CHAPTER - 2

RANDOM VARIABLES AND DISTRIBUTION FUNCTIONS

2.1 INTRODUCTION

Previously we considered the concept of random experiment, events, sample space and sample points. The events described on a random experiment may be numerical or non-numerical (descriptive). For example, the outcomes that we obtain, when we throw a die are numerical. (We get the outcomes as 1, 2,3, 4,5 or 6). Hence the sample space is numerical. But, the outcomes we obtain, when we toss a coin are non-numerical. We get the outcomes as head or tail. Here the sample - space is non-numerical or descriptive.

It is inconvenient to deal with these descriptive outcomes mathematically. Hence, for ease of manipulation, we may assign a real number to each of the outcomes using a fixed rule or mapping. For example, when we toss a coin we get two outcomes, namely head or tail; we can assign numerical values, say '1' to head and '0' to tail. This interpretation is easy and attractive from mathematical point of view and also practically meaningful.

This rule or mapping from the original sample space (numerical or non-numerical) to a numerical (real) sample space, subjected to certain constraints is called a random variable. Thus Random variable is a real valued function which maps the numerical or non-numerical sample space (domain) of the random experiment to real values (codomain or range).

2.2 RANDOM VARIABLE

We know that a variable is a quality which changes or varies - the change may occur due to time factor or any factor. For example, the height of a person vary with age, but the age of a person changes with time. Every variable has a range in which it can take any value. Examples are: marks scored by a student in an examination, number of children in a family, height and weight of a person, etyc. The variable could be **continuous** or **discrete**. A continuous variable takes all possible values in its range. For example, the height of a person can take any value in a certain range, but for the sake of convenience, it is measured only up to the accuracy of inches or cms. Similarly, the weight of a person can be any value, but it is usually expressed in kgs. A discrete variable takes only certain values in a range. For example, number of children in a family, number on a dice, etc can take only integral values.

A variable is called a random variable when there is a chance factor associated with the various values, which it can take in its range. For example, the life of a person is a random variable and there are probabilities associated with various age values.

Def. : *A real variable X whose value is determined by the outcome of a random experiment is called a random variable.* A random variable X can also be regarded as a real - value function defined on the sample space S of a random experiment such that for each point x of the sample space, $f(x)$ is the probability of occurrence of the event represented by x.

Illustration: The sample space corresponding to tossing of two coins.

Who two coins are tossed, its outcomes or sample points can be (Head, Head), (Head, Tail), (Tail, Head), (Tail, Tail) *i.e.*, S = {HH, HT, TH, TT}. After the performance of the experiment, we count the number of tails and denote it by X. The first outcome HH has 0 Tail, so X = 0. Similarly X = 1, denote the outcomes HT or TH and X = 2, represents the outcome TT.

Thus X takes the values 0,1,2 i.e., X = 0,1,2.

A random variable is also called a stochastic variable or simply a variable or a chance variable. The name random variable is given to the variable X because it is defined on a sample space, the outcomes of which are uncertain and hence depend on chance.

The random variable assigns a real value $X(w)$ such that

1. The set $\{w \mid X(w) \leq x\}$ is an event for every $x \in R$, for which a probability is defined. This condition is called as measurability.
2. The probabilities of the events $\{w \mid X(w) = \infty\}$ and $\{w \mid X(w) = -\infty\}$ are equal to zero. *i.e.* $P(X = \infty) = P(X = -\infty) = 0$
3. For every set $A \subset S$ there corresponds a set $T \subset R$ called the image of A. Also for every $T \subset R$ there exists in S the inverse image $X^{-1}(T)$.
   i.e. $X^{-1}(T) = \{w \in S \mid X(w) \in T\}$

This set is an event in S which has a probability $P[X^{-1}(T)]$.

Notation : Random variables are usually denoted by capital letters of English alphabets and particular values which the random variable takes are denoted by the corresponding small letters.

Ex. 1. Cosider a random experiment consisting of tossing a coin twice. The sample space $S = \{s_1, s_2, s_3, s_4\}$, where $s_1 = HH$, $s_2 =$ HT, $s_3 = TH$ and $s_4 = TT$ consists of four elements (sample points). Define a function $X : S \to R$ by $X(s)$ = "number of heads".

Then $X(s_1) = 2$, $X(s_2) = 1$, $X(s_3) = 1$ and $X(s_4) = 0$

$$\begin{aligned} \text{Range of } X &= \{X(s) : s \in S\} \\ &= \{X(s_1), X(s_2), X(s_3), X(s_4)\} \\ &= \{0, 1, 2\} \end{aligned}$$

Ex. 2. Consider the random experiment of throwing a pair of dice and noting the sum. The sample space consists of points S = {(1, 1), (1, 2), ..., (1, 6), (2, 1), (2, 2), ... (2, 6), ..., (6, 1), (6, 2), ..., (6, 6)}.

If X is the "sum of numbers appearing on the faces of dice", X is a random variable. Here $X : S \to R$ is defined through $X((i, j)) = i + j$ for $(i, j) \in S$.

If in a random experiment, the event corresponding to a number a occurs, then the corresponding random variable X is said to assume the value a and the probability of the event is denoted by $P(X = a)$. Similarly, the probability of the event X assuming any value between a and b is denotd by $P(a < X < b)$. The probability of the event $X \leq c$ (X assumes any value smaller than or equal to c) is denoted by $P(X \leq c)$ and the probability of the event $X > c$ is denoted by $P(X > c)$. That is, $P(X > c)$ is simply the probability of the set of outcomes s for which $X(s) > c$ or $P(X > c) = P\{s \mid X(s) > c\}$.

For the example (1) given above, we have

$$P(X \leq 1) = P\{HH, HT, TH\} = \frac{3}{4}$$

If X and Y are random variables on the sample space S, then $X + Y$, $X + K$, KX and KY (K is a constant) are also random variabels on S and are defined by $(X + Y)(s) = X(s) + Y(s)$, $(X + K)(s) = X(s) + K$, $K X (s) = K \cdot X(s)$, etc.

Note :
1. $P(X \leq c) + P(X > c) = P(-\infty < X < \infty)$
2. $P(X > c) = 1 - P(X \leq c)$
3. $P(X = a \text{ or } X = b) = P\{(X = a) \cup (x = b)\}$
4. $P(X = a \text{ and } X = b) = P\{(X = a) \cap (X = b)\}$

2.3 TYPES OF RANDOM VARIABLES

Random variable is of two types :

(*i*) Discrete Random Variable

(*ii*) Continuous Random Variable

1. Discrete Random Variable : A random variable X which can take only a finite number of discrete values in an interval of domain is called a *discrete random variable.* In other words, if the random variable takes the values only on the set $\{0,1,2,\ldots\ldots,n\}$ is called a Discrete Random variable.

Tossing of a coin, throwing a dice, the number of defectives in a sample of electric bulbs, the number of printing mistakes in each page of a book, the number of telephone calls received by the telephone operator are examples of Discrete Random variables.

Thus to each outcome 's' of a random experiment there corresponds a real number $X(s)$ which is defined for each point of the sample S.

A few examples are :

(*i*) In the example (1), $X(s) = \{s : s = 0, 1, 2\}$ or Range of $X = \{0, 1, 2\}$.

The random variable X is a discrete random variable.

(*ii*) The random variable denoting the number of students in a class is

$$X(x) = \{x : x \text{ is a positive integer}\}$$

2. Continuous Random Variable : A random variable X which can take values continuously i.e., which takes all possible values in a given interval is called a *continuous random variable.*

For example, the height, age and weight of individuals are examples of continuous random variable. Also temperature and time are continuous random variables.

2.4 PROBABILITY FUNCTION

Illustration 1 : Distribution of Probability over varous numbers on Dice

When a single dice is thrown, the probability associatged with each number on the top face of the dice can take the values with probability being same is 1/6 for all the numbers 1 to 6. This can be depicted with the help of a diagram as follows:

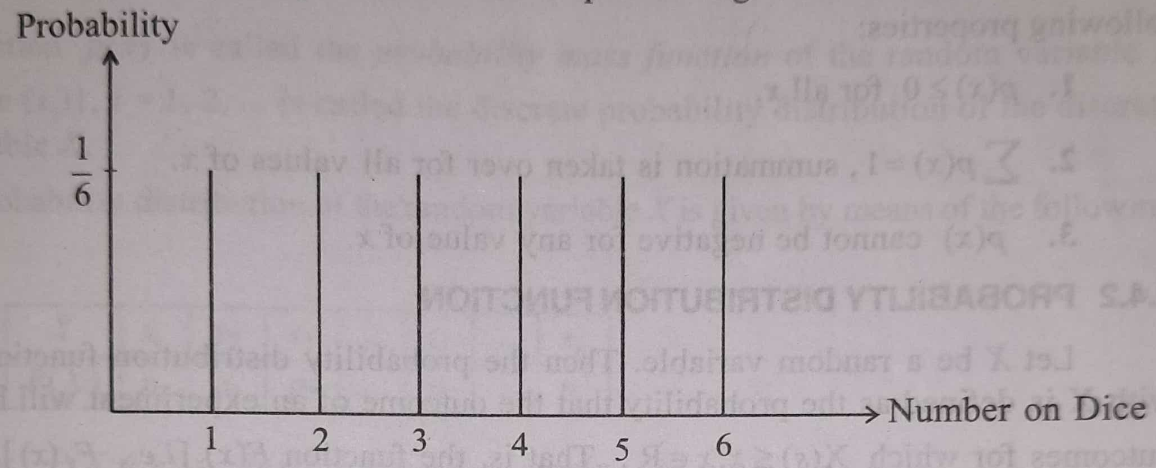
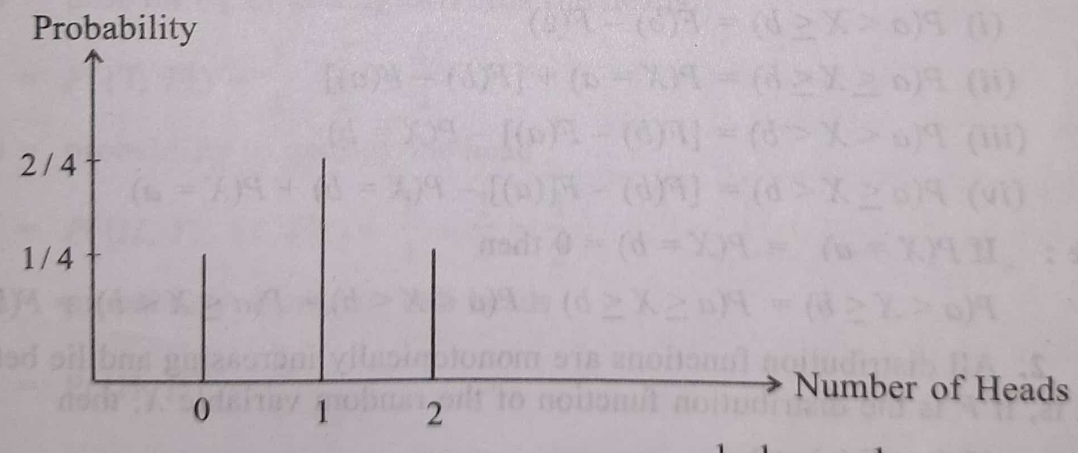


Illustration 2: Distribution of Probability over number of Heads

When two coins are tossed, the number of Heads is a random variable which can take the values 0,1 or 2. The associated probabilities are $\frac{1}{4}, \frac{1}{2}$ and $\frac{1}{4}$ and is shown below:



It may be noted that total of all the three probabilities $\frac{1}{4}, \frac{1}{2}$ and $\frac{1}{4}$ is 1. This is so because the random variable is bound to take one of the three values 0,1 or 2 in its range.

The above diagrams shows how the total probability (= 1) is distributed over the range of the variable. So, these diagrams are referred to graphs of probability distributions. In many situations, these graphs are described by a mathematical function which is referred to as a Probability Function or Statistical distribution.

2.4.1 PROBABILITY FUNCTION OF A DISCRETE RANDOM VARIABLE

If for a discrete random variable X, the real valued function $p(x)$ is such that

$$P(X = x) = p(x)$$

then $p(x)$ is called probability function or probability density function of a discrete random variable X. Probability function $p(x)$ gives the measure of probability for different values of X.

Properties of a Probability Function

If $p(x)$ is a probability function of a random variable X, then it possesses the following properties:

1. $p(x) \geq 0$ for all x.
2. $\sum p(x) = 1$, summation is taken over for all values of x.
3. $p(x)$ cannot be negative for any value of x.

2.4.2 PROBABILITY DISTRIBUTION FUNCTION

Let X be a random variable. Then the probability distribution function associated with X is defined as the probability that the outcome of an experiment will be one of the outcomes for which $X(s) \leq x, x \in R$. That is, the function $F(x)$ [*i.e.*, $F_X(x)$] defined by $F_X(x) = P(X \leq x) = P\{s : X(s) \leq x\}, -\infty < x < \infty$ is called the distribution function of X.

Properties of Distribution Function : **[JNTU (K) Dec. 2013 (Set No. 3)]**

1. If F is the distribution function of a random variable X and if $a < b$, then

(*i*) $P(a < X \leq b) = F(b) - F(a)$

(*ii*) $P(a \leq X \leq b) = P(X = a) + [F(b) - F(a)]$

(*iii*) $P(a < X < b) = [F(b) - F(a)] - P(X = b)$

(*iv*) $P(a \leq X < b) = [F(b) - F[(a)] - P(X = b) + P(X = a)$

Note : If $P(X = a) = P(X = b) = 0$ then

$$P(a < X \leq b) = P(a \leq X \leq b) = P(a < X < b) = P(a \leq X < b) = F(b) - F(a)$$

2. All distribution functions are monotonically increasing and lie between 0 and 1. That is, if F is the distribution function of the random variable X, then

(*i*) $0 \leq F(x) \leq 1$

(*ii*) $F(x) < F(y)$ when $x < y$.

3. (*i*) $F(-\infty) = \underset{x \to -\infty}{\text{Lt}} F(x) = 0$

(*ii*) $F(\infty) = \underset{x \to \infty}{\text{Lt}} F(x) = 1$

2.5 DISCRETE PROBABILITY DISTRIBUTION (PROBABILITY MASS FUNCTION)

Probability distribution of a random variable is the set of its possible values together with their respective probabilities. Suppose X is a discrete random variable with possible outcomes (values) $x_1, x_2, x_3, ...$ The probability of each possible outcome x_i is $p_i = P(X = x_i) = p(x_i)$ for $i = 1, 2, 3, ...$

If the numbers $p(x_i)$, $i = 1, 2, 3, ...$ satisfy the two conditions

(i) $p(x_i) > 0$ for all values of i ; $0 < p_i \leq 1$

(ii) $\Sigma p(x_i) = 1, i = 1, 2, 3, ...,$

then the function $p(x)$ is called the *probability mass function* of the random variable X and the set $\{p(x_i)\}$, $i = 1, 2, ...$ is called the discrete probability distribution of the discrete random variable X.

The probability distribution of the random variable X is given by means of the following table :

X	x_1	x_2	x_3	....	x_i	....	x_n
$P(X)$	p_1	p_2	p_3	....	p_i	....	p_n

Further $P(X < x_i) = p(x_1) + p(x_2) + ... + p(x_{i-1})$

$P(X \leq x_i) = p(x_1) + p(x_2) + ... + p(x_{i-1}) + p(x_i)$

and $P(X > x_i) = 1 - P(X \leq x_i)$

Ex. 1. In tossing a coin two times, $S = \{TT, HT, TH, HH\}$

$P(X = 0)$ = probability of getting two tails (no heads)

$$= P(\{T, T\}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$P(X = 1)$ = probability of getting one head

$$= P(\{H, T\}, \{T, H\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$P(X = 2)$ = probability of getting two heads

$$= P(\{H, H\}) = \frac{1}{4}$$

Thus the total probability 1 is distributed into three parts $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$ according to whether $X = 0$ or 1 or 2. This probability distribution is given in the following table :

$X = x_i$	0	1	2
$P(X = x_i)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Ex. 2. Let X be the random variable which represents the sum of numbers of points on throwing two unbiased dice. If the point shown in each die is equal to one, the minimum sum of numbers is equal to two. If both dice show six, then the sum of numbers at the maximum is 12.

Thus if two unbiased dice are thrown, then the sum X of the two numbers which turn up must be an integer between 2 and 12. For $X = 2$, there is only one favourable point (1, 1) and hence $P(X = 2) = 1/36$, since there are 36 sample points in all. For $X = 3$, there are two favourable sample points (1, 2) and (2, 1) and hence $P(X = 3) = 2/36$. Similarly for $X = 4$, there are three favourable sample points (1, 3,), (2, 2), (3, 1) and hence $P(X = 4) = 3/36$ and so on. Now the probability distribution in this case is given by the following table :

$X = x_i$	2	3	4	5	6	7	8	9	10	11	12
$P(X = x_i)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Clearly $\Sigma p(x_i) = p_1 + p_2 + ... + p_{11} = 1$

From the above table, we have

$$P(X \geq 10) = \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{6}{36} = \frac{1}{6}$$

$$P(X \leq 6) = \frac{1}{36} + \frac{2}{36} + \frac{3}{36} + \frac{4}{36} + \frac{5}{36} = \frac{15}{36} = \frac{5}{12}$$

and $$P(7 < X < 12) = \frac{5}{36} + \frac{4}{36} + \frac{3}{36} + \frac{2}{36} = \frac{14}{36} = \frac{7}{18}$$

In the above example, X takes only a finite number of values as such it is a discrete random variable.

Note : The concept of **probability distribution function** is analogous to that of **frequency distribution**. A frequency distribution tells us how the total frequency is distributed among different values of the variable, whereas a probability distribution tells us how total probability 1 is distributed among the values which the random variable can take.

Cumulative Distribution Function of a Discrete Random Variable

There are many occasions in which it is of interest to know the probability that the value of a random variable is less than or equal to some real number x.

Suppose that X is a discrete random variable. Then the Discrete Distribution function or Cumulative Distribution function $F(x)$ is defined by

$$F(x) = P(X \leq x) = \sum_{(i:\, x_i \leq x)} p(x_i) = \sum_{i=1}^{x} p(x_i) \text{ where } x \text{ is any integer.}$$

(OR) $F(x) = P(X \leq x) = \sum_{t \leq x} f(t)$, where $-\infty < x < \infty$ and $f(t)$ is the value of the probability distribution of X at t.

Aliter : If $p(x)$ *i.e.,* $f(x)$ is the probability function or probability distribution, then the value of $\sum_{x=0}^{x} p(x)$ *i.e.,* $\sum_{x=0}^{x} f(x)$ denoted by $F(x)$ is called the **Cumulative Distribution Function** or simply **Distribution Function**. It is similar to cumulative frequency *i.e.*, the cumulative frequency upto a given point or value. It gives the probability that the random variable takes any value from 0 (lowest value that the variable can take) to a given value.

For example,

$$F(2) = \sum_{x=0}^{2} f(x) = f(0) + f(1) + f(2) = p(0) + p(1) + p(2)$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1 \text{ (in Ex. 1 above)}$$

Suppose if X takes only a finite number of values $x_1, x_2, \ldots, x_n$ then the distribution function is given by

$$F(x) = \begin{cases} 0, -\infty < x < x_1 \\ p(x_1), x_1 \le x < x_2 \\ p(x_1) + p(x_2), x_2 \le x < x_3 \\ .. \\ .. \\ .. \\ p(x_1) + p(x_2) + \ldots + p(x_n), x_n \le x < \infty \end{cases}$$

Probability Density Function :

The probability density function $f_X(x)$ is defined as the derivative of the probability distribution function, $F_X(x)$ of the random variable X.

Thus $f_X(x) = \frac{d}{dx}[F_X(x)]$

2.6 EXPECTATION, MEAN, VARIANCE AND STANDARD DEVIATION OF A PROBABILITY DISTRIBUTION

The behaviour of a random variable is completely characterized by the distribution function $F(x)$ or density function $f(x)$ or $P(x_i)$. Instead of a function, a more compact description can be made by a single numbers such as mean, median and mode known as measures of central tendency of the random variable X.

EXPECTATION

In this section, We discuss the application of the concepts of probability theory to real life situations when the decisions are based on expectations about the value of a variable like life of an item, say electric bulb, steel, cement, scooter battery, etc.

When a dice is thrown, we know that the variable representing the top number on the dice can be any value from 1 to 6 with probability 1/6. Now, suppose a person is not interested in listening to this statement giving the range of the variable, but all we wants to know that single number which is expected to come up when the dice is thrown. The answer to these types of situations when one wants the reply as a single value, is provided by the theory of expectation. The theory is discussed separately for discrete and continuous variables.

Expectation theory plays a very important role in decision - making because most of the time we take decisions based on what is expected to happen.

(1) Expectation of a Discrete Variable : 1(a)

As defined earlier, a discrete variable takes only some finite values like number on a dice, number of children in a family, etc.

Suppose a random variable X assumes the values $x_1, x_2, ..., x_n$ with respective probabilities $p_1, p_2, ..., p_n$. Then the mathematical expectation or mean or expected value of X, denoted by $E(X)$, is defined as the sum of products of different values of x and the corresponding probabilities.

$$\therefore \ E(X) = x_1 p_1 + x_2 p_2 + ... + x_n p_n$$

$$\boxed{i.e., \quad E(X) = \sum_{i=1}^{n} p_i \, x_i}$$

Similarly, $E(x^r) = \sum_{i=1}^{n} p_i \cdot x_i^r$

In general, the expected value of any function $g(x)$ of a random variable X is defined as

$$\boxed{E[g(x)] = \sum_{i=1}^{n} p_i \, g(x_i)}$$

Note : Expected value of X is a population mean. If population mean is μ then $E(X) = \mu$.

Some Important Results on Expectation :

(*i*) If X is a random variable and K is a constant, then

$$E(X + K) = E(X) + K$$

[JNTU 2005S, 2005, 2007S, (H) Nov. 2009 (Set No.1)]

Proof : By definition, we have

$$E(X + K) = \sum_{i=1}^{n} (x_i + K) p_i = \sum_{i=1}^{n} x_i \, p_i + K \sum_{i=1}^{n} p_i$$

$$= E(X) + K(1) = E(X) + K \qquad \left[\because \sum_{i=1}^{n} p_i = 1\right]$$

Note 1 : Expected value of constant term is constant, that is, if K is constant, then $E(K) = K \quad [\because E(K) = \sum p_i K = K \sum p_i = K(1) = K]$.

Note 2 : If K is constant, then $E(KX) = KE(X)$

$$E(KX) = \sum_{i=1}^{n} (K x_i) p_i = K \sum_{i=1}^{n} x_i p_i = K E(X), \text{ where } K \text{ is a constant.}$$

(*ii*) If X is a random variable and a and b are constants, then

$$E(aX \pm b) = a E(X) \pm b$$

Proof : Proceed as in (*i*)

(*iii*) If X and Y are any two random variables, then

$E(X + Y) = E(X) + E(Y)$ provided $E(X)$ and $E(Y)$ exist.

[JNTU 2005 S, 2005, 2007S, (H) Nov. 2009 (Set No.1)]

Proof : Let X assume the values $x_1, x_2, \ldots, x_n$ and Y assume the values $y_1, y_2, \ldots, y_m$. Then by definition,

$$E(X) = \sum_{i=1}^{n} p_i x_i \text{ and } E(Y) = \sum_{j=1}^{m} p_j y_j$$

Let $p_{ij} = P(X = x_i \cap Y = y_j) = p(x_i, y_i)$

[This is called the joint probability function of X and Y]

The sum $(X + Y)$ is also a random variable which can take $m \times n$ values $(x_i + y_i)$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

$\therefore$ By definition,

$$E(X + Y) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_{ij}(x_i + y_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} p_{ij} x_i + \sum_{i=1}^{n} \sum_{j=1}^{m} p_{ij} y_j$$

$$= \sum_{i=1}^{n} \left[x_i \sum_{j=1}^{m} p_{ij} \right] + \sum_{j=1}^{m} \left[y_j \sum_{i=1}^{n} p_{ij} \right]$$

$$= \sum_{i=1}^{n} x_i p_i + \sum_{j=1}^{m} y_j p_j = E(X) + E(Y)$$

Note : 1. $E(X + Y + Z) = E(X + (Y + Z)) = E(X) + E(Y + Z) = E(X) + E(Y) + E(Z)$

2. $E(aX + bY) = a\,E(X) + b\,E(Y)$, where a and b are constants

3. $E(X - \overline{X}) = 0$

(*iv*) If X and Y are two independent random variables, then

$$E(XY) = E(X)\,E(Y)$$

Proof : Let X assume the values $x_1, x_2, \ldots, x_n$ with respective probabilities $p_1, p_2, \ldots, p_n$ and Y assume the values $y_1, y_2, \ldots, y_m$ with probabilities $p'_1, p'_2, \ldots, p'_m$.

Then by definition,

$$E(X) = \sum_{i=1}^{n} p_i x_i \text{ and } E(Y) = \sum_{j=1}^{m} p'_j y_j$$

Let $p_{ij} = P(X = x_i \cap Y = y_j)$

$= P(X = x_i) \cdot P(Y = y_j)$ (since X and Y are independent)

$= p_i \, p'_j$

The product XY is a random variable which can assume $m \times n$ values $x_i y_j$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$.

$\therefore$ By definition,

$$E(XY) = \sum_{i=1}^{n}\sum_{j=1}^{m} x_i \, y_j \, p_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{m} x_i \, y_j \, p_i \, p_j'$$

$$= \left(\sum_{i=1}^{n} p_i \, x_i\right)\left(\sum_{j=1}^{n} p_j' \, y_j\right) = E(X)\,E(Y)$$

Note : $E(XYZ) = E(X.YZ) = E(X) \cdot E(YZ) = E(X)\,E(Y)\,E(Z)$

Most of the concepts discussed with the frequency distributions apply equally well to distribution functions.

Important Observation : $E\left(\frac{1}{X}\right)$ and $\frac{1}{E(X)}$ are not same.

(2) Mean :

The mean value μ of the discrete distribution function is given by

$$\mu = \frac{\Sigma \, p_i x_i}{\Sigma \, p_i} = \Sigma \, p_i \, x_i = E(X)$$

Note : If $E(X) = \mu$, then $E(X - \mu) = 0$.

(3) Variance :

Variance characterizes the variability in the distributions since two distributions with same mean can still have different dispersion of data about their means.

Variance of the probability distribution of a random variable X is the mathematical expectation of $[X - E(X)]^2$. Then

$$\text{Var}(X) = E[X - E(X)]^2 \text{ i.e., } \text{Var}(X) = \sum_{i=1}^{n}\left\{[x_i - E(X)]^2 \times p(x_i)\right\}$$

Another Form of Variance :

If X is a random variable, then the mathematical expectation of $(X - \mu)^2$ is defined to be the variance of the random variable X. Then

$$\text{Var}(X) = E(x - \mu)^2 = \sum_{i=1}^{n} p_i \, (x_i - \mu)^2$$

$$= \Sigma \, (x_i^2 - 2\mu x_i + \mu^2) p_i$$

$$= \Sigma x_i^2 p_i - 2\mu \Sigma x_i p_i + \mu^2 \Sigma p_i$$

$= E(X^2) - 2\mu.\mu + \mu^2.1 \quad [\because \mu = \sum x_i p_i, \sum p_i = 1]$

$\therefore \sigma^2 = E(X^2) - \mu^2$

i.e., $\boxed{\text{Var}(X) = E(X^2) - [E(X)]^2}$

Note : The variance of a random variable X is also denoted by V(X).

(4) **Standard Deviation :** It is the positive square root of the variance.

$$\therefore \text{S.D} = \sigma = \sqrt{\sum_{i=1}^{n} p_i x_i^2 - \mu^2} = \sqrt{E(X^2) - \mu^2} = \sqrt{E[X - E(X)]^2}$$

Some Important Results on Variance :

1. Variance of constant is zero *i.e.,* $V(K) = 0$

2. If K is a constant, then $V(KX) = K^2 V(X)$

3. If X is a random variable and K is a constant, then $V(X + K) = V(X)$

4. If X is a discrete random variable, then $V(aX + b) = a^2 V(X)$, where $V(X)$ is variance of X and a, b are constants.

Proof. Let $Y = aX + b$ ... (1)

Then $E(Y) = E(aX + b) = aE(X) + b$ ... (2)

(1) – (2) gives $Y - E(Y) = a[X - E(X)]$

Squaring and taking expectation of both sides, we get

$$E[\{Y - E(Y)\}^2] = a^2 E[\{X - E(X)\}^2] \quad i.e., \quad V(Y) = a^2 V(X)$$

or $V(aX + b) = a^2 V(X)$

Cor. (*i*) If $b = 0$, then $V(aX) = a^2 V(X)$

(*ii*) If $a = 0$, then $V(b) = 0$

(*iii*) If $a = 1$, then $V(X + b) = V(X)$

5. If X and Y are two independent random variables, then $V(X \pm Y) = V(X) \pm V(Y)$.

SOLVED EXAMPLES

Example 1 : Let X denote the number of heads in a single toss of 4 fair coins. Determine (*i*) $P(X < 2)$ (*ii*) $P(1 < X \leq 3)$. **[JNTU 2006 (Set No. 4)]**

Solution : The required probability distribution is

X	0	1	2	3	4
$P(X)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

(*i*) $P(X < 2) = P(X = 0) + P(X = 1) = \frac{1}{16} + \frac{4}{16} = \frac{5}{16}$

(*ii*) $P(1 < X \leq 3) = P(X = 2) + P(X = 3) = \frac{6}{16} + \frac{4}{16} = \frac{10}{16} = \frac{5}{8}$

Example 2 : Two dice are thrown. Let X assign to each point (a, b) in S the maximum of its numbers i.e., $X(a, b) = max.(a, b)$. Find the probability distribution. X is a random variable with $X(s) = \{1, 2, 3, 4, 5, 6\}$. Also find the mean and variance of the distribution.

[JNTU 2004, 2007, 2008S, (A) Dec. 2009, Nov. 2010 (Set No. 4)]

(OR) A random variable X has the following distribution ?

x	1	2	3	4	5	6
P(x)	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

Find (a) the mean (b) variance (c) $P(1 < x < 6)$ **[JNTU (H) Apr. 2012 (Set No. 1)]**

Solution : The total number of cases are $6 \times 6 = 36$.

The maximum number could be 1,2, 3, 4, 5, 6 i.e., $X(s) = X(a, b) = \max(a, b)$.

The number 1 will appear only in one case (1, 1). So $p(1) = P(X = 1) = P(1, 1) = \frac{1}{36}$

For maximum 2, favourable cases are (2, 1), (2, 2), (1, 2)

So $p(2) = P(X = 2) = 3/36$

For maximum 3, favourable cases are (1, 3), (3, 1), (2, 3), (3, 2), (3, 3).

So $p(3) = P(X = 3) = 5/36$

For maximum 4, favourable cases are (1, 4),(4, 1),(2, 4),(4, 2),(3, 4),(4, 3),(4, 4)

So $p(4) = P(X = 4) = 7/36$

$$\begin{aligned}\text{Similarly, } p(5) &= P(X = 5)\\ &= P((1, 5), (5, 1), (2, 5), (5, 2), (3, 5), (5, 3), (4, 5), (5, 4),(5, 5))\\ &= \frac{9}{36}\\ p(6) &= P(X = 6)\\ &= P((1, 6), (6, 1), (2, 6), (6,2), (3, 6), (6, 3), (4, 6), (6, 4), (5, 6),\\ &\qquad (6, 5), (6, 6))\\ &= \frac{11}{36}\end{aligned}$$

$\therefore$ The required discrete probability distribution is

$X = x_i$	1	2	3	4	5	6
$P(X = x_i) = p(x_i)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

(*i*) $$\text{Mean, } \mu = \sum_{i=1}^{6} p_i x_i = 1 \cdot \frac{1}{36} + 2.\frac{3}{36} + 3.\frac{5}{36} + 4.\frac{7}{36} + 5.\frac{9}{11} + 6.\frac{11}{36}$$

$$= \frac{1}{36}(1 + 6 + 15 + 28 + 45 + 66) = \frac{161}{36} = 4.47$$

(*ii*) $$\text{Variance, } \sigma^2 = \sum_{i=1}^{6} p_i x_i^2 - \mu^2$$

$$= \frac{1}{36}(1)^2 + \frac{3}{36}(2)^2 + \frac{5}{36}(3)^2 + \frac{7}{36}(4)^2 + \frac{9}{36}(5)^2 + \frac{11}{36}(6)^2 - (4.47)^2$$

$$= \frac{1}{36}(1 + 12 + 45 + 112 + 225 + 396) - (4.47)^2$$

$$= \frac{791}{36} - 19.9808 = 21.97 - 19.981 = 1.9912$$

Example 3 : A random variable X has the following probability function :

x	0	1	2	3	4	5	6	7
$p(x)$	0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2 + K$

(*i*) Determine K (*ii*) Evaluate $P(X < 6)$, $P(X \geq 6)$, $P(0 < X < 5)$ and $P(0 \leq X \leq 4)$

(*iii*) if $P(X \leq K) > \frac{1}{2}$, find the minimum value of K and, (*iv*) Determine the distribution function of X (*v*) Mean (*vi*) variance.

[JNTU 04S, 05S, 08S, (A) Nov. 11, (H) Dec. 11, (K) May 10S, Nov. 2012, Mar. 2014 (Set No. 2)]

Solution :

(*i*) Since $\sum_{x=0}^{7} p(x) = 1$, we have

$$K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$$

i.e., $10K^2 + 9K - 1 = 0$ i.e., $(10K - 1)(K + 1) = 0$

$\therefore$ $K = \frac{1}{10} = 0.1$ (since $p(x) \geq 0$, so $K \neq -1$)

(*ii*) $P(X < 6) = P(X = 0) + P(X = 1) + ... + P(X = 5)$

$= 0 + K + 2K + 2K + 3K + K^2 = 8K + K^2 = 0.8 + 0.01 = 0.81$

$[\because K = 0.1]$

$P(X \geq 6) = 1 - P(X < 6) = 1 - 0.81 = 0.19$

$P(0 < X < 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$

$= K + 2K + 2K + 3K = 8K = \frac{8}{10} = 0.8$

$P(0 \leq X \leq 4) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$

$= 0 + K + 2K + 2K + 3K = 8K = 8(0.1) = 0.8$

Note : $P(X \leq 5) = P(X = 0) + P(X = 1) + ... + P(X = 5)$

$= 0.81$ (Refer (*ii*))

$P(X > 5) = 1 - P(X \leq 5) = 1 - 0.81 = 0.19$

$P(0 < X < 6) = P(X = 1) + P(X = 2) + ... + P(X = 5) = 0.81$

(*iii*) The required minimum value of K is obtained as below.

$P(X \leq 1) = P(X = 0) + P(X = 1) = 0 + K = \frac{1}{10} = 0.1$

$$P(X \le 2) = [P(X=0)+P(X=1)]+P(X=2)$$

$$= \frac{1}{10}+\frac{2}{10}=\frac{3}{10}=0.3$$

$$P(X \le 3) = [P(X=0)+P(X=1)+P(X=2)]+P(X=3)=0.3+0.2=0.5$$

$$P(X \le 4) = P(X \le 3)+P(X=4) = 0.5+\frac{3}{10}=0.8>0.5=\frac{1}{2}$$

$\therefore$ The minimum value of K for which $P(X \le K) > \frac{1}{2}$ is $K = 4$

(*iv*) The distribution function of X is given by the following table :

X	$F(x) = P(X \le x)$
0	0
1	$K = 1/10$
2	$3K = 3/10$
3	$5K = 5/10$
4	$8K = 8/10$
5	$8K + K^2 = 81/100$
6	$8K + 3K^2 = 83/100$
7	$9K + 10K^2 = 1$

(*v*) Mean, $\mu = \sum_{i=0}^{7} p_i x_i$

$$= 0(0)+1(K)+2(2K)+3(2K)+4(3K)+5(K^2)+6(2K^2)+7(7K^2+K)$$

$$= 66K^2+30K = \frac{66}{100}+\frac{30}{10}=0.66+3=3.66\left(\because K=\frac{1}{10}\right)$$

(*vi*) Variance $= \sum_{i=0}^{7} p_i x_i^2 - \mu^2$

$$= K+8K+18K+48K+25K^2+72K^2+343K^2+49K-(3.66)^2$$

$$= 440K^2+124K-(3.66)^2 = \frac{440}{100}+\frac{124}{10}-(3.66)^2$$

$$= 4.4+12.4-13.3956 = 3.4044$$

Example 4 : A random variable X has the following probability distribution.

X :	1	2	3	4	5	6	7	8
P(X) :	K	2K	3K	4K	5K	6K	7K	8K

Find the value of

(*i*) K (*ii*) $P(X \le 2)$ (*iii*) $P(2 \le X \le 5)$ **[JNTU 2008 (Set No. 1)]**

Solution : (*i*) Since $\sum_{i=1}^{\infty} p(x_i) = 1$, we have

$$K + 2K + 3K + 4K + 5K + 6K + 7K + 8K = 1 \Rightarrow 36K = 1.$$

$$\therefore K = \frac{1}{36}$$

(*ii*) $P(X \leq 2) = P(X = 1) + P(X = 2)$

$$= K + 2K = 3K = \frac{3}{36} = \frac{1}{12}$$

(*iii*) $P(2 \leq X \leq 5) = P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)$

$$= 2K + 3K + 4K + 5K = 14K = \frac{14}{36} = \frac{7}{18}$$

Example 5 : A random variable X has the following probability distribution.

Values of x	0	1	2	3	4	5	6	7	8
P(x)	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

(*i*) Determine the value of a.

(*ii*) Find $p(x<3)$, $p(x \geq 3)$ and $p(0<x<5)$

(*iii*) Find the distribution function $F(x)$. **[JNTU(K) Dec. 2013 (Set No.1)]**

<u>**Solution :**</u> (*i*) Since $\sum_{i=0}^{8} p(x_i) = 1$, we have

$$a+3a+5a+7a+9a+11a+13a+15a+17a=1$$

i.e., $81a = 1$ or $a = \dfrac{1}{81}$

(*ii*) $p(x<3) = p(x=0) + p(x=1) + p(x=2)$

$$= a + 3a + 5a = 9a = \frac{9}{81} = \frac{1}{9}$$

$$p(x \geq 3) = 1 - p(x<3) = 1 - \frac{1}{9} = \frac{8}{9}$$

$$p(0<x<5) = p(x=1) + p(x=2) + p(x=3) + p(x=4)$$

$$= 3a + 5a + 7a + 9a = 24a = \frac{24}{81} = \frac{8}{27}.$$

(*iii*) The distribution function $F(x)$ of the discrete variate X is defined by

$$F(x) = P(X \leq x) = \sum_{i=1}^{x} p(x_i) \text{ where } x \text{ is any integer}$$

X	$F(X) = P(X \le x)$
0	$a = \frac{1}{81}$
1	$4a = \frac{4}{81}$
2	$9a = \frac{9}{81}$
3	$16a = \frac{16}{81}$
4	$25a = \frac{25}{81}$
5	$36a = \frac{36}{81}$
6	$49a = \frac{49}{81}$
7	$64a = \frac{64}{81}$
8	$81a = \frac{81}{81} = 1$

Example 6 : The probability density function of a variate X is

X	0	1	2	3	4	5	6
P(X)	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

(*i*) Find k (*ii*) Find P(X<4), $P(X \ge 5)$, $P(3 < X \le 6)$ (*iii*) What will be the minimum value of k so that $P(X \le 2) > 0.3$? **[JNTU (K) Nov. 2009, Dec. 2013 (Set No. 3, 4)]**

Solution : (*i*) Since $\sum_{i=0}^{6} p(x_i) = 1$ so $k + 3k + 5k + 7k + 9k + 11k + 13k = 1$

i.e., $49k = 1$ or $k = \frac{1}{49}$

(*ii*) $P(X<4) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$

$$= k + 3k + 5k + 7k = 16k = \frac{16}{49} \quad \left[\because k = \frac{1}{49}\right]$$

$P(X \ge 5) = P(X = 5) + P(X = 6) = 11k + 13k = 24k = \frac{24}{49}$

$P(3 < X \le 6) = P(X = 4) + P(X = 5) + P(X = 6) = 9k + 11k + 13k = 33k = \frac{33}{49}$

(*iii*) $P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2) = k + 3k + 5k = 9k$

$P(X \le 2) > 0.3 \Rightarrow 9k > 0.3$ or $k > \frac{1}{30}$

$\therefore$ The minimum value of k is $\frac{1}{30}$.

Example 7 : For the discrete probability distribution

X	0	1	2	3	4	5	6
F	0	2K	2K	3K	K^2	$2K^2$	$7K^2+K$

Find *i)* k *ii)* mean *iii)* Variance **[JNTU (H) 2009 (Set No. 1)]**

Solution : (*i*) Since the total probability is unity, we have $\sum_{x=0}^{6} p(x) = 1$

i.e., $0+2K+2K+3K+K^2+2K^2+(7K^2+K)=1$

i.e., $10K^2+8K-1=0$

$$\therefore K = \frac{-8\pm\sqrt{64+40}}{20} = \frac{-8\pm\sqrt{104}}{20} = \frac{-8\pm2\sqrt{26}}{20} = \frac{-4\pm\sqrt{26}}{10}$$

Since, $p(x)\geq 0$,

$$\therefore K = \frac{-4+\sqrt{26}}{10} = 0.1099$$

(*ii*) Mean, $\mu = \sum_{i=0}^{6} p_i x_i$

$$= (0)(0)+(1)(2K)+(2)(2K)+(3)(3K)+(4)(K^2)+(5)(2K^2)+(6)(7K^2+K)$$

$$= 2K+4K+9K+4K^2+10K^2+42K^2+6K$$

$$= 56K^2+21K = K(56K+21)$$

$$= (0.1099)[56(0.1099)+21] = 2.9842$$

(*iii*) Variance $= \sum_{i=0}^{6} p_i x_i^2 - \mu^2$

$$= 0+2K(1)^2+2K(2)^2+3K(3)^2+K^2(4)^2+2K^2(5)^2+(7K^2+K)(6)^2-(2.9842)^2$$

$$= 2K+8K+27K+16K^2+50K^2+252K^2+36K-8.9054$$

$$= 318K^2+73K-8.9054 = K(318K+73)-8.9054$$

$$= (0.1099)[318(0.1099)+73]-8.9054 = 2.9581$$

Example 8 : A random variables X has the following probability function

X_i	–3	–2	–1	0	1	2	3
$P(X_i)$	K	0.1	K	0.2	$2k$	0.4	$2k$

Find (*i*) K (*ii*) Mean (*iii*) Variance

[JNTU (H) 2009 (Set No. 2)]

Solution : (*i*) We know that, if X is a random variable, then $\sum p(x_i)=1$

i.e., $K+0.1+K+0.2+2K+0.4+2K=1$

i.e., $6K+0.7=1$ *i.e.*, $6K=0.3$ or $K=\dfrac{0.3}{6}=\dfrac{0.1}{2}$

(*ii*) Mean, $\mu=\sum_{i=0}^{6} p_i x_i$

$=(-3)(K)+(-2)(0.1)+(-1)(K)+0(0.2)+1(2K)+2(0.4)+3(2K)$

$=-3K-0.2-K+0+2K+0.8+6K$

$=4K+0.6=4\left(\dfrac{0.1}{2}\right)+0.6 \ =0.8$

(*iii*) Variance $=\sum_{i=0}^{6} p_i x_i^2-\mu^2$

$=K(-3)^2+(0.1)(-2)^2+K(-1)^2+0.2(0)^2+2K(1)^2+$

$(0.4)(2)^2+2K(3)^2-(0.8)^2$

$=9K+0.4+K+0+2K+1.6+18K-0.64$

$=30K+2-0.64 \ =30\left(\dfrac{0.1}{2}\right)+1.36 \ =1.5+1.36=2.86$

Example 9 : From a lot of 10 items containing 3 defectives, a sample of 4 items is drawn at random. Let the random variable X denote the number of defective items in the sample. Find the probability distribution of X when the sample is drawn without replacement.

[JNTU (K) May 2013 (Set No.4)]

Solution : Obviously X can take the values 0, 1, 2 or 3.

Given total number of items = 10

No. of good items = 7

No. of defective items = 3

$P(X=0)=P(\text{no defective})=\dfrac{{}^{7}C_4}{{}^{10}C_4}=\dfrac{7!}{4!3!}\times\dfrac{4!6!}{10!}=\dfrac{1}{6}$

$P(X=1)=P(\text{one defective and 3 good items})$

$=\dfrac{{}^{3}C_1\times{}^{7}C_3}{{}^{10}C_4}=\dfrac{3\times 7!}{3!\,4!}\times\dfrac{4!\,6!}{10!}=\dfrac{1}{2}$

$P(X=2)=P(\text{2 defective and 2 good items})$

$=\dfrac{{}^{3}C_2\times{}^{7}C_2}{{}^{10}C_4}=\dfrac{3}{10}$

$$P(X=3) = P(3 \text{ defective and } 1 \text{ good item})$$

$$= \frac{{}^3C_3 \times {}^7C_1}{{}^{10}C_4} = \frac{7}{{}^{10}C_4} = \frac{4!}{8\times 9\times 10} = \frac{1}{30}$$

$\therefore$ The probability distribution of random variable X is as follows :

X	0	1	2	3
$P(X)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

Example 10 : Let X denote the minimum of the two numbers that appear when a pair of fair dice is thrown once. Determine the

(*i*) Discrete probability distribution (*ii*) Expectation

(*iii*) Variance **[JNTU 2006, (K) Nov. 2009, 2010S (Set No. 1)]**

Solution : When two dice are thrown, total number of outcomes is $6 \times 6 = 36$

In this case, sample space $S = \begin{Bmatrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6) \\ (2,1),(2,2),(2,3),(2,4),(2,5),(2,6) \\ (3,1),(3,2),(3,3),(3,4),(3,5),(3,6) \\ (4,1),(4,2),(4,3),(4,4),(4,5),(4,6) \\ (5,1),(5,2),(5,3),(5,4),(5,5),(5,6) \\ (6,1),(6,2),(6,3),(6,4),(6,5),(6,6) \end{Bmatrix}$

If the random variable X assigns the minimum of its number in S, then the sample space $S = \begin{Bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{Bmatrix}$

The minimum number could be 1, 2, 3, 4, 5, 6.

For minimum 1, favourable cases are (1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (1, 4), (4, 1), (1, 5), (5, 1), (1, 6), (6, 1).

So $P(X=1) = \frac{11}{36}$

For minimum 2, favourable cases are (2, 2), (2, 3), (3, 2), (2, 4), (4, 2), (2, 5) , (5, 2), (2, 6), (6, 2).

So $P(X=2) = \frac{9}{36}$

Similarly, $P(X=3) = P((3, 3), (3, 4), (4, 3), (3, 5), (5, 3), (3, 6), (6, 3)) = \frac{7}{36}$

$$P(X=4) = P((4, 4), (4, 5), (5, 4), (4, 6), (6, 4)) = \frac{5}{36}$$

$$P(X=5) = P((5, 5), (5, 6), (6, 5)) = \frac{3}{36}$$

$$P(X=6) = P((6, 6)) = \frac{1}{36}$$

∴ The probability distribution is

X	1	2	3	4	5	6
$P(X)$	$\frac{11}{36}$	$\frac{9}{36}$	$\frac{7}{36}$	$\frac{5}{36}$	$\frac{3}{36}$	$\frac{1}{36}$

(*ii*) Expectation= Mean $= \Sigma\, p_i\, x_i$

i.e., $E(X) = 1.\frac{11}{36} + 2.\frac{9}{36} + 3.\frac{7}{36} + 4.\frac{5}{36} + 5.\frac{3}{36} + 6.\frac{1}{36}$

or $\mu = \frac{1}{36}(11 + 18 + 21 + 20 + 15 + 6) = \frac{91}{36} = 2.5278$

(*iii*) Variance $= \Sigma\, p_i\, x_i^2 - \mu^2$

$$= \frac{11}{36}.1 + \frac{9}{36}.4 + \frac{7}{36}.9 + \frac{5}{36}.16 + \frac{3}{36}.25 + \frac{1}{36}.36 - \left(\frac{91}{36}\right)^2$$

$$= \frac{1}{36}(11 + 36 + 63 + 80 + 75 + 36) - \left(\frac{91}{36}\right)^2$$

i.e., $\sigma^2 = 8.3611 - 6.3898 = 1.9713$

Note : Standard deviation, $\sigma = \sqrt{1.9713} = 1.404$

Example 11 : Calculate expectation and variance of X, if the probability distribution of the random variable X is given by

X	-1	0	1	2	3
f	0.3	0.1	0.1	0.3	0.2

[JNTU 2006 (Set No.2)]

Solution :

(*i*) Expectation = Mean $= \Sigma\, f_i\, x_i$

i.e., $E(X) = (-1)(0.3) + 0(0.1) + 1(0.1) + 2(0.3) + 3(0.2)$

or $\mu = -0.3 + 0.1 + 0.6 + 0.6 = 1$

(ii) Variance $= \Sigma f_i x_i^2 - \mu^2 = \Sigma x_i^2 f_i - [E(X)]^2$

$= (-1)^2 (0.3) + 0(0.1) + 1(0.1) + 2^2(0.3) + 3^2(0.2) - 1$

$= 0.3 + 0.1 + 1.2 + 1.8 - 1 = 2.4$

Example 12 : Calculate the mean for the following distribution.

X = x :	0.3	0.2	0.1	0	1	2	3
P(X = x) :	0.05	0.10	0.30	0	0.30	0.15	0.1

[JNTU 2008 (Set No. 4)]

Solution : The mean value of the probability distribution of a variate X is commonly known as its expectation and is denoted by E(X). It is given by

$$E(X) = \sum_i x_i f(x_i)$$

$$= 0.3(0.05) + 0.2(0.10) + 0.1(0.30) + 0(0) + 1(0.30) + 2(0.15) + 3(0.1)$$

$$= 0.015 + 0.02 + 0.03 + 0 + 0.30 + 0.30 + 0.3 = 0.965$$

Example 13 : A random variable X is defined as the sum of the numbers on the faces when two dice are thrown. Find the mean of X. [JNTU (A) 2009 (Set No. 4)]

Solution : Let x be the sum of the numbers on the faces when two dice are thrown. x is a discrete random variable whose probability distribution is given by

x_i	2	3	4	5	6	7	....	11	12
$p(x_i)$	1/36	2/36	3/36	4/36	5/36	6/36	....	2/36	1/36

$$\therefore \text{ Mean of } X = E(X) = \sum_i p_i x_i$$

$$= 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + 4\left(\frac{3}{36}\right) + 5\left(\frac{4}{36}\right) + 6\left(\frac{5}{36}\right) + 7\left(\frac{6}{36}\right)$$

$$+ 8\left(\frac{5}{36}\right) + 9\left(\frac{4}{36}\right) + 10\left(\frac{3}{36}\right) + 11\left(\frac{2}{36}\right) + 12\left(\frac{1}{36}\right)$$

$$= \frac{1}{36}(2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12)$$

(1) Assignment $= \frac{252}{36} = 7$

Example 14 : Find the mean and variance of the uniform probability distribution given by $f(x) = \frac{1}{n}$ for $x = 1, 2, 3, ..., n$.

[JNTU 2001, (H) Nov. 2009, Nov. 2010, Dec. 2011 (Set No. 3)]

Solution : The probability distribution is

x	1	2	3	...	n
$f(x)$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	...	$\frac{1}{n}$

(*i*) Mean $= \sum_{i=1}^{n} x_i\, f(x_i) = 1.\frac{1}{n} + 2.\frac{1}{n} + ... + n.\frac{1}{n}$

or $E(X) = \mu = \frac{1}{n}(1 + 2 + 3 + \ldots + n) = \frac{1}{n}.\frac{n(n+1)}{2} = \frac{n+1}{2}$

(*ii*) Variance $= \sum_{i=1}^{n} x_i^2 f(x_i) - \mu^2$

$$= 1^2.\frac{1}{n} + 2^2.\frac{1}{n} + 3^2.\frac{1}{n} + \ldots + n^2.\frac{1}{n} - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{1}{n}(1^2 + 2^2 + 3^2 + \ldots + n^2) - \frac{1}{4}(n+1)^2$$

$$= \frac{1}{n}.\frac{n(n+1)(2n+1)}{6} - \frac{1}{4}(n+1)^2$$

$$= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{(n+1)}{2}\left(\frac{2n+1}{3} - \frac{n+1}{2}\right)$$

$$= \frac{n+1}{12}(4n + 2 - 3n - 3) = \frac{(n+1)(n-1)}{12} = \frac{n^2-1}{12}$$

Example 15 : A sample of 4 items is selected at random from a box containing 12 items of which 5 are defective. Find the expected number E of defective items.

[JNTU (H) Nov. 2009 (Set No. 4), (A) Nov. 2010 (Set No.1)]

Solution : Let X denote the number of defective items among 4 items drawn from 12 items.

Obviously X can take the values 0, 1, 2, 3, or 4.

No. of good items = 7

No. of defective items = 5

$$P(X = 0) = P(\text{no defective}) = \frac{{}^7C_4}{{}^{12}C_4} = \frac{{}^7C_4}{495} = \frac{35}{495} = \frac{7}{99}$$

$P(X = 1) =$ P(one defective and 3 good items)

$$= \frac{{}^7C_3 \times {}^5C_1}{{}^{12}C_4} = \frac{175}{495} = \frac{35}{99}$$

$P(X=2) = P(2 \text{ defective and } 2 \text{ good items})$

$$= \frac{{}^7C_2 \times {}^5C_2}{{}^{12}C_4} = \frac{210}{495} = \frac{42}{99}$$

$P(X=3) = P(3 \text{ defective and } 1 \text{ good item})$

$$= \frac{{}^7C_1 \times {}^5C_3}{{}^{12}C_4} = \frac{70}{495} = \frac{14}{99}$$

$P(X=4) = P(\text{all are defective})$

$$= \frac{{}^5C_4}{{}^{12}C_4} = \frac{5}{495} = \frac{1}{99}$$

Discrete probability distribution is

$X = x_i$	0	1	2	3	4
$P(X = x_i) = f(x_i)$	$\frac{7}{99}$	$\frac{35}{99}$	$\frac{42}{99}$	$\frac{14}{99}$	$\frac{1}{99}$

Expected number of defective items $= E(X) = \Sigma\, x_i f(x_i)$

$$= 0.\frac{7}{99} + 1.\frac{35}{99} + 2.\frac{42}{99} + 3.\frac{14}{99} + 4.\frac{1}{99} = \frac{165}{99}$$

Example 16 : Show that the variance of a random variable X is given by

$$\sigma^2 = E(X^2) - [E(X)]^2$$

Solution : We know that

$$\sigma^2 = E[(X-\mu)^2] = E(X^2 - 2X\mu + \mu^2) = E(X^2) - 2\mu\, E(X) + E(\mu^2)$$

$$= E(X^2) - 2\mu.\mu + \mu^2 = E(X^2) - \mu^2 = E(X^2) - [E(X)]^2$$

Example 17 : Given that $f(x) = K/2x$, is a probability distribution for a random variable X that can take on the values $x = 0,1,2,3$ and 4.

(*i*) Find K (*ii*) Mean and variance of X. **[JNTU (A) Sup., 2010]**

Solution : Given $f(x) = K/2x$, where $x = 0,1,2,3,4$. If $x = 0, f(x)$ is not defined.

Hence the problem is not correct.

We shall delete $x = 0$ and workout the problem.

(*i*) If $f(x)$ is a probability distributive function, then

1. $f(x) \geq 0 \Rightarrow f(x) = \dfrac{K}{2x} \geq 0 \Rightarrow K \geq 0$

2. $\displaystyle\sum_{\forall x} f(x) = 1 \Rightarrow \sum_{x=1}^{4} \frac{K}{2x} = 1 \Rightarrow \frac{K}{2}.\sum_{x=1}^{4}\frac{1}{x} = 1$

$$\Rightarrow \frac{K}{2}\left[1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right]=1 \Rightarrow \frac{K}{2}\left(\frac{25}{12}\right)=1$$

$$\therefore K=\frac{24}{25}$$

The probability distribution is given in the following table :

x	1	2	3	4
$f(x)$	$\frac{K}{2}$	$\frac{K}{4}$	$\frac{K}{6}$	$\frac{K}{8}$

(ii) Mean $=E(X)=\sum_{x=1}^{4} x.f(x)$

$$=1\times\frac{K}{2}+2\times\frac{K}{4}+3\times\frac{K}{6}+4\times\frac{K}{8}$$

$$=\frac{K}{2}+\frac{K}{2}+\frac{K}{2}+\frac{K}{2}=4\left(\frac{K}{2}\right)=2K$$

$$=2\left(\frac{24}{25}\right)=\frac{48}{25}\quad\left[\because K=\frac{24}{25}\right]$$

Now $E(X^2)=\sum_{x=1}^{4} x^2.f(x)$

$$=1\times\frac{K}{2}+4\times\frac{K}{4}+9\times\frac{K}{6}+16\times\frac{K}{8}$$

$$=\frac{K}{2}+K+\frac{3K}{2}+2K=5K$$

$\therefore$ Variance $=E(X^2)-[E(X)]^2$

$$=5K-(2K)^2=5K-4K^2=K(5-4K)$$

$$=\frac{24}{25}\left(5-\frac{96}{25}\right)\quad\left[\because K=\frac{24}{25}\right]$$

$$=\frac{24}{25}\left(\frac{29}{25}\right)=\frac{696}{625}=1.1136$$

Example 18 : For the following probability distribution

x	-3	6	9
$p(x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

Find (i) $E(X)$ (ii) $E(X^2)$ (iii) $E[(2X+1)^2]$

Solution :

(i) $E(X) = \sum_{i=1}^{3} p_i x_i = (-3)\left(\frac{1}{6}\right) + 6.\frac{1}{2} + 9.\frac{1}{3} = -\frac{1}{2} + 3 + 3 = \frac{11}{2}$

(ii) $E(X^2) = \sum_{i=1}^{3} p_i x_i^2 = (-3)^2\left(\frac{1}{6}\right) + (6)^2 . \frac{1}{2} + (9)^2 . \frac{1}{3}$

$= \frac{9}{6} + \frac{36}{2} + \frac{81}{3} = \frac{93}{2}$

(iii) $E[(2X+1)^2] = E(4X^2 + 4X + 1) = 4E(X^2) + 4E(X) + E(1)$

$= 4.\frac{93}{2} + 4.\frac{11}{2} + 1 = \frac{418}{2} = 209$

Example 19 : A player wins if he gets 5 on a single throw of a die, he loses if he gets 2 or 4. If he wins, he gets Rs. 50, if he loses he gets Rs. 10, otherwise he has to pay Rs. 15. Find the value of the game to the player. Is it favourable ?

Solution : Range of $X = \{-15, 10, 50\}$

Since there are six numbers in a die, out of these 5 is only one number, the probability of getting Rs. 50 is

$$P(X = 50) = \frac{1}{6}$$

Similarly, the probability of getting two numbers (2 or 4) to win Rs. 10 is

$$P(X = 10) = \frac{2}{6} = \frac{1}{3}$$

The probability of getting three numbers (1 or 3 or 6) to loose Rs. 15 is

$$P(X = -15) = \frac{3}{6} = \frac{1}{2}$$

Discrete probability distribution is

$X = x$	-15	10	50
$P(X) = p(x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

Expected value of the game $= E(X) = \sum x_i\, p(x_i) = (-15).\frac{1}{2} + 10.\frac{1}{3} + 50.\frac{1}{6} = \text{Rs. } \frac{25}{6}$

Game is favourable to the player since $E > 0$.

Example 20 : A player tosses 3 fair coins. He wins Rs. 500 if 3 heads appear, Rs. 300 if 2 heads appear, Rs. 100 if 1 head occurs. On the other hand, he loses Rs. 1500 if 3 tails occur. Find the expected gain of the player.

Solution : Let X denote the gain. Then the range of X is $\{-1500, 100, 300, 500\}$

The sample space $S = n(s) = 2^3 = 8$

$= \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

The probability of all 3 heads (getting Rs. 500) = $P(X = 3) = \frac{1}{8}$

The probability of getting 2 heads (getting Rs. 300) = $P(X = 2) = \frac{3}{8}$

The probability of getting one head (getting Rs. 100) = $P(X = 1) = \frac{3}{8}$

The probability of getting 3 tails (lossing Rs. 1500) = $P(X = 0) = \frac{1}{8}$

Discrete probability distribution is

$X = x_i$	-1500	100	300	500
$P(X = x_i) = p(x_i)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$\therefore$ Expected value of $X = E(X) = \Sigma\, x_i\, p(x_i)$

$$= (-1500)\,.\,\frac{1}{8} + 100.\frac{3}{8} + 300.\frac{3}{8} + 500.\frac{1}{8}$$

$$= \frac{1}{8}(-1500 + 300 + 900 + 500) = \text{Rs. } 25.$$

Example 21 : A fair coin is tossed until a head or five tails occurs. Find the expected number E of tosses of the coin. **[JNTU (H) Nov. 2009 (Set No.3)]**

Solution : If head occurs first time there will be only one toss. On the other hand, if first one is tail, second occurs. If head occurs there will be only two tosses. Suppose second one is also tail third occurs. If head occurs there will be three tosses and so on.

$$\therefore\; p(1) = p(H) = \frac{1}{2},\quad p(2) = p(TH) = \frac{1}{4},$$

$$p(3) = p(TTH) = \frac{1}{8},\quad p(4) = p(TTTH) = \frac{1}{16}$$

$$p(5) = p(TTTTH) + p(TTTTT) = \frac{1}{32} + \frac{1}{32} = \frac{1}{16}$$

The probability distribution function of X is

x	1	2	3	4	5
$p(x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$

Hence $E(X) = \sum_i p_i x_i$

$$= 1\times\frac{1}{2} + 2\times\frac{1}{4} + 3\times\frac{1}{8} + 4\times\frac{1}{16} + 5\times\frac{1}{16} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{16}$$

$$= \frac{8+8+6+4+5}{16} = \frac{31}{16} = 1.9375$$

Example 22 : A player tosses two fair coins. He wins Rs.100/- if head appears, Rs. 200/– if two heads appear. On the other hand he loses Rs.500/- if no head appears. Determine the expected value E of the game and is the game favourable to the player?

[JNTU (K) Nov. 2009 (Set No. 3)]

Solution : Let X denote the number of heads occuring in tosses of two fair coins. The sample space S is

$$S = \{H,T\} \times \{H,T\} = \{HH, HT, TH, TT\}$$

$$\text{Probability of all 2 leads} = P(X = 2) = \frac{1}{4}$$

$$\text{Probability of all 2 tails} = P(X = 0) = \frac{1}{4}$$

$$\text{Probability of one head} = P(X = 1) = \frac{2}{4}$$

Discrete probability distribution is

$X = x_i$	0	1	2
$P(x_i)$	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

$$\text{Expected value of the game} = 100\times\frac{2}{4}+200\times\frac{1}{4}-500\times\frac{1}{4}$$

$$= \frac{200+200-500}{4} = \frac{-100}{4} = -25 \text{ rupees}$$

$\therefore$ Game is not favourable to the player since $E < 0$

Example 23 : A discrete random variable X has the following distribution function :

$$F(x) = \begin{cases} 0, \text{ for } x < 1 \\ 1/3, \text{ for } 1 \le x < 4 \\ 1/2, \text{ for } 4 \le x < 6 \\ 5/6, \text{ for } 6 \le x < 10 \\ 1, \text{ for } x \ge 10 \end{cases}$$

Find (i) $P(2 < X \le 6)$ (ii) $P(X = 5)$ (iii) $P(X = 4)$(iv) $P(X \le 6)$ (v) $P(X = 6)$

[JNTU (K) Nov. 2009 (Set No. 3)]

Solution :

(i) $P(2 < X \le 6) = F(6) - F(2) = P(X \le 6) - P(X \le 2) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

(ii) $P(X = 5) = P(X \le 5) - P(X < 5) = F(5) - P(X < 5) = \frac{1}{2} - \frac{1}{2} = 0$

(iii) $P(X = 4) = P(X \le 4) - P(X < 4) = F(4) - P(X < 4) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

(iv) $P(X \le 6) = F(6) = \frac{5}{6}$

(v) $P(X = 6) = F(6) - P(X < 6) = \frac{5}{6} - \frac{1}{2} = \frac{2}{6} = \frac{1}{3}$

Example 24 : Find the distribution function which corresponds to the probability distribution defined by $f(x) = \dfrac{x}{15}$ *for* $x = 1, 2, 3, 4, 5.$ **[JNTU (K) May 2013 (Set No.3)]**

Solution :

Given $f(x) = \dfrac{x}{15}$. So $f(1) = \dfrac{1}{15}, f(2) = \dfrac{2}{15}, f(3) = \dfrac{3}{15}, f(4) = \dfrac{4}{15}$ and $f(5) = \dfrac{5}{15}$

Now $F(1) = f(1) = \dfrac{1}{15},\ F(2) = F(1) + f(2) = \dfrac{1}{15} + \dfrac{2}{15} = \dfrac{1}{5}$

$F(3) = F(2) + f(3) = \dfrac{1}{5} + \dfrac{1}{5} = \dfrac{2}{5},\ F(4) = F(3) + f(4) = \dfrac{2}{5} + \dfrac{4}{15} = \dfrac{2}{3}$

and $F(5) = F(4) + f(5) = \dfrac{2}{3} + \dfrac{1}{3} = 1$

Example 25 : A sample of 3 items is selected at random from a box containing 10 items of which 4 are defective. Find the expected number of defective items?

Solution : The probability of defective is $p = \dfrac{4}{10} = \dfrac{2}{5}$

Number of items chosen, $n = 3$

The expected number of defective is

$$E(X) = \mu = np = 3\left(\frac{2}{5}\right) = \frac{6}{5} = 1.2 \approx 1$$

Alternative Method :

No. of Exhaustive cases $= {}^{10}C_3 = \dfrac{10!}{3!\,7!} = 120$

Probability that there are no defective items $= p\,(x = 0) = \dfrac{{}^6C_3}{120} = \dfrac{1}{6}$

Probability that there is one defective item $= p\,(x = 1) = \dfrac{{}^6C_2 \cdot {}^4C_1}{120} = \dfrac{1}{2}$

Probability that there are two defective items $= p\,(x = 2) = \dfrac{{}^6C_1 \cdot {}^4C_2}{120} = \dfrac{3}{10}$

Probability that there are three defective items $= p\,(x = 3) = \dfrac{{}^4C_3}{120} = \dfrac{1}{30}$

x	0	1	2	3
$p(x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

$\therefore$ Expected number of defective items $= E(x) = \Sigma\, p_i x_i$

$$= 0.\frac{1}{6} + 1.\frac{1}{2} + 2.\frac{3}{10} + 3.\frac{1}{30}$$

$$= 0 + \frac{1}{2} + \frac{3}{5} + \frac{1}{10} = \frac{12}{10} = 1.2$$

Example 26 : Let X denote the sum of the two numbers that appear when a pair of fair dice is tossed. Determine the (*i*) Distribution function, (*ii*) mean and (*iii*) variance.

[JNTU (K) May 2010 (Set No.4)]

(or) Two dice are rolled at random. Find the probability distribution of the sum of the numbers on them. Also find the mean of the distribution ? **[JNTU (K) Nov. 2011 (Set No.4)]**

Solution : If two unbiased dice are thrown, then the sum X of the two numbers which turn up must be an integer between 2 and 12.

For $X = 2$, there is only one favourable point (1, 1) and hence $P(X = 2) = 1/36$, since there are 36 sample points in all.

For $X = 3$, there are two favourable sample points (1, 2) and (2, 1) and hence $P(X = 3) = 2/36$.

Similarly for $X = 4$, there are three favourable sample points (1, 3,), (2, 2), (3, 1) and hence $P(X = 4) = 3/36$ and so on.

Now the probability distribution in this case is given by the following table :

$X = x_i$	2	3	4	5	6	7	8	9	10	11	12
$P(X = x_i)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Mean, $\mu = \Sigma x_i \; p(x_i)$

$$= 2 \times \frac{1}{36} + 3 \times \frac{2}{36} + 4 \times \frac{3}{36} + ... + 12 \times \frac{1}{36}$$

$$= \frac{1}{36}(2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12)$$

i.e., $E(x) = \frac{252}{36} = 7$

Variance, $\sigma^2 = E(x^2) - [E(x)]^2$

$$= \Sigma x_i^2 p(x_i) - (7)^2$$

$$= 2^2 \times \frac{1}{36} + 3^2 \times \frac{2}{36} + 4^2 \times \frac{3}{36} + 5^2 \times \frac{4}{36} + ... + 12^2 \times \frac{1}{36}$$

$$= \frac{1}{36}[4 + 18 + 48 + 100 + 180 + 294 + 320 + 324 + 300 + 242 + 144] - 49$$

$$= \frac{1974}{36} - 49 = 54.83 - 49 = 5.83$$

Example 27 : A fair die is tossed. Let the random variable X denote the twice the number appearing on the die :

(*i*) Write the probability distribution of X

(*ii*) The mean (*iii*) The variance [JNTU (H) Apr. 2012 (Set No.2)]

Solution : Let x denote twice the number appearing on the face when a die is thrown. Then x is a discrete random variable whose probability distribution is given by

(*i*)

x_i	2	4	6	8	10	12
$p(x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

(*ii*) Mean $= E(X) = p_i x_i$

$$= 2\times\frac{1}{6}+4\times\frac{1}{6}+6\times\frac{1}{6}+8\times\frac{1}{6}+10\times\frac{1}{6}+12\times\frac{1}{6}$$

$$= \frac{1}{6}(2+4+6+8+10+12) = \frac{42}{6} = 7$$

Now $E(x^2) = \Sigma x_i^2 p(x_i)$

$$= 2^2\times\frac{1}{6}+4^2\times\frac{1}{6}+6^2\times\frac{1}{6}\times 8^2\times\frac{1}{6}\times 10^2\times\frac{1}{6}\times 12^2\times\frac{1}{6}$$

$$= \frac{1}{6}(2^2+4^2+6^2+8^2+10^2+12^2)$$

$$= \frac{1}{6}(4+16+36+64+100+144) = \frac{364}{6} = 60.67$$

(*iii*) Variance $= E(x^2)-[E(x)]^2$

$$= 60.67-(7)^2 = 60.67-49 = 11.67$$

Example 28 : A random sample with replacement of size 2 is taken from $S=\{1,2,3\}$. Let the random variable X denote the sum of the two numbers taken:

(*i*) Write the probability distribution of X.

(*ii*) Find the mean.

(*iii*) Find the variance.

[JNTU (H) Apr. 2012 (Set No.3)]

Solution : Sample space $S=\{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$

$X(S) = (2,3,4,5,6)$

$$p(2) = P(X=2) = p[(1,1)] = \frac{1}{9}$$

$$p(3) = P(X=3) = p[(1,2),(2,1)] = \frac{2}{9}$$

$$p(4) = P(X=4) = p\,[(1,3),(2,2),(3,1)] = \frac{3}{9}$$

$$p(5) = P(X=5) = p\,[(2,3),(3,2)] = \frac{2}{9}$$

$$p(6) = P(X=6) = p\,[(3,3)] = \frac{1}{9}$$

(*i*) The probability distribution of X is

x	2	3	4	5	6
$p(x)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

(*ii*) Mean $= E(x) = \Sigma p_i x_i = 2\times\frac{1}{9} + 3\times\frac{2}{9} + 4\times\frac{3}{9} + 5\times\frac{2}{9} + 6\times\frac{1}{9}$

$$= \frac{1}{9}(2+6+12+10+6) = \frac{1}{9}(36) = 4$$

(*iii*) $E(x^2) = \Sigma p_i x_i^2$

$$= \frac{1}{9}\times 4 + \frac{2}{9}\times 9 + \frac{3}{9}\times 16 + \frac{2}{9}\times 25 + \frac{1}{9}\times 36$$

$$= \frac{1}{9}(4+18+48+50+36) = \frac{156}{9} = 17.33$$

Hence variance $= E(x^2) - [E(x)]^2$

$$= 17.33 - (4)^2 = 17.33 - 16 = 1.33$$

REVIEW QUESTIONS

1. Write the definitions of (*i*) Random variable (*ii*) Discrete Random Variable (*iii*) Continuous Random Variable and (*iv*) Probability Distribution function

[JNTU (K) Nov. 2011 (Set No. 2)]

2. Define random variable, discrete probability distribution, continuous probability distribution and cumulative distribution. Give an example of each.

[JNTU 2007, 2008S (Set No.4), (K) Nov. 2011 (Set No. 1)]

3. List the properties of probability distribution function.

[JNTU (K) Dec. 2013 (Set No. 3)]

4. Define mathematical expectation
5. Define (*i*) Probability density function (*ii*) Probability mass function (*iii*) Discrete random variable (*iv*) Continuous random variable

[JNTU (K) Dec. 2013 (Set No. 3)]

EXERCISE 2(A)

1. (*a*) The probability density function of a variate X is as follows :

$X = x$	0	1	2	3	4	5	6
$P(x)$	K	$3K$	$5K$	$7K$	$9K$	$11K$	$13K$

(*i*) Find $P(X < 4)$, $P(X \geq 5)$, $P(3 < X \leq 6)$

(*ii*) What will be the minimum value of K so that $P(X \leq 2) > 0.3$.

[JNTU (K) Nov. 2009, Mar. 2014 (Set No. 4)]

(*b*) A random variable X has the following probability function :

X	0	1	3	4	5	6	7
$P(X)$	0	K	$2K$	$2K$	$3K$	K^2	$7K^2 + K$

(*i*) Find the value of K (*ii*) Evaluate $P(X < 6), P(X \geq 6)$

(*iii*) Evaluate $P(0 < X < 5)$ **[JNTU (K) Mar. 2013 (Set No. 1)]**

(*c*) A discrete random variable X has the following probability distribution

Value of X	0	1	2	3	4	5	6	7	8
$P(X = x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$	$15k$	$17k$

(*i*) Find the value of 'k' (*ii*) Find $P(X \leq 3), P(0 < X < 3), P(X \geq 3)$

[JNTU (K) Mar. 2014 (Set No.1)]

(*d*) A discrete random variable X has the following probability distribution

Value of X	1	2	3	4	5	6	7	8
$P(X = x)$	$2k$	$4k$	$6k$	$8k$	$10k$	$12k$	$14k$	$4k$

(*i*) Find the value of 'k' (*ii*) Find $P(X < 3)$ and $P(X \geq 5)$

(*iii*) Find the distribution function of X. **[JNTU (K) Mar. 2014 (Set No.3)]**

2. A random varibale X has the following probability function :

$X = x$	-2	-1	0	1	2	3
$P(X)$	0.1	K	0.2	$2K$	0.3	K

Find (*i*) K (*ii*) Mean (*iii*) Variance

(*iv*) $P(x \geq 2)$ (*v*) $P(x < 2)$ (*vi*) $P(-1 < x < 3)$ **[JNTU 2007S, (A) May 2012]**

3. A random variable X has the following probability function :

$X = x$	4	5	6	8
$P(X)$	0.1	0.3	0.4	0.2

Determine (*i*) Expectation (*ii*) Variance (*iii*) Standard deviation

4. Find mean and variance for the following discrete distribution

$X = x$	-3	-2	-1	0	1	2	3
$P(X)$	K	0.1	K	0.2	$2K$	0.4	$2K$

[JNTU (H) Dec. 2009 (Set No.2)]

5. (*a*) A random variable X has the following probability function :

$X=x$	1	2	3	4	5	6
$P(X)$	K	$3K$	$5K$	$7K$	$9K$	$11K$

Determine (*i*) K (*ii*) Expectation (*iii*) Variance

(*b*) A random variable X has the following probability function :

X	0	1	2	3	4	5	6	7	8
$P(X)$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

(*i*) Determine the value of a,

(*ii*) Evaluate $P(X<3), P(X \geq 3), P(2 \leq X<5)$ **[JNTU (K) Nov. 2012 (Set No. 2)]**

6. Find the expected value of x and standard deviation for the following discrete distribution :

X	8	12	16	20	24
$P(X)$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{12}$

[JNTU Jan. 2007]

7. A random variable X has the following probability function :

X	0	1	2	3	4	5	6	7	8
$P(X)$	$\frac{K}{45}$	$\frac{K}{15}$	$\frac{K}{9}$	$\frac{K}{5}$	$\frac{2K}{45}$	$\frac{6K}{45}$	$\frac{7K}{45}$	$\frac{8K}{45}$	$\frac{4K}{45}$

Determine (*i*) K (*ii*) Mean (*iii*) Variance (*iv*) Standard deviation.

[JNTU (K) Nov. 2011 (Set No. 3)]

8. For the following probability distribution

X	–3	–2	–1	0	1	2	3
$P(X)$	0.001	0.01	0.1	?	0.1	0.01	0.001

Find (*i*) the missing probability (*ii*) mean

(*iii*) variance (*iv*) $E(X^2+2X+3)$

9. A fair coin is tossed until a head or five tails occurs.
Find (*i*) the discrete probability distribution (*ii*) mean of the distribution.

10. A box contains 8 items of which 2 are defective. A man draws 3 items from the box. Find the expected number of defective items he has drawn.

11. A die is thrown at random. What is the expectation of the number on it ?

12. A box contains 4 white and 6 black balls. A man draws 2 balls and is given Rs. 140 for every white ball and Rs. 70 for every black ball. What is his expectation ?

13. Five defective bolts are accidentally mixed with 20 good ones. Find the probability distribution of the number of defective bolts, if four bolts are drawn at random from this lot.

14. Two dice are rolled at random. Find the probability distribution of the sum of the numbers on them. Also find the mean of the distribution.

15. If 3 cars are selected from a lot of 6 cars containing 2 defective cars, find the probability distribution of the number of defective cars. [JNTU 2008S (Set No.1)]

16. Find the expected value and variance of the following distribution :

X	-10	-20	30	75	80
$p(x)$	$\frac{1}{5}$	$\frac{3}{20}$	$\frac{1}{2}$	$\frac{1}{10}$	$\frac{1}{20}$

17. The probability distribution of a random variable X is given below :

X	-2	-1	0	1	2
$p(x)$	0.2	0.1	0.3	0.3	0.1

Find (*i*) $E(X)$ (*ii*) Var (X) (*iii*) $E(2X-3)$ (*iv*) Var $(2X-3)$

18. Let X denotes the minimum of the two numbers that appear when a pair of fair dice is thrown once. Determine the (*i*) Discrete probability distribution (*ii*) Expectation (*iii*) Variance. [JNTU (K) May 2013 (Set No. 2)]

ANSWERS

1. (*a*) (*i*) $\frac{33}{49}$ (*ii*) $\frac{1}{30}$ 2. (*i*) 0.1 (*ii*) 0.8 (*iii*) 2.16

3. (*i*) 5.9 (*ii*) 1.49 (*iii*) 1.22 4. 0.8, $\frac{143}{50}$

5. (*a*) (*i*) $\frac{1}{36}$ (*ii*) 4.46 (*iii*) 2.08 6. 16, $2\sqrt{5}$

7. (*i*) 1 (*ii*) 4.622 (*iii*) 4.9971 (*iv*) 2.24

8. (*i*) 0.778 (*ii*) 0.2 (*iii*) 0.258 (*iv*) 3.698

9.

x	1	2	3	4	5
$f(x)$	1/2	1/4	1/8	1/16	1/16

; 1.9

10. $\frac{3}{4}$ 11. $\frac{7}{2}$ 12. 196

13.

x	0	1	2	3	4
$p(x)$	$\frac{969}{2530}$	$\frac{1140}{2530}$	$\frac{380}{2530}$	$\frac{40}{2530}$	$\frac{1}{2530}$

14.

$X = x_i$	2	3	4	5	6	7	8	9	10	11	12
$p_i = p(x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

; 7

15.

x	0	1	2
$p(x)$	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{2}{5}$

16. 21.5, 950.24 17. (*i*) 0 (*ii*) 1.6 (*iii*) -3 (*iv*) 6.4