**Exercise 7.** Use exercises 1-5 to prove the following:

**Theorem.** *The axiom of choice is equivalent to the well-ordering theorem.*

*Proof.* Let$X$ be a set; let $c$ be a fixed choice function for the nonempty subsets of $X$. If $T$ is a subset of $X$ and $<$ is a relation on $T$, we say that $(T, <)$ is a *tower* in $X$ if $<$ is a well-ordering of $T$ and if for each $x \in T$,

$$x = c\left(X - S_x\left(T\right)\right)$$

where $S_x(T)$ is the section of $T$ by $x$.

(a) Left$(T_1, <_1)$ and $(T_2, <_2)$ be two towers on $X$. Show that either these two ordered sets are the same, or one equals a section of the other. (*Hint:* Switching indices if necessary, we can assume that $h\colon T_1 \to T_2$ is order-preserving and and $h(T_1)$ is either $T_2$ or a section of $T_2$. Use Exercise 2 to show that $h(x) = x$ for all $x$.)

(b) If $(T, <)$ is a tower in $X$ and $T \neq X$, show that there is a tower in $X$ of which $(T, <)$ is a section.

(c) Let $\{(T_k, <_k) \mid k \in K\}$ be the collection of all towers in $X$. Let

$$T = \bigcup_{k\in K} T_k \quad \text{and} \quad <= \bigcup_{k\in K} (<_k)$$

Show that $(T, <)$ is a tower in $X$. Conclude that $T = X$.

□

*Proof.*

First, if $X$ is empty, it is vacuously well-ordered; and the choice function $c$ is not defined since there are no nonempty subsets of $X$. Therefore we consider only nonempty sets.

(a) Since $(T_1, <_1)$ and $(T_2, <_2)$ are well-ordered sets, either they have the same order type, or one of them has the order type of a section of the other. Switching the roles of $T_1$ and $T_2$ if necessary, we can suppose that $(T_1, <_1)$ has the order type of $(T_2, <_2)$ or a section of $(T_2, <_2)$, so that there exists an order-preserving map $h\colon T_1 \to T_2$ whose image is $(T_2, <_2)$ or a section of $(T_2, <_2)$.

From exercise 2, we know that there is at most one such map, and that it satisfies the following properties for all $x \in T_1$:

$$h(x) = \min\left\{T_2 - h\big(S_x(T_1)\big)\right\}$$
$$h\big(S_x(T_1)\big) = S_{h(x)}(T_2)$$

Let $J = \{x \in T_1 \mid h(x) = x\}$, and suppose that $S_\alpha(T_1) \subset J$ for some $\alpha \in T_1$. For all $y \in J$, we have, since $h\colon T_1 \to h(T_1)$ is bijective:

$$x <_1 y \iff x = h(x) <_2 h(y) = y \tag{1}$$

for all $x \in J$, so that $S_y(T_1) = S_y(T_2)$. Two cases arise:

- If $\alpha$ has a direct predecessor $u \in T_1$, then $S_\alpha(T_1) = \{u\} \cup S_u(T_1)$, $h(u) = u$ and

$$\begin{aligned} h\big(S_\alpha(T_1)\big) &= h\big(\{u\} \cup S_u(T_1)\big) \\ &= h(\{u\}) \cup h\big(S_u(T_1)\big) \\ &= \{u\} \cup S_u(T_2) \\ &= \{u\} \cup S_u(T_1) \\ &= S_\alpha(T_1) \end{aligned}$$

  from which we deduce that $h(\alpha) = \alpha$, so that $\alpha \in J$.

- If $\alpha$ does not have a direct predecessor, then the section $S_\alpha(T_1)$ equals $\cup_{\beta <_1 \alpha} S_\beta(T_1)$, and for all $\beta <_1 \alpha$ we have $h(\beta) = \beta$. Therefore

$$\begin{aligned} h\left(S_\alpha\left(T_1\right)\right) &= h\left(\bigcup_{\beta <_1 \alpha} S_\beta(T_1)\right) \\ &= \bigcup_{\beta <_1 \alpha} h\left(S_\beta(T_1)\right) \\ &= \bigcup_{\beta <_1 \alpha} S_{h(\beta)}(T_2) \\ &= \bigcup_{\beta <_1} S_\beta(T_1) \\ &= S_\alpha(T_1) \end{aligned}$$

  and $h(\alpha) = \alpha$. Thus $\alpha \in J$.

From the above we conclude that $(J, <_1)$ is an inductive subset of $(T_1, <_1)$, and thus equals $(T_1, <_1)$. The equation (1) holds then for all $x, y \in (T_1, <_1)$. This implies that $(T_1, <_1)$ is equal to $(T_2, <_2)$ or a section of $(T_2, <_2)$.

(b) If $(T, <)$ is a tower in $X$ and $X - T \neq \varnothing$, then $c(X - T) = x_0 \in X - T$. Let $T_0 = T \cup \{x_0\}$ and let $<_0$ be the relation on $T_0$ defined by

$$\begin{aligned} \text{for all } x, y \in T, &\quad x < y \iff x <_0 y \\ \text{for all } x \in T, &\quad x <_0 x_0 \end{aligned}$$

With these definitions, $(T_0, <_0)$ is a tower in $X$, and

$$\begin{aligned} h_0 \colon T &\to S_{x_0}(T_0) \\ x &\mapsto x \end{aligned}$$

is an order-preserving bijection. Therefore $(T_0, <_0)$ is a tower in $X$ of which $(T, <)$ is a section.

(c) For all $x \in T$, there exists $k \in K$ such that $x \in (T_k, <_k)$, and we have $x = c(X - T_k)$. For all $l \in K$, we know from item (a) that one of the three following cases is true:

- $(T_l, <_l) = (T_k, <_k)$, in which case $x = c(X - T_k) = c(X - T_l)$
- $(T_l, <_l)$ is a section of $(T_k, <_k)$, in which case $x = c(X - T_k)$ as an element of $(T_k, <_k)$, and $x = c(X - T_l)$ as an element of $(T_l, <_l)$; so that we again have $c(X - T_k) = c(X - T_l)$.
- $(T_k, <_k)$ is a section of $(T_l, <_l)$, in which case we exchange the roles of $k$ and $l$ in the above and arrive again at $c(X - T_k) = c(X - T_l) = x$.

(d) First, let us show that $<$ is a simple order on $T$. For all $x, y$ different elements of $T$, there exist $j, k \in K$ such that $x \in (T_j, <_j)$ and $y \in (T_k, <_k)$. If $(T_j, <_j) = (T_k, <_k)$, then $x$ and $y$ are comparable by $<_j$, and thus by $<$. Otherwise, assume that $(T_j, <_j)$ is a section of $(T_k, <_k)$; then $x \in (T_k, <_k)$ and $x, y$ are comparable as elements of $(T_k, <_k)$. They are therefore comparable by $<$. Otherwise, $(T_k, <_k)$ is a section of $(T_j, <_j)$, and the same reasoning leads to the comparability of $x$ and $y$ by $<$.

Furthermore, suppose that both $x < y$ and $y < x$. Then there are $j, k \in K$ such that $x <_j y$ and $y <_k y$. The towers $(T_j, <_j)$ and $(T_k, <_k)$ are either equal, or one is a section of the other; this leads to either $x <_j y$ and $y <_j x$, or $x <_k y$ and $y <_k x$, which are both impossible. So exactly one of $x < y$ or $y < x$ is true.

Suppose that there exists $x \in (T, <)$ such that $x < x$. Then there exists $k \in K$ such that $x <_k x$. Since $<_k$ is defined on $T_k$, we have $x \in T_k$. And since $(T_k, <_k)$ is well-ordered, we cannot have $x <_k x$. From this we deduce that $<$ is nonreflexive.

Let $x, y, z \in T$ such that $x < y$ and $y < z$; there exist $j, k \in K$ such that $x <_j y$ and $y <_k z$. Thus $x, y \in T_j$ and $y, z \in T_k$. The towers $(T_j, <_j)$ and $(T_k, <_k)$ are either equal, or one is a section of the other; this leads to either $x <_j y <_j z$ or $x <_k y <_k z$, and, from the transitivity of $<_j$ and $<_k$, to $x <_j z$ or $x <_k z$, both implying $x < z$.

The relation $<$ is a well-order on $T$. Let $A$ be a nonempty subset of $T$, and let $x \in A$; there exists $k \in K$ such that $x \in T_k$. The set $T_k \cap A$ is nonempty and is well-ordered by $<_k$, so has a smallest element $m$. Then $m$ is also the smallest element of $A$ for $<$. Suppose instead that there exists $y \in A$ satisfying $y < m$; there exists $j \in K$ such that $y \in T_j$. If $(T_j, <_j) = (T_k, <_k)$, then $y$ and $m$ are comparable by $<_k$, so that $m <_k y$, which implies that we also have $m < y$. But this is impossible since $<$ is a simple order on $T$.

Suppose then that $(T_j, <_j)$ is a section of $(T_k, <_k)$; then we have $y <_j m$, which implies $y <_k m$, and the latter is impossible. Otherwise $(T_k, <_k)$ is a section of $(T_j, <_j)$, so that $y <_j m$, and since $<_k$ and $<_j$ are equal on $(T_k, <_k)$, we find again that $y <_k m$, which is absurd.

Let us now show that for all $x \in T$ we have $x = c\big(X - S_x(T)\big)$. Let $x \in T$, there exists $k \in K$ such that $x \in T_k$. For all $y \in S_x(T)$, there exists $j \in K$ such that $y \in T_j$; we have then $y <_j x$. If $T_j = T_k$, then $y <_k x$. Otherwise, if $(T_j, <_j)$ is a section of $(T_k, <_k)$, then $y <_j x$ implies $y <_k x$; and if $(T_k, <_k)$ is a section of $(T_j, <_j)$, then $<_j$ and $<_k$ agree on $T_k$, and $y <_k x$.

From this we deduce that $S_x(T) = S_x(T_k)$ (the inclusion from right to left comes from $T_k \subset T$). Since $x = c\big(X - S_x(T_k)\big)$, we find that $x = c\,(X - S_x(T))$.

All the above conditions make $T$ a tower in $X$. If $T \neq X$, then there exists a tower $(T', <')$ in $X$ of which $(T, <)$ is a section. Since $T$ is the union of all towers in $X$, there exists some $n \in K$ such that $(T', <') = (T_n, <_n)$, and then we have $(T_n, <_n) \subset (T, <)$. $(T, <)$ is thus a section of one of its subsets, which is impossible since it is nonempty (for example, $(\{c(X)\}, <)$ is a tower in $X$). Therefore $T = X$; $X$ is then well-ordered by $<$, and thus the axiom of choice implies the well-ordering theorem.

Conversely, suppose that the well-ordering theorem holds, and let $\mathscr{A}$ be a collection of disjoint nonempty sets. Let $<$ be a well-ordering of the set $X = \cup_{A \in \mathscr{A}} A$, and let

$$\begin{aligned} c\colon \mathscr{A} &\to X \\ A &\mapsto \text{smallest element of } A \text{ for } < \end{aligned}$$

Since for all $A \in \mathscr{A}$, $A$ is nonempty and well-ordered by $<$ (as a subset of the well-ordered set $(X, <)$), $c(A)$ exists. If $A, B$ are distinct elements of $\mathscr{A}$, then $A \cap B = \varnothing$ and therefore $c(A) \neq c(B)$. $c$ is then a choice function on $\mathscr{A}$, and the well-ordering theorem implies the axiom of choice. $\square$