Exercise 5. Show that is $\mathscr{A}$ is a basis for a topology on X, then the topology generated by $\mathscr{A}$ equals the intersection of all topologies on X that contain $\mathscr{A}$. Prove the same if $\mathscr{A}$ is a subbasis.

Proof. Let $\tau(\mathscr{A})$ be the topology generated by the basis $\mathscr{A}$, and let T be the intersection of all topologies containing $\mathscr{A}$. As a topology containing $\mathscr{A}$, $\tau(\mathscr{A})$ contains T. Conversely, a topology T' containing A must contain all unions of elements of $\mathscr{A}$; $\tau(\mathscr{A})$ is the set of such unions, so $\tau(\mathscr{A}) \subset T'$. This is true for all topologies $T' \in T$, so $\tau(\mathscr{A}) \subset T$.

If $\mathscr{A}$ is a subbasis of X, let $\sigma(\mathscr{A})$ be the topology generated by $\mathscr{A}$, and let S be the intersection of all topologies containing $\mathscr{A}$. As in the previous case, $\mathscr{A} \subset \sigma(\mathscr{A})$, so $S \subset \sigma(\mathscr{A})$. Conversely, a topology S' containing $\mathscr{A}$ must contain all finite intersections $(I_\alpha)_{\alpha \in J}$ of elements of $\mathscr{A}$, and all unions of its elements; therefore it must contain all $\cup_{\beta \in K} I_\beta$, for all $K \subset J$. The set of all such unions is $\sigma(\mathscr{A})$, so $\sigma(\mathscr{A}) \subset S'$. This inclusion is true for all S', so $\sigma(\mathscr{A}) \subset S$.

$\square$