Mathematical Logic

→ **Statement :-**

A statement is a declarative sentence that is either true or false but not both. Sometimes called as propositions.

→ **Proposition :-**

→ Proposition is a collection of a propositional variables and connectives

→ we denote the propositional variables by capital letters. (A, B, C, ----)

→ The connectives connect the propositional variables.

* **Predicate :-**

→ Predicate is an expression of one or more variables defined on some specific domain.

→ A predicate with variables can be made a proposition by either Assigning a value to a variable or by quantifying the variable.

Examples:-

Let $E(x, y)$ denote "$x = y$"

Let $X(a, b, c)$ denote "$a + b + c = 0$"

Let $M(x, y)$ denote "x married to y"

* Connectives :-

→ A logical connective is a symbol which is used to connect two or more propositions (or) predicate logics in such a manner that resultant logic depends only on the input logic and the meaning of the connective used.

→ Generally, there are 5 connectives, which are

1. OR ($\vee$)
2. AND ($\wedge$)
3. NOT ($\neg$) or negation
4. Implication (or) If then ($\rightarrow$)
5. If and only if (or) double implies ($\Longleftrightarrow$)

* OR ($\vee$) :- The OR operation of two propositions A and B written as "$A \vee B$" is True if atleast any one of the propositional variable A or B is True.

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

* ~~AND~~

$\neg$

$\neg$

$\neg$ $\sim$

* **AND ($\wedge$)** :- The AND operation of two propositions A and B written as "$A \wedge B$" is True if both the propositions A and B is True.

* Truth Table :-

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

* **Negation ($\neg$)** :- The negation of proposition A writtens as "$\neg A$" is false when A is True and "$\neg A$" is True when A is false.

* Truth Table :-

A	$\neg A$
T	F
F	T

* **Implication** or **If then** ($\rightarrow$) :-

→ An implication $A \rightarrow B$ is the proposition "if A then B". It is false if A is True and B is false, the rest cases are True.

* Truth Table :-

A	B	$A \rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

* If and only if or double implies ($\Longleftrightarrow$) :-

→ $A \Longleftrightarrow B$ is bidirectional logical connective which is true when both the propositions are same i.e. both are false and both are True

Truth Table:-

A	B	$A \Longleftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

* Tautologies :-

→ A Tautology is a formula which is always True for every value of its propositional variables.

Ex:- ① Prove $[(A \to B) \land A] \to B$

A	B	$(A \to B)$	$(A \to B) \land A$	$[(A \to B) \land A] \to B$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Given Statement is Tautology.

Ex: (2) $[(A \land B) \rightarrow C] \Leftrightarrow [(\neg A \lor \neg C) \land (\neg B \lor \neg C)]$

A	B	C	$\neg A$	$\neg B$	$\neg C$	$A \land B$	$(A \land B) \rightarrow C$	$(\neg A \lor \neg C)$	$(\neg B \lor \neg C)$	$(\neg A \lor \neg C) \land (\neg B \lor \neg C)$
T	T	T	F	F	F	T	T	F	F	F
T	T	F	F	F	T	T	F	T	T	T
T	F	T	F	T	F	F	T	F	T	F
T	F	F	F	T	T	F	T	T	T	F
F	T	T	T	F	F	F	T	T	F	T
F	T	F	T	F	T	F	T	T	T	F
F	F	T	T	T	F	F	T	T	T	T
F	F	F	T	T	T	F	T	T	T	T

$[(A \land B) \rightarrow C] \Leftrightarrow [(\neg A \lor \neg C) \land (\neg B \lor \neg C)]$

F
F
F
T
F
T
T
T

Given statement is Contingency.

→ A Compound preposition that is always True, no matter what the truth value of the propositional Variable that occur in it is called a "Tautalogy".

→ A Compound proposition that is always false, is called "Contradiction".

→ A Compound proposition that is neither True nor False, is called "Contigency".

③ $[(A \to B) \land A] \to B$ (Already Solved)

③ $[(p \to q) \land (q \to r)] \to (p \to r)$

p	q	r	$p \to q$	$q \to r$	$p \to r$	$(p \to q) \land (q \to r)$	$[(p \to q) \land (q \to r)] \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Given Statement is "Tautology".

* If expression contains a single variable, then that exp. is called as Contigency. Negation of variable is also Contigency.

p	
T	} Contigency.
F	

④ $[(p \wedge \neg q) \to r] \to [p \to (q \vee r)]$

p	q	r	$\neg q$	$p \wedge \neg q$	$(p \wedge \neg q) \to r$	$q \vee r$	$p \to (q \vee r)$	$(p \wedge \neg q) \to r \to p \to (q \vee r)$
T	T	T	F	F	T	T	T	T
T	T	F	F	F	T	T	T	T
T	F	T	T	T	T	T	T	T
T	F	F	T	T	F	F	F	T
F	T	T	F	F	T	T	T	T
F	T	F	F	F	T	T	T	T
F	F	T	T	F	T	T	T	T
F	F	F	T	F	T	F	T	T

The given logical statement is "Tautology".

⑤ $[(p \to q) \wedge (r \to s) \wedge (p \vee r)] \to (q \vee s)$

p	q	r	s	$p \to q$	$r \to s$	$p \vee r$	$(p \to q) \wedge (r \to s)$	$(p \to q) \wedge (r \to s) \wedge (p \vee r)$	$q \vee s$	P
T	T	T	T	T	T	T	T	T	T	T
T	T	T	F	T	F	T	F	F	T	T
T	T	F	T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T	T	T	T
T	F	T	T	F	T	T	F	F	T	T
T	F	T	F	F	F	T	F	F	F	T
T	F	F	T	F	T	T	F	F	T	T
T	F	F	F	F	T	T	F	F	F	T
F	T	T	T	T	T	T	T	T	T	T
F	T	T	F	T	F	T	F	F	T	T
F	T	F	T	T	T	F	T	F	T	T
F	T	F	F	T	T	F	T	F	T	T
F	F	T	T	T	T	T	T	T	T	T
F	F	T	F	T	F	T	F	F	F	T
F	F	F	T	T	T	F	T	F	T	T
F	F	F	F	T	T	F	T	F	F	T

* Well-Formed Formulas :-

→ Well-formed formulas are used to avoid the drawbacks of Truth Tables like priority scheduling and processing of connectives either from left to right or right to left.

→ The following are the characterstics of well-formed Formulas :-

(1) A statement expression contains a single variable is known as a well-formed formula (WFF)

(2) Applying Negation to a single variable is also a WFF

$P \rightarrow WFF$

$\sim P \rightarrow WFF$

(3) A stmt expression contains single connective and two or more stmt variables is also a WFF

$$\left.\begin{array}{l} P \wedge Q \\ P \vee Q \\ P \rightarrow Q \\ P \leftrightarrow Q \end{array}\right\} WFF$$

(4) If the statement two or more connectives and two or more stmt variables, then it must contains "Paranthesis" to assign priorities to the connectives.

(6) * Identify the following stmts are WFF or not ?

1. P - WFF
2. $\sim P$ - WFF
3. $P \wedge Q$ - WFF

4. $P \vee Q \vee S \vee T$ - WFF

5. $P \vee (Q \forall S \rightarrow T)$ - Not a WFF

6. $P \wedge Q \wedge R \wedge S \rightarrow T$ - Not a WFF

7. $P \vee (Q \wedge S) \rightarrow T$ - Not a WFF

8. $P \wedge (Q \vee S) \rightarrow (T \vee S)$ - Not a WFF

9. $P \wedge Q \wedge R \rightarrow S$ - Not

10. $P \rightarrow Q \rightarrow R \rightarrow S$ - WFF

* Equivalence formulas :-

→ In Mathematical Logic, equivalence formulas are used to identify the similarity b/w two stmt expressions.

→ The following are the different equivalence formula in propositional logic

1. Commutative Law :- ($a \times b = b \times a$; $a+b = b+a$)

(i) $P \wedge Q \Leftrightarrow Q \wedge P$ ($\Leftrightarrow$ - equivalence symbol)

(ii) $P \vee Q \Leftrightarrow Q \vee P$

Truth Table :-

P	Q	$P \wedge Q$	$Q \wedge P$	$P \vee Q$	$Q \vee P$
T	T	T	T	T	T
T	F	F	F	T	T
F	T	F	F	T	T
F	F	F	F	F	F

2. Associative Law :-

$$\left[\begin{array}{l} a\times(b\times c) = (a\times b)\times c \\ a+(b+c) = (a+b)+c \end{array}\right]$$

(i) $P\wedge(Q\wedge R) \Leftrightarrow (P\wedge Q)\wedge R$

(ii) $P\vee(Q\vee R) \Leftrightarrow (P\vee Q)\vee R$

Truth Table :-

P	Q	R	$Q\wedge R$	$P\wedge(Q\wedge R)$	$P\wedge Q$	$(P\wedge Q)\wedge R$	$Q\vee R$	$P\vee(Q\vee R)$	$P\vee Q$	$(P\vee Q)\vee R$
T	T	T	T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	T	T	T	T
T	F	T	F	F	F	F	T	T	T	T
T	F	F	F	F	F	F	F	T	T	T
F	T	T	T	F	F	F	T	T	T	T
F	T	F	F	F	F	F	T	T	T	T
F	F	T	F	F	F	F	T	T	F	T
F	F	F	F	F	F	F	F	F	F	F

3. <u>Distributive Law</u>:- $a \times (b + c) = (a \times b) + (a \times c)$
$a + (b \times c) = (a + b) \times (a + c)$

(i) $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$

(ii) $P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$ ✓	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$ ✓	$Q \wedge R$	$P \vee (Q \wedge R)$ ✓	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$ ✓
T	T	T	T	T	T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T	F	T	T	T	T
T	F	T	T	T	F	T	T	F	T	T	T	T
T	F	F	F	F	F	F	F	F	T	T	T	T
F	T	T	T	F	F	F	F	T	T	T	T	T
F	T	F	T	F	F	F	F	F	F	F	F	F
F	F	T	T	F	F	F	F	F	F	T	T	T
F	F	F	F	F	F	F	F	F	F	F	F	F

* Absorption Law :-

(i) $P \cap (P \vee Q) \Leftrightarrow P$

(ii) $P \vee (P \wedge Q) \Leftrightarrow P$

P	Q	$P \vee Q$	$P \wedge (P \vee Q)$	$P \wedge Q$	$P \vee (P \wedge Q)$
T	T	T	T	T	T
T	F	T	T	F	T
F	T	T	F	F	F
F	F	F	F	F	F.

* General Law :-

(i) $P \rightarrow Q \Leftrightarrow \neg P \vee Q$

(ii) $P \leftrightarrow Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$

P	Q	$\neg P$	$\neg P \vee Q$	$P \rightarrow Q$	$P \leftrightarrow Q$	$Q \rightarrow P$	$(P \rightarrow Q) \wedge (Q \rightarrow P)$
T	T	F	T	T	T	T	T
T	F	F	F	F	F	T	F
F	T	T	T	T	F	F	F
F	F	T	T	T	T	T	T

* Idempodant Law :-

(i) $P \wedge P \Leftrightarrow P$

$P \vee P \Leftrightarrow P$

P	$P \wedge P$	$P \vee P$
T	T	T
F	F	F

* Identity Law :-

(i) $P \wedge T \Leftrightarrow P$

(ii) $P \vee F \Leftrightarrow P$

P	$P \wedge T$	$P \vee F$
T	T	T
F	F	F

* Dominant Law :-

(i) $P \wedge F \Leftrightarrow F$

(ii) $P \vee T \Leftrightarrow T$

P	$P \wedge F$	$P \vee T$
T	F	T
F	F	T

* Negation Law :-

(i) $P \wedge \neg P \Leftrightarrow F$

(ii) $P \vee \neg P \Leftrightarrow T$

P	$\neg P$	$P \wedge \neg P$	$P \vee \neg P$
T	F	F	T
F	T	F	T

* Double Negation :-

(i) $\sim(\sim P) \Leftrightarrow P$

P	$\sim P$	$\sim(\sim P)$
T	F	T
F	T	F

* Demorgan's Law:-

(i) $\sim(P \wedge Q) \Leftrightarrow \sim P \vee \sim Q$

(ii) $\sim(P \vee Q) \Leftrightarrow \sim P \wedge \sim Q$

P	Q	$\sim P$	$\sim Q$	$P \wedge Q$	$\sim(P \wedge Q)$	$P \vee Q$	$\sim(P \vee Q)$	$(\sim P \vee \sim Q)$	$(\sim P \wedge \sim Q)$
T	T	F	F	T	F	T	F	F	F
T	F	F	T	F	T	T	F	T	F
F	T	T	F	F	T	T	F	T	F
F	F	T	T	F	T	F	T	T	T

* Normal Forms :-

→ Normal forms are used to reduce or transform a given expression statement into standard form.

→ The standard statement contains only two Connective ~~(and or operator)~~ '$\wedge$' and '$\vee$' operator

→ Normal forms are defined with the help of elementary product and elementary Sum.

⇒ Let us consider P and Q are two statement Variables, then their Elementary Product is defined as '$P \wedge Q$'

⇒ Let us consider P and Q are two stmt Variables, then their Elementary Sum is defined as "$P \vee Q$"

→ Normal forms are categorized into 4 categories.

1. Conjuctive Normal form (CNF)
2. Disjuctive Normal form (DNF)
3. Principle Conjuctive Normal form (PCNF)

4. Principle Disjuctive Normal Form (PDNF)

1. Conjuctive Normal Form (CNF):-

→ CNF is defined as the product of elementary sums of given stmt. Variables ie.

the stmt. Variables are connected using " $\vee$ " operator and variable pairs are connected using " $\wedge$ " operator.

→ The General Syntax of CNF is

$$\underbrace{(var_1 \vee var_2)}_{\text{variable pair}} \wedge (var_3 \vee var_4) \wedge (var_5 \vee var_6)$$

Ex:- $(P \vee Q) \wedge (R \vee S) \wedge (T \vee W)$

⇒ Procedure for CNF :-

Step 1:- If the given expression stmt. contains " $\rightarrow$ " operator or " $\leftrightarrow$ " operator. Then apply "General law" equivalance formula.

$$P \rightarrow Q \Leftrightarrow \neg P \vee Q$$

$$P \leftrightarrow Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$$

Step 2:- If the expression stmt. contains two or more connectives then apply "distributive law" equivalence formulae.

$$P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$$

③ If the expression stmt. contains "$\sim$" operator, then apply Demorgan's law equivalence formulae.

$$\sim(P \wedge Q) \Leftrightarrow \sim P \vee \sim Q$$

$$\sim(P \vee Q) \Leftrightarrow \sim P \wedge \sim Q$$

④ Apply the remaining equivalence formulae, if necessary to convert the given expression stmt into CNF!

Ex:- ① Convert $P \to Q$ into CNF.

Sol:- Here, we apply General law. equivalence formulae

$$\boxed{P \to Q \Leftrightarrow \neg P \vee Q}$$

	P	Q	$\neg P$	$P \to Q$	$\neg P \vee Q$
(No need)	T	T	F	T	
	T	F	F	F	
	F	T	T	T	
	F	F	T	T	

'$\neg P \vee Q$' is in the form of Elementary Sum. It is in the form of a CNF.

② Convert $P \leftrightarrow Q$ into CNF.

Here, we apply General law equivalence formulae.

$$P \leftrightarrow Q \Leftrightarrow (P \to Q) \wedge (Q \to P)$$

$$(\because P \to Q \Leftrightarrow \neg P \vee Q)$$

$= (P \to Q) \wedge (Q \to P) \Leftrightarrow (\neg P \vee Q) \wedge (\neg Q \vee P)$

(3) Convert $\sim(P \to (Q \wedge R))$

Sol: Here, we apply General law

$$\boxed{P \to Q \Leftrightarrow \neg P \vee Q}$$

$\therefore P \to (Q \wedge R) \Leftrightarrow \neg P \vee (Q \wedge R)$

Then, $\sim(\neg P \vee (Q \wedge R)) \to (1)$

here we apply Demorgan's law.

$$\boxed{\sim(P \vee Q) \Leftrightarrow \sim P \wedge \sim Q}$$

Then eq (1) is

$\sim(\neg P \vee (Q \wedge R)) \Leftrightarrow \sim(\sim P) \wedge \sim(Q \wedge R)$

$\Leftrightarrow P \wedge (\sim(Q \wedge R))$ ✓

x $\left(\boxed{\sim(P \wedge Q) \Leftrightarrow \cancel{\sim}\sim P \vee \sim Q}\right.$

$\left.\Leftrightarrow \cancel{P}R \wedge (\sim Q \vee \sim R)\right)$ x

2. Disjuctive Normal Form (DNF) :—

→ DNF is defined as the Sum of elementary products of given stmt. variables ie.

stmt. variables are connected with "$\wedge$" operator and variable pairs are connected with "$\vee$" operator.

→ General Syntax of DNF is

$(var_1 \wedge var_2) \vee (var_3 \wedge var_4) \vee (var_5 \wedge var_6)$

Ex:- $(P \wedge Q) \vee (R \wedge S) \vee (T \wedge W)$

Procedure for DNF :-

(1) If the given expression stmt. contains '$\rightarrow$' operator or "$\leftrightarrow$" operator, then apply General law equivalence formula.

$P \rightarrow Q \Leftrightarrow \neg P \vee Q$

$P \leftrightarrow Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$

(2) If the expression stmt. contains two or more connectives, then apply distributive law equivalence formulae.

$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

(3) If the expression stmt. contains "$\sim$" operator, then apply Demorgans law equivalence formula.

$\sim(P \wedge Q) \Leftrightarrow \sim P \vee \sim Q$

$\sim(P \vee Q) \Leftrightarrow \sim P \wedge \sim Q$.

④ Apply the remaining equivalence formulae, if necessary to convert the given expression stmt into 'DNF'.

Ex ① Convert $P \rightarrow Q$ into DNF.

General law, $P \rightarrow Q \Leftrightarrow \neg P \vee Q$

we have to get terms in Sum of Elem. Product forms.

from Idempotant law, $P \wedge P \Leftrightarrow P$

$P \rightarrow Q \Leftrightarrow \neg P \vee Q$ (convert single variable into Elem Prod)

$(\neg P \wedge \neg P) \vee (Q \wedge Q)$

It is form of DNF.

② Convert $P \leftrightarrow Q$ into DNF.

General law, $P \leftrightarrow Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$

($\because P \rightarrow Q \Leftrightarrow \neg P \vee Q$)

then $P \leftrightarrow Q \Leftrightarrow (\neg P \vee Q) \wedge (\neg Q \vee P)$

It is in form of product of Elem. Sum.

But we need in Sum of Elem. prod (DNF)

$\because$ Distributive law $\Rightarrow \boxed{P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)}$

$\Leftrightarrow ((\neg P \vee Q) \wedge \neg Q) \vee ((\neg P \vee Q) \wedge P)$

$\Leftrightarrow (\neg P \wedge \neg Q) \vee (Q \wedge \neg Q) \vee (\neg P \wedge P) \vee (Q \wedge P)$

$\Leftrightarrow (\neg P \wedge \neg Q) \vee (P \wedge Q)$

(3) Convert $\sim(P \to (Q \wedge R))$ into DNF

Soln: Given, $\sim(P \to (Q \wedge R))$

General law, $P \to Q \Leftrightarrow \neg P \vee Q$

$\Leftrightarrow \sim[\neg P \vee (Q \wedge R)]$

Demorgans law, $\sim(P \vee Q) \Leftrightarrow \sim P \wedge \sim Q$

$\Leftrightarrow \sim(\neg P) \wedge \sim(Q \wedge R)$

$\Leftrightarrow P \wedge \sim(Q \wedge R)$

$\Leftrightarrow P \wedge (\sim Q \vee \sim R)$

$\Leftrightarrow (P \wedge \sim Q) \vee (P \wedge \sim R)$

3. Principle Conjective Normal form (PCNF):-

→ Principle Conjective Normal form is defined as a "Canonical product of Elementary Sums." i.e. the statement variables are combined with 'V' operator and Variable pairs are combined with "Λ" operator.

→ The General Syntax of PCNF is

$(Var_1 \vee Var_2) \wedge (Var_3 \vee Var_4) \wedge (Var_5 \vee Var_6)$

Ex:- × ~~(P V Q) Λ (R V S) Λ (T V W)~~ ×

[∵ Each elementary sum contains all Variables]

→ PCNF can be calculated for given expression stament in two ways.

1) By using Truth Tables
2) By using Equivalence formulae.

By using Truth Tables :-

→ Draw the Truth Table for the given expression stmt

→ Identify Contradiction or False Values from the Truth Table.

→ Form Max. terms for all the contradiction Values of Truth Table. Let P and Q are two stmt Variables, then their max. terms are defined as $P \vee Q$, $\sim P \vee Q$, $P \vee \sim Q$, $\sim P \vee \sim Q$.
If 'n' is the no. of stmt variables of a given stmt. expression, then the possible max terms are "2^n".

→ Combine all the Contradiction max terms by using 'AND' operator.

Ex:- ① Convert $P \leftrightarrow Q$ into PCNF by using Truth Table.

Sol:-

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

P	Q	Max terms
T	F	$\sim P \vee Q$
F	T	$P \vee \sim Q$

$(\sim P \vee Q) \wedge (P \vee \sim Q)$

This is required PCNF.

Ex: ② Convert $(\sim P \leftrightarrow R) \wedge (Q \leftrightarrow P)$

P	Q	R	$\sim P$	$\sim P \leftrightarrow R$	$Q \leftrightarrow P$	$(\sim P \leftrightarrow R \wedge (Q \leftrightarrow P)$
T	T	T	F	F	T	F
T	T	F	F	T	T	T
T	F	T	F	F	F	F
T	F	F	F	T	F	F
F	T	T	T	T	F	F
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	F	T	F

false values

P	Q	R	Max terms
T	T	T	$\sim P \vee \sim Q \vee \sim R$
T	F	T	$\sim P \vee Q \vee \sim R$
T	F	F	$\sim P \vee Q \vee R$
F	T	T	$P \vee \sim Q \vee \sim R$.
F	T	F	$P \vee \sim Q \vee R$
F	F	F	$P \vee Q \vee R$

PCNF form $= (\sim P \vee \sim Q \vee \sim R) \wedge (\sim P \vee Q \vee \sim R) \wedge (\sim P \vee Q \vee$
$\wedge (P \vee \sim Q \vee \sim R) \wedge (P \vee \sim Q \vee R) \wedge (P \vee Q \vee R)$

Ex: ③ Convert $(P \leftrightarrow (Q \wedge R)) \wedge (\neg P \rightarrow (\neg Q \rightarrow \neg R))$

$(P \rightarrow (Q \wedge R)) \wedge (\neg P \rightarrow (Q \rightarrow \neg R))$

$P \to (Q \wedge R)$

P	Q	R	$\neg P$	$\neg Q$	$\neg R$	$Q \wedge R$	$P \to (Q \wedge R)$	$\neg Q \to \neg R$	$\neg P \to (\neg Q \to \neg R)$	Prob.
T	T	T	F	F	F	T	T	T	T	T
T	T	F	F	F	T	F	F	T	T	F
T	F	T	F	T	F	F	F	F	T	F
T	F	F	F	T	T	F	F	T	T	F
F	T	T	T	F	F	T	T	T	T	T
F	T	F	T	F	T	F	T	T	T	T
F	F	T	T	T	F	F	T	F	F	F
F	F	F	T	T	T	F	T	T	T	T

False values.

P	Q	R	Max terms
T	T	F	$\sim P \vee \sim Q \vee R$
T	F	T	$\sim P \vee Q \vee \sim R$
T	F	F	$\sim P \vee Q \vee R$
F	F	T	$P \vee Q \vee \sim R$.

PCNF $\Rightarrow (\sim P \vee \sim Q \vee R) \wedge (\sim P \vee Q \vee \sim R) \wedge (\sim P \vee Q \vee R) \sim (P \vee Q \vee \sim R$

(ii) **By using Equivalence formulas :-**

① If the given expression stmt. contains "$\to$" operator or "$\leftrightarrow$" operator. Then apply 'General law' equivalence formula.

$P \to Q \Leftrightarrow \neg P \vee Q$

$P \leftrightarrow Q \Leftrightarrow (P \to Q) \wedge (Q \to P)$

(2) If the expression stmt. contains two or more connectives, then apply distributive law equivalence formulae.

$$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$$

(3) If the expression stmt. contains "$\sim$" operator, then apply Demorgans law equivalence formula

$$\sim(P \wedge Q) \Leftrightarrow \sim P \vee \sim Q$$

$$\sim(P \vee Q) \Leftrightarrow \sim P \wedge \sim Q.$$

(4) Apply remaining equivalence formulae, if necessary to convert the given expression stmt. into "PCNF".

(5) In PCNF, each elementary sum must contain all the variables of given expression stmt.

If any variable is missing in any elementary sum then add that variable by using 'v' variable and $\vee \sim$ variable.

(6) If any repeated or duplicate elementary sum in the CNF, then avoid that repeated elementary sum.

① Convert $P \wedge Q$ into PCNF

Sol^n Idempotent law :- $\boxed{P \vee P \Leftrightarrow P}$

$\Leftrightarrow (P \vee P) \wedge (Q \vee Q)$

(write missing terms) $\Leftrightarrow (P \vee P \vee Q) \wedge (P \vee P \vee \sim Q) \wedge (P \vee Q \vee Q) \wedge (\sim P \vee Q \vee Q)$

$\Leftrightarrow (P \vee Q) \wedge (P \vee \sim Q) \wedge (P \vee Q) \wedge (\sim P \vee Q)$

Ignore duplicate term $\Leftrightarrow (P \vee Q) \wedge (P \vee \sim Q) \wedge (\sim P \vee Q)$

② Convert $(P \to Q) \wedge (Q \to R)$ into PCNF.

Sol^n $\because \boxed{P \to Q \Leftrightarrow \neg P \vee Q}$

$\Leftrightarrow (\neg P \vee Q) \wedge (\neg Q \vee R)$

$(\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee R)$

③ Convert $(P \wedge Q) \vee (Q \wedge R)$

Distributive law :- $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

$\Leftrightarrow ((P \wedge Q) \vee Q) \wedge ((P \wedge Q) \vee R)$

$\Leftrightarrow ((P \vee Q) \wedge (Q \vee Q)) \wedge ((P \vee R) \wedge (Q \vee R))$

$\Leftrightarrow$ ~~PCNF~~.

$(P \vee Q \vee R) \wedge (P \vee Q \vee \sim R) \wedge (P \vee Q \vee Q) \wedge (\sim P \vee Q \vee Q) \wedge$

$($

* Principle Disjective Normal Form :- (PDNF)

→ PDNF is defined as "Canonical sum of Elementary Products" ie the statement variables are combined with '$\wedge$' operator and variable pairs are combined with '$\vee$' operator.

→ The General Syntax of PDNF is

$(var_1 \wedge var_2) \vee (var_1 \wedge var_2) \vee (var_1 \wedge var_2)$

→ PDNF can be calculated for given expression ~~comparison~~ statement in two ways.

i) By using Truth Tables

ii) By using Equivalence formulas.

By using Truth Tables :-

→ Draw the Truth Table for the given expression stmt.

→ Identify ~~extract~~ True values from the Truth Table.

→ Form Min terms for all the True values of Truth Table. Let P and Q are two stmt variables, then their min terms are defined as $P \wedge Q$, $P \wedge \sim Q$, $\sim P \wedge Q$ and $\sim P \wedge \sim Q$.

If 'n' is the no. of stmt variables of a given stmt expression, then the possible min terms are "2^n".

→ combine all the True value min terms by using 'V' operator.

Ex:- ① Convert $P \leftrightarrow Q$ into PDNF using truth Table.

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

P	Q	Min terms
T	T	T $(P \wedge Q)$
F	F	T $(\sim P \wedge \sim Q$

$(P \wedge Q) \vee (\sim P \wedge \sim Q)$

This is req. PDNF.

(ii) By using Equivalence formulas:—

① If the given expression stmt contains "$\rightarrow$" operator or "$\leftrightarrow$" operator. Then apply General law equivalence formula.

Ex ① Convert $P \to Q$ into PDNF using Truth Table Method

P	Q	$P \to Q$	Miniterms
T	T	T	$P \wedge Q$
T	F	F	
F	T	T	$\sim P \wedge Q$
F	F	T	$\sim P \wedge \sim Q$

PDNF form $\Rightarrow (P \wedge Q) \vee (\sim P \wedge Q) \vee (\sim P \wedge \sim Q)$

② Convert $(P \to Q) \wedge (Q \to P)$ into PDNF using Truth Table

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \wedge (Q \to P)$	Miniterms
T	T	T	T	T	$P \wedge Q$
T	F	F	T	F	
F	T	T	F	F	
F	F	T	T	T	$\sim P \wedge \sim Q$

PDNF form $\Rightarrow (P \wedge Q) \vee (\sim P \wedge \sim Q)$

③ Find the PDNF of $(\sim P \leftrightarrow R) \wedge (Q \leftrightarrow P)$ using Truth Table

P	Q	R	$\sim P$	$\sim P \leftrightarrow R$	$Q \leftrightarrow P$	$(\sim P \leftrightarrow R) \wedge (Q \to P)$	Miniterms
T	T	T	F	F	T	F	
T	T	F	F	T	T	T	$P \wedge Q \wedge \sim R$
T	F	T	F	F	F	F	
T	F	F	F	T	F	F	
F	T	T	T	T	F	F	
F	T	F	T	F	F	F	
F	F	T	T	T	T	T	$\sim P \wedge \sim Q \wedge R$
F	F	F	T	F	T	F	

PDNF form $\Rightarrow (P \wedge Q \wedge \sim R) \vee (\sim P \wedge \sim Q \wedge R)$

(4) Find the PDNF of $(P \to (Q \wedge R)) \wedge (\neg P \to (\neg Q \to \neg R))$

P	Q	R	$\neg P$	$\neg Q$	$\neg R$	$Q \wedge R$	$P \to (Q \wedge R)$	$\neg Q \to \neg R$	$\neg P \to (\neg Q \to \neg R)$
T	T	T	F	F	F	T	T	T	T
T	T	F	F	F	T	F	F	T	T
T	F	T	F	T	F	F	F	F	T
T	F	F	F	T	T	F	F	T	T
F	T	T	T	F	F	T	T	T	T
F	T	F	T	F	T	F	T	T	T
F	F	T	T	T	F	F	T	F	F
F	F	F	T	T	T	F	T	T	T

P	Q	R	Miniterms
T	T	T	$P \wedge Q \wedge R$
F	T	T	$\sim P \wedge Q \wedge R$
F	T	F	$\sim P \wedge Q \wedge \sim R$
F	F	F	$\sim P \wedge \sim Q \sim \wedge R$

PDNF form $\Rightarrow (P \wedge Q \wedge R) \vee (\sim P \wedge Q \wedge R) \vee (\sim P \wedge Q \wedge \sim R) \vee (\sim P \wedge$

By using Equivalence law

(1) convert $P \to Q$ into PDNF form.

$P \to Q$ $\quad [\because P \wedge P \Leftrightarrow P]$

$\Leftrightarrow \neg P \vee Q$

$\Leftrightarrow (\neg P \wedge \neg P) \vee (Q \wedge Q)$

$\Leftrightarrow (\neg P \wedge \neg P \wedge Q) \vee (\neg P \wedge \neg P \wedge \sim Q) \vee (Q \wedge Q \wedge P) \vee (Q \wedge Q \wedge$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \sim Q) \vee (P \wedge Q) \vee (\sim P \wedge Q)$

$\Leftrightarrow (P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \sim Q)$

② Find PDNF for $(\sim P \leftrightarrow R) \wedge (Q \leftrightarrow P)$

Sol: Given, $(\sim P \leftrightarrow R) \wedge (Q \leftrightarrow P)$

$P \leftrightarrow Q \Leftrightarrow (P \to Q) \wedge (Q \to P)$
$P \to Q \Leftrightarrow \neg P \vee Q$
} General Law

$\Leftrightarrow (\sim P \leftrightarrow R) \wedge (Q \leftrightarrow P)$

$\Leftrightarrow ((\sim P \to R) \wedge (R \to \sim P)) \wedge ((Q \to P) \wedge (P \to Q))$

$\sim(\sim P) \equiv P$

$\Leftrightarrow ((P \vee R) \wedge (\sim R \vee \sim P)) \wedge ((\sim Q \vee P) \wedge (\sim P \vee Q))$

$\Leftrightarrow$ It is in form prod. of elem sum.

But we need sum of elem. prod.

Apply distributive Law $\Rightarrow$ $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee$

$\Leftrightarrow ((P \vee R \wedge \sim R) \vee (P \vee R \wedge \sim P)) \wedge (\sim Q \vee P.$

~~Taut~~

* Duality Law:-

→ Duality laws are based on Basic connectors - negation (~), ∧ and ∨ operators. So we will restrict the dual operations of any expression stmt by using these operators only.

→ Statement:- Two expression stmts, A and A* are said to be Duals of each other. If either one can be obtained from the other by replacing '∧' by 'V' and 'V' by '∧'.

The Connectives '∧' and 'V' operators are also called as Duals of each other.

→ If any expression stmt 'A' contains special variable T and F, then 'A*' its dual is obtained by replacing T by F and 'F by T' in addition to above mentioned interchanges.

Ex:- ① Write Duals of $(P \wedge Q) \vee R$; $(P \wedge Q) \vee T$; $\sim(P \vee Q) \wedge (P \vee (\sim Q \wedge S))$

Dual of $(P \wedge Q) \vee R = (P \vee Q) \wedge R$

Dual of $(P \wedge Q) \vee T = (P \vee Q) \wedge F$

Dual of $\sim(P \vee Q) \wedge (P \vee (\sim Q \wedge S)) = \sim(P \wedge Q) \vee (P \wedge (\sim Q \vee S$

* Principles of Duality (or) Equivalence formulae of Duality :-

① Let A and A^* be Dual Formulas and let $P_1, P_2, \ldots P_n$ are automic Variables occur in A and A^*. i.e. we may write 'A' as $A(P_1, P_2, \ldots P_n)$ and A^* as $A^*(P_1, P_2, \ldots P_n)$.

then the equivalence formula of Duality law is

$$\boxed{\neg A(P_1, P_2, \ldots P_n) \Leftrightarrow A^*(\neg P_1, \neg P_2, \ldots \neg P_n)}$$

② from the Negation formula, it is equivalent to its Dual in which every variable is replaced by its Negation.

As a Consequence of Negation formula, we can also have the

$$\boxed{A(\neg P_1, \neg P_2 \ldots \neg P_n) \Leftrightarrow \neg A^*(P_1, P_2, \ldots P_n)}$$

Ex:- Verify Equivalence of duality law, if
$A(P, Q, R) = \neg P \wedge \neg(Q \vee R)$

Sol:
$A^*(P, Q, R) = \neg P \vee \neg(Q \wedge R)$

$\neg A(P, Q, R) = \neg(\neg P \wedge \neg(Q \vee R))$ $(\because \neg(P \wedge Q) = \neg P \vee \neg Q)$

$= \neg(\neg P) \vee \neg(\neg(Q \vee R))$

$\neg A(P, Q, R) = P \vee (Q \vee R) \rightarrow ①$

$\because A^*(P, Q, R) = \neg P \vee \neg(Q \wedge R) \rightarrow ②$

for $A^*(\neg P, \neg Q, \neg R)$ Replace P, Q, R by $\neg P, \neg Q, \neg R$ in ②

$$A^*(\neg P, \neg Q, \neg R) = \neg(\neg P) \vee \neg(\neg Q \wedge \neg R)$$
$$= P \vee (\neg(\neg Q) \vee \neg(\neg R))$$
$$= P \vee (Q \vee R)$$

$\boxed{\neg A(P,Q,R) = A^*(\neg P, \ldots}$

$\because A(P,Q,R) = \neg P \wedge \neg(Q \vee R)$

$A^*(P,Q,R) = \neg P \vee \neg(Q \wedge R)$

$A(\neg P, \neg Q, \neg R) = \neg(\neg P) \wedge \neg(\neg Q \vee \neg R)$ ($\because$ Demorgan

$= P \wedge (Q \wedge R)$

$\neg A^*(P,Q,R) = \neg(\neg P \vee \neg(Q \wedge R))$

$= \neg(\neg P) \wedge \neg(\neg(Q \wedge R))$

$= P \wedge (Q \wedge R)$

$$\therefore \boxed{A(\neg P, \neg Q, \neg R) \Leftrightarrow \neg A^*(P,Q,R)}$$

Principle.

2) Let us consider $P_1, P_2, \ldots P_n$ be all atomic variables in expression statements A and B, given that $A \Leftrightarrow B$ then we have,

$$\boxed{A(\neg P_1, \neg P_2, \ldots \neg P_n) \Leftrightarrow B(\neg P_1, \neg P_2, \ldots \neg P_n)}$$

(or)

$$\neg A^*(P_1, P_2, P_3, \ldots P_n) \Leftrightarrow \neg B^*(P_1, P_2, \ldots P_n)$$

(or)

$$A^*(P_1, P_2, \ldots P_n) \Leftrightarrow B^*(P_1, P_2, \ldots P_n)$$

Sol: ① Show that $\neg(P \wedge Q) \to (\neg P \vee (\neg P \vee Q)) \Leftrightarrow \neg P \vee Q$

$\Rightarrow \neg(P \wedge Q) \to (\neg P \vee (\neg P \vee Q))$ $\quad (\because P \to Q \Leftrightarrow \neg P \vee Q)$

$\Rightarrow \neg(\neg(P \wedge Q)) \vee (\neg P \vee (\neg P \vee Q))$

$\Rightarrow (P \wedge Q) \vee ((\neg P \vee \neg P) \vee Q)$

$\Rightarrow (P \wedge Q) \vee (\neg P \vee Q)$ $\quad (\because P \vee P \Leftrightarrow P)$

$\Rightarrow (P \wedge Q) \vee (\underbrace{\neg P \vee Q}_{R})$ $\quad [\because ((P \wedge Q) \vee R) \Leftrightarrow (P \vee R) \wedge (Q \vee R)]$

$\Rightarrow (P \vee (\neg P \vee Q)) \wedge (Q \vee (\neg P \vee Q))$

$\Rightarrow ((P \vee \neg P) \vee Q) \wedge ((Q \vee Q) \vee \neg P)$ $\quad (\because$ Associative law$)$

$(T \vee Q) \wedge (Q \vee \neg P)$

$\Rightarrow (T \vee Q) \wedge (\neg P \vee Q)$

$\Rightarrow T \wedge (\neg P \vee Q)$ $\quad (\because$ Identity law$)$

$\Rightarrow \neg P \vee Q$

② Show that $(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Leftrightarrow (\neg P \wedge Q)$

Sol:

Let $A = (P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q))$

Dual of $A = A^* = (P \wedge Q) \vee (\neg P \vee (\neg P \vee Q))$

$B = \neg P \wedge Q$

$B^* = \neg P \vee Q$

from the above example, $A^* \Leftrightarrow B^*$

$\because A^* \Leftrightarrow B^* \Rightarrow \boxed{A \Leftrightarrow B}$

* Tautological Implications:-

→ Let us consider A and B are two compound expression stmts, then A is said to be Tautologically implies to B, if and only if "$A \to B$" is a Tautology.

→ The Tautological Implication b/w A and B is de[noted] by "$A \Rightarrow B$".

Here, the symbol "$\Rightarrow$" is not Commitative i[.e.]

$$A \Rightarrow B \neq B \Rightarrow A$$

Ex: ① Verify $(P \wedge Q) \Rightarrow (P \to Q)$ using Truth Table

P	Q	$P \wedge Q$	$P \to Q$	$(P \wedge Q) \to (P \to Q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

Tautology

$\therefore (P \wedge Q) \Rightarrow (P \to Q)$

② Verify $(P \to Q) \wedge (Q \to R) \Rightarrow (P \to R)$

$(P \to Q) \wedge (Q \to R) \to (($

P	Q	R	$P \to Q$	$Q \to R$	$(P \to Q) \wedge (Q \to R)$	$P \to R$	~~Ans.~~
T	T	T	T	T	T	T	T
T	T	F	T	F	F	T	T
T	F	T	F	T	F	F	T
T	F	F	F	T	F	T	T
F	T	T	T	T	T	F	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

$\therefore (P \to Q) \wedge (Q \to R) \Rightarrow (P \to R)$

③ $(P \wedge Q) \Rightarrow P$

④ $\neg P \Rightarrow (P \to Q)$

Sol:

P	Q	$\neg P$	$P \to Q$	$\neg P \to (P \to Q)$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

Hence, $\neg P \Rightarrow (P \to Q)$

③ $(P \wedge Q) \Rightarrow P$

P	Q	$(P \wedge Q)$	$(P \wedge Q) \to P$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

$\therefore (P \wedge Q) \Rightarrow P$

* **Converse Statement :-**

→ Let us consider P and Q are two stmt. variables and $(P \to Q)$ is a stmt. expression, then $Q \to P$ is said to be its converse stmt.

* **Inverse Statement :-**

→ Let us consider P and Q are two stmt. Variables and $(P \rightarrow Q)$ is an expression stmt, then $(\neg P \rightarrow \neg Q)$ is called its Inverse stmt.

* **Contrapositive Statement :-**

Let us consider P and Q are two stmt. Variables, and $(P \rightarrow Q)$ is an expression statement, then $(\neg Q \rightarrow \neg P)$ is called its contrapositive stmt.

* **Theory of Inference :-**

→ Inference theory is concerned with identifying conclusion from a certain hypothesis or Basic assumptions called "Premises", by applying certain principles of reasoning, called "Rules of Inference"

→ When a conclusion is derived from a set of premises by using Rules of Inference, the process of such derivation is called a 'formal Proof'

→ Let us consider $H_1, H_2, \ldots H_n$ are 'n' basic assumptions and 'c' is a conclusion Variable, Then the theory of inference is defined as

$$\boxed{H_1 \cap H_2 \cap H_3 \ldots \cap H_n \Rightarrow C}$$

where $H_1, H_2 \ldots H_n$ are called Premises

c is a Conclusion Variable.

* Argument :-

→ An Argument is a Sequence of propositions, the final proposition in the argument is Called a Conclusion and the remaining Previous propositions are Called as "Premises"

→ An Argument is Valid if the Truth Value of all its premises implies that the Conclusion is True

→ Any Conclusion which is arrived at, by following the rules of Inference is Called a 'Valid Conclusion' and the argument is Called a 'Valid Argument'.

→ Theory of Inference Contains four formal methods to Verify the Validity of Conclusion Variable and Argument.

(1) Truth Table Method :-

→ When A and B are two statement formulas, then 'B' is said to follow 'A' (or) 'B' is a Valid Conclusion of the premise 'A' If and only if $A \to B$ is a Tautalogy.

→ Extending, a Conclusion 'C' is Said to follow from a Set of Premises $H_1, H_2, \ldots H_n$ if and only if $\boxed{H_1 \cap H_2 \cap \ldots \cap H_n \Rightarrow C}$

→ If a set of premises and a Conclusion are given, it is possible to find whether the Conclusion is Valid or not, by constructing Relevant Truth Tables.

→ from the Truth Table, Identify the truth value of premises in which conclusion is True.

→ All the respective premises are true, then the Possibility is said to be a Valid possibility.

→ In Truth Table, if it contains atleast one Valid possibility, then the Conclusion is Valid and the Argument is also valid.

Ex:- Verify Validity of 'C' if $H_1 : P \to Q$, $H_2 : Q$, C

Sol: $H_1 : P \to Q$

$H_2 : Q$

$C : P$

C P	H_2 Q	H_1 $P \to Q$	
T	T	T	Valid
T	F	F	× (Invalid)
F	T	T	
F	F	T	

~~Here it is V~~

H_1 & H_2 are True only in first row in which case 'c' is also true. Hence it is a Valid Conclusion and the argument is Valid argument.

② $H_1: P \to Q$; $H_2: \sim(P \wedge Q)$ $C: \sim P$

P	Q	C $\sim P$	H_1 $P \to Q$	$P \wedge Q$	H_2 $\sim(P \wedge Q)$	
T	T	F	T	T	F	
T	F	F	F	F	T	
F	T	T	T	F	T	valid
F	F	T	T	F	T	valid

H_1 and H_2 are True in 3rd and 4th row in which case 'C' is also True. Hence it is valid conclusion and the argument is valid argument.

③ $H_1: \sim P$, $H_2: P \leftrightarrow Q$, and $C: \sim(P \wedge Q)$

P	Q	H_1 $\sim P$	H_2 $P \leftrightarrow Q$	$P \wedge Q$	C $\sim(P \wedge Q)$
T	T	F	T	T	F
T	F	F	F invalid	F	T
F	T	T	F	F	T
F	F	T	T valid	F	T

H_1 and H_2 are True in 4th row in which case 'c' is also True. Hence it is valid conclusion and the argument is valid argument.

* Tautological Implication formulas:—

I_1 : $P \wedge Q \Rightarrow P$
I_2 : $P \wedge Q \Rightarrow Q$
} Simplification/ formulae

I_3 : $P \Rightarrow P \vee Q$
I_4 : $Q \Rightarrow P \vee Q$
} Additions

I_5 : $\sim P \Rightarrow P \to Q$

I_6 : $Q \Rightarrow P \to Q$

I_7 : $\neg(P \to Q) \Rightarrow P$

I_8 : $\neg(P \to Q) \Rightarrow \neg Q$

I_9 : $P, Q \Rightarrow P \wedge Q$ [here P,Q means $P \wedge Q$]

I_{10} : $\neg P, P \vee Q \Rightarrow Q$ –(disjunctive Syllogism)

I_{11} : $P, P \to Q \Rightarrow Q$ [Modus Ponens]

I_{12} : $\neg Q, P \to Q \Rightarrow \neg P$ [Modus Tollens]

I_{13} : $P \to Q, Q \to R \Rightarrow P \to R$ [Hypothetical Syllogism]

I_{14} : $P \vee Q, P \to Q, Q \to R \Rightarrow R$ [Dilema Implificatio

* Rule of Inference (or) Direct Method of Proof :-

→ Rule of Inference is one of the most prominent method to verify whether the conclusion variable is valid or not.

→ In Rule of Inference method, we should follow the theory of Inference, by using two fundamental rule

① Rule P ② Rule T

→ If an existing premise or given premise is introduced into the problem derivation, then that rule is said to be "Rule P"

→ A new stmt. expression is derived from two or more introduced premises, then that rule is said to be " Rule T "

→ while introducing premises into derivation, we have to follow these constraints

* All premises must be introduced in derivation.
* The order of premises which are introduced into derivation may or may not same given in the problem.
* Any premise can be used in derivation in any no. of times if required.

Ex: ① Show that $R \vee S$ follow the premises $C \vee D$, $(C \vee D) \rightarrow \neg H$, $(A \wedge \neg B) \rightarrow R \vee S$, $\neg H \rightarrow A \wedge \neg B$

S.NO.	Premises	Rule
1	$C \vee D$	Rule P
2	$(C \vee D) \rightarrow \neg H$	Rule P
3	$\neg H$	Rule T [from ① & ② with I_{11} (Modus Ponens)]
4.	$\neg H \rightarrow (A \wedge \neg B)$	Rule P
5	$A \wedge \neg B$	Rule T (from ③ and ④ with I_{11} (Modus Ponens)]
6	$(A \wedge \neg B) \rightarrow R \vee S$	Rule P
7	$R \vee S$	Rule T [from ⑤ & ⑥ with I_{11} (Modus Ponens)]

∴ $R \vee S$ follows the given premises, then it is a valid conclusion.

② Show that $T \wedge S$ follow the premises. $P \rightarrow Q$, $Q \rightarrow \neg R$, R, $P \vee (T \wedge S)$

S.NO.	Premise	Rules
1	$P \rightarrow Q$	Rule P
2	$Q \rightarrow \neg R$	Rule P
3	$P \rightarrow \neg R$	Rule T [from ① & ② with I_{13} (Hypothetical syllogism)]
4	R	Rule P

5	$\neg P$	Rule T [from (3) and (4) with I_{12} (Modus Tollens)]
6	$P \vee (T \wedge S)$	Rule P
7	$T \wedge S$	Rule T (from (5) and (6) with I_{10} (Disjunctive Syllogism)]

$\therefore$ $T \wedge S$ follows the given premises, then it is a valid conclusion.

(3) Write symbolic notations for the following stmts. and apply the rule of Inference method.

Stmt 1 : Rohan is watching TV

Stmt 2 : If Rohan watching TV, then he is not studying well

Stmt 3 : If Rohan not studying well, then his father will not buy a bike.

Stmt 4 : Therefore, His father will not buy a bike

Sol:

$H_1 : R$

$H_2 : R \rightarrow \neg S$

$H_3 : \neg S \rightarrow \neg B$

$C : \neg B$

S.NO	Premises	Rules
1	R	Rule P
2	$R \rightarrow \neg S$	Rule P
3	$\neg S$	Rule T [from (1) & (2) with I_{11} (Modus Ponens)]
4	$\neg S \rightarrow \neg B$	Rule P
5	$\neg B$	Rule T [from (3) and (4) ... (Modus Ponens)]

$\therefore$ $\neg B$ follows given premises, then it is a valid conclusion.

(4) $H_1 : P \to Q$, $H_2 : P \to \neg Q$, $H_3 : R$, $C : \neg P$

(5) $H_1 : (P \to Q) \wedge (R \to S)$, $H_2 : (Q \to T) \wedge (S \to U)$,

$H_3 : \sim(T \wedge U)$, $H_4 : P \to R$; $C : \neg P$

(8)

Sol:-

S No.	Premises	Rule
1	$(P \to Q) \wedge (R \to S)$	Rule P
2	$P \to Q$	R_T (from (1) with I_1 (simplification)
3	$R \to S$	R_T (from (1) with I_2 "
4	$(Q \to T) \wedge (S \to U)$	Rule P
5	$Q \to T$	RT (from (4) with I_1 (simplification
6	$S \to U$	RT (from (4) with I_2 (")
7	$P \to T$	RT (from (2) and (5) with I_{13})
8	$R \to U$	RT (from (3) and (6) with I_{13})
9	$P \to R$	Rule P
10	$P \to U$	RT (from (9) and (8) with I_{13})
11	$\neg(T \wedge U)$	Rule P
12	$\neg U$	Rule T (from (11) with I_2 (simplification)
13	$\neg P$	Rule T (from (12) and (10) with I_{12} (Modus Tollens)

(4) Sol:

S.No.	Premises	Rule

(6) Show that 'S' is a valid inference from the premises $P \to Q$, $Q \to R$, $R \to S$, P.

SNO	Premises	Rule
1	$P \to Q$	Rule P
2	$Q \to R$	Rule P
3	$P \to R$	Rule T (from ① & ② with I_{13})
4	$R \to S$	Rule P
5	$P \to S$	Rule T (from ① & ② with I_{13})
6	P	Rule P
7	S	Rule T [from ⑤ and ⑥ with I_{11} (Modus Ponens)]

(7) Show That $S \vee R$ is a valid Conclusion. for The premises $P \vee Q$, $P \rightarrow Q$, $Q \rightarrow S$

Sol:

S.NO.	Premises	Rule
1		

* **Rule of Conditional Proof Method:-**

→ Rule of Conditional is used to identify whether the conclusion variable is valid or not, if and only if, if the Conclusion variable contains Conditional Connective ('→')

→ The following are the rules followed by rule of conditional proof Method

1) Rule P
2) Rule T
3) Rule CP

Rule CP:- If the conclusion variables contains the conditional part, then identify the LHS premise of conditional connective as "Additional premise" and RHS variable as conclusion variable.

$$C_1 \rightarrow C_2$$

C_1 ⇓ Additional Premise

C_2 ⇓ Conclusion Variable.

→ In Rule CP method, the additional premise is introduced first and then introduce the premises as in the rule of inference method.

→ from this, the conditional proof is defined as

$$H_1, H_2, \ldots H_n \Rightarrow C_1 \rightarrow C_2$$

here we have to prove that

$$H_1, H_2, \ldots H_n \Rightarrow C_2$$

where C_1 is an Additional premise

C_2 is a Conclusion Variable.

① Show that $R \rightarrow S$ is logically derived from the premises $P \rightarrow (Q \rightarrow S)$, $\neg R \vee P$, Q.

Sol: Here $R \rightarrow S$ is conclusion containing conditional connective.

R is the Additional premise and Conclusion Variable is "S"

We have to prove that

$R, P \to (Q \to S), \neg R \vee P, Q \Rightarrow S$

S.NO	Premise	Rule
1	R	Rule CP (Additional Premise)
2	$\neg R \vee P$	Rule P
3	P	Rule T (from ① & ② with I_{10} (disjunctive
4	$P \to (Q \to S)$	Rule P
5	$Q \to S$	Rule T (from ③ & ④ with I_{11} (Modus Ponens
6	Q	Rule P
7	S	Rule T (from ⑤ & ⑥ with I_{11} (")
8	$R \Rightarrow S$	Rule CP (from ① & ⑦)

② Show that $P \to (Q \to R)$ derived from the premises $A \to B$, $B \to S$, $S \to R$, A

S NO.	Premises	Rule
1	P	Rule CP (Additional Premise)
2	Q	Rule CP (" ")
3	$A \to B$	Rule P
4	$B \to S$	Rule P
5	$A \to S$	Rule T (from ③ & ④ with I_{13} Hypothetical Syllogism)

6	$S \to R$	Rule P
7	$A \to R$	Rule T (from (5) & (6) with I_{13})
8	A	Rule P
9	R	Rule T (from (8) & (7) with I_{11} (Modus pones))
10	$Q \to R$	Rule CP (from (2) & (10))
11	$P \to (Q \to R)$	Rule CP (from (1) & (10))

* Method of Indirect proof:-

→ Indirect method of proof is an alternative method to verify the conclusion variable in inference theory.

→ Indirect method follow these basic steps to identify the validity of conclusion variable.

1) Identify the conclusion variable from the given problem statement.
2) Identify an assumed premise by applying negation to the conclusion variable.
3) Introduce the assumed premise in the derivation and then introduce the remaining premises.

→ The result of problem derivation is a contradiction, then we can conclude that negation of conclusion variable does not follow the given premises.

→ From the indirect method of proof, if negation of conclusion doesn't follow the premises, hence the conclusion variable follow the given premises

Ex: ① $P \to \neg S$ follows premises $P \to (Q \vee R)$; $Q \to \neg P$; $S \to \neg R$ and P. by using Indirect method of proof.

S.NO.	Premises	Rule.
1	$P \to \neg S$	Conclusion Variable
2	$\neg(P \to \neg S)$	Rule of Indirect proof (Assumed Premise)
3	$P \to (Q \vee R)$	Rule 'P'
4	P	Rule 'P'
5	$Q \vee R$	Rule T [from ④ & ③ with I_{11} - Modus Ponens)]
6	$\neg Q \to R$	R_T (from ⑤ apply General Law) $P \to Q \Leftrightarrow \neg P \vee Q$.
7	$\neg R \to Q$	R_T (from ⑥ Contrapositive form $(P \to Q) \Rightarrow (\neg Q \to$
8	$Q \to \neg P$	Rule 'P'
9	$\neg R \to \neg P$.	Rule T (from ⑦ & ⑧ with I_{13} hypothetical syllogism)
10	$S \to \neg R$	Rule 'P'
11	$S \to \neg P$	Rule T (from ⑩ & ⑨ with I_{13})
12	$P \to \neg S$	R_T (from ⑪ apply contrapositive formula $P \to Q \Rightarrow \neg Q \to \neg P$)
13	$\neg(P \to \neg S) \wedge (P \to \neg S)$	R_T (from ② and ⑫ apply Conjuctive addition)

14.	F	R_T [from (13), Negation law $P \wedge \neg P \Leftrightarrow F$)

Hence $P \rightarrow \neg S$ does not follow the given premises

(2) Show that R follows $P \rightarrow Q$, $Q \rightarrow R$, $\neg(P \wedge R)$, $P \vee R$ by using ~~indirection~~ method.

SNO	Premises	Rule.
1	R	Conclusion Variable
2	$\neg R$	Rule of Indirect proof (Assumed Premise)
3	$P \rightarrow Q$	Rule P
4	$Q \rightarrow R$	Rule P
5	$P \rightarrow R$	Rule T { from (3) & (4) with I_{13} hypothetical syllogism)
6	$\neg(P \wedge R)$	Rule P
7.	$\neg P \vee \neg R$	R_T (from (6) apply demorgans law)
8	$P \rightarrow \neg R$	R_T [from (7) apply general law $\neg P \vee Q \Leftrightarrow P \rightarrow Q$]
9	$\neg R \rightarrow P$	R_T (from (8) apply Converse law $P \rightarrow Q$ converse is $Q \rightarrow P$)
10	$\neg R \rightarrow R$	R_T (from (9) & (5) with I_{13})
11	R	R_T [from (2) & (10) with I_{11} (modus ponens)
12.	$\neg R \wedge R$	conjunctive addition. R_T (from (2)&(11) ~~modus ponens~~.
13	False	R_T (from (13) negation law)

③ $R \to \neg Q$, $R \vee S$, $S \to \neg Q$, $P \to Q \Rightarrow \neg P$ by using indirect method of proof.

SN	Premises	Rule
1	$\neg P$	Conclusion Variable.
2	P	Rule of Indirect proof (Assume Premise)
3	$P \to Q$	Rule P
4	Q	R_T (from ② & ③ with I_{11} modus Ponens)
5	$R \to \neg Q$	Rule P
6	$\neg R$	R_T (from ④ & ⑤ with I_{12}) Modus Tollens
7	$R \vee S$	Rule P
8	S	R_T (from ⑥ & ⑦ with I_{10} disjunctives)
9	$S \to \neg Q$	Rule P
10	$\neg Q$	R_T (from ⑧ & ⑨ with I_{11})
11	$P \to Q$	Rule P
12	$\neg P$	R_T from ⑩ & ⑪ with I_{12} (modus Tollens)
13	$P \wedge \neg P$	R_T from ② & ⑫ (conjunctive addition)
14	False	R_T (⑭ Negation law)

$\neg P$ doesnot follow premises.

④ $(P \to Q) \wedge (R \to S)$, $(Q \to T) \wedge (S \to U)$, $\neg(T \wedge U)$, $P \to R \Rightarrow \neg P$ by using indirect method of proof.

* Consistent Premises and Inconsistent Premises:-

① Show that $P \to Q$, $P \to R$, $Q \to \neg R$ and P are inconsistent premises.

S.NO.	Premises	Rule.
1.	$P \to Q$	Rp
2	P	Rp
3	Q	RT (from ② & ① I_{11}
4	$Q \to \neg R$	Rp
5	$\neg R$	RT (from ③ & ④ with I_{11})
6	$P \to R$	Rp
7	$\neg P$	RT (from ⑤ & ⑥ with I_{12})

(negation of premise P)

The Given Premises are Inconsistent premises.

② $P_1 : R$, $P_2 : R \to \neg S$, $P_3 : \neg S \to \neg T$, $P_4 : \neg T$, verify the given premises are Consistent or Inconsistent.

S.NO.	Premises	Rule.
1	R	Rp
2	$R \to \neg S$	Rp
3	$\neg S$	RT (from ① & ② with I_{11}
4	$\neg S \to \neg T$	Rp
5	$\neg T$	RT (from ③ & ④ with I_{11})

The Given premises are Consistent.

Consistent Premises:- If the conclusion variable is valid then the premises are called consistent Premises.

Inconsistent Premises:- If the conclusion variable is invalid, then. The premises are called as "Inconsistent premises".

* Automatic Theorem Proof:-

→ It is introduced by mathematician HAO WANG for the purpose of inference theory. The following are the basic terminologies in Automatic Theorem proofing.

***Variables:-**

→ The upper case characters A, B, C, ... P, Q, R ... X, Y, Z are used as statement variables.

***Connectives:-**

→ $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ and $\neg$ are the connectives in Automatic theorem. '$\neg$' is the highest precedence connective and followed by '$\wedge$' operator and so on.

*** String of formulae:-**

→ A string of formula is defined as

1) Any formula is a string of formula.
2) If α and β are string of formulas, then α, β and β, α are also string of formulas.

3) Empty string is also a string of formula.

<u>Note</u>: order of string of formula is not an important. A, B, C is same as B, C, A (or) A, C, B ...

* <u>Sequent</u> :-

→ If α and β are string of formulas, then $\alpha \rightarrow \beta$ is called a 'Sequent'. If and only if $\alpha \rightarrow \beta$ is a Tautology.

→ In '$\alpha \rightarrow \beta$', α is called as "<u>antecedent</u>" and β is called as "<u>Consequent</u>"

→ A sequent $\alpha \rightarrow \beta$ is True. Iff either atleast one of the formulas of the antecedent is False (or) atleast one of the formulas of consequent is true.

ie. A, B, C → D, E, F is True.

If and only if

$A \wedge B \wedge C \rightarrow D \vee E \vee F$ is True

→ The symbol "$\xrightarrow{s}$" is called as logical sequent ie. α logically sequent to β ($\alpha \xrightarrow{s} \beta$) means that $\alpha \rightarrow \beta$ is True

* Axiom:-

→ A Sequent $\alpha \xrightarrow{s} \beta$ is an Axiom, If and only if, α and β have atleast one Variable in Common.

Ex:- ① $A, B, C \xrightarrow{s} P, Q, R$ - (Not an Axiom)

Ex:- ② $A, B, C \xrightarrow{s} P, Q, B$ - (Axiom.)

* Theorem:-

1) Every Axiom is a Theorem.
2) If a Sequent 'α' is a Theorem and 'β' is derived from 'α' through the use of one of the rules of antecedent and Consequent of the system, then β is a theorem.
3) Sequence obtained by ① and ② steps are the only theorems.

* Anticedent Rules:-

(1) Rule $\neg \xrightarrow{s}$: If $\alpha, \beta \xrightarrow{s} x, \gamma$ Then $\alpha, \neg x, \beta \xrightarrow{s} \gamma$

~~(If $\alpha, \beta \xrightarrow{s} x$)~~

① Rule $\neg \xrightarrow{s}$: if $\alpha, \beta \xrightarrow{s} x, \gamma$ then $\alpha, \neg x, \beta \xrightarrow{s} \gamma$

② Rule $\wedge \xrightarrow{s}$: if $\alpha, x, \beta \xrightarrow{s} \gamma$ and if $\alpha, y, \beta \xrightarrow{s} \gamma$ then $\alpha, x \wedge y, \beta \xrightarrow{s} \gamma$

③ Rule $\vee \xrightarrow{S}$: if $\alpha, x, \beta \xrightarrow{S} \gamma$
if $\alpha, y, \beta \xrightarrow{S} \gamma$ } then $\alpha, x \vee y, \beta \xrightarrow{S} \gamma$

④ Rule $\rightarrow \xrightarrow{S}$: if $\alpha, x, \beta \xrightarrow{S} \gamma$
if $\alpha, \beta \xrightarrow{S} y, \gamma$ } then $\alpha, x \rightarrow y, \beta \xrightarrow{S} \gamma$

⑤ Rule $\leftrightarrow \xrightarrow{S}$: if $\alpha, x, y, \beta \xrightarrow{S} \gamma$
if $\alpha, \beta \xrightarrow{S} x, y, \gamma$ } then $\alpha, x \leftrightarrow y, \beta \xrightarrow{S} \gamma$

(or)

$\alpha, \beta \xrightarrow{S} x, y, \gamma$

* Consequent Rules :-

① Rule $\xrightarrow{S} \neg$: if $\alpha, x \xrightarrow{S} \beta, \gamma$ then $\alpha \xrightarrow{S} \beta, \neg x, \gamma$

② Rule $\xrightarrow{S} \wedge$: if $\alpha \xrightarrow{S} \beta, x, \gamma$
if $\alpha \xrightarrow{S} \beta, y, \gamma$ } and then $\alpha \xrightarrow{S} \beta, x \wedge y, \gamma$

③ Rule $\xrightarrow{S} \vee$: if $\alpha \xrightarrow{S} \beta, x, \gamma$
if $\alpha \xrightarrow{S} \beta, y, \gamma$ } and then $\alpha \xrightarrow{S} \beta, x \vee y, \gamma$

④ Rule $\xrightarrow{S} \rightarrow$: if $\alpha, x \xrightarrow{S} \beta, y, \gamma$ then $\alpha \xrightarrow{S} \beta, x \rightarrow y, \gamma$

⑤ Rule $\xrightarrow{S} \leftrightarrow$: if $\alpha, x \xrightarrow{S} \beta, y, \gamma$
if $\alpha, y \xrightarrow{S} \beta, x, \gamma$ } then $\alpha \xrightarrow{S} \beta, x \leftrightarrow y, \gamma$

show that

Ex:- $(\neg Q \wedge (P \rightarrow Q)) \xrightarrow{S} \neg P$

Sol: The given problem statement is

$[\neg Q \wedge (P \rightarrow Q)] \xrightarrow{S} \neg P$

Step(1):- $\neg Q, P \rightarrow Q \xrightarrow{S} \neg P$ [By using rule anticedent '$\wedge$']

Step(2):- $P \rightarrow Q \xrightarrow{S} \neg P, Q$ (by using rule anticedent '$\neg$")

Step (3) :- $P, P \rightarrow Q \xrightarrow{S} Q$ (by using consequent rule of '$\neg$')

Step 4:- $P, Q \xrightarrow{S} Q$; $P \xrightarrow{S} P, Q$ [by using antieced $\rightarrow$ rule]

Ex:-② show that $P \vee Q \xrightarrow{S} P$

The given problem stmt is

~~Step~~ $P \vee Q \xrightarrow{S} P$

step1:-

* Predicate Calculus (or) Predicate Logic :-

→ A stmt expression contains subject part and Subject reference part is known as a "Predicate Calculus".

→ The Subject part is denoted with lower case alphabet and subject reference part is denoted with upper case alphabet.

Ex:- 1. Srisha is a Good girl
 (Srisha = subject, s; Good girl = Subject Reference, G)

2. Ramu and Laxman are good brothers
 (Ramu = r; Laxman = l; good brothers = G)

3. Raji, Leela, Tanu are good friends
 (Raji = r; Leela = l; Tanu = t; good friends = G)

→ The predicate logic is classified into several Categories.

① One subject Predicate logic :-

The predicate logic stmt contains only one subject is known as One Subject predicate logic

Ex: Raji is a good girl
 (Raji = r; good girl = G)

$G(r)$

② Two Subject Predicate logic:-

The predicate logic stmt contains exactly two subjects is known as Two Subject predicate logic

Ex: Raji and Leela are good friends
r l G
$G(r, l)$

③ Three Subject Predicate Logic :-

The predicate logic stmt contains three subjects is known as Three subject predicate logic

Ex: Raji, Sup and Piyo are good friends
r s p G
$\rightarrow G(r, s, p)$

④ n- Subject Predicate logic :-

The predicate logic stmt contains n-subjects is known as n-subject predicate logic.

Ex: Leela, Sup, Piyo, Raji, Ammutha, ... are good friends
l s p r a G
$G(l, s, p, r, a)$

* Statement function :-

→ A predicate logic stmt can be represented as Subject reference along with subject inside the paranthesis is known as "Statement function".

ie. Subject refrence (subject)
$G(r_1, r_2)$

→ The stmt functions are clasified into 2 categories.

① Automic statement functions
② Molecular " "

→ Any predicate logic stmt has no Connectives is known as "Atomic statement function"

Ex:- Leela is a good girl
l G

$G(l)$

Leela, Raji are good friends
l, r G $G(l, r)$

→ Any Predicate logic stmt has connectives is known as "Molecular statement function"

Ex:- 1) Ramu is a good boy and Ramu secured first cl
r G S

$G(r) \wedge S(r)$

2) If ramu is a good boy then ramu secured first class.
r G f

$G(r) \longrightarrow f(r)$

* Variables :-

→ Variables are used to assign a value. The value of the variable may or may not be change during the execution of a function.

Ex:- a, b, A, B, - - - -

→ Variables are classified into two categories.

1) Free Variables

2) Bound Variables.

* Free Variables :-

→ Any variable 'x' is appears in the set 'S' is known as Free variable. If any variable is not appears in, the set 'S' is known as Un-free variables.

Ex:- Let us consider $S = \{1, 2, 3, 4 \ldots 9\}$

$x_1 = 5$, $x_2 = 9$, $x_3 = 10$, $x_4 = 15$.

Identify Free and unfree variables.

Sol: Given. $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$x_1 = 5$; appears in the 'S', so x_1 is free-var

$x_2 = 9$; appears in set. x_2 is free variable

$x_3 = 10$ not appear in the set. x_3 is a un-free variable

$x_4 = 15$; not appears in the set. x_4 is Unfree var

* Bound Variables :-

→ Let $f(x)$ and $g(x)$ are two boundary regions and x is said to be a Bound variable if x is appears in b/w two boundary regions $f(x)$ and $g(x)$.

→ Otherwise, if x is not in between $f(x)$ and $g(x)$, then 'x' is called a Unbound - Variable

Ex: $f(x) = 1$ and $g(x) = 9$, $x_1 = 5$, $x_2 = 9$, $x_3 = 10$, $x_4 = 15$. Verify Bound variables and Un-bound variables.

Sol:

$$\boxed{f(x) < x < g(x)}$$

$x_1 = 5 \Rightarrow 1 < 5 < 9 \Rightarrow \checkmark$ x_1 is a bound-va

$x_2 = 9 \Rightarrow 1 < 9 < 9$

Condition is false, Here. x_2 is Un-bound Va

$x_3 = 10 \Rightarrow 1 < \underline{10 < 9}$ $\times$ unbound vari

$x_4 = 15 \Rightarrow$ unbound-variable.

Ex: (2) Given $f(x) = x^2+1$ and $g(x) = x^3+2x^2+2$

$x_1 = 2$, $x_2 = 5$, $x_3 = 9$, verify Bound and Unbound Variables.

Sol: for $x_1 = 2 \Rightarrow f(x) = x^2+1$

$= 2^2+1 = 4+1 = 5$

$g(x) = x^3+2x^2+2$

$= 8+2(4)+2 = 8+8+2 = 18$

$f(x) < x < g(x)$

$5 < 2 < 18$ $\times$

here Condition fails. x_1 is un-bound Variable

for $x_2 = 5 \Rightarrow f(x) = 5^2+1 = 26$

$g(x) = (5)^3+2(5)^2+2 = 125+50+2$

$26 < 5 < 177$ $\times$

$\therefore x_2$ is un-bound Variable

for $x_3 = 9 \Rightarrow f(x) = (9)^2+1 = 82$

$g(x) = 900$ $\Rightarrow 82 < 9 < 900$ $\times$

x_3 is Unbound Variable.

* Quantifiers :-

→ Any predicate Calculus statement contains all, everybody, everyone, none, some, few, one english words is known as Quantifier statements.

Ex:- 1, All Apples are in red colour.

2, Some Apples are in red colour.

3, All women are good.

4, All girls are clever.

5, All girls are Intelligent.

6, All kings are Rulers.

7, Some dancers are rocking.

8, All players are playing.

9, All Indians are my brothers.

→ Quantifiers are classified into 2 categories.

1. Universal Quantifier

2. Existential Quantifier.

* Universal Quantifier :-

Any Predicate Calculus stmt contains all, Everybody, no, none english words, then it is a Universal Quantifier.

→ The Universal Quantifier is denoted with "for all" and symbol "for all" in mathematical representation is "$\forall$"

→ The Universal Quantifier is connected with Conditional Connective.

* **Existential Quantifier :-**

→ Any predicate calculus stmt contains some few, one English words is known as Existential Quantifier.

→ The Existential quantifier is denoted with "there exist" in english and symbol is "$\exists$" in mathematical repres

→ The Existential quantifiers are connected with AND ($\wedge$) connective.

Ex:- ① Write symbolic notation for "All apples in red colour!"

For all, if x is an $\underbrace{\text{apple}}_{A}$ then 'x' is in $\underbrace{\text{red colour}}_{R}$.

$$\forall x, [A(x) \rightarrow R(x)]$$

Ex ② Some apples are in red colour

There exists x is an $\underbrace{\text{apple}}_{A}$ and x is in $\underbrace{\text{red colour}}_{R}$.

$$\exists x, [A(x) \wedge R(x)]$$

Ex: (2) All girls are clever.

For all, if x is $\underbrace{\text{a girl}}_{G}$ then x is $\underbrace{\text{clever}}_{C}$.

$$\forall x, [G(x) \rightarrow C(x)]$$

Ex: (3) All Girls are Intelligent

for all, if x is $\underbrace{\text{a girl}}_{G}$ then x is an $\underbrace{\text{Intelligent}}_{I}$.

$$\forall x, [G(x) \rightarrow I(x)].$$

Ex: (4) All kings are Rulers.

for all, if x is $\underbrace{\text{a king}}_{K}$ then x is a $\underbrace{\text{Ruler}}_{R}$.

$$\forall x, [K(x) \rightarrow R(x)]$$

Ex: (5) Some dancers are rocking.

There exist, x is $\underbrace{\text{a dancer}}_{D}$ and x is $\underbrace{\text{rocking}}_{R}$

$$\exists x, [D(x) \wedge R(x)]$$

Ex: (6) All players are playing well.

for all, if x is $\underbrace{\text{a player}}_{P}$ then x is $\underbrace{\text{Playing well}}_{W}$

$$\forall x, [P(x) \rightarrow W(x)]$$

Ex: (7) All Indians are my Brothers.

for all, if x is $\underbrace{\text{an Indian}}_{I}$, Then x is my $\underbrace{\text{brother}}_{B}$.

$$\forall x, [I(x) \rightarrow B(x)]$$

Ex:- ⑧ All women are clever : Write the Universal Quantifier and find Converse, Inverse Contrapositive.

Sol:

All Women are clever

for all, if 'x' is $\underbrace{\text{women}}_{W}$ then x is $\underbrace{\text{clever}}_{C}$

$$\forall x, [W(x) \rightarrow C(x)]$$

Converse :-

$$\forall x [C(x) \rightarrow W(x)]$$

Inverse :-

$$\forall x [\sim W(x) \rightarrow \sim C(x)]$$

Contrapositive :-

$$\forall x [\sim C(x) \rightarrow \sim W(x)]$$

Ex ⑨ Find the Contrapositive, Converse and Inverse for the following expression. $\forall x [P(x) \wedge \neg Q(x)]$

Sol: $\Rightarrow \forall x \neg [\neg P(x) \vee Q(x)]$

$\forall x \neg (P(x) \rightarrow Q(x)]$

$\forall x [\neg P(x) \rightarrow \neg Q(x)]$

[∴ Apply negation and then use Demorgan's law

$\boxed{\sim(P \wedge Q) \Leftrightarrow \sim P \vee \sim}$

Converse $\forall x [\neg Q(x) \rightarrow \neg P(x)]$

Inverse $\forall x [P(x) \rightarrow Q(x)]$

Contrapositive $\forall x [Q(x) \rightarrow P(x)]$

* Theory of Inference for Predicate Calculus:-

→ The theory which is associated with set of rules in predicate calculus is known as "Theory of Inference for Predicate Calculus".

→ The following are the rules in theory of inference in predicate calculus.

1. **Universal Specification:-**

It converts Universal Quantifier predicate calculus statement into normal predicate calculus statement ie., $\boxed{\forall x, P(x) \Rightarrow P(y)}$

2. **Universal Generalisation:-**

It converts normal predicate calculus stmt into Universal quantifier predicate calculus stmt.
ie,

$$\boxed{P(y) \Rightarrow \forall x, P(x)}$$

3. **Existential Specification:-**

It converts an existential quantifier predicate calculus stmt into normal predicate calculus stmt.
ie.

$$\boxed{\exists x, P(x) \Rightarrow P(y)}$$

4. **Existential Generalisation:-**

It converts normal predicate calculus stmt into an existential quantifier predicate calculus stmt. ie.

$$\boxed{P(y) \Rightarrow \exists x, P(x)}$$

Ex: (1) Show that the following statements are valid by using Inference Theory.

1. All men are mortal
2. Ravana is a man
3. Ravana is mortal.

Sol: Given that,

1. All men are mortal

for all, if 'x' is $\underset{M}{\underline{\text{men}}}$ Then x is $\underset{I}{\underline{\text{mortal}}}$

$$\forall x, [M(x) \rightarrow I(x)]$$

2. Ravana is a Man

for all, if 'x' is $\underset{r}{\underline{\text{ravana}}}$ Then x is $\underset{m}{\underline{\text{man}}}$

$$\forall x, [r(x) \rightarrow m(x)]$$

3. Ravana is mortal,

for all, if x is $\underset{V}{\underline{\text{Ravana}}}$ Then x is $\underset{T}{\underline{\text{mortal}}}$

$$\forall x, [V(x) \rightarrow T(x)]$$

2. $\underset{r}{\underline{\text{Ravana}}}$ is a $\underset{m}{\underline{\text{Man}}}$

$m(r)$

3. $\underset{r}{\underline{\text{Ravana}}}$ is $\underset{I}{\underline{\text{mortal}}}$

$I(r)$

(i) $\cancel{\forall x, [M_1(x) \rightarrow M_2(x)]}$ $\forall x\, [M(x) \rightarrow I(x)]$

(ii) $M(r)$.

(iii) $I(r)$

S.NO	Premises	Rule.
1.	$\forall x [M_1(x) \rightarrow M_2(x)]$	Rule P
2.	$M_1(r) \rightarrow M_2(r)$	Universal Specification
3.	$M(r)$	Rule P
4.	$M_2(r)$ $I(r)$	Rule T (I11 Modus Ponens)

Ex:- (2) Apply theory of Inference for the following stmt & verify validity

1. All dogs are cats
2. All cats are rats
3. All dogs are rats

Given, All dogs are cats.

$\forall x, [D(x) \rightarrow C(x)]$

All cats are rats

$\forall x [C(x) \rightarrow r(x)]$

All dogs are rats

$\forall x [D(x) \rightarrow r(x)]$

S.NO.	Premises	Rule.
1.	$\forall x [D(x) \rightarrow C(x)]$	Rule P
2.	$D(y) \rightarrow C(y)$	R_T (Universal Specification)
3.	$\forall x [C(x) \rightarrow r(x)]$	Rule P
	$C(y) \rightarrow r(y)$	

S.NO	Premises	Rule
5.	$D(y) \rightarrow r(y)$	R_T from ② & ④ I13 (Hypothetical Syllogism)
6.	$\forall x [D(x) \rightarrow r(x)]$	R_T (By applying Universal Generalisation) I1

Ex:③ Verify validation of given stmts using Theory of Inference.

1. All dogs are Carnivals.
2. Some dogs are animals
3. Some Animals are Carnivals

Sol:

1. All dogs are Carnivals

$\forall x, [D(x) \rightarrow C(x)]$.

2. Some dogs are animals.

$\exists x, [D(x) \wedge A(x)]$

3. Some animals are Carnivals

$\exists x, [A(x) \wedge C(x)]$

S.NO.	Premises	Rule
1.	$\forall x, [D(x) \rightarrow C(x)]$	Rule P
2.	$D(y) \rightarrow C(y)$	R_T (Universal Specification)
3.	$\exists x; [D(x) \wedge A(x)]$	Rule P
4.	$D(y) \wedge A(y)$	R_T (Existential Specification)

S.NO.	Premises	Rule
5	$D(y)$	R_T (Conjunctive Simplification I_1 from (4))
6.	$A(y)$	R_T (from (4) with I_2)
7	$C(y)$	R_T (from (2) & (5) I_{11} Modus ponens)
8	$A(y) \land C(y)$	R_T (from (6) & (7) with I_9)
9	$\exists x, A(x) \land C(x)$	R_T (from 8, apply Existential generalisation)

Ex:- (4) 1. All dogs are Cats
2. Some dogs are Herbivorous
3. Some cats are Herbivorous

Sol:- 1. All dogs are cats
$\forall x, [D(x) \rightarrow C(x)]$

2. Some dogs are herbivorous
$\exists x, [D(x) \land H(x)]$

3. Some cats are Herbivorous
$\exists x, (C(x) \land H(x))$

S.NO	Premises	Rule
1.	$\forall x, [D(x) \rightarrow C(x)]$	Rule P
2.	$D(y) \rightarrow C(y)$	Rule T (Universal Specification)

S.NO	Premises	Rule
3.	$\exists x, D(x) \wedge H(x)$	Rule P
4	$D(y) \wedge H(y)$	RT (Existential Specification
5	$D(y)$	RT (from (4) with I_1
6	$H(y)$	RT (from (4) with I_2
7	$C(y)$	RT (from (2) & (5) with I_{11} (modus pon
8	$H(y) \wedge C(y)$	RT (from (6) & (7) by I_9]
9.	$\exists x[H(x) \wedge C(x)]$	RT [from (8) Existential generalisation]

(5) (i) $\forall x[P(x) \rightarrow Q(x)]$

(ii) $\exists x[Q(x) \rightarrow R(x)]$

(iii) $\exists x[P(x) \rightarrow R(x)]$ verify the validity.

S-NO	Premises	Rules
1.	$\forall x[P(x) \rightarrow Q(x)]$	Rule P.
2.	$P(y) \rightarrow Q(y)$	RT [from (1) universal speciali-sation].
3.	$\exists x[Q(x) \rightarrow R(x)]$	RT

4.	$Q(y) \rightarrow R(y)$	RT [from ③ Existential specialization]
5.	$P(y) \rightarrow R(y)$	RT [from ④ & ② with I_{13} hypothetical syllogism]
6.	$\exists x [P(x) \rightarrow R(x)]$	RT [from ⑤ Existential generalisation].

⑥ (i) $\forall x [P(x) \vee Q(x)]$

(ii) $\neg [P(x)]$

(iii) $\forall x [Q(x)]$

sol:-

S.NO	Premises	Rules
1.	$\forall x [P(x) \vee Q(x)]$	Rule P.