Exercise 5. Show that Zorn's lemma implies the following:

Lemma (Kuratowski)**.** *Let $\mathscr{A}$ be a collection of sets. Suppose that for every subcollection $\mathscr{B}$ of $\mathscr{A}$ that is simply ordered by proper inclusion, the union of the elements of $\mathscr{B}$ belongs to $\mathscr{A}$. Then $\mathscr{A}$ has an element that is properly contained in no other element of $\mathscr{A}$.*

Proof. For all $\mathscr{B}$ subcollection of $\mathscr{A}$ that is simply ordered by proper inclusion, the set $B = \cup_{b \in \mathscr{B}} b$ is an upper bound of $\mathscr{B}$ in $\mathscr{A}$. Zorn's lemma gives us the existence of a maximal element $M \in \mathscr{A}$. Since M is maximal, for all $A \in \mathscr{A}$ such that $A \neq M$, we have $A \subset M$. Therefore M is not a proper subset of any other element of $\mathscr{A}$.

□