Exercise 3. Let $A = A_1 \times A_2 \times \cdots$ and $B = B_1 \times B_2 \times \cdots$.

(a) Show that if $B_i \subset A_i$ for all i, then $B \subset A$. (Strictly speaking, if we are given a function mapping the index set $\mathbb{Z}_+$ into the union of the sets B_i, we must change its range before it can be considered as a function mapping $\mathbb{Z}_+$ into the union of the sets A_i. We shall ignore this technicality when dealing with cartesian products).

(b) Show the converse of (a) holds if B is nonempty.

(c) Show that if A is nonempty, each A_i is nonempty. Does the converse hold?

(d) What is the relation between the set $A \cup B$ and the cartesian product of the sets $A_i \cup B_i$? What is the relation between the set $A \cap B$ and the cartesian product of the sets $A_i \cap B_i$?

Proof.

(a) For all $x \in B$, $x = (x_1, x_2, \dots)$ with $x_i \in B_i$ for all i. Since $B_i \subset A_i$, we have $x_i \in A_i$, so that $(x_1, x_2, \dots) \in A$. Therefore $B \in A$. Note that the above does not suppose the existence of *some* x in B and remains thus true if $B = \varnothing$.

(b) Suppose $B \subset A$ and $B \neq \varnothing$. Let $x = (x_1, x_2, \dots) \in B$. Since $B \subset A$, we have $x \in A$, which by definition of the cartesian product, implies that $x_i \in A_i$ for all i, so that $\forall i \in \mathbb{Z}_+,\ A_i \subset B_i$.

(c) Suppose that $A \neq \varnothing$ and let $x \in A$. By definition of the cartesian product, x is a function with domain $\mathbb{Z}_+$ and image $\cup_{i \in \mathbb{Z}_+} A_i$ such that $x(i) = x_i \in A_i$ for all i. Suppose that for some $j \in \mathbb{Z}_+$ we have $A_j = \varnothing$. Then we must have $x_j \in A_j$, which is impossible. So the function x cannot be defined at j, which is a contradiction with x having the whole of $\mathbb{Z}_+$ as its domain. Therefore $\forall i \in \mathbb{Z}_+,\ A_i \neq \varnothing$.

Conversely, suppose that $\forall i \in \mathbb{Z}_+, A_i \neq \varnothing$. To build an ω-tuple $(x_1, x_2, \dots)$ as an element of A, we need to exhibit, for each $i \in \mathbb{Z}_+$, some $x_i \in A_i$. If we admit the axiom of choice, then we can build the required ω-tuple, and conclude that $A \neq \varnothing$. Otherwise, we cannot justify the existence of the ω-tuple, and thus cannot conclude about A being nonempty.

(d) For all $x \in A \cup B$, then:

- either $x \in A$, in which case $x = (x_1, x_2, \dots)$ with $x_i \in A_i \subset A_i \cup B_i$ for all i
- or $x \in B$, in which case $x = (x_1, x_2, \dots)$ with $x_i \in B_i \subset A_i \cup B_i$ for all i

Since

$$\prod_{i\in\mathbb{Z}_+} A_i \cup B_i = \big\{(x_1, x_2, \dots) \mid x_i \in A_i \cup B_i \text{ for all } i\big\}$$

we conclude that $A \cup B \subset \prod_{i\in\mathbb{Z}_+} A_i \cup B_i$. Let $A_i = \{2i-1\}$ and $B_i = \{2i\}$ for all $i \in \mathbb{Z}_+$. Then $x = (1, 2, 3, \dots) \in \prod_{i\in\mathbb{Z}_+} A_i \cup B_i$, but $x \notin A \cup B$, since to be part of $A \cup B$, the coordinates of x would need to be either all even, or all odd. So the reverse inclusion is false.

For all $x \in \prod_{i\in\mathbb{Z}_+} A_i \cap B_i$, $x = (x_1, x_2, \dots)$ with $x_i \in A_i \cap B_i$ for all i, so that $x_i \in A_i$ and $x_i \in B_i$ and thus $x \in A$ and $x \in B$. From this we conclude that $\prod_{i\in\mathbb{Z}_+} A_i \cap B_i \subset A \cap B$. Conversely, let $x \in A \cap B$; we have

- $x \in A$, so that $x = (x_1, x_2, \dots)$ with $x_i \in A_i$ for all i
- $x \in B$, so that $x = (y_1, y_2, \dots)$ with $y_i \in B_i$ for all i

and both expressions are equal, so that $x_i = y_i$ for all i. From this we deduce that $x_i \in A_i \cap B_i$ for all i, so that $A \cap B \subset \prod_{i\in\mathbb{Z}_+} A_i \cap B_i$.

$\square$