Exercise 2.

(a) Let J and E be well-ordered sets; let $h : J \to E$. Show the following two statements are equivalent:

(i) h is order-preserving and its image is E or a section of E.

(ii) $h(\alpha) = \text{smallest of } [E - h(S_\alpha)]$ for all α.

[*Hint:* Show that each of these conditions implies that $h(S_\alpha)$ is a section of E; conclude that it must be the section by $h(\alpha)$.]

(b) If E is a well-ordered set, show that no section of E has the order type of E, nor do two different sections of E have the same order type. [*Hint:* Given J, there is at most one order-preserving map of J into E whose image is E or a section of E.]

Lemma 1. Let W be a nonempty well-ordered set, and let $\alpha \in W$. If α has an immediate successor β, then $S_\alpha \cup \{\alpha\} = S_\beta$. Otherwise, $\alpha = \max W$ and $S_\alpha \cup \{\alpha\} = W$.

Proof. If $\alpha = \max W$, then for all $x \in W$, either $x = \alpha$ or $x < \alpha$, and thus $W = S_\alpha \cup \{\alpha\}$. Otherwise, since α is not the largest element of W, it has an immediate successor β, and $S_\alpha \cup \{\alpha\} \subset S_\beta$. Conversely, for all $x \in S_\beta$, if $x > \alpha$, then $\alpha < x < \beta$, so that β is not the immediate successor of α, a contradiction. Therefore either $x < \alpha$ or $x = \alpha$, and $S_\beta \subset S_\alpha \cup \{\alpha\}$. □

Lemma 2. Let W be a well-ordered set, and let I be a subset of W. Then $\cup_{\alpha \in I} S_\alpha$ is either W or a section of W.

Proof. Let $U = \cup_{\alpha \in I} S_\alpha$.

Suppose that U has un upper bound $\beta \in W$, and let m be the smallest such upper bound. We have $U \subset S_m$. Suppose there exists $x \in S_m - U$. If there exists $\alpha \in I$ such that $x < \alpha$, then $x \in S_\alpha \subset U$, a contradiction; therefore x is an upper bound for U in W, since m is the smallest such upper bound, we have $x > m$ or $x = m$, which contradicts $x \in S_m$. Therefore $U = S_m$.

Otherwise, U does not have an upper bound in W. Let $x \in W$, there exists $\alpha \in I$ such that $x \in S_\alpha$, so that $W \subset U$, and $U = W$. □

Proof.

(a) If J is empty, then there is only one function from J to E and it vacuously satisfies both conditions. In the rest of the proof, we suppose then that J is nonempty. We will show that both propositions are equivalent to

$$\forall \alpha \in J, \quad h(S_\alpha) = S_{h(\alpha)} \tag{1}$$

(i) $\implies$ (1)

Suppose that (i) holds. Let $\alpha \in J$; for all $\beta \in J$ such that $\beta < \alpha$, we have $h(\beta) < h(\alpha)$, so $h(\beta) \in S_{h(\alpha)}$. Conversely, for all $\beta > \alpha$, $h(\beta) > h(\alpha)$, so $h(\beta) \notin S_{h(\alpha)}$. Therefore

$$\forall \beta \in J, \quad \beta < \alpha \iff h(\beta) \in S_{h(\alpha)} \tag{2}$$

Suppose that $h(S_\alpha) \neq S_{h(\alpha)}$, and let $z \in S_{h(\alpha)} - h(S_\alpha)$. If there exists $x \in J$ such that $h(x) = z$, then from (2) we deduce that $x < \alpha$, so that $h(x) = z \in h(S_\alpha)$, which is absurd. Therefore $z \notin h(J)$.

- if $h(J) = E$, then from the above $z \notin E$, which is absurd.
- otherwise, $h(J) = S_m$ for some $m \in E$, and since $h(\alpha) \in S_m$, we have $h(\alpha) < m < z$. This contradicts the fact that $z \in S_{h(\alpha)}$.

Therefore z does not exist, and we conclude that (1) is true.

(1) $\implies$ (i)

Suppose that (1) holds. For all $x, y \in J$ such that $x < y$, we have $x \in S_y$ and therefore $h(x) \in h(S_y) = S_{h(y)}$. From this we deduce that $h(x) < h(y)$, so h is increasing.

Suppose that $h(J) \neq E$, and let m be the smallest element of $E - h(J)$. If there exists $z \in J$ such that $h(z) > m$, then $m \in S_{h(z)} = h(S_z)$, contradicting the existence of m. Therefore for all $z \in J$, $h(z) \in S_m$, and for all $y \in S_m$, there exists $x \in J$ such that $h(x) = y$ by definition of m. From this we conclude that $h(J) = S_m$, so that (1) $\implies$ (i).

(1) $\implies$ (ii)

Let $\alpha \in J$, from (1) we have $h(\alpha) \notin h(S_\alpha)$. For all $y \in E$, $y < h(\alpha)$ implies $y \in h(S_\alpha)$, so $h(\alpha)$ is the smallest element of $E - h(S_\alpha)$, and (1) $\implies$ (ii).

(ii) $\implies$ (1)

Suppose that condition (ii) holds, and let J_0 be the subset of J such that for all $\alpha \in J_0$, $h(S_\alpha) = S_{h(\alpha)}$. For all $\beta \in J$ such that $S_\beta \subset J_0$:

- If β has an immediate predecessor u, then $S_\beta = S_u \cup \{u\} \subset J_0$, and $u \in J_0$. Furthermore,

  $$h(S_\beta) = h(S_u \cup \{u\}) = h(S_u) \cup \{h(u)\} = S_{h(u)} \cup \{h(u)\}$$

  Since $h(\beta) \in E - h(S_\beta)$, we do not have $h(S_\beta) = E$, and from lemma 1 we deduce that $h(u)$ has an immediate successor s in

E. Since $h(\beta) = \min\big(E - h(S_\beta)\big) = \min\{x \in E \mid x > h(u)\}$, we deduce that $s = h(\beta)$, and $h(S_\beta) = S_{h(\beta)}$, so that $\beta \in J_0$.

- If β does not have an immediate predecessor, then $S_\beta = \cup_{\alpha<\beta} S_\alpha$. Since $S_\beta \in J_0$, we have $\alpha \in J_0$ for all $\alpha < \beta$, and

$$h(S_\beta) = h\left(\bigcup_{\alpha<\beta} S_\alpha\right) = \bigcup_{\alpha<\beta} h(S_\alpha) = \mathscr{S}$$

From lemma 2, we deduce that $\mathscr{S}$ is either E or a section of E. And since $h(\beta) \in E - h(S_\beta)$ exists, $\mathscr{S} = S_m$ for some $m \in E$. The element m is the smallest upper bound for $h(S_\beta)$ in E, and therefore $m = \beta$, and $\beta \in J_0$.

From the above, we deduce that J_0 is an inductive subset of J, and therefore $J_0 = J$.

(b) If $E = \varnothing$, then for all J section of E there is no function from J to E, so J cannot have the same order type as E.

Suppose therefore that E is nonempty and let J be a section of E. From item (a), we deduce that the recursive definition in (ii) gives a unique order-preserving function $h : J \to E$ whose image is either E or a section of E.

Suppose that $h(J) = S_m$ for some $m \in E$, then $m \notin h(J)$ and therefore h is not bijective. Otherwise, $h(J) = E$, so that h is bijective. As a section of E, $J = S_\alpha$ for some $\alpha \in E$. Then there exists $\beta < \alpha$ such that $h(\beta) = \alpha$. And since $\beta \in S_\alpha$, from (1) we deduce that $h(\beta) \in h(S_\alpha) = S_{h(\alpha)}$, which implies $h(\beta) < h(\alpha)$, a contradiction. Therefore h is not bijective.

From the above, we conclude that there is no order-preserving bijection from a section of E to E, so that every section of E has an order type different from that of E.

Let S_α and S_β for $\alpha, \beta \in E$ be two distinct sections of E. Since E is well-ordered, the elements α and β are comparable. Swapping the roles of α and β, we can suppose that $\alpha < \beta$. Then $S_\alpha \subset S_\beta$ and $S_\alpha = \{x \in E \mid x < \alpha\} = \{x \in S_\beta \mid x < \alpha\}$, and therefore S_α is a section of S_β.

From item (a) we deduce that there is a unique order-preserving function $h : S_\alpha \to S_\beta$, defined as in (ii), and that for this function $h(S_\alpha)$ is either S_β or a section of S_β. As a subset of the well-ordered set E, S_β is well-ordered, and we can apply the previous reasoning on J and E to deduce that S_α and S_β have different order types. $\square$