Problem Sheet: Chain Rule and Substitutions 1 Solution

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Exercise 1. Differentiate the following functions.

1. $\ln(5x)$
2. e^{-20x^2}
3. $\sin(16x^3)$
4. $\cos\left(\frac{2}{x}\right)$

Solution.

1. Differentiating $\ln(5x)$

   Let
   $$y = \ln(5x), \qquad u = 5x$$
   so that $y = \ln u$.

   We compute
   $$\frac{dy}{du} = \frac{1}{u} \qquad \text{(using the derivative of } \ln u\text{)}$$
   $$\frac{du}{dx} = 5 \qquad \text{(using the derivative of } 5x\text{)}$$

   Therefore
   $$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= \frac{1}{u} \cdot 5 \\ &= \frac{5}{5x} \\ &= \frac{1}{x} \end{aligned}$$

2. Differentiating e^{-20x^2}

   Let
   $$y = e^{-20x^2}, \qquad u = -20x^2$$
   so that $y = e^u$.

   We compute
   $$\frac{dy}{du} = e^u \qquad \text{(using the derivative of } e^u\text{)}$$
   $$\frac{du}{dx} = -40x \qquad \text{(using the derivative of } -20x^2\text{)}$$

Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= e^u(-40x)\\ &= -40xe^{-20x^2}\end{aligned}$$

3. Differentiating $\sin(16x^3)$

   Let

   $$y = \sin(16x^3), \qquad u = 16x^3$$

   so that $y = \sin u$.

   We compute

   $$\begin{aligned}\frac{dy}{du} &= \cos u && \text{(using the derivative of } \sin u\text{)}\\ \frac{du}{dx} &= 48x^2 && \text{(using the derivative of } 16x^3\text{)}\end{aligned}$$

   Therefore

   $$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= \cos u \cdot 48x^2\\ &= 48x^2\cos(16x^3)\end{aligned}$$

4. Differentiating $\cos\left(\frac{2}{x}\right)$

   Let

   $$y = \cos\left(\frac{2}{x}\right), \qquad u = \frac{2}{x}$$

   so that $y = \cos u$.

   We compute

   $$\begin{aligned}\frac{dy}{du} &= -\sin u && \text{(using the derivative of } \cos u\text{)}\\ \frac{du}{dx} &= -\frac{2}{x^2} && \text{(using the derivative of } 2x^{-1}\text{)}\end{aligned}$$

   Therefore

   $$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx}\\ &= (-\sin u)\left(-\frac{2}{x^2}\right)\\ &= \frac{2\sin\left(\frac{2}{x}\right)}{x^2}\end{aligned}$$

$\square$

Exercise 2. Evaluate the following indefinite integrals.

1. $\displaystyle\int \sin^2 x\,dx$
2. $\displaystyle\int \cos^2 x\,dx$

Solution.

1. Integrating $\sin^2 x$

   We use the identity

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

   Therefore

$$\begin{aligned}\int \sin^2 x\, dx &= \int \frac{1 - \cos(2x)}{2}\, dx \\ &= \frac{1}{2}\int 1\, dx - \frac{1}{2}\int \cos(2x)\, dx \\ &= \frac{1}{2}x - \frac{1}{2}\cdot\frac{1}{2}\sin(2x) + c \\ &= \frac{x}{2} - \frac{1}{4}\sin(2x) + c\end{aligned}$$

2. Integrating $\cos^2 x$

   We use the identity

$$\cos^2 x = \frac{1 + \cos(2x)}{2}$$

   Therefore

$$\begin{aligned}\int \cos^2 x\, dx &= \int \frac{1 + \cos(2x)}{2}\, dx \\ &= \frac{1}{2}\int 1\, dx + \frac{1}{2}\int \cos(2x)\, dx \\ &= \frac{1}{2}x + \frac{1}{2}\cdot\frac{1}{2}\sin(2x) + c \\ &= \frac{x}{2} + \frac{1}{4}\sin(2x) + c\end{aligned}$$

□

Exercise 3. For each integral below, state a suitable substitution before evaluating the indefinite integral.

1. $\displaystyle\int \ln(5x)\, dx$
2. $\displaystyle\int e^{-20x}\, dx$
3. $\displaystyle\int \sin(16x)\, dx$
4. $\displaystyle\int \cos(2x)\, dx$

Solution.

1. Integrating $\int \ln(5x)\, dx$

   We use the substitution

$$u = 5x$$

   so that $du = 5\, dx$ and $dx = \frac{1}{5}du$.

   Then

$$\begin{aligned}\int \ln(5x)\, dx &= \frac{1}{5}\int \ln u\, du \\ &= \frac{1}{5}(u \ln u - u) + c \\ &= \frac{1}{5}(5x \ln(5x) - 5x) + c \\ &= x \ln(5x) - x + c\end{aligned}$$

2. Integrating $\int e^{-20x}\,dx$

   We use the substitution
$$u = -20x$$
   so that $du = -20\,dx$ and $dx = -\frac{1}{20}du$.

   Then
$$\begin{aligned}\int e^{-20x}\,dx &= -\frac{1}{20}\int e^{u}\,du\\ &= -\frac{1}{20}e^{u} + c\\ &= -\frac{1}{20}e^{-20x} + c\end{aligned}$$

3. Integrating $\int \sin(16x)\,dx$

   We use the substitution
$$u = 16x$$
   so that $du = 16\,dx$ and $dx = \frac{1}{16}du$.

   Then
$$\begin{aligned}\int \sin(16x)\,dx &= \frac{1}{16}\int \sin u\,du\\ &= -\frac{1}{16}\cos u + c\\ &= -\frac{1}{16}\cos(16x) + c\end{aligned}$$

4. Integrating $\int \cos(2x)\,dx$

   We use the substitution
$$u = 2x$$
   so that $du = 2\,dx$ and $dx = \frac{1}{2}du$.

   Then
$$\begin{aligned}\int \cos(2x)\,dx &= \frac{1}{2}\int \cos u\,du\\ &= \frac{1}{2}\sin u + c\\ &= \frac{1}{2}\sin(2x) + c\end{aligned}$$

$\square$