Exercise 3. Let X be the two-element set $\{0, 1\}$. Show there is a bijective correspondence between the set $\mathscr{P}(\mathbb{Z}_+)$ and the cartesian product X^ω.

Proof. Let

$$\begin{aligned} f : \mathscr{P}(\mathbb{Z}_+) &\to X^\omega \\ A &\mapsto (x_1, x_2, \dots) \qquad \text{with} \qquad x_i = \begin{cases} 1 & \text{if } i \in A \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Let $(x_1, x_2, \dots) \in X^\omega$. The expression $S = \{i \in \mathbb{Z}_+ \mid x_i = 1\}$ defines a subset of $\mathbb{Z}_+$. If $S \neq \varnothing$, then we have $f(S) = (x_1, x_2, \dots)$. Otherwise, $x_i = 0$ for all i, and $f(\varnothing) = (0, 0, \dots)$. From this we deduce that f is surjective.

Let now A, B be distinct subsets of $\mathbb{Z}_+$. By switching the roles of A and B if needed, we can suppose that $A \neq \varnothing$. Since A and B are distinct, there exists $j \in A - B$. Then for $f(A) = (a_1, a_2, \dots, a_j, \dots)$ and $f(B) = (b_1, b_2, \dots, b_j, \dots)$, we have $a_j = 1$ but $b_j = 0$, so $f(A) \neq f(B)$ and f is injective.

□