Exercise 6. Use exercises 1 and 5 to prove the following:

Theorem. *The maximum principle is equivalent to the well-ordering theorem.*

Proof. Suppose that the maximum principle holds. If X has at most one element, then it is vacuously well-ordered. Otherwise, X has at least 2 elements x and y. The subset $\{x\}$ (resp. $\{x, y\}$) of X is well-ordered by the relation $\varnothing$ (resp. $\{(x, y)\}$), so the collection $\mathscr{A}$ of exercise 5 and the relation $\prec$ are well-defined and $\mathscr{A}$ is nonempty. There exists a maximal chain C in $\mathscr{A}$ that is linearly ordered by $\prec$. Suppose that there exists $x_0 \in X$ such that for all $A \in C$, $x_0 \notin A$. For all $A \in C$, define a relation $<'_A$ by

$$\begin{gathered} \forall y \in A, \quad x_0 <'_A y \\ \forall x, y \in A, x <'_A y \iff x <_A y \end{gathered}$$

where $<_A$ is the order relation on A, as in the hypotheses from exercise 5. The set $A' = \{x_0\} \cup A$ is well-ordered by $<'_A$, and if $A, B \in \mathscr{A}$ satisfy $(A, <_A) \prec (B, <_B)$, then $(A', <'_A) \prec (B', <'_B)$ (since x_0 is the smallest element of $(A', <'_A)$ and of $(B', <'_B)$). Thus by adding the set $\{x_0\}$ to the family C, we create a new family C' that is linearly ordered by $\prec$, and strictly contains C. This contradicts the hypothesis that C is maximal; therefore $x_0 \in B'$ (as defined in exercise 5), from which we conclude that $B' = X$.

Therefore the order relation $<'$ from exercise 5 makes X well-ordered, so that the maximum principle implies the well-order theorem.

Suppose now that the well-ordering theorem holds, and let X be a set. If X has at most one element, then the only strict partial order on X is $\varnothing$, and the maximum principle holds vacuously.

Otherwise, let $\prec$ be a strict partial order on X, and let $<$ be a well-order on X. For all $\alpha \in X$, let $S_\alpha = \{\beta \in X \mid \beta < \alpha\}$.

Define a relation $\mathscr{R}$ on X by

$$x \mathscr{R} y \iff x \prec y \text{ or } y \prec x$$

and, for all $A \subset X$,

$$\mathscr{R}(A) = \{y \in X \mid x \mathscr{R} y \text{ for all } x \in A\}$$

Let x_0 be an element of X, and let $C = \{0, 1\}$. With the notations from exercise 1, define

$$\begin{aligned} \rho\colon \mathscr{F} &\to C \\ (f\colon S_\alpha \to C) &\mapsto \begin{cases} 0 & \text{if } \alpha = x_0 \\ 0 & \text{if } \alpha \in \mathscr{R}\left(\{x_0\} \cup f^{-1}\left(\{0\}\right)\right) \\ 1 & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} h\colon X &\to C \\ \alpha &\mapsto \rho(h|S_\alpha) \end{aligned}$$

From exercise 1 we know that h is well-defined and unique. We have $h(x_0) = 0$, so $H = h^{-1}(\{0\})$ is nonempty. The set H is simply ordered by $\prec$: this is trivial if H has only one element. Otherwise let $\alpha, \beta \in h^{-1}(\{0\})$ be distinct elements of H such that that $\alpha < \beta$. If either of them equals x_0, then they are comparable by definition of h. Otherwise, β is comparable with every element of $(h|S_\beta)^{-1}(\{0\}) = h^{-1}(\{0\}) \cap S_\beta$, which contains α. The same reasoning holds in the case $\beta < \alpha$.

The set H is maximal for $\prec$. Suppose that there exists $B \subset X$ that is simply ordered by $\prec$ and strictly contains H, and let α be the smallest element of $B - H$ for $<$. Then $S_\alpha \subset H$; furthermore, $\alpha \mathscr{R} x_0$ since $x_0 \in B$. We also have $\alpha \in \mathscr{R}\left(\{x_0\} \cup (H \cap S_\alpha)\right)$. From this we deduce that $h(\alpha) = 0$, a contradiction.

From the above we conclude that the well-ordering theorem implies the maximum principle.

$\square$