Exercise 4. Let A be a nonempty finite simply ordered set.

(a) Show that A has a largest element. [Hint: proceed by induction on the cardinality of A.]

(b) Show that A has the order type of a section of the positive integers.

Proof.

(a) Let n be the cardinality of A. Since A is nonempty, we have $n \geq 1$. Suppose that $n = 1$. Then $A = \{a\}$, and since the element a is the only element of A, it is also its largest element. The subset $\mathscr{A}$ of $\mathbb{Z}_+$ of elements n such that any nonempty finite simply ordered set of cardinality n has a largest element, is nonempty since $1 \in \mathscr{A}$.

Suppose that $n \in \mathscr{A}$ and let A be a nonempty finite simply ordered set of cardinality $n+1$ (such sets exist: for example consider the section S_n of positive integers). Let $a_0 \in A$. The set $B = A - \{a_0\}$ is nonempty (since $n \geq 1$), finite (since it is a proper subset of a finite set), simply ordered (since elements of B can be compared as elements of A). From this and $n \in \mathscr{A}$ we deduce that B has a largest element b_0. As elements of A, a_0 and b_0 are comparable, and

- either $b_0 < a_0$, in which case a_0 is the largest element of A
- or $a_0 < b_0$, in which case b_0 is the largest elements of A

In both cases, we conclude to the existence of a largest element in A, so that $n+1 \in \mathscr{A}$. As a nonempty inductive subset of $\mathbb{Z}_+$, $\mathscr{A} = \mathbb{Z}_+$ and the result is proved.

(b) Suppose that A has the same order type as a section S_n of the positive integers. Then there exists a bijection $f : A \to S_n$ which respects order. Since S_n is finite, we conclude that A has the cardinality of S_n. Therefore when supposing that A has the order type of a section of the positive integers, it is enough to consider the section of positive integers with the same cardinality as A.

Suppose that A has cardinality 1, then there is a bijection $f : A \to \{1\}$, and f necessarily respects order. So A has the order type of the section of the positive integers $S_0 = \{1\}$.

Suppose now that the result is true for any A of cardinality n, and let A be a nonempty finite simply ordered set of cardinality $n+1$. From the previous point, A has a largest element a_0. Then $B = A - \{a_0\}$ is a nonempty finite simply ordered set of cardinality n, so by hypothesis, it has the same order type as the section of positive integers $S_{n-1} =$

$\{1, 2, \ldots, n\}$. Let $f : B \to \{1, 2, \ldots, n\}$ be a bijection that respects order, and let

$$\begin{aligned} g : A &\to \{1, 2, \ldots, n+1\} \\ x &\mapsto \begin{cases} f(x) & \text{if} \quad x \in B \\ n+1 & \text{if} \quad x = a_0 \end{cases} \end{aligned}$$

Then g is bijective, since f is bijective and $a_0 \notin B$ has a different image by g than any element of B. Further we have $\forall x \in B$, $f(x) < f(a_0) = n+1$ so g respects the order of A, so that A and S_n have the same order type.

$\square$