = p \end{aligned}$$
From $x^2 > a$ we deduce that $p > 0$, and we can choose h such that $0 < h < \sup\{p, 1\}$. By the contrapositive of the implication shown above, we have, for such a choice of h, $(x-h)^2 > a$.

(c) Let $a > 0$ and $B = \{x \in \mathbb{R} \mid 0 < x^2 < a\}$. Taking $y = \sup\{1, a\}$, we have $y^2 \geq a$, so B is bounded above by y. If $a \geq 1$, then $0 < (1/2)^2 < 1$, so $1/2 \in B$. Otherwise, $0 < a^2 \leq a$, so that $a \in B$. Thus B is nonempty. From this we deduce that $b = \sup B$ exists and is positive.

Suppose that $b^2 < a$. Then there exists $h > 0$ such that $(b+h)^2 < a$, so that b is not an upper bound of B, which is a contradiction. Therefore $b^2 \geq a$. Suppose now that $b^2 > a$, then there exists $h > 0$ such that $(b-h)^2 > a$, so that $b - h$ is an upper bound of B that is smaller than b, which is again a contradiction. From this we deduce that $b^2 = a$, so that any positive real number has at least one square root.

(d) Let b and c be positive reals such that $b^2 = c^2$. Suppose that $0 < b < c$, then we have $0 < b^2 < c^2$, which is a contradiction. Similarly if $0 < c < b$ we arrive at $0 < c^2 < b^2$, which is again a contradiction. Therefore $b = c$, and the square root of any positive real number is unique.

□