Exercise 4. Show that the collection $\mathscr{T}_c$ given in example 4 of §12 is a topology on the set X. Is the collection

$$\mathscr{T}_\infty = \{U \mid X - U \text{ is infinite or empty or all of } X\}$$

a topology on X?

Example 4. *Let X be a set; let $\mathscr{T}_c$ be the collection of all subsets U of X such that $X - U$ is either countable or is all of X. Then $\mathscr{T}_c$ is a topology on X.*

Proof. Let us show that $\mathscr{T}_c$ satisfies the definition of a topology.

(a) $X - X = \varnothing$, which is countable, so $X \in \mathscr{T}_c$.

(b) $X - \varnothing = X$, which is all of X, so $\varnothing \in \mathscr{T}_c$.

(c) For all nonempty family $\{U_\alpha\}_{\alpha\in J}$ of elements of X, we have

$$X - \bigcup_{\alpha\in J} U_\alpha = \bigcap_{\alpha\in J} (X - U_\alpha)$$

and the latter is countable as a subset of $X - U_\alpha$ for all $\alpha \in J$. Thus the union $\cup_{\alpha\in J} U_\alpha$ is also an element of X.

(d) For all finite set $\{U_1, \ldots, U_n\}$ of elements of X, we have

$$\bigcap_{k=1}^{n} (X - U_k) = X - \bigcup_{k=1}^{n} U_k$$

From the previous point, the union $\cup_{k=1}^n U_k$ of elements of X is an element U of X, and so $X - U$ is either countable or equal to X. Thus the finite intersection $\cap_{k=1}^n U_k$ is an element of X.

The family $\mathscr{T}_\infty = \{U \in X \mid X - U \text{ is infinite or empty or all of } X\}$ is not in general a topology on X: let $X = \mathbb{R}$, the family

$$F = \{(-\infty, -1/n) \cup (1/n, +\infty), n \in \mathbb{Z}_+\}$$

satisfies $X - U$ is infinite for all $U \in F$ and is therefore a subset of $\mathscr{T}_\infty$. However,

$$\bigcap_{U\in F} (X - U) = \bigcap_{k=1}^{\infty} [-\frac{1}{k}, \frac{1}{k}] = \{0\}$$

which is not infinite, not empty, and not equal to $\mathbb{R}$, so $\cup_{U\in F} U$ is not an element of $\mathscr{T}_\infty$.

□