Exercise 1.
Theorem (General principle of recursive definition)**.** *Let J be a well-ordered set; let C be a set. Let $\mathscr{F}$ be the set of all functions mapping sections of J into C. Given a function $\rho : \mathscr{F} \to C$, there exists a unique function $h : J \to C$ such that $h(\alpha) = \rho(h|S_\alpha)$ for each $\alpha \in J$.*

[*Hint:* Follow the pattern outlined in exercise 10 of §10.]

Proof. For future reference in the proof, the property we wish to have a function $f : J \to C$ satisfy is

$$\forall \alpha \in J, \quad h(\alpha) = \rho(f|S_\alpha) \tag{1}$$

The existence of a function $f : J \to C$ implies that C is nonempty. Furthermore, if J is empty, then there is a unique function from J to C, and it vacuously verifies (1). Therefore we assume in the rest of the proof that J is nonempty.

Lemma 1. *If h and k map sections of J into C and satisfy* (1) *for all α in their respective domains, then that $h(x) = k(x)$ for all x in both domains.*

Proof. Let $\mathscr{D}_h$ (resp. $\mathscr{D}_k$) be the domain of h (resp. of k). The set $\mathscr{D}_h \cap \mathscr{D}_k$, as the intersection of two sections of J, is equal to one of these sections. Possibly switching the roles of h and k, we can suppose that $\mathscr{D}_h \subset \mathscr{D}_k$. Let $\mathscr{A}$ be the subset of $\mathscr{D}_h$ such that for all $\alpha \in \mathscr{A}$, $h(\alpha) \neq k(\alpha)$. If $\mathscr{A}$ is nonempty, then, as a subset of the well-ordered set J, it has a smallest element β. For all $\alpha < \beta$, α is an element of the section $\mathscr{D}_h$ and $h(\alpha) = k(\alpha)$, so that $h|S_\beta = k|S_\beta$. Since h and k satisfy (1), we deduce that $h(\beta) = \rho(h|S_\beta) = \rho(k|S_\beta) = k(\beta)$, which contradicts the existence of β. Therefore for all $\alpha \in \mathscr{D}_h$, we have $h(\alpha) = k(\alpha)$.

□

Lemma 2. *If there exists a function $h : S_\alpha \to C$ which satisfies* (1)*, then there exists a function $k : S_\alpha \cup \{\alpha\} \to C$ which satisfies* (1)*.*

Proof. Let $\alpha \in J$ and suppose there exists $h : S_\alpha \to C$ satisfying (1). Let

$$\begin{aligned} k : S_\alpha \cup \{\alpha\} &\to C \\ x &\mapsto \begin{cases} h(x) & \text{if } x \in S_\alpha \\ \rho(h) & \text{otherwise} \end{cases} \end{aligned}$$

For all $x \in S_\alpha$, $k(x) = h(x)$, so that for all $y \in S_x$, $h|S_x = k|S_x$. Since h satisfies (1), we have $k(x) = h(x) = \rho(h|S_x) = \rho(k|S_x)$. Thus k satisfies (1) on S_α. Moreover, $k(\alpha) = \rho(h) = \rho(k|S_\alpha)$, so k satisfies (1) on $S_\alpha \cup \{\alpha\}$.

□

Lemma 3. *If $K \subset J$ and for all $\alpha \in K$, there exists a function $h_\alpha : S_\alpha \to C$ satisfying* (1)*, there exists a function*

$$k : \bigcup_{\alpha \in K} S_\alpha \to C$$

satisfying (1)*.*

Proof. Let $I = \cup_{\alpha \in K} S_\alpha$. If K is empty or has only one element, then I is empty and there is no function from I to C. Therefore K has at least two elements.

Let R be the subset of $I \times C$ containing all the tuples $(x, h_\alpha(x))$ for all $\alpha \in K$ and all $x \in S_\alpha$. Let $\alpha, \beta \in K$ such that $\alpha < \beta$; we have $S_\alpha \subset S_\beta$, and from lemma 1, $h_\beta | S_\alpha = h_\alpha$. From this we deduce that for all $x \in I$, there is a unique $y \in C$ such that $(x, y) \in R$. Therefore R is the rule of a function $k : I \to C$.

Let $x \in I$, there exists $\alpha \in K$ such that $x \in S_\alpha$. The function h_α satisfies (1) on its domain S_α, so $k(x) = h_\alpha(x) = \rho(h_\alpha | S_x)$. For all $y \in S_x$, $h_\alpha(y)$ is the unique element of R with y as its first coordinate, so $h_\alpha(y) = k(y)$. From this we deduce that $h_\alpha | S_x = k | S_x$, and therefore k satisfies (1).

$\square$

Lemma 4. *For all $\beta \in J$, there exists a function $h_\beta : S_\beta \to C$ satisfying* (1)*.*

Proof. Let J_0 be the subset of J such that for all $\alpha \in J_0$, there exists a function $h_\alpha : S_\alpha \to C$ satisfying (1). Since J is nonempty and well-ordered, it has a smallest element m, and $S_m = \varnothing$. There is only one function from $\varnothing$ to C, and it vacuously satisfies (1), so $S_m \subset J_0$, $m \in J_0$ and J_0 is nonempty.

Suppose that for some $\beta \in J$, we have $S_\beta \subset J_0$.

- If β has an immediate predecessor α, then $\alpha \in S_\beta$, so that $\alpha \in J_0$. Therefore there exists a function $h_\alpha : S_\alpha \to C$ satisfying (1). The fact that α is the immediate predecessor of β implies that $S_\beta = S_\alpha \cup \{\alpha\}$. From lemma 2 we deduce that there exists a function $h_\beta : S_\beta \to C$ satisfying (1). Therefore $\beta \in J_0$.

- Otherwise, β does not have an immediate predecessor, and thus $S_\beta = \cup_{\alpha < \beta} S_\alpha$. For all $\alpha \in J$ such that $\alpha < \beta$, we have $\alpha \in S_\beta$ and therefore $\alpha \in J_0$. From this we deduce that there exists a function $h_\alpha : S_\alpha \to C$ satisfying (1). We are within the hypotheses of lemma 3, and can conclude to the existence of a function $h_\beta : S_\beta \to C$ satsifying (1). From this we deduce that $\beta \in J_0$.

The above shows that J_0 is an inductive subset of the well-ordered set J, and therefore $J_0 = J$.

$\square$

If J has a largest element M, then $J = S_M \cup \{M\}$. From lemma 4, we deduce the existence of $h_M : S_M \to C$ satisfying (1); and from lemma 2, we deduce the existence of a function $h : J \to C$ satisfying (1).

Otherwise, $J = \cup_{\alpha \in J} S_\alpha$: for all $\alpha \in J$, $S_\alpha \subset J$, so that $\cup_{\alpha \in J} S_\alpha \subset J$. Conversely, for all $\alpha \in J$, there exists $\beta \in J$ such that $\alpha < \beta$, since J does not have a largest element. From this we deduce that $\alpha \in S_\beta$ and finally $J \subset \cup_{\alpha \in J} S_\alpha$.

For all $\alpha \in J$, we deduce from lemma 4 the existence of a function $h_\alpha : S_\alpha \to C$ satisfying (1). Let then R be the subset of the cartesian product $J \times C$ containing all the tuples $(x, h_\alpha(x))$ for all $\alpha \in J$ and all $x \in S_x$. From lemma 1, we deduce that, for all $\alpha, \beta \in J$ such that $\alpha < \beta$, $h_\beta | S_\alpha = h_\alpha$. The set R is therefore the rule of a function $h : J \to C$.

Let $\alpha \in J$. For all $x \in S_\alpha$, $(x, h_\alpha(x))$ is the unique tuple in R with x as its first coordinate, so that $h_\alpha | S_x = h | S_x$. Furthermore, since h_α satisfies (1) on S_α, we have

$$h(x) = h_\alpha(x) = \rho(h_\alpha | S_x) = \rho(h | S_x)$$

so that the function $h : J \to C$ satisfies (1).

Suppose that $h, k : J \to C$ both satisfy (1), and let $\mathscr{A}$ be the subset of J such that for all $\alpha \in \mathscr{A}$, $h(\alpha) \neq k(\alpha)$. If $\mathscr{A}$ is nonempty, then it has a smallest element β. The functions h and k are both defined on the section S_β, and for all $\alpha \in S_\beta$, $h(\alpha) = k(\alpha)$. From this we deduce that $h | S_\beta = k | S_\beta$, and, since h and k satisfy (1) on S_β, we have $h(\beta) = \rho(h | S_\beta) = \rho(k | S_\beta) = k(\beta)$, which contradicts the existence of β. Therefore the function h defined in the preceding paragraph is unique.

$\square$