3. Recurrence Relations

* Generating Function Sequence :-

→ A Generating Function sequence contains numeric constants and statement variable.

→ Generating function is denoted with $G(x)$.

→ The generation function sequence is defined as

$$G(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 - - - + a_n x^n$$

$$G(x) = \sum_{i=0}^{n} a_i x^i$$

where $a_0, a_1, - - a_n$ are Numeric Constants

$x^0, x^1 . . . x^n$ are Statement Variables

* Number Function Sequence :-

→ Number function sequence contains a collection of numeric constants. Number function sequence is denoted by $N(x)$ and it can be defined as

$$N(x) = \{a_0, a_1, a_2, - - a_n\}$$

* Taylor Series formulas :-

1) $x^0 + x^1 + x^2 + \cdots + x^n = \frac{1}{1-x}$

2) $1.x^0 + 2.x^1 + 3.x^2 + \cdots (n+1)x^n = \frac{1}{(1-x)^2}$

3) $x^0 - x^1 + x^2 - x^3 + \cdots + (-1)^n . x^n = \frac{1}{1+x}$

4) $1.x^0 - 2.x^1 + 3.x^2 - \cdots \cancel{(-1)^n \times (n+1) x^n} = \frac{1}{(1+x)^2}$

$(-1)^n (n+1) x^n$

5) $1+2+3+\cdots+n = \frac{n(n+1)}{2}$

6) $1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$

7) $1^3+2^3+3^3+\cdots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Examples :-

① Let $N(x) = \{2,2,2,2,\ldots\}$ then find $G(x)$.

Sol: Given, $N(x) = \{2,2,2,2,\ldots\}$

$\because G(x) = a_0x^0 + a_1x^1 + a_2x^2 + \cdots + a_nx^n$

Here, $a_0 = 2$, $a_1 = 2$, $a_2 = 2$, $a_3 = 2$, ...

$$G(x) = 2x^0 + 2.x^1 + 2.x^2 + \cdots + 2x^n$$

$$= 2(x^0 + x^1 + x^2 + \cdots x^n)$$

$$= 2\left[\frac{1}{1-x}\right]$$

$$= \frac{2}{1-x}$$

Substitute constant values of $a_0, a_1, \ldots a_n$ into Generating function.

$$\therefore \boxed{G(x) = \frac{2}{1-x}}$$

② Given $N(x) = \{0,0,0,1,1,1,1,1,\ldots\}$ then find $G(x)$.

Sol: Given, $N(x) = \{0,0,0,1,1,1,1,1,\ldots\}$

here, $a_0 = 0,\ a_1 = 0,\ a_2 = 0,\ a_3 = 1,\ a_4 = 1, \ldots a_n = 1$

$\because G(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots + a_n x^n$

Substitute Constant Values of $a_0, a_1, \ldots a_n$ into Generating function.

~~where,~~

$$G(x) = 0\,x^0 + 0.x^1 + 0.x^2 + 1.x^3 + 1.x^4 + 1.x^5 + \ldots x^n$$

$$G(x) = x^3 + x^4 + x^5 + \cdots + x^n$$

$$= [x^0 + x^1 + x^2 + x^3 + x^4 + \ldots + x^n] - (x^0 + x^1 + x^2)$$

$$= \left(\frac{1}{1-x}\right) - (x^2 + x + 1)$$

(or)

$$G(x) = x^3 + x^4 + x^5 + \cdots + x^n$$

$$= x^3 [x^0 + x^1 + x^2 \cdots + x^{n-3}]$$

$$= x^3 \left[\frac{1}{1-x}\right]$$

$$= \frac{x^3}{1-x}$$

③ Let $N(x) = \{1,2,3,4 \ldots\}$ then find $G(x)$

Sol: Given, $N(x) = \{1,2,3,4,\ldots\}$.

here $a_0 = 1,\ a_1 = 2,\ a_2 = 3,\ a_3 = 4$.

$$G(x) = a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots + a_n x^n$$

Substitute Constant values of $a_0, a_1, \ldots a_n$ into Generating function.

$G(x) = 1x^0 + 2 \cdot x^1 + 3 \cdot x^2 + 4 \cdot x^3 +$ $(n+1)x^{n-1}$

$G(x) = \frac{1}{(1-x)^2}$

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④ Let $N(x) = \{0,1,0,1,0, \ldots\}$ then find $G(x)$

Sol: Given, $N(x) = \{0,1,0,1,0 \ldots\}$

here $a_0 = 0, a_1 = 1, a_2 = 0, a_3 = 1, \ldots$

$\therefore \boxed{G(x) = a_0 x^0 + a_1 x^1 + \ldots + a_n x^n}$

Substitute constant values into Generating function

$G(x) = 0x^0 + 1 \cdot x^1 + 0 \cdot x^2 + 1 \cdot x^3 + 0 \cdot x^4 + 1 \cdot x^5 + \ldots$

$= x + x^3 + x^5 + x^7 + \ldots$

We have,

$x^0 + x^1 + x^2 + x^3 \ldots + x^n = \frac{1}{1-x} \rightarrow ①$

$x^0 - x^1 + x^2 - x^3 + \ldots + (-1)^n x^n = \frac{1}{1+x} \rightarrow ②$

eq ① - eq ②

$\Rightarrow 2x + 2x^3 + 2x^5 + 2x^7 + \ldots = \frac{1}{1-x} - \frac{1}{1+x}$

$\Rightarrow 2[x + x^3 + x^5 + x^7 + \ldots] = \frac{1+x-1+x}{(1-x^2)}$

$\Rightarrow \cancel{2}(x + x^3 + x^5 + \ldots) = \frac{\cancel{2}x}{(1-x)^2}$

$x + x^3 + x^5 + \ldots = \frac{x}{(1-x)^2}$

(5) Given $G(x) = \frac{5}{1-x}$ then find $N(x)$.

Sol:

Given, $G(x) = \frac{5}{1-x}$

$= 5\left(\frac{1}{1-x}\right)$

$= 5\left[x^0 + x^1 + x^2 + \dots + x^n\right]$

$G(x) = 5x^0 + 5x^1 + 5x^2 + \dots + 5x^n$

here $a_0 = 5,\ a_1 = 5,\ a_2 = 5, \dots\ a_n = 5$

$\therefore N(x) = \{a_0, a_1, a_2, \dots\}$

$= \{5, 5, 5, \dots\}$

(6) Given $G(x) = \frac{x^2}{1-x}$ then find $N(x)$.

Sol:

Given, $G(x) = \frac{x^2}{1-x}$

$= x^2\left(\frac{1}{1-x}\right)$

$= x^2(x^0 + x^1 + x^2 + \dots + x^n)$

$= x^2 + x^3 + x^4 + \dots + x^{2+n}$

$G(x) = 0.x^0 + 0.x^1 + 1.x^2 + 1.x^3 + 1.x^4 + \dots$

here, it is in the form of $a_0x^0 + a_1x^1 + \dots$

$a_0 = 0,\ a_1 = 0,\ a_2 = 1,\ a_3 = 1 \dots,\ a_n = 1$

$\therefore N(x) = \{0, 0, 1, 1, 1, \dots\}$

(7) $G(x) = \frac{1}{(1-x)^2}$ then find $N(x)$

Sol:- $G(x) = \frac{1}{(1-x)^2}$

$G(x) = 1.x^0 + 2.x^1 + 3.x^2 + \cdots (n+1).x^n$

here $a_0 = 1$, $a_1 = 2$, $a_2 = 3$, ...

$N(x) = \{1, 2, 3, \cdots\}$

* Partial Fractions :-

⇒ The Partial Fractions are Calculated for any fraction by following these steps.

1. Find the characterstic roots if necessary
2. Partition the given expression into Several parts depends on denominator and find Partial fraction Variable values.

Ex:- (1) Find the partial fraction for $\frac{1+6x}{(4x+1)(5x+1)}$

Sol:- The Given, $\frac{1+6x}{(4x+1)(5x+1)}$ it can be written as

$$\frac{(1+6x)}{(4x+1)(5x+1)} = \frac{A}{4x+1} + \frac{B}{5x+1}$$

$$\frac{1+6x}{(4x+1)(5x+1)} = \frac{A(5x+1) + B(4x+1)}{(4x+1)(5x+1)}$$

$1+6x = A(5x+1) + B(4x+1)$

compare Coeff of 'x' , compare Const.

$6 = 5A + 4B \rightarrow (1)$ $\qquad 1 = A + B \rightarrow (2)$

Solve (1) & (2)

$$\begin{aligned} \cancel{5A} + 4B &= 6 \\ \cancel{5A} + 5B &= 5 \\ (-)\quad (-) \quad & \quad (-) \\ \hline -B &= 1 \end{aligned} \Rightarrow \boxed{B=-1}$$

$A + B = 1$

$A = 1 - B = 1-(-1) = 2 \Rightarrow \boxed{A=2}$

$$\therefore \frac{1+6x}{(4x+1)(5x+1)} = \frac{2}{4x+1} - \frac{1}{5x+1}$$

(2) Find the Partial fractions for $\frac{1+7x}{1+5x+6x^2}$

Sol: Given, $\frac{1+7x}{1+5x+6x^2}$

$$= \frac{1+7x}{(3x+1)(2x+1)}$$

$(\because 6x^2+5x+1$

$6x^2+3x+2x+1$

$3x(2x+1)+2x+1$

$(3x+1)(2x+1))$

$$\frac{1+7x}{(3x+1)(2x+1)} = \frac{A}{(3x+1)} + \frac{B}{(2x+1)}$$

$$\frac{1+7x}{(3x+1)(2x+1)} = \frac{A(2x+1) + B(3x+1)}{(3x+1)(2x+1)}$$

$$1+7x = A(2x+1) + B(3x+1)$$

$7 = 2A + 3B \rightarrow (1)$ $\qquad 1 = A + B \rightarrow (2)$

Sol (1) & (2) $\Rightarrow$

$$\begin{aligned} \cancel{2A} + 3B &= 7 \\ \cancel{2A} + 2B &= 2 \\ (-)\quad (-) \quad & \quad (-) \\ \hline B &= 5 \end{aligned}$$

$A + 5 = 1$

$A = -4$

$\boxed{A=-4, B=5}$

(3) Find the partial fractions $\frac{1+5x}{x^2-x-1}$

Sol:

x^2-x-1

$$root\ 1 = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$root\ 1 = \frac{-(-1)+\sqrt{1-4(1)(-1)}}{2(1)}$$

$$= \frac{1+\sqrt{5}}{2}$$

$$root\ 2 = \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{1-\sqrt{5}}{2}$$

$$x^2-x-1 = \left[x-\left(\frac{1+\sqrt{5}}{2}\right)\right]\left[x-\left(\frac{1-\sqrt{5}}{2}\right)\right]$$

$$\frac{1+5x}{x^2-x+1} = \frac{1+5x}{\left(x-\left(\frac{1+\sqrt{5}}{2}\right)\right)}$$

(4) Find the partial fraction for $\frac{1+7x}{1+6x+9x^2}$

Sol:

$$9x^2+6x+1 = 9x^2+3x+3x+1$$

$$= 3x(3x+1)+1(3x+1)$$

$$= (3x+1)(3x+1) = (3x+1)^2$$

$$\frac{1+7x}{1+6x+9x^2} = \frac{A}{3x+1} + \frac{B}{(3x+1)^2}$$

$$1+7x = A(3x+1) + \cancel{B(3x+1)}$$

$$1+7x = A(3x+1) + B$$

$7 = 3A$ $\quad\quad$ $1 = A + B$

$A = \frac{7}{3}$ $\quad\quad$ $B = 1 - A = 1 - \frac{7}{3} = -\frac{4}{3}$

$$\boxed{A = \frac{7}{3},\ B = -\frac{4}{3}}$$

(5) Find the P.F for $\dfrac{1+7x}{1+4x+4x^2}$

Sol

$$1+4x+4x^2 = 4x^2+2x+2x+1$$
$$= 2x(2x+1) + 1(2x+1)$$
$$= (2x+1)^2$$

$$\frac{1+7x}{1+4x+4x^2} = \frac{A}{2x+1} + \frac{B}{(2x+1)^2}$$

$$1+7x = A(2x+1) + B$$

$2A = 7$ $\quad\quad$ $1 = A + B$

$A = \frac{7}{2}$ $\quad\quad$ $B = 1 - \frac{7}{2} = \frac{2-7}{2} = -\frac{5}{2}$

$$\boxed{A = \frac{7}{2},\ B = -\frac{5}{2}}$$

* Recurrence Relation:-

→ A Recurrence Relation is an equation that defines a sequence based on a rule that gives the next term as a function of previous terms.

Ex: $x_{n'} = x_n + x_{n-1}$ is a Recurrence relation to produce fibonacci Series.

→

→ Recurrence Relation can be classified into 2 categories

1. Homogenous Recurrence Relation
2. Non-Homogenous " "

* **Homogenous Recurrence Relation:-**

→ A Recurrence Relation is said to be Homogenous if the right-hand side production term is equal to zero ie. $\boxed{a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots a_n x^n = 0}$

* **Non-Homogenous Recurrence Relation:-**

→ A Recurrence Relation is said to be Non-Homo if the RHS production term is not equal to zero ie. $\boxed{a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots + a_n x^n \neq 0}$

→ Identify the Recurrence Relations either Homo or Non-Homo.

(i) $a_n - a_{n-1} + a_{n-2} = 0$ → Homogenous

(ii) $a_n - a_{n-1} = a_{n-2}$ ⇒ Homogenous.

⇒ $a_n - a_{n-1} - a_{n-2} = 0$

(iii) $a_n - a_{n-1} = 5$ ⇒ Non-Homogenous

(iv) $a_n - a_{n-1} + 5a_{n-2} = n + n^2$ → Non-Homo

(v) $a_n - a_{n-1} + 5a_{n-2} - r^n = 0$ → Non-Homo

* The Homogenous Recurrence Relations are Solved by using three methods.

1. Substitution Method (or) Iterative Method
2. Characterstic Roots Method
3. Generating function Method.

* 1. Substitution (or) Iterative Method :-

The Substitution Method solve the given Recurrence relation eqn by Substituting Initial Condition values in given Recurrence Relation.

Ex: Find the first 5 terms for the Recurrence relation $a_n = 2a_{n-1}$ with Initial Condition $a_0 = 5$

Sol: Given, $a_n = 2.a_{n-1}$ and $a_0 = 5$

$n = 1, 2, 3, 4, 5$

$n=1, \; a_1 = 2a_0 = 2(5) = 10$

$n=2, \; a_2 = 2a_1 = 2 \times 10 = 20$

$n=3, \; a_3 = 2a_2 = 2 \times 20 = 40$

$n=4, \; a_4 = 2a_3 = 2 \times 40 = 80$

$n=5, \; a_5 = 2 \times a_4 = 2 \times 80 = 160$

Ex:(2) Find the first 5 terms for the recurrence relation $a_n = 2a_{n-2} + a^2_{n-1}$ and Initial condition values are $a_0 = 1, \; a_1 = 2$

Sol:- Given, $a_n = 2a_{n-2} + a_{n-1}^2$; $a_0 = 1, a_1 = 2$

$n=2$, $a_2 = 2a_0 + a_1^2$

$= 2(1) + (2)^2 = 2 + 4 = 6$

$n=3$, $a_3 = 2a_1 + a_2^2$

$= 2(2) + (6)^2 = 4 + 36 = 40$

$n=4$, $a_4 = 2a_2 + a_3^2$

$= 2(6) + (40)^2 = 12 + 1600 = 1612$

$n=5$, $a_5 = 2a_3 + a_4^2$

$= 2\times 40 + (1612)^2 = 80 + 2598544$

$= 2598624.$

$n=6$, $a_6 = 2a_4 + a_5^2$

$= 2\times 1612 + (2598624)^2$

$= 3224 + (6.7\times 10^{12})$

Ex:- 3 Solve the Recurrence relation for the eqn $a_n = a_{n-1} + n$, $n \geq 1$ and $a_0 = 5$ by using Iterative approach.

Sol:- Given $a_n = a_{n-1} + n$, $a_0 = 5$

Let $n=1$ then $a_n = a_{n-1} + 1$

$a_1 = a_0 + 1 = 5 + 1 = 6$

$a_1 = a_0 + 1 \longrightarrow 1$

Let $n=2$ then $a_2 = a_1 + 2$

$a_2 = a_0 + 1 + 2 \longrightarrow 2$

Let $n=3$, then $a_3 = a_2 + 3$

$a_3 = a_0 + 1 + 2 + 3 \longrightarrow 3$

Let $n=4$, $a_4 = a_3 + 4$

$a_4 = a_0 + 1 + 2 + 3 + 4 \longrightarrow (4)$

Let $n=5$, $a_5 = a_4 + 5$

$a_5 = a_0 + 1 + 2 + 3 + 4 + 5$

for $n=n \Rightarrow a_n = a_0 + (1 + 2 + 3 + \cdots + n)$

$$a_n = 5 + \left[\frac{n(n+1)}{2}\right]$$

$$a_n = \frac{10 + n^2 + n}{2}$$

(4) Solve $a_n = a_{n-1} + n^2$; $n \geq 1$ and $a_0 = 5$ by using Iterative approach.

Sol: Given, $a_n = a_{n-1} + n^2$, $a_0 = 5$

Let $n=1$, $a_1 = a_0 + 1^2$

$a_1 = a_0 + 1 \longrightarrow (1)$

Let $n=2$, $a_2 = a_1 + 2^2$

$a_2 = a_0 + 1^2 + 2^2 \longrightarrow (2)$

Let $n=3$, $a_3 = a_2 + 3^2$

$= a_0 + 1^2 + 2^2 + 3^2 \longrightarrow (3)$

$n=4$, $a_4 = a_3 + 4^2$

$a_4 = a_0 + 1^2 + 2^2 + 3^2 + 4^2 \longrightarrow (4)$

$n=5$, $a_5 = a_4 + 5^2$

$= a_0 + 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \longrightarrow (5)$

for $n=n \Rightarrow a_n = a_0 + (1^2 + 2^2 + 3^2 + \cdots + n^2)$

$$= a_0 + \frac{n(n+1)(2n+1)}{6}$$

$$= 5 + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{5(6) + (n^2+n)(2n+1)}{6}$$

$$= \frac{30 + 2n^3 + n^2 + 2n^2 + n}{6}$$

$$a_n = \frac{2n^3 + 3n^2 + n + 30}{6}$$

(5) Given $a_n = a_{n-1} + a_{n-3}$ & $a_0 = 1, a_1 = 2, a_2 = 0$: solve eqn by using Iterative appr

Sol

$$a_n = a_{n-1} + a_{n-3}$$

for $n=3$, $a_3 = a_2 + a_0$

$= 0 + 1 = 1 \Rightarrow \boxed{a_3 = 1}$

let $n=4$, $a_4 = a_3 + a_1$

$= 1 + 2 = 3 \Rightarrow \boxed{a_4 = 3}$

let $n=5$, $a_5 = a_4 + a_2$

$= 3 + 0 = 3 \Rightarrow \boxed{a_5 = 3}$

let $n=6$, $a_6 = a_5 + a_3$

$= 3 + 1 = 4 \Rightarrow \boxed{a_6 = 4}$

Let $n=7$, $a_7 = a_6 + a_4$

$= 4 + 3 = 7 \Rightarrow \boxed{a_7 = 7}$

(Imp)

Characterstic Roots Method :-

→ characterstic Roots Method is used to solve the Homogenous Recurrence relation.

→ The following are the steps involved in the characterstic roots method.

Step 1: Find the order of given recurrence relation
ie. Order = Highest subscript − Lowest subscript of given recurrence relation

Step 2: Substitute order of given recurrence relation in place of the variable 'n' in a given recurrence relation.

Step 3: Convert the subscript value into power for the given recurrence relation. The converted eqn is also called as "Characterstic Equation"

Step 4: Find the characterstic roots for the characterstic equation.

(i) If the characterstic roots are similar then

$$a_n = r^n [c_1 + c_2.n + c_3.n^2 + \ldots]$$

(ii) If the characterstic roots are different, then

$$a_n = c_1.r_1^n + c_2.r_2^n + c_3.r_3^n \ldots$$

(iii) From the characterstic roots, ~~same~~ (some) roots are equal but not all, Then

$$a_n = r_1^n [c_1 + c_2.n + c_3.n^2 + \ldots] + d_1 r_2^n + d_2 r_3^n + \ldots$$

PTO

Step 5:- Find the values of $c_1, c_2, c_3, \ldots$ by using Initial condition values.

Ex:- Solve the recurrence relation $a_n - 7a_{n-1} + 12a_{n-2} = 0$ with Initial condition values $a_0 = 1, a_1 = 2$ by using characteristic roots method.

Sol: Given,

$$a_n - 7 \cdot a_{n-1} + 12 \cdot a_{n-2} = 0$$

$$\boxed{\text{order} = \text{Highest subscript} - \text{Lowest subscript}}$$

$$= n - (n-2)$$

$$\text{order} = 2$$

Substitute 'order value' in place of n

$$a_2 - 7a_1 + 12a_0 = 0$$

Convert given R.R into characteristic eqn

$$a^2 - 7a^1 + 12a^0 = 0$$

$$a^2 - 7a + 12 = 0$$

$$a^2 - 3a - 4a + 12 = 0$$

$$a(a-3) - 4(a-3) = 0$$

$$(a-3)(a-4) = 0$$

$$\boxed{a = 3, 4}$$

$$r_1 = 3 \text{ and } r_2 = 4$$

Roots are different $\Rightarrow a_n = c_1 \cdot r_1^n + c_2 \cdot r_2^n$

$$a_n = c_1(3)^n + c_2(4)^n$$

For $n=0$, $a_n = c_1(3)^n + c_2(4)^n$

$a_0 = c_1(3)^0 + c_2(4)^0$

$\boxed{1 = c_1 + c_2}$ $\quad (\because$ Given $a_0 = 1)$

For $n=1$, $a_1 = c_1(3)^1 + c_2(4)^1$

$\boxed{2 = 3c_1 + 4c_2}$

Solve these two eqns

$3c_1 + 4c_2 = 2$

$3c_1 + 3c_2 = 3$

$(-) \quad (-) \quad (-)$

$c_2 = -1$

$c_1 - 1 = 1$

$\boxed{c_1 = 2}$, $c_2 = -1$

$a_n = 2(3)^n - 1(4)^n$

$\Rightarrow \boxed{a_n = 2(3)^n - 4^n}$

(2) Solve recurrence relation $a_n - 9a_{n-1} + 20a_{n-2} = 0$ with initial values $a_0 = 1$, $a_1 = 2$ by using char. roots methods.

Sol: Given, $a_n - 9a_{n-1} + 20a_{n-2} = 0$

Order = Highest precedence - lowest prece

$= n-(n-2) = \cancel{n} - \cancel{n} + 2 = 2$

$\boxed{\text{Order} = 2}$

Sub order value in place of 'n'

$a_2 - 9a_1 + 20a_0 = 0$

convert given R.R into characterstic eqn

$a^2 - 19a^1 + 20a^0 = 0$

$a^2 - 19a + 20 = 0$

$a^2 - 4a - 5a + 20 = 0$

$a(a-4) - 5(a-4) = 0$

$a = 4,\ a = 5$

$\therefore \boxed{r_1 = 4} \quad \boxed{r_2 = 5}$

$\because$ Roots are different then

$$\boxed{a_n = c_1 \cdot r_1^n + c_2 r_2^n}$$

$n = 0,$

$a_0 = c_1(4)^0 + c_2(5)^0$

$1 = c_1 + c_2 \longrightarrow (1)$

$n = 1,\quad a_1 = c_1(4)^1 + c_2(5)^1$

$2 = 4c_1 + 5c_2 \longrightarrow (2)$

Solve (1) & (2)

$\cancel{4c_1} + 5c_2 = 2$

$\cancel{4c_1} + 4c_2 = 4$

$(-) \quad (-) \quad (-)$

$c_2 = -2$

$\boxed{c_2 = -2}$

$c_1 - 2 = 1 \Rightarrow \boxed{c_1 = 3}$

$a_n = (3)(4)^n + (-2)(5)^n.$

3) Solve R.R $a_n - 4a_{n-1} + 4a_{n-2} = 0$, $a_0 = 1$ and $a_1 = 2$ by using characterstic roots method.

Sol: Given, $a_n - 4a_{n-1} + 4a_{n-2} = 0$

order $= n-(n-2) = 2$

sub 'order' in place of 'n'

$$a_2 - 4a_1 + 4a_0 = 0$$

convert R.R into char. eqn

$$a^2 - 4a^1 + 4a^0 = 0$$

$$a^2 - 4a + 4 = 0$$

$$a^2 - 2a - 2a + 4 = 0$$

$$a(a-2) - 2(a-2) = 0$$

$$a = 2, 2$$

$$r_1 = r_2 = \cancel{x} = 2$$

Roots are Equal then

$$\boxed{a_n = r^n[c_1 + c_2 . n + c_3 n^2 + \dots]}$$

$n=0$,

$$a_0 = 2^0 [c_1 + c_2(0) + 0 + \dots]$$

$$1 = 1[c_1]$$

$$\boxed{c_1 = 1}$$

$n=1$,

$$a_1 = 2^1 [c_1 + c_2(1) + c_3(1) + \dots]$$

$$\cancel{2} = \cancel{2}[1 + c_2]$$

$$c_2 = 1-1 = 0 \Rightarrow \boxed{c_2 = 0}$$

$\therefore$ eqn is $a_n = 2^n[1 + 0.(n)]$

$$a_n = 2^n$$

(4) Solve R.R $a_n - 10a_{n-1} + 33a_{n-2} - 36a_{n-3} = 0$

with $a_0 = 1, a_1 = 1, a_2 = -23$

Given, $a_n - 10a_{n-1} + 33a_{n-2} - 36a_{n-3} = 0$

order $= n - (n-3)$

$\boxed{\text{order} = 3}$

Sub 'order' in place of 'n'

$a_3 - 10a_2 + 33a_1 - 36a_0 = 0$

Convert R.R into char. eqn

$\cdot\ a^3 - 10a^2 + 33a^1 - 36a^0 = 1$

Find roots by Trail & Error Method.

$$\begin{array}{c|cccc} 1 & 1 & -10 & 33 & -36 \\ & 0 & 1 & -9 & 24 \\ \hline & 1 & -9 & 24 & 12 \end{array} \times \qquad \begin{array}{c|cccc} -1 & 1 & -10 & 33 & -36 \\ & 0 & -1 & 11 & -22 \\ \hline & 1 & -11 & 22 & 58 \end{array} \times$$

$1, -1 \times$

$2, -2 \times$

$$\begin{array}{c|cccc} 3 & 1 & -10 & 33 & -36 \\ & 0 & 3 & -21 & 36 \\ \hline & 1 & -7 & 12 & \boxed{0} \end{array}$$

$(a-3)(a^2 - 7a + 12) = 0$

$(a-3)(a^2 - 3a - 4a + 12) = 0$

$(a-3)[(a-3)(a-4)] = 0$

$a = 3, \quad a = 3 \quad a = 4$

$r_1 = 3, \quad r_2 = 3, \quad r_3 = 4$

Some roots are equal & some are different.

$$\boxed{a_n = r^n[c_1 + c_2 . n] + c_3 r_3^n}$$

$$a_n = 3^n[c_1 + c_2 . n] + c_3 [4]^n$$

$\underline{n=0},$

$$a_0 = r^n[c_1 + c_2 . n] + c_3 r_3^n$$

$$1 = 3^0[c_1 + c_2(0)] + c_3(4)^0$$

$$= 1[c_1] + c_3$$

$$\boxed{1 = c_1 + c_3} \rightarrow (1) \quad c_3 = 1 - c_1$$

$\underline{n=1}$

$$a_1 = 3^1[c_1 + c_2(1)] + c_3(4)^1$$

$$= 3[c_1 + c_2] + 4c_3$$

$$= 3[1 - c_3 + c_2] + 4c_3$$

$$1 = 3c_2 + c_3 + 3$$

$$\boxed{3c_2 + c_3 = -2} \rightarrow (2) \quad 3c_2 = c_1 - 3$$

$\underline{n=2}$

$$a_2 = 3^2[c_1 + c_2[2]] + c_3[4]^2$$

$$-23 = 9[c_1 + 2c_2] + c_3(16)$$

$$-23 = 9[c_1 + 2c_2] + 16c_3$$

$$\boxed{9c_1 + 18c_2 + 16c_3 = -23} \longrightarrow (3)$$

$$\times[9(1 - c_3) + 6[-c_3 - 2] + 16c_3 = -23$$

$$9 - 9c_3 - 6c_3 - 12 + 16c_3 = -23$$

$$\cancel{+}27c_3 = \cancel{+}14 \Rightarrow \boxed{c_3 = \frac{14}{27}}$$

$$c_1 = 1 - c_3 = 1 - \frac{14}{27} = \frac{27-14}{27} = \boxed{\frac{13}{27}} = c_1$$

$$3c_2 = -\frac{14}{27} - 2 = -\left(\frac{14+54}{27}\right) = -\left(\frac{68}{27}\right)$$

$$c_2 = -\frac{68}{27 \times 3} = \frac{-68}{81} \Rightarrow \boxed{c_2 = \frac{-68}{81}}$$

$c_3 = 1 - c_1$

$3c_2 = c_1 - 3$

Eq(3) $\Rightarrow$ $9c_1 + 18c_2 + 16c_3 = -23$

$9c_1 + 6(c_1 - 3) + 16(1 - c_1) = -23$

$9c_1 + 6c_1 - 18 + 16 - 16c_1 = -23$

$15c_1 - 16c_1 - 2 = -23$

$-c_1 = -23 + 2 \quad = -21$

$\boxed{c_1 = 21}$

$c_3 = 1 - c_1$

$= 1 - 21 = -20 \Rightarrow \boxed{c_3 = -20}$

$3c_2 = c_1 - 3$

$= 21 - 3$

$3c_2 = 18 \Rightarrow \boxed{c_2 = 6}$

$a_n = 3^n[21 + 6 \cdot n] + (-20)[4]^n$

(5) Solve the R.R $a_{n+2} = a_n + a_{n+1}$, $a_2 = 2$ $a_3 = 3$, by using char. root method.

Sol: Given, $a_{n+2} = a_n + a_{n+1}$

order $= n + 2 - n$

$\boxed{\text{order} = 2}$

Sub 'order' in place of 'n'.

$a_4 = a_2 + a_3$

$a_4 - a_3 - a_2 = 0$

Convert RR into char. Eqn

$a^4 - a^3 - a^2 = 0$

$$\begin{array}{c|ccccc} 2 & 1 & -1 & -1 & 0 & 0 \\ & 0 & -2 & -6 & & \\ \hline & 1 & -3 & & & \end{array}$$

$a^2(a^2 - a - 1) = 0$

$a^2 = 0$ $\quad$ $a^2 - a - 1 = 0$

$r_1 = 0, r_2 = 0$ $\quad$ Roots $= \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{1 \pm \sqrt{1 - 4(1)(-1)}}{2(1)}$

$= \dfrac{1 \pm \sqrt{5}}{2}$

$r_1 = 0, r_2 = 0,$ $\quad r_3 = \dfrac{1+\sqrt{5}}{2}$, $r_4 = \dfrac{1-\sqrt{5}}{2}$

$r = r_1 = r_2 = 0$

Some roots are equal and some are different.

$$\boxed{a_n = r^n[c_1 + c_2 . n] + d_1 . r_2^n + d_2 . r_3^n}$$

$a_n = 0[c_1 + c_2 . n] + d_1\left[\dfrac{1+\sqrt{5}}{2}\right]^n + d_2\left(\dfrac{1-\sqrt{5}}{2}\right)^n$

$\underline{n = 2}$,

$a_2 = 0[c_1 + c_2(2)] + d_1\left(\dfrac{1+\sqrt{5}}{2}\right)^2 + d_2\left(\dfrac{1-\sqrt{5}}{2}\right)^2$

$2 = d_1\left[\dfrac{1+\sqrt{5}}{2}\right]^2 + d_2\left[\dfrac{1-\sqrt{5}}{2}\right]^2$

$= d_1\left(\dfrac{1+5+2\sqrt{5}}{4}\right) + d_2\left(\dfrac{1+5-2\sqrt{5}}{4}\right)$

$8 = (6+2\sqrt{5})d_1 + (6-2\sqrt{5})d_2$

$$8 = (6 + 2\sqrt{5})d_1 + (6 - 2\sqrt{5})d_2 \longrightarrow (i)$$

$n=3$,

$$a_3 = d_1\left(\frac{1+\sqrt{5}}{2}\right)^3 + d_2\left(\frac{1-\sqrt{5}}{2}\right)^3$$

$$3 = d_1\left[\frac{1+5\sqrt{5}+3\sqrt{5}+15}{8}\right] +$$

$$d_2\left[\right.$$

(6) Solve the fibonacci Series recurrence relation by using char. root method.

Sol: Fibonacci $\Rightarrow$ 0 1 1 2 3 5 8

a_{n-2} a_{n-1} a_n

$$a_n = a_{n-1} + a_{n-2}$$

Here $a_0 = 0$; $a_1 = 1$

$\Rightarrow$ $a_n = a_{n-1} + a_{n-2}$

$a_n - a_{n-1} - a_{n-2} = 0$

Order $= n-(n-2)$

$\boxed{\text{order} = 2}$

Sub 'order' in place of 'n'

$a_2 = a_1 + a_0$

$a_2 - a_1 - a_0 = 0$

Convert RR into char. Eqn.

$a^2 - a^1 - a^0 = 0$

$a^2 - a - 1 = 0$

Roots $= \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$= \dfrac{1 \pm \sqrt{1-4(1)(-1)}}{2(1)} = \dfrac{1 \pm \sqrt{5}}{2}$

$r_1 = \dfrac{1+\sqrt{5}}{2}$, $r_2 = \dfrac{1-\sqrt{5}}{2}$

Roots are Different.

$$\boxed{a_n = c_1 \cdot r_1^n + c_2 r_2^n + c_3 r_3^n + \ldots}$$

$\underline{n=0}$, $a_n = c_1 r^n + c_2 r^n$

$a_0 = c_1 r_1^0 + c_2 . r_2^0$

$0 = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^0 + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^0$

$0 = c_1(1) + c_2(1)$

$\boxed{c_1 = -c_2}$

$\underline{n=1}$, $a_1 = c_1(r_1)^1 + c_2(r_2)^1$

$1 = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^1 + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^1$

$1 = c_1 \left(\frac{1+\sqrt{5}}{2}\right) - c_1 \left(\frac{1-\sqrt{5}}{2}\right)$

$= \frac{c_1 + \sqrt{5} c_1}{2} - \left[\frac{c_1 - \sqrt{5} c_1}{2}\right]$

$2 = c_1 + \sqrt{5} c_1 - c_1 + \sqrt{5} c_1$

$\cancel{2} = \cancel{2} \sqrt{5} c_1 \Rightarrow \boxed{c_1 = \frac{1}{\sqrt{5}}}$

$c_2 = -1/\sqrt{5}$

$\therefore a_n = \left(\frac{1}{\sqrt{5}}\right)\left[\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{-1}{\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$

* Generating Function Method:-

- Generating function Method is also used to solve the homogenous R.R.
- The following are the steps involved in Generating function Method.

Step 1: Find the order of given recurrence relation

order = highest subscript - Lowest subscript of given r.r.

step 2: Multiply with $\sum_{n=order}^{\infty} x^n$ on both sides of given R.R.

step 3: Find the generating function for each production term in R.R.

step 4: Find the value of $G(x)$

step 5: calculate partial fractions of $G(x)$ if necessary.

① Solve the R.R $a_n - 7a_{n-1} + 10a_{n-2} = 0$; $a_0 = 1$, $a_1 = 2$ by using Generating function Method.

Sol: Given, $a_n - 7a_{n-1} + 10a_{n-2} = 0$

$$\text{order} = n - (n-2) = 2$$

Multiply given R.R with $\sum_{n=2}^{\infty} x^n$ on both sides

$$\sum_{n=2}^{\infty} x^n [a_n - 7a_{n-1} + 10a_{n-2}] = 0$$

$$\sum_{n=2}^{\infty} x^n a_n - 7\sum_{n=2}^{\infty} x^n a_{n-1} + 10\sum_{n=2}^{\infty} x^n a_{n-2} = 0 \longrightarrow (1)$$

$$\sum_{n=2}^{\infty} x^n a_n = x^2 a_2 + x^3 a_3 + x^4 a_4 + \cdots$$

$$= (a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \cdots) - a_0 x^0 - a_1 x^1$$

$$= G(x) - a_0 x^0 - a_1 x^1$$

$$= G(x) - 1x^0 - 2x^1$$

$$= G(x) - 1 - 2x \longrightarrow (2)$$

$$\sum_{n=2}^{\infty} x^n a_{n-1} = (a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots)$$

$$= x[a_1 x + a_2 x^2 + a_3 x^3 + \cdots]$$

$$= x\big[[a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots] - a_0 x^0\big]$$

$$= x[G(x) - 1] \qquad (\because a_0 = 1)$$

$$= xG(x) - x \longrightarrow (3)$$

$$\sum_{n=2}^{\infty} x^n a_{n-2} = a_0 x^2 + a_1 x^3 + a_2 x^4 + a_3 x^5 + \cdots$$

$$= x^2[a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots]$$

$$= x^2[G(x)] \longrightarrow (4)$$

Substitute (2), (3), (4) in equation (1)

$$G(x) - 1 - 2x - 7[xG(x) - x] + 10[x^2(G(x))] = 0$$

$$G(x) - 1 - 2x - 7xG(x) + 7x + 10x^2 G(x) = 0$$

$$G(x)[1 - 7x + 10x^2] = 1 + 2x - 7x$$

$$G(x)[1 - 7x + 10x^2] = 1 - 5x$$

$$G(x) = \frac{1-5x}{1-7x+10x^2}$$

$$= \frac{1-5x}{(5x-1)(2x-1)}$$

$= 10x^2 - 7x + 1 -$

$10x^2 - 2x - 5x + 1$

$2x(5x-1) - 1(5x-1)$

$(5x-1)(2x-1)$

$$= \frac{\cancel{(1-5x)}^{-1}}{\cancel{(5x-1)}(2x-1)} = \frac{-1}{2x-1}$$

$$\therefore \boxed{G(x) = \frac{1}{1-2x}}$$

② Solve the R.R. $a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = 0$ with $a_0 = 1, a_1 = 2, a_2 = 3$ by using Generating func. Method

Sol: Given, $a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3} = 0$

order $= n - (n-3) = 3 \Rightarrow \boxed{\text{order} = 3}$

Multiply given R.R with $\sum_{n=\text{order}}^{\infty} x^n$ on Both sides

$$\sum_{n=3}^{\infty} x^n [a_n - 6a_{n-1} + 12a_{n-2} - 8a_{n-3}] = 0$$

$$\sum_{n=3}^{\infty} x^n a_n - 6\sum_{n=3}^{\infty} x^n a_{n-1} + 12\sum_{n=3}^{\infty} x^n a_{n-2} - 8\sum_{n=3}^{\infty} x^n a_{n-3} = 0 \longrightarrow ①$$

$$\sum_{n=3}^{\infty} x^n a_n = x^3 a_3 + x^4 a_4 + x^5 a_5 + \ldots$$

$$= (a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + a_4x^4 + \ldots) - a_0x^0 - a_1x^1 - a_2x^2$$

$$= G(x) - a_0x^0 - a_1x^1 - a_2x^2$$

$$= G(x) - 1x^0 - 2x^1 - 3x^2$$

$$= G(x) - 1 - 2x - 3x^2 \longrightarrow ②$$

$$\sum_{n=3}^{\infty} x^n a_{n-1} = a_2 x^3 + a_3 x^4 + a_4 x^5 + \cdots$$

$$= x\left(a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots\right)$$

$$= x\left[\left(a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots\right] - a_0 x^0 - a_1 x\right]$$

$$= x\left[G(x) - 1x^0 - 2x\right] = x\left(G(x) - 1 - 2x\right)$$

$$= xG(x) - x - 2x^2 \longrightarrow (3)$$

$$\sum_{n=3}^{\infty} x^n a_{n-2} = a_1 x^3 + a_2 x^4 + a_3 x^5 + \cdots$$

$$= x^2\left[a_1 x^1 + a_2 x^2 + a_3 x^3 + \cdots\right]$$

$$= x^2\left[\left(a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots\right] - a_0 x^0\right]$$

$$= x^2\left[G(x) - 1(x^0)\right] = x^2\left[G(x) - 1\right)$$

$$= x^2 G(x) - x^2 \longrightarrow (4)$$

$$\sum_{n=3}^{\infty} x^n a_{n-3} = a_0 x^3 + a_1 x^4 + a_2 x^5 + \cdots$$

$$= x^3\left(a_0 x^0 + a_1 x^1 + a_2 x^2 + \cdots\right)$$

$$= x^3\left(G(x)\right) \longrightarrow (5)$$

Sub (2), (3), (4) & (5) in Eq (1)

$$G(x) - 1 - 2x - 3x^2 - 6\left(x(G(x) - x - 2x^2\right) + 12\left(x^2(G(x) - x^2\right) - 8\left(x^3(G(x)\right) = 0$$

$$G(x) - 1 - 2x - 3x^2 - 6xG(x) + 6x^2 + 12x^3 + 12x^2 G(x) - 12x^4 - 8x^3 G(x) = 0$$

$$G(x)\left[1 - 6x + 12x^2 - 8x^3\right] = 1 + 3x^2 - 4x$$

$$G(x) = \frac{1+4x+3x^2}{1-6x+12x^2-8x^3}$$

$$= \frac{3x^2-4x+1}{-8x^3+12x^2-6x+1}$$

$$= \frac{3x^2-4x+1}{(x-1/2)^3}$$

Partial fractions :- $\frac{A}{(x-1/2)} + \frac{B}{(x-1/2)^2} + \frac{C}{(x-1/2)^3}$

$$\frac{3x^2-4x+1}{(x-1/2)^3} = \frac{A}{(x-1/2)} + \frac{B}{(x-1/2)^2} + \frac{C}{(x-1/2)^3}$$

$$3x^2-4x+1 = A(x-1/2)^2 + B(x-1/2) + C$$

put $x = 1/2$

$$3\left(\frac{1}{4}\right) - 4\left(\frac{1}{2}\right) + 1 = A(0)^2 + B(0) + C$$

$$\frac{3}{4} - 1 = C \Rightarrow C = \frac{3-4}{4} = \frac{-1}{4}$$

$$\boxed{C = \frac{-1}{4}}$$

$$3x^2-4x+1 = A\left[x^2 + \frac{1}{4} - 2(x)\left(\frac{1}{2}\right)\right] + B(x-1/2) + C$$

Compare x^2 coeffs $\boxed{A = 3}$

Compare x - coeff $\Rightarrow -4 = -A + B$

$$-4 + A = B$$

$$B = 3 - 4 = -1$$

$$\boxed{B = -1}$$

$$G(x) = \frac{3}{(x-1/2)} - \frac{1}{(x-1/2)^2} - \frac{1}{4(x-1/2)^3}$$

③ Solve RR $a_n - 9a_{n-1} + 26a_{n-2} - 24a_{n-3} = 0$ for $n \geq 3$ by using Generating function method

④ Solve R.R $F_n = F_{n-1} + F_{n-2}$ by using G.F.M (fibonacci)

⑤ Solve RR $F_n = 2F_{n-1}$; $F_0 = 1$ by using G.F.M

③ Given, $a_n - 9a_{n-1} + 26a_{n-2} - 24a_{n-3} = 0$

Sol:

Order $= n-(n-3) = 3$

Multiply with $\sum_{n=\text{order}}^{\infty} x^n$

$$\sum_{n=3}^{\infty} x^n \left[a_n - 9a_{n-1} + 26a_{n-2} - 24a_{n-3}\right] = 0$$

$$\sum_{n=3}^{\infty} x^n a_n - 9\sum_{n=3}^{\infty} x^n a_{n-1} + 26\sum_{n=3}^{\infty} x^n a_{n-2} - 24\sum_{n=3}^{\infty} x^n a_{n-3} = 0 \longrightarrow ①$$

$$\sum_{n=3}^{\infty} x^n a_n = a_3x^3 + a_4x^4 + a_5x^5 + \dots$$

$$= (a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + \dots) - (a_0x^0 + a_1x^1 + a_2x^2)$$

$$= G(x) - a_0x^0 - a_1x^1 - a_2x^2 \longrightarrow ②$$

$$\sum_{n=3}^{\infty} x^n a_{n-1} = a_2x^3 + a_3x^4 + a_4x^5 + \dots$$

$$= x(a_2x^2 + a_3x^3 + \dots)$$

$$= x\left[(a_0x^0 + a_1x^1 + a_2x^2 + \dots) - a_0x^0 - a_1x^1\right]$$

$$= x\left[G(x) - a_0x^0 - a_1x^1\right]$$

$$= xG(x) - a_0x - a_1x^2 \longrightarrow ③$$

$$\sum_{n=3}^{\infty} x^n a_{n-2} = a_1x^3 + a_2x^4 + a_3x^5 + \cdots$$

$$= x^2[a_1x^1 + a_2x^2 + a_3x^3 + \cdots]$$

$$= x^2[(a_0x^0 + a_1x^1 + \cdots) - a_0x^0]$$

$$= x^2[G(x) - a_0x^0]$$

$$= x^2G(x) - a_0x^2 \longrightarrow (4)$$

$$\sum_{n=3}^{\infty} x^n a_{n-3} = a_0x^3 + a_1x^4 + \cdots$$

$$= x^3[a_0x^0 + a_1x^1 + \cdots]$$

$$= x^3[G(x)] \longrightarrow (5)$$

Sub eq (2), (3), (4), (5) in (1)

$$G(x) - a_0x^0 - a_1x^1 - a_2x^2 - 9xG(x) + 9a_0x + 9a_1x^2 + 26x^2G(x) - 26x^2a_0 - 24x^3G(x) = 0$$

$$G(x) - 9xG(x) + 26x^2G(x) - 24x^3G(x) =$$

$$= a_0 + a_1x + a_2x^2 - 9a_0x - 9a_1x^2 + 26a_0x^2$$

$$= a_0(1-9x+26x^2) + a_1(x-9x^2) + a_2x^2$$

$$G(x)[1-9x+26x^2-24x^3]$$

$$= a_0(1-9x+26x^2) + a_1(x-9x^2) + a_2x^2$$

$$G(x) = \frac{a_0(1-9x+26x^2) + a_1(x-9x^2) + a_2x^2}{1-9x+26x^2-24x^3}$$

(14) Solve the R.R. $f_n = f_{n-1} + f_{n-2}$
by using GFM (Fibonacci)

Sol

0 1 1 2 3 5 8 13

Given, $f_n = f_{n-1} + f_{n-2}$.

It is in Fibonacci series.

So we have to take $F_0 = 1, F_1 = 1$

order $= (n) - (n-2)$

$= 2$

$\boxed{\text{order} = 2}$

Multiply with $\sum_{n=\text{order}}^{\infty} x^n$ on Both sides

$$\sum_{n=2}^{\infty} x^n [f_n - f_{n-1} - f_{n-2}] = 0$$

$$\sum_{n=2}^{\infty} x^n f_n - \sum_{n=2}^{\infty} x^n f_{n-1} - \sum_{n=2}^{\infty} f_{n-2} x^n = 0 \longrightarrow (1)$$

i) $\sum_{n=2}^{\infty} x^n f_n = F_2 x^2 + F_3 x^3 + \dots$

$= (F_0 x^0 + F_1 x^1 + F_2 x^2 + \dots)$
$- (F_0 x^0 + F_1 x^1)$

$= G(x) - F_0 x^0 - F_1 x^1$

$= G(x) - 0(1) - 1(x)$

$= G(x) - x \longrightarrow (2)$

$$\sum_{n=2}^{\infty} x^n f_{n-1} = f_1 x^2 + f_2 x^3 + \cdots$$

$$= x[f_1 x^1 + f_2 x^2 + \cdots]$$

$$= x[[f_0 x^0 + f_1 x^1 + f_2 x^2 + \cdots] - f_0 x^0]$$

$$= x[G(x) - f_0 x^0]$$

$$= x[G(x) - 0(1)]$$

$$= xG(x) \longrightarrow (3)$$

$$\sum_{n=2}^{\infty} x^n f_{n-2} = f_0 x^2 + f_1 x^3 + f_2 x^4 + \cdots$$

$$= x^2 [f_0 x^0 + f_1 x^1 + f_2 x^2 + \cdots]$$

$$= x^2 [G(x)]$$

$$= x^2 G(x) \longrightarrow (4)$$

Sub (2), (3), (4) in Eq (1)

$$G(x) - x - xG(x) - x^2 G(x) = 0$$

$$G(x) - xG(x) - x^2 G(x) = x$$

$$G(x)[1 - x - x^2] = x$$

$$G(x) = \frac{x}{1 - x - x^2}$$

$$\frac{x}{-x^2 - x + 1} = \frac{A}{\left[x - \left(\frac{-1+\sqrt{5}}{2}\right)\right]} + \frac{B}{\left[x - \left(\frac{-1-\sqrt{5}}{2}\right)\right]}$$

$$= A\left[x - \left(\frac{-1-\sqrt{5}}{2}\right)\right] + B\left[x - \left(\frac{-1+\sqrt{5}}{2}\right)\right]$$

$x = A\left[x + \frac{1}{2} + \frac{\sqrt{5}}{2}\right] + B\left[x + \frac{1}{2} - \frac{\sqrt{5}}{2}\right]$

$x = Ax + \frac{A}{2} + \frac{\sqrt{5}}{2}A + Bx + \frac{B}{2} - \frac{\sqrt{5}B}{2}$

$A + B = 1 \rightarrow (5)$ $\quad \frac{A}{2} + \frac{\sqrt{5}A}{2} + \frac{B}{2} - \frac{\sqrt{5}}{2}B = 0$

$\boxed{A = 1 - B}$

$\frac{1-B}{2} + \frac{\sqrt{5}}{2}(1-B) + \frac{B}{2} - \frac{\sqrt{5}}{2}B = 0$

$\frac{1}{2} - \cancel{\frac{B}{2}} + \frac{\sqrt{5}}{2} - \frac{\sqrt{5}B}{2} + \cancel{\frac{B}{2}} - \frac{\sqrt{5}}{2}B = 0$

$\frac{1}{2} + \frac{\sqrt{5}}{2} - \frac{\sqrt{5}B}{2} - \frac{\sqrt{5}B}{2} = 0$

$\frac{1+\sqrt{5}}{2} = \frac{2\sqrt{5}B}{\cancel{2}}$

$\boxed{\frac{1+\sqrt{5}}{2\sqrt{5}} = B}$

$A = 1 - \frac{1+\sqrt{5}}{2\sqrt{5}}$

$= \frac{2\sqrt{5} - 1 - \sqrt{5}}{2\sqrt{5}}$

$\boxed{A = \frac{\sqrt{5}-1}{2\sqrt{5}}}$

(5) $f_n = 2f_{n-1}$ & $f_0 = 1$ Solve by GFM

Sol: Given, $f_n = 2f_{n-1}$

$f_n - 2f_{n-1} = 0$

order $= n - (n-1) = 1$

Multiply with $\sum_{n=1}^{\infty} x^n$ on B.S.

$$\sum_{n=1}^{\infty} x^n (f_n - 2f_{n-1}) = 0$$

$$\sum_{n=1}^{\infty} x^n f_n - 2\sum_{n=1}^{\infty} f_{n-1} x^n = 0$$

$$\sum_{n=1}^{\infty} x^n f_n = f_1 x^1 + f_2 x^2 + \dots$$

$$= (f_0 x^0 + f_1 x^1 + f_2 x^2 + \dots) - f_0 x^0$$

$$= G(x) - f_0 x^0$$

$$= G(x) - 1 \longrightarrow (2)$$

$$\sum_{n=1}^{\infty} x^n f_{n-1} = f_0 x^1 + f_1 x^2 + f_2 x^3 + \dots$$

$$= x[f_0 x^0 + f_1 x^1 + f_2 x^2 + \dots]$$

$$= x[G(x)] \longrightarrow (3)$$

(2), (3) in (1)

$$G(x) - 1 - 2xG(x) = 0$$

$$G(x) - 2xG(x) = 1$$

$$G(x)[1-2x] = 1$$

$$G(x) = \frac{1}{1-2x}$$

* Non-Homogenous Recurrence Relation:-

→ A Recurrence relation is said to be a non-homogenous, if Right hand side production term is not equal to zero.

→ The following are the steps involved in getting solution of non-homo recurrence relation

Step 1 : Find the Solution for the left hand side recurrence eqn by using characterstic roots method.

Step 2:- Find the particular solution for the right hand side part of recurrence equations.

Step 3:- Combine the results of left hand side part and right hand side part with '+' operator.

* Particular Solution Method:-

→ In Non-Homogenous recurrence relations, Particular soln's are classified into three categories.

1. Constant particular soln.
2. Exponential " "
3. Polynomial " "

1. Constant Particular Solution :-

→ In Non-Homogenous Recurrence Relation, the Right Hand Side production contains constant value then the constant particular soln is defined as $\boxed{a_n = n^0 d}$ where d is a constant particular solution. to calculate.

→ If d is not possible, Then apply $a_n = n^1 d$

If d is also not possible, then apply $a_n = n^2 d$

$\vdots$

$a_n = n^n d$

Ex: ① Solve R.R. $a_n - 3a_{n-1} = 3$ where $a_0 = 1$

Given, $a_n - 3a_{n-1} = 3$, $a_0 = 1$

consider $a_n - 3a_{n-1} = 0$

order $= n-(n-1) = 1$ ⇒ $\boxed{\text{Order} = 1}$ ⇒ $a_1 - 3a_0 = 0$

Convert RR into char. Eqn

$a^1 - 3a^0 = 0$

$a - 3(1) = 0$ ⇒ $a = 3$

$r = 3$

$a_n = c_1 . r^n$

$n = 0$, $a_0 = c_1 . 3^0$

$\boxed{1 = c_1}$

$n = 1$, $a_1 = c_1 (3)^1$

$a_1 = 1(3)$

$\boxed{a_1 = 3}$

$a_n = c_1 . r^n$

$$\boxed{a_n = 1 . 3^n}$$

$a_n - 3a_{n-1} = 3$

$a_n = n^0 d$

$a_{n-1} = (n-1)^0 d$

ie. $n^0 d - 3(n-1)^0 d = 3$

$d - 3d = 3$

$$\boxed{d = -3/2}$$

$a_n = 3^n + (-3/2)$

$\therefore$ $$\boxed{a_n = 3^n - 3/2}$$

② Solve the R.R $a_n - 7a_{n-1} + 12a_{n-2} = 8$ with $a_0 = 1, a_1 = 2$

Sol: Given, $a_n - 7a_{n-1} + 12a_{n-2} = 8$

~~Order~~ $\Rightarrow$

Consider LHS :- $a_n - 7a_{n-1} + 12a_{n-2} = 0 \rightarrow$ (1)

order $= n-(n-2)$

order $= 2$

$a_2 - 7a_1 + 12a_0 = 0$

characterstic eqn is

$a^2 - 7a + 12a^0 = 0$

$a^2 - 7a + 12 = 0$

$(a-3)(a-4) = 0 \Rightarrow a = 3 \quad a = 4$

$r_1 = 3 \quad r_2 = 4$

Roots are Different,

$a_n = c_1 \cdot r_1^n + c_2 r_2^n$

$a_n = c_1 3^n + c_2 4^n$

$n=0 \Rightarrow a_0 = c_1(3)^0 + c_2(4)^0$

$\boxed{1 = c_1 + c_2}$

$a_1 = c_1(3)^1 + c_2(4)^1$

$\boxed{2 = 3c_1 + 4c_2}$

Solve Eqns
$$3c_1 + 3c_2 = 3$$
$$3c_1 + 4c_2 = 2$$
$$(-) \quad (-) \quad (-)$$
$$-c_2 = 1$$
$$\boxed{c_2 = -1}$$

$c_1 - 1 = 1$

$\boxed{c_1 = 2}$

$\therefore \boxed{a_n = 2(3)^n - 1(4)^n.}$

RHS

Constant Particular Solution

$a_n - 7a_{n-1} + 12a_{n-2} = 8$

$a_n = n^0 d$

$a_{n-1} = (n-1)^0 d$

$a_{n-2} = (n-2)^0 d$

$n^0 d - 7(n-1)^0 d + 12(n-2)^0 d = 8$

$d - 7d + 12d = 8$

$\boxed{d = 4/3}$

$a_n = n^0 d$
$= 1 \cdot \frac{4}{3}$

$\boxed{a_n = 4/3}$

Soln: $a_n =$
$\boxed{2(3)^n - 4^n + 4/3}$

Exponential Particular Solution:-

→ In Non-homogenous recurrence relation, the Right hand side production term is an Exponent then the exponential particular soln is $\boxed{a_n = n^0 d r^n}$ where 'r' is the base value of given exponent term.

→ If 'd' is not possible to calculate then

$a_n = n^1 d r^n$

→ If 'd' is again not possible to calculate then

$a_n = n^2 d r^n$

|
|
|
|

$a_n = n^n d r^n$.

Ex:- ① Solve the R.R $a_n - 9a_{n-1} + 20a_{n-2} = 3^n$ where $a_0 = 1$, $a_1 = 2$

Sol:

Given, $a_n - 9a_{n-1} + 20a_{n-2} = 3^n$

Consider LHS,

$a_n - 9a_{n-1} + 20a_{n-2} = 0$

order $= n-(n-2)$

order $= 2$

$a_2 - 9a_1 + 20a_0 = 0$

Convert RR into char. eqn

$a^2 - 9a^1 + 20a^0 = 0 \Rightarrow a^2 - 9a + 20 = 0$

$a^2 - 4a - 5a + 20 = 0$

$(a-4)(a-5) = 0$

$a = 4, \quad a = 5$

$r_1 = 4 \quad r_2 = 5$

$a_n = c_1 r^n + c_2 . r^n$

$a_n = c_1 . 4^n + c_2 . 5^n$

$n = 0 \Rightarrow a_0 = c_1 . 4^0 + c_2 . 5^0$

$\boxed{1 = c_1 + c_2}$

$n = 1 \Rightarrow a_1 = c_1 . (4)^1 + c_2 (5)^1$

$\boxed{2 = 4c_1 + 5c_2}$

Solve two eqns $\Rightarrow$ $\cancel{4c_1} + 4c_2 = 4$

$\cancel{4c_1} + 5c_2 = 2$

$(-) \quad (-) \quad (-)$

$-c_2 = 2$

$\boxed{c_2 = -2}$

$c_1 - 2 = 1$

$\boxed{c_1 = 3}$

$\therefore \boxed{a_n = 3(4)^n - 2(5)^n}$

RHS Exponential particular soln.

$a_n - 9a_{n-1} + 20a_{n-2} = 3^n$

$a_n = n^0 d 3^n , \quad r = 3$

$a_{(n-1)} = (n-1)^0 d 3^{n-1}$

$a_{n-2} = (n-2)^0 d 3^{n-2}$

$n^0 d . 3^n - 9(n-1)^0 d . 3^{n-1} + 20(n-2)^0 d . 3^{n-2} = 3^n$

$d . 3^n - 9d . \frac{3^n}{3} + 20d . \frac{3^n}{9} = 3^n$

$$d \cdot 3^n - 9d \cdot \frac{3^n}{3} + 20 \cdot d \frac{3^n}{3^2} = 3^n$$

$$3^n \left(d - 3d + \frac{20d}{9} \right) = 3^n \cdot 1$$

$$d \left(1 - 3 + \frac{20}{9} \right) = 1$$

$= \frac{20}{9} - 2$

$= \frac{20 - 18}{9}$

$= \frac{2}{9}$

$$d = \frac{1}{\left(\frac{2}{9}\right)} = \frac{9}{2}$$

$$\boxed{d = \frac{9}{2}}$$

$$a_n = n^0 d \cdot 3^n$$

$$a_n = (1)\left(\frac{9}{2}\right)(3)^n$$

$$= \frac{3^2 \, 3^n}{2} = \frac{3^{n+2}}{2}$$

$$\boxed{a_n = \frac{3^{n+2}}{2}}$$

$$a_n = 3(4)^n - 2(5)^n + \frac{3^{n+2}}{2}$$

---

(3) Solve the RR $a_n - 7a_{n-1} + 12a_{n-2} = 5 \times 2^n$ with $a_0 = 1$, $a_1 = 2$

Sol: Given, $a_n - 7a_{n-1} + 12a_{n-2} = 5 \times 2^n$

Consider LHS,

$$a_n - 7a_{n-1} + 12a_{n-2} = 0$$

$$\text{Order} = n - (n-2)$$

$$= 2 \Rightarrow a_2 - 7a_1 + 12a_0 = 0$$

Convert into char. eqn

$$a^2 - 7a^1 + 12a^0 = 0$$

$$a^2 - 7a + 12 = 0$$

$a^2 - 3a - 4a + 12 = 0$

$(a-3)(a-4) = 0$

$a = 3, \quad a = 4$

$r_1 = 3 \quad r_2 = 4$

$a_n = c_1 r^n + c_2 r^n$

$a_n = c_1(3)^n + c_2(4)^n$

$n=0 \Rightarrow a_0 = c_1(3)^0 + c_2(4)^0$

$\boxed{1 = c_1 + c_2}$

$n=1 \Rightarrow a_1 = c_1(3)^1 + c_2(4)^1$

$\boxed{2 = 3c_1 + 4c_2}$

Solve Eqns $\Rightarrow$ $3c_1 + 3c_2 = 3$

$3c_1 + 4c_2 = 2$

$(-) \quad (-) \quad (-)$

$-c_2 = 1 \Rightarrow \boxed{c_2 = -1}$

$c_1 - 1 = 1$

$\boxed{c_1 = 2}$

$\therefore \boxed{a_n = 2(3)^n - 4^n}$

RHS $a_n - 7a_{n-1} + 12a_{n-2} = 5 \times 2^n$

Consider, constant particular sol^n.

$a_n = n^0 d$

$a_{n-1} = (n-1)^0 d$

$a_{n-2} = (n-2)^0 d$

$n^0 d - 7(n-1)^0 d + 12(n-2)^0 d = 5$

$d - 7d + 12d = 5$

$6d = 5 \Rightarrow \boxed{d = \frac{5}{6}}$

$a_n = n^0 d$

$= (1)\frac{5}{6}$

$\boxed{a_n = 5/6}$

Consider exponential particular sol^n

$a_n - 7a_{n-1} + 12a_{n-2} = 5 * 2^n \quad ; r = 2$

$a_n = n^0 d r^n \Rightarrow a_n = n^0 d 2^n$

$a_{n-1} = (n-1)^0 d.(2)^{n-1}$

$a_{n-2} = (n-2)^0 d.(2)^{n-2}$

$n^0 d.2^n - 7(n-1)^0 d(2)^{n-1} + 12(n-2)^0 d(2)^{n-2}$
$= 2^n$

$d 2^n - 7d\frac{(2)^n}{2} + 12(d)\frac{2^n}{4} = 2^n$

$2^n\left[d - \frac{7d}{2} + \frac{12d}{4}\right] = 2^n$

$d\left[1 - \frac{7}{2} + 3\right] = 1$

$d\left(4 - \frac{7}{2}\right) = 1$

$d\left(\frac{1}{2}\right) = 1 \Rightarrow \boxed{d = 2}$

$= 7 - \frac{7}{2}$

$= \frac{14-7}{2}$

$= \frac{7}{2}$

$4 - \frac{7}{2}$

$\frac{8-7}{2}$

$a_n = n^0 d 2^n$

$a_n = (1)(2)2^n$

$a_n = 2^{n+1} = 2^{n+1}$

$= \frac{5}{6} * 2^{n+1}$

$= \frac{5}{2(3)} * 2^{n+1}$

$= \frac{5}{3} * 2^n$

$\therefore \boxed{a_n = 2(3)^n - 4^n + \left(\frac{5}{6} * 2^{n+1}\right)}$

$\Rightarrow a_n = 2(3)^n - 4^n + \frac{5}{3} * 2^n$

Polynomial Particular

→ In Non-homogenous recurrence relation, the Right hand side production term is a Polynomial, then the polynomial particular solⁿ is $\boxed{a_n = n^0 d_1 + n^1 d_2 + n^2 d_3 + \cdots}$

→ If 'd' is not possible to calculate then

$$a_n = n^1 d_1 + n^2 d_2 + n^3 d_3 + \cdots$$

If 'd' is again not possible to calculate then

$$a_n = n^2 d_1 + n^3 d_2 + n^4 d_3 + \cdots$$

(1) Solve the RR $a_n - 7a_{n-1} + 12a_{n-2} = 2^n + 8 + 6n$

Sol: Given, $a_n - 7a_{n-1} + 12a_{n-2} = 2^n + 8 + 6n$

Consider LHS

$$a_n - 7a_{n-1} + 12a_{n-2} = 0$$

$$\text{Order} = n-(n-2) = 2$$

$$a_2 - 7a_1 + 12a_0 = 0$$

Char Eqn $\Rightarrow a^2 - 7a^1 + 12a^0 = 0$

$$a^2 - 7a + 12 = 0$$

$$(a-3)(a-4) = 0$$

$$a = 3,\ a = 4$$

$$r_1 = 3,\ r_2 = 4$$

$$a_n = c_1 \cdot r^n + c_2 \cdot r^n$$

$$a_n = c_1(3)^n + c_2(4)^n$$

$n=0 \Rightarrow a_0 = c_1(3)^0 + c_2(4)^0$

$$\boxed{1 = c_1 + c_2}$$

$a_1 = c_1(3)^1 + c_2(4)^1$

$$\boxed{2 = 3c_1 + 4c_2}$$

Solve

$\cancel{3}c_1 + 3c_2 = 3$

$\cancel{3}c_1 + 4c_2 = 2$

$(-)\quad(-)\quad\quad(-)$

$-c_2 = 1 \Rightarrow \boxed{c_2 = -1}$

$c_1 - 1 = 1$

$$\boxed{c_1 = 2}$$

$$\boxed{a_n = 2(3)^n - 4^n}$$

RHS

$a_n - 7a_{n-1} + 12a_{n-2} = \cancel{8}^n + 8 + 6n$

Consider, $a_n - 7a_{n-1} + 12a_{n-2} = 8 + 6n$

Polynomial PS :-

$a_n = n^0 d_1 + n^1 d_2$

$a_{n-1} = (n-1)^0 d_1 + (n-1)^1 d_2$

$a_{n-2} = (n-2)^0 d_1 + (n-2)^1 d_2$

$n^0 d_1 + n^1 d_2 - 7((n-1)^0 d_1 + (n-1)^1 d_2)$

$+ 12((n-2)^0 d_1 + (n-2)^1 d_2) = 6n + 8$

$d_1 + nd_2 - 7d_1 - 7(n-1)d_2 + 12d_1 + 12(n-2)d_2$

$= 8 + 6n$

$d_1 + nd_2 - 7d_1 - 7nd_2 + 7d_2 + 12d_1 + 12nd_2$
$- 24d_2 = 8 + 6n$

$d_1 - 7d_1 + 12d_1 + nd_2 - 7nd_2 + 12nd_2 + 7d_2 - 24d_2$
$= 8 + 6n$

$6d_1 + 6nd_2 - 17d_2 = 8 + 6n$

$(6d_1 - 17d_2) + 6nd_2 = 8 + 6n$

$6d_1 - 17d_2 = 8$ | $6nd_2 = 6n$

$6d_1 - 17 = 8$ | $\boxed{d_2 = 1}$

$d_1 = \frac{25}{6}$

$a_n = n^0\left(\frac{25}{6}\right) + n^1(1)$

$$\boxed{a_n = \frac{25}{6} + n}$$

Consider Exponential P.S :-

$a_n - 7a_{n-1} + 12a_{n-2} = 2^n$, $r = 2$

$a_n = n^0 d 2^n$

$a_{n-1} = (n-1)^0 d 2^{n-1}$

$a_{n-2} = (n-2)^0 d 2^{n-2}$

$n^0 d 2^n - 7(n-1)^0 d \cdot 2^{n-1} + 12(n-2)^0 d \cdot 2^{n-2} = 2^n$

$d 2^n - 7d \frac{2^n}{2} + 12(d)\frac{2^n}{4} = 2^n$

$\cancel{2^n}\left[d - \frac{7d}{2} + 3d\right] = \cancel{2^n}\ 1$

$d\left[1 - \frac{7}{2} + 3\right] = 1$

$d\left(4 - \frac{7}{2}\right) = 1 \Rightarrow \boxed{d = 2}$

$a_n = n^0 d \cdot 2^n$

$= 1(2)(2)^n$

$\boxed{a_n = 2^{n+1}}$

$\therefore \boxed{a_n = 2(3)^n - 4^n + 2^{n+1} + \frac{25}{6} + n}$

② Solve RR. $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1$

with $a_0 = 1, a_1 = 1$

Sol: Given, $a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1$

order $= n-(n-2)$

$= 2$

Consider LHS,

$a_n + 5a_{n-1} + 6a_{n-2} = 0$

$a_2 + 5a_1 + 6a_0 = 0$

char. Eqn is

$a^2 + 5a^1 + 6a^0 = 0$

$a^2 + 5a + 6 = 0 \Rightarrow (a+3)(a+2) = 0$

$a = -2, \quad a = -3$

$r_1 = -2, \quad r_2 = -3$

$a_n = c_1(-2)^n + c_2(-3)^n$

$a_0 = c_1(-2)^0 + c_2(-3)^0$

$\boxed{1 = c_1 + c_2}$

$a_1 = c_1(-2)^1 + c_2(-3)^1$

$\boxed{1 = -2c_1 - 3c_2}$

$-2c_1 - 3c_2 = 1$

$2c_1 + 2c_2 = 2$

$-c_2 = 3$

$\boxed{c_2 = -3}$

$c_1 = 4$

$$\boxed{a_n = 4(-2)^n - 3(-3)^n}$$

$a_n + 5a_{n-1} + 6a_{n-2} = 3n^2 - 2n + 1$

Polynomial particular Solution

$$a_n = n^0 d_1 + n^1 d_2 + n^2 d_3$$

$$a_{n-1} = (n-1)^0 d_1 + (n-1)^1 d_2 + (n-1)^2 d_3$$

$$a_{n-2} = (n-2)^0 d_1 + (n-2)^1 d_2 + (n-2)^2 d_3$$

$$n^0 d_1 + n^1 d_2 + n^2 d_3 + 5[(n-1)^0 d_1 + (n-1)^1 d_2 + (n-1)^2 d_3]$$
$$+ 6[(n-2)^0 d_1 + (n-2)^1 d_2 + (n-2)^2 d_3] = 3n^2 - 2n + 1$$

$$d_1 + n d_2 + n^2 d_3 + 5(n-1) d_2 + 5\underbrace{(n-1)^2}_{(n^2+1-2n)} d_3 +$$
$$6(n-2) d_2 + 6\underbrace{(n-2)^2}_{n^2+4-4n} d_3 = 3n^2 - 2n + 1$$

$$d_1 + n d_2 + n^2 d_3 + 5n d_2 - 5d_2 + 5n^2 d_3 + 5d_3 - 10n d_3 + 5d_1 + 6d_1$$
$$+ 6n d_2 - 12 d_2 + 6n^2 d_3 + 24 d_3 - 24 n d_3 = 3n^2 - 2n + 1$$

$$6n^2 d_3 + 5n^2 d_3 + n^2 d_3 + n d_2 + 5n d_2 + 6n d_2$$
$$- 10n d_3 - 24 n d_3 + 12 d_1 - 5d_2 - 12 d_2 + 5d_3 + 24 d_3$$

$$12n^2 d_3 + 12 n d_2 - 34 n d_3 + 12 d_1 - 17 d_2 + 29 d_3$$
$$= 3n^2 - 2n + 1$$

Compare coeff's

$12 d_3 n^2 = 3n^2$

$d_3 = 3/12$

$\boxed{d_3 = 1/4}$

$\cancel{(12n) d_2 = -2n}$

$\cancel{d_2 = -2/12}$

$\boxed{d_2 = -1/6}$

$12 d_2 n - 34 n d_3 = -2n$

$12 d_2 - 34 d_3 = -2 \Rightarrow 12 d_2 - 34\left(\frac{1}{4}\right) = -2$

$12 d_2 = -2 + \frac{17}{2} = \frac{-4+17}{2}$

$$12d_2 = \frac{13}{2} \Rightarrow \boxed{d_2 = \frac{13}{24}}$$

$$12d_1 - 17d_2 + 29d_3 = 1$$

$$12d_1 - 17\left(\frac{13}{24}\right) + 29\left(\frac{1}{4}\right) = 1$$

$$12d_1 = \frac{-29}{4} + \frac{17(13)}{24}$$

$$= \frac{\quad}{24}$$

$\frac{42}{29}$ / 26

$$\boxed{d_1 = \frac{71}{288}}$$

$$a_n = n^0 d_1 + n^1 d_2 + n^2 d_3$$

$$a_n = (1)\left(\frac{71}{288}\right) + n^1\left(\frac{13}{24}\right) + n^2\left(\frac{1}{4}\right)$$

Soln $\Rightarrow$ $a_n = 4(-2)^n - 3(-3)^n + \frac{n^2}{4} + \frac{13n}{24} + \frac{71}{288}$

③ Solve the RR. $a_n + 5a_{n-1} + 6a_{n-2}$
$= 3n^2 - 2n + 1 + 42 \cdot 4^n$; $a_0 = 1$, $a_1 = 1$

Sol: Consider, LHS $= a_n + 5a_{n-1} + 6a_{n-2}$

order $= n-(n-2) = 2$

$a_2 + 5a_1 + 6a_0 = 0$

Char-eqn is

$a^2 + 5a^1 + 6 = 0$

$(a+3)(a+4) = 0$

$a = -3, \quad a = -2$

$r_2 = -3, \quad r_1 = -2$

$a_n = c_1(-2)^n + c_2(-3)^n$

$a_0 = c_1(-2)^0 + c_2(-3)^0$

$\boxed{1 = c_1 + c_2}$

$a_1 = c_1(-2)^1 + c_2(-3)^1$

$\boxed{1 = -2c_1 - 3c_1}$

$2c_1 + 2c_2 = 2$
$-2c_1 - 3c_2 = 1$

$-c_2 = 3$

$\boxed{\begin{matrix} c_2 = -3 \\ c_1 = 4 \end{matrix}}$