Economy I

compound interest

Some financial companies and government organizations, have instruments that allow us to lend money to the company in exchange for a return (interest) at the time of repayment of the money lent. Once this mechanism is established, the question arises: if at the end of the term, we invest again in the same instrument. This time the same amount, plus the return and we do this successively. How much money will we have after a certain time? Even further, if at the end of each term, we consistently add a certain amount of money, how much money would I have after a certain time? How long do I need to wait or what return is necessary to be a millionaire? This is known as compound interest and in this letter we show how to calculate it.

What's the deal?

by J. López

Let us consider a specific case. Say that the deal offered by the company/government-institution is the following: If we lend them a certain amount of money (I_0), at the end of a month they pay us the money lent (I_0), plus a percentage of the same money that we lent them (αI_0). Let us also assume that month after month, we make a constant contribution (I_m) to our savings and invest them in the same financial instrument(I_m). Then how much money (M) will we have saved after a certain amount of time (t)?

Let us first consider the increase in money (ΔM_1) that we would have to receive after the first month. This would be proportional to the sum of: our monthly contribution of the first month and our initial investment:

$$\Delta M_1 = \alpha \left(I_0 + I_m \right), \tag{1}$$

where α is the monthly return on investment offered by the company/government-institution.

Consequently, the increase that we should be paid in the second month will be proportional to the total money we have. The total amount of money will now be given by; our initial investment, plus the monthly contributions, plus the return paid last month for the investment. That is,

$$\begin{aligned} \Delta M_2 =& \alpha \left(I_0 + I_m + I_m + \Delta M_1 \right) \\ =& \alpha \left(I_0 + I_m \right) + \alpha \left(I_m + \Delta M_1 \right). \end{aligned} \tag{2}$$

But from Eq. 1 we get that;

$$\Delta M_2 = \Delta M_1 \left(1 + \alpha \right) + \alpha I_m. \tag{3}$$

Similarly for the third month we have that;

$$\Delta M_3 = \alpha \left(I_0 + 3 I_m + \Delta M_1 + \Delta M_2 \right) \tag{4}$$

But from Eq. 2 we see that;

$$\Delta M_3 = \Delta M_2 \left(1 + \alpha \right) + \alpha I_m \tag{5}$$

Similarly for the fourth month and using Eq 4 we see that:

$$\begin{aligned} \Delta M_4 &= \alpha \left(I_0 + 4 I_m + \Delta M_1 + \Delta M_2 + \Delta M_3 \right) \\ &= \Delta M_3 \left(1 + \alpha \right) + \alpha I_m. \end{aligned} \tag{6}$$

From the equations 3, 5 and 6 we see that in general

$$\Delta M_n = \Delta M_{n-1} \left(1 + \alpha \right) + \alpha I_m. \tag{7}$$

The above equation can be proved by induction.

Now, let us take $\Delta M_3, \Delta M_4$ that we have calculated and put it in terms of ΔM_1. For one hand, we can combine equations 5 and 3 to get

$$\Delta M_3 = \Delta M_1 \left(1+\alpha\right)^2 + \alpha I_m \left(1 + (1+\alpha)\right). \tag{8}$$

Then we can combine this equation with equation 6 to get

$$\Delta M_4 = \Delta M_1 \left(1+\alpha\right)^3 + \alpha I_m \left(1 + (1+\alpha) + (1+\alpha)^2\right). \tag{9}$$

Using Eq 7 to calculate ΔM_5 and then combine the result with the above equation we can show that:

$$\begin{aligned}\Delta M_5 =& \Delta M_1 \left(1+\alpha\right)^4 + ...\\ &... + \alpha I_m \left(1 + (1+\alpha) + (1+\alpha)^2 + (1+\alpha)^3\right).\end{aligned} \tag{10}$$

By looking at equations 8, 9, 10 and also 3 we see that in general

$$\Delta M_n = \Delta M_1 \left(1+\alpha\right)^{n-1} + \alpha I_m \sum_{l=0}^{n-2} \left(1+\alpha\right)^l. \tag{11}$$

In theory, the above equation gives us a direct way to calculate the monthly payments of our investments (ΔM_n), only knowing the data given at the beginning of our investment (α, I_m, I_0, n). However, the series of the term $\left(1+\alpha\right)^l$ can become difficult to calculate for large values of l (In this case l parameterizes time, we will go into this aspect in more depth later). Therefore, we will calculate the sum of the series present in the previous equation. The procedure is the standard one for studying power series. Let us consider the series in question and its partial sum to the element $n-2$

$$\sum_{l=0}^{n-2} \left(1+\alpha\right)^l = 1 + (1+\alpha) + (1+\alpha)^2 + ... + (1+\alpha)^{n-2}$$

$$\rightarrow \left[1 - (1+\alpha)\right] \sum_{l=0}^{n-2} \left(1+\alpha\right)^l = 1 - \left(1+\alpha\right)^{n-1}$$

$$\sum_{l=0}^{n-2} \left(1+\alpha\right)^l = \frac{\left(1+\alpha\right)^{n-1} - 1}{\alpha}$$

Now we can use this result in the Eq 11 to get

$$\Delta M_n = \left(1+\alpha\right)^{n-1} \left(\Delta M_1 + I_m\right) - I_m. \tag{12}$$

Now with the above equation, we have a method to calculate ΔM_n in a fast way.

The total amount of money (M) that we would have, It is given by the sum of our initial money (I_0), plus all the savings that we have made monthly (I_m) and the payments of our investments (ΔM_n). Then

$$M(t) = I_0 + \sum_{n=1}^{t} I_m + \sum_{n=1}^{t} \Delta M_n.$$

Combining the above equation with 12 and 1 we get

$$M(t) = \left(\alpha I_0 + I_m(1+\alpha)\right) \sum_{n=1}^{t} \left(1+\alpha\right)^{n-1}.$$

Again the series in the above equation can be calculated as follows

$$\left[1 - (1+\alpha)\right] \sum_{n=1}^{t} \left(1+\alpha\right)^{n-1} = 1 - \left(1+\alpha\right)^t$$

$$\rightarrow \sum_{n=1}^{t} \left(1+\alpha\right)^{n-1} = \frac{(1+\alpha)^t - 1}{\alpha}.$$

Then with this result and the equation for $M(t)$. We finally have the function that describes how our money grows with time

$$M(t) = I_0 \left(1+\alpha\right)^t + \left(\frac{1+\alpha}{\alpha} I_m\right) \left(\left(1+\alpha\right)^t - 1\right), \tag{13}$$

with t be the amount of months that have passed since we start the investment.

Something to keep in mind, is that most of the real world deals gives the return on investment in percentage per year β. Just then taking into account that $\alpha = \beta/1200$. Another note that we can make is that this equation has not take into account for inflation "ι". If you would like to take into account the inflation then the definition of α in the equation will be $\alpha = (\beta - \iota)/1200$.

We focus now on study the $\sum \Delta M_n$ function which as investors we would like to maximize

$$\sum_{n=1}^{t} \Delta M_n = \left(I_0 + \frac{1+\alpha}{\alpha} I_m\right) \left((1+\alpha)^t - 1\right) - I_m t.$$

From the above equation, we see that the investment increase linearly with our savings, but exponentially! with the time and the monthly return on investment, offered by the company/government-institution. We could make a lot of more conclusions and refine even more our model. For that, we should first see the model in action. Let's end this article leaving the invitation to prove the simulator inversions that we made with this model.

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