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Statistical theory of the polarization on the Poincaré sphere

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The random nature of light emitters requires that the theory to study polarization must adopt a statistical approach. Although the Stokes parameters are proposed in this sense, these are implemented as temporal averages, appealing to an ergodic hypothesis, without discussing which probability distributions are treated. Therefore, here the polarization dynamics on the Poincaré sphere of random light sources, whose phases and amplitudes of the electric fields are statistically independent, is described through the von Mises–Fisher distribution. This allows us to relate the degree of polarization and the normalized Stokes parameters with the concentration parameter and the mean direction of the von Mises–Fisher distribution. Also, it is found that their marginal distributions have the same statistical behavior as the Eliyahu–Brosseau distribution.

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The polarization state of a light source is defined by the averages of the instantaneous Stokes parameters, and their representation on the Poincaré sphere corresponds to a point on the sphere or an inner region, according to the degree of polarization of the source [1,2]. The Poincaré sphere is a handy visual tool for both problems involving random light sources and those related to polarized light [3–5], but random light sources are studied by means of the statistics of the Stokes parameter [1,6–8]. There are several systems where it is necessary to work with partially polarized light due to the effects of interaction between light and materials, e.g., in optical fiber systems where the polarization dispersion phenomenon is important [9], and in bio-medicine to study and characterize samples of cancerous and non-cancerous tissues [10,11].

Considering that a partially polarized light source can be understood as one whose polarization state changes with time, we can see that the polarization dynamics on the Poincaré sphere constitutes a random process that characterizes the source statistics. The characterization of this polarization dynamics is particularly useful in problems where polarization fluctuations are present, e.g., in optimizing the sensitivity of optical fiber sensors where external disturbances lead to changes in the polarization state of the light that propagates through the fiber [12]; in astronomy, the contribution of polarization fluctuations to the microwave power spectrum has been estimated by a statistical study of the polarized emission of extragalactic radio sources [13]; in optical fibers, the polarization dispersion has been studied as a random change of birefringence along the fiber [14]; and multiple dispersions in turbid media such as biological tissues [15].

Recently, a new approach to study the polarization dynamics on the Poincaré sphere has been introduced, where partially polarized light sources are characterized by a polarization time, a time where it is assumed that the polarization state does not change [16,17]. Although this characteristic represents a direct measure of the fluctuations of the polarization state of a light source, its observation also requires the measurement of events as fast as the electromagnetic field vibrates, being femtoseconds for usual sources.

However, until now, the use of probability density distributions on the Poincaré sphere has not been formalized for the study of the polarization dynamics of random light sources. So, in this Letter is described the polarization state of stationary random light sources on the Poincaré sphere through the von Mises–Fisher (vMF) distribution for electric fields whose phase and amplitude are statistically independent. Also, we study the behavior of the marginal distributions on the axes of the Poincaré sphere and its association with the Eliyahu–Brosseau distributions [18,19]. Finally, we find a relation between the degree of polarization and the concentration parameter κ and the polarization state of the partially polarized sources with mean direction $\boldsymbol{\mu}$ of the vMF distribution.

The vMF distribution is a normal distribution on the surface of the sphere of unitary radius [20,21]. On the other hand, the Poincaré sphere is a graphical method that through points represents the polarization state of a light source [1,22] and how the polarization state from partially polarized sources fluctuates [6,23]. Therefore, in this Letter, we propose the vMF distribution to characterize the polarization dynamics of stationary random light sources on the Poincaré sphere.

The probability density distribution of vMF for a random unit vector $\mathbf{x} = (S_1, S_2, S_3)$ is defined as

$$f(\mathbf{x}; \boldsymbol{\mu}, \kappa) = \frac{\kappa}{4\pi \sinh \kappa} e^{\kappa \boldsymbol{\mu}^T \mathbf{x}}, \qquad \kappa \geq 0, \tag{1}$$

where S_2 and S_3 are statistically independent random variables, $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$ is a unit vector called the mean direction, and κ is the concentration parameter.

We write $\mathbf{x}$ and $\boldsymbol{\mu}$ in spherical coordinates:

$$\mathbf{x} = (\cos 2\chi \cos 2\alpha, \cos 2\chi \sin 2\alpha, \sin 2\chi), \tag{2}$$

$$\boldsymbol{\mu} = (\cos 2\beta \cos 2\theta, \cos 2\beta \sin 2\theta, \sin 2\beta), \tag{3}$$

with $-\pi/4 \leq (\chi, \beta) < \pi/4$ and $0 \leq (\alpha, \theta) < \pi$. Using the Jacobian of the transformation for conservation of probability [see Fig. 1(a)], Eq. (1) is written in the form

$$f(2\alpha, 2\chi) = \frac{\kappa \cos 2\chi}{4\pi \sinh \kappa} e^{\kappa(\cos 2\beta \cos 2\chi \cos(2\alpha - 2\theta) + \sin 2\beta \sin 2\chi)}. \tag{4}$$

The first moment is given by [24]

$$E\{\mathbf{x}\} = \left(\coth \kappa - \frac{1}{\kappa}\right)\boldsymbol{\mu}, \tag{5}$$

where E represents the expected value, and this allows relating the degree of polarization ρ as a function of κ [ver Fig. 1(b)]:

$$\rho = \left(\coth \kappa - \frac{1}{\kappa}\right). \tag{6}$$

Thus, this concentration parameter κ constitutes a measure of the degree of correlation between the Stokes parameters proposed by Setälä *et al.* [17]. It is important to note that when $\kappa \to \infty$, the degree of polarization is equal to one, and the vMF distribution of Eq. (4) tends to Dirac distribution or a pure state of polarization $f(\boldsymbol{x}; \boldsymbol{\mu}, \infty) = \delta(\boldsymbol{x} - \boldsymbol{\mu})$, and when $\kappa \to 0$, the degree of polarization is zero, which corresponds to an unpolarized light source, and the vMF distribution is equal to $f(\boldsymbol{x}; \boldsymbol{\mu}, 0) = 1/4\pi$, indicating that all the polarization states of the sources on the sphere are equally probable.

Figure 2 shows computational simulations for 10,000 samples with three different parameters of the concentration and mean direction of the vMF distribution. Through these parameters and Eqs. (6) and (5), we obtain the degree of polarization ρ and the temporal averages of the normalized Stokes parameters $\langle \mathbf{x} \rangle = (\omega_1, \omega_2, \omega_3)$, being $\omega_i = \langle S_i \rangle$ with $i = 1, 2, 3$. In Figs. 2(a)–2(c), the parameters are $\rho = 0$ and $\langle \mathbf{x} \rangle = (0, 0, 0)$, $\rho = 0.95$ and $\langle \mathbf{x} \rangle = (0.55, 0.55, 0.54)$, and $\rho = 0.98$ and $\langle \mathbf{x} \rangle = (0.98, 0, 0)$, respectively. In summary, the parameters $(\boldsymbol{\mu}, \kappa)$ characterize the vMF distribution of the polarization state of light sources on the Poincaré sphere [see Figs. 1(a) and 2], thus allowing the representation of the partially polarized light on the Poincaré sphere.

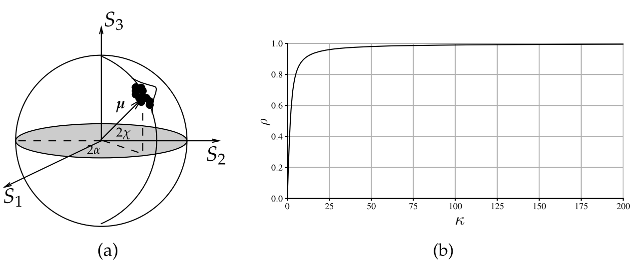

Fig. 1. (a) Schematic of vMF distribution on the Poincaré sphere with mean direction $\boldsymbol{\mu}$; (b) relation between the degree of polarization ρ and the concentration parameter κ.

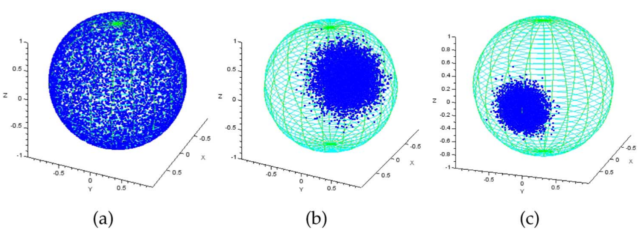

Fig. 2. vMF distribution: (a) $\kappa = 0$, (b) $\kappa = 20$, $\boldsymbol{\mu} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, and (c) $\kappa = 50$, $\boldsymbol{\mu} = (1, 0, 0)$.

Barakat studied the statistics of the Stokes parameters associated with the random fluctuations of the polarization state [25], presenting probability density functions for Stokes parameters. However, the normalized Stokes parameters proposed by Eliyahu–Brosseau [18,19] for Gaussian stochastic light sources is of more interest for our study, since these have a significance on the Poincaré sphere and therefore can be related to the vMF distribution.

Under the hypothesis of statistical independence of the phase and amplitude of the electric field vector of a partially polarized light, the Eliyahu–Brosseau probability distribution function of the normalized Stokes parameter S_1, corresponding to the Gaussian stochastic plane-wave field, has the following form [18]:

$$p_{S_1}(S_1) = \frac{(1 - P^2)}{2} \left\{ \frac{1 - S_1 \omega_1}{[(1 - S_1 \omega_1)^2 - (1 - S_1^2)(P^2 - \omega_1^2)]^{3/2}} \right\}, \tag{7}$$

where S_1 is the Stokes parameter ($|S_1| \leq 1$), P is the degree of polarization, and ω_1 represents the expected value of S_1. The probability density functions $p_{S_2}(S_2)$ and $p_{S_3}(S_3)$ are evaluated from the expression of $p_{S_1}(S_1)$ by making use of symmetry transformation. A consequence of statistical independence between amplitudes and phases is $E\{S_1 S_2\} = 0$, i.e., S_2 and S_3 also are statically independent; thus, its probability distribution of normalized Stokes parameters on the Poincaré sphere corresponds to a vMF distribution, and consequently, its marginal distributions are the Eliyahu–Brosseau distributions.

The continuous curve in Fig. 3 describes the probability density function of the normalized Stokes parameters derived by Eliyahu–Brosseau, where the degree of polarization P and

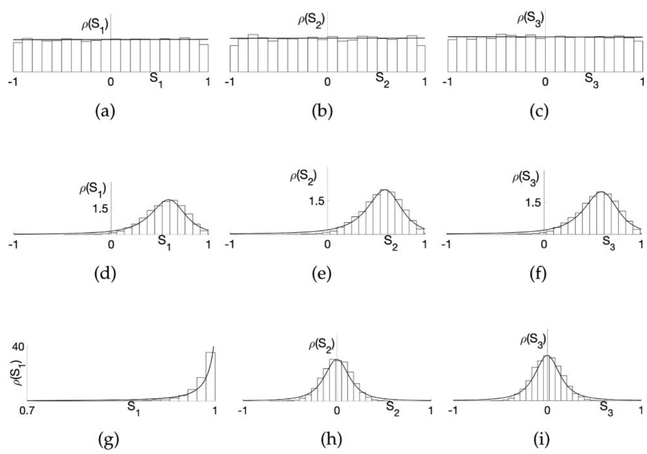

Fig. 3. (a)–(c), (d)–(f), (g)–(i) Histograms representing projections on the axes of distribution points on the Poincaré sphere associated with the vMF distribution of the graphs in Figs. 2(a), 2(b), and 2(c), respectively. The continuous curve represents the Eliyahu–Brosseau distribution.

the expected values ω_1, ω_2, and ω_3 are the same as those obtained in the previous section for the three cases studied above. So, the histograms show the same statistical behavior of the Eliyahu–Brosseau distributions for the three cases. Then, the statistical independence of the phase and amplitude of the electric field vector of a partially polarized light source entails statistically independent random variables of the normalized Stokes parameters.

The configuration used in the laboratory is schematized in Fig. 4. Since the Eliyahu–Brosseau distributions correspond to a light source with uncorrelated amplitudes and phases, and given that these correlations are given for the electric field components $E_x(t_1)$ and $E_y(t_2)$ with $\tau = t_2 - t_1$ less than the coherence time, for partially coherent sources and according to the results of the Ref. [26], each birefringent plate introduces delays that depolarize, and if these plates are interposed with their fast axes arbitrarily oriented, finally we can delay the amplitudes and the phases until breaking the coherence between them and reaching a distribution of vMF whose marginal distributions correspond to those of Eliyahu–Brosseau.

The experimental measures were carried out using a commercial Thorlabs polarimeter (PAN5710VIS), which is based on a rotating waveplate and a polarizer [27]. The input beam was prepared from a partially coherent and unpolarized diode with a center wavelength of 625 nm, which was polarized for a dichroic linear polarizer (P). Subsequently, the beam passed through appropriately arranged n-retarders; these introduced a decrease in the degree of polarization of the source and allowed generation of a light source whose output polarization states formed on the Poincaré sphere a circular distribution, i.e., the n-retarders introduced decorrelations between amplitudes and phases. Thus, for an incoherent source, the output polarization state corresponds not to a rotation on the Poincaré sphere but to depolarization.

The time τ of capture of the polarimeter (333 polarization states per second) is much greater than the so-called polarization time (some tens of femtoseconds) [17], which means the range of the state recorded in a catch is constituted in a representative sample of the random process, which is assumed to be stationary and ergodic. Therefore, in each polarimeter detection period, this divides the signal random into long segments compared to the time of polarization, and under the hypothesis of ergodicity, each segment is a representative set of the entire random signal, so the statistical information is contained in a single sample signal of the random signal. It has been proven that the statistical independence of each portion of the segmented signal is sufficient to guarantee that a process can be estimated from a single sample signal [28]. This is what allows the theoretical predictions based on an ergodic hypothesis to coincide with the experimental results [29].

We measured 10,000 values through the polarimeter corresponding to the three normalized Stokes parameters, i.e., corresponding to the average over integration time of the detector.

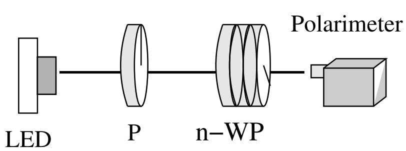

Fig. 4. Optical system for the recording of experimental data.

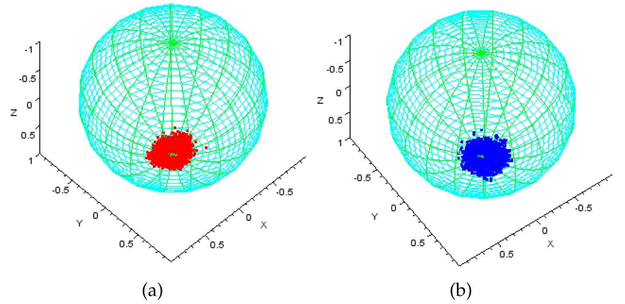

Fig. 5. (a) Experimental data obtained by the polarimeter. (b) Simulation data of the vMF distribution.

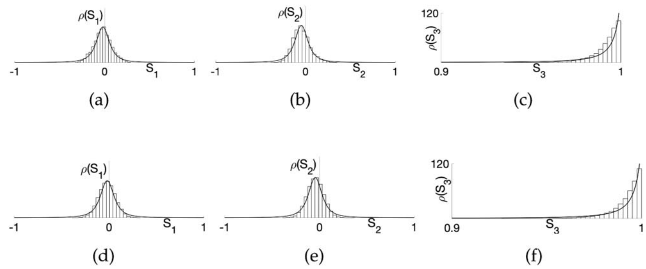

Fig. 6. (a)–(c), (d)–(f) Histograms associated with the vMF distribution of experimental and simulated data in Figs. 5(a) and 5(b), respectively. The continuous curve represents the Eliyahu–Brosseau distribution.

In our case, this integration time was enough to be able to sample the dynamics of the polarization and thus estimate the distribution of vMF. Then, we obtained the degree of polarization $\rho = 0.994$ and the expected value of the Stokes parameters $E[\mathbf{x}] = (-0.019, -0.045, 0.993)$. The experimental data obtained are represented on the Poincaré sphere as shown in Fig. 5(a). On the other hand, Fig. 5(b) represents the simulated data of a vMF distribution. Using Eqs. (5) and (6), from the experimental data, we calculate the mean direction $\boldsymbol{\mu} = (-0.020, -0.045, 0.999)$ and the concentration parameter $\kappa = 166.667$, which are used as the input parameters for the simulation.

The histograms in Figs. 6(a)–6(c) and 6(d)–6(f) represent the projections onto the axes of the distribution of the experimental data in Fig. 5(a) and simulated data 5(b), respectively, showing the same statistical behavior as Eliyahu–Brosseau distributions. This result implies the partially polarized light sources with statistically independent amplitude and phase of the random electric field corresponding to normalized Stokes parameter distributions with circular distribution on the Poincaré sphere. So, the vMF distribution of statistically independent random normalized Stokes parameters is well adapted to the study of polarization dynamics on the Poincaré sphere of light sources with statistically independent amplitude and phase of the random electric field.

In summary, vMF distribution has been adapted to the study of polarization dynamics of partially polarized random light sources. We found a relation between the degree of polarization ρ and the normalized Stokes parameters $\mathbf{S}$ with the concentration parameter κ and mean direction $\boldsymbol{\mu}$, respectively. This result allowed us to characterize the polarization state of the partially polarized light source through the parameters

$(\rho, \mathbf{S}) \rightarrow (\kappa, \boldsymbol{\mu})$ and to couple it with the Poincaré graphical method. Also, because of the uncorrelated amplitudes and phases, we showed that the marginal distributions of vMF distributions on the axes of the Poincaré sphere have the same statistical behavior as Eliyahu–Brosseau distributions.

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