Exercise 7. Let A and B be two nonempty sets. If there is an injection of B into A, but no injection of A into B, we say that A has *greater cardinality* than B.

(a) Conclude from Theorem 9.1 that every uncountable set has greater cardinality than $\mathbb{Z}_+$.

(b) Show that if A has greater cardinality than B, and B has greater cardinality than C, the A has greater cardinality than C.

(c) Find a sequence $A_1, A_2, \dots$ of infinite sets such that for each $n \in \mathbb{Z}_+$, the set A_{n+1} has greater cardinality than A_n.

(d) Find a set that for every n has a greater cardinality than A_n.

Proof.

(a) Let A be an uncountable set. Since every finite set is countable, A must be infinite. From theorem 9.1, we deduce that there exists an injection $f : \mathbb{Z}_+ \to A$. Since A is uncountable, there is no injection from A into $\mathbb{Z}_+$, and therefore A has greater cardinality than $\mathbb{Z}_+$.

(b) Suppose that A has greater cardinality than B and B has greater cardinality than C, and let $f : B \to A$ and $g : C \to B$ be injections. Then $f \circ g$ is an injection of C into A. Suppose that there exists an injection $h : A \to C$, and let $h' = h|f(B)$ and $f' : B \to f(B)$, $x \mapsto f(x)$. f' is a bijection, so $h' \circ f' : B \to C$ is an injection. This is a contradiction with B having greater cardinality than C, so such an h does not exist, and therefore A has greater cardinality than C.

(c) Define a sequence of sets by

$$\begin{aligned} A_1 &= \mathbb{Z}_+ \\ A_{n+1} &= \mathscr{P}(A_n) \qquad\qquad \text{for all } n \in \mathbb{Z}_+ \end{aligned}$$

For all n, we have $A_n \in A_{n+1}$, so that $f_n : A_n \to A_{n+1}$, $x \mapsto \{x\}$ is an injection of A_n into A_{n+1}. However, there is no injection of $\mathscr{P}(A_n)$ into A_n, so A_{n+1} has greater cardinality than A_n.

(d) Let

$$A = \cup_{n \in \mathbb{Z}_+} A_n$$

For all $n \in \mathbb{Z}_+$, $A_n \subset A$, so the function $i_n : A_n \to A$, $x \mapsto x$ is an injection. Suppose that there is an injection $h : A \to A_m$ for some $m \in \mathbb{Z}_+$. Then $h \circ i_{m+1}$ is an injection of A_{m+1} into A_m, which contradicts the fact that A_{m+1} has greater cardinality than A_m. Therefore A has greater cardinality than every A_n.

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