ELASTICITY

Elasticity is the property of material by virtue of which they regain their original shape and size after the removal of deforming forces.

The body which obeys the property of elasticity is called elastic body.

Plasticity: It is the property of material by virtue of which they do not regain their original shape and size after the removal of deforming forces.

The body which obeys the property of plasticity is called Plastic body.

Ex:-wet clay, putty etc.

When an external deforming force acts on the system of two atoms producing a change in inter atomic distance, an internal force comes into play to counteract the deforming force which is called **Stress.** The change in the interatomic distance represents strain.

Stress:

When a force acts on a body, producing deformation, the internal reaction which tends to restore the original condition is called ***Stress***.

It is measured in terms of force applied per unit area

$$stress = \frac{F}{A} \quad \boldsymbol{nm^{-2}}$$

There are three types of stress:

i. Longitudinal Stress
ii. Volumetric Stress
iii. Shear Stress

Longitudinal stress: it is the force per unit area to change the length of the body.

Volumetric stress: it is the normal force per unit area to change the volume of the body.

Shear stress: it is the tangential force per unit area to change the shape of the body.

Strain:

The ratio of change in length 'l' to original length 'l' when the force applied along its length

$$\text{Longitudinal strain } = \frac{l}{L}$$

It is dimension less quantity.

Volumetric strain:- it is defined as ratio of change in volume v to original volume V when the force applied normally over a surface.

$$\text{Volumetric strain} = \frac{\text{v}}{\text{V}}$$

Shear strain: it is the ratio of displacement of the layer to the perpendicular distance from the fixed layer

$$\theta = \frac{dd^{\text{l}}}{ad}$$

Elastic limit: The maximum value of stress above which the linear relationship between stress and strain causes to be valid is termed as elastic limit.

Hooke's law

"Within the elastic limit, the stress is directly proportional to strain.

i.e., $stress \propto strain$

$$\frac{stress}{strain} = constant\left(e\right)$$

Where, e-modulus of elasticity, which depends upon the material of the body
Its unit is **Nm^{-2}**

Modulli of elasticity

1. **Young's modulus:** it is defined as the ratio of longitudinal stress to longitudinal strain within the elastic limit.

$$q = \frac{longitudinal\ stress}{longitudinal\ strain}$$

$$q = \frac{F/A}{l/L} \Rightarrow q = \frac{FL}{lA}$$

If l = 1 m, a = 1 m^2 and L = 1 m then ***q = F***

Then ***"the young's modulus of the material of a wire of unit area of cross section is numerically equal to the force required to double the length of the wire."***

2. **Bulk modulus:** it is defined as the ratio of volume of stress to volume strain within elastic limit.

$$k = \frac{volume\ stress}{volume\ strain}$$

$$k = \frac{F/A}{v/V} \Rightarrow q = \frac{FV}{vA}$$

If v = 1 m^3, A = 1 m^2 and V = 1 m^3 Then, ***k = F*** i.e., ***"the bulk modulus of the material of unit area of cross section is numerically equal to the force required to double the length of the wire"***

Compressibility:

the reciprocal of bulk modulus is called compressibility which is defined as the change in volume strain per unit change of pressure.

Its unit is $\mathbf{N^{-1}m^{2}}$

3. **Rigidity modulus**: it is the ratio of shear stress to the shear strain under elastic limit.

$$\eta = \frac{shear\ \ stress}{shear\ \ strain}$$

$$\eta = \frac{F/A}{\theta} \Rightarrow \eta = \frac{F}{A\theta}$$

Poison's ratio:

"Within the elastic limit the ratio of lateral strain to the longitudinal strain is called Poisson's ratio of material of the body"

$$\text{Poisson's ratio } (\sigma) = \frac{\text{lateral strain } (\beta)}{\text{longitudinal strain } (\alpha)}$$

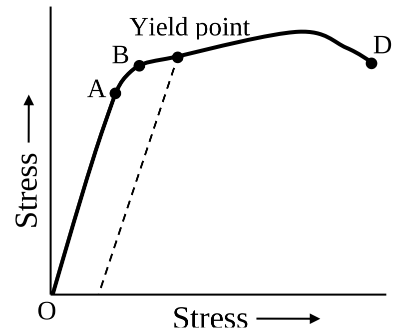
Differential form: $\sigma \ = \dfrac{\beta}{\alpha} = \dfrac{r/R}{l/L}$

$$\sigma \ = \frac{rL}{R\ l}$$

It is dimension less quantity.

Stress strain graph

The curve plotted between stresses applied to the material along y- axis verses strain produced in the material along x – axis is known as stress - strain graph



Imagine a wire being loaded gradually and the extension produced is plotted against the load as shown in figure.

Along the path OA, which is straight, obeys Hooke's law. OA is in perfectly **elastic range** and A is the **elastic limit**. Beyond OA the extension is no longer proportional to load. at point B even the small addition of load causes enormous elongation. This point is called yield point. After this point the extension increases very rapidly and depends on the time for which the load acts. The specimen exhibits a phenomenon known as **necking.** i.e., as the extension increases, the area of cross section decreases until the breaking point is reached and finally the wire breaks. The load at which the wire breaks is called breaking load. The breaking load per original cross-sectional area is called **breaking stress** or **ultimate strength** or **tensile strength.**

When a body is continuously subject to stress and strain, it gets elastic fatigue.

The ratio of ultimate strength to working stress is called **factor of safety**

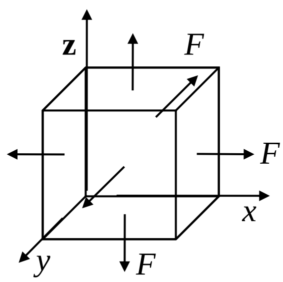
$$Factor\ \ of\ \ safety = \frac{ultimate\ \ strength}{working\ \ stress}$$

Usually, allowed factor safety values lies between 10 to 5.

Relation between elastic constants:

i) Relation between Bulk modulus (*k*), Youngs modulus (*q*) in terms of α and β

Consider a solid cube of unit side. let the force *F,* act outwards perpendicular to each faces parallel to *x, y* and *z* axes as shown in figure. α is the elongation per unit length per unit stress. β is the lateral contraction per unit length per unit stress.



The elongation along *x*-axis is, = $F \cdot \alpha$

lateral contraction along *y*, or *z* axis = $F \cdot \beta$

longitudinal strain increases the dimension where as lateral contractions decreases the dimensions

$\therefore$ The net elongation along *x*-axis is

$$= F\alpha - F\beta - F\beta = F\left(\alpha - 2\beta\right)$$

Then the new volume of the cube $= \left(1 + F(\alpha - 2\beta)\right)^3$ simplifying and neglecting higher order terms we get the new volume as $\left(1 + 3F\left(\alpha - 2\beta\right)\right)$

$\therefore$ The increase in the volume of the cube to the initial volume is volumetric strain equal to $3F\left(\alpha - 2\beta\right)$ (since initial volume is unity)

Bulk modulus $k = \dfrac{volumetric\ stress}{volumetric\ strain} = \dfrac{F}{3F\left(\alpha - 2\beta\right)}$

$$\boldsymbol{k = \frac{1}{3\left(\alpha - 2\beta\right)}} \qquad \boldsymbol{k = \frac{1}{3\alpha\left(1 - 2\left(\frac{\beta}{\alpha}\right)\right)}}$$

$$\boldsymbol{k = \frac{q}{3\left(1 - 2\sigma\right)}} \quad \text{..... (a)} \qquad \because q = \frac{1}{\alpha}$$

ii) **Relation between all the thre easic modulus**

From equation (A)

$$q = 3k\left(1 - 2\sigma\right) \quad \Rightarrow 2\sigma = \left(1 - \frac{q}{3k}\right) \quad \text{....... 1}$$

From equation $\eta = \dfrac{q}{2\left(1 + \sigma\right)}$

$$q = 2\eta\left(1 + \sigma\right)$$

$$\frac{q}{\eta} = 2 + 2\sigma \Rightarrow 2\sigma = \frac{q}{\eta} - 2 \quad \text{....... 2}$$

comparing (5) and (6)

$$1 - \frac{q}{3k} = \frac{q}{\eta} - 2 \qquad \Rightarrow \qquad \frac{3k - q}{3k} = \frac{q - 2\eta}{\eta}$$

$$\eta\left(3k - q\right) = 3k\left(q - 2\eta\right)$$

$$3kq + q\eta = 3k\eta + 6k\eta$$

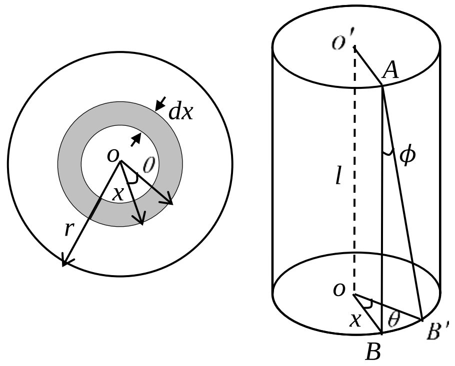
$$q\left(3k + \eta\right) = 9k\eta$$

$$\boldsymbol{q = \frac{9k\eta}{3k + \eta}}$$

Torsion of a cylinder:

Expression for couple per unit twist:

Consider a cylindrical rod of length *l* and radius *r*, rigidly clamped at the upper end and twisted by applying a couple at the other end. Imagine the cylinder to consist of an infinite number of hollow coaxial cylinders. Consider one such hollow cylinder of radius *x* and thickness *dx*. let the twist at the lower end be θ .



Shear angle = $\angle BAB' = \phi$

$BB' = x\theta = l\phi \quad \Rightarrow$ *Shear strain*, $\phi = \frac{x\,\theta}{l}$

Rigidity modulus, $\eta = \dfrac{shear\ stress}{shear\ strain}$

Shear stress = η x *shear strain*

$$= \eta\ \phi\ = \frac{\eta\ x\ \theta}{l}$$

Area over which shearing force acts = 2πxdx

Shearing force = shear stress × area on which force acts

$$\therefore \text{Shearing force} \ = \frac{\eta\ x\ \theta}{l} \times 2\pi x dx$$

$$= \frac{2\pi\eta\ x^2\theta\ dx}{l}$$

Moment about $OO' = \dfrac{2\pi\eta\ x^2\theta\ dx}{l} \times x$

$$= \frac{2\pi\eta\ x^3\theta\ dx}{l}$$

Therefore Torque due to twisting couple,

$$C = \int_0^r \frac{2\pi\eta\ x^3\theta\ dx}{l}$$

Couple per unit twist, $c = \dfrac{C}{\theta} = \displaystyle\int_0^r \frac{2\pi\eta\ x^3 dx}{l}$

$$c = \frac{2\pi\eta}{l}\int_0^r x^3 dx$$

$$c = \frac{2\pi\eta}{l}\left[\frac{x^4}{4}\right]_0^r \quad \Rightarrow \quad c = \frac{\pi\eta\ r^4}{2l}$$

Torsional oscillations:

Consider a wire which is clamped vertically with the help of a chunk nut at the top and carrying a uniform body like disc, bar or cylinder at the other end. When the body is given slight twist and let go, the body execute oscillation about the wire as axis. These oscillations are called torsional oscillations.

If I is the moment of inertia of the body about the wire as axis and α, the angular acceleration of the system,

Then the external deflection couple = $I\alpha$

The internal deflecting couple $= c\theta$, where c couples per unit twist of the suspension wire and θ is the angular displacement

For equilibrium $I\alpha = -c\theta$ but $\alpha = \frac{d^2\theta}{dt}$

$$I\frac{d^2\theta}{dt} + c\theta = 0 \quad \Rightarrow \quad \frac{d^2\theta}{dt} + \frac{c}{I}\theta = 0$$

This equation represents a simple harmonic motion and the period of oscillation is given by

$$T = 2\pi\sqrt{\frac{I}{c}}$$

Expression for η in terms of torsionals oscillation:

We know that couple per unit twist is given by

$$c = \frac{\pi\eta\ r^4}{2l}$$

And the period of torsional oscillation is given by

$$T = 2\pi\sqrt{\frac{I}{c}}$$

$$\text{so } T^2 = 4\pi^2\left(\frac{I}{c}\right)$$

$$c = 4\pi^2\left(\frac{I}{T^2}\right)$$

$$\frac{\pi\eta r^4}{2l} = 4\pi^2\left(\frac{I}{T^2}\right)$$

$$\eta = \frac{8\pi l}{r^4}\left(\frac{I}{T^2}\right)$$

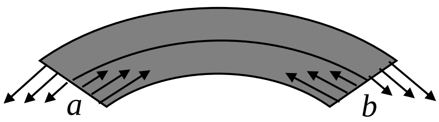
Bending of beams:

A beam is defined as a rod or bar of uniform cross-section whose length is very larger than its thickness so that the shearing stresses over any cross section are small and may be neglected. it may be considered to be made up of a large number of thin plane horizontal layers kept one over the other.

When a straight bar is bent as shown in figure, the outer layer get elongated while the inner layers get contracted. However in between there will be a layer which is neither elongated nor contracted. This layer is known as **neutral surface.** The line intersection of the neutral surface and the plane of bending is known as the **neutral axis.**

Bending moment:

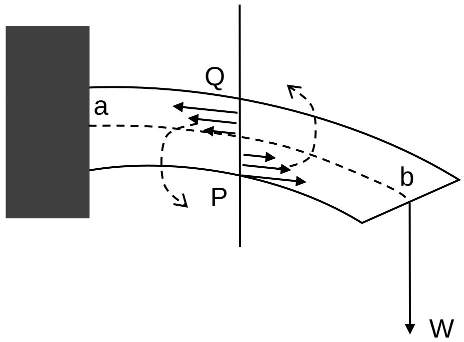
Let a beam is fixed at one end and loaded at the other so that it bends as shown in figure. Consider a section of beam cut by plane PQ at right angles to its length. the load W acting vertically downwards at the free end and its reaction R at O acting vertically upwards constitute a couple which is the bending couple. This couple tends to bend the beam in clock wise direction.



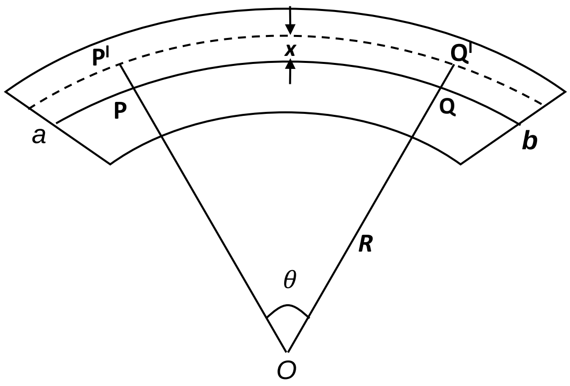
Since there is no rotation of the beam, this external bending couple must be balanced by another equal and opposite couple. The moment of this elastic couple is called the internal bending moment.

Expression for the bending moment:

Let a portion of beam bent in the form of arc as shown in the figure. R is the radius of curvature the neutral axis *ab* which subtends an angle θ at the centre of the curvature *O*.



Consider a filament $P'Q'$ at a distance x from the neutral axis. The length of the filament in the unstrained condition is the same as PQ. Increase in length in stained condition it is given by



Original length $PQ = R\theta$

Extended length $P'Q' = (R+x)\theta$

Increase in the length $= P'Q' - PQ = (R+x)\theta - R\theta$

$= x\theta$

$\therefore$ Linear strain, $\dfrac{x\theta}{R\theta} = \dfrac{x}{R}$

If *q* is the Young's modulus of the material of the bar,

Stress on $P'Q' = q \times linear\,strain = q \times \dfrac{x}{R}$

If *f* is the force which produces the stress

$\dfrac{f}{a} = \dfrac{qx}{R}$ Where *a* is the cross section

$\therefore\ f = q\dfrac{ax}{R}$

Moment of *f* about neutral axis $= f \times x$

$= q\dfrac{ax^2}{R}$

The sum of the moments of all such internal elastic forces acting over the whole cross-section of the beam is known as the bending moment

$$Bending\ \ moment = \sum q\frac{ax^2}{R} = \frac{q}{R}\sum ax^2$$

$$Bending\ \ moment = \frac{qI}{R}$$

Where $I = \sum ax^2$ is the geometric moment if inertia of the cross section of the beam about an axis through its center and perpendicular to the plane of bending

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