2.7 CONTINUOUS PROBABILITY DISTRIBUTION

When a random variable X takes every value in an interval, it gives rise to continuous distribution of X. The distributions defined by the variates like temperature, heights and weights are continuous distributions.

Probability Density Function **[JNTU 2001]**

For continuous variable, the probability distribution is called **Probability Density function** because it is defined for every point in the range and not only for certain values.

Consider the small interval $\left[x-\frac{dx}{2}, x+\frac{dx}{2}\right]$ of length dx round the point x. Let $f(x)$ be any continuous function of x so that $f(x)$ dx represents the probability that the variable X falls in the infinitesimal interval $\left[x-\frac{dx}{2}, x+\frac{dx}{2}\right]$. Symbolically it can be expressed as $P\left(x-\frac{dx}{2} \le X \le x+\frac{dx}{2}\right) = f(x)\,dx$. Then $f(x)$ is called the *probability denisty function* or simply *density function* of the variate X and the continuous curve $y = f(x)$ is known as the *probability denisty curve* or simply *probability curve.*

As the probability for a variate value to lie in the interval dx is $f(x)\,dx$, so the probability for a variate value to fall in the finite interval (a, b) is $\int_a^b f(x)\,dx$ which represents the area between the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$. Since the total probability is unity, we have $\int_a^b f(x)\,dx = 1$, where [a, b] is the range of the variate X. The range of the variable may be finite or inifinite. But even when the range is finite, it is convenient to consider it as infinite by supposing the density function to be zero outside the given interval.

Properties of the probability density function $f(x)$

(*i*) $f(x) \ge 0,\ \forall\ x \in R$ (*ii*) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(*iii*) The probability $P(E)$ is given by

$P(E) = \int_E f(x)\,dx$ is well defined for any event E.

Note : In the case of continuous random variable, we associate the probabilities with intervals. In this case the probability of the variable at a particular point is always zero.

Thus $P(a < X \le b) = P(a \le X < b) = P(a < X < b) = P(a \le X \le b) = F(b) - F(a)$

$[\because P(X = a) = 0,\ P(X = b) = 0]$

That is, inclusion or non-inclusion of end points, does not change the probability, which is not the case in the discrete distributions.

4 (b) **Cumulative Distribution Function of A Continuous Random Variable :**

The *cumulative distribution* function or simply the *distribution function* of a continuous random variable X is denoted by $F(x)$ and is defined as

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\, dx$$

Thus $F(x)$ gives the probability that the value of the variable X will be $\leq x$.

Properties of $F(x)$:

(*i*) $0 \leq F(x) \leq 1, -\infty < x < \infty$.

(*ii*) $F'(x) = f(x) \geq 0$, so that $F(x)$ is a non-decreasing function.

(*iii*) $F(-\infty) = 0$

(*iv*) $F(\infty) = 1$

(*v*) $F(x)$ is a continuous function of x on the right.

(*vi*) The discontinuities of F(x) are countable.

(*vii*) $P(a \leq X \leq b) = \int_a^b f(x)\, dx = F(b) - F(a)$

(*viii*) Since $F'(x) = f(x)$, we have $\frac{d}{dx}[F(x)] = f(x) \Rightarrow dF = f(x)\, dx$

This is known as probability differential of X.

2.8 MEASURES OF CENTRAL TENDENCY FOR CONTINUOUS PROBABILITY DISTRIBUTION

If a variable is continuous then it takes all possible values in its range like height of an individual, life of a car battery, etc., the expectation of the variable is defined as $E(x) = \int x\, f(x)\, dx$ whhere $f(x)$ is the probability function of the variable x.

On replacing p_i by $f(x)\, dx$, x_i by x and the summation over 'i' by integration over the specified range of the variate X in the formulae of discrete probability distribution, we obtain the corresponding formulae for continuous probability distribution.

Let $f(x)$ be the probability density function of a continuous random variable X.

Then

(*i*) **Mean** of a distribution is given by $\mu = E(X) = \int_{-\infty}^{\infty} x\, f(x) dx$

If X is defined from a to b, then $\boxed{\mu = E(X) = \int_a^b x\, f(x) dx}$

In general, mean or expectation of any function $\phi(x)$ is given by

$$\boxed{E[\phi(x)] = \int_{-\infty}^{\infty} \phi(x) f(x) dx}$$

(*ii*) **Median:** Median is the point which divides the entire distribution into two equal parts. In case of continuous distribution, median is the point which divides the total area into two equal parts. Thus if X is defined from a to b and M is the median, then

$$\int_a^M f(x) dx = \int_M^b f(x)\, dx = \frac{1}{2}$$

Solving for M, we get the median.

***(iii)* Mode :** Mode is the value of x for which $f(x)$ is maximum. Mode is thus given by $f'(x)=0$ and $f''(x)<0$ *for* $a<x<b$.

***(iv)* Variance** of a distribution is given by

$$\sigma^2 = \int_{-\infty}^{\infty}(x-\mu)^2 f(x)\,dx \text{ or } \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

Suppose that the variate X is defined from a to b. Then

$$\sigma^2 = \int_a^b (x-\mu)^2 f(x)\,dx \quad \text{or} \quad \sigma^2 = \int_a^b x^2 f(x)\,dx - \mu^2$$

***(v)* Mean deviation :** Mean deviation about the mean (μ) is given by

$$\int_{-\infty}^{\infty} |x-\mu|\, f(x)\,dx$$

SOLVED EXAMPLES

Example 1 : *If a random variable has the probability density f (x) as*

$$f(x) = \begin{cases} 2e^{-2x}, \text{for } x>0 \\ 0, \text{ for } x \le 0 \end{cases}$$ find the probabilities that it will take on a value

(*i*) between 1 and 3 (*ii*) greater than 0.5. **[JNTU 2001, 2006S(Set No. 4)]**

Solution :

(*i*) The probability that a variate takes a value between 1 and 3 is given by

$$P(1 \le X \le 3) = \int_1^3 f(x)\,dx = \int_1^3 2e^{-2x}\,dx$$

$$= 2\left(\frac{e^{-2x}}{-2}\right)_1^3 = -(e^{-6}-e^{-2}) = e^{-2}-e^{-6}$$

(*ii*) The probability that a variable takes a value greater than 0.5 is

$$P(X \ge 0.5) = \int_{0.5}^{\infty} f(x)\,dx = \int_{0.5}^{\infty} 2e^{-2x}\,dx$$

$$= 2\left(\frac{e^{-2x}}{-2}\right)_{0.5}^{\infty} = -\left(e^{-\infty}-e^{-1}\right) = -(0-e^{-1}) = e^{-1}$$

Example 2 : If the probability density of a random variable is given by

$$f(x) = \begin{cases} k(1-x^2), \text{ for } 0<x<1 \\ 0, \text{ otherwise} \end{cases}$$

find the value of k and the probabilities that a random variable having this probability density will take on a value (*i*) between 0.1 and 0.2 (*ii*) greater than 0.5. **[JNTU 1999S]**

Solution : Given $f(x) = \begin{cases} k(1-x^2), & 0 < x < 1 \\ 0, \text{ otherwise} \end{cases}$

We know that $\int_{-\infty}^{\infty} f(x)\, dx = 1$

i.e., $\int_{-\infty}^{0} f(x)\, dx + \int_{0}^{1} f(x)\, dx + \int_{1}^{\infty} f(x)\, dx = 1$

i.e., $0 + \int_{0}^{1} k(1-x^2)\, dx + 0 = 1$

i.e., $k\left(x - \frac{x^3}{3}\right)_0^1 = 1$ or $k\left(1 - \frac{1}{3}\right) = 1$

$\therefore \quad k = \frac{3}{2}$

(*i*) The probability that the variate will take on a value between 0.1 and 0.2 is

$$P(0.1 < X < 0.2) = \int_{0.1}^{0.2} f(x)\, dx = \int_{0.1}^{0.2} k(1-x^2)dx$$

$$= \frac{3}{2}\left(x - \frac{x^3}{3}\right)_{0.1}^{0.2} \quad \left(\because k = \frac{3}{2}\right)$$

$$= \frac{3}{2}\left[\left(0.2 - \frac{0.008}{3}\right) - \left(0.1 - \frac{0.001}{3}\right)\right]$$

$$= \frac{3}{2}\left[0.1 - \frac{0.007}{3}\right] = 0.2965$$

(*ii*) The probability that the variate will take on a value greater than 0.5 is

$$P(X > 0.5) = \int_{0.5}^{\infty} f(x)\, dx = \int_{0.5}^{1} f(x)\, dx + \int_{1}^{\infty} f(x)\, dx$$

$$= \frac{3}{2}\int_{0.5}^{1} (1-x^2)\, dx + 0 = \frac{3}{2}\left(x - \frac{x^3}{3}\right)_{0.5}^{1}$$

$$= \frac{3}{2}\left[\left(1 - \frac{1}{3}\right) - \left(0.5 - \frac{0.125}{3}\right)\right]$$

$$= \frac{3}{2}\left(\frac{2}{3} - 0.4583\right) = 0.3125$$

Example 3 : The probability density $f(x)$ of a continuous random variable is given by $f(x) = c\,e^{-|x|}$, $-\infty < x < \infty$. Show that $c = 1/2$ and find that the mean and variance of the distribution. Also find the probability that the variate lies between 0 and 4.

[JNTU Jan. 2007, (A) Nov. 2010 (Set No. 2), (K) May 2013 (Set No. 4)]

Solution : Given $f(x) = c\,e^{-|x|}$, $-\infty < x < \infty$

We have $\int_{-\infty}^{\infty} f(x)\,dx = 1$ [since the total probability is unity]

$$\text{i.e.,}\quad \int_{-\infty}^{\infty} c\,e^{-|x|}dx = 1 \text{ i.e., } c\int_{-\infty}^{\infty} e^{-|x|}dx = 1$$

$$\text{i.e.,}\quad 2c\int_{0}^{\infty} e^{-|x|}dx = 1 \quad \left[\because e^{-|x|} \text{ is an even function}\right]$$

$$\text{i.e.,}\quad 2c\int_{0}^{\infty} e^{-x}dx = 1 \qquad [\because \text{ in } 0 \le x \le \infty,\ |x| = x]$$

$$\Rightarrow \quad 2c\left(-e^{-x}\right)_0^{\infty} = 1 \Rightarrow -2c\,(0-1) \Rightarrow 2c = 1 \quad \therefore c = \frac{1}{2}$$

Hence $f(x) = c\,e^{-|x|} = \dfrac{1}{2}e^{-|x|}$

(*i*) Mean of the distribution,

$$\mu = \int_{-\infty}^{\infty} x\,f(x)\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x\,e^{-|x|}dx = 0, \text{ since integrand is odd.}$$

(*ii*) Variance of the distribution,

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx = \int_{-\infty}^{\infty} (x-0)^2 \cdot \frac{1}{2}e^{-|x|}dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x^2\,e^{-|x|}\,dx$$

$$= 2\cdot\frac{1}{2}\int_{0}^{\infty} x^2 e^{-|x|}\,dx \text{ , since integrand is even}$$

$$= \int_0^\infty x^2 e^{-x} dx = \left(x^2 \frac{e^{-x}}{-1} - 2x \frac{e^{-x}}{1} + 2 \frac{e^{-x}}{-1} \right)_0^\infty$$

$$= [0 - (-2)] = 2$$

(*iii*) The probability between 0 and 4 $= P(0 \leq X \leq 4)$

$$= \frac{1}{2} \int_0^4 e^{-|x|} dx = \frac{1}{2} \int_0^4 e^{-x} dx$$

$$[\because \text{in } 0 < x < 4, |x| = x]$$

$$= -\frac{1}{2} \left(e^{-x} \right)_0^4 = -\frac{1}{2} \left(e^{-4} - 1 \right)$$

$$= \frac{1}{2} (1 - e^{-4}) = 0.4908 \text{ (nearly)}$$

Example 4 : Probability density function of a random variable X is

$$f(x) = \begin{cases} \frac{1}{2} \sin x, \text{ for } 0 \leq x \leq \pi \\ 0, \text{elsewhere} \end{cases}.$$

Find the mean, mode and median of the distribution and also find the probability between 0 and $\pi/2$. **[JNTU 2004, 2008S, (A) Nov. 2010 (Set No. 4)]**

<u>**Solution :**</u> (*i*) Mean of the distribution $= \int_{-\infty}^{\infty} x f(x) dx$

$$= \int_{-\infty}^{0} x(0) dx + \int_0^\pi x . \frac{1}{2} \sin x \, dx + \int_\pi^\infty x(0) \, dx$$

$$= \frac{1}{2} \int_0^\pi x \sin x \, dx = \frac{1}{2} (-x \cos x + \sin x)_0^\pi$$

$$= \frac{\pi}{2}$$

(*ii*) Mode is the value of x for which $f(x)$ is maximum

Now $f'(x) = \frac{1}{2} \cos x$

For $f(x)$ to be maximum, $f'(x) = 0$

i.e., $\cos x = 0 \;\; \therefore \; x = \frac{\pi}{2}$

$f''(x) = -\frac{1}{2}\sin x$. At $x = \frac{\pi}{2}$, $f''(x) = -\frac{1}{2} < 0$

Hence $f(x)$ is maximum at $x = \frac{\pi}{2}$

$\therefore$ Mode of the distribution is given by $x = \frac{\pi}{2}$

(*iii*) If M is the median of the distribution, then

$$\int_a^M f(x)\,dx = \int_M^b f(x)\,dx = \frac{1}{2}$$

i.e., $$\int_0^M \frac{1}{2}\sin x\,dx = \int_M^\pi \frac{1}{2}\sin x\,dx = \frac{1}{2}$$

Solving $\int_0^M \frac{1}{2}\sin x\,dx = \frac{1}{2}$, we get

$$-\frac{1}{2}(\cos x)_0^M = \frac{1}{2} \Rightarrow -\frac{1}{2}(\cos M - 1) = \frac{1}{2}$$

$$\Rightarrow \quad 1 - \cos M = 1 \Rightarrow \cos M = 0 \quad \therefore M = \frac{\pi}{2}$$

$\therefore$ Median of the distribution $= \frac{\pi}{2}$

Thus Mean = Mode = Median $= \frac{\pi}{2}$

(*iv*) $$P(0 < x < \frac{\pi}{2}) = \int_0^{\pi/2} f(x)\,dx = \int_0^{\pi/2} \frac{1}{2}\sin x\,dx$$

$$= -\frac{1}{2}(\cos x)_0^{\pi/2} = -\frac{1}{2}(0-1) = \frac{1}{2}$$

Example 5 : A continuous random vaiable has the probability density function

$$f(x) = \begin{cases} k\,x\,e^{-\lambda x}, \text{ for } x \geq 0, \lambda > 0 \\ 0, \text{ otherwise} \end{cases}$$

Determine (*i*) k (*ii*) Mean (*iii*) Variance

[JNTU 2003S, 2004, (A) Dec. 2009, Nov. 2010, Dec. 2011, (H) May 2011, (K) May 2013 (Set No. 1)]

<u>**Solution :**</u> (*i*) Since the total probability is unity, we have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

i.e., $\int_{-\infty}^{0} 0.\,dx + \int_{0}^{\infty} kx\,e^{-\lambda x}\,dx = 1$ i.e., $k\int_{0}^{\infty} x\,e^{-\lambda x}\,dx = 1$

i.e., $k\left[x\left(\frac{e^{-\lambda x}}{-\lambda}\right) - 1.\left(\frac{e^{-\lambda x}}{\lambda^2}\right)\right]_0^{\infty} = 1$

i.e., $k\left[(0-0) - \left(0 - \frac{1}{\lambda^2}\right)\right] = 1$ or $k = \lambda^2$

Now $f(x)$ becomes

$$f(x) = \begin{cases} \lambda^2 x e^{-\lambda x}, & \text{for } x \geq 0, \lambda > 0 \\ 0, & \text{otherwise} \end{cases}$$

(ii) Mean of the distribution, $\mu = \int_{-\infty}^{\infty} x\,f(x)\,dx$

i.e., $\mu = \int_{-\infty}^{0} 0.\,dx + \int_{0}^{\infty} x.\lambda^2 x\,e^{-\lambda x}\,dx = \lambda^2 \int_{0}^{\infty} x^2\,e^{-\lambda x}\,dx$

$$= \lambda^2\left[x^2\left(\frac{e^{-\lambda x}}{-\lambda}\right) - 2x\left(\frac{e^{-\lambda x}}{\lambda^2}\right) + 2\left(\frac{e^{-\lambda x}}{-\lambda^3}\right)\right]_0^{\infty}$$

$$= \lambda^2\left[(0-0+0) - (0-0-\frac{2}{\lambda^3})\right] = \frac{2}{\lambda}$$

(iii) Variance of the distribution, $\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$

i.e., $\sigma^2 = \int_{0}^{\infty} x^2 f(x)\,dx - \left(\frac{2}{\lambda}\right)^2 = \lambda^2 \int_{0}^{\infty} x^3 e^{-\lambda x}\,dx - \frac{4}{\lambda^2}$

$$= \lambda^2\left[x^3\left(\frac{e^{-\lambda x}}{-\lambda x}\right) - 3x^2\left(\frac{e^{-\lambda x}}{\lambda^2}\right) + 6x\left(\frac{e^{-\lambda x}}{-\lambda^3}\right) - 6\left(\frac{e^{-\lambda x}}{\lambda^4}\right)\right]_0^{\infty} - \frac{4}{\lambda^2}$$

$$= \lambda^2\left[(0-0+0-0) - \left(0-0+0-\frac{6}{\lambda^4}\right)\right] - \frac{4}{\lambda^2}$$

$$= \frac{6}{\lambda^2} - \frac{4}{\lambda^2} = \frac{2}{\lambda^2}$$

Example 6 : A continuous random variable X is defined by

$$f(x)=\begin{cases}\frac{1}{16}(3+x)^2, \text{ if } -3\le x<-1\\ \frac{1}{16}(6-2x^2), \text{ if } -1\le x<1\\ \frac{1}{16}(3-x)^2, \text{ if } 1\le x\le 3\\ 0, \text{ elsewhere}\end{cases}$$

Verify that $f(x)$ is a density function and also find the mean of X.

[JNTU 2003S, (K) May 2013 (Set No. 3)]

Solution : $f(x)$ is clearly ≥ 0 for every x in $[-3, 3]$ and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-3} f(x)\,dx+\int_{-3}^{-1} f(x)\,dx+\int_{-1}^{1} f(x)\,dx+\int_{1}^{3} f(x)\,dx+\int_{3}^{\infty} f(x)\,dx$$

$$= \int_{-3}^{-1}\frac{1}{16}(3+x)^2\,dx+\int_{-1}^{1}\frac{1}{16}(6-2x^2)dx+\int_{1}^{3}\frac{1}{16}(3-x)^2\,dx$$

$$= \frac{1}{16}\cdot\left[\frac{(3+x)^3}{3}\right]_{-3}^{-1}+\frac{1}{16}\cdot 2\int_{0}^{1}(6-2x^2)\,dx+\frac{1}{16}\cdot\left[\frac{(3-x)^3}{-3}\right]_{1}^{3}$$

$$= \frac{1}{48}(8-0)+\frac{1}{8}\left(6x-\frac{2x^3}{3}\right)_0^1-\frac{1}{48}(0-8)$$

$$= \frac{1}{6}+\frac{1}{8}\left(6-\frac{2}{3}\right)+\frac{1}{6}=\frac{1}{3}+\frac{1}{4}\left(3-\frac{1}{3}\right)=\frac{1}{3}+\frac{2}{3}=1$$

Hence the function $f(x)$ satisfies the requirements for a density function.

$$\text{Mean of } f(x) = \int_{-\infty}^{\infty} x\,f(x)\,dx$$

$$= \frac{1}{16}\int_{-3}^{-1}x(3+x)^2\,dx+\frac{1}{16}\int_{-1}^{1}x(6-2x^2)dx+\frac{1}{16}\int_{1}^{3}x(3-x)^2\,dx$$

$$= \frac{1}{16}\int_{-3}^{-1} x(9+x^2+6x)\,dx + 0 + \frac{1}{16}\int_{1}^{3} x(9-6x+x^2)\,dx$$

[$\because$ the integrand of the second integral is odd function]

$$= \frac{1}{16}\int_{-3}^{-1}(9x+6x^2+x^3)dx + \frac{1}{16}\int_{1}^{3}(9x-6x^2+x^3)\,dx$$

$$= \frac{1}{16}\left(\frac{9x^2}{2}+\frac{6x^3}{3}+\frac{x^4}{4}\right)_{-3}^{-1} + \frac{1}{16}\left(\frac{9x^2}{2}-\frac{6x^3}{3}+\frac{x^4}{4}\right)_{1}^{3}$$

$$= \frac{1}{16}\left[\left(\frac{9}{2}-2+\frac{1}{4}\right)-\left(\frac{81}{2}-54+\frac{81}{4}\right)\right] + \frac{1}{16}\left[\left(\frac{81}{2}-54+\frac{81}{4}\right)-\left(\frac{9}{2}-2+\frac{1}{4}\right)\right]$$

$$= 0$$

Example 7 : Is the function defined by

$$f(x) = \begin{cases} 0, x < 2 \\ \frac{1}{18}(2x+3), 2 \le x \le 4 \\ 0, \ x > 4 \end{cases}$$

a probability density function? Find the probability that a variate having $f(x)$ as density function will fall in the interval $2 \le x \le 3$.

[JNTU (K) Nov. 2009, May 2013 (Set No. 2)]

Solution : (*i*) For all points x in $-\infty \le x \le \infty, f(x) \ge 0$ and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{2} 0.\,dx + \int_{2}^{4}\frac{1}{18}(2x+3)\,dx + \int_{4}^{\infty} 0.\,dx$$

$$= \frac{1}{18}\int_{2}^{4}(2x+3)\,dx = \frac{1}{18}\left[\frac{(2x+3)^2}{4}\right]_{2}^{4}$$

$$= \frac{1}{72}(121-49) = 1$$

Hence $f(x)$ is a probability density function.

(*ii*)The probability that the density will fall in the interval $2 \le x \le 3$ is

$$P(2 \le x \le 3) = \int_{2}^{3} f(x)\,dx = \frac{1}{18}\int_{2}^{3}(2x+3)\,dx$$

$$= \frac{1}{18}\left(x^2+3x\right)_{2}^{3} = \frac{1}{18}(18-10) = \frac{8}{18} = \frac{4}{9}$$

Example 8 : A random variable X gives measurements x between 0 and 1 with a probability function $f(x)=\begin{cases}12x^3-21x^2+10x, \text{ for } 0\le x\le 1\\ 0, \text{ elsewhere}\end{cases}$

(*i*) Find $P\left(X\le\frac{1}{2}\right)$ and $P\left(X>\frac{1}{2}\right)$

(*ii*) Find a number k such that $P(X\le k)=\frac{1}{2}$ **[JNTU 2003]**

Solution : (*i*) $P\left(X\le\frac{1}{2}\right) = \int_0^{1/2} f(x)\,dx = \int_0^{1/2}(12x^3-21x^2+10x)\,dx$

$$= \left(12.\frac{x^4}{4}-21.\frac{x^3}{3}+10.\frac{x^2}{2}\right)_0^{1/2} = \left(3x^4-7x^3+5x^2\right)_0^{1/2}$$

$$= \left(\frac{3}{16}-\frac{7}{8}+\frac{5}{4}\right)-0=\frac{1}{16}(3-14+20)=\frac{9}{16}$$

Now $P\left(X>\frac{1}{2}\right) = 1-P\left(X\le\frac{1}{2}\right)=1-\frac{9}{16}=\frac{7}{16}$

(*ii*) Given $P(X\le k) = \frac{1}{2}$

i.e., $\int_0^k f(x)\,dx = \frac{1}{2}$ i.e., $\int_0^k(12x^3-21x^2+10x)\,dx=\frac{1}{2}$

i.e., $(3x^4-7x^3+5x^2)_0^k = \frac{1}{2}$

i.e., $3k^4-7k^3+5k^2=\frac{1}{2}$ or $6k^4-14k^3+10k^2-1=0$

$\therefore$ $k=0.452$

Example 9 : A continuous random variable X has the distribution function

$$F(x) = \begin{cases}0, \text{ if } x\le 1\\ k(x-1)^4, \text{ if } 1<x\le 3\\ 1, \text{ if } x>3\end{cases}$$

Determine (*i*) $f(x)$ (*ii*) k (*iii*) Mean **[JNTU 2004S, 2007S, (A) Apr. 2012 (Set No. 2)]**

Solution : (*i*) We know that $f(x)=\frac{d}{dx}[F(x)]$

$$\therefore \quad f(x) = \begin{cases} 0, \text{if } x \le 1 \\ 4k(x-1)^3, \text{if } 1 < x \le 3 \\ 0, \text{if } x > 3 \end{cases}$$

(*ii*) Since total probability is unity, we have

$$\int_1^3 f(x)\,dx = 1 \text{ i.e., } 4k \int_1^3 (x-1)^3\,dx = 1$$

$$\text{i.e.,} \quad 4k\left[\frac{(x-1)^4}{4}\right]_1^3 = 1 \quad \text{i.e., } k\,(16-0) = 1 \quad \text{or } k = \frac{1}{16}$$

$$\text{Hence } f(x) = \begin{cases} 0, \text{ if } x \le 1 \\ \frac{1}{4}(x-1)^3, \text{ if } 1 < x \le 3 \\ 0, \text{ if } x > 3 \end{cases}$$

(*iii*) Mean of X, $\mu = \int_{-\infty}^{\infty} x\, f(x)\,dx$

$$= \int_{-\infty}^{1} x.f(x)\,dx + \int_1^3 x.f(x)\,dx + \int_3^{\infty} x.f(x)\,dx$$

$$= 0 + \int_1^3 x.\frac{1}{4}(x-1)^3\,dx + 0 = \frac{1}{4}\int_1^3 x(x-1)^3\,dx$$

$$= \frac{1}{4}\int_0^2 (t+1)\,t^3\,dt \text{ (Putting } x-1 = t)$$

$$= \frac{1}{4}\int_0^2 (t^4+t^3)\,dt = \frac{1}{4}\left(\frac{t^5}{5}+\frac{t^4}{4}\right)_0^2$$

$$= \frac{1}{4}\left(\frac{2^5}{5}+\frac{2^4}{4}\right) = \frac{2^4}{4}\left(\frac{2}{5}+\frac{1}{4}\right)$$

$$= 4\left(\frac{13}{20}\right) = \frac{13}{5} = 2.6$$

Example 10 : If X is a continuous random variable and $Y = aX + b$, prove that $E(Y) = a\,E(X) + b$ and $V(Y) = a^2.V(X)$, where V stands for variance and a, b are constants.

[JNTU 2000, (A) Dec. 2009 (Set No. 2), (H), (A) Nov. 2010, Nov. 2011, Apr. 2012 (Set No. 1)]

Solution : (*i*) By definition,

$$E(Y) = E(aX + b) = \int_{-\infty}^{\infty} (ax+b)\, f(x)\, dx \qquad \left[\because E[\phi(x)] = \int_{-\infty}^{\infty} \phi(x)\, f(x)\, dx\right]$$

$$= a\int_{-\infty}^{\infty} x\, f(x)\, dx + b\int_{-\infty}^{\infty} f(x)\, dx$$

$$= a\,E(X) + b\,(1) \qquad \text{[since total probability is unity]}$$

$$= a\,E(X) + b$$

(*ii*) From (*i*), we have $E(Y) = a\,E(X) + b$ ... (1)

where $Y = aX + b$ ... (2)

(2) – (1) gives $Y - E(Y) = a\,[X - E(X)]$

Squaring, $[Y - E(Y)]^2 = a^2\,[X - E(X)]^2$

Taking Expectation of both sides, we get

$$E\{[Y - E(Y)]^2\} = a^2\,E\{[X - E(X)]^2\}$$

$$\therefore \quad V(Y) = a^2 V(X)$$

Example 11 : If X is a continuous random variable and k is a constant, then prove that (*i*) $Var\,(X + k) = Var\,(X)$ (*ii*) $Var\,(kX) = k^2\,Var\,(X)$ [JNTU 2006, 2007(Set No. 1)]

Solution : By definition,

$$\text{Var}(X) = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \left[\int_{-\infty}^{\infty} x\, f(x)\, dx\right]^2$$

$$(i)\ \text{Var}(X + K) = \int_{-\infty}^{\infty} (x+k)^2\, f(x)\, dx - \left[\int_{-\infty}^{\infty} (x+k)\, f(x)\, dx\right]^2$$

$$= \int_{-\infty}^{\infty} (x^2 + 2kx + k^2)\, f(x)\, dx - \left[\int_{-\infty}^{\infty} x\, f(x)\, dx + k\int_{-\infty}^{\infty} f(x)\, dx\right]^2$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\, dx + 2k\int_{-\infty}^{\infty} x\, f(x)\, dx + k^2 - \left[\int_{-\infty}^{\infty} x\, f(x)\, dx + k\right]^2$$

$$\left[\because \int_{-\infty}^{\infty} f(x)\, dx = 1\right]$$

$$= E(X^2) + 2k\,E(X) + k^2 - [E(X) + k)]^2$$
$$= E(X^2) + 2k\,E(X) + k^2 - [E(X)]^2 - 2k\,E(X) - k^2$$
$$= E(X^2) - [E(X)]^2$$
$$= \text{Var}\,(X)$$

(*ii*) $\text{Var (kX)} = \int_{-\infty}^{\infty} k^2x^2 f(x)\,dx - \left[\int_{-\infty}^{\infty} kx\,f(x)\,dx\right]^2$

$$= k^2 \int_{-\infty}^{\infty} x^2 f(x)\,dx - k^2\left[\int_{-\infty}^{\infty} f(x)\,dx\right]^2$$

$$= k^2\left[E(X)^2 - \{E(X)\}^2\right] = k^2\,\text{Var}\,(X)$$

Example 12 : For the continuous random variable X whose probability density function is given by $f(x) = \begin{cases} cx\,(2-x), \text{ if } 0 \le x \le 2 \\ 0, \text{ otherwise} \end{cases}$ where c is a constant.

Find c, mean and variance of X. **[JNTU 2003, 2004 (Set No.3)]**

(OR) The frequency function of a continuous random variable X is given by $f(x) = y_0 x\,(2-x), 0 \le x \le 2$. Find the value of y_0, mean and variance of X.

[JNTU (A) Dec. 2009 (Set No.3)]

__Solution :__ (*i*) Since the total probability is unity, we have

$\int_{-\infty}^{\infty} f(x)\,dx = 1$. So $\int_0^2 f(x)\,dx = 1$

i.e., $\int_0^2 cx\,(2-x)\,dx = 1$ *i.e.,* $c\left(x^2 - \frac{x^3}{3}\right)_0^2 = 1$

i.e., $c\left(4 - \frac{8}{3}\right) = 1$ *i.e.,* $\frac{4c}{3} = 1$ or $c = \frac{3}{4}$

$$\therefore \quad f(x) = \begin{cases} \frac{3x}{4}(2-x), \text{ if } 0 \le x \le 2 \\ 0, \text{otherwise} \end{cases}$$

(*ii*) Mean of $X = E(X) = \int_{-\infty}^{\infty} x\,f(x)\,dx$

$$i.e., \quad \mu = \int_0^2 x.\frac{3x}{4}(2-x)\,dx = \frac{3}{4}\int_0^2 (2x^2 - x^3)\,dx$$

$$= \frac{3}{4}\left(\frac{2x^3}{3} - \frac{x^4}{4}\right)_0^2 = \frac{3}{4}\left(\frac{2^4}{3} - \frac{2^4}{4}\right) = \frac{3}{4}(2^4)\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{12}{12} = 1$$

(*iii*) Variance of $X = V(X) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2 = \int_0^2 x^2.\frac{3x}{4}(2-x)\,dx - (1)^2$

$$= \frac{3}{4}\int_0^2 (2x^3 - x^4)\,dx - 1 = \frac{3}{4}\left(2.\frac{x^4}{4} - \frac{x^5}{5}\right)_0^2 - 1$$

$$= \frac{3}{4}\left[\frac{32}{4} - \frac{32}{5}\right] - 1 = 24\left(\frac{1}{4} - \frac{1}{5}\right) - 1 = \frac{6}{5} - 1 = \frac{1}{5}$$

Example 13 : For the continuous probability function $f(x) = k\,x^2\,e^{-x}$ when $x \geq 0$, find (*i*) k (*ii*) Mean (*iii*) Variance **[JNTU 2005, 2005S, 2007, (K) Dec. 2013 (Set No. 4)]**

Solution : (*i*) We have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\therefore \quad \int_0^{\infty} kx^2 e^{-x}\,dx = 1 \qquad (\because x \geq 0)$$

$$i.e., \quad k\left[x^2(-e^{-x}) - 2x\,(e^{-x}) + 2(-e^{-x})\right]_0^{\infty} = 1$$

$$i.e., \quad k\left\{-e^{-x}(x^2 + 2x + 2)\right\}_0^{\infty} = 1$$

$$i.e., \quad k\,(0 + 2) = 1 \qquad \text{or} \qquad k = \frac{1}{2}$$

$$(ii) \quad \text{Mean} = \int_{-\infty}^{\infty} x\,f(x)\,dx = \int_0^{\infty} kx^3 e^{-x}\,dx$$

$$= k\left[x^3(-e^{-x}) - 3x^2(e^{-x}) + 6x(-e^{-x}) - 6e^{-x}\right]_0^{\infty}$$

$$= k\left[-e^{-x}(x^3 + 3x^2 + 6x + 6)\right]_0^{\infty} = k\,[0 + 6] = 6k$$

$$\therefore \mu = 6\left(\frac{1}{2}\right) = 3 \quad \left(\because k = \frac{1}{2}\right)$$

(*iii*) Variance $= \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2 = \int_0^{\infty} x^2 . kx^2 e^{-x} dx - (3)^2$

$$= k \int_0^{\infty} x^4 e^{-x} dx - 9$$

$$= k \left[x^4(-e^{-x}) - 4x^3(e^{-x}) + 12x^2(-e^{-x}) - 24x(-e^{-x}) + 24\,e^{-x}\right]_0^{\infty} - 9$$

$$= \frac{1}{2}\left[-e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24\right]_0^{\infty} - 9$$

$$= \frac{1}{2}[0 + 24] - 9 = 12 - 9 = 3$$

Example 14 : The trouble shooting capability of an IC chip in a circuit is a random variable X whose distribution function is given by

$$F(X) = \begin{cases} 0, \text{ for } x \leq 3 \\ 1 - \dfrac{9}{x^2}, \text{ for } x > 3 \end{cases}$$

where x denote the number of years.

Find the probability that the IC chip will work properly

(*i*) Less than 8 years

(*ii*) Beyond 8 years

(*iii*) Between 5 to 7 years

(*iv*) Anywhere from 2 to 5 years

[JNTU (K) Nov. 2009 (Set No.4)]

Solution : We have $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$

$$\therefore \quad F(X) = \int_0^x f(t)\,dt \; (\because x > 0) = \begin{cases} 0, \text{ if } x \leq 3 \\ 1 - \dfrac{9}{x^2}, \text{ if } x > 3 \end{cases}$$

(*i*) $P(X \leq 8) = \int_0^8 f(t)\,dt = 1 - \dfrac{9}{8^2} = 0.8594$

(*ii*) $P(X > 8) = 1 - P(X \leq 8) = 1 - 0.8594 = 0.1406$

(*iii*) $P(5 \leq x \leq 7) = F(7) - F(5) = \left(1 - \dfrac{9}{7^2}\right) - \left(1 - \dfrac{9}{5^2}\right)$

$$= 9\left(\frac{1}{25} - \frac{1}{49}\right) = \frac{24 \times 9}{25 \times 49} = 0.1763$$

(*iv*) $P(2 \leq x \leq 5) = F(5) - F(2) = \left(1 - \dfrac{9}{5^2}\right) - 0 = \dfrac{16}{25} = 0.64$

Example 15 : If the probability density function of a random variable X is given by

$$f(x) = \begin{cases} 2k\,x\,e^{-x^2}, \text{for } x > 0 \\ 0, \text{ for } x \le 0 \end{cases}$$

Determine (*i*) k (*ii*) the distribution function for X.

Solution : (*i*) We have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

i.e., $\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = 1$ *i.e.,* $0 + \int_{0}^{\infty} 2k\,x\,e^{-x^2}\,dx = 1$

i.e., $k\int_{0}^{\infty} 2x.e^{-x^2}\,dx = 1$ *i.e.,* $k\int_{0}^{\infty} e^{-t}\,dt = 1$ (Putting $x^2 = t$)

or $k\left(-e^{-t}\right)_0^{\infty} = 1$ or $-k\,(0-1) = 1$

$\therefore$ $k = 1$

(*ii*) The distribution function is

$$F(X) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = 0 \text{ if } x \le 0$$

and $$F(X) = \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{x} f(t)\,dt, \text{ if } x > 0$$

$$= 0 + \int_{0}^{x} 2t\,e^{-t^2}\,dt = \int_{0}^{x^2} e^{-u}\,du \quad (\text{Putting } t^2 = u)$$

$$= -\left(e^{-u}\right)_0^{x^2} = -(e^{-x^2} - 1) = 1 - e^{-x^2}$$

$$\therefore F(x) = \begin{cases} 1 - e^{-x^2}, \text{if } x > 0 \\ 0, \text{ otherwise} \end{cases}$$

Example 16 : Suppose a continuous random variable X has the probability density $f(x) = K(1-x^2)$ for $0 < x < 1$, and $f(x) = 0$ otherwise. Find (*i*) K (*ii*) Mean (*iii*) Variance

[JNTU 2007S(Set No.1)]

Solution : (*i*) Since the total probability is unity,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

i.e., $$\int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx = 1$$

i.e., $0 + \int_0^1 K(1-x^2)\,dx + 0 = 1$

i.e., $K\left(x - \frac{x^3}{3}\right)_0^1 = 1$ or $K\left(1 - \frac{1}{3}\right) = 1$

$\therefore K = \frac{3}{2}$

(ii) Mean of $X = \int_{-\infty}^{\infty} x.f(x)\,dx = \int_0^1 x.f(x)\,dx$

$= \int_0^1 x.K(1-x^2)dx = K\int_0^1 (x - x^3)dx$

i.e., $\mu = K\left(\frac{x^2}{2} - \frac{x^4}{4}\right)_0^1 = K\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{K}{4} = \left(\frac{3}{2}\right)\left(\frac{1}{4}\right) = \frac{3}{8}$ $\left[\because K = \frac{3}{2}\right]$

(iii) Variance of $X = \int_0^1 x^2 f(x)dx - \mu^2$

$= \int_0^1 x^2 . K(1-x^2)\,dx - \left(\frac{3}{8}\right)^2$

$= K\int_0^1 (x^2 - x^4)dx - \frac{9}{64} = K\left(\frac{x^3}{3} - \frac{x^5}{5}\right)_0^1 - \frac{9}{64}$

i.e., $\sigma^2 = K\left(\frac{1}{3} - \frac{1}{5}\right) - \frac{9}{64} = \frac{2K}{15} - \frac{9}{64} = \frac{2}{15}\left(\frac{3}{2}\right) - \frac{9}{64}$

$= \frac{1}{5} - \frac{9}{64} = \frac{19}{320} = 0.06$

Example 17 : The daily consumption of electric power (in millions of kW-hours) is a random variable having the probability density function (p.d.f)

$$f(x) = \begin{cases} \frac{1}{9} x e^{-x/3}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

If the total production is 12 million KW-hours, determine the probability that there is power cut (shortage) on any given day. **[JNTU Jan. 2008, (K) May 2010S]**

Solution : Probability that the power consumed is between 0 to 12 is

$$P(0 \le x \le 12 \text{ million KW-hours}) = \int_0^{12} f(x)\,dx = \frac{1}{9}\int_0^{12} x e^{-x/3} dx$$

$$= \frac{1}{9}\left[x\frac{e^{-x/3}}{(-1/3)} - 1.\frac{e^{-x/3}}{1/9}\right]$$

$$= \frac{1}{9}[-36\, e^{-4} - 9\, e^{-4} + 9]$$

$$= \frac{1}{9}(9 - 45e^{-4})$$

$$= 1 - 5e^{-4}$$

Power supply is inadequate if daily consumption exceeds 12 million kW, *i.e.*,

$$P(x > 12) = 1 - P(0 \le x \le 12) = 1 - (1 - 5\, e^{-4})$$

$$= 5e^{-4} = 0.0915781$$

Example 18 : If probability density function

$$f(x) = \begin{cases} Kx^3 \text{ in } 0 \le x \le 3 \\ 0 \quad \text{elsewhere} \end{cases}$$

Find the value of K and find the probability between $x = 1/2$ and $x = 3/2$.

[JNTU Nov. 2008 (Set No. 1)]

Solution : Given $f(x) = \begin{cases} Kx^3 \text{ in } 0 \le x \le 3 \\ 0 \quad \text{elsewhere} \end{cases}$

We have $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\Rightarrow \int_0^3 Kx^3 dx = 1 \Rightarrow K\left(\frac{x^4}{4}\right)_0^3 = 1$$

$$\Rightarrow K = \frac{4}{81}$$

Now $P\left(\frac{1}{2} < X < \frac{3}{2}\right) = \int_{1/2}^{3/2} f(x)\, dx = \int_{1/2}^{3/2} \frac{4}{81} x^3 dx$

$$= \frac{4}{81}\left(\frac{x^4}{4}\right)_{1/2}^{3/2} = \frac{1}{81}\left[\left(\frac{3}{2}\right)^4 - \left(\frac{1}{2}\right)^4\right]$$

$$= \frac{1}{81 \times 16}[81 - 1] = \frac{80}{81 \times 16} = \frac{5}{81}$$

Example 19 : Find the constant K such that

$$f(x)=\begin{cases} Kx^2, \text{ if } 0<x<3 \\ 0, \text{ otherwise} \end{cases}$$

is a probability function.

i) Find the distribution function F(x)

ii) $P(1 < X \le 2)$

[JNTU Nov. 2008 (Set No. 2)]

Solution : (*i*) Since the total probability is unity,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow K\int_0^3 x^2\,dx = 1 \Rightarrow K\left(\frac{x^3}{3}\right)_0^3 = 1 \Rightarrow K = \frac{1}{9}$$

The distribution function is

$$F(X) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = 0 \text{ if } x \le 0$$

and $$F(X) = \int_{-\infty}^{0} f(t)\,dt + \int_0^x f(t)\,dt \text{ if } x > 0$$

$$= 0 + \int_0^x \frac{1}{9}t^2\,dt = \frac{1}{9}\left(\frac{t^3}{3}\right)_0^3 = \frac{1}{27}x^3$$

$$\therefore \quad F(x)=\begin{cases} \dfrac{1}{27}x^3, \text{ if } x>0 \\ 0, \quad \text{otherwise} \end{cases}$$

(*ii*) $$P(1 < X \le 2) = \int_1^2 f(x)\,dx = \int_1^2 Kx^2\,dx$$

$$= \frac{1}{9}\int_1^2 x^2\,dx \quad \left[\because k = \frac{1}{9}\right]$$

$$= \frac{1}{9}\left(\frac{x^3}{3}\right)_1^2 = \frac{1}{27}(8-1) = \frac{7}{27}$$

Example 20 : If the probability density function of X is given by

$$f(x)=\begin{cases} x/2, \text{ for } 0<x\le 1 \\ 1/2, \text{ for } 1<x\le 2 \\ (3-x)/2, \text{ for } 2<x<3 \\ 0, \text{ elsewhere} \end{cases}$$

Find the expected value of $f(x) = x^2 - 5x + 3$

[JNTU Nov. 2008 (Set No. 2)]

Solution : The expectation of any function $\phi(x)$ is given by

$$E[\phi(x)] = \int_{-\infty}^{\infty} \phi(x)\, f(x)\, dx$$

$$\therefore \quad E(x^2 - 5x + 3) = \int_{-\infty}^{\infty} (x^2 - 5x + 3)\, f(x)\, dx$$

$$= \int_0^1 (x^2 - 5x + 3).\frac{x}{2} dx + \int_1^2 (x^2 - 5x + 3).\frac{1}{2} dx + \int_2^3 (x^2 - 5x + 3).\left(\frac{3-x}{2}\right) dx$$

$$= \frac{1}{2}\int_0^1 (x^3 - 5x^2 + 3x)\, dx + \frac{1}{2}\int_1^2 (x^2 - 5x + 3)\, dx + \frac{1}{2}\int_2^3 (-x^3 + 8x^2 - 18x + 9)\, dx$$

$$= \frac{1}{2}\left(\frac{x^4}{4} - \frac{5x^3}{3} + \frac{3x^2}{2}\right)_0^1 + \frac{1}{2}\left(\frac{x^3}{3} - \frac{5x^2}{2} + 3x\right)_1^2 + \frac{1}{2}\left(-\frac{x^4}{4} + \frac{8x^3}{3} - \frac{18x^2}{2} + 9x\right)_2^3$$

$$= \frac{1}{24} - \frac{13}{12} - \frac{19}{24} = \frac{-44}{24} = \frac{-11}{6}$$

Example 21 : Let the continuous random variable X have the probability density function,

$$f(x) = \begin{cases} 2/x^3, & if\ 1 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

Find F(x). **[JNTU Apr. 2009 (Set No. 3)]**

Solution : We are given

$$f(x) = \begin{cases} 2/x^3, & if\ 1 < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

$$\text{i.e., } f(x) = \begin{cases} 2/x^3, & \text{if } x > 1 \\ 0, & \text{if } x \le 1 \end{cases}$$

We have $F(x) = \int_{-\infty}^{x} f(x)\, dx$

If $x > 1$

$$F(x) = \int_1^x f(x)\, dx$$

$$= \int_1^x \frac{2}{x^3} dx = 2\left(\frac{x^{-2}}{-2}\right)_1^x = -\left(\frac{1}{x^2}\right)_1^x$$

$$= -\left(\frac{1}{x^2} - 1\right) = 1 - \frac{1}{x^2}$$

If $x < 1$, $F(x) = 0$

$$\therefore \quad F(x) = \begin{cases} 0, & \text{if } x \le 1 \\ 1 - \frac{1}{x^2}, & \text{if } 1 < x < \infty \end{cases}$$

Example 22 : The probability density function is

$$y = \begin{cases} k(3x^2 - 1), & -1 \le x \le 2 \\ 0, & \text{else where} \end{cases}$$

Find the value of k and find $P(-1 \le x \le 0)$. **[JNTU Apr. 2009 (Set No. 4)]**

<u>**Solution :**</u> Since $\int_{-\infty}^{\infty} f(x)dx = 1$, we have $\int_{-\infty}^{-1} f(x)dx + \int_{-1}^{2} f(x)dx + \int_{2}^{\infty} f(x)dx = 1$

$$i.e., \quad \int_{-\infty}^{-1} 0 \cdot dx + \int_{-1}^{2} K(3x^2 - 1)dx + \int_{2}^{\infty} 0 \cdot dx = 1$$

$$i.e., \quad K\left(3 \cdot \frac{x^3}{3} - x\right)_{-1}^{2} = 1 \qquad i.e., \quad K\left(x^3 - x\right)_{-1}^{2} = 1$$

$$i.e., \quad K[(8-2) - (-1+1)] = 1 \text{ or } 6K = 1. \quad \therefore K = \frac{1}{6}.$$

$$\text{Thus } y = \begin{cases} \frac{1}{6}(3x^2 - 1), & -1 \le x \le 2 \\ 0, & \text{else where} \end{cases}$$

$$\text{Hence } P(-1 \le x \le 0) = \int_{-1}^{0} f(x)dx = \frac{1}{6}\int_{-1}^{0}(3x^2 - 1)dx = \frac{1}{6}\left[3 \cdot \frac{x^3}{3} - x\right]_{-1}^{0}$$

$$= \frac{1}{6}\left(x^3 - x\right)_{-1}^{0} = \frac{1}{6}[(0-0) - (-1+1)] = 0$$

Example 23 : The density function of a random variable X is $f(x) = \begin{cases} e^{-x}, x \ge 0 \\ 0, \text{otherwise} \end{cases}$.

Find $E(X)$, $E(X^2)$, Var (X). **[JNTU (K) Nov. 2009 (Set No. 1)]**

<u>**Solution :**</u> Given $f(x) = \begin{cases} e^{-x}, x \ge 0 \\ 0, \text{otherwise} \end{cases}$

$$(i) \quad E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx = \int_{-\infty}^{\infty} x(0)dx + \int_{0}^{\infty} xe^{-x}dx = \int_{0}^{\infty} xe^{-x}dx$$

$$=\left[-(x+1)e^{-x}\right]_0^{\infty}=(-1)\ \underset{x\to\infty}{\text{Lt}}\ \frac{(x+1)}{e^x}+(0+1)=0+1=1.$$

(ii) $$E(X^2)=\int_{-\infty}^{\infty}x^2 f(x)dx=\int_0^{\infty}x^2e^{-x}dx$$

$$=\left[x^2(-e^{-x})-2x(e^{-x})+2(-e^{-x})\right]_0^{\infty}=\left[-e^{-x}(x^2+2x+2)\right]_0^{\infty}$$

$$=(-1)\ \underset{x\to\infty}{\text{Lt}}\ \frac{x^2+2x+2}{e^x}+2=2$$

(iii) $$\text{Var}(X)=E(X^2)-\{E(X)\}^2=2-(1)^2=2-1=1$$

Example 24 : Is the function defined as follows a density function $f(x)=\begin{cases}e^{-x}, x\geq 0\\ 0, x<0\end{cases}$

If so determine the probability that the variate having this density will fall in the interval (1,2)? Find the cumulative probability F(2)?

[JNTU (K) Nov. 2009 (Set No. 2)]

Solution : Given $f(x)=\begin{cases}e^{-x}, x\geq 0\\ 0, x<0\end{cases}$

(i) Clearly $f(x)\geq 0, \forall\ x$ in (1,2) and

$$\int_{-\infty}^{\infty}f(x)dx=\int_{-\infty}^{0}0\cdot dx+\int_0^{\infty}e^{-x}dx=\int_0^{\infty}e^{-x}dx=-(e^{-x})_0^{\infty}=-(0-1)=1$$

Hence the function $f(x)$ is a density function.

(ii) Required probability $=P(1\leq x\leq 2)=\int_1^2 f(x)dx$

$$=\int_1^2 e^{-x}dx=-(e^{-x})_1^2=-(e^{-2}-e^{-1})=e^{-1}-e^{-2}$$

$$=0.368-0.135=0.233\ .$$

(iii) Cumultative probability function

$$F(2)=\int_{-\infty}^{2}f(x)dx=\int_{-\infty}^{0}0\cdot dx+\int_0^2 e^{-x}dx$$

$$=-(e^{-x})_0^2=-(e^{-2}-1)=1-e^{-2}$$

$$=1-0.135=0.865\ .$$

Example 25 : The cumulative distribution function for a continuous random variable X is $F(x)=\begin{cases}1-e^{-2x}, & x\geq 0\\ 0, & x<0\end{cases}$

Find (*i*) the density function $f(x)$, (*ii*) mean and (*iii*) variance of the density function.

[JNTU (K) May 2010 (Set No. 2)]

Solution : (*i*) The density function $f(x)$ is given by $f(x)=\frac{d}{dx}[F(x)]$

$$\therefore f(x)=\begin{cases}\frac{1}{2}e^{-2x}, x\geq 0\\ 0, x<0\end{cases}$$

(*ii*) Mean $=E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{-\infty}^{0} x\, f(x)\, dx + \int_{0}^{\infty} x\, f(x)\, dx$

i.e. $\mu = 0+\int_0^\infty x.\frac{1}{2}e^{-2x}dx = \frac{1}{2}\int_0^\infty x\, e^{-2x}\, dx$

$$=\frac{1}{2}\left[x\left(\frac{e^{-2x}}{-2}\right)-1.\left(\frac{e^{-2x}}{4}\right)\right]_0^\infty = -\frac{1}{8}\left[e^{-2x}(2x+1)\right]_0^\infty$$

$$=-\frac{1}{8}[0-(0+1)]=\frac{1}{8}$$

(*iii*) Variance $= \int_{-\infty}^{\infty} x^2 f(x)dx-\mu^2 = \int_{-\infty}^{0} x^2 f(x)\, dx + \int_{0}^{\infty} x^2 f(x)\, dx - \mu^2$

$$=0+\int_0^\infty x^2.\frac{1}{2}e^{-2x}dx-\left(\frac{1}{8}\right)^2 = \frac{1}{2}\int_0^\infty x^2 e^{-2x}dx-\frac{1}{64}$$

$$=\frac{1}{2}\left[x^2\left(\frac{e^{-2x}}{-2}\right)-2x\left(\frac{e^{-2x}}{4}\right)+2\left(\frac{e^{-2x}}{-8}\right)\right]_0^\infty-\frac{1}{64}$$

$$=-\frac{1}{2}\left[e^{-2x}\left(\frac{1}{2}x^2-\frac{1}{2}x+\frac{1}{4}\right)\right]_0^\infty-\frac{1}{64}$$

$$=-\frac{1}{8}\left[e^{-2x}\left(2x^2-2x+1\right)\right]_0^\infty-\frac{1}{64}$$

$$=-\frac{1}{8}[0-(0-0+1)]-\frac{1}{64}=\frac{1}{8}-\frac{1}{64}=\frac{8-1}{64}=\frac{7}{64}$$

Example 26 : If X is a continuous random variable with p .d .f $f(x)=\begin{cases}x^2, 0\leq x\leq 1,\\ 0, \text{elsewhere}\end{cases}$

If $p(a\leq x\leq 1)=\frac{19}{81}$, find the value of '$a$'. **[JNTU (K) May 2010 (Set No. 3)]**

Solution : Given $p(a \le x \le 1) = \frac{19}{81} \Rightarrow \int_a^1 f(x)\,dx = \frac{19}{81} \Rightarrow \int_a^1 x^2\,dx = \frac{19}{81}$

$$\Rightarrow \left(\frac{x^3}{3}\right)_a^1 = \frac{19}{81} \Rightarrow \frac{1-a}{3} = \frac{19}{81}$$

$$\Rightarrow 1 - a = \frac{19}{27}.$$

$$\therefore a = 1 + \frac{19}{27} = \frac{46}{27}$$

Example 27 : If a random variable has the probability density function

$$f(x) = \begin{cases} k(x^2-1), -1 \le x \le 3 \\ 0, \text{elsewhere} \end{cases}$$

Find the value of 'k' and $p\left(\frac{1}{2} \le x \le \frac{5}{2}\right)$ **[JNTU (K) May 2010 (Set No. 4)]**

Solution : In order that $f(x)$ should be a probability density function

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

i.e., $\int_{-\infty}^{-1} f(x)\,dx + \int_{-1}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_{-1}^{3} k(x^2-1)\,dx = 1 \Rightarrow k\left(\frac{x^3}{3} - x\right)_{-1}^{3} = 1$$

$$\Rightarrow k\left[(9+1) - \left(-\frac{1}{3}+1\right)\right] = 1 \Rightarrow k\left(9 + \frac{1}{3}\right) = 1$$

$$\Rightarrow \frac{28}{3}k = 1. \quad \therefore k = \frac{3}{28}$$

Now $p\left(\frac{1}{2} \le x \le \frac{5}{2}\right) = \int_{1/2}^{5/2} \frac{3}{28}(x^2-1)\,dx$

$$= \frac{3}{28}\left(\frac{x^3}{3} - x\right)_{1/2}^{5/2} = \frac{3}{28}\left[\left(\frac{(5/2)^3}{3} - \frac{5}{2}\right) - \left(\frac{(1/2)^3}{3} - \frac{1}{2}\right)\right]$$

$$= \frac{3}{28}\left[\frac{65}{24} + \frac{11}{24}\right]$$

$$= \frac{3}{28} \times \frac{76}{24} = \frac{19}{56}$$

Example 28 : If X is the continuous random variable whose density function is

$$f(x)=\begin{cases} x & \text{if } 0<x<1 \\ 2-x & \text{if } 1\le x<2 \\ 0, & \text{elsewhere} \end{cases}$$

Find $E(25X^2+30X-5)$. **[JNTU (H) Nov. 2010 (Set No. 2)]**

Solution : Given $f(x)=\begin{cases} x & \text{if } 0<x<1 \\ 2-x & \text{if } 1\le x<2 \\ 0, & \text{elsewhere} \end{cases}$

$$\therefore \text{E}(\text{X})=\int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$=\int_0^1 x.x\,dx+\int_1^2 x(2-x)\,dx=\int_0^1 x^2 dx+\int_1^2 (2x-x^2)\,dx$$

$$=\left(\frac{x^3}{3}\right)_0^1+\left(2.\frac{x^2}{2}-\frac{x^3}{3}\right)_1^2=\frac{1}{3}+\left(x^2-\frac{x^3}{3}\right)_1^2$$

$$=\frac{1}{3}+\left(4-\frac{8}{3}\right)-\left(1-\frac{1}{3}\right)=4+\frac{1}{3}-\frac{8}{3}-\frac{2}{3}$$

$$=4-1=3$$

$$\text{E}(\text{X}^2)=\int_{-\infty}^{\infty} x^2 f(x)\,dx=\int_0^1 x^2.x\,dx+\int_1^2 x^2(x-2)\,dx$$

$$=\int_0^1 x^3 dx+\int_1^2 (x^3-2x^2)\,dx=\left(\frac{x^4}{4}\right)_0^1+\left(\frac{x^4}{4}-2.\frac{x^3}{3}\right)_1^2$$

$$=\frac{1}{4}+\left(4-\frac{16}{3}\right)-\left(\frac{1}{4}-\frac{2}{3}\right)=-\frac{2}{3}$$

$$\therefore \text{E}(25\text{X}^2+30\text{X}-5)=25.\,\text{E}(\text{X}^2)+30.\,\text{E}(\text{X})-5$$

$$=25\left(-\frac{2}{3}\right)+30(3)-5=-\frac{50}{3}+85$$

$$=\frac{-50+255}{3}=\frac{205}{3}=68.33$$

Example 29 : If a random variable has the probability density function

$$f(x)=\begin{cases}k(x^2-1), -1\le x\le 3\\ 0, \text{elsewhere}\end{cases}$$

Find the value of 'k' and $p\left(\frac{1}{2}\le x\le\frac{5}{2}\right)$. **[JNTU (K) May 2010 (Set No. 4)]**

Solution : In order that $f(x)$ should be a probability density function

$$\int_{-\infty}^{\infty} f(x)\,dx=1$$

i.e., $$\int_{-\infty}^{-1} f(x)\,dx+\int_{-1}^{3} f(x)\,dx+\int_{3}^{\infty} f(x)\,dx=1$$

$$\Rightarrow \int_{-1}^{3} k(x^2-1)\,dx=1\Rightarrow k\left(\frac{x^3}{3}-x\right)_{-1}^{3}=1$$

$$\Rightarrow k\left[(9+1)-\left(-\frac{1}{3}+1\right)\right]=1\Rightarrow k\left(9+\frac{1}{3}\right)=1$$

$$\Rightarrow \frac{28}{3}k=1. \quad \therefore k=\frac{3}{28}$$

Now $$P\left(\frac{1}{2}\le X\le\frac{5}{2}\right)=\int_{1/2}^{5/2}\frac{3}{28}(x^2-1)\,dx$$

$$=\frac{3}{28}\left(\frac{x^3}{3}-x\right)_{1/2}^{5/2}=\frac{3}{28}\left[\left(\frac{(5/2)^3}{3}-\frac{5}{2}\right)-\left(\frac{(1/2)^3}{3}-\frac{1}{2}\right)\right]$$

$$=\frac{3}{28}\left[\frac{65}{24}+\frac{11}{24}\right]=\frac{3}{28}\times\frac{76}{24}=\frac{19}{56}$$

Example 30 : Find the value of K and the distribution function F(x) given the probability density function of a random variable X as :

$$f(x)=\frac{K}{x^2+1}, -\infty<x<\infty$$ **[JNTU(K) March 2014 (Set No. 3)]**

Solution : Since total probability is unity, we have $\int_{-\infty}^{\infty} f(x)dx=1$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{K}{x^2+1}dx=1\Rightarrow K\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx=1\Rightarrow K(\tan^{-1}x)_{-\infty}^{\infty}=1$$

$$\Rightarrow K(\tan^{-1}\infty-\tan^{-1}(-\infty))=1\Rightarrow K\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]=1\Rightarrow K\pi=1.$$

$$\therefore K=\frac{1}{\pi}.$$

By definition, the distribution function is given by

$$F(x)=\int_{-\infty}^{x} f(x)dx=\int_{-\infty}^{x}\frac{K}{1+x^2}dx=\frac{1}{\pi}\int_{-\infty}^{x}\frac{1}{1+x^2}dx \qquad \left[\because K=\frac{1}{x}\right]$$

$$=\frac{1}{\pi}(\tan^{-1}x)_{-\infty}^{x}=\frac{1}{\pi}[\tan^{-1}x-\tan^{-1}(-\infty)]=\frac{1}{\pi}\left(\tan^{-1}x+\frac{\pi}{2}\right).$$

Example 31 : The length of time (in minutes) that a certain lady speaks on the telephone is found to be random phenomenon, with a probability function specified by the function

$$f(x)=\begin{cases}A\,e^{-x/5}, \text{ for } x\geq 0\\ 0, \text{ otherwise}\end{cases}$$

(*i*) Find the value of A that makes $f(x)$ a probability density function.

(*ii*) What is the probability that the number of minutes that she will take over the phone is more than 10 minutes? **[JNTU(K) Dec. 2013 (Set No. 2)]**

Solution : (*i*) In order that $f(x)$ should be a probability density function,

$$\int_{-\infty}^{\infty} f(x)\,dx=1 \text{ i.e., } \int_{0}^{\infty} f(x)\,dx=1 \text{ or } \int_{0}^{\infty} A\,e^{-x/5}\,dx=1 \Rightarrow A\left(\frac{e^{-x/5}}{-1/5}\right)_{0}^{\infty}=1$$

$$\Rightarrow -5A\,(e^{-\infty}-e^{-0})=1 \Rightarrow -5A\,(0-1)=1 \Rightarrow 5A=1 \text{ or } A=\frac{1}{5}.$$

(*ii*) $$P(X>10)=\int_{10}^{\infty} f(x)\,dx=\int_{10}^{\infty} A\,e^{-x/5}\,dx=A\left(\frac{e^{-x/5}}{-1/5}\right)_{10}^{\infty}$$

$$=-5A\,(e^{-\infty}-e^{-2})=-5\left(\frac{1}{5}\right)(0-e^{-2})=\frac{1}{e^2}.$$

REVIEW QUESTIONS

1. Define expectation for discrete and continuous random variables.

EXERCISE 2(B)

1. The probability density function $f(x)$ of a continuous random variable is given by

$$f(x)=\begin{cases}k\,x^3, \text{ for } 0<x<1\\ 0, \text{ elsewhere}\end{cases}$$

Find the value of k and the probability that the random variable takes on a value (*i*) between $\frac{1}{4}$ and $\frac{3}{4}$ (*ii*) greater than $\frac{2}{3}$.

2. (a) If the probability density of a random variable is given by

$$f(x) = \begin{cases} x, \text{ for } 0 < x < 1 \\ 2 - x, \text{ for } 1 \le x < 2 \\ 0, \text{ elsewhere} \end{cases}$$

Find the probabilities that a random vairable having this probability density will take on a value (i) between 0.2 and 0.8 (ii) between 0.6 and 1.2.

[JNTU (K) March 2014 (Set No.2)]

(b) If the probability density of a random variable is given by $f(x) = \begin{cases} kx^2, 0 < x < 1 \\ 0, \text{elsewhere} \end{cases}$

Find the value of k and the probabilities that a random variable having this probability density will take on a value (i) between 0.25 and 0.75, (ii) greater than 2/3.

[JNTU (K) March 2014 (Set No. 1)]

3. Show that the function $f(x) = \dfrac{1}{\pi(1+x^2)}, -\infty \le x \le \infty$ is a probability density function.

[JNTU (K) Nov. 2011 (Set No. 4)]

4. Find the constant k so that the function $f(x)$ defined by

(i) $f(x) = \begin{cases} \dfrac{1}{k}, \text{ if } a \le x \le b \\ 0, \text{ elsewhere} \end{cases}$ (ii) $f(x) = \begin{cases} k\, e^{-3x}, x > 0 \\ 0, \text{ elsewhere} \end{cases}$ is a density function.

5. "The diameter of an electric cable is assumed to be continuous random variable with probability density function $f(x) = 6x(1-x), 0 \le x \le 1$". Justify.

Find the mean and variance of the distribution.

6. The probability density function of the continuous random variable X is given by

(i) $f(x) = \begin{cases} \dfrac{1}{4}, \text{ for } 4 \le x \le 8 \\ 0, \text{ elsewhere} \end{cases}$ (ii) $f(x) = \begin{cases} \dfrac{1}{16}(3+x)^2, -3 \le x \le -1 \\ \dfrac{1}{16}(2-6x^2), -1 \le x < 1 \\ \dfrac{1}{16}(3-x)^2, 1 \le x \le 3 \end{cases}$

Show that the area under the cure above x-axis is unity. Also find the mean of the distribution.

7. X is a continuous random variable with probability density function given by

(i) $f(x) = \begin{cases} \dfrac{1}{4}e^{-x/4}, \text{ for } x > 0 \\ 0, \text{ elsewhere} \end{cases}$ (ii) $f(x) = \begin{cases} \dfrac{1}{b}e^{-x/b}, x > 0, b > 0 \\ 0, \text{ elsewhere} \end{cases}$

Find the mean and variance of X.

8. X is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} kx, \text{ for } 0 \le x < 2 \\ 2k, \text{ for } 2 \le x < 4 \\ k(6-x), \text{ for } 4 \le x < 6 \end{cases}$$

[JNTU 2003, 2004]

Find k and mean of the density function.

9. Find the standard deviation of the probability density function

(i) $f(x) = \begin{cases} x^3, 0 \le x \le 1 \\ (2-x)^3, 1 \le x \le 2 \end{cases}$ (ii) $f(x) = \begin{cases} 30x^4(1-x), 0 \le x \le 1 \\ 0, \text{elsewhere} \end{cases}$

10. Find the mean and variance of the density function $f(x) = \dfrac{2x}{9}, 0 \le x \le 3.$

11. The frequency function of a continuous random variable X is given by

$f(x) = y_0\, x(2-x), 0 \le x \le 2.$

Find mean, median and variance of X. [**Hint** : Refer Solved Example 12]

12. X is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} k\, x^{\alpha-1}(1-x)^{\beta-1}, \text{ for } 0 < x < 1, \alpha > 0, \beta > 0 \\ 0, \text{ otherwise} \end{cases}$$

Find k and mean value of X. **[JNTU (K) Nov. 2011 (Set No.3)]**

13. Let $f(x) = 3x^2$, when $0 \le x \le 1$ be the probability density function of a continuous random variable X. Determine 'a' and 'b' such that

(i) $P(X \le a) = P(X > a)$ (ii) $P(X \ge b) = 0.05$.

[JNTU 2004 (Set No. 2), (K) Nov. 2011 (Set No. 1)]

14. If $f(x) = \begin{cases} \frac{1}{2}(x+1), -1 < x < 1 \\ 0, \text{elsewhere} \end{cases}$ represents the density of a random variable X, find the mean and standard deviation of X. **[JNTU 2004S (Set No. 4)]**

15. The probability density function of a random variable X is given by

$$f(x) = \begin{cases} kx(x-1), \text{ for } 1 \le x \le 4 \\ 0, \text{ otherwise} \end{cases}$$

Given that $P(1 \le X \le 3) = \dfrac{28}{3}$. Find the value of k.

16. Let X be a continuous random variable with distribution : $f(x) = \begin{cases} \frac{1}{8}, \text{if } 0 \le x \le 8 \\ 0, \text{elsewhere} \end{cases}$

Find (i) $P(2 \le X \le 5)$ (ii) $P(3 \le X \le 7)$ (iii) $P(X \le 6)$ (iv) Determine and plot the graph of the cumulative distribution function F of X.

[JNTU (K) Nov. 2012 (Set No. 2)]

ANSWERS

1. (i) 4 (ii) $\frac{5}{16}$ (iii) $\frac{65}{81}$

2. (a) (i) 0.3 (ii) 0.5

4. (i) $b - a$ (ii) 3

5. $\frac{1}{2}, \frac{1}{20}$

6. (i) 6 (ii) 0

7. (i) 4, 80 (ii) b, b^2

8. $\frac{1}{8}, 3$

9. (i) $\frac{1}{\sqrt{15}}$ (ii) $\frac{\sqrt{5}}{14}$

10. $2, \frac{1}{2}$

11. $1, 1, \frac{1}{5}$

12. $\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)}, \frac{\alpha}{\alpha+\beta}$

13. $\left(\frac{1}{2}\right)^{1/3}, \left(\frac{19}{20}\right)^{1/3}$

14. $\frac{1}{3}, \frac{\sqrt{2}}{3}$

15. 2