Exercise 3. Find terminals in $\mathbf{Set}^2$, $\mathbf{Set}^{\to}$ and the poset $\mathbf{n}$.

Proof.

1. In $\mathbf{Set}^2$, any object of the form $\langle\{a\},\{b\}\rangle$ where $\{a\}$ and $\{b\}$ are singleton sets, is terminal: let $\langle A, B\rangle$ be an object in $\mathbf{Set}^2$, there exists a unique function $f\colon A \to \{a\}$ and a unique function $g\colon B \to \{b\}$, since both $\{a\}$ and $\{b\}$ are terminal in $\mathbf{Set}$. Then $\langle f, g\rangle\colon \langle A, B\rangle \to \langle\{a\},\{b\}\rangle$ is an arrow of $\mathbf{Set}^2$. For all arrows $\langle f', g'\rangle$ of $\mathbf{Set}^2$ between the same objects, f' is a function $A \to \{a\}$ and g' is a function $B \to \{b\}$. By terminality of $\{a\}$ and $\{b\}$ in $\mathbf{Set}$, we must then have $f' = f$ and $g' = g$, so that $\langle\{a\},\{b\}\rangle$ is terminal in $\mathbf{Set}^2$.

2. In $\mathbf{Set}^{\to}$, any object of the form $\{a\} \to \{b\}$ where $\{a\}$ and $\{b\}$ are singleton sets, is terminal: let $\phi\colon A \to B$ be an object of $\mathbf{Set}^{\to}$, there exist unique functions $f\colon A \to \{a\}$ and $g\colon B \to \{b\}$. These functions make the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi"] & \{a\} \arrow[d] \\ B \arrow[r, "g"] & \{b\} \end{tikzcd} \tag{1}$$

   so that $\langle f, g\rangle$ is the unique arrow of $\mathbf{Set}^{\to}$ between ϕ and $\{a\} \to \{b\}$.

3. In the poset $\mathbf{n}$ for $n \geq 1$, for all objects m we have $m \leq n-1$, so that there is an arrow $m \to n-1$. As an arrow in a poset, it is unique. Therefore, $n-1$ is terminal. The poset $\mathbf{0}$ has a unique object 0 and an unique arrow $\mathbf{1}_0\colon 0 \to 0$; 0 is therefore terminal.

$\square$