**Exercise 3.**

(a) Prove by induction that given $n \in \mathbb{Z}_+$, every nonempty subset of $\{1, 2, \ldots, n\}$ has a largest element

(b) Explain why you cannot conclude from (a) that every nonempty subset of $\mathbb{Z}_+$ has a largest element

*Proof.*

(a) Let $A$ be the subset of $\mathbb{Z}_+$ such that $\forall n \in A$, every nonempty subset of $\{1, 2, \ldots, n\}$ has a largest element. Then $1 \in A$ since the only nonempty subset of $\{1\}$ is itself, and 1 is thus its largest element. Suppose that $n \in A$, and consider a nonempty subset $A_0$ of $\{1, 2, \ldots, n+1\}$. If $n+1 \in A_0$, then it is the largest element of $A_0$. Otherwise, $A_0 \cap \{1, 2, \ldots, n\}$ is nonempty and $n \in A$, so that $A_0$ has a largest element. By induction we deduce that $A = \mathbb{Z}_+$.

(a) The previous point showed that every nonempty subset of $\mathbb{Z}_+$ that has an upper bound has a largest element. There are nonempty subsets of $\mathbb{Z}_+$ that do not have an upper bound, for example $\mathbb{Z}_+$ itself, so the result would not hold for any nonempty subset of $\mathbb{Z}_+$. $\square$