Differential Dynamics and Mesh-Based Inertia

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1 Recursive Dynamics

A recursive Newton-Euler dynamics that describes an open serial chain of rigid bodies is revisited. We assume that the base is fixed, i.e., $V_0 = 0, \dot{V}_0 = -g$, and denote by $\mathcal{F}_{\text{ext}}$ the external force applied to the end-effector. $\lambda(i)$ and $\xi(i)$ denote the indices of the parent (preceding) and child (succeeding) links of the i-th link, respectively.

The forward iterations evaluate the kinematic variables of each link (e.g., $T_{i-1,i}$, V_i, $\dot{V}_i$) as follows:

$$T_{\lambda(i),i} = M_i e^{[A_i]\theta_i}, \tag{1}$$

$$V_i = A_i\dot{\theta}_i + \mathrm{Ad}_{T^{-1}_{\lambda(i),i}} V_{\lambda(i)}, \tag{2}$$

$$\dot{V}_i = A_i\ddot{\theta}_i + \mathrm{Ad}_{T^{-1}_{\lambda(i),i}} \dot{V}_{\lambda(i)} + \mathrm{ad}_{V_i} A_i\dot{\theta}_i. \tag{3}$$

while the forces are given by the backward iterations (i.e., $i = n, \cdots, 1$):

$$\mathcal{F}_i = \sum_{j\in\xi(i)} \mathrm{Ad}^T_{T^{-1}_{i,j}} \mathcal{F}_j + G_i\dot{V}_i - \mathrm{ad}^T_{V_i} G_i V_i, \tag{4}$$

$$\tau_i = A_i^T \mathcal{F}_i. \tag{5}$$

We assume that $M_i = \mathrm{I}$ for notational simplicity.

2 Differential Dynamics with respect to Joint Twists

First, we re-parameterize the joint twists $A_i \in se(3)$ with large adjoint operator as the adjoint preserves the joint constraints, e.g., $\|w\| = 1,\ w^T v = 0$ for revolute joints.[1] Otherwise, the joint constraints must be enforced by explicit constraints while solving the optimization problem. That is, a joint twist is parameterized by a vector $\eta_i \in se(3)$ as

$$A_i = \mathrm{Ad}_{e^{[\eta_i]}}\left(\overline{A}_i\right), \tag{6}$$

around the neighborhood of $\overline{A}_i \in se(3)$ which are the current estimates of the twists (Thus, updated for every iteration). Based on the derivative of the large adjoint mapping

$$\frac{d}{dt}\mathrm{Ad}_T = \mathrm{Ad}_T \mathrm{ad}_{T^{-1}\dot{T}}, \tag{7}$$

the derivative of (6) is given as

$$\delta A_i = -\mathrm{ad}_{\overline{A}_i}\delta\eta_i. \tag{8}$$

Extending the conventional differential dynamics (with respect to $\theta, \dot{\theta}, \ddot{\theta}$) to a generalized version (with respect to $\lambda = [\boldsymbol{\eta}, \theta, \dot{\theta}, \ddot{\theta}] \in \mathbb{R}^{9n+6}$), the derivatives of the i-th link variables are denoted by $P_i, Q_i, R_i \in \mathbb{R}^{6\times(9n+6)}$:

$$P_i = \frac{\partial V_i}{\partial \lambda}, \quad Q_i = \frac{\partial \dot{V}_i}{\partial \lambda}, \quad R_i = \frac{\partial \mathcal{F}_i}{\partial \lambda}. \tag{9}$$

[1] Li, Cheng, et al. "POE-based robot kinematic calibration using axis configuration space and the adjoint error model." IEEE Transactions on Robotics 32.5 (2016): 1264-1279.

To zero-pad the non-recursive terms ($\overline{P}, \overline{Q}, \overline{R}$), the subscript $[\cdot]_{(:,i)}$ denotes the i-th column block, e.g., $[X]_{(:,i)} \in \mathbb{R}^{m\times k}$ if $X \in \mathbb{R}^{m\times nk}$ for some m, k. Though it appears complicated, it is just detailed so that one can write a code directly from it.

Algorithm 1 Differential Dynamics

Input: joint states $(\theta, \dot{\theta}, \ddot{\theta})$, joint twist A_i and inertia tensor G_i.
Output: P_i, Q_i, R_i (i.e., the derivatives of $T, V, \dot{V}, F$).
$P_1 = \big[-\mathrm{ad}_{A_1\dot{\theta}_1},\ \mathbf{0}_{6\times 6n}\big]$
$Q_1 = \big[-\mathrm{ad}_{A_1\ddot{\theta}_1} + \mathrm{Ad}_1^{-1}\mathrm{ad}_{\dot{V}_0}(\mathrm{I} - \mathrm{Ad}_1),\ \mathbf{0}_{6\times 6n}\big]$
for $i = 1 : n$ **do**
  $\overline{P}^{\eta} = \overline{Q}^{\eta} = \mathbf{0}_{6\times 6(n+1)}$
  $[\overline{P}^{\eta}]_{(:,i)} = \mathrm{Ad}_i^{-1}\mathrm{ad}_{V_{i-1}}(\mathrm{I} - \mathrm{Ad}_i) - \mathrm{ad}_{A_i\dot{\theta}_i}$
  $[\overline{Q}^{\eta}]_{(:,i)} = \mathrm{Ad}_i^{-1}\mathrm{ad}_{\dot{V}_{i-1}}(\mathrm{I} - \mathrm{Ad}_i) - \mathrm{ad}_{V_i}\mathrm{ad}_{A_i\dot{\theta}_i} - \mathrm{ad}_{A_i\ddot{\theta}_i}$
  $\overline{P}^{\text{pos}} = \overline{P}^{\text{vel}} = \overline{P}^{\text{acc}} = \overline{Q}^{\text{pos}} = \overline{Q}^{\text{vel}} = \overline{Q}^{\text{acc}} = \mathbf{0}_{6\times n}$
  $[\overline{P}^{\text{pos}}]_{(:,i)} = \mathrm{Ad}_i^{-1}\mathrm{ad}_{V_{i-1}}A_i$
  $[\overline{Q}^{\text{pos}}]_{(:,i)} = \mathrm{Ad}_i^{-1}\mathrm{ad}_{\dot{V}_{i-1}}A_i$
  $[\overline{P}^{\text{vel}}]_{(:,i)} = A_i$
  $[\overline{Q}^{\text{vel}}]_{(:,i)} = \mathrm{ad}_{V_i}A_i$
  $[\overline{P}^{\text{acc}}]_{(:,i)} = \mathbf{0}_{6\times 1}$
  $[\overline{Q}^{\text{acc}}]_{(:,i)} = A_i$
  $\overline{P} = [\overline{P}^{\eta}\ \overline{P}^{\text{pos}}\ \overline{P}^{\text{vel}}\ \overline{P}^{\text{acc}}]$
  $\overline{Q} = [\overline{Q}^{\eta}\ \overline{Q}^{\text{pos}}\ \overline{Q}^{\text{vel}}\ \overline{Q}^{\text{acc}}]$
  $P_i = \mathrm{Ad}_i^{-1}P_{i-1} + \overline{P}$
  $Q_i = \mathrm{Ad}_i^{-1}Q_{i-1} - [\mathrm{ad}_{A_i\dot{\theta}_i}]P_i + \overline{Q}$
end for
$R_n = -\big(\mathrm{ad}_{V_n}^T G_n + \mathrm{ad}^{\star}_{G_n V_n}\big)P_n + G_n Q_n$
for $i = n-1 : 1$ **do**
  $\overline{R}^{\eta} = \mathbf{0}_{6\times 6(n+1)}$
  $[\overline{R}^{\eta}]_{(:,i+1)} = -(\mathrm{Ad}_{i+1}^{-1})^T\mathrm{ad}^{\star}_{\mathcal{F}_{i+1}}(\mathrm{Ad}_{i+1}^{-1} - \mathrm{I})$
  $\overline{R}^{\text{pos}} = \overline{R}_{\text{vel}} = \overline{R}_{\text{acc}} = \mathbf{0}_{6\times n}$
  $[\overline{R}^{\text{pos}}]_{(:,i+1)} = -(\mathrm{Ad}_{i+1}^{-1})^T\mathrm{ad}^{\star}_{\mathcal{F}_{i+1}}A_{i+1}$
  $[\overline{R}^{\text{vel}}]_{(:,i+1)} = \mathbf{0}_{6\times 1}$
  $[\overline{R}^{\text{acc}}]_{(:,i+1)} = \mathbf{0}_{6\times 1}$
  $\overline{R} = [\overline{R}^{\eta}\ \overline{R}^{\text{pos}}\ \overline{R}^{\text{vel}}\ \overline{R}^{\text{acc}}]$
  $R_i = \mathrm{Ad}_{i+1}^{-T}R_{i+1} - (\mathrm{ad}_{V_i}^T G_i - \mathrm{ad}^{\star}_{G_i V_i})P_i + G_i Q_i + \overline{R}$
end for

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where we compactly denote the large adjoint as

$$\mathrm{Ad}_i = \mathrm{Ad}_{e^{A_1\theta_1}e^{A_2\theta_2}\cdots e^{A_i\theta_i}}.$$

and $\mathrm{ad}^{\star}_{\mathcal{F}}$ is defined for any $F = (m, f)$ as

$$\mathrm{ad}^{\star}_{\mathcal{F}} = \begin{bmatrix} [m] & [f] \\ [f] & 0 \end{bmatrix} \tag{10}$$

by which A and $\mathcal{F}$ in the dual adjoint can be reversed:

$$\mathrm{ad}_A^T\mathcal{F} = \mathrm{ad}^{\star}_{\mathcal{F}}A.$$

This is identical to $d[ad]^T/dt$, the time derivative of the dual adjoint.

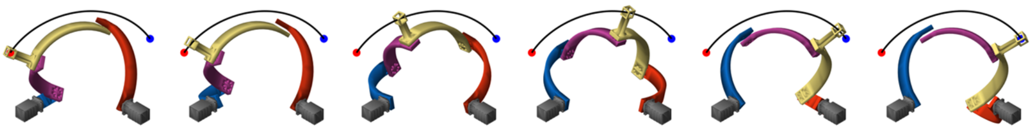

Figure 1: As a parallel robot, the five-bar linkage system with four joints driven by two motors is executing a point-to-point reaching motion.

3 Differential Dynamics of The Spherical Five-Bar System

Let the links are numbered from the left to the right, e.g., the first blue link and fourth orange links in Figure 1. Let two branches grow from each motor: the first branch of link 1, 2, 3 each rotated by joint 1, 2, 3, and the second branch of link 4, 3 rotated by joint 4, 5. The active joints are $\theta_a = (\theta_1, \theta_4)$, and the passive joints are $\theta_p = (\theta_2, \theta_3, \theta_5)$. Since all of the joint axes intersect at the origin, each of the joint twists are in the form of $A_i = (w_i, 0)$. As the (yellow) link 3 appears in both branches, the closed-loop constraint is given by

$$e^{w_1\theta_1(t_i)}e^{w_2\theta_2(t_i)}e^{w_3\theta_3(t_i)} = e^{w_4\theta_4(t_i)}e^{w_5\theta_5(t_i)}, \quad ^\forall t_i \tag{11}$$

where the position terms are omitted since $v_i = 0$ for all twists. Differentiating (11) leads to $\dot\theta_p = J_p\dot\theta_a$ where

$$\begin{gathered} J_p = -g_p^{-1}g_a, \\ g_p = \begin{bmatrix} w_2' & w_3' & -w_5' \end{bmatrix}, \quad g_a = \begin{bmatrix} w_1' & -w_4' \end{bmatrix}, \end{gathered} \tag{12}$$

where $w_i' \in \mathbb{R}^3$ is the twist at configuration θ (i.e., w_i rotated by θ); for example, $w_3' = e^{w_1\theta_1}e^{w_2\theta_2}w_3, \quad w_5' = e^{w_4\theta_4}w_5$.

Denote by $\tau^{\text{pr}} \in \mathbb{R}^2$ and $\tau^{\text{tr}} = [\tau_a^{\text{tr}},\ \tau_p^{\text{tr}}] \in \mathbb{R}^5$ the joint torque of the parallel robot and its equivalent tree, respectively. Substituting the kinematics $\dot\theta_p = J_p\dot\theta_a$ into the virtual work equation $\dot\theta_a^T\tau^{\text{pr}} = \dot\theta^T\tau^{\text{tr}}$,

$$\tau^{\text{pr}} = \tau_a^{\text{tr}} + J_p^T\tau_p^{\text{tr}} = \begin{bmatrix} \mathrm{I} & J_p^T \end{bmatrix}\tau^{\text{tr}}, \tag{13}$$

where τ^{tr} is directly given by the recursive dynamics. This can be differentiated as

$$\begin{gathered} \frac{\partial\tau^{\text{pr}}}{\partial\lambda} = -G_a(\hat\tau) + g_a^T g_p^{-T}G_p(\hat\tau) + [\mathrm{I},\ J_p^T]S^{\text{tr}}, \\ G_a(\hat\tau) = \begin{bmatrix} \hat\tau^T(\partial w_1'/\partial\lambda) \\ \hat\tau^T(\partial w_4'/\partial\lambda) \end{bmatrix}, G_p(\hat\tau) = \begin{bmatrix} \hat\tau^T(\partial w_2'/\partial\lambda) \\ \hat\tau^T(\partial w_3'/\partial\lambda) \\ \mathbf{0} \end{bmatrix}, \end{gathered} \tag{14}$$

and $\hat\tau = g_p^{-T}\tau^{\text{tr}}$. Assuming that the joint 5 is virtually cut in the tree (i.e., link 3 attached to joint 3), its torque should be zero (i.e., $\phi_5 = 0,\ \tau_5^{\text{tr}} = 0$) and the last row of G_p becomes zero.

4 Mesh-Based Inertia

While it is presented in several studies[2], full (10-dim.) inertial parameters and its derivatives are not completely given in a single paper or written in some complex notations. Thus, we just gather and re-write them here with simpler notations.

Each face $F = (\mathrm{v}_1^F, \mathrm{v}_2^F, \mathrm{v}_3^F) \in \mathbb{F}$ is composed of three vertices arranged in an order that its normal vector always points outwards as

$$n_F = (\mathrm{v}_2^F - \mathrm{v}_1^F) \times (\mathrm{v}_3^F - \mathrm{v}_1^F),$$

[2]Mirtich, Brian. "Fast and accurate computation of polyhedral mass properties." Journal of graphics tools 1.2 (1996): 31-50.

of which the magnitude is twice the face area. $\mu \in \mathbb{R}$ is the density and let the center $\vec{f}(F) = (\mathrm{v}_1^F + \mathrm{v}_2^F + \mathrm{v}_3^F)/3$.

Let (i, j, k) be any cyclic order of $(1, 2, 3)$ (i.e., $(i, j, k) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$), so that the normal vector and the determinant written as

$$n_F = (\mathrm{v}_j - \mathrm{v}_i) \times (\mathrm{v}_k - \mathrm{v}_i), \quad \det(F) = \det([\mathrm{v}_i \, \mathrm{v}_j \, \mathrm{v}_k]) \tag{15}$$

are always constant over the cyclic orders. We omit the super-script of v^F for notational simplicity. Then, their derivatives are

$$\frac{\partial n_F}{\partial \mathrm{v}_i} = [\mathrm{v}_k - \mathrm{v}_j], \quad \frac{\partial \det(F)}{\partial \mathrm{v}_i} = \mathrm{v}_j^T [\mathrm{v}_k]. \tag{16}$$

Remind that $[\cdot]$ denotes the skew-symmetric matrix for cross product.

First, the mass m of the mesh can be given by

$$m = \frac{\mu}{18} \sum_{F \in \mathbb{F}} (\mathrm{v}_1 + \mathrm{v}_2 + \mathrm{v}_3)^T n_F, \tag{17}$$

which can be differentiated by

$$\frac{\partial m}{\partial \mathrm{v}_i} = \frac{\mu}{18} \sum_{F \in \mathbb{F}} \sum_{i=1}^{3} (n_F^T + (\mathrm{v}_1 + \mathrm{v}_2 + \mathrm{v}_3)^T \frac{\partial n_F}{\partial \mathrm{v}_i}), \tag{18}$$

where i can be any of $1, 2, 3$ by switching the orders.

Let $\circ$ indicates the Hadamard (i.e., element-wise) product and a function $g(F) \in \mathbb{R}^3$ be defined as

$$g(F) = \sum_{i=1}^{3} \sum_{j=i}^{3} \mathrm{v}_i \circ \mathrm{v}_j, \tag{19}$$

of which the derivative is

$$\frac{\partial g(F)}{\partial \mathrm{v}_i} = \mathrm{diag}(\mathrm{v}_1 + \mathrm{v}_2 + \mathrm{v}_3 + \mathrm{v}_i).$$

Then, the second term $h \in \mathbb{R}^3$ (center-of-mass multiplied by the mass) is given by

$$h = \frac{\mu}{24} \sum_{F} g(F) \circ n_F, \tag{20}$$

of which the derivative is

$$\frac{\partial h}{\partial \mathrm{v}_i} = \frac{\mu}{24} \sum_{F \in \mathbb{F}} \Big(n_F \odot \frac{\partial g(F)}{\partial \mathrm{v}_i} + g(F) \odot \frac{\partial n_F}{\partial \mathrm{v}_i} \Big), \tag{21}$$

where $\odot$ is a row-wise (vector-matrix) multiplication, i.e.,

$$a \odot A = (a \cdot \mathbf{1}^T) \circ A,$$

for any vector $a \in \mathbb{R}^m$ and matrix $A \in \mathbb{R}^{m \times k}$.

Finally, the rotational inertia I is separated into the diagonal and off-diagonal terms as

$$I_{\text{diag}} = [I_{xx}, I_{yy}, I_{zz}]^T, \quad \mathrm{I}_{\text{off}} = [I_{xy}, I_{xz}, I_{yz}]^T.$$

Then, the diagonal components are given by

$$I_{\text{diag}} = \frac{\mu}{60} \sum_{F \in \mathbb{F}} \det(F) (\mathbf{1} \cdot \mathbf{1}^T - \mathrm{I}) g(F), \tag{22}$$

which can be differentiated by

$$\frac{\partial I_{\text{diag}}}{\partial \text{v}_i} = \frac{\mu}{60} \sum_{F \in \mathbb{F}} \sum_{i=1}^{3} (\mathbf{1} \cdot \mathbf{1}^T - \text{I}) \frac{\partial I_{F,\text{diag}}}{\partial \text{v}_i}, \tag{23}$$

where $(\partial I_{F,\text{diag}}/\partial \text{v}_i)$ is

$$g(F) \frac{\partial \det(F)}{\partial \text{v}_i} + \det(F) \frac{\partial g(F)}{\partial \text{v}_i}. \tag{24}$$

The off-diagonal term I_{off} is

$$I_{\text{off}} = -\frac{\mu}{120} \sum_{F \in \mathbb{F}} \det(F) \sum_{i=1}^{3} \sum_{j=i}^{3} r(\text{v}_i) \text{v}_j, \tag{25}$$

where $r(\text{v})$ is defined for any vertex $\text{v} = (x, y, z)$ as

$$r(\text{v}) = \begin{bmatrix} y & x & 0 \\ z & 0 & x \\ 0 & z & y \end{bmatrix}, \tag{26}$$

and the derivative is

$$\frac{\partial I_{\text{off}}}{\partial \text{v}_i} = -\frac{\mu}{120} \sum_{F \in \mathbb{F}} \sum_{i=1}^{3} \frac{\partial I_{F,\text{off}}}{\partial \text{v}_i}, \tag{27}$$

where $(\partial I_{F,\text{off}}/\partial \text{v}_i)$ is

$$\Big(\sum_{i=1}^{3} \sum_{j=i}^{3} r(\text{v}_i) \text{v}_j\Big) \frac{\partial \det(F)}{\partial \text{v}_i} + \det(F) r(\text{v}_i + \sum_{j=1}^{3} \text{v}_j). \tag{28}$$

5 Derivative of The Translated Inertia

The coordinate transformation of inertia $\overline{G}_i \in \mathbb{R}^{6 \times 6}$ by any rigid body motion $T \in SE(3)$ is given by

$$G_i = (\text{Ad}_T)^{-T} \overline{G}_i (\text{Ad}_T)^{-1}, \tag{29}$$

which can be differentiated by $[\delta\eta] = \delta T T^{-1}$ as

$$\delta G_i = -\text{ad}_{\delta\eta}^T G_i - G_i \text{ad}_{\delta\eta}, \tag{30}$$

using the adjoint derivative (7).

Letting $\delta\eta = (\delta\omega, \delta v)$, then it is rewritten in the vector form $\partial\phi^{\text{motor}}/\partial\eta$ as

$$\begin{bmatrix} [\delta\omega] & [\delta v] \\ \mathbf{0} & [\delta\omega] \end{bmatrix} \begin{bmatrix} I & [h] \\ [h]^T & m\text{I} \end{bmatrix} - \begin{bmatrix} I & [h] \\ [h]^T & m\text{I} \end{bmatrix} \begin{bmatrix} [\delta\omega] & \mathbf{0} \\ [\delta v] & [\delta\omega] \end{bmatrix} \tag{31}$$

which individually indicates

$$\delta I = [\delta\omega] I - I [\delta\omega] - [\delta v][h] - [h][\delta v], \tag{32}$$
$$\delta h = -[h]\delta\omega + m\delta v, \tag{33}$$
$$\delta m = 0. \tag{34}$$

From (32), the diagonal term $\partial I_{\text{diag}}/\partial \eta \in \mathbb{R}^{3\times 6}$ is

$$\begin{bmatrix} 2\begin{bmatrix} 0 & I_{xz} & -I_{xy} \\ -I_{yz} & 0 & I_{xy} \\ I_{yz} & -I_{xz} & 0 \end{bmatrix} & 2\begin{bmatrix} 0 & h_y & h_z \\ h_x & 0 & h_z \\ h_x & h_y & 0 \end{bmatrix} \end{bmatrix}$$

while the off-diagonal term $\partial I_{\text{off}}/\partial \eta \in \mathbb{R}^{3\times 6}$ is

$$\begin{bmatrix} \begin{bmatrix} -I_{xz} & I_{yz} & I_{xx} - I_{yy} \\ I_{xy} & I_{zz} - I_{xx} & -I_{yz} \\ I_{yy} - I_{zz} & -I_{xy} & I_{xz} \end{bmatrix} & -\begin{bmatrix} h_y & h_x & 0 \\ h_z & 0 & h_x \\ 0 & h_z & h_y \end{bmatrix} \end{bmatrix}$$

Then, the derivative of the inertia tensor in the vector form is given by

$$\frac{\partial \phi^{\text{motor}}}{\partial \eta} = \begin{bmatrix} \partial m/\partial \eta \\ \partial h/\partial \eta \\ \partial I_{\text{diag}}/\partial \eta \\ \partial I_{\text{off}}/\partial \eta \end{bmatrix} \in \mathbb{R}^{10\times 6}.$$