Exercise 8. Show that the relation defined in Example 7 is an order relation. For reference, Example 7 is reproduced below.

Define xCy if $x^2 < y^2$, or if $x^2 = y^2$ and $x < y$, for $x, y \in \mathbb{R}$.

Proof.

comparability Let $x, y \in \mathbb{R}$ such that $x \neq y$. Suppose that $x^2 = y^2$. Then either $x < y$ or $y < x$, since they are different, so that x and y are comparable by C. Otherwise, $x^2 \neq y^2$; the comparability of the real numbers x^2 and y^2 for the usual order relation $<$ on $\mathbb{R}$ gives us either $x^2 < y^2$ or $y^2 < x^2$. From this we deduce that either xCy or yCx, so that x and y are comparable by C.

non-reflexivity Let $x \in \mathbb{R}$. The non-reflexivity of $<$ implies that we have neither $x^2 < x^2$ nor $x < x$. From this we deduce that we do not have xCx either.

transitivity Let $x, y, z \in \mathbb{R}$ such that xCy and yCz. Suppose $x^2 < y^2$. Then if we also have $y^2 < z^2$, we deduce that xCz by transitivity of $<$ on $\mathbb{R}$. Otherwise, $y^2 = z^2$ and $y < z$. Then we have $x^2 < z^2$ so that xCz.

Suppose that $x^2 = y^2$ and $x < y$. If $y^2 < z^2$, then $x^2 < z^2$ and thus xCz. Otherwise $x^2 = y^2 = z^2$ and $x < y < z$, from which we deduce again xCz

From the above we conclude that C is an order relation on $\mathbb{R}$.

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