Exercise 7. If A and B are finite, show that the set of all functions $f : A \to B$ is finite.

Proof. Let D be the set of all functions from A to B. If $B = \varnothing$, then there is no function from A to B, so D is empty, and thus finite. If $A = \varnothing$, then there is one function f from A to B (the function whose rule of assignment is $\varnothing$), so $D = \{f\}$ and is therefore finite.

Otherwise, let

$$\begin{aligned} \phi : D &\to \mathscr{P}(A \times B) \\ f &\mapsto \big\{\big(a, f(a)\big) \mid a \in A\big\} \end{aligned}$$

Let $f, g : A \to B$ be distinct functions. There exists $x \in A$ such that $f(x) \neq g(x)$, from which we deduce that $\phi(f) \neq \phi(g)$, so that ϕ is injective. Since A and B are finite, the cartesian product $A \times B$ is finite, and so is $\mathscr{P}(A \times B)$; let $h : \mathscr{P}(A \times B) \to \{1, 2, \dots, n\}$ be a bijection, for some n. Then $h \circ \phi$ is an injection from D to $\{1, 2, \dots, n\}$, so that D is finite. $\square$