Exercise 4. Use exercises 1-3 to prove the following:

(a) If A and B are well-ordered sets, then exactly one of the following three conditions hold: A and B have the same order type, or A has the order type of a section of B, or B has the order type of a section of A. [*Hint:* Form a well-ordered set containing both A and B, as in exercise 8 of §10; then apply the preceding exercise.]

(b) Suppose that A and B are well-ordered sets that are uncountable, such that every section of A and of B is countable. Show that A and B have the same order type.

Proof.

(a) If A is empty, then either B is also empty and $A = B$, or B is nonempty and A is equal to a section of B. If B is empty, then we switch the roles of A and B in the previous sentence and find that either $B = A$ or B is equal to a section of A. Therefore we suppose in the rest of the proof that A and B are both nonempty.

The sets $A' = A \times \{1\}$ and $B' = B \times \{2\}$ are disjoint and well-ordered for the relations:

$$\begin{aligned} &\forall (x,1),(y,1) \in A', \quad (x,1) \prec_A (y,1) \iff x <_A y \\ &\forall (x,2),(y,2) \in B', \quad (x,2) \prec_B (y,2) \iff x <_B y \end{aligned}$$

where $<_A$ (resp. $<_B$) is the order relation on A (resp. B).

Let $E = A' \cup B'$, and define an order relation $\prec$ on E by: $x \prec y$ if and only if one of the following conditions holds:

- $x, y \in A'$ and $x \prec_A y$
- $x, y \in B'$ and $x \prec_B y$
- $x \in A'$ and $y \in B'$

for all $x, y \in E$. From exercise 8 of section §10, the set E is well-ordered by $\prec$.

Let $g : B' \to E$, $\alpha \mapsto \alpha$. The function g preserves the order: for all $\alpha, \beta \in B'$, we have

$$\alpha \prec_B \beta \iff g(\alpha) \prec_B g(\beta) \iff g(\alpha) \prec g(\beta)$$

Then from exercise 3, we deduce that B' has the order type of E or a section of E.

Similarly, let $f : A' \to E$, $\alpha \to \alpha$. The function f is also order-preserving, and therefore A' has the order type of E or a section of E.

To find the relation between the order types of A and B, we need to examine the following cases:

- if A' and B' have the same order type as E, then there exist increasing bijections $\varphi : A' \to E$ and $\psi : B' \to E$. The bijection $\psi^{-1} \circ \varphi : A' \to B'$ is also increasing, so that A' and B' have the same order type. The function f below

$$\begin{tikzcd} A' \arrow[r, "\psi^{-1}\circ\varphi"] & B' \arrow[d, "(y,2)\mapsto y"] \\ A \arrow[u, "x\mapsto(x,1)"] \arrow[r, "f"] & B \end{tikzcd}$$

  is the composition of 3 order-preserving bijections, and therefore an order-preserving bijection itself. Therefore f is an order-preserving bijection, and A and B have the same order type.

- if A' has the order type of a section of E, and B' that of E, then there exist increasing bijections $\varphi : A' \to S_\alpha$ for some α in E, and $\psi : B' \to E$. The subset $\psi^{-1}(S_\alpha)$ of B' satisfies

$$\forall \beta \in S_\alpha, \quad \psi^{-1}(\beta) \prec_B \psi^{-1}(\alpha)$$

  since ψ^{-1} is order-preserving, and is therefore a subset of $S_{\psi^{-1}(\alpha)}$. Conversely,

$$\forall \beta \in S_{\psi^{-1}(\alpha)}, \quad \psi(\beta) \prec \psi \circ \psi^{-1}(\alpha) = \alpha$$

  since ψ is order-preserving. From this we deduce that

$$\psi^{-1}(S_\alpha) = S_{\psi^{-1}(\alpha)}$$

  so that A' has the order type of a section of B'. Let $\theta : A' \to S_\beta$ for $\beta \in B'$ be an order-preserving bijection. The function f below

$$\begin{tikzcd} A' \arrow[r, "\theta"] & S_\beta \arrow[d, "\pi:(y,2)\mapsto y"] \\ A \arrow[u, "x\mapsto(x,1)"] \arrow[r, "f"] & \pi(S_\beta) \subset B \end{tikzcd}$$

  is the composition of order-preserving bijections, and is therefore an order-preserving bijection. From this we deduce that

$$\pi(S_\beta) = S_{\pi(\beta)}$$

  so that A has the order type of a section of B. By switching the roles of A and B in the reasoning above, we deduce that if A' has the order type of E and B' that of a section of E, then B has the order type of a section of A.

- otherwise, both A' and B' have the order type of a section of E. Let $\alpha, \beta \in E$ be such that there exist order-preserving bijections $\varphi : A' \to S_\alpha$ and $\psi : B' \to S_\beta$, and suppose that $\alpha \prec \beta$. Then S_α is a section of S_β, and we have the following diagram:

$$\begin{tikzcd}
A' \arrow[r, "\varphi"] & S_\alpha \arrow[r, "i\colon x \mapsto x"] & S_\beta \arrow[r, "\psi^{-1}"] & B' \arrow[d, "(y,2)\mapsto y"] \\
A \arrow[u, "x\mapsto(x,1)"] \arrow[rrr, "f"] & & & B
\end{tikzcd} \tag{1}$$

  The function f, as the composition of order-preserving functions, is order-preserving. We deduce from exercice 3 that A has the order type of B or a section of B. Since $\alpha \prec \beta$, the function i in (1) is not bijective. Besides i, the functions on the top and sides of the diagram are bijective; from this we deduce that f is not bijective, and its image is not B. Therefore A has the order type of a section of B.

  If $\beta \prec \alpha$, then by switching the roles of A and B above, we deduce that B has the order type of a section of A.

  If $\alpha = \beta$, then i is a bijection, and so is f as the composition of bijections. In that case, we deduce that A has the order type of B.

- Since A and B are well-ordered, we know from point (a) that one of the following cases is true:

  - A and B have the same order type: the result is then trivially true.

  - A has the order type of a section of B: then there exists an order-preserving bijection $\varphi\colon A \to S_\alpha$ for some $\alpha \in B$. Since S_α is countable, we deduce that A is also countable, contrary to the hypothesis. Therefore A does not have the order type of a section of B.

  - B has the order type of a section of A: by switching the roles of A and B in the previous point, we deduce that this case si also impossible.

□