Exercise 7. Let J be a well-ordered set. A subset J_0 of J is said to be *inductive* if for every $\alpha \in J$,

$$(S_\alpha \subset J_0) \implies \alpha \in J_0$$

Theorem. *(The principle of transfinite induction). If J is a well-ordered set and J_0 is an inductive subset of J, then $J_0 = J$.*

Proof. Let $x \in J$. If x has an immediate predecessor y, then $S_x = S_y \cup \{y\}$. Otherwise, let $A_x = \cup_{y<x} S_y$. If $x_0 \in S_x$, then there exists $y \in S_x$ such that $x_0 < y$, since x does not have an immediate predecessor. From this we deduce that $x_0 \in A_x$. Conversely, if $x_0 \in A_x$, then there exists $y < x$ such that $x_0 \in S_y \subset S_x$. Therefore $A_x = S_x$.

If $J = \varnothing$, then the only subset of J is $J_0 = J = \varnothing$, which is inductive. Suppose then that $J \neq \varnothing$, and that there exists $x \in J - J_0$. Since J is well-ordered, there exists a smallest such element α.

If α has an immediate predecessor β, then $S_\alpha = S_\beta \cup \{\beta\}$. For all $x < \alpha$, we have $x \in J_0$, so that $S_\beta \subset J_0$ and $\beta \in J_0$. Since J_0 is inductive, we deduce that $\alpha \in J_0$, which is a contradiction.

Suppose then that α does not have an immediate predecessor, then $S_\alpha = \cup_{x<\alpha} S_x$. For all $x < \alpha$, we have $x \in J_0$ so that $\cup_{x<\alpha} S_x \subset J_0$, and, from the inductivity of J_0, $\alpha \in J_0$, which is again a contradiction.

Therefore such an α does not exist. From this we conclude that $J_0 = J$.

$\square$