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Cosmology I

The expansion of the universe

In this letter we explore the curvature and expansion rate of the universe using the principles of general relativity. By applying Einstein's field equations to a cosmological model, assuming homogeneity and isotropy at large scales, the Robertson-Walker metric is introduced as a solution that describes the universe's geometry. The Friedmann-Lemaître (FL) equations are then derived to relate the universe's expansion rate, $\dot{R}(t)$, to its energy densities: matter, radiation, dark energy, and curvature. These equations are solved numerically to model the universe's evolution, yielding an estimate for the current age of the universe at approximately 13.8 billion years. This study highlights the critical role of density parameters in predicting the future dynamics of the universe while noting several areas for further exploration, such as deriving the Robertson-Walker metric and measuring the Hubble parameter.

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1 The curvature of the universe and the expansion rate

To know the curvature and expansion rate of the Universe, $\dot{R}(t)$, it is necessary to have a model that describes the dynamics of the Universe on a large scale. For this, let's assume that the Universe on a large scale is electrically neutral and that the contributions of the nuclear forces are very small. Then the dynamics of the Universe on a large scale is solely determined by the gravity of the Universe itself. Now we can use the field equations of the theory of General Relativity (GR),

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{1}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, $g_{\mu\nu}$ is the metric tensor, Λ is the cosmological constants, G is the Newton constant for gravity, c is the speed of the light and $T_{\mu\nu}$ is the Energy-momentum tensor. To get to know how the universe will evolve. We need to solve Eq 1 for the Energy-momentum distribution ($T_{\mu\nu}$) of our universe.

As we want to find the solution that describes our universe, we can restrict our analysis of solutions, to those that satisfy the symmetries that we observe today in the universe. These symmetries are stated in what is known as the cosmological principle; Based on the evidence offered by observations such as: the cosmic microwave background, galaxy distribution maps and the experience previously obtained when searching for the Earth's place in the Universe. The cosmological principle (Peterson et al., 2004; Schutz, 2009) is proposed (and commonly accepted by the scientific community). The cosmological principle proposes 2 aspects that the Universe fulfills on a large scale:

- **Homogeneity:** It proposes that on large scales (structures of at least $100\ Mpc$) the properties of the Universe are the same at any point.
- **Isotropy:** It proposes that the Universe is approxi-

mately the same in all directions for any observer.

Assuming the existence of a standard cosmic time and introducing Gaussian coordinates, Robertson (1935) and Walker (1936) found the most general 4-dimensional metric that is consistent with the cosmological principle (Equation 2). This metric is known as the Robertson-Walker (RW) metric

$$ds^2 = -c^2dt^2 + R(t)^2\left[d\chi^2 + S_k(\chi)^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]$$
$$S_k(\chi) = \left\{\begin{array}{lll} \chi & if & k=0 \\ \sin\chi & if & k=1 \\ \sinh\chi & if & k=-1 \end{array}\right\}, \tag{2}$$

where $R(t)$ is the scale factor of the Universe and k is the curvature factor of the Universe. k can take values of $-1, 0$ or 1 depending on whether the Universe is hyperbolic, flat or spherical, respectively. Both the functional form of $R(t)$ and the value of k depends on: the energy density of matter, the energy density of radiation and the dark energy density of the Universe.

Considering that the Universe on a large scale behaves like a perfect fluid, where the pressure P of this fluid is the sum of the individual pressures of: matter P_M, radiation P_r and dark energy P_Λ. Furthermore the energy density ρ of this fluid is the sum of the individual densities of: matter ρ_M, radiation ρ_r and dark energy ρ_Λ. Then the total pressure and density

$$\begin{aligned} P =& P_M + P_r + P_\Lambda \\ \rho =& \rho_M + \rho_r + \rho_\Lambda. \end{aligned} \tag{3}$$

The introduction of a "dark fluid" can be done due to the cosmological constant term $\Lambda g_{\mu\nu}$ in Eq. 1. In general, both the pressure and the density of the Universe do not have to be constant in time, so they are functions of time, $P(t), \rho(t)$. Considering the above definitions of the pressure and density of the Universe, Eq. 3, the expression for the energy-momentum tensor of the Universe as a perfect fluid takes the form:

$$T_{\mu\nu} = Pg_{\mu\nu} + \left(P + \rho c^2\right)\frac{dx^\sigma}{d\tau}g_{\sigma\mu}\frac{dx^\epsilon}{d\tau}g_{\epsilon\nu} \tag{4}$$

Combining the field equations (eq.1) with the Robertson Walker metric (eq. 2) and modeling the Universe as a perfect fluid (eq. 4), we obtain the Fridman-Lemaître (FL) equations. These equations describe the functional form of the Universe scale factor $R(t)$ and the Universe curvature factor k, in terms of the Universe energy densities. The first equation comes from considering the temporal, t, t, component of the field equations (1). It is known as the first FL equation, and has the form of the second equation in 5.

$$\begin{aligned} R_{tt} - \frac{1}{2}Rg_{tt} &= \frac{8\pi G}{c^4}T_{tt} \\ \rightarrow \left(\frac{\dot{R}(t)}{R(t)}\right)^2 &= \frac{8\pi G}{3}\rho(t) - \frac{kc^2}{R(t)^2}, \end{aligned} \tag{5}$$

where $\dot{R}(t)$ is the time derivative of the scale factor of the Universe, that is, the speed at which the Universe expands.

By introducing the following definitions: the Hubble parameter $H(t)$, the critical density (ρ_c) and the density parameters $\Omega, \Omega_M, \Omega_r, \Omega_\Lambda, \Omega_k$

$$\begin{aligned} H(t) &\equiv \frac{\dot{R}(t)}{R(t)}, \\ \rho_c &\equiv \frac{3H(t)^2}{8\pi G}, \\ \Omega &\equiv \frac{\rho}{\rho_c} = \frac{\rho_M}{\rho_c} + \frac{\rho_r}{\rho_c} + \frac{\rho_\Lambda}{\rho_c} = \Omega_M + \Omega_r + \Omega_\Lambda, \\ \Omega_k &\equiv \frac{-kc^2}{R(t)^2H(t)^2}. \end{aligned} \tag{6}$$

Substituting the definitions 6 into the second equation in 5 we get that:

$$\begin{aligned} &\frac{3H(t)^2}{8\pi G} = \rho_c\left(\Omega_M + \Omega_r + \Omega_\Lambda\right) - \frac{3kc^2}{8\pi GR(t)^2} \\ &\rightarrow \rho_c = \rho_c\left(\Omega_M + \Omega_r + \Omega_\Lambda\right) + \rho_c\Omega_k \\ &\rightarrow \Omega_M + \Omega_r + \Omega_\Lambda + \Omega_k = 1. \end{aligned}$$

And if we define the index $n = \{M, r, \Lambda, k\}$. Then the above equation can be written as

$$\sum_n \Omega_n = 1 \tag{7}$$

The above equation it is known as the first Friedman equation in terms of the density parameters Ω_n. To solve this equation we need to have some references of the values of the solution. Let's use the present values (values at t_0).

$$\begin{aligned} &\rho_c(t = t_0) = \rho_{c,0} = \frac{3H_0^2}{8\pi G} \rightarrow \rho_c = \rho_{c,0}\left(\frac{H}{H_0}\right)^2 \\ &\Omega_n(t = t_0) = \Omega_{n,0} = \frac{\rho_{n,0}}{\rho_{c,0}}. \end{aligned}$$

Combining the above equations with the definition of Ω_n we have that

$$\Omega_n = \frac{\rho_n}{\rho_c} = \Omega_{n,0}\left(\frac{\rho_n}{\rho_{n,0}}\right)\left(\frac{H_0}{H}\right)^2.$$

Combining the above result with Eq 7 and the definition of $H(t)$ we have that:

$$\begin{aligned} H(t) &= H_0\sqrt{\sum_n \Omega_{n,0}\left(\frac{\rho_n}{\rho_{n,0}}\right)} \\ \dot{R}(t) &= H_0R(t)\sqrt{\sum_n \Omega_{n,0}\left(\frac{\rho_n}{\rho_{n,0}}\right)}. \end{aligned} \tag{8}$$

In the above equation $H_0, \Omega_{n,0}$ are constants whose value we can consult but the factor $\rho_n/\rho_{n,0}$ it is a function of

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time. To solve the above equation we still need how the energies densities ρ_n relate with the scale factor, $R(t)$, of the universe.

To go further in our analysis we need the results of applying the conservation of the four-momentum to our energy-momentum tensor that is:

$$\nabla_\mu T^\mu_\nu = 0.$$

From the previous equation, developing the covariant derivative, using our expression for the momentum energy tensor of the universe Eq 4 and taking the temporal component, we can obtain the so-called fluid equation of the universe or energy dilution equation.

$$\dot{\rho}_n + 3\frac{\dot{R}}{R}\left(\rho_n + \frac{P_n}{c^2}\right) = 0. \tag{9}$$

To solve this equation we need to assume some equation of estate, $\rho(P)$, The discussion of the solution of this equation is not hard. Although needs some other physics, so it deserves a letter on his own. Lets just take the final result and say that from the fluid equation it can be show that

$$\sum_n \Omega_{n,0}\left(\frac{\rho_n}{\rho_{n,0}}\right) = \frac{\Omega_{M,0}}{R^3} + \frac{\Omega_{r,0}}{R^4} + \frac{\Omega_{k,0}}{R^2} + \Omega_{\Lambda,0}. \tag{10}$$

Combining this result with Ec 8 we get an explicit differential equation for $R(t)$.

$$\dot{R}(t) = H_0\sqrt{\frac{\Omega_{M,0}}{R} + \frac{\Omega_{r,0}}{R^2} + \Omega_{k,0} + \Omega_{\Lambda,0}R^2}. \tag{11}$$

Unfortunately, there is not an analytic function, that could be written in terms of fundamental functions, that describe the solution of the Eq 11. Although we do not find and exact solution for the above equation we can solved numerically. In this letter we use the Euler method to solve Eq 11. The solutions of this equation for different values of the density parameters in the present time $(\Omega_{M,0}, \Omega_{r,0}, \Omega_{k,0}, \Omega_{\Lambda,0})$ are shown in the as a blue line in this dashboard.

Another feature that we can get from our modelling is the age of the universe that our model will predict. Let's define the $E(R)$ function as

$$E(R) \equiv \sqrt{\frac{\Omega_{M,0}}{R} + \frac{\Omega_{r,0}}{R^2} + \Omega_{k,0} + \Omega_{\Lambda,0}R^2}.$$

Then Eq. 11 can be written as

$$\begin{aligned} &\frac{1}{E(R)}\dot{R}(t) = H_0 \\ &\rightarrow \int \frac{1}{E(R)}\dot{R}(t)(dt) = \int H_0(dt) \end{aligned} \tag{12}$$

The solution of the integral on the left is given by a function $f(R)$ whose derivative respect to time is equal to the integrand, that is, if $f(R)$ is a solution of the integral it satisfies that $\dot{f}(R) = E(R)^{-1}\dot{R}(t)$. Since f is a function of R and R in turn is a function of time, this means that by the chain rule $\dot{f}(R) = \dot{R}f'(R)$, where $f'(R)$ is the derivative of f respect to R. This means that

$$\begin{aligned} &\frac{1}{E(R)}\dot{R}(t) = \dot{R}(t)f'(R) \\ &\rightarrow \int f'(R)(dR) = \int \frac{dR}{E(R)} \end{aligned}$$

Then by the fundamental theorem of calculus we have that the solution f it is

$$f(R) = \int \frac{dR}{E(R)}.$$

The integral on the right in Eq. 12 is direct. Therefore, considering the corresponding integration limits, it follows from Eq. 12 that

$$t_0 - t = \frac{1}{H_0}\int_R^{R_0} \frac{dR}{E(R)}$$

Where t being some moment in the past. Unfortunately the above integral can not be calculated in terms of fundamental functions. Those we need to calculated by numeric approximations.

To calculate the age of the universe, we make the integral from the beginning, $t \simeq 0, R \simeq 0$ and until the present day, $R_0 = 1$. Then the age of the universe, t_0, will be given by the next integral

$$t_0 = \frac{1}{H_0}\int_0^1 \frac{dR}{E(R)} \simeq 13,800[My] \tag{13}$$

We can solve the above integral using the Simpson method for numerical integrals. If we consider standard values of $H_0 \simeq 69.6[km\ Mpc^{-1}s^{-1}], \Omega_{M,0} \simeq 0.28, \Omega_{r,0} \simeq 10^{-16}, \Omega_{\Lambda,0} \simeq 0.72 \rightarrow \Omega_{k,0} \simeq 0.0$. We get that the age of the universe it's about 13.8 Billion years. The result of solving the above integral (for different values of the parameters at the present time) it's shown in the dashboard as a red point in the graph. The value of the age of the universe it's on the labels of the graph in units of billion of years.

So that is it. We have seen, in general steps, how to describe the expansion of the universe (11) from the Gilbert-Einstein field equations (1). And also how to calculate the age of our universe.

Although four important steps have been left out, these are:

- How to deduce the R.W. metric from the field equations.
- How to deduce the first Friedman-Lemaitre equation from the field equations and the R.W. metric.
- How to solve the fluid equation.

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- How to measure the Hubble parameter H_0 and the density parameters $\Omega_{n,0}$.

Each of these topics deserves a letter for itself and we will leave those procedures for future letters. In this first letter our objective was to analyze the origin and solution of the Friedman-Lemaitre equation. And use it to describe the past and future expansion of our universe, the age of the universe, thus illustrating one of the most spectacular applications of Einstein's theory of relativity.

References

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