Exercise 5. Let X be a set; let $\mathscr{A}$ be the collection of all pairs $(A, <)$ where A is a subset of X and $<$ is a well-ordering of A. Define

$$(A, <) \prec (A', <')$$

if $(A, <)$ *equals* a section of $(A', <')$.

(a) Show that $\prec$ is a strict partial order on $\mathscr{A}$.

(b) Let $\mathscr{B}$ be a subcollection of $\mathscr{A}$ that is simply ordered by $\prec$. Define B' to be the union of the sets B, for all $(B, <)$ in $\mathscr{B}$; and define $<'$ to be the union of the relations $<$, for all $(B, <)$ in $\mathscr{B}$. Show that $(B', <')$ is a well-ordered set.

Proof.

(a) For all $(A, <_A), (B, <_B) \in \mathscr{A}$ such that $(A, <_A) \prec (B, <_B)$, there exists $\beta \in B$ such that $(A, <_A) = (S_\beta, <_B)$, so that $B \neq \varnothing$.

Lemma 1. *Let $(A, <_A), (B, <_B) \in \mathscr{A}$ such that $(A, <_A) \prec (B, <_B)$. For all $x, y \in A$, $x <_A y$ if and only if $x <_B y$.*

Proof. Let J be the set of elements $y \in A$ such that

$$\forall x \in A, \quad x <_A y \iff x <_B y$$

Suppose that for some $\alpha \in A$, we have $S_\alpha \subset J$.

- If α has an immediate predecessor u in $(A, <_A)$, then $S_\alpha = S_u \cup \{u\}$. Suppose that $u >_B \alpha$. Since $u \in J$, we also have $u >_A \alpha$, which is a contradiction. Therefore $u <_B \alpha$. For all $x \in (S_u, <_A)$, we have $x <_A u <_A \alpha$, and, since $u \in J$, $x <_B u <_B \alpha$. From this we deduce that $\alpha \in J$.

- If α does not have an immediate predecessor in $(A, <_A)$, then $S_\alpha = \cup_{\beta <_A \alpha} S_\beta$. Suppose that there exist $x \in (S_\alpha, <_A)$ such that $\alpha <_B x$. There exists $\beta \in (S_\alpha, <_A)$ such that $x \in (S_\beta, <_A)$, and since $(S_\alpha, <_A) \in J$, we deduce that $x <_B \beta <_B \alpha$, a contradiction. Therefore $\alpha \in J$.

From the above, we conclude that the set J is an inductive subset of the well-ordered set $(A, <_A)$, and $J = A$.

□

The relation $\prec$ is a strict partial order on $\mathscr{A}$:

Non-reflexivity

For all nonempty subsets A of X, A has the order type of itself, and therefore, from exercise 4, it does not have the order type of one of its sections. Therefore the relation $(A, <_A) \prec (A, <_A)$ never holds. Moreover, if A is empty, then it does not have any section, and again $(A, <_A) \prec (A, <_A)$ does not hold.

Transitivity

Suppose that for some $(A, <_A), (B, <_B), (C, <_C) \in \mathscr{A}$, we have $(A, <_A) \prec (B, <_B) \prec (C, <_C)$. Then there exist $\beta \in B$ and $\gamma \in C$ such that $(A, <_A) = (S_\beta, <_B)$ and $(B, <_B) = (S_\gamma, <_C)$. From lemma 1, we deduce that for all $x, y \in A$,

$$x <_A y \iff x <_B y \iff x <_C y \tag{1}$$

so that $(A, <_A) = (S_\beta, <_B) = (S_\beta, <_C)$ is a section of $(C, <_C)$ and $(A, <_A) \prec (C, <_C)$.

(b) If B' has at most one element, then it is well-ordered. Otherwise, let us show that $<'$ is a linear order on B'.

Comparability

Let x, y be distinct elements of B'; there exist $B_1, B_2 \in \mathscr{B}$ such that $x \in B_1$ and $y \in B_2$. If $B_1 = B_2$, then x and y are comparable by $<_{B_1}$, and thus by $<'$. Otherwise, since $\mathscr{B}$ is linearly ordered, we have either $(B_1, <_{B_1}) \prec (B_2, <_{B_2})$ or $(B_2, <_{B_2}) \prec (B_1, <_{B_1})$.

In the first case, $(B_1, <_{B_1})$ is a section of $(B_2, <_{B_2})$, and therefore $x \in B_2$. B_2 being well-ordered, x and y are comparable by $<_{B_2}$. If $x <_{B_2} y$, then $(x, y) \in\ <_{B_2}\ \subset\ <'$, so that $x <' y$. Otherwise, $y <_{B_2} x$, so that $y <' x$.

By switching the roles of B_1 and B_2, we deduce also, in the second case, that $x <' y$ or $y <' x$.

Conversely, suppose that we have both $x <' y$ and $y <' x$. Then there exist $<_{B_1}$ and $<_{B_2}$ such that $x <_{B_1} y$ and $y <_{B_2} x$. If $B_1 = B_2$, then we have both $x <_{B_1} y$ and $y <_{B_1} x$, which is absurd. Otherwise, since $\mathscr{B}$ is linearly ordered, then $(B_1, <_{B_1}) \prec (B_2, <_{B_2})$ (resp. $(B_2, <_{B_2}) \prec (B_1, <_{B_1})$), so that $x <_{B_2} y$ (resp. $y <_{B_1} x$), a contradiction.

Therefore either $x <' y$ or $y <' x$.

Non-reflexivity

Suppose that $x <' x$ for some $x \in B'$, there exists $<_C$ such that $x <_C x$, which is impossible since C is well-ordered.

Transitivity

For all $x, y, z \in B'$ such that $x <' y <' z$, there exist $B_1, B_2 \in \mathscr{B}$ such that $x <_{B_1} y$ and $y <_{B_2} z$. If $B_1 = B_2$, then $x <_{B_1} <_{B_1} z$. Otherwise, B_1 and B_2 are comparable for $\prec$, so that either $x <_{B_1}$

$y <_{B_1} z$ or $x <_{B_2} y <_{B_2} z$. The transitivity of $<_{B_1}$ and $<_{B_2}$ then leads to either $x <_{B_1} z$ or $x <_{B_2} z$, which in turn implies that $x <' z$.

In particular, we have the following result: for all $C \in \mathscr{B}$, and for all $x, y \in C$,

$$x <_C y \iff x <' y$$

Let us now prove that B' is well-ordered by $<'$. Let C be a nonempty subset of B', and let $x \in C$. There exists $D_x \in \mathscr{B}$ such that $x \in D_x$. The set $C \cap D_x$ is a nonempty subset of the well-ordered set $(D_x, <')$, and therefore has a smallest element x_0.

For all $y \in C$, there exists $D_y \in \mathscr{B}$ such that $y \in D_y$. If $(D_y, <') \preceq (D_x, <')$, then $D_y \cap C \subset D_x \cap C$, so that $x_0 \leq' y$. Otherwise, $(D_x, <') \preceq (D_y, <')$. If $y \in D_x$, then $y \in D_x \cap C$ and thus $x_0 \leq' y$. Otherwise, for all $z \in D_x$, $z <' y$ since D_x is a section of D_y, and thus $x_0 <' y$.

From the above, we conclude that x_0 is the smallest element of C, and therefore C is well-ordered.

$\square$