Asian Option Pricing with R/Rmetrics

Diethelm Würtz

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Chapter 1

Introduction

An arithmetic Asian option is a path-dependent option where the payoff function depends on the history in form of the arithmetic average of the underlying asset.
Asian options are extremely useful in the financial industry for two reasons. First, they can be used to hedge a thinly traded asset over a certain period of time cheaper compared to plain vanilla options, and second they can be used effectively to protect against price manipulations by either contract party on the maturity date.
However, pricing and hedging an arithmetic Asian option is difficult because of the intractability of the arithmetic average of the log-Normal process. The challenge to theorists is that the valuation of Asian options is a tricky problem which is just on the edge of solvability by analytic methods. A reasonable simple closed-form solution does not exist and thus effort has been concentrated on approximate and numerical valuation techniques.
We start with the standard assumption in option valuations that the asset price $S(t)$ driven under the risk neutral measure Q satisfies the dynamics defined by Geometric Brownian motion

$$
\begin{aligned}
dS(t) &= (r-q)S(t)dt + \sigma S(t)dW_t \ , \qquad (1.1)\\
S(t) &= Se^{(r-\frac{\sigma^2}{2})t+\sigma W_t} \ , \qquad (1.2)
\end{aligned}
$$

where r denotes the instantaneous interest rate, σ the instantaneous volatility, t the time, and W_t is a Wiener process. We further assume that the average A_t over $[0,t]$ can be represented by the integral

$$A_t = \int_0^t S(u)du \ . \qquad (1.3)$$

where $\frac{1}{t}A_t$ satisfies the scaling relation

$$\frac{1}{t} A_t = \frac{4}{\sigma^2 t} \mathscr{A}^{(\nu)}_{\frac{\sigma^2 t}{4}} \tag{1.4}$$

with

$$\frac{1}{\tau} \mathscr{A}^{(\nu)}_{\tau} = \frac{1}{\tau} \int_0^{\tau} e^{2(\nu s + W_s)} ds \,. \tag{1.5}$$

The value of the Asian call option c is then expressed by arbitrage arguments as

$$c = e^{-rT} S \, \mathbb{E} \left[\frac{1}{\tau} \mathscr{A}^{(\nu)}_{\tau} - \overline{X} \right]^{+} , \tag{1.6}$$

where $\tau = \sigma^2 t/4$ and $\overline{X} = X/S$ are dimensionless quantities and $S \equiv S(0)$. The rest of this document deals with the numerical evaluation of the expectation value and related functions. The goal is twofold, first we give a review on the used methods to valuate Asian options, and second we discuss the numerical algorithms and formulas underlying the several different approaches.

We implement the algorithms as “R-functions”. R is a language and environment for statistical computing and graphics. It is a GNU project which is similar to the S language and environment which was developed at Bell Laboratories by John Chambers [1998] and colleagues. R provides a wide variety of statistical and graphical techniques, and is highly extensible. R runs out of the box on a wide variety of platforms including Windows, Linux and and MacOS. The software and the source codes written for this report are also public available under the GNU License. The R-Package can be downloaded from *http://www.rmetrics.org*. It is worth to mention that the functions can be transformed with some minor modifications to the commercial available S-Plus environment.

The library contains functions to valuate Asian Options, to evaluate state price densities, and to calculate related tasks. The methods include density approaches based on *Moment Matching Methods*, on *Statistical Series Expansions*, on *Partial Differential Equation* approaches, on *Laplace Inversions* and *Fourier Transforms*, on *Spectral Expansions*, on methods calculating *Bounds on Option Prices*, and finally on *Symmetry and Equivalence Relations*. All the functions are tested on benchmark parameter sets which were also used in other publications. The software package comes with a few dozens of examples which investigate and compare the results and the associated accuracies for these benchmark data sets.

Chapter 2

Moment Matched Distributions

The expectation value and thus the prices for the call and put options can be evaluated if we know the density $f_{\mathscr{A}_\tau^{(\nu)}}$ of the exponential Brownian motion $\mathscr{A}_\tau^{(\nu)}$:

$$\mathbb{E}\left[\frac{1}{\tau}\mathscr{A}_\tau^{(\nu)} - \overline{X}\right]^+ = \frac{1}{\tau}\int_{\overline{X}}^{\infty}(x-\overline{X})f_{\mathscr{A}_\tau^{(\nu)}}(x)dx\,. \tag{2.1}$$

However the calculation of the density $f_{\mathscr{A}_\tau^{(\nu)}}(x)$, in the following named "state price density", is a rather nontrivial mathematical task. Nevertheless, it is possible to derive astonishing simple expressions for the raw moments of the density and some more complex expressions for the density itself together with results describing the asymptotic behavior.

On the other hand it would be quite natural to derive approximations for the state price density from which we can calculate option prices. This was the first attempt to tackle Asian options. Guided by the approximative nature of log-Normality of the underlying asset returns Jarrow and Rudd [1982] suggested already in 1982 a general approach which simplifies option valuation. They proposed to replace the density $f(x)$ with an approximate one which has the same first few moments as the true state price density. We call this approach "moment matching".

In the case of Asian option pricing this scheme was followed by several approaches. As a first step we will present as typical candidates for moment matched distributions the "log-Normal" approximation introduced by Levy [1992], Turnbull and Wakeman [1991], the "reciprocal-Gamma" approximation introduced by Milevsky and Posner [1998, 1999] and the "Johnson-Type-I" approximation introduced by Posner and Milevsky [1998].

In addition, we show from the papers of Yor [1992], Geman and Yor [1993], Dufresne [1990] and Schroeder [2001] how to calculate the moments and the true density.

After the seminal paper of Jarrow and Rudd [1982], several authors have proposed to use different statistical series expansions to price options when the risk-neutral density is asymmetric and leptokurtic. As discusssed by Jurczenko, Maillet and Negrea [2002] one can distinguish the Gram-Charlier and the Edgeworth statistical series expansions. They compared these different multi-moment approximations for the valuation of options and derived analytical formulae for the four-moment approximations and also for the analytical expressions of implied probability densities, implied volatility smile functions and several hedging parameters of interest. Datey, Gauthier and Simonato [1999] calculated option prices for Edgeworth expansions around the log-Normal and reciprocal-Gamma state price densities.

2.1 LOG-NORMAL DISTRIBUTION

Levy [1992] argues that the sum of correlated log-Normal (LN) random variables can be well approximated by another log-Normal distribution with the first and second moment of the correct distribution as a surrogate. In this context fall also the approaches of Turnbull and Wakeman [1992] and Levy and Turnbull [1992]. We approximate the state price density by a moment matched log-Normal density

$$f_{\mathscr{A}_\tau^{(\nu)}}(x)dx \approx \ell_{\mu,\varsigma}(x) = \frac{1}{\sqrt{2\pi}\varsigma}\,\frac{1}{x}\,e^{\frac{(lnx-\mu)^2}{2\varsigma^2}}dx\,, \tag{2.2}$$

where μ and ς are related to the raw moments M_n of the true density by

$$M_n = n\mu + n^2\frac{\varsigma^2}{2}\,. \tag{2.3}$$

Note, when the first two moments are determined, then the others are fixed. If skewness and kurtosis are not deviating too much from the relations given in equation (2.3) we expect, that this approximation would yield reasonable well approximated values for the option prices. Performing the integrations we end up with

$$\begin{aligned} c^{LTW} &\approx e^{-rT}\left[M_1 S\Phi(d_+) - X\Phi(d_-)\right]\,, \qquad (2.4)\\ d_\pm &= \frac{log\frac{S}{X} + (\beta \pm \frac{\varsigma^2}{2})T}{\varsigma\sqrt{T}}\,,\\ \beta &= \frac{lnM_1}{T}\,,\\ \varsigma^2 &= \frac{1}{T}ln\frac{M_2}{M_1^2}\,. \end{aligned}$$

Comparison with the Black-Scholes Formula

The previous expression corresponds closely to the Black and Scholes formula

$$c^{BS} = e^{-rT}\left[e^{bT} S\Phi(d_+) - X\Phi(d_-)\right] , \tag{2.5}$$

where the cost-of-carry term b (mean) is substituted by β and the volatility σ (variance) by ς. Cost-of-carry and volatility are determined from the first two moments $M_{1,2}$, explicitly given by

$$\begin{aligned} M_1 &= \frac{e^{rT}-1}{rT} , \\ M_2 &= \frac{2e^{(2r+\sigma^2)T}}{(r+\sigma^2)(2r+\sigma^2)T^2} + \frac{2}{rT^2}\left[\frac{1}{2r+\sigma^2} - \frac{e^{rT}}{r+\sigma^2}\right] . \end{aligned}$$

2.2 RECIPROCAL-GAMMA DISTRIBUTION

Milevsky and Posner [1997] obtained a closed-form expression for the price of the arithmetic Asian option by using the reciprocal-Gamma (RG) distribution as the state price density function. Their method converges for the case when the distribution is an infinite sum of log-Normal random variables and they use this technique to approximate the finite case. The justification to use RG densities originates from the fact, that the integral of exponential Brownian motion converges to an RG distribution.
Introducing the density $g^R_{\alpha,\beta}(x)$ and probability function $G^R_{\alpha,\beta}(x)$ defining the reciprocal-Gamma distribution and its relation to the density $g_{\alpha,\beta}(x)$ and probability function $G^R_{\alpha,\beta}(x)$ of the Gamma distribution

$$\begin{aligned} f_{\mathscr{A}^{(v)}_\tau}(x)dx &\approx g^R_{\alpha,\beta}(x)dx , \\ g^R_{\alpha,\beta}(x) &= \frac{1}{x^2} g_{\alpha,\beta}\left(\frac{1}{x}\right) , \\ G^R_{\alpha,\beta)}(x) &= 1 - G_{\alpha,\beta}\left(\frac{1}{x}\right) , \\ g_{\alpha,\beta}(x) &= \frac{x}{(\alpha-1)\beta}\ g_{\alpha-1,\beta}(x) , \\ g_{\alpha,\beta}(x) &= \frac{dG_{\alpha,\beta}(x}{dx} = \frac{x^{\alpha-1}e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^\alpha} , \end{aligned} \tag{2.6}$$

we can perform the integration and obtain for the approximative price for an Asian call

$$
\begin{aligned}
c^{MP} &\approx e^{-rT}\left[M_1 S\, G_{\alpha-1,\beta}\left(\frac{S}{X}\right) - X\, G_{\alpha,\beta}\left(\frac{S}{X}\right)\right], \qquad (2.7)\\
\alpha &= \frac{2M_2 - M_1^2}{M_2 - M_1^2},\\
\beta &= \frac{M_2 - M_1^2}{M_1 M_2}.
\end{aligned}
$$

α and β are positive quantities defined via the first two moments $M_{1,2}$. The name "reciprocal-Gamma" stems from the fact that the derivation of the call price is based on "reciprocal" random variables $Y = 1/X$. Nevertheless, the final price can be re-expressed in form of Gamma distribution functions, since reciprocal-Gamma and Gamma distributions are related to each other. The moments

$$M_n = E[1/X] = [\beta^n(\alpha-1)(\alpha-2)\cdots(\alpha-n)]^{-1} \qquad (2.8)$$

were also derived with the help of this "duality".

2.3 Johnson Type Distributions

For greater accuracy Posner and Milevsky [1998] matched the moments of the state price density to Johnson type distributions. Johnson [1949] described these distributions as transformations of the standard normal curve. By applying these transformations to a standard normal variable, a wide variety of non-normal distributions can be approximated, including distributions which are bounded on either one or both sides. The advantage of this approach is that once a particular Johnson distribution has been fitted, the normal integral can be used to compute the option prices. Methods for fitting Johnson distributions by moments are described in Hahn and Shapiro [1967] and in Hill, Hill, and Holder [1976].
Following Hill, Hill and Holder, but using notations of Posner and Milevsky, the Johnson distributions describe a family of three distributions consisting of

$$
\begin{aligned}
Type\ I: \quad z(x) &= a + b\, ln\frac{(x-c)}{d}, \quad c < x, \qquad (2.9)\\
Type\ II: \quad z(x) &= a + b\, sinh^{-1}\frac{x-c}{d},\\
Type\ III: \quad z(x) &= a + b\, ln\frac{x-c}{c+d-x}, \quad c < x < b,
\end{aligned}
$$

where $z(x)$ is a standardized normal variate. Fitting the moments of x has two aspects, first to determine which distribution is necessary and second

to estimate the parameters a, b, c, and d. Here we consider the case of Type I distributions.
Posner and Milevsky performed the integration with the state space density

$$f_{\mathscr{A}_\tau^{(\nu)}}(x)dx \approx \frac{1}{\sqrt{2\pi}} e^{\frac{-z(x)^2}{2}} \, dz(x) \,, \tag{2.10}$$

and derived a closed-form expression for the call price

$$\begin{aligned} C^{PM} &\approx e^{-rT}\left[M_1 S - dSe^{\frac{1-2a}{b^2}}\Phi(D_1) - X + (X - cS)\Phi(D_2)\right] \,, \quad (2.11) \\ D_2 &= a + b\, ln \frac{\frac{X}{S} - c}{d} \,, \\ D_1 &= D_2 - \frac{1}{b} \,, \end{aligned}$$

where a, b, c and d, are the parameters which have to be matched to the moments. We can follow the moment matching technique introduced in Hill, Hill, and Holder and described by Posner and Milevsky. For the *Type I* formulation we have

$$\begin{aligned} a &= \frac{1}{\sqrt{lnw}} \,, \quad (2.12) \\ b &= \frac{b}{2}\, ln \frac{w(w-1)}{\varsigma^2} \,, \\ c &= sign(\eta) \,, \\ d &= dM_1 - e^{\frac{1-2ab}{b^2}} \,, \end{aligned}$$

where w solves the cubic equation $(w-1)(w+2)^2 = \eta^2$. By definition of the kurtosis κ we expect that $\kappa \approx w^4 + 2w^3 + 3w^2 - 3$. When the value of the polynomial in w differs to much from the kurtosis, we should use another formulation, e.g. that one of Johnson-Type-II. Mean μ, standard deviation ς, skewness η and kurtosis κ are defined through the moments M_i

$$\begin{aligned} \mu &= M_1 \,, \quad (2.13) \\ \varsigma &= \sqrt{M_2 - M_1^2} \,, \\ \eta &= (M_3 - 3M_2M_1 + 2M_1^3)/\varsigma^3 \,, \\ \kappa &= (M_4 - 4M_3M_1 + 6M_2M_1^2 - 3M_1^4)/\varsigma^4 \,. \end{aligned}$$

Milevsky and Posner [1999] gave explicit expressions for the first and second moment and remark that the remaining two moments are easy to derive but lengthy to print. We wan't do this ugly work here and present in the next chapter a formula for the moments of general degree M_n.

X S=100	r	σ T=1	RST-LB	TW-LN	MP-RG	PM-JI	T-UB
95	0.05	0.05	7.1777	7.1780	7.1776	7.1777	7.1778
100			2.7162	2.7182	2.7148	2.7162	2.7162
105			0.3372	0.3355	0.3385	0.3373	0.3374
95	0.09		8.8088	8.8089	8.8088	8.8088	8.8089
100			4.3082	4.3097	4.3072	4.3082	4.3084
105			0.9583	0.9582	0.9585	0.9584	0.9585
95	0.15		11.0941	11.0941	11.0941	11.0941	11.0941
100			6.7944	6.7946	6.7942	6.7944	6.7945
105			2.7444	2.7464	2.7430	2.7445	2.7446
90	0.05	0.10	11.9511	11.9533	11.9497	11.9510	11.9522
100			3.6413	3.6475	3.6370	3.6415	3.6416
110			0.3311	0.3240	0.3362	0.3312	0.3322
90	0.09		13.3852	13.3863	13.3845	13.3852	13.3860
100			4.9151	4.9231	4.9094	4.9151	4.9154
110			0.6301	0.6234	0.6350	0.6303	0.6310
90	0.15		15.3988	15.3991	15.3986	15.3988	15.3992
100			7.0277	7.0348	7.0225	7.0276	7.0285
110			1.4133	1.4108	1.4154	1.4138	1.4143
90	0.05	0.20	12.5956	12.6298	12.5729	12.5953	12.6007
100			5.7627	5.7828	5.7475	5.7643	5.7644
110			1.9892	1.9711	2.0012	1.9910	1.9927
90	0.09		13.8312	13.8617	13.8108	13.8306	13.8372
100			6.7770	6.8035	6.7572	6.7782	6.7787
110			2.5455	2.5345	2.5525	2.5476	2.5485
90	0.15		15.6416	15.6653	15.6255	15.6408	15.6489
100			8.4085	8.4408	8.3843	8.4091	8.4104
110			3.5547	3.5564	3.5528	3.5571	3.5578
90	0.05	0.30	13.9524	14.0381	13.8934	13.9530	13.9620
100			7.9444	7.9925	7.9064	7.9495	7.9505
110			4.0701	4.0572	4.0748	4.0767	4.0786
90	0.09		14.9828	15.0670	14.9240	14.9823	14.9928
100			8.8276	8.8858	8.7822	8.8318	8.8333
110			4.6949	4.6951	4.6904	4.7017	4.7027
90	0.15		16.5120	16.5908	16.4561	16.5103	16.5237
100			10.2087	10.2781	10.1545	10.2115	10.2141
110			5.7282	5.7480	5.7094	5.7348	5.7355
			Inside Bounds:	4	1	22	
			Outside Bounds:	32	35	14	

FIGURE 2.1:
Comparison of analytical approximations with lower and upper bounds for Asian call prices. The parameters are $S = 100$, and $T = 1$, the three strike values X are ranging from 90 (95 for the lowest volatility) to 110, interest rates r are given by 0.05, 0.09 and 0.15, and volatilities by 0.05, 0.10, 0.20 and 0.30. LTW-LN denotes the Levy-Turnbull-Wakenman log-Normal approximation, MP-RG denotes the Milevsky-Posner reciprocal-Gamma approximation, PM-JI denotes the Posner-Milevsky Johnson-Type-I approximation. RST-LB gives the lower bound derived by Roger-Shi-Thompson, and T-UB gives the upper bound derived by Thompson. (The formulas for the bounds will be presented later.)
Note that only 4 values out of 36 of the log-Normal and only 1 value of the reciprocal-Gamma approximation are lying inside the bounds! In the case of the Johnson-Type-I distribution more than half of the values are lying inside. It is worth to note, that for higher volatilities and/or longer maturities moment matched approximations loose more and more their accuracy.

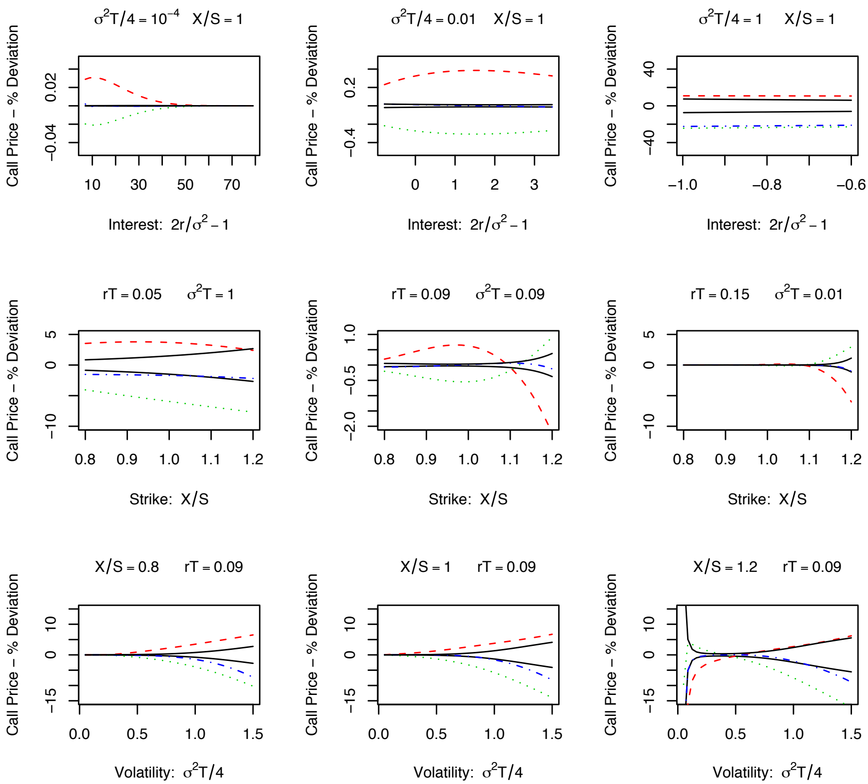

FIGURE 2.2:
The graphs show the call price deviations of the moment matched approximations LT-LN (dashed), MP-RG (dotted), PM-JI (dashdotted), together with the lower, RST-LB (solid), and upper bounds, T-UB (solid), for various option parameter settings.
The first row of the graphs shows for at-the-money options the increasing call price deviations as a function of normalized interest rates $2r/\sigma^2 - 1$ for low, intermediate and high volatility scenarios. The graphs in the middle row show the variations of the call price deviations for fixed interest rate and high, intermediate and low volatility scenarios as a function of normalized moneyness X/S. Note, that prices for options which are high in-the-money are not very precisely evaluated by all three approximations. The graphs in the lower row show the deviations as function of the normalized volatility $\sigma^2 T/4$ for a fixed interest rate $rT = 0.09$ and options which are in-the-money ($X/S = 1.2$), at-the-money and out-of-the-money (0.8).
Here, Johnson Type I fits the option prices best, log-Normal and reciprocal-Gamma behave in most of the situations contrary, when one approximation overestimates the price, then the other one underestimates it, and vice versa.

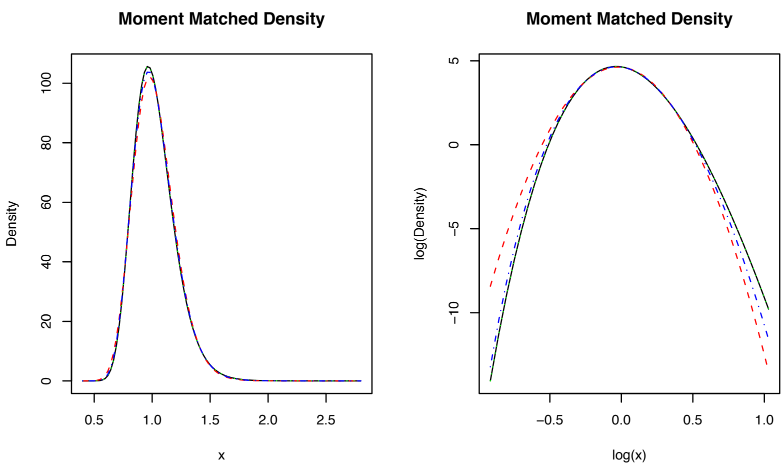

FIGURE 2.3:
The two graphs show the log-Normal (dashed), the reciprocal-Gamma (dotted) and the Johnson-Type-I (dashdotted) moment matched state price densities. The reciprocal-Gamma density has the highest peak and the log-Normal the lowest one. To see the behavior in the tails more clearly, the right graph is on a double-logarithmic scale. In the lower tail the log-Normal density dominates, whereas in the upper tail the reciprocal-Gamma density shows the highest values. The Johnson-Type-I lies in between the other two approximations.

2.4 R-FUNCTIONS

The function `MomentMatchedAsianOption()`

The function `MomentMatchedAsianOption()` computes the moment matched option prices by selecting the appropriate model through the argument `method`. `"LN"` calculates the call and put price of Asian Options within the log-Normal approximation of Levy, Turnbull and Wakeman using the Black and Scholes formula with mean and volatility substituted from the state price density. `"RG"` calculates call and put prices within the reciprocal-Gamma approximation of MilevskyPosner. The Johnson-Type-I approximation of Posner and Milevsky can be computed by setting `method="JI"`.

The function `MomentMatchedAsianOption()`

The approximations for the state price densities can be calculated in the same way calling the function `MomentMatchedAsianDensity()`. The meaning of the argument `method` is the same as for the computation of the prices.

The functions `lnorm()`, `rgamma()` and `johnson()`

In addition, `dlnorm()`, `plnorm()`, `drgamma()`, `prgamma()`, and `djohnson()`, `pjohnson()`, allow to calculate the density and probability functions for the distributions discussed in this section.

Chapter 3

Statistical Series Expansions

We can systematically improve an approximation to the state price density by a series expansion of the "true" density $f(x)$ around a fitted or approximative density $a(x)$

$$f(x) = a(x) + \sum_{i=0}^{\infty} \frac{1}{i!} \left[\sum_{j=1}^{N-1} (-1)^j \frac{[\kappa_j(f) - \kappa_j(g)]}{j!} \frac{d^j}{dx^j} \right]^i a(x) + \varepsilon(x) \tag{3.1}$$

where the $\kappa_j(\cdot)$ are the cumulants of the distributions $f(x)$ and $a(x)$. Specific orderings of the terms and special choices of the density function $a(x)$ lead to several different final forms of this series expansion known under the names Gram-Charlier and Edgeworth expansions. Jurczenko, Maillet and Negrea [2002] gave an overview on these approaches.

3.1 Gram-Charlier Series Expansion

Successive derivatives of $a(x)$ up to the fourth order yield

$$f(x) = a(x) + P_1(k) \frac{d\,a(x)}{dx} + P_2(k) \frac{d^2 a(x)}{dx^2} + P_3(k) \frac{d^3 a(x)}{dx^3} + P_4(k) \frac{d^4 a(x)}{dx^4} + \dots , \tag{3.2}$$

where the Polynomials $P_n(k)$ are defined as

$$\begin{aligned}
P_1(k) &= k_1 , \qquad (3.3) \\
P_2(k) &= \frac{1}{2!}[k_2 + k_1^2] , \\
P_3(k) &= \frac{1}{3!}[k_3 + 3k_2k_1 + 3k_1^3] , \\
P_4(k) &= \frac{1}{4!}[k_4 + 4k_3k_1 + 3k_2^2 + 6k_2k_1^2 + k_1^4] ,
\end{aligned}$$

and $k_i = \kappa_i(f) - \kappa_i(g)$. This type of ordering and collection of terms is called *Gram-Charlier* series expansion.

3.2 EDGEWORTH SERIES EXPANSION

If x in equation ... is a normalized sum of *iid* variables one can set up a proper asymptotic series expansion. The type of ordering is based on the fact that for a sum of *iid* standardized random variables the j-th cumulant is proportional to $n^{1-j/2}$, with $j \geq 2$. Developing and collecting terms of equal order in $n^{-1/2}$ in equation (3.1), say up to order n^{-1}, $f(x)$ can then be expressed as

$$f(x) = a(x) - n^{-1/2}\frac{k_{i,3}}{3!}\frac{d^3a(x)}{dx^3} + n^{-1}\left[\frac{k_{i,4}}{4!}\frac{d^4a(x)}{dx^4} + 10\frac{k_{i,3}^2}{6!}\frac{d^6a(x)}{dx^6}\right] + \dots, \tag{3.4}$$

where $k_{i,j} = \kappa_{i,j}(f) - \kappa_{i,j}(a)$. $\kappa_{i,j}$ is the j-th cumulant of the standardized random variable $\sigma^{-1}(x_i - 1)$, $k_{i,1} = 0$ and $k_{i,2} = 1$ with $(1 \times j) = (1, \dots, n) \times (3, 4)$. This grouping of terms is known as *Edgeworth* series expansion. Second and third terms in equation (3.3) allow to adjust $a(x)$ according to the gap between the skewness and the kurtosis of the density.
Here we consider the log-Normal, the reciprocal-Gamma and the Johnson-Type-I as approximative distributions $a(x)$ and perform Gram-Charlier series expansions around these distributions. An investigation on these three distributions can also be found in the work of Datey, Gauthier, and Simonate [2002]. The case of the log-Normal distribution goes back to Jarrow and Rudd [1982].

Gram-Charlier log-Normal Model

We assume that the approximative state price density is the log-Normal distribution $\ell(x)$. Applying a Gram-Charlier series expansion, the state price density reads

$$f(x) \approx \ell(x) - \frac{m_3^{(f)} - m_3^{(\ell)}}{3!}\frac{d^3\ell(x)}{dx^3} + \frac{m_4^{(f)} - m_4^{(\ell)}}{4!}\frac{d^4\ell(x)}{dx^4}, \tag{3.5}$$

where $m_n^{(f,\ell)}$ are the central moments of the “true” Asian and the “approximative” log-Normal density respectively. With the help of the identity, used by Jarrow and Rudd,

$$\int_{-\infty}^{\infty}(x - X)\frac{d^ja(x)}{dx^j}dx = \left[\frac{d^{j-2}a(x)}{dx^{j-2}}\right]_{x=X}, \tag{3.6}$$

valid for $j \geq 2$, we can calculate the call price of an Asian option as

$$c^{GCLN} = c^{LTW} - e^{-rT} \left[\frac{\kappa_3}{3!} \frac{d\ell(x)}{dx} - \frac{\kappa_4}{4!} \frac{d^2\ell(x)}{dx^2} \right]_{x=X} . \tag{3.7}$$

The first term c^{LTW} counts for the Levy-Turnbull-Wakeman Asian call price, given by equation (2.4), with the first two Asian density moments matched to the log-Normal density. The next two terms take care for the corrections due to skewness and kurtosis effects which are not fully captured by the log-Normal distribution.

Gram Charlier Reciprocal-Gamma Model

In the same way we can develop the Gram-Charlier expansion for the reciprocal-Gamma density. In expression (3.5) we replace $\ell(x)$ with the reciprocal-Gamma distribution $g^R_{\alpha,\beta}(x)$ as defined in (2.6) and calculate again by partial integration the Asian call price

$$c^{GCRG} = c^{MP} - e^{-rT} \left[\frac{\kappa_3}{3!} \frac{d g^R_{\alpha,\beta}(x)}{dx} - \frac{\kappa_4}{4!} \frac{d^2 g^R_{\alpha,\beta}(x)}{dx^2} \right]_{x=X} . \tag{3.8}$$

The first term c^{MP} is the call price derived by Milevsky and Posner from the reciprocal-Gamma distribution as given in equation (2.7) and the remaining third and fourth order terms correct for remaining skewness and kurtosis effects.

Gram-Charlier Johnson-Type-I Model

Since for the moment matched Johnson Type I distribution the first three moments are in agreement between the “true” and the “approximated”, the third order term vanishes and we are left with the fourth order term. Again density and call price can be derived, yielding

$$c^{GCJI} = c^{PM} + e^{-rT} \left[\frac{\kappa_4}{4!} \frac{d^2 z(x)}{dx^2} \right]_{x=X} , \tag{3.9}$$

where c^{PM} counts for the call price derived by Posner and Milevsky based on the 3-moment matched Johnson Type I distribution. The additional term corrects for remaining kurtosis effects.
In all three cases the derivatives can be computed in a straightforward way, but lead to some lengthy expressions and therefore they will be not listed here. The expressions can be found in the implemented R functions for the option pricing formulae.

X S=100	r	σ T=1	RST-LB	GC-LN	GC-RG	GC-JI	T-UB
95	0.05	0.05	7.1777	7.1777	7.1777	7.1777	7.1778
100			2.7162	2.7161	2.7162	2.7162	2.7162
105			0.3372	0.3373	0.3372	0.3373	0.3374
95	0.09		8.8088	8.8088	8.8088	8.8088	8.8089
100			4.3082	4.3082	4.3082	4.3082	4.3084
105			0.9583	0.9584	0.9584	0.9584	0.9585
95	0.15		11.0941	11.0941	11.0941	11.0941	11.0941
100			6.7944	6.7944	6.7943	6.7944	6.7945
105			2.7444	2.7444	2.7445	2.7444	2.7446
90	0.05	0.10	11.9511	11.9513	11.9509	11.9511	11.9522
100			3.6413	3.6409	3.6419	3.6413	3.6416
110			0.3311	0.3315	0.3309	0.3312	0.3322
90	0.09		13.3852	13.3854	13.3850	13.3852	13.3860
100			4.9151	4.9146	4.9158	4.9150	4.9154
110			0.6301	0.6307	0.6298	0.6303	0.6310
90	0.15		15.3988	15.3988	15.3987	15.3988	15.3992
100			7.0277	7.0275	7.0281	7.0277	7.0285
110			1.4133	1.4138	1.4132	1.4137	1.4143
90	0.05	0.20	12.5956	12.5881	12.6066	12.5949	12.6007
100			5.7627	5.7585	5.7676	5.7624	5.7644
110			1.9892	1.9965	1.9818	1.9908	1.9927
90	0.09		13.8312	13.8257	13.8395	13.8307	13.8372
100			6.7770	6.7703	6.7856	6.7763	6.7787
110			2.5455	2.5515	2.5389	2.5469	2.5485
90	0.15		15.6416	15.6394	15.6449	15.6415	15.6489
100			8.4085	8.3996	8.4212	8.4075	8.4104
110			3.5547	3.5575	3.5518	3.5559	3.5578
90	0.05	0.30	13.9524	13.8956	14.0354	13.9453	13.9620
100			7.9444	7.9295	7.9625	7.9428	7.9505
110			4.0701	4.0991	4.0349	4.0752	4.0786
90	0.09		14.9828	14.9254	15.0691	14.9756	14.9928
100			8.8276	8.8019	8.8617	8.8244	8.8333
110			4.6949	4.7174	4.6662	4.6991	4.7027
90	0.15		16.5120	16.4586	16.5954	16.5054	16.5237
100			10.2087	10.1690	10.2667	10.2036	10.2141
110			5.7282	5.7377	5.7150	5.7306	5.7355
			Inside Bounds:	15	6	21	
			Outside Bounds:	21	30	15	

FIGURE 3.1:
This figure shows the results for Asian call prices obtained from Gram-Charlier series expansions around the log-Normal, GC-LN, the reciprocal-Gamma, GC-RG, and the Johnson-Type-I, GC-JI, distributions. The option parameters are the same as used in Table 1. We achieve a significant improvement in the precision of the prices, especially for low volatilities and/or short maturities. Much more prices are falling inside the lower and upper bounds, or at least they are moving towards the bounds.
For the log-Normal Gram-Charlier case the prices are in agreement with those calculated by Fusai and Tagliani [2002]. They show also prices calculated from a Gram-Charlier series expansion around the Normal distribution.

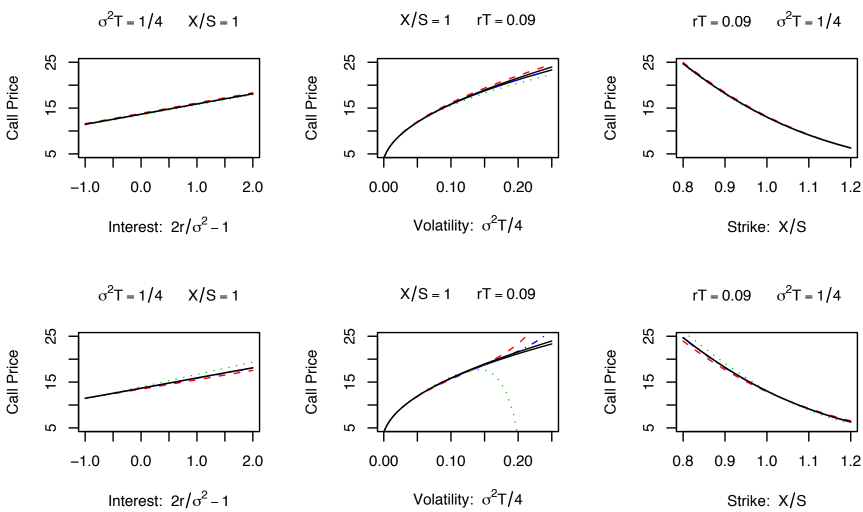

FIGURE 3.2:
The graphs show the call price deviations of the moment matched approximations improved by the Gram-Charlier series expansion. Prices are shown for the log-Normal GC-LN (dashed), the reciprocal-Gamma GC-RG (dotted), the Johnson-Type-I GC-JI (dashdotted) approaches, together with the lower, RST-LB (solid), and upper bounds, T-UB (solid), for various option parameter settings.
The first row of the graphs shows for at-the-money options the increasing call price deviations as a function of normalized interest rates $2r/\sigma^2 - 1$ for low, intermediate and high volatility scenarios. The graphs in the middle row show the variations of the call price deviations for fixed interest rate and high, intermediate and low volatility scenarios as a function of normalized moneyness X/S. Note, that prices for options which are high in-the-money are not very precisely evaluated by all three approximations. The graphs in the lower row show the deviations as function of the normalized volatility $\sigma^2 T/4$ for a fixed interest rate $rT = 0.09$ and options which are in-the-money ($X/S = 1.2$), at-the-money and out-of-the-money (0.8).
Here, Johnson Type I fits the option prices best, log-Normal and reciprocal-Gamma behave in most of the situations contrary, when one approximation overestimates the price, then the other one underestimates it, and vice versa.

3.3 R-FUNCTIONS

The function `GramCharlierAsianOption()`

The function `GramCharlierAsianOption()` allows to calculate Asian option prices from the Gram-Charlier series expansions for the log-Normal, the reciprocal-Gamma and the Johnson-Type-I distributions. The distributions can be selected by choosing one element from the functions argument `method=c("LN", "RG", "JI")`.

Chapter 4

Moments of Exponential Brownian Motion

In the following we briefly report the work of Dufresne [1989], of Abrahamson [2002], of Geman and Yor [1993], and of Schroeder [2002] concerned with the calculation of the moments of exponential Brownian motion and the computation of the full density.

4.1 Evaluation of Moments

The moments of the continuous arithmetic average $A_\tau^{(\nu)}$ for Asian options can be calculated analytically. The expression for the calculation of the moments can be found in Dufresne [1989, 1990] and Geman and Yor [1993]

$$\begin{aligned}\mathscr{M}_n &:= \mathbb{E}\left[(A_\tau^{(\nu)})^n\right] = \frac{n!}{2^{2n}} \sum_{j=0}^{n} d_j^{(\nu/2)} e^{2(j^2+j\nu)\tau} \,, \\ d_j^{(\beta)} &= 2^n \prod_{0\le i\ne j\le n} \frac{1}{(\beta+j)^2-(\beta+i)^2} \,,\end{aligned} \tag{4.1}$$

where $\nu = 2r/\sigma^2 - 1$.

Abrahamson [2002] derived expressions for the moments at an arbitrary number, N, of discrete times. The expressions are closed summations, which entail only the N-th powers of, and the successive differences between, the moments of the log-Normal finite dimensional distribution of the processŠ values at the time of the first averaging. By passing to the continuum limit he obtained the expression

$$\begin{aligned} \mathcal{M}_n &= n!\left[\sum_{j=1}^{n} \frac{e^{c_j \tau}}{c_j \prod_{i=1, i\neq m}^{n}(c_j - c_i)} + \frac{(-1)^n}{\prod_{i=1}^{n} c_i}\right], \\ c_j &= 2(j^2 + j\nu)\,. \end{aligned} \tag{4.2}$$

4.2 ASYMPTOTIC BEHAVIOR OF MOMENTS

An expression for the moments for $\nu = 0$ in the limit $n \to \infty$ was derived by Tolmatz [2001]. The moments behave like

$$\mathcal{M}_{n\to\infty}|_{\nu=0} \to \frac{2^{-(2n-1)}}{(2n-1)!!} e^{2n^2\tau}\,. \tag{4.3}$$

We add the expression for ν different from zero

$$\mathcal{M}_{n\to\infty} \to \frac{1}{2^n} \frac{\Gamma(\nu+n)}{\Gamma(\nu+2n)} e^{2(n^2+n\nu)\tau}\,. \tag{4.4}$$

Setting $\nu = 0$ we obtain equation (4.3).

4.3 PDE APPROACH TO COMPUTE MOMENTS

In addition it is worth to note that Little and Pant [2001] have presented an partial differential equation approach to calculate the moments of A_t for a wide class of functionals of an underlying asset. They have illustrated their method for Asian Options and have shown that it is highly accurate. Their method is not restricted to Asian options and can be useful because it provides an uniform framework to calculate moments for functionals for which there may be no analytical results.

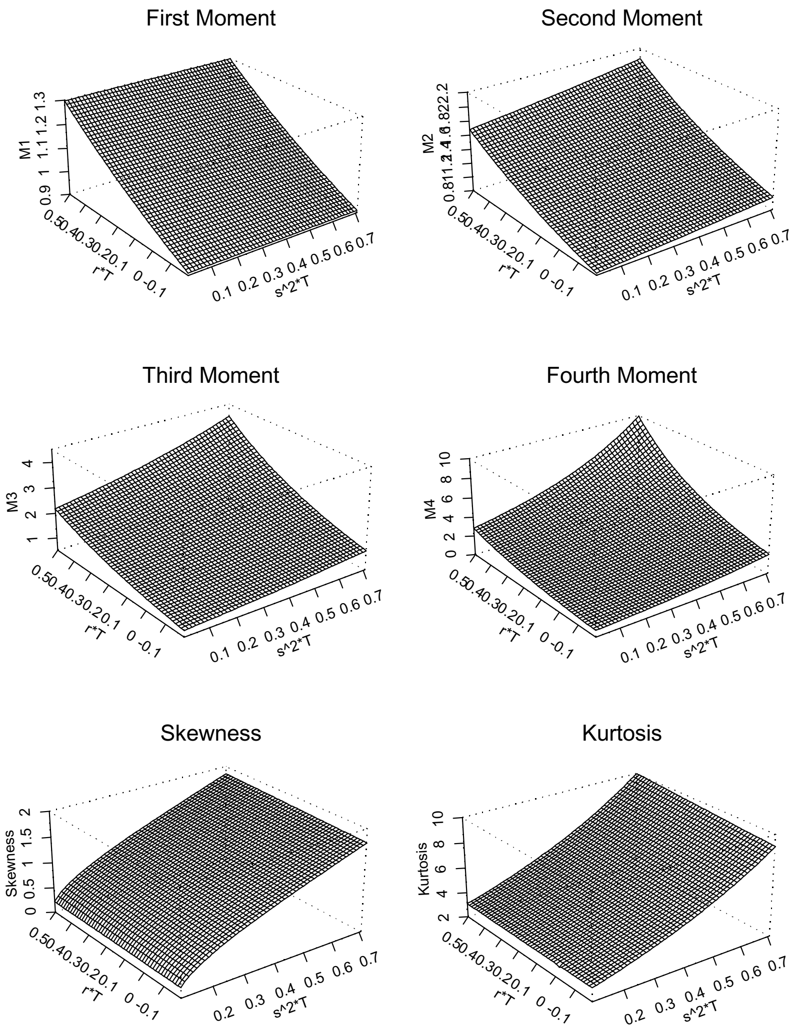

FIGURE 4.1:
First four moments, skewness and kurtosis as functions of rT and $\sigma^2\sqrt{T}$). In general, higher moments are growing fast with increasing time to maturity and/or for high interest rates and volatilities. Skewness and kurtosis are increasing with volatility, whereas they remain almost constant with varying interest rates.

4.4 R-FUNCTIONS

The function `AsianOptionMoments()` computes the moments of exponential Browning motion by different methods. The argument `method=c("D", "A", "TW", "T")` allows to select the desired method.

Selecting `method="D"` computes the moments of $A_t^{(\nu)}$ with the help of Dufresne's and Geman-Yor's formula, and `method="A"` uses the formula derived by Abrahamson. Furthermore, for `method="TW"` the function returns the first two moments as calculated by Turnbull and Wakeman. In addition `method="T"` calculates the asymptotics for large moments from the expression derived by Tolmatz together with the generalization to arbitrary ν as presented here.

Chapter 5

Evaluation of the Asian Density

Schroeder studied the law of any power of the integral of geometric Brownian motion over any finite time interval. As his main result he derived integral representations for this. To achieve this he enhanced the Laplace transform ansatz of Yor [1980] with complex analytic methods, which are the main methodological contributions of his paper. One of his integrals has a similar structure to that obtained in Yor [1980], while the other is in terms of Hermite functions as those of Dufresne [1999]. Performing or not performing a certain Girsanov transformation is identified as the source of these two forms of the laws.

5.1 Schroeder's Integral Representations

Guided by these concepts the Asian density can be either written as

$$\begin{aligned} f_{\mathscr{A}_t^{(\nu)}}(u) &= \frac{c_{\nu,t}(u)e^{\frac{\pi^2}{8t}}}{\pi}\int_0^\infty dx\, x^\nu e^{-x^2} \\ &\times \int_0^\infty dy\; e^{\frac{y^2}{2t}} cosh(y) cos(\frac{2x}{\sqrt{2u}}\sinh(y) - \frac{\pi y}{2t}) , \end{aligned} \tag{5.1}$$

$$\begin{aligned} f_{\mathscr{A}_t^{(\nu)}}(u) &= \frac{c_{\nu,t}(u)e^{\frac{\pi^2}{2t}}}{\pi}\int_0^\infty dx\, x^\nu e^{-x^2} \\ &\times \int_0^\infty dy\; e^{\frac{y^2}{2t}} \sinh(y)\sin(\frac{\pi y}{t})e^{\frac{2x}{\sqrt{2u}}\sinh(y)} , \end{aligned} \tag{5.2}$$

where the two formulas are specializations of a more general theorem. The pre-factor is given by

$$c_{\nu,t}(u) = \frac{2^{\frac{\nu}{2}} u^{\frac{\nu-3}{2}}}{\sqrt{\pi t}} \exp\left(-\frac{1}{2}(\nu^2 t + \frac{1}{u})\right) \tag{5.3}$$

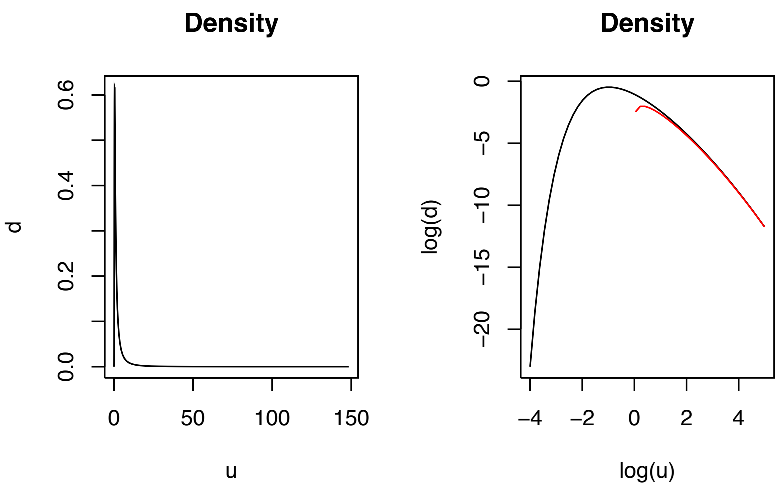

FIGURE 5.1:
Linear and double logarithmic plot of the density of $f_{\mathscr{A}_{\sqcup}^{(\nu)}}$ for $t = 1$. The thin curve displays the asymptotic behavior derived from Tolmatz' formula (5.6).

The second formula in equation (5.1) is seen to specialize to the double integral for the Asian density resulting from Yor's [1992] investigation. The second approach expresses the Asian density in form of integrals over Hermite functions. The result is

$$
\begin{aligned}
f_{\mathscr{A}_t^{(\nu)}}(u) &= \Gamma(\nu+1)\frac{c_{\nu,t}(u)e^{\frac{\pi^2}{8t}}}{\pi} \\
&\times \int_0^\infty dy\; e^{-\frac{y^2}{2t}} cosh(y) Re\left[e^{-\frac{\pi y i}{2t}} H_{-(\nu+1)}\left(-i\frac{\sinh(y)}{\sqrt{2u}}\right)\right]
\end{aligned}
\tag{5.4}
$$

$$
\begin{aligned}
f_{\mathscr{A}_t^{(\nu)}}(u) &= \Gamma[\nu+1]\frac{c_{\nu,t}(u)e^{\frac{\pi^2}{2t}}}{\pi} \\
&\times \int_0^\infty dy\; e^{-\frac{y^2}{2t}} \sinh(y)\sin(\frac{\pi y}{t}) H_{-(\nu+1)}\left(\frac{cosh(y)}{\sqrt{2u}}\right),
\end{aligned}
\tag{5.5}
$$

Note that the trigonometric function sin causes serious numerical problems because of the increasing oscillations of the integrand when t becomes larger. Unfortunately, the factor $\exp(\pi^2/2t)$ gets at the same time bigger. We have investigated the limits of small t's in this formulation and have compared our numerical results with those obtained by Gould [2002].

5.2 ASYMPTOTICS OF THE ASIAN DENSITY

Tolmatz [2002] has derived the asymptotic behavior of the density and probability functions of $\mathscr{A}_\tau(u)$ for large u. Let $f_{\mathscr{A}_\tau}(u)$ be the density function and $F_{\mathscr{A}_\tau}(u)$ be the probability function than for fixed $t > 0$ and $u \to +\infty$ we have

$$\begin{aligned} f_{\mathscr{A}_\tau}(u) &\sim \frac{\sqrt(\tau)e^{\tau/2}}{u\sqrt{u}} \frac{\exp(-\frac{\alpha_t \tau^2(u)}{8\tau})}{\alpha_t \tau(u)\Gamma(\frac{\alpha_\tau(u)}{4\tau})}\beta_\tau(u) \\ F_{\mathscr{A}_\tau}(u) &\sim 1 - 4\tau\sqrt{\tau}e^{\tau/2} \frac{\exp(-\frac{\alpha_t \tau^2(u)}{8\tau})}{\sqrt{u}\alpha_\tau^2(u)\Gamma(\frac{\alpha_\tau(u)}{4\tau})}\beta_\tau(u)(1+o(1)) \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} \alpha_\tau(u) &:= ln(8ue^{-2\tau}), \\ \beta_\tau(u) &:= \exp\left(-\frac{1}{8\tau} ln^2 \frac{\alpha_\tau(u)}{4\tau}\right). \end{aligned} \tag{5.7}$$

In addition the logarithmic asymptotics of the tail distribution $1 - F_{\mathscr{A}_\tau}(u)$ can be written in the form

$$-ln(1 - F_{\mathscr{A}_t \tau}(u)) \sim \frac{1}{8\tau} ln^2 u . \tag{5.8}$$

as $u \to +\infty$.

5.3 R-FUNCTIONS

The function `Schroeder[12]AsianDensity()`

The functions `Schroeder2AsianDensity()` and `Schroeder1AsianDensity` implement the single "1" and double "2" integrated densities as given by equations (5.4) and (5.3), respectively.

The function `Yor[12]AsianDensity()`

`Yor1AsianDensity()` implements the density for $\nu = 4$ along the formulas given in Yor [1980]. `Yor2AsianDensity` reexpresses Yor's $\nu = 0$ result, but now we have performed the outer integration over x analytically, yielding a single integral over y, where the integrand is expressed in form of the normal density and normal probability functions with sinh() and cosh() related arguments. We noticed that this expression can also be obtained from the Hermite formula for $\nu = 0$ using the relationship $2/\sqrt{\pi}H_{-1}(z) = \exp(z^2)Erfc(z)$.

The function `Tolmatz[12]AsianDensity()`

The functions `TolmatzAsianDensity()` and `TolmatzAsianProbability()` implement the density and probability function of the asymptotic behavior for $\nu = 0$, respectively.

Chapter 6

Bounds on Option Prices

Several publications consider the problem of finding lower and upper bounds on Asian call prices. This includes the early work of Curran [1992], Rogers and Shi [1995] and the most recent work of Thompson [2002].

6.1 Lower Bounds

Curran and Thompson's Approximation: Curran [1992] has developed a discrete approximation based on the geometric conditioning approach for Asian options. Thompson [2001] derived the continuous time limit as an approximation to his lower bound. This lower bound can be calculated from

$$c^{CT} = e^{-rT}\left[\int_0^\infty Se^{\alpha t+\sigma^2 t/2}\Psi\left(\frac{-\gamma^* + \sigma t(1-t/2)}{1/\sqrt{3}}\right)dt - X\Psi\left(\frac{-\gamma^*}{1/\sqrt{3}}\right)dt\right] \tag{6.1}$$

with $\alpha = r - \sigma^2/2$. γ^* is given by

$$\gamma^* = \left[log\left(\frac{2X}{S} - \int_0^\infty e^{3(log(X/S)-\alpha/2)t(1-t/2)+\alpha t+\frac{1}{2}\sigma^2(t-3t^2(1-t/2)^2)}dt\right) - \frac{\alpha}{2}\right] \tag{6.2}$$

Rogers-Shi-Thompson's Lower Bound

Rogers and Shi [1995] derived a lower bound for Asian calls in form of a double integral expression. Thompson [2001] simplified Rogers and Shi's result, now writing it as a one dimensional integral expression rather than in form of a more numerically intensive two-dimensional integral. The lower bound can be calculated from

$$c^{RST} = e^{-rT} S \tag{6.3}$$

$$\left[\int_0^\infty e^{\alpha t+\sigma^2 t/2}\Phi\left(\frac{-\gamma^\star+\sigma t(1-t/2)}{1/\sqrt{3}}\right)dt - \frac{X}{S}\Phi\left(\frac{-\gamma^\star}{1/\sqrt{3}}\right)dt\right] \tag{6.4}$$

which has exactly the same form as c^{CT} but now γ^* replaced by $\gamma^\star$, which has to be determined from the implicit relationship

$$\int_0^\infty e^{3\gamma^\star\sigma t(1-t/2)+\alpha t+\frac{1}{2}\sigma^2(t-3t^2(1-t/2)^2)}dt = \frac{X}{S}\,. \tag{6.5}$$

6.2 Upper Bounds

Two methods deal with upper bounds, the derivation of Rogers and Shi [1995] and the alternative one of Thompson [2001] which yields a more precise expression.

Roger and Shi's Upper Bound

The derivation of Rogers and Shi [1995] on the lower bound of Asian call prices leads to the inequality

$$c^{RS} \le e^{-rT} S \int_{-\infty}^\infty \sqrt{3}\phi(\sqrt{3}z)\left[\int_0^\infty e^{3\sigma t(1-t/2)z+\alpha t+\frac{1}{2}\sigma^2(t-3t^2(1-t/2)^2)}dt - \frac{X}{S}\right]^+ dz\,. \tag{6.6}$$

This formula is not easy to calculate since the outer integration has a non smooth integrand.

Thompson's Upper Bounds

Thompson [2001] derived a formula which is much better suited for numerical issues

$$c^T \le e^{-rT}\int_0^\infty\int_{-\infty}^\infty 2v\phi(w)\left[a(t,x)\Phi\left(\frac{a(t,x)}{b(t,x)}\right)+b(t,x)\phi\left(\frac{a(t,x)}{b(t,x)}\right)\right]dw\,dv\,. \tag{6.7}$$

where the functions $a(t,x)$ and $b(t,x)$ are given by

$$\begin{aligned} a(t,x) &= S\exp(\sigma x+\alpha t) - X(\mu_t+\sigma x) + X\sigma(1-t/2)x\,, \\ b(t,x) &= X\sigma\sqrt{\frac{1}{3}-t(1-t/2)^2}\,. \end{aligned} \tag{6.8}$$

μ_t can be deduced from

$$\mu_t = \frac{1}{X}(S\exp(\alpha t) + \gamma\sqrt{v_t}) , \tag{6.9}$$

where

$$\begin{aligned} v_t &= c_t^2 + 2(X\sigma)c_t t(1-t/2) + (X\sigma)^2/3 , \\ c_t &= S\exp(\alpha t)\sigma - X\sigma , \\ \gamma &= \frac{X - S(e^\alpha - 1)/\alpha}{\int_0^1 \sqrt{v_t}dt} . \end{aligned} \tag{6.10}$$

These equations constitute Thompson's upper bound for Asian calls.
In Table 1 and table 2 we have already compared the lower bounded call prices with call prices obtained from moment matched log-Normal, reciprocal-Gamma and Johnson-Type-I state price densities and with those obtained from the Gram-Charlier series expansion. The figures in these tables were compute from equations (6.3) and (6.7).
Note, that for low volatilities and/or short maturities $\sigma^2 T < 0.01$ the lower bound price c^{RST} is very close to the exact Asian call price and can be considered for most practical applications as accurate enough.

6.3 R-FUNCTIONS

The function `CurranThompsonLowerBound()`

The function `CurranThompsonLowerBound()` calculates lower bounds for Asian calls from equation (6.1).

The function `RogerShiThompsonLowerBound()`

The function `RogerShiThompsonLowerBound()` calculates lower bounds for Asian calls from equation (6.3).

The function `ThompsonUpperBound()`

The function `ThompsonUpperBound()` allows to calculate an upper price bound for Asian calls according to equation (6.7).
In all three cases, values for put options can be derived via the call-put-parity relationship. Additionally we have implemented a common function called `BoundOnAsianOption()` with an argument `method=c("CT, "RST", "T")` which allows to select the desired method for calculating the bound.

Chapter 7

Partial Differential Equation Approach

For an exact solution, several numerical approaches are available. One of the most natural ones is to solve directly the partial partial differential equation, PDE. This is a widespread approach. In general, as shown by Ingersoll [1987], the price of an Asian option can be found by solving a 2-dimensional PDE. With $A_T = \int_0^T S(u)du$ the value of an Asian option is given by the following PDE in the two space dimensions S and A

$$u_t + \frac{1}{2}\sigma^2 S^2 u_{SS} + rSu_S + Su_I A - ru = 0\,. \tag{7.1}$$

Unfortunately, this PDE is prone to oscillatory solutions. Alternatively, an equivalent formulation can be given in terms of the average $\hat{A}_T = A_T/T$ rather than A_T. Following Barraquand and Pudet [1996] we have

$$u_t + \frac{1}{2}\sigma^2 S^2 u_{SS} + rSu_S + \frac{1}{t}(S-\hat{A})u_{\hat{A}} - ru = 0\,. \tag{7.2}$$

Note, that the problem is formulated as a degenerate advection-diffusion PDE. Barraquand and Pudet have implemented a forward shooting grid finite difference scheme to provide numerical approximations.
Methods for solving two-dimensional PDEs were extensively investigated by Zvan, Forsyth and Vetzal [1997]. Their review paper can serve as a reference for solving two-dimensional PDEs. Alziary, Décamps, and Koehl [1997], provided a generalized PDE approach using an explicit finite difference scheme. Moreover, they include an analysis of the deltas and gammas used to hedge these instruments. Dewynne and Wilmott [19XX] formulated the pricing model as a linear complementary problem and provided a detailed description of the weighted finite difference method used to obtain numerical solutions.
Note, that a reduction to a one-dimensional PDE would result in a great advantage from the numerical point of view. Rogers and Shi [1995] were the first who have formulated a one-dimensional PDE to valuate Asian

options. They reduced the dimension of the problem by dividing $X - S(t)$ by the stock price $S(t)$. Unfortunately, their one-dimensional PDE is difficult to solve numerically since the diffusion term is very small for values of interest on the finite difference grid.

7.1 Zhang's one-dimensional PDE

A major step forward was done by the work of Zhang [2001]. He presented a semi-analytical method to price Asian options. He derived an analytical approximation formula by using a singularity removal technique, and the correction to this approximate formula is shown to be governed by a one-dimensional PDE.
The approximate call value reads

$$c^Z = \frac{S}{T}\left[-\xi\Phi\left(-\frac{\xi}{\sqrt{2\eta}}\right) + \sqrt{\left(\frac{\eta}{\pi}\right)}e^{-\frac{\xi^2}{4\eta}}\right] + u(\xi,\tau) \tag{7.3}$$

with $\tau = T - t$ and

$$\begin{aligned} \xi &= \frac{TX - I}{S}e^{-r\tau} - \frac{1}{r}(1 - e^{-r\tau})\,, \\ \eta &= \frac{\sigma^2}{4r^3}(-3 + 2r\tau + 4e^{-r\tau} - e^{-2r\tau})\,. \end{aligned} \tag{7.4}$$

The correction $u(\xi,\tau)$ to this value satisfies the following PDE

$$\begin{aligned} u_\tau - \frac{1}{2}\sigma^2\left[\xi + \frac{1}{r}(1 - e^{-r\tau})\right]^2 u_{\xi\xi} &= \frac{\sigma^2\xi}{4\sqrt{\pi\eta}}e^{-\frac{\xi^2}{4\eta}}\left[\xi + \frac{1}{r}(1 - e^{-r\tau})\right]\,, \\ u(\xi,\tau = 0) &= 0 \end{aligned} \tag{7.5}$$

This PDE is an inhomogeneous linear diffusion equation with a variable coefficient.

7.2 Vecer's one-dimensional PDE

Vecer [2001] derived a simple one-dimensional PDE by a similar space reduction as used by Rogers and Shi [1995] based on the observation that the arithmetic average Asian option, for both floating and fixed strike), is a special case of an option on a traded account. As discussed by Shreve and Vecer [2000] options on a traded account generalize the concept of many particular options like passport, European, American, vacation options and the same pricing techniques could be applied to price the Asian option. The PDE and its boundary condition can be written as

$$
\begin{aligned}
u_t &= r(z-q_t)u_z - (z-q_t)^2\sigma^2 u_{zz}\,, && (7.6)\\
u(T,z) &= z^+\,. && (7.7)
\end{aligned}
$$

The price of the Asian option is then given in terms of u by

$$c^V = S\,u(0,\frac{X}{S})\,. \tag{7.8}$$

7.3 ONE-DIMENSIONAL PDE SOLVERS

Zhang used the Crank-Nicholson Scheme to solve the one-dimensional PDE. We used the standard one-dimensional PDE solver "PDECOL", a Fortran program developed and written by Madson and Sincovic [1979]. The software code is available from the ACM library, TOMS algorithm 540.

Crank-Nicholson Scheme

To find the solution of the inhomogeneous linear diffusion equation with variable coefficient

$$f_\tau - c(\xi,\tau)f_{\xi\xi} = R(\xi,\tau) \tag{7.9}$$

Zhang used in his numerical investigations the Crank-Nicholson scheme, an implicit scheme which reads

$$\frac{u_i^{n+1}-u_i^n}{\Delta\tau} - c_i^{n+1/2}\left(\frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{2(\Delta x)^2} + \frac{u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}}{2(\Delta x)^2}\right) = R_i^{n+1/2}\,. \tag{7.10}$$

A rearrangement yields the algorithm

$$
\begin{aligned}
-\frac{1}{2}s_i^{n+1/2}u_{i-1}^{n+1} + (1+s_i^{n+1/2})u_i^{n+1} - \frac{1}{2}s_i^{n+1/2}u_{i+1}^{n} = \\
\frac{1}{2}s_i^{n+1/2}u_{i-1}^{n} + (1-s_i^{n+1/2})u_i^{n} + \frac{1}{2}s_i^{n+1/2}u_{i+1}^{n} + \Delta\tau R_i^{n+1/2}\,,
\end{aligned} \tag{7.11}
$$

where the discretization parameter is $s_i^{n+1/2} = c_i^{n+1/2}\Delta\tau/(\Delta x)^2$. To obtain the value of f at the n-th time level, one needs to solve a $I\times I$ tridiagonal system of equations which can be straightforward achieved by a standard algorithm. Finally we are left with some technical issues. In numerical computation we have to truncate the domain to be finite. Since the right hand side of equation () goes for large ξ to zero as $\xi^2\exp[-\xi^2/(4\eta)]$.

Therefore we can derive a boundary value ξ_m from $\exp(-\xi^2/(4\eta)) = \epsilon$, ($\epsilon \approx 10^{-16}$), where η is evaluated at $\tau = T$. From the definition of η we have for small *r*

$$\begin{aligned} \eta &\approx \frac{1}{6}\sigma^2\tau^3 , \\ \xi_m &\approx \sqrt{64 l n 10\eta} \approx 5\sigma T^{3/2} .. \end{aligned} \tag{7.12}$$

Thus in the numerical computation the spatial domain is bounded by $[-\xi_m, \xi_m]$. For the boundary conditions Zhang tested that all three choices, Derichlet $u = 0$, Neumann $u_\xi = 0$, or $u_{\xi\xi}$, gave indifferent results in agreement with the observation that the solution of u vanishes rapidly as ξ increases while the spatial boundary is large enough to ensure that u, u_ξ, or $u\_\xi\xi$ vanish. The influence of grid size was also investigated by Zhang. For $\Delta\tau$ of about 10^{-3} he achieved an accuracy of about 10^{-7}. This approach can be implemented for numerical investigations in a straightforward way.

General Collocation Algorithm

Here we used the general collocation software PDECOL for PDE's. PDECOL implements finite element collocation methods based on piecewise polynomials for the spatial discretization techniques. The time integration process is then accomplished by the "numerical method of lines" for the solution of non-linear partial differential equations. PDECOL solves

$$u_\tau = \{(\tau, x, u, u_x, u_{xx}) . \tag{7.13}$$

Depending on the particular type of equation the boundary conditions have to be supplied.
We have investigated the Asian option prices for a wide range of option parameters. The results are presented in Table 3.

Strike (K)	Volatility (σ)	ADK–PDE	RS–PDE	AA	AAc
95	0.05	8.8107	8.823	8.8091638	8.8088392
100	0.05	4.3176	4.185	4.3173467	4.3082350
105	0.05	1.0104	1.011	0.9561651	0.9583841
90	0.1		13.382	13.3935810	13.3851974
95	0.1	8.9195		8.9453288	8.9118509
100	0.1	4.9486	4.887	4.9597032	4.9151167
105	0.1	2.1209		2.0660011	2.0700634
110	0.1		0.659	0.5840749	0.6302713
90	0.2		13.825	14.0072426	13.8314996
95	0.2	10.0126		10.1775718	9.9956567
100	0.2	6.8116	6.773	6.8915575	6.7773481
105	0.2	4.3298		4.2915208	4.2964626
110	0.2		2.551	2.4269417	2.5462209
90	0.3		14.981	15.3963998	14.9839595
95	0.3	11.6803		11.9899632	11.6558858
100	0.3	8.8509	8.827	9.0147076	8.8287588
105	0.3	6.5504		6.5183423	6.5177905
110	0.3		4.698	4.5161603	4.6967089
95	0.4	13.5293		13.9983130	13.5107083
100	0.4	10.9514		11.1893414	10.9237708
105	0.4	8.7612		8.7454895	8.7299362
95	0.5	15.4746		16.0914792	15.4427163
100	0.5	13.0626		13.3849520	13.0281555
105	0.5	11.0021		10.9727670	10.9296247

FIGURE 7.1:
Comparison of PDE solutions. $S = 100$, $r = 0.09$, and $T = 1$. ADKŰPDE denotes values of PDE of Alziary, Decamps and Koehl [1997]; RSŰPDE denotes values of PDE of Roger and Shi [1995]; AAc denotes PDE values from Zhang [2001]. Missing values are due to lack of original data. Source Zhang [2001].

7.4 R-FUNCTIONS

We have implemented the Madsen-Sincovec PDE solver for the solution of Zhangs's and Vecer's one dimensional partial differential equations.

The functions `ZhangAsianOption()` and `VecerAsianOption()` solve the PDE's defined by equations (7.5) and (7.6) using the PDE solver of Madsen and Sincovec. The implementation of Vecer's PDE was done for call and put options. So one calculate for both call and put options the prices through the PDE and can compare them by the call-put parity relationship.

The function `CallPutParityAsianOption()` valuates puts when the call prices are available and calls when the put prices are available.

CHAPTER 8

LAPLACE INVERSION

Geman and Yor [1993] computed an expression for the Laplace transform of the Asian option price. Thus numerical inversion remains to obtain the price. Unfortunately the result has a quite complicated structure from which one has to determine the final price by a nontrivial Laplace inversion. Geman and Eydeland [1995] attempted to invert the Geman-Yor expression numerically using a fast Fourier transform. Fu, Madan, and Wang [1998] and Craddock, Heath, and Platen [2000] used numerical Laplace inversion methods on the same expression, although both report on problems for short maturities and/or small volatilities. In addition Sudler [1999] valuated Asian options making use of the relationship to Fourier transforms.

8.1 GEMAN AND YOR'S FORMALISM

According to Geman and Yor [1993] the price of an Asian Call option can be written as

$$e^{-rT}\mathbb{E}[(\mathscr{A}_T - X)^+] = e^{-rT}\left(\frac{4S}{\sigma^2 T}\right) C^{(\nu)}(h,q)\,, \tag{8.1}$$

where, in the notation of Geman and Yor,

$$\nu = \frac{2r}{\sigma^2} - 1\,,\ \ h = \frac{\sigma^2 T}{4}\,,\ \ q = \frac{\sigma^2 T}{4}\frac{X}{S}\,. \tag{8.2}$$

The Laplace transform of $C^{(\nu)}(h,q)$ in the first parameter h is given by

$$\hat{C}(\lambda,q) = \int_0^\infty e^{-\lambda h} C^{(\nu)}(h,q)dh = \frac{\int_0^{1/(2q)} e^{-x} x^{(\mu-\nu)/2-2}(1-2qx)^{(\mu+\nu)/2+1}dx}{\lambda(\lambda-2-2\nu)\Gamma((\mu-\nu)/2-1)}\,, \tag{8.3}$$

with $\mu = \sqrt{2\lambda+\nu^2}$.

8.2 Numerical Laplace Inversion

Geman and Yor [1993] notice, that the inversion of this Laplace transform for a fixed h is not easy, since the parameter λ also appears in the Gamma function. They remark that "there is no of-the-shelf software for inverting Laplace transforms, which may be useful for a numerical solution of this problem." Public available software prior to 1993 included the TOMS 619 package of Piessens and Huysens [1984], the TOMS 662 package of Garbow, Giunta, Lyness and Murli [1988], and the TOMS 682 package by Murli and Rizzardi [1990]. Fu, Madan and Wang [1997] also claim that a naive attempt at inverting this transform may lead to erroneous results. They employed the methods of Euler and Post-Widder, as proposed by Abate and Whitt [1995].
A more recent Laplace inversion method, going back to D'Amore, Laccetti, and Murli [1999a, 1999b], seems to us better suited for solving the problems with the Laplace inversion. Their method is based on a Fourier series expansion of the inverse transform. They also analyzed the corresponding discretization error and demonstrated how the expansion can be used in the development of an "automatic routine", one in which the user needs to specify only the required accuracy. This is a big advantage in comparison to the Euler and Post-Widder implementations, where several convergence parameters have to be specified.

8.3 Equivalence to the Fourier Transform

In the spirit of this approach, and following Sudler [1999], we express the Laplace transform of $C^{(\nu)}(h,q)$ by Kummer's confluent hypergeometric function $M(a,b,z)$, i.e.

$$C^{(\nu)}(\lambda,q)=\frac{(\frac{1}{2q})^a}{\lambda(\lambda-2-2\nu)}\frac{\Gamma(b-a)}{\Gamma(b)}M\left(\frac{1}{2}(\mu-\nu),\mu,-\frac{1}{2q}\right)\,. \tag{8.4}$$

Note, that the indexes of the Kummer function M are functions of λ. In order to make the function regular in the upper-half complex plane, we introduce the function $\widetilde{f}(h,q)=e^{-\alpha h}f(h,q)$, where $\alpha=2(1+\nu)/\sigma^2$. Then $\widetilde{C}(\lambda,q)=C(\alpha+i\lambda,q)$ and the inverse Fourier Transform is given by

$$\widetilde{f}(h,q)=\frac{1}{2pi}\int_{-\infty}^{\infty}e^{i\lambda h}\widetilde{C}(i\lambda,q)d\lambda\,. \tag{8.5}$$

Hence we finally arrive at

$$C^{\nu}(h,q)=e^{\alpha h}\widetilde{f}(h,q)\,, \tag{8.6}$$

which allows us to derive the call price via equation (8.1).

8.4 COMPLEX GAMMA AND CONFLUENT HYPERGEOMETRIC FUNCTIONS

For the evaluation of the Asian option the “Gamma function” $\Gamma(z)$ with complex argument z and the “log-Gamma function” $\ln\Gamma(z)$ for “extreme” values are needed. For an implementation one has to take care to the complicated branch cut structure inherited from the logarithm function. In addition the confluent hypergeometric functions $M(a,b,z)$ and $U(a,b,z)$ with complex argument z and indexes a and b are required. These are degenerate forms of the hypergeometric function which arise as a solution of the “confluent hypergeometric differential equation” $xy''+(c-x)y'-ay=0$. The functions are also called Kummer's function $M(a,b,z)$ of the first and $U(a,b,z)$ of the second kind. These functions are given by

$$
\begin{aligned}
M(a,b,z) &= \frac{\Gamma(b)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)z^n}{\Gamma(b+n)n!}\,, \\
U(a,b,z) &= \frac{\pi}{sin(\pi b)} \left[\frac{M(a,b,z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b} \frac{M(1+a-b,2-b,z)}{\Gamma(a)\Gamma(2-b)} \right].
\end{aligned}
\tag{8.7}
$$

Since the complex Gamma and Kummer functions are not available in R's base package we have implemented them from Fortran routines. It's worth to note, that Kummer's functions were implemented from the TOMS 707 algorithm of Nardin, Berger and Bhalla [1992]. Their Fortran routines are suited for complex arguments and complex indexes with large magnitudes. In the case of small volatilities and/short times to maturity it is very important to prevent the calculations from numerical overflows and thus the Kummer functions must also support the direct calculation of their logarithms. The algorithm makes use of a direct summation of the Kummer series. As further sources to confluent hypergeometric functions and their relationships to other complex functions, we refer to the textbooks of Abramowitz and Stegun [1972], and Slater [1960], as well as to Weinstein's [2003] MathWorld web resource.

Before we valuate Asian options by Laplace inversion, we take a closer look at the function which has to be integrated. For low volatilities and/or short times to maturity the function values get extremely small and oscillates with a very slow decaying slope. Note, that the real parts of the indexes of Kummer's function $\mathscr{R}(a)$ and $\mathscr{R}(b)$ can take values larger than 1000, which mean that $M(a,b,z)$ takes on extremely large values. These facts limit the evaluation of Asian call prices to values $\sigma^2 T \gtrsim 0.002$.

Table 4 lists the results for Asian call prices obtained from our numerical implementation in comparison with some results we found in the literature.

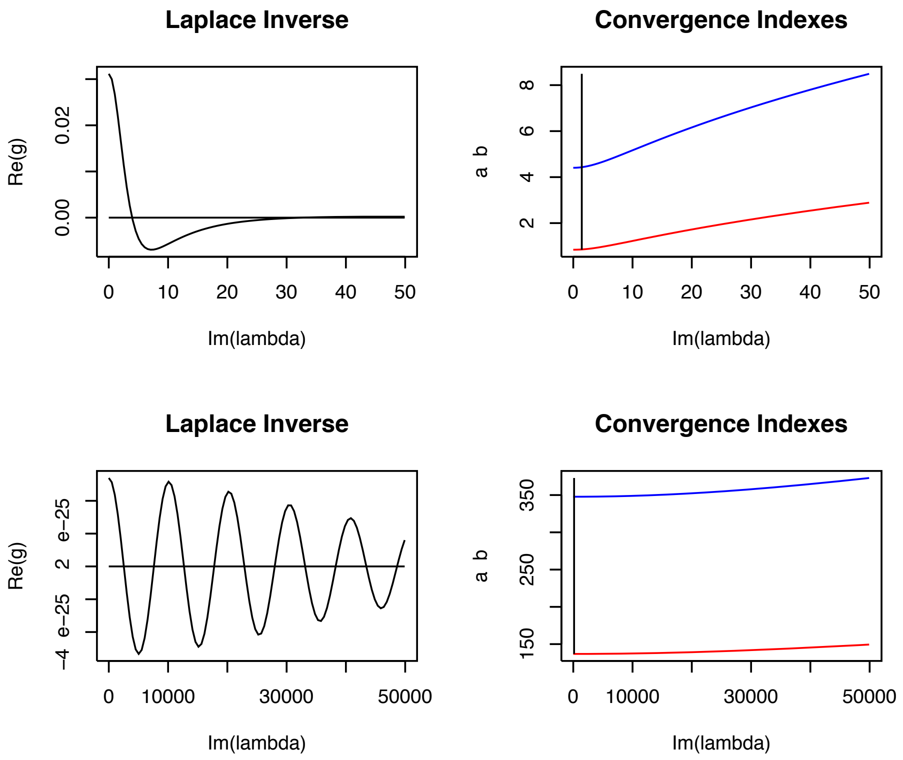

FIGURE 8.1:
The figures to the left show the Real part of the function g to be Laplace inverted for two settings, "high volatility - long maturity" and "low volatility - short maturity", i.e. $\sigma^2 T = 0.25$ and 0.0025, respectively. The figures to the right display the "big numbers" for the real part of the two indexes a and b, appearing in Kummer's confluent hypergeometric function.

X S=100	T σ=0.02	Euler FMW	CPU Sparc-10	PostWidder FMW	CPU Sparc-10	FT our Results	CPU Pentium4
90	0.4	11.5293	53	11.5176	640	11.5249	14
95		7.2131	44	7.1981	631	7.2106	11
100		3.8087	42	3.8196	628	3.8073	11
105		1.6465	45	1.6623	623	1.6459	11
110		0.5761	38	0.5728	625	0.5759	11
90	0.5	11.9247	34	11.9241	613	11.9251	10
95		7.7249	33	7.7185	591	7.7252	9
100		4.3696	33	4.3759	601	4.3698	9
105		2.1175	32	2.1290	585	2.1176	8
110		0.8734	32	0.8753	595	0.8735	9
90	1.0	13.8372	26	13.8439	518	13.8315 *	5
95		9.9998	26	10.0029	513	9.9957 *	6
100		6.7801	26	6.7823	514	6.7773 *	5
105		4.2982	25	4.3010	503	4.2965 *	5
110		2.5473	25	2.5450	503	2.5462 *	5
90	2.0	17.1212	24	17.1297	475	17.0988	3
95		13.6763	23	13.6830	473	13.6584	4
100		10.6319	23	10.6370	473	10.6180	4
105		8.0436	23	8.0474	469	8.0331	3
110		5.9267	22	5.9295	474	5.9189	4
90	3.0	19.8398	21	19.8495	468	19.7960	3
95		16.6740	21	16.6822	467	16.6372	4
100		13.7974	21	13.8042	467	13.7669	3
105		11.2447	20	11.2502	462	11.2199	3
110		9.0316	20	9.0360	465	9.0116	4
90	5.0	24.0861	19	24.0861	453	23.9780	3
95		21.3774	22	21.3774	442	21.2814	4
100		18.8399	19	18.8399	449	18.7553	3
105		16.4917	21	16.4917	441	16.4176	4
110		14.3442	21	14.3442	453	14.2798	3

FIGURE 8.2:
Comparison of numerical techniques for pricing continuous Asian options by Laplace inversion and direct Fourier transformation: The table summarizes results from the Euler and Post-Widder approach with settings recommended by Abate and Whitt as calculated by Fu, Madan and Wang [1997] and our results obtained by direct Fourier transformation. Note, for $T=1$ our results market by the star are identical to those presend by Zhang []. The values obtained by the Euler and Post-Whitter approach overestimate for all applied parameters the call prices, in same cases significantly.

8.5 R-FUNCTIONS

The function `GemanYorAsianOption()`

The function `GemanYorAsianOption()` valuates Asian options from the Laplace transform via the Fourier transform and `GemanYorFun()` calculates the function itself to be transformed. For the Fourier Transform the R-function `integrate()` from R's base package is used, which allows for an adaptive quadrature of functions. This function uses routines written by Piessens and deDoncker-Kapenga [1983].

The function `Gamma()`

The function `Gamma()` allows to calculate the complex Gamma function.

The functions `kummerM()` and `kummerM()`

The functions `kummerM()` and RfunkummerU allow to calculate the Kummer functions of the first and second kind with complex argument and complex indexes.

CHAPTER 9

SPECTRAL EXPANSION

Linetzky [2002a] derived two analytical formulas for the valuation of Asian options. He expressed Asian put values

$$e^{-rT} E[(X - \mathscr{A}_T)^+] = e^{-rT} \left(\frac{4S}{\sigma^2 T} \right) P_{\diamond}^{(\nu)}(\chi, \tau) , \tag{9.1}$$

in terms of spectral expansions $P_b^{(\nu)}(\chi, \tau)$ and $P^{(\nu)}(\chi, \tau)$ associated with the diffusion infinitesimal generator. The "diamond" index denotes that there exist two representations for the spectral expansions. The first formula for $P_b^{(\nu)}(\chi, \tau)$ is an infinite series of terms involving Whittaker functions M and W. The second formula for $P^{(\nu)}(\chi, \tau)$ is a single real integral of an expression involving the Whittaker function $W_{\kappa,\mu}(z)$ plus (for some parameter values) a finite number of additional terms involving incomplete Gamma functions $\Gamma(a, z)$ and Laguerre polynomials.
The underlying spectral theory of one-dimensional diffusion is described by Linetzky [2002b] and goes back to McKean [1956] and Ito and McKean [1974].

9.1 SPECTRAL SERIES REPRESENTATION

Linetzky's Proposition 1: For real ν, real and positive τ, k, and b the numbers $p_{i,b}$ are the ordered positive simple zeros, $0 < p_{1,b} < p_{2,b} < \cdots < p_{n,b} \uparrow \infty$ as $n \uparrow \infty$, of the Whittaker function $W_{\kappa,\mu}(z)$ with the fixed argument $z = \frac{1}{2b}$, and the purely imaginary second index $\mu = \frac{ip}{2}$, where $p > 0$. Note, that the Whittaker function is a real function of p since $W_{\kappa,\mu}(z) = W_{\kappa,-\mu}(z)$

$$W_{\frac{1-\nu}{2}, \frac{ip}{2}}(\frac{1}{2b}) = 0 . \tag{9.2}$$

We introduce the notation

$$\xi_{n,b}^{(\nu)} := \frac{\partial}{\partial p} W_{\frac{1-\nu}{2}, \frac{ip}{2}}(\frac{1}{2b})|_{p=p_{n,b}} . \tag{9.3}$$

For $v > 0$, let $0 \leq N_v(b) \leq [|(v)|/2] + 1$, where $[\cdot]$ denotes the integer part of "$\cdot$", be the total number of roots of the equation

$$W_{\frac{1-v}{2}, \frac{q}{2}}(\frac{1}{2b}) = 0 \tag{9.4}$$

in the interval $q \in [0, |v|)$ and for $N_v(b) > 0$, let $\{q_{n,b},\ n = 1, \dots, N_v(b)\}$ be the corresponding roots ordered in descending order, $0 \leq q_{N_{v(b)},b} < \cdots < q_{1,b} < |v|$. Furthermore, we introduce the notation

$$\eta^{(v)}_{n,b} := \frac{\partial}{\partial q} W_{\frac{1-v}{2}, \frac{q}{2}} \left(\frac{1}{2b} \right) |_{q = q_{n,b}} . \tag{9.5}$$

Note, that for $v \geq 0$ equation () has no roots in the interval $q \in [0, v)$ and $N_v(b) \equiv 0$. Then the function $P_b^{(v)(k,\tau)}$ is represented by the series

$$\begin{aligned} & P_b^{(v)(k,\tau)} \\ = & \sum_{n=1}^{\infty} e^{-\frac{(v^2 + p^2_{n,b})\tau}{2}} \frac{p_{n,b} \Gamma(\frac{v + ipn,b}{2})}{4 \eta_{n,b}(v) \Gamma(1 + i p_{n,b})} (\frac{1}{2k})^{-\frac{v+3}{2}} e^{-\frac{1}{2k}} W_{-\frac{v+3}{2}, \frac{ip_{n,b}}{2}} \left(\frac{1}{2k} \right) M_{-\frac{1-v}{2}, \frac{ip_{n,b}}{2}} \left(\frac{1}{2b} \right) \\ + & \sum_{n=1}^{N_v(b)} e^{-\frac{(v^2 + -q^2_{n,b})\tau}{2}} \frac{q_{n,b} \Gamma(\frac{v + qn,b}{2})}{4 \eta^{(v)}_{n,b} \Gamma(1 + q_{n,b})} (\frac{1}{2k})^{-\frac{v+3}{2}} e^{-\frac{1}{2k}} W_{-\frac{v+3}{2}, \frac{q_{n,b}}{2}} \left(\frac{1}{2k} \right) M_{-\frac{1-v}{2}, \frac{q_{n,b}}{2}} \left(\frac{1}{2b} \right) \end{aligned} \tag{9.6}$$

Now we are left to find the roots p and q which has to be done numerically. Although there exists an infinite sequence of roots the higher ones are suppressed with the factors $e^{-p^2_{n,b}\tau/2}$.

9.2 SPECTRAL INTEGRAL REPRESENTATION

In the limit $b \uparrow \infty$ the series expansion converges to an integral representation

$$P^{(v)}(\chi, \tau) = \tag{9.7}$$

$$\frac{1}{8\pi^2} \int_0^{\infty} e^{-\frac{(v^2 + p^2)\tau}{2}} (\frac{1}{2\chi})^{-\frac{v+3}{2}} e^{-\frac{z}{2}} W_{-\frac{v+3}{2}, \frac{ip}{2}} (\frac{1}{2\chi}) |\Gamma(\frac{v + ip}{2})|^2 sinh(\pi p) p dp + \dots \tag{9.8}$$

where the indicator function $1_{\{v<0\}}$ takes the value 1 if $v < 0$ and zero otherwise, $v \geq 0$.

It is worth to note, that Linetzky also derived an expression to calculate the Asian density $f_{\mathscr{A}_\tau^{(v)}}$ from the spectral representations.

9.3 WHITTAKER AND RELATED FUNCTIONS

The Whittaker functions $M_{\kappa,\mu}(z)$ and $W\kappa,\mu(z)$ are solutions to Whittaker's PDE

$$u_{zz} - \frac{1}{4} + \frac{\kappa}{z} + \frac{(\frac{1}{4} - \mu^2)}{z^2} = 0\,. \tag{9.9}$$

and can be defined via Kummer's confluent hypergeometric functions $M(a,b,z)$ and $U(a,b,z)$

$$\begin{aligned} W_{\kappa,\mu}(z) &= e^{\frac{1}{2}z} z^{\mu+\frac{1}{2}} \Psi(\frac{1}{2} + \mu - \kappa, 1 + 2\mu, z)\,, \\ M_{\kappa,\mu}(z) &= e^{\frac{1}{2}z} z^{\mu+\frac{1}{2}} F_1(\frac{1}{2} + \mu - \kappa, 1 + 2\mu, z)\,. \end{aligned} \tag{9.10}$$

Since Whittaker functions are not available in R's base packages we have implemented them from Kummer's confluent hypergeometric function.
Again we take first a closer look at the function which has to be integrated. As expected for low volatilities and/or short times to maturity the functions take extremely large values due to the exponential, Gamma and Whittaker functions combining together. These facts limit the evaluation of Asian call prices to values $\sigma^2 T \gtrsim 0.002$.
Linetzky used the software package Mathematica and computed Asian options to a very high precision. Table 5 shows his and our results and compares the figures to those obtained from the methods we have discussed in the previous chapters.

								Sudler	Laplace Inversion	Laplace Inversion	Laplace Inversion	Spectral Representation
S	**X**	**T**	**r**	**σ**	**nu**	**σ2 T/4**	**GE**	**GY**	**Euler**	**PW**	**FT**	**Linetzky**
2.00	2.00	1	0.0200	0.10	3.000	0.00250	0.058	0.056	0.056	0.062	0.05599	NA
2.00	2.00	1	0.1800	0.30	3.000	0.02250	0.227	0.218	0.219	0.219	0.21839	0.21839
2.00	2.00	2	0.0125	0.25	-0.600	0.03125	0.1722	0.1723	0.172	0.172	0.17227	0.17227
1.90	2.00	1	0.0500	0.50	-0.600	0.06250	0.195	0.193	0.194	0.194	0.19317	0.19315
2.00	2.00	1	0.0500	0.50	-0.600	0.06250	0.248	NA	0.247	0.247	0.24642	0.24640
2.10	2.00	1	0.0500	0.50	-0.600	0.06250	0.308	0.306	0.307	0.307	0.30622	0.30621
2.00	2.00	2	0.0500	0.50	-0.600	0.12500	0.351	0.350	0.352	0.352	0.35010	0.35222

FIGURE 9.1:
Comparison of numerical techniques for pricing continuous Asian options by Laplace inversion, LI, or Fourier transformation, FT: The table collects results from various authors and compares them with ours. Geman-Eydeland (GE), Shaw, Euler, Post-Widder (PW), Monte Carlo (MC), and the Turnbull-Wakeman (TW) approximation; $K = 2.0$ and $t = 0$ throughout, T measured in years. From Fu, Madan and Wang [1997].

9.4 R-FUNCTIONS

The function `LinetzkyAsianOption()`

The function `LinetzkyAsianOption()` valuates Asian options from the spectral integral representation. In addition `gLinetzky()` calculates the function to be integrated.

The function `LinetzkyAsianDensity()`

With the function `LinetzkyAsianDensity()` we can calculate the Asian state price density from the spectral representation.

The function `whittakerM()` and `whittakerW()`

The functions `whittakerM()`, and `whittakerW()` allow to calculate the Whittaker functions with complex argument and complex indexes.

Chapter 10

Symmetry and Equivalence Relations

Symmetry and equivalence relations are useful in relating the prices of two Asian options to each other under special conditions. Such relationships allow for example to compute Asian options in the case of continuously payed dividends or to compute call prices from put prices and vice versa,

10.1 Continuous Dividends

The case of continuously payed dividends q can easily be added to Asian option valuation. Performing the transformations

$$\begin{aligned} r^{(q)} &= r - q\,, \\ S^{(q)} &= S e^{-rT}\,, \\ X^{(q)} &= X e^{-rT}\,, \end{aligned} \tag{10.1}$$

and replacing r, S, and X by $r^{(q)}$, $S^{(q)}$, and $X^{(q)}$ gives the Asian prices for options with the parameter set $\{S, X, T, r, q, \sigma\}$.

10.2 Call-Put Parity

The result for the Call-Put-Parity for Asian options is

$$c(X,S,r,q,T) - p(X,S,r,q,T) = \frac{1}{(r-q)T}(e^{-qT} - e^{-rT})S - e^{-rT}X\,. \tag{10.2}$$

This relationship can be used to valuate Asian puts from calls , and vice versa.

10.3 CALL-PUT DUALITY

The Call-Put-Duality is another simple relationship between call and put options introduce by Peskir and Shiryaev [2000]

$$p(X, S, r, q, T) = c(-X, -S, r, q, T) . \tag{10.3}$$

This relationship may be useful for the numerical evaluation of Asian options and for numerical test purposes.

Appendix A

Packages Required for this eBook

```
> library(fOptions)
> library(fAsianOptions)
```

In the following we briefly describe the packages required for this eBook.

A.1 Rmetrics Package: fOptions

`fOptions` contains `R` functions to price Plain Vanilla options.

```
> listDescription(fOptions)
 fOptions Description:

  Package:            fOptions
  Version:            2110.78
  Revision:           4844
  Date:               2010-04-23
  Title:              Basics of Option Valuation
  Author:             Diethelm Wuertz and many others, see the SOURCE
                        file
  Depends:            R (>= 2.4.0), methods, timeDate, timeSeries,
                        fBasics
  Suggests:           RUnit, tcltk
  Maintainer:         Rmetrics Core Team <Rmetrics-core@r-project.org>
  Description:        Environment for teaching "Financial Engineering and
                        Computational Finance"
  NOTE:               SEVERAL PARTS ARE STILL PRELIMINARY AND MAY BE
                        CHANGED IN THE FUTURE. THIS TYPICALLY INCLUDES
                        FUNCTION AND ARGUMENT NAMES, AS WELL AS DEFAULTS
                        FOR ARGUMENTS AND RETURN VALUES.
  LazyLoad:           yes
  LazyData:           yes
  License:            GPL (>= 2)
  URL:                http://www.rmetrics.org
```

```
  Packaged:            2010-04-27 07:14:14 UTC; yankee
  Repository:          CRAN
  Date/Publication:    2010-04-27 20:02:44
  Built:               R 2.11.1; i386-pc-mingw32; 2010-07-06 11:41:11 UTC;
                         windows
```

A.2 RMETRICS PACKAGE: fAsianOptions

fAsianOptions contains R functions to price Asian options.

```
> listDescription(fAsianOptions)
 fAsianOptions Description:

  Package:             fAsianOptions
  Version:             2100.76
  Revision:            4057
  Date:                2009-09-28
  Title:               EBM and Asian Option Valuation
  Author:              Diethelm Wuertz and many others, see the SOURCE
                         file
  Depends:             R (>= 2.4.0), timeDate, timeSeries, fBasics,
                         fOptions
  Suggests:            RUnit
  Maintainer:          Rmetrics Core Team <Rmetrics-core@r-project.org>
  Description:         Environment for teaching "Financial Engineering and
                         Computational Finance"
  NOTE:                SEVERAL PARTS ARE STILL PRELIMINARY AND MAY BE
                         CHANGED IN THE FUTURE. THIS TYPICALLY INCLUDES
                         FUNCTION AND ARGUMENT NAMES, AS WELL AS DEFAULTS
                         FOR ARGUMENTS AND RETURN VALUES.
  LazyLoad:            yes
  LazyData:            yes
  License:             GPL (>= 2)
  URL:                 http://www.rmetrics.org
  Packaged:            2009-09-28 15:21:34 UTC; yankee
  Repository:          CRAN
  Date/Publication:    2009-09-30 19:40:52
  Built:               R 2.11.1; i386-pc-mingw32; 2010-07-06 11:42:40 UTC;
                         windows
```

Appendix B

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