Unit 5. Graph Theory

* Graph:-

→ A graph consists of non-empty set of Vertices and set of edges. ie. graph is denoted by G(V, E) where 'G' represents a Graph, 'V' represents non-empty set of vertices and 'E' represents set of Edges.

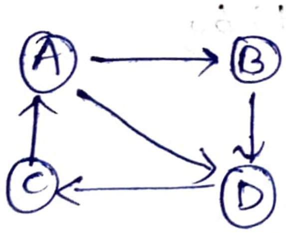
→ In Graph, vertices are Symbollically denoted by dot or circle (•/o) and Edges are denoted with —/→
undirected

→ Graph can be represented in two ways. ie.
1. Directed Graph
2. Undirected Graph.

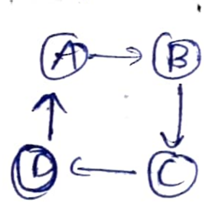
* Directed Graph:-

Any pair of vertices are combined with Arrows (→) is known as Directed Graph.

Example: 1)



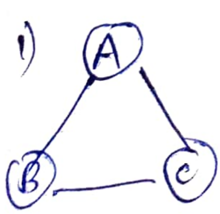
2)



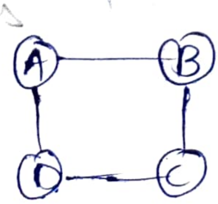
* Undirected Graph:-

→ Any pair of vertices are combined with Undirected lines is known as Undirected Graph

Ex: 1)

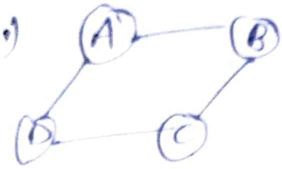
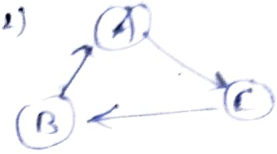


2)

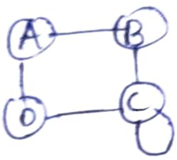
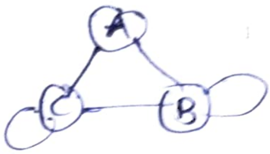


Types of Graphs :-

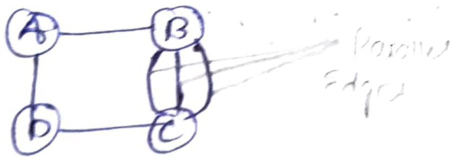
1. **Simple Graph :** A Graph has no self loop, or no parallel edges is known as Simple Graph

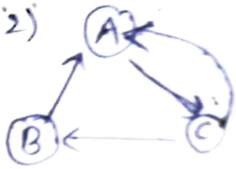
Ex:- 1)  2)  3) 

2. **Loop Graph :** A Graph contains atleast one self loop is known as Loop Graph.

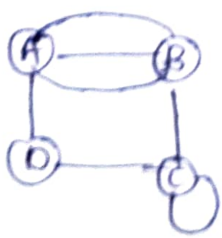
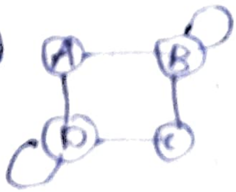
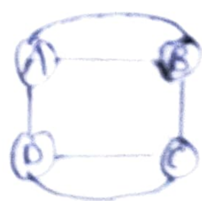
Ex:- 1)  2) 

3. **Parallel Graph :** A Graph contains atleast
In a Graph, any pair of vertices has two or more Edges is known as a Parallel Graph.

Ex:- 1) 

2) 

4. **Multi Graph :-** A Graph has either Parallel or Loop or both is known as Multi Graph.

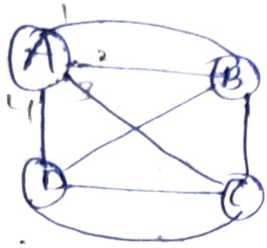
Ex:- 1)  2)  3) 

Degree of a Vertex :-

The degree of vertex is defined as the no. of edges are connected to a Specific Vertex.

→ The Degree of a Graph is defined as sum of Degrees of all vertices of a graph.

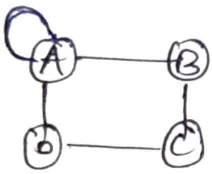
Ex: find The degree of The Graph.



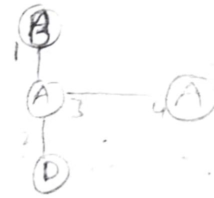
Deg(A) = 4
Deg(B) = 4
Deg(C) = 4
Deg(D) = 4

Deg(Graph) = Deg(G) = sum of Deg of all vertices
= 4+4+4+4 = 16

Ex: (2) Find The degree of The Graph.



Deg(A) = 4 (how many no. of edges are connected is called Degree)
Deg(B) = 2
Deg(C) = 2
Deg(D) = 2

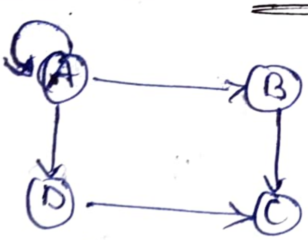


Deg(Graph) = 4+2+2+2 = 10

$\boxed{\text{Deg(G)} = 10}$

* Degree of a directed Graph:-

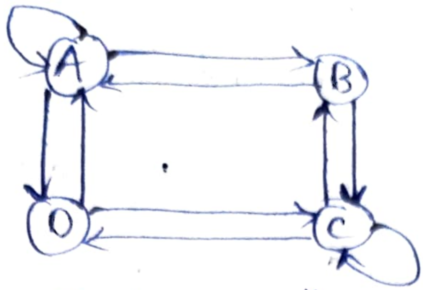
Ex:



Vertex	Indegree	outdegree
A	1	3
B	1	1
C	2	0
D	1	1
	5	5

Deg(G) = Indegree + outdegree
= 5+5 = 10 ⇒ Deg(G) = 10.

Ex:- ②



Vertex	Indegree	Outdegree
A	3	3
B	2	2
C	3	3
D	2	2
	10	10

$\deg(G) = \text{Indeg} + \text{outdeg} = 10 + 10 = 20$

* **Null Graph** :- A Null graph is defined as no edges in a Graph. The Null graph is represented with "N". The Subscript representation of Null graph is "v" where v is no. of vertices and the null graph with v is denoted as "N_v"

Ex: N_3 – (Null Graph with 3 vertices)

(i)

(ii) $N_5 \Rightarrow$

(iii) N_2

(iv) N_0

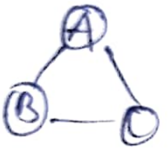
✗ (not possible)

* **Regular Graph** :- A Graph is said to be a Regular graph, if each and every vertex has same degree (or) each and every vertex has same no. of edges.

→ The Regular graph reperesentation is " R ".

→ The Subscript representation of Regular graph is "d", where d indicates degree of Vertex.

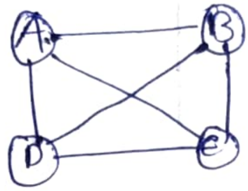
Ex: (i) R_2



Deg(A) = 2
Deg(B) = 2
Deg(C) = 2

∴ It is Regular graph

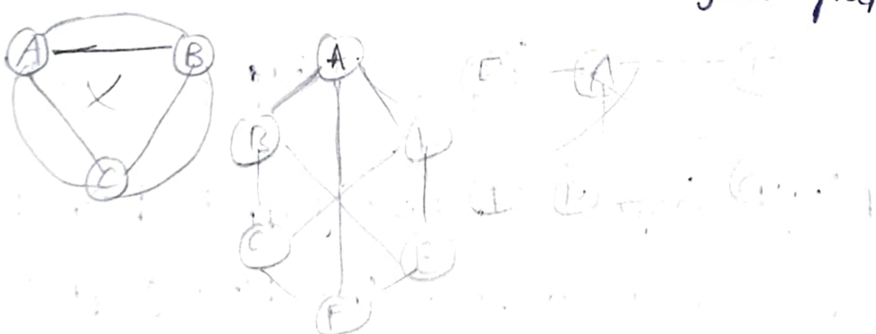
(ii) R_3



Deg(A) = 3
Deg(B) = 3
Deg(C) = 3
Deg(D) = 3

∴ It is a Regular graph

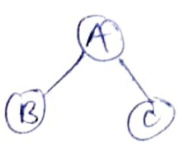
(iii) R_5



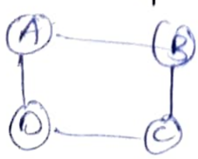
* <u>Cycle Graph</u>:— A graph is said to be a Cycle Graph, if it contains equal no. of edges and vertices.

→ The cycle graph is denoted by C and the Subscript representation of Cycle graph is 'v' i.e. C_v where v is no. of vertices.

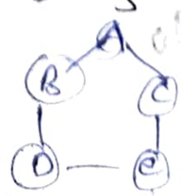
Ex: (i) C_3



(ii) C_4



(iii) C_5



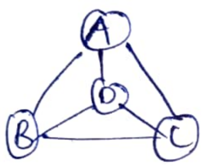
<u>Note</u>:- ~~If all the vertices~~

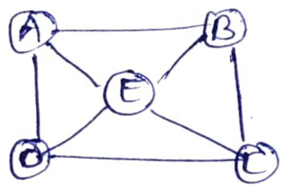
" All cycle graphs are Regular Graphs."

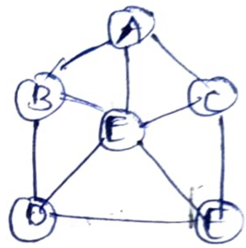
* **Wheel Graph:-** A graph is said to be a wheel graph, if a new vertex is adding inside the cycle graph and connect with all other vertices.

→ The wheel graph is denoted by 'W' and subscript representation is 'v', where v is no. of vertices ie. " W_v "

Ex: (i) W_4 (ii) W_5 (iii) W_6



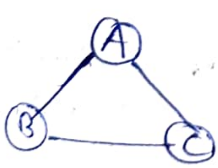


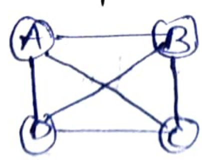


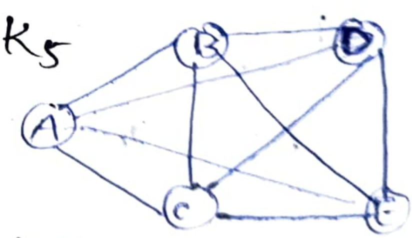
* **Complete Graph:-** A Graph is said to be a Complete graph, if each and every vertex is connected with each and every other vertex.

→ The Complete graph is denoted by 'K' and subscript representation is 'v' ie. K_v.

Ex: (i) K_3 (ii) K_4 (iii) K_5







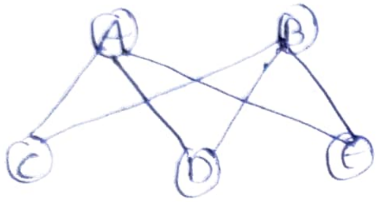
(iv) K_6

* **Complete Biparted Graph:-**

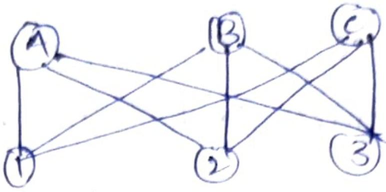
→ A Graph is said Complete Biparted Graph, if all the vertices in first vertex set is connected with " " " " second " " and no two vertices in the same set has an edge.

→ Complete Biparted Graph is denoted by 'K' and subscript representation is v_1, v_2, ie. K_{v_1, v_2}.

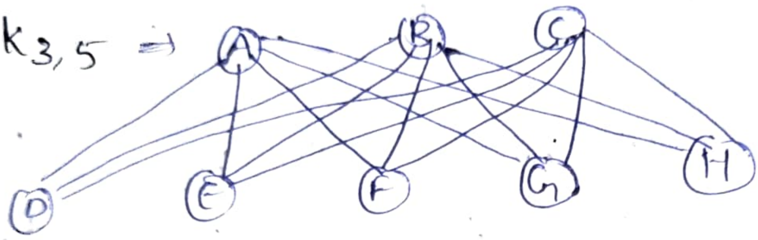
Ex: (i) $K_{2,3}$



(ii) $K_{3,3}$



(iii) $K_{3,5}$ →



* Isomorphic Graph :-

→ Two graphs are said to be Isomorphic graph, if it satisfies the following property

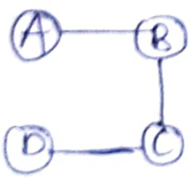
(i) No. of vertices in first graph (G_1) = No. of vertices in 2nd graph (G_2)

(ii) No. of Edges in 1st graph = No. of Edges in 2nd gra

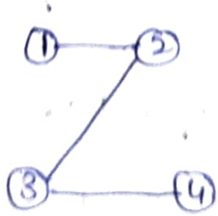
(iii). Degree Sequence of G_1 = Deg. Sequence of G_2

Ex: Check the following two graphs are isomorph or not.

G_1 =



; G_2 =



(i) No. of vertices of G_1 = No. of vertice of G_2

<A, B, C, D> = <1, 2, 3, 4>

4 = 4

Step 1 is Proved.

(ii) No. of edges of G_1 = No. of edges of G_2

<(A,B), (B,C), (C,D)> = <(1,2), (2,3), (3,4)>

3 = 3

Step 2 is Proved.

(iii) Degree sequence of G_1 = Degree sequence of G_2.

$\langle 1, 2, 2, 1 \rangle = \langle 1, 2, 2, 1 \rangle$

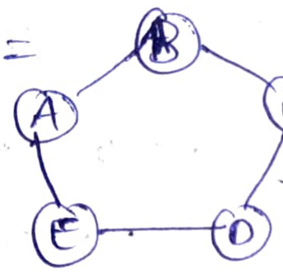
Step 3 is proved.

$\because$ deg(A) = 1 = deg(1)
(B) = 2
(C) = 2
(D) = 1

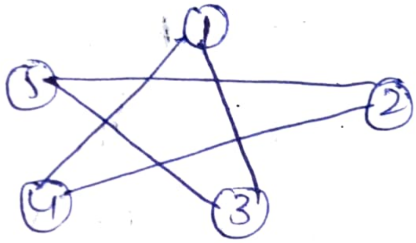
$\therefore$ G_1 and G_2 are Isomorphic.

Ex: ② Check given graphs are Isomorphic or not.

G_1 =



G_2 =



Sol: (i) No. of vertices in G_1 = No. of vertices in G_2

$\langle A, B, C, D, E \rangle = \langle 1, 2, 3, 4, 5 \rangle$

$5 = 5$

Step 1 is proved.

(ii) No. of Edges in G_1 = No. of Edges in G_2

$\langle (A,B), (B,C), (C,D), (D,E), (E,A) \rangle = \langle (1,4), (1,3), (5,2), (5,3), (4,2) \rangle$

$5 = 5$

Step 2 is proved.

(iii) Deg Seq. in G_1 = Degree sequence of G_2

$\langle 2, 2, 2, 2, 2 \rangle = \langle 2, 2, 2, 2, 2 \rangle$

Step 3 is proved.

$\therefore$ G_1 and G_2 are Isomorphic graphs

* Euler's Graph (or) Eulerian Graph:-

Path:- Alternative Sequence of vertices & edges.

Circuit:- Starting and ending vertices must be same in path.

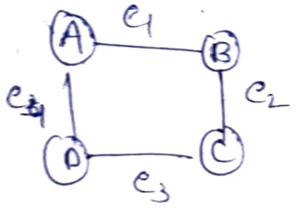
Euler's Graph:- A graph is Said to be Euler's graph, if it Satisfies Euler's path and Euler's Circuit.

Euler's path:- The path touches all the edges of the graph is known as Euler's path.

Euler's Circuit:- In a circuit, no two edges are repeated then the circuit is known as Euler's Circuit

Ex:- Check whether the following graph is Euler's graph or not.

$G_1 =$



edges $= \{e_1, e_2, e_3, e_4\}$

Sol:- (i) path of G_1

A e_1 B e_2 C e_3 D e_4 A

Path touches all the edges of G_1 $\{e_1, e_2, e_3, e_4\}$

$\therefore$ G_1 has Euler's path.

(ii) Circuit of G_1

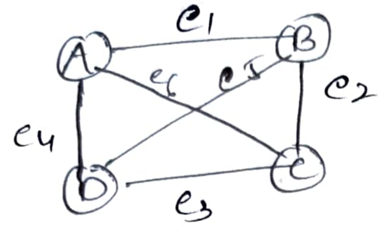
A $\breve{e}_1$ B $\breve{e}_2$ C $\breve{e}_3$ D $\breve{e}_4$ A

Edges are not repeated in circuit.

$\therefore$ G_1 has Euler's Circuit.

∴ G_1 is a Euler's Graph.

Ex: (2) check whether graph is Euler's or not.



Sol: (i) Path of G_1

A e_1 B e_2 C e_3 D e_4 A e_6 C e_3 D e_5 B

path touches all the Edges of G_1 $\{e_1, e_2, e_3, e_4, e_5, e_6\}$

∴ G_1 has Euler's path.

(ii) Circuit of G_1

A e_1 B e_2 C e_3 D e_4 A e_6 C e_3 D e_5 B

Edge '3' is repeated in the Circuit.

∴ G_1 has no Euler's Circuit.

∴ Graph is not Euler's Graph

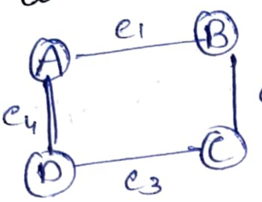
* Hamilton Graph:-

→ A Graph is said to be a Hamilton Graph, if it satisfies Hamilton path and Hamilton Circuit.

Hamilton Path:- In a path, if it touches all the vertices, then that path is known as Hamilton Path

Hamilton Circuit:- In a Circuit, no vertices are repeated except starting and Ending vertex, then circuit is known as Hamilton Circuit.

Ex: check whether the graph is Hamilton or not.

$G_1 =$ 

vertex set $= \{A, B, C, D\}$

Sol:- (i) path of G_1

$A\ e_1\ B\ e_2\ C\ e_3\ D\ e_4\ A$

Path touches all the vertices

$\therefore$ G_1 has hamilton path.

(ii) Circuit of G_1

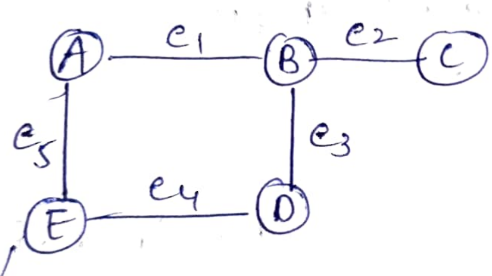
$A\ e_1\ B\ e_2\ C\ e_3\ D\ e_4\ A$

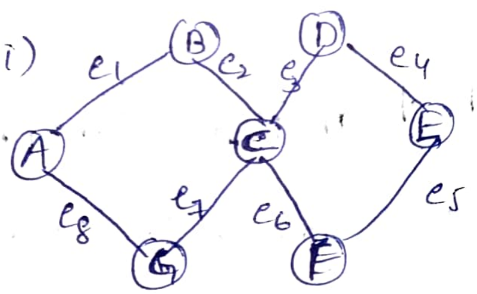
vertices are not repeated except starting and Ending vertices.

$\therefore$ G_1 has hamilton Circuit

$\therefore$ G_1 is a Hamilton Graph.

Ex:(2) check whether the graph are Hamilton or Euler

(i) 

(ii) 

Sol: (i) path of G_1 :-

$A\ e_1\ B\ e_2\ C\ e_2\ B\ e_3\ D\ e_4\ E\ e_5\ A$

Path touches all the vertices.

$\therefore$ G_1 has Euler's path & Hamilton path.

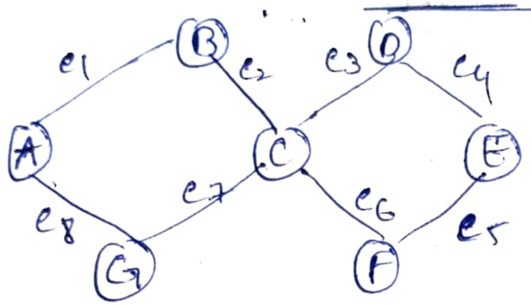
(ii) ~~Path~~ Circuit of G_1

$A\ e_1\ B\ \underline{e_2}\ C\ \underline{e_2}\ \underline{B}\ e_3\ D\ e_4\ E\ e_5\ A$

It is not Hamilton circuit because ~~edge E_2~~ vertex B is repeated ~~and~~ (B is not starting or Ending vertex, and It is not Euler's circuit (E_2 repeated)

$\therefore$ G_1 is niether Euler nor Hamilton.

(ii)



(i) path Of G_1

A e_1 B e_2 C e_3 D e_4 E e_5 F e_6 C e_7 G e_8 A

path touches all the Edges.

G_1 has Euler's path & Hamilton path

(ii) Circuit of G_1

A e_1 B e_2 C e_3 D e_4 E e_5 F e_6 C e_7 G e_8 A

G_1 has Euler's ~~path~~ circuit.

$\because$ Edges are not repeated.

G_1 has no Hamilton circuit.

Becoz, here 'c' is repeated which is neither Starting nor Ending vertex.

$\therefore$ G_1 is Euler's Graph.

* Planar Graph :-

→ A graph can be modified as without crossing edges is known as Planar graph.

Ex:

(modify to remove crossing edge).

* Graph Colouring :-

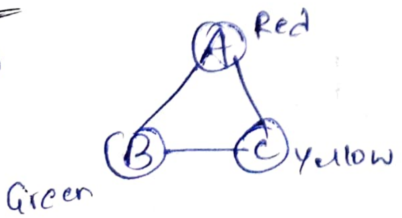
→ The Assignment of colours to the vertices and no two adjacent vertices will have same colour.

→ The Minimum no. of Colours required to perform the Graph Colouring is called as "Chromatic Number" and it is denoted by $X(G)$.

→ $\boxed{X(G) = \text{Minimum no. of colours for Graph colouring.}}$

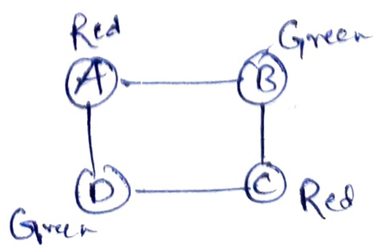
Ex: Find the chromatic no. for Graph

(i)



$X(G) = 3$

(ii)



$X(G) = 2$