UNIT-2
SET THEORY

Basic Concepts :-

i, Set :- Any collection of well defined objects is called as Set

The objects in a Set can be numbers, people, letters etc. Any object belonging to a set is called a member or Element of that Set. A Set is Said to be well-defined if it is possible to determine whether any given object is a member of the set.

Example :- 1. The Set of Vowels of the alphabet

2. The set of people living on the Earth.

Notations :- Capital letters A, B, C . . . X, Y, Z will be used to denotes Sets. Lower case letters a, b, c . . . x, y, z will be used to denote elements

Standard Symbols :-

N :- The Set of all natural numbers $N = \{1, 2, 3 \ldots\}$

Z :- The Set of all integers $Z = \{\ldots -2, -1, 0, 1, 2, \ldots\}$

Z^+ :- The Set of all positive integers $Z^+ = \{1, 2, 3 \ldots\}$

Z^* :- The Set of all non-zero integers $Z^* = \{\ldots -3, -1, 0, 1, 2 \ldots\}$

E :- The Set of all Even integers $E = \{0, \pm 2, \pm 4, \pm 6 \ldots\}$

Q :- The Set of all rational numbers $Q = \{P/q \cdot / q \neq 0, \& P, q\}$

Q^* :- The Set of all non-zero rational numbers

Q^+ :- The Set of all positive rational numbers

R :- The Set of all real numbers

R^+ :- The Set of all positive real numbers

R^* :- The Set of all non-zero real numbers

C :- The Set of all Complex numbers.

Subsets & Set inclusion :- Let x and y be any two Sets. If Every Element of x is an Element of y then 'x' is Said to be a Subset of y and is written as $x \subseteq y$

Example:- Given $x = \{a, b, c\}$ $y : \{a, b\}$ $z = \{a, c\}$ & $w = \{c\}$ provides the relation between these Sets

$\{a, b\} \subseteq \{a, b, c\} \Rightarrow y \subseteq x$

$\{a, c\} \subseteq \{a, b, c\} \Rightarrow z \subseteq x$

$\{c\} \subseteq \{a, c\} \Rightarrow w \subseteq z$

$\{c\} \subseteq \{a, b, c\} \Rightarrow w \subseteq x$

Proper Sub Set :- Let x and y be any two Sets x is Said to be a proper Subset of y if x is a Subset of y and there is atleast one Element in y that is not in x. In other words a set x is a proper Subset of a Set y if $x \subseteq y$ but $x \neq y$ Symbolically it is written as $x \subset y$

Example:- Given $x = \{1, 3, 5\}$ and $y = \{1, 2, 3, 4, 5\}$ is x is a proper Subset of y?

Sol :- $x \subseteq y$ but $x \neq y$

$x \subset y$ that is x is a proper subset of y.

Equal Sets :- Two sets x and y are Said to be Equal if $x \subseteq y$ and $y \subseteq x$ that is $x = y$ if both x and y have the Same numbers.

Example :- Given $x : \{1, 2, 3, 4, 5\}$ and $y = \{3, 5, 1, 2, 4\}$ Are they Equal Sets?

Sol. Since x and y have the Same members $x = y$

Power Sets :- The collection of all Subsets of Set x is called the power set of x and is denoted by $p(x)$

Example :- Find the power Set of a Set $x = \{1, 2, 3\}$

Sol :- The power Set of a Set x is written as $p(x) = \{\emptyset, \{1\}, \{2\}, \{3\}$ $\{1, 2\} \{1, 3\} \{2, 3\} \{1, 2, 3\}\}$

Finite and Infinite Sets :- If the number of distinct Elements in a Set x is finite or countable then the Set x is Said to be a finite Set.

Infinite Set :- of the number of Elements in a Set x is not finite or uncountable. then the Set x is Said to be an infinite Set.

Venn diagrams :-

The diagrams used to represent Sets in a simple manner are known as Venn diagrams.

$\Rightarrow x < y$

Operations on sets :-

i, union of two sets :- The union of two sets is defined as the Set of all the Elements that are members of Set A, Set B or both and is denoted by $A \cup B$ read as A union B. It is written as

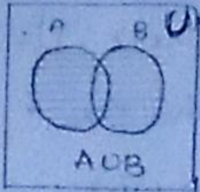
$$A \cup B = \{x / x \in A \text{ or } x \in B\}$$

Example :- Given $A = \{1, 2, 3, 4, 5, a, b\}$ and $B = \{a, b, c, d, e\}$

Find $A \cup B$?

Sol. $A \cup B = \{1, 2, 3, 4, 5, a, b, c, d, e\}$

The Venn diagrams for $A \cup B$ is given in



Properties of union operation :-

Let A, B, C be any 3 sets then ① $A \cup A = A$

Idempotent property.

② $A \cup \phi = A$ ③ $A \cup U = U$ ④ $A \cup A' = U$ ⑤ $A \subseteq A \cup B$ & $B \subseteq A \cup B$

⑥ $A \cup B = B \cup A$ (commutative property)

⑦ $(A \cup B) \cup C = A \cup (B \cup C)$ (Associative property)

Intersection of two Sets :- The intersection of any two sets A and B is the set consisting of all the elements that belong to both A and B and is denoted by $A \cap B$ read as A intersection B. Symbolically it is written as

$$A \cap B = \{x / x \in A \text{ and } x \in B\}$$

Example :- Given $A = \{a, b, c, d, e, f, g, h\}$ and $B = \{a, f, d, i, j\}$ then find $A \cap B$.

Sol. Given $A = \{a, b, c, d, e, f, g, h\}$

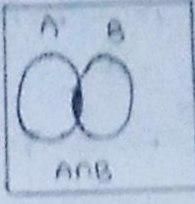
$B = \{a, d, f, i, j\}$

$A \cap B = \{a, d, f\}$

Properties of intersection operation :- Let A, B & C be 3 sets then ① $A \cap A = A$ ② $A \cap \phi = \phi$ ③ $A \cup U = A$ where 'U' is universal set.

④ $A \cap B \subseteq A$ and $A \cap B \subseteq B$ ⑤ $A \cap B = B \cap A$

⑥ $(A \cap B) \cap C = A \cap (B \cap C)$



* Mutually disjoint sets :- A collection of sets is called a disjoint collection if every pair of sets in the collection is disjoint the elements of disjoint collection are called mutually disjoint.

Example :- If $X = \{\{a,b\} \{c\}\}$ $Y = \{\{a\} \{b,c\}\}$ and $Z = \{\{a,b,c\}\}$ then show that X, Y & Z are mutually disjoint.

Sol :- $X \cap Y = \phi$ $Y \cap Z = \phi$ $X \cap Z = \phi$

$\therefore X, Y, Z$ are mutually disjoint.

* Difference of two sets :- The difference of any two sets A and B is the set of elements that belong to A but not to B. It is denoted by A-B and is read as A difference B.

ie, $A - B = \{x / x \in A$ and $x \notin B\}$ or $B - A = \{x / x \notin A$ and $x \in B\}$

Example :- If $A = \{1,2,3,4,5,6\}$ $B = \{3,5,7,9\}$ the find A-B and B-A.

Sol :- $A - B = \{1,2,4,6\}$ ie $x \in A$ and $x \notin B$

$B - A = \{7,9\}$ ie $x \in A$ and $x \in B$

Properties of the difference operation :-

let A, B and C be any 3 sets and 'U' be the universal set then, ① $A' = U - A$ ② $A - B = A \cap B'$ ③ $A - A = \phi$ ④ $A - B = A \iff A \cap B = \phi$

⑤ $A - \phi = A$ ⑥ $A - B = B - A \iff A = B$ ⑦ $A - B = \phi \iff A \subseteq B$.

* Complement of a set :- let U be the universal set for any set A, $U - A$ is called the absolute complement of A. The absolute complement of a set A is also known as the complement of A and is denoted by $\sim A$ or A^c or A'.

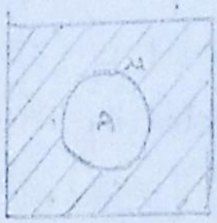
Example :- If $U = Z^+$ and $A = \{1, 3, 5 \ldots\}$ then find A^c

Sol :- $U = \{1, 2, 3, 4, 5 \ldots\}$ $A = \{1, 3, 5 \ldots\}$

$A^c = \{2, 4, 6 \ldots\}$

Properties of complements :-

1. $A \cup A' = U$
2. $A \cap A' = \phi$
3. $U' = \phi$
4. $\phi' = U$
5. $(A')' = A$
6. $(A \cup B)' = A' \cap B'$
7. $(A \cap B)' = A' \cup B'$



Venn diagram of A^c

* Demorgon laws :- For any two sets A and B

(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

* Distributive laws :- For any three sets A, B and C

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$\therefore A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$

Symmentric difference of two sets :-

let A and B be any two sets the symmetric difference A and B is the consisting of all the elements that belong to A (or) B but not to both A and B. It is denoted by $A \oplus B$, $A \Delta B$ (or) $A + B$

i.e, $A+B = \{x / x\in A \text{ and } x\notin B\}$ (or) $\{x\in B \text{ and } x\notin A\} = (A-B)\cup(B-A)$

Properties of Symmetric difference :- Let A and B are any two sets then ① $A+A = \phi$ ② $A+\phi = A$ ③ $A+B = B+A$

④ $A+B = (A\cup B)-(A\cap B)$

Problems : ① Show that for any two sets A & B

$A-(A\cap B) = A-B$

Sol: Given L.H.S $\Rightarrow A-(A\cap B)$

Let $x\in A-(A\cap B) = x\in A$ and $x\notin(A\cap B)$

$= x\in A$ and $(x\notin A$ (or) $x\notin B)$

$= (x\in A$ and $x\notin A)$ (or) $(x\in A$ and $x\notin B)$

$= x\in A$ and $x\notin B$

$= x\in A-B$

$\therefore A-(A\cap B)\subseteq A-B$ — ①

Let $y\in A-B \Rightarrow y\in A$ and $y\notin B$

$= (y\in A$ and $y\notin A)$ (or) $(y\in A$ and $y\notin B)$

$\Rightarrow y\in A$ and $(y\notin A$ (or) $y\notin B)$

$\Rightarrow y\notin(A\cap B)$ and $y\in A$

$\Rightarrow y\in A-(A\cap B)$

$\therefore A-B \subseteq A-(A\cap B)$ — ②

From Equations ① and ②

$A-(A\cap B) = A-B$.

② If A, B and c are any three sets then prove that

i) $A\cup(B-A) = A\cup B$ ii) $A-(B\cup C) = (A-B)\cap(A-C)$

Sol. $A\cup(B-A) = A\cup(B\cap A')$ $[\because B-A = B\cap A']$

$= (A\cup B)\cap(A\cup A')$ [By distributive laws]

$= (A\cup B)\cap U$ $[\because A\cup A' = U]$

$= A\cup B$ $[\because X\cap U = X]$

$\therefore A\cup(B-A) = A\cup B$.

ii) $A-(B\cup C) = A\cap(B\cup C)'$ $[\because X-Y = X\cap Y']$

$= A\cap(B'\cap C')$

$= A\cap B'\cap C'$ — ①

Now $(A-B)\cap(A-C) = (A\cap B')\cap(A\cap C')$

$[\because X-Y = X\cap Y']$

$= A\cap B'\cap C'$ — ②

From Equations ① & ② $A-(B\cup C) = (A-B)\cap(A-C)$

③ Show that $A\cup(B-C) = (A\cup B)-(C-A)$

Sol. Given $A\cup(B-C) = (A\cup B)-(C-A)$

Take $A\cup(B-C) = A\cup(B\cap C')$ $[\because B-C = B\cap C']$

$= (A\cup B)\cap(A\cup C')$ [By distributive law]

$= (A\cup B)\cap((A')'\cup C')$ [Since $(A')' = A$]

$= (A\cup B)\cap(A'\cap C)'$ [By demorgons law]

$= (A\cup B)\cap(C\cap A')'$ [By commutative law]

$= (A\cup B)\cap(C-A)'$

$= (A\cup B) - (C-A) \quad [\because A\cap B' = A-B]$

(4) For any 3 sets A, B & C

① $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

② $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$

Proof: 1. $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

It is Enough to prove that

$A\cup(B\cap C) \subseteq (A\cup B)\cap(A\cup C)$

and $(A\cup B)\cap(A\cup C) \subseteq A\cup(B\cap C)$

Let $x\in A\cup(B\cap C) \Rightarrow x\in A$ or $x\in(B\cap C)$

$\Rightarrow x\in A$ or $(x\in B$ and $x\in C)$

$\Rightarrow (x\in A$ or $x\in B)$ and $(x\in A$ or $x\in C)$

$= x\in(A\cup B)$ and $x\in(A\cup C)$

$= x\in(A\cup B)\cap(A\cup C)$

$\therefore A\cup(B\cap C) \subseteq (A\cup B)\cap(A\cup C)$ — ①

Let $y\in(A\cup B)\cap(A\cup C) \Rightarrow y\in(A\cup B)$ and $y\in(A\cup C)$

$\Rightarrow (y\in A$ or $y\in B)$ and $(y\in A$ or $y\in C)$

$\Rightarrow y\in A$ or $(y\in B$ and $y\in C)$

$\Rightarrow y\in A\cup(B\cap C)$

$\therefore (A\cup B)\cap(A\cup C) \subseteq A\cup(B\cup C)$ — ②

From ① & ② $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

(ii) Given $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$

It is Enough to prove that

$A\cap(B\cup C) \subseteq (A\cap B)\cup(A\cap C)$

and $(A\cap B)\cup(A\cap C) \subseteq A\cap(B\cup C)$

Let $x \in A \cap (B \cup C) \Rightarrow x \in A$ and $x \in (B \cup C)$

$\Leftrightarrow x \in A$ and $(x \in B$ or $x \in C)$

$\Leftrightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

$\Leftrightarrow x \in (A \cap B)$ or $x \in (A \cap C)$

$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$

$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

We can similarly prove that

$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

Hence $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Principle of inclusion and Exclusion:-

principle:- Let A and B be any two finite sets. Then the number of Elements in the union of the two sets A and B is the sum of the number of Elements in the sets minus the number of elements in their intersection that is

$|A \cup B| = |A| + |B| - |A \cap B|$

① Show that $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

Proof:- We know by the principle of inclusion and Exclusion that $|A \cup B| = |A| + |B| - |A \cap B|$

$\therefore |A \cup B \cup C| = |A \cup (B \cup C)|$

$= |A| + |B \cup C| - |A \cap (B \cup C)|$

$= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)|$

$= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |(A \cap B) \cap (A \cap C)|$

$$= |A|+|B|+|C|-|A\cap B|-|B\cap C|-|A\cap C|+|A\cap B\cap C|$$

General form of the principle of inclusion and Exclusion:-

If $A_1, A_2 \dots A_n$ are any finite sets then

$$|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{1\le p\le n} |A_i| - \sum_{1\le p< q\le n} |A_i \cap A_j| + \sum_{\substack{1\le i<j \\ <k\le n}}$$

$$\dots |A_i \cap A_j \cap A_k| + \dots + (-1)^{n-1} |A_1 \cap A_2 \cap \dots \cap A_n|$$

① In a class of 50 students, 20 students play foot ball and 16 students play hockey. It is found that 10 students play both the games. Find the number of students who play neither

Sol:- Let A represent the set of students playing football and B represents the set of students playing hockey

$\therefore A\cap B$ represents the set of students playing both the games

Given $|A| = 20$ $|B| = 16$ & $|A\cap B| = 10$

By the principle of inclusion and Exclusion we have

$|A\cup B| = |A|+|B|-|A\cap B|$ $= 20+16-10 = 26$

Thus, the number of students who play neither hockey nor foot ball.

$= 50 - 26 = 24.$

② Among 60 students in a class, 45 passed in the first Semester Examinations and 30 passed in the second Semester Examinations. If 12 did not pass in either Semesters how many passed in both Semesters?

Sol:- Let A represent the set of students who passed in the first Semester Examinations and B represents the set of students who passed in the Second semester Examinations. then $|A| = 45$ $|B| = 30$.

Given that 12 students did not pass in either Semester we can write $|A' \cap B'| = 12$

$60 - 12 = 48$ students have passed in either the first or the second Semester or both

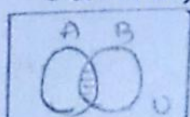
$$\Rightarrow |A \cup B| = 48.$$

We know that $|A \cup B| = |A| + |B| - |A \cap B|$ [$\because$ By the principle of Inclusion and Exclusion]

$$48 \Rightarrow 45 + 30 - |A \cap B|$$

$$|A \cap B| = 45 + 30 - 48$$

$$= 27.$$



Hence, the number of students who passed in both the Semester is 27.

③ A Survey of 260 television viewers produced the following information, 64 watch Spots, 58 watch flims, 94 watch news, 28 watch both Spots and films, 26 watch both Spots and news items, 22 watch both flims and news items and 14 are interested in all the 3 types. How many people in the Survey are not interested in any of these 3 types? How many people in the Survey are interested in Seeing only news items?

Sol:- Let 'S' represent the set of people who watch spots items 'F' represent the set of people who see films and 'N' represent the set of people who see news items

then $|S| = 64$ $|F| = 58$ $|N| = 94$ $|S \cap F| = 28$ $|S \cap N| = 26$

$|F \cap N| = 22$ $|S \cap F \cap N| = 14$

number of people interested in any of these 3 types

$= |S \cup F \cup N|$

∴ By the principle of Inclusion and Exclusion we have

$$|S \cup F \cup N| = |S| + |F| + |N| - |S \cap F| - |S \cap N| - |F \cap N| - |S \cap F \cap N|$$

$$= 64 + 58 + 94 - 28 - 26 - 22 + 14$$

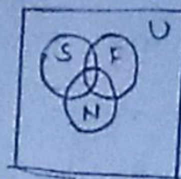
$$= 230 - 76$$

$$= 154.$$

Hence the number of people in the Survey not interested in any of these 3 types is $|U| - |S \cup F \cup N|$

$= 260 - 154 = 106.$

number of people interested in seeing only news items and flim $= |F \cap N| - |S \cap F \cap N|$



$= 22 - 14 = 8$

Number of people interested in seeing only sports and news items $= |S \cap N| - |S \cap F \cap N|$

$= 26 - 14 = 12$

$\therefore$ Number of people interested in seeing only news items $= |N| - 12 - 4 - 8$

$= 94 - 34 = 60.$

(4) A total of 1,232 students have taken a course in Spanish, 879 have taken a course in French and 114 have taken a course in Russian. Further 100 have taken courses in both spanish and Russian. and 12 have taken courses in both French and Russian. If 2,092 students have taken at least one of Spanish, French and Russian courses, how many students have taken a course in all 3 languages?

Sol:- Let 'A' bet the set of students who have taken a course in Spanish, B be the set of students who have taken a course in French and c be the set of students who have taken a course in Russian. Then $|A| = 1{,}232$

$|B| = 879$, $|C| = 114$, $|A \cap B| = 100$, $|A \cap C| = 25$, $|B \cap C| = 12$

$|A \cup B \cup C| = 2{,}092$

By the principle of Inclusion and Exclusion we have

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| + |A \cap B \cap C|$$

$$2{,}092 = 1232 + 879 + 114 - 100 - 25 - 12 + |A \cap B \cap C|$$

$$|A \cap B \cap C| = 2092 - 2088$$

$$= 4$$

Thus, the number of students who have taken a Course in Spanish, French and Russian is 4.

Relations:- Any Set of ordered pairs defines a binary relation (or) Simply a relation Binary relations represent relation ships between elements of two sets, If 'R' is a relation, a particular ordered pair Say $(x, y) \in R$ can be written as xRy and can be read as "x is in relation R to y.

Ex.- Give an Example of a relation.

Sol. The relation "greater than" for real numbers is denoted by $>$, If x and y are any two real numbers Such that $x > y$ then we Say that $(x, y) \in >$ Thus the relation $>$ is

$$> = \{(x, y) \mid x \text{ and } y \text{ are real numbers and } x > y\}$$

Domain and range of a Relation:-

Let 'S' be a binary relation. The domain of the relation 'S' is defined as the Set of all first Elements of the ordered pairs that belong to S and is denoted by D or D(S)

Thus $D(S) = \{x \mid (\exists y)\, (x, y) \in S\}$

The range of the relation S is defined as the set of all the Second elements of the ordered pairs that belong to S and is denoted by R or R(S)

Thus $R(S) = \{ y \mid (\exists x)(x,y) \in S \}$

Ex:- Consider the relation $S = \{(1,2)(3,4)(a,\ )(p,q)\}$ find the domain and range of S?

Sol:- Given $S = \{(1,2)(3,4)(a,t),(p,q)\}$

The domain of S is written as $D(S) = \{1,3,a,p\}$

The Range of S is written as $R(S) = \{2,4,t,q\}$

Properties of Binary relations in a set:-

1. Reflexive relation:- A Binary relation R in a set X is said to be reflexive if for every $x \in X$, $x R x$ that is for every $x \in X$ the ordered pair $(x,x) \in R$

Ex:- 1. The relation $\leq$ is reflexive in the set of real numbers since for any $x \in R$, $x \leq x$

2. The relation $<$ is not reflexive in the set of real numbers

2. Symmetric relation:- A relation R in a set X is said to be symmetric if given $x R y$ then $y R x$, for every x and y in X. In other words if $(x,y) \in R$ then $(y,x) \in R$ for every $x,y \in X$.

Ex: 1. The Relation of Similarity in the set of triangles in a plane is symmetric and reflexive.

2. The Relations $\leq$ and $<$ are not symmetric in the set of real numbers.

3. Transitive Relation:- A Relation R in a X is Said to be transitive if given xRy and yRz then xRz for Every x, y and z in X ie, if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$ for Every x, y

Ex:- 1. The relations $\leq$, $<$ and $=$ are transitive in the set of real numbers.

2. The relation of similarity of triangles in a plane is transitive

4. irreflexive Relation:- A Relation R in a set X is Said to be irreflexive if for Every $x \in X$, the ordered pair $(x, x) \notin R$

Ex:- 1. The relation $<$ in the set of real numbers is irreflexive.

5. Anti Symmetric Relation:- A Relation R in a set X is Said to be anti Symmetric if xRy and yRx then $x = y$ for $x, y \in X$ ie, $(x, y) \in R$ and $(y, x) \in R \Rightarrow x = y$ for Every $x, y \in X$

Ex:- The relation of Equality in R. is both Symmetric and anti Symmetric.

Relation Matrix:- A Relation R from a finite Set A to a finite Set B can be represented by a matrix called the relation matrix R.

Let $A = \{a_1, a_2 \dots a_m\}$ and $B = \{b_1, b_2, \dots b_n\}$ be finite Sets containing m and n Elements respectively and R be the relation from A to B. Then R can be represented by an $m \times n$ matrix $M_R = [r]$

$$= \begin{cases} 1 & \text{if } a_i R b_j \\ 0 & \text{if } a_i \not R b_j \end{cases}$$

The Matrix M_R has the Elements as 1's and 0's

Properties of a relation in a set :-

1. If a relation is reflexive then all the diagonal Entries must be 1

If a relation is Symmetric then the relation matrix is Symmetric ie. $r_{ij} = r_{ji}$ for Every i & j

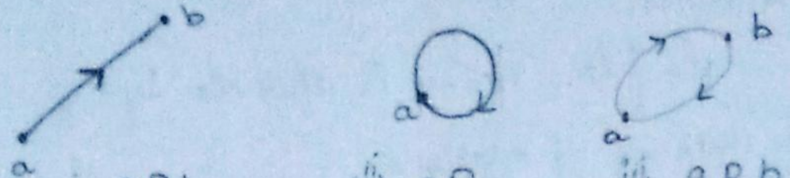
If a relation is anti- Symmetric then its matrix is Such that if $r_{ij} = 1$ then $r_{ij} = 0$ for $i \neq j$

Graph of a Relation :- A relation defined in a finite set can also be represented pictorially with the help of a graph, Let 'R' be a relation in a finite Set $A = \{a_1, a_2, \ldots a_n\}$. The Elements of A are represented by points or circles. called nodes. These nodes are called Vertices. if $(a_i, a_j) \in R$ then we connect the nodes a_i and a_j by means of an arc and put an arrow on the arc in the direction from a_i to a_j. This is called an Edge. If all nodes corresponding to the ordered pairs in R are connected by arcs with proper arrows then we get a graph of the relation R.

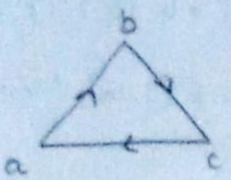
Note :- 1. If $a_i R a_j$ and $a_j R a_i$ then we draw two arcs between a_i & a_j with arrows pointing in both directions.

If $a_i R a_j$ then we get an arc which starts from node a_i and returns to node a_j. This arc is called a loop.

Ex:-



(i) aRb (ii) aRa (iii) aRb∧bRa



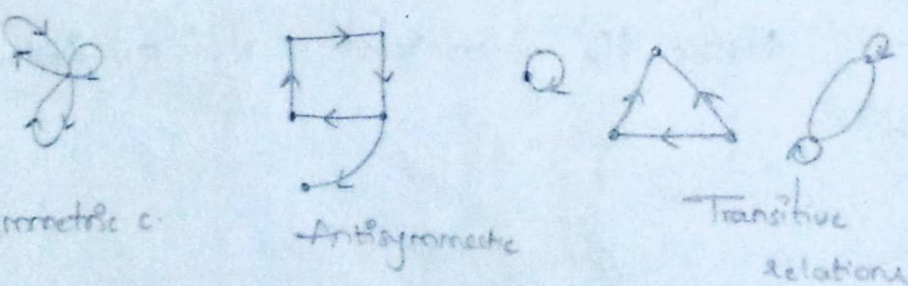
(iv) aRb∧bR & cRA ε.

Properties of relations:- 1. If a relation is reflexive then there is no loop at any node

2. If a relation is Symmetric and if one node is connected to another then there must be a return arc from the Second node to the first.

3. For anti symmetric relations no direct return paths Should Exist.

4. If a relation is transitive then the Situation is not So Simple.



Symmetric Antisymmetric Transitive relations

Problems on relations -

① Let $X = \{1, 2, 3, 4, 5, 6, 7\}$ and $R = \{(x,y) / x - y$ is divisible by $3\}$. Show that 'R' is an Equivalence relation. Draw the graph of R

Sol. Given relation

$$R = \{(x,y) / x - y \text{ is divisible by } 3\}$$

∴ The number of relations.

$R = \{(1\,1)(2\,2)(3\,3)(4\,4)(5\,5)(6\,6)(7\,7)(1\,7)$
$(7,1)(6,3)(3,6)(5,2)(2,5)(1,4)(4,1)(7,4)$
$(4,7)\}$

1. Reflexive :- $(1,1)(2,2)(3,3)(4,4)(5,5)(6,6)(7,7) \in R$

∴ R is Satisfied reflexive property.

2. Symmetric :- $\{(1,7)(7,1)(6,3)(3,6)(5,2)(2,5)(1,4)$
$(4,1)(7,4)(4,7)\} \in R$.

∴ R is Satisfied Symmetric property

3. Transitive :- $(5,2)(2,5) \in R \Rightarrow (5,5) \in R$.

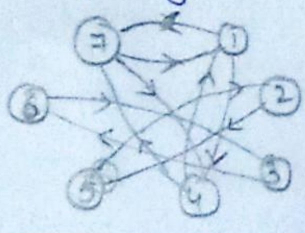
$(1,4)(4,1) \in R \Rightarrow (1,1) \in R$

∴ R is Satisfied Transitive property.

∴ The given 'R' is Satisfied about reflexive, transitive Symmetric property.

Hence the given relation 'R' is an Equivalence relation.

Graph:



Matrix form :-

$$\begin{array}{c|ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 4 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 5 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 6 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 7 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}$$

(2) Let R be a relation define on the set of all real numbers by if x, y are real numbers such that $xRy \iff x-y$ is a rational number show that the relation R is an Equivalence relation.

Sol: 1. Reflexive :- $xRx \ \forall x \in R$ then $x-x=0$, $0 \in Q$

$\therefore$ R is a reflexive relation

2. Symmetric :- xRy, & $yRx \ \forall x, y \in R$

$x-y$ and $y-x \in Q$

$\therefore$ R is a symmetric relation.

3. Transitive :- xRy, & yRz then $xRz \ \forall x, y, z \in R$.

then $x-y$ & $y-x$ & $x-z \in Q$.

$\therefore$ R is a transitive relation.

conclusion :- R is satisfied a reflexive, symmetric, transitive property

Hence R is an Equivalence relation.

(3) Let $X = \{1, 2, 3, 4\}$ and $R = \{(1,1) (1,4) (4,1) (4,4) (2,2) (2,3)$

(3,2) (3,3)} prove that R is an Equivalence relation draw the graph & matrix.

Sol:- Given relation

$R = \{(1,1)\ (1,4)\ (4,1)\ (4,4)\ (2,2)\ (2,3)\ (3,2)\ (3,3)\}$

1. Reflexive :- $\{(1,1)\ (2,2)\ (3,3)\ (4,4)\} \in R$.

'R' is Satisfied a reflexive relation.

2. Symmetric: $\{(1,4)\ (4,1)\ (2,3)\ (3,2)\} \in R$

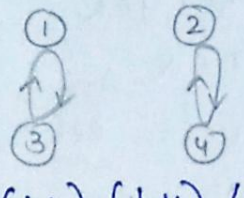
'R' is Satisfied a Symmetric relation.

3. Transitive :- $\left\{\begin{array}{l}(1,4)\ (4,1) \Rightarrow (1,1) \\ (2,3)\ (3,2) \Rightarrow (2,2)\end{array}\right\} \in R$.

'R' is Satisfied a transitive relation.

conclusion:- R is Satisfied about reflexive, Symmetric, Transitive properties and R is Said to be an Equivalence relation.

graph:-



Matrix: $\{(1,1)\ (1,4)\ (4,1)\ (4,4)\ (2,2)\ (2,3)\ (3,2)\ (3,3)\}$

	1	2	3	4
1	1	0	0	1
2	0	1	1	0
3	0	1	1	0
4	1	0	0	1

(4) Let $A = \{5, 6, 7, 8\}$ and $R = \{(x, y) / x > y\}$ Draw the graph of R and also give its matrix.

Sol:- $A = \{(8,5)(8,6)(8,7)(7,5)(7,6)(6,5)\}$

Here $(5,5) \notin R$ or $5 \not R 5$

$\therefore$ 'R' does not Satisfy reflexive, transitive & symmetric Property.

So, it is not a Equivalence relation.

Matrix:

$$\begin{array}{c|cccc} & 5 & 6 & 7 & 8 \\ \hline 5 & 0 & 0 & 0 & 0 \\ 6 & 1 & 0 & 0 & 0 \\ 7 & 1 & 1 & 0 & 0 \\ 8 & 1 & 1 & 1 & 0 \end{array}$$

graph.

* Equivalence relation:- A relation R on a Set X is said to be an Equivalence relation if it is reflexive, Symmetric and transitive.

* Partial order relation:- A relation 'R' on a set 'X' is said to be a partial order relation if it is reflexive, anti symmetric and transitive.

* Inverse relation:- A relation R from a set A to B has an inverse relation R^{-1} from B to A which is defined by

$$R^{-1} = \{(b, a) \mid (a, b) \in R\}$$

* Equivalence classes :- Let 'R' be an Equivalence relation on a set A. The Set of all Elements that are related to an Element "a" of A is called the Equivalence class of "a"

ie $[a]_R = \{ s \in A \mid (a,s) \in R \}$

* Covering and partition of a Set :-

Covering of a set :- Let S be a given set and $A = \{A_1, A_2 \ldots A_n\}$ where Each A_i, $i = 1, 2 \ldots m$ is a Subset of S and $\bigcup_{i=1}^{n} A_i = S$

Then the Set A is called a covering of S and the Sets $A_1, A_2 \ldots A_m$ are Said to be covers.

Ex :- Let $S = \{a, b, c\}$ and consider the following collection of Subsets as

$A_1 = \{\{a,b\}, \{b,c\}\}$ $A_2 = \{\{a\} \{a,c\}\}$ $A_3 = \{\{a\}\{b,c\}\}$

$A_4 = \{\{a,b,c\}\}$ $A_5 = \{\{a\}\{b\}\{c\}\}$ $A_6 = \{\{a\} \{a b\}\{a,c\}\}$

Which of the above Sets are Covering.

Sol :- the sets A_1, A_3, A_4, A_5, A_6 are covering of S but the Set A_2 is not a covering of 'S' Since their union is not S.

Partition of a set :- A partiation of a set "S" is a collection of disjoint non Empty Subsets of S that have S as their union.

Let 'S' be a given Set and $A = \{A_1 A_2 \ldots A_n\}$ where

Each A_i $i = 1, 2 \ldots m$ is a Subset of S.

i, $A_i \neq \emptyset$ for Each i (2) $A_i \cap A_j = \emptyset$ for $i \neq j$ (3)

$$\bigcup_{i=1}^{M} A_i = S$$

Then A is Called partitions of "S"

Partial ordering :- A binary relation R on a set is called Partial ordering or partial order (1) if it is reflexive, anti-symmetric and transitive.

A set s together it with a partial ordering 'R' is called a partially ordered set and poset and is denoted by (S, R)

Problems on Partial ordering :-

(1) Show that the relation ($\geq$ or $=$) is a partial ordering on the set of integers.

Show that the relation " greater than or Equal to " is a partial ordering on the Set of integers.

Sol. Let Z be the Set of all integers and the relation

$R = \geq$

Reflexive :- Since $a \geq a$ for Every integer 'a' the relation $\geq$ is reflexive.

Anti-Symmetric :- Let a and b be any two integers.

Let aRb and $bRa \Rightarrow a \geq b$ and $b \geq a$

$\Rightarrow a = b$.

$\therefore$ The relation $\geq$ is anti Symmetric.

Transitive :- Let a, b and c be any 3 integers

Let aRb, bRc, $\Rightarrow aRc$

$\Rightarrow a \geq b$, $b \geq c \Rightarrow a \geq c$

$\Rightarrow$ The relation $\geq$ is Transitive

Since the relation $\geq$ is reflexive, anti symmetric and transitive.

Hence, $\geq$ is a partial ordering on the set of integers

$\therefore$ ($\geq$) is a poset

(2) Show that the divisibility relation "/" is a partial ordering on the Set of positive integers ?

Sol. Let Z^+ be the Set of positive integers.

Let the relation R be divisibility that is "/"

i, Since a/a for any positive integer, the relation "/" is reflexive.

ii, Let a and b be any two positive integers, if a/b and b/a then $a=b \Rightarrow$ / is a antisymmetric

iii, Let a, b, & c be any 3 positive integers. If a/b & b/c then $a/c \Rightarrow$ The relation "/" is transitive.

Since the relation "/" is reflexive, anti-symmetric and transitive the given relation "/" is a partial ordering on Z^+

* Hasse Diagrams:- In a partially ordered Set $(P, \leq)$ an Element $y \in P$ is Said to cover an Element $x \in P$ if $x < y$ and if there does not Exist any Element $z \in P$ Such that $x \leq z$ and $z \leq y$ that is y covers x

$$\Rightarrow x < y \wedge (x \leq z \leq y \Rightarrow x = z \vee z = y)$$

A Partial ordering $\leq$ on a Set P can be represented by means of a diagram known as Hasse diagram or a partially ordered Set diagram of $(P, \leq)$

To draw a Hasse diagram the following Steps need to be adopted.

1. Each Element is represented by a small circle or dot.
2. The circle for $x \in P$ is drawn below the circle for $y \in P$ if $x < y$ and a line is drawn between x and y if y covers x.
3. If $x < y$ but y does not cover x, then x and y are not connected directly by a Single line. But they are connected through one or more Elements of P.

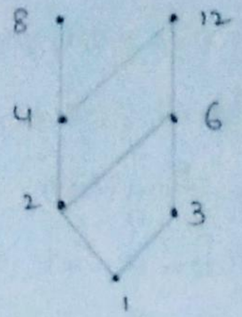
Hence the Set of ordered pairs in $\leq$ Can be Obtained from Such a diagram.

① Draw the Hasse diagram representing the partial ordering $\{(a,b) / a \text{ divides } b\}$ on $\{1, 2, 3, 4, 6, 8, 12\}$

Sol: Let $P = \{1, 2, 3, 4, 6, 8, 12\}$

$(P, \leq) = \{(1,2)(1,3)(1,4)(1,6)(1,8)(1,12)(2,4)(2,6)$
$(2,8)(2,12)(3,6)(3,12)(4,8)(4,12)(6,12)\}$

Hasse diagram of $\{1,2,3,4,6,8,12\}$ as shaon belao.



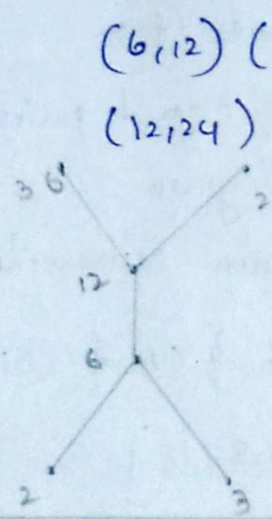
② Let $x = \{2,3,6,12,24,36\}$ and a relation on "$\leq$" be Such that $x \leq y$ if x divides y, Draw the Hasse diagram of $(x, \leq)$

Sol. Given $x = \{2,3,6,12,24,36\}$ $\leq$ is a relation.

The Hasse diagram of $(x, \leq)$ is shaon below.

Given relation $= \{x \leq y$ if x divisible $y\}$

$\{P, \leq\} = \{(2,6)(2,12)(2,24)(2,36)(3,6)$
$(3,12)(3,24)(3,36)$
$(6,12)(6,24)(6,36)$
$(12,24)(12,36)\}$



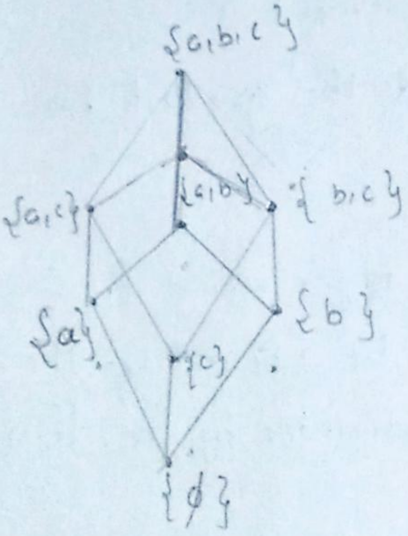
④ Draw the Hasse diagram for the partial ordering $\{(A,B) / A \subseteq B\}$ on the power set $P(S)$ where $S = \{a,b,c\}$

Sol:- $S = \{a, b, c\}$

$P(S) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

$(P(S), \subseteq) = \{(\phi\ \{a\})(\phi\ \{b\})(\phi\ \{c\})(\phi\ \{a,b\})$
$(\phi\ \{a,c\})(\phi\ \{b,c\})(\phi\ \{a,b,c\})$
$(\{a\}\ \{a,b\})(\{a\}\ \{a,c\})$
$(\{b\}\ \{a,b\})(\{b\}\ \{b,c\})(\{b\}, \{a,b,c\})$
$(\{c\}, \{a,c\})(\{c\}\ \{b,c\})(\{c\}\ \{a,b,c\})$

Hasse diagram for the partially ordered Set is shown below.



Hasse diagram for $(P(S), \subseteq)$

* Least member and greatest member :- Let $(P, \leq)$ denote a partially ordered set. If there exists an element $y \in P$ such that $y \leq x$ $\forall x \in P$ then y is called the least member in P relative to the partial ordering $\leq$. If there exist an element $y \in P$ such that $x \leq y$ $\forall x \in P$ then y is called the greatest member in P relative to $\leq$

① Find the greatest lower bound and least upper bound of the sets $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$ if they exist in the poset $(Z^+, /)$.

Sol:- ① An integer is a lower bound of $\{3, 9, 12\}$ if $3, 9, 12$ are divisible by this integer, the only such integers are 1 and 3 since $1/3$ '3' is the greatest lower bound of $\{3, 9, 12\}$

ii, The only lower bound for the set $\{1, 2, 4, 5, 10\}$ with respect to "/" is the element "1"

Hence "1" is the greatest lower bound of $\{1, 2, 4, 5, 10\}$

iii, An integer is an upper bound for $\{3, 9, 12\}$ if and only if it is divisible by 3, 9 and 12. The integers with this property are those divisible by the least common multiple of $3, 9, 12$

$$\text{LCM} = 3 \underline{|\,3, 9, 12}$$
$$\quad 1, 3, 4$$