Exercise 2. Determine which of the following statements are true for all sets A, B, C, D.

(a) $A \subset B$ and $A \subset C \iff A \subset (B \cup C)$

(b) $A \subset B$ or $A \subset C \iff A \subset (B \cup C)$

(c) $A \subset B$ and $A \subset C \iff A \subset (B \cap C)$

(d) $A \subset B$ or $A \subset C \iff A \subset (B \cap C)$

(e) $A - (A - B) = B$

(f) $A - (B - A) = A - B$

(g) $A \cap (B - C) = (A \cap B) - (A \cap C)$

(h) $A \cup (B - C) = (A \cup B) - (A \cup C)$

(i) $(A \cap B) \cup (A - B) = A$

(j) $A \subset C$ and $B \subset D \implies A \times B \subset C \times D$

(k) The converse of (j)

(l) The converse of (j), assuming that A and B are nonempty

(m) $(A \times B) \cup (C \times D) = (A \cup C) \times (B \cup D)$

(n) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

(o) $A \times (B - C) = (A \times B) - (A \times C)$

(p) $(A - B) \times (C - D) = (A \times C - B \times C) - A \times D$

(q) $(A \times B) - (C \times D) = (A - C) \times (B - D)$

Proof.

(a) False.

$A \subset B$ and $B \subset B \cup C$, so $A \subset (B \cup C)$, and we have ($A \subset B$ and $A \subset C \implies A \subset (B \cup C)$). Take $A = \{1, 2, 3\}, B = \{1, 2\}, C = \{3\}$. Then $A \subset (B \cup C)$, but neither $A \subset B$ nor $A \subset C$, so the converse is false.

(b) False.

$A \subset B$ and $B \subset B \cup C$, so $A \subset (B \cup C)$, and we have ($A \subset B$ or $A \subset C \implies A \subset (B \cup C)$). The same counter-example shows that the converse is false here too.

(c) True.

Let $x \in A$. From $A \subset B$ we deduce $x \in B$. From $A \subset C$ we deduce $x \in C$. From both previous statements, we deduce $x \in B \cap C$. This being true for all $x \in A$, we deduce $(A \subset B \text{ and } A \subset C \implies A \subset (B \cap C))$. Conversely, from $A \subset (B \cap C)$ and $B \cap C \subset B$, we deduce $A \subset B$. From $A \subset (B \cap C)$ and $B \cap C \subset C$, we deduce $A \subset C$. From both, we deduce $(A \subset B \cap C) \implies A \subset B$ and $A \subset C$.

(d) False.

Take $A = \{1\}, B = \{1\}, C = \{2\}$. We have $(A \subset B$ or $A \subset C)$, but $A \not\subset B \cap C$ since $B \cap C = \varnothing$ and $A \neq \varnothing$. The converse is true, though. From $A \subset B \cap C$ and $B \cap C \subset B$ we deduce $A \subset B$. From it, we deduce $A \subset B$ or $A \subset C$.

(e) False.

Take $A = \{1, 2\}, B = \{2, 3\}$. Then $A - B = \{1\}$ and $A - (A - B) = \{2\} \neq B$. However $A - (A - B) \subset B$. Let $x \in A - (A - B)$. This is equivalent to $x \in A$ and $x \notin A - B$. Also, $x \notin A - B$ is equivalent to $x \notin A$ or $x \in B$. From both of these, we deduce that $A - (A - B) = A \cap B \subset B$.

(f) False.

Let $x \in B - A$. By definition of $B - A$, $x \in B$ and $x \notin A$. From this we deduce that $A \cap (B - A) = \varnothing$. Since the sets A and $B - A$ are disjoint, $A - (B - A) = A$. Noting that $A - B \subset A$, we deduce that $A - B \subset A - (B - A)$.

(g) True.

Let $x \in A \cap (B - C)$. This is equivalent to $x \in A$ and $x \in B$ and $x \notin C$, so $A \cap (B - C) = (A \cap B) - C$. Let $x \in (A \cap B) - (A \cap C)$. This is equivalent to $x \in A$ and $x \in B$ and $(x \notin A$ or $x \notin C)$, which simplifies to $x \in A$ and $x \in B$ and $x \notin C$, which is again $(A \cap B) - C$.

(h) False.

Let $x \in (A \cup B) - (A \cup C)$. This is equivalent to $(x \in A$ or $x \in B)$ and $(x \notin A$ and $x \notin C)$, which simplifies to $x \in B$ and $x \notin A$ and $x \notin C$, so that $(A \cup B) - (A \cup C) = (B - C) - A \subset A \cup (B - C)$. Taking $A = \{1\}, B = \{2, 3\}, C = \{3\}$, we have $A \cup (B - C) = \{1, 2\}$ and $(A \cup B) - (A \cup C) = \{1, 2, 3\} - \{1, 3\} = \{2\}$, so the reverse inclusion is false.

(i) True.

Both $A \cap B$ and $A - B$ are subsets of A, so their union is, too, and we have $(A \cap B) \cup (A - B) \subset A$. Conversely, let $x \in A$. If we also have $x \in B$, then $x \in A \cap B$. Otherwise, we have $x \in A$ and $x \notin B$, so that $x \in (A - B)$. It follows that $A \subset (A \cap B) \cup (A - B)$. With both inclusions, we conclude that $(A \cap B) \cup (A - B) = A$.

(j) True.

Let $(x, y) \in A \times B$. Then we have $x \in A$ and $y \in B$. From $x \in A$ and $A \subset C$ we deduce $x \in C$. From $y \in B$ and $B \subset D$ we deduce $y \in D$. From both, we deduce that $(x, y) \in C \times D$, so that $A \times B \subset C \times D$.

(k) False.

The converse of proposition (i) is $A \times B \subset C \times D \implies A \subset C$ and $B \subset D$. Take $A = \{a, b\}, B = \varnothing, C = \{a\}, D = \{1\}$. Then $A \times B = \varnothing$, $C \times D = \{(a, 1)\}$, so that $A \times B \subset C \times D$. However, $A \not\subset C$.

(l) True.

Suppose $A \times B \subset C \times D$, and that neither A nor C are empty. Let $(x, y) \in A \times B$. Then $x \in A$ and $y \in B$. From $A \times B \subset C \times D$ we deduce that $(x, y) \in C \times D$, so that $x \in C$ and $y \in D$. Summing up, we have $x \in A \implies x \in C$ and $y \in B \implies y \in D$, which is the definition of $A \subset C$ and $B \subset D$.

(m) False.

Since $A \subset A \cup C$ and $B \subset B \cup D$, we have $(A \times B) \cup (C \times D) \subset (A \cup C) \times (B \cup D)$ Take $A = \{a\}, B = \{1\}, C = \{a, b\}, D = \{1, 2\}$. Then $(b, 1) \in (A \cup C) \times (B \cup D)$, but it is not an element of $(A \times C) \cup (B \times D)$.

(n) True.

Note that if any of the sets A, B, C, or D is empty, then the formula reduces to $\varnothing = \varnothing$, which is trivially true. Let $(x, y) \in (A \times B) \cap (C \times D)$. From $(x, y) \in A \times B$, we deduce $x \in A$ and $y \in B$. From $(x, y) \in C \times D$ we deduce $x \in C$ and $y \in D$. From both, we deduce $x \in A \cap C$ and $y \in B \cap D$, so that $(A \times B) \cap (C \times D) \subset (A \cap C) \times (B \times D)$.

Conversely, let $(x, y) \in (A \cap C) \times (B \cap D)$. We have $x \in A \cap C$ and $y \in B \cap D$. From $A \cap C \subset A$ and $B \cap D \subset B$, we deduce $x \in A$ and $y \in C$, so that $(x, y) \in A \times B$. From $A \cap C \subset C$ and $B \cap D \subset D$, we deduce $x \in C$ and $y \in D$, so that $(x, y) \in C \times D$. From both, we deduce $(A \cap C) \times (B \cap D) \subset (A \times B) \cap (C \times D)$.

(o) True.

If $A = \varnothing$ or $B = \varnothing$, the left-hand side is $\varnothing$, and the right-hand side translates to $\varnothing - D$ for some $D \subset X$. This expression is again equal to $\varnothing$ since $\forall x \in X,\ x \notin \varnothing$. So the equality is true.

If $C = \varnothing$, then $A \times C = \varnothing$ and $B - C = B$, so the equality is again trivially true.

Otherwise, let $x \in A \times (B - C)$. This is equivalent to having $x \in A$ and $x \in B$ and $x \notin C$, and since $x \in A$ is again equivalent to ($x \in A$ and $x \in A$), we have, by commutativity of "and", ($x \in A$ and $x \in B$) and ($x \in A$ and $x \notin C$). So the equality is true.

(p) True.

Let $(x, y) \in (A \times C - B \times C) - A \times D$. We have $(x, y) \in (A \times C - B \times C)$ and $(x, y) \notin A \times D$. $(x, y) \notin A \times D$ gives ($x \notin A$ or $y \notin D$). $(x, y) \in (A \times C - B \times C)$ gives $x \in A$ and $x \notin B$ and $y \in C$.

Combining both, we get: ($x \notin A$ or $y \notin D$) and $x \in A$ and $x \notin B$ and $y \in C$, which simplifies to $y \notin D$ and $x \in A$ and $x \notin B$ and $y \in C$, so that $(x, y) \in (A - B) \times (C - D)$

All the above transformations are equivalences, so $(A - B) \times (C - D) = (A \times C - B \times C) - A \times D$

(q) False.

Take $A = \{a\}, B = \{1, 2\}, C = \varnothing, D = \{1\}$. Then $A \times B - C \times D = \{(a, 1), (a, 2)\}$, but $(A - C) \times (B - D) = \{a\} \times \{2\} = \{(a, 2)\}$.

Let $(x, y) \in (A - C) \times (B - D)$. We have $x \in A$ and $x \notin C$ and $y \in B$ and $y \notin D$, from which we deduce $x \in A$ and $y \in B$ and $x \notin C$ and $y \notin D$. From $x \notin C$ and $y \notin D$, we deduce $(x, y) \notin C \times D$. Putting both parts together we get $(A - C) \times (B - D) \subset (A \times B) - (C \times D)$.

$\square$