Exercise 3.

(a) Show that if $\mathscr{A}$ is a collection of inductive sets, then the intersection of the elements of $\mathscr{A}$ is an inductive set.

(b) Prove the basic properties of $\mathbb{Z}_+$:

- $\mathbb{Z}_+$ is inductive
- (Principle of induction) If A is an inductive set of positive integers, then $A = \mathbb{Z}_+$

Proof.

(a) Let $\mathscr{A}$ be a collection of inductive sets, let $I = \cap_{A\in\mathscr{A}} A$, and let $A_0 \in \mathscr{A}$. Then A_0 is inductive, from which we deduce that $1 \in A_0$. The previous proposition does not depend on any specific choice of A_0, so $\forall A_0 \in \mathscr{A}, 1 \in A_0$, which is the definition of $1 \in I$. In particular, $I \neq \varnothing$. Let then $x \in I$; we have $\forall A \in \mathscr{A}, x \in A$, and since every such A is inductive, $x + 1 \in A$. We have shown $\forall A \in \mathscr{A}, x + 1 \in A$, which is by definition of the intersection $x + 1 \in I$. Thus $\forall x \in I, x + 1 \in I$, and I is inductive.

(a) Let $\mathscr{A}$ denote the family of inductive sets of $\mathbb{R}$. By definition, $\mathbb{Z}_+ = \cap_{A\in\mathscr{A}} A$, so that $\mathbb{Z}_+$ is the intersection of inductive sets, and thus inductive. Let $B \in \mathscr{A}$. We have $\mathbb{Z}_+ \subset B \subset \mathbb{R}$, and since $\mathbb{Z}_+$ is inductive, $\mathbb{Z}_+ \in \mathscr{A}$. Any inductive set of positive integers must therefore be a subset of $\mathbb{Z}_+$. Conversely, if $A_0 = \cap_{A\in\mathscr{A}} A$, then A_0 is an inductive subset of $\mathbb{R}$, so that $A_0 \in \mathscr{A}$ and $\mathbb{Z}_+ \subset A$. $\square$