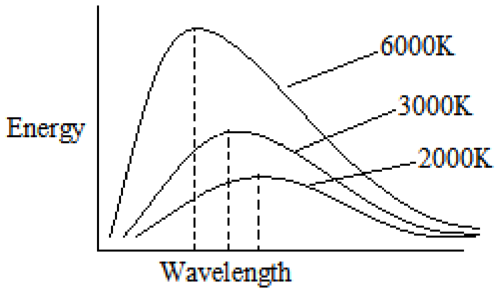
Modern Physics

Blackbody Radiation spectrum:

A Blackbody is one which absorbs the entire radiation incident on it and emits all the absorbed radiation when it is hotter. A true blackbody does not exist practically. A blackbody designed by Wein has features very close to the true blackbody. Ferry has also constructed blackbody called Ferry's blackbody. A blackbody at a particular temperature found to emit a radiation of all possible wavelengths. It is a continuous spectrum starting from certain minimum wavelength to maximum wavelength. The maximum intensity corresponds to a particular wavelength. For different temperatures of the black body, there are different curves. As the temperature of the body increases, the wavelength corresponding to maximum intensity shifts towards lower wavelength side. The distribution of energy in black body radiation is shown in the following fig.



Wein's, Rayleigh-Jeans and Planck have given their explanations to account these observed experimental facts as follows:

Wein's Displacement Law:

The law states that "*the wavelength of maximum intensity is inversely proportional to the absolute temperature of the emitting body*".

i.e. $\lambda_m \,\alpha \left(\frac{1}{T}\right)$ or $\lambda_m T = \text{constant}$ = 2.898×10^{-3} mK

Wein showed that the maximum energy of the peak emission is directly proportional to the fifth power of absolute temperature.

$E_m \,\alpha\, T^5$ or $E_m = \text{constant} \times T^5$

Wein's law:

The relation between the wavelength of emission and the temperature of the source is

$$U_\lambda \, d\lambda = C_1 \lambda^{-5} e^{-\left(\frac{C_2}{\lambda T}\right)} d\lambda$$

Where $U_\lambda \, d\lambda$ is the energy per unit volume in the range of wavelength λ and $\lambda + d\lambda$, C_1 and C_2 are constants. This is called Wein's law of energy distribution in the black body radiation spectrum.

Drawbacks of Wein's law:

Wein's law holds good for the shorter wavelength region and high temperature of the source. It failed to explain gradual drop in intensity of radiation corresponding to longer wavelength greater than the peak value.

Rayleigh-Jeans Law:

Rayleigh derived an equation for the blackbody radiation on the basis of principle of equipartition of energy. The principle of equipartition of energy suggests that an average energy kT is assigned to each mode of vibration. The number of vibrations per unit volume whose wavelength is in the range of λ and $\lambda + d\lambda$ is given by $8\pi\lambda^{-4} \, d\lambda$.

The energy per unit volume in the wavelength range λ and $\lambda + d\lambda$ is

$$U_\lambda \, d\lambda = 8\pi kT \, \lambda^{-4} \, d\lambda$$

Where k is Boltzmann constant = 1.38 x 10^{-23}JK^{-1}.

This is Rayleigh-Jeans equation. Accordingly energy radiated by the blackbody decreases with increasing wavelength.

Drawbacks of Rayleigh-Jeans Law: (or Ultra Violet Catastrophe)

Rayleigh-Jeans Law predicts to radiate all the energy at shorter wavelength side but it does not happen so. A black body radiates mainly in the infra-red or visible region of electromagnetic spectrum and intensity of radiation decreases down steeply for shorter wavelengths. Thus, the Rayleigh-Jeans Law fails to explain the lower wavelength side of the spectrum. This is referred to as ultra-violet Catastrophe.

Assumption of Quantum theory of radiation

1. A black body is made up of large number of oscillatory particles, these particles can vibrate in all possible frequencies
2. An oscillator can have discrete set of energies which are integral multiples of a finite quantum of energy
3. The atomic oscillators can absorb or emit energy in discrete units by making transitions from one quantum state to another. The amount of radiant energy in each unit is called quantum of energy

$$E = h\nu$$

Where h → Planck's constant and ν → frequency of radiation

Planck's Law:

Planck assumed that walls of the experimental blackbody consists larger number of electrical oscillators. Each oscillator vibrates with its own frequency.

i) Each oscillator has an energy given by integral multiple of $h\nu$ where h is Planck's constant and ν is the frequency of vibration.
$E = nh\nu$ where n = 1, 2, 3 . . . etc.

ii) An oscillator may lose or gain energy by emitting or absorbing respectively a radiation of frequency ν, where $\nu = \frac{\Delta E}{\Delta h}$, ΔE is difference in energies of the oscillator before and after the emission or absorption take place.

Planck derived the law which holds good for the entire spectrum of the blackbody radiation as

$$U_\lambda \, d\lambda = \frac{8\pi hc}{\lambda^5}\left[\frac{1}{e^{h\nu/kT} - 1}\right] d\lambda$$

This is Planck's Radiation Law.

Reduction of Planck's law to Wein's law and Rayleigh Jeans law:

1) For Wein's law: For shorter wavelengths, $\nu = c/\lambda$ is large.

When ν is large, $e^{\frac{h\nu}{kT}}$ is very large.

$$\therefore e^{\frac{h\nu}{kT}} >> 1$$

$$\therefore (e^{\frac{h\nu}{kT}} - 1) \approx e^{\frac{h\nu}{kT}} = e^{\frac{hc}{\lambda kT}}$$

Substituting in Planck's Radiation Law

$$U_\lambda \, d\lambda = \frac{8\pi hc}{\lambda^5}\left[\frac{1}{e^{hc/\lambda kT}}\right] d\lambda$$

$$U_\lambda \, d\lambda = C_1 \, \lambda^{-5} \, e^{\left(\frac{-C_2}{\lambda T}\right)} d\lambda$$

Where $C_1 = 8\pi hc$ and $C_2 = hc/k$

This is the Wein's law of radiation.

2) For Rayleigh Jeans law: For longer wavelengths $v = c/\lambda$ is small.

When v is small $\frac{hv}{kT}$ is very small.

Expanding $e^{\frac{hv}{kT}}$ as power series:

$$e^{\frac{hv}{kT}} = 1 + \frac{hv}{kT} + \left(\frac{hv}{kT}\right)^2 + ...$$

Since hv/kT is small, its higher order powers can be neglected.

$$\therefore e^{\frac{hv}{kT}} = 1 + \frac{hv}{kT} = 1 + \frac{hc}{\lambda kT}$$

Substituting in Planck's Radiation Law

$$U_\lambda \, d\lambda = \frac{8\pi hc}{\lambda^5 \left(\frac{hc}{\lambda kT}\right)} d\lambda$$

$$U_\lambda \, d\lambda = \left[\frac{8\pi kT}{\lambda^4}\right] d\lambda = 8\pi kT \lambda^{-4} d\lambda$$

This is Rayleigh Jeans Law of Radiation.

Dual nature of matter (de-Broglie Hypothesis)

Light exhibits the phenomenon of interference, diffraction, photoelectric effect and Compton Effect. The phenomenon of interference, diffraction can only be explained with the concept that light travels in the form of waves. The phenomenon of photoelectric effect and Compton Effect can only be explained with the concept of *Quantum theory of light.* It means to say that light possess particle nature. Hence it is concluded that light exhibits dual nature namely wave nature as well as particle nature.

de-Broglie's Wavelength:

A particle of mass '*m*' moving with velocity '*c*' possess energy given by

Einstein's Equation $E = mc^2$ .... (1)

According to Planck's quantum theory the energy of quantum of frequency 'υ' is

$E = hv$ .... (2)

From (1) and (2)

$$E = mc^2 = hv = \frac{hc}{\lambda}$$

$$mc = \frac{h}{\lambda} \qquad \text{or} \qquad \mathrm{mv} = \frac{h}{\lambda}$$

Relation between de-Broglie wavelength and kinetic energy

Consider an electron in an electric potential V, the energy acquired is given by

$$E = eV = \frac{1}{2}m\mathrm{v}^2 = \frac{p^2}{2m} \Rightarrow E = \frac{p^2}{2m} \Rightarrow p = \sqrt{2mE}$$

Where '*m*' is the mass, 'v' is the velocity and 'p' is the momentum of the particle. 'e' is charge of an electron. The expression for de-Broglie wavelength is given by

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}}$$

de-Broglie wavelength of an accelerated electron:

An electron accelerated with potential difference 'V' has energy 'eV'. If 'm' is the mass and 'v' is the velocity of the electron.

Then $eV = \frac{1}{2}m\mathrm{v}^2$

If 'p' is the momentum of the electron, then p = mv

$$E = eV = \frac{1}{2}m\mathrm{v}^2 = \frac{p^2}{2m} \qquad \Rightarrow p = \sqrt{2meV}$$

According to de-Broglie $\lambda = \frac{h}{p}$

Therefore $\lambda = \frac{h}{\sqrt{2meV}} = \frac{1}{\sqrt{V}}\left(\frac{h}{\sqrt{2me}}\right)$

$$\lambda = \frac{1}{\sqrt{V}}\left[\frac{6.626\times10^{-34}}{\sqrt{2\times9.11\times10^{-31}\times1.602\times10^{-19}}}\right]$$

$$\lambda = \frac{12.26}{\sqrt{V}}\dot{\mathrm{A}}$$

Characteristics of matter waves:

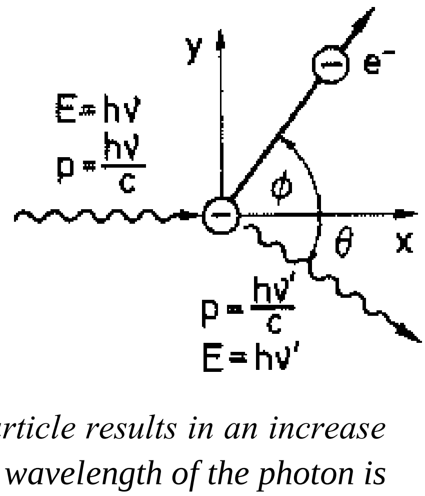
1. Waves associated with moving particles are called matter waves. The wavelength 'λ' of a de-Broglie wave associated with particle of mass 'm' moving with velocity 'v' is

   λ = h/(mv)
2. Matter waves are not electromagnetic waves because the de Broglie wavelength is independent of charge of the moving particle.
3. The velocity of matter waves (v_P) is not constant. The wavelength is inversely proportional to the velocity of the moving particle.
4. Lighter the particle, longer will be the wavelength of the matter waves, velocity being constant.
5. For a particle at rest the wavelength associated with it becomes infinite. This shows that only moving particle produces the matter waves.

The Compton Effect:

The Compton Effect (also called *Compton scattering*) is the result of a high-energy photon colliding with a target, which releases loosely bound electrons from the outer shell of the atom or molecule. The scattered radiation experiences a wavelength shift that cannot be explained in terms of classical wave theory, thus explained with the support of Einstein's photon theory. The effect was first demonstrated in 1923 by Arthur Holly Compton (for which he received a 1927 Nobel Prize).

A high-energy photon (generally X-ray or gamma-ray) collides with a target, which has loosely-bound electrons on its outer shell. The incident photon gives part of its energy to one of the almost-free electrons, in the form of kinetic energy. The wavelength of scattered photon increases since the energy of photon energy is decreased. Expression for the change in the wavelength ($\Delta\lambda$) between incident and scattered photon is derived from concept of law of conservation of energy and momentum.



$$\Delta\lambda = \lambda^{\prime} - \lambda = \frac{h}{m_o c}(1-\cos\theta)$$

The quantity $\frac{h}{m_o c}$ is called Compton wavelength

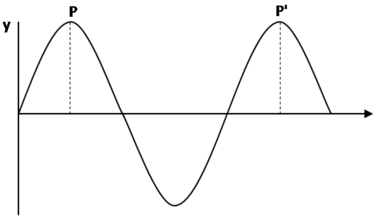
"A phenomenon in which a collision between a photon and a particle results in an increase in the kinetic energy of particle and a corresponding increase in wavelength of the photon is called Compton Effect"

Phase velocity and group velocity:

A wave is represented by the equation:

$$y = A\sin(\omega t - kx)$$

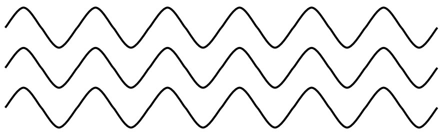
Where 'y' is the displacement along Y-axis at an instant t, 'ω' is the angular frequency, 'k' is propagation constant or wave number. 'x' is the displacement along x-axis at the instant 't'.



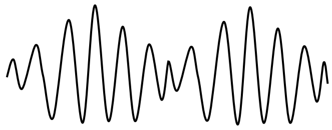
If 'p' is the point on a progressive wave, then it is the representative point for a particular phase of the wave, *the velocity with which it is propagated owing to the motion of the wave is called phase velocity.*

The phase velocity of a wave is given by $v_{phase} = \dfrac{\omega}{k}$

Group velocity



Individual Waves



Amplitude variation after Superposition

A group of two or more waves, slightly differing in wavelengths are super imposed on each other. The resultant wave is a packet or wave group. The velocity with which the envelope enclosing a wave group is transported is called *Group Velocity*.

Expression for group Velocity:

Let $y_1 = A\sin(\omega t - kx)$ ...... (1) and $y_2 = A\sin\left[(\omega+\Delta\omega)t - (k+\Delta k)x\right]$ .... (2)

The two waves having same amplitude & slightly different wavelength, where y_1 & y_2 are the displacements at any instant t, 'A' is common amplitude, $\Delta\omega$ & Δk is difference in angular velocity and wave number are assumed to be small. 'x' is the common displacement at time 't'

By the principle of superposition

$$y = y_1 + y_2 = A\sin(\omega t - kx) + A\sin\left[(\omega+\Delta\omega)t - (k+\Delta k)x\right]$$

But, $\sin a + \sin b = 2\cos\left(\dfrac{a-b}{2}\right)\sin\left(\dfrac{a+b}{2}\right)$

$$y = 2A\cos\left[\left(\frac{\Delta\omega}{2}\right)t - \left(\frac{\Delta k}{2}\right)x\right]\sin\left[\left(\frac{2\omega+\Delta\omega}{2}\right)t - \left(\frac{2k+\Delta k}{2}\right)x\right]$$

Since $\Delta\omega$ and Δk are small

$2\omega + \Delta\omega \approx 2\omega$ and $2k + \Delta k \approx 2k$

$$\therefore y = 2A\cos\left[\left(\frac{\Delta\omega}{2}\right)t - \left(\frac{\Delta k}{2}\right)x\right]\sin(\omega t - kx) \quad \text{.......................... (3)}$$

From equations (1) & (3) it is seen the amplitude becomes

$$2A\cos\left[\left(\frac{\Delta\omega}{2}\right)t - \left(\frac{\Delta k}{2}\right)x\right]$$

The velocity with which the variation in amplitude is transmitted in the resultant wave is the group velocity.

$$v_{group} = \frac{(\Delta\omega/2)}{(\Delta k/2)} = \frac{\Delta\omega}{\Delta k}$$

In the limit $\dfrac{\Delta\omega}{\Delta k} \to \dfrac{d\omega}{dk}$

Relation between group velocity and phase velocity:

The equations for group velocity and phase velocity are:

$$v_{group} = \frac{d\omega}{dk} \quad \text{............ 1}$$

and $$v_{phase} = \frac{\omega}{k} \quad \text{............2}$$

Where ω the angular frequency of the wave and k is the wave number

From 2, $\omega = k v_{phase}$

$$v_{group} = \frac{d\omega}{dk} = \frac{d(k v_{phase})}{dk}$$

Applying product rule in differentiating

$$v_{group} = v_{phase} + k\frac{dv_{phase}}{dk}$$

$$v_{group} = v_{phase} + k\left(\frac{dv_{phase}}{d\lambda}\right)\left(\frac{d\lambda}{dk}\right) \quad \text{....... 3}$$

We have $k = \frac{2\pi}{\lambda}$ differentiating

$$\frac{dk}{d\lambda} = -\frac{2\pi}{\lambda^2} \text{ or } \frac{d\lambda}{dk} = \frac{\lambda^2}{2\pi}$$

$$k\frac{d\lambda}{dk} = \left(\frac{2\pi}{\lambda}\right)\left(-\frac{\lambda^2}{2\pi}\right) = -\lambda$$

Using this in equation (3)

$$v_{group} = v_{phase} - \lambda\left(\frac{dv_{phase}}{d\lambda}\right)$$

This is the relation between group velocity and phase velocity.

Relation between group velocity and particle velocity:

The equation for group velocity is

$$v_{group} = \frac{d\omega}{dk} \quad \text{...... 1}$$

But $$\omega = 2\pi v = 2\pi\frac{E}{h}$$

$$d\omega = \frac{2\pi}{h}dE \quad \text{......... 2}$$

We have $k = \frac{2\pi}{\lambda} = 2\pi\frac{p}{h}$

$$dk = \frac{2\pi}{h}dP \quad \text{.......... 3}$$

Dividing equation 2 by 3 we have

$$\frac{d\omega}{dk} = \frac{dE}{dP} \quad \text{......... 4}$$

But we have $E = {p^2}/{2m}$ Where 'p' is the momentum of the particle.

$$\frac{dE}{dP} = \frac{2P}{2m} = \frac{P}{m}$$

Using the above in equation 4

$$\frac{d\omega}{dk} = \frac{P}{m}$$

But $p = m v_{particle}$,

Where $v_{particle}$ is the velocity of the particle.

$$\frac{d\omega}{dk} = \frac{m v_{particle}}{m} = v_{particle} \quad \text{.......... 5}$$

From equation 1 and 5, we have

$$v_{group} = v_{particle}$$

The de Broglie's wave group associated with a particle travels with a velocity equal to the velocity of the particle itself.

Relation between velocity of light, group velocity and phase velocity:

$$v_{group} \times v_{phase} = c^2$$

Heisenberg's Uncertainty Principle:

According to classical mechanics a particle occupies a definite place in space and possesses a definite momentum. If the position and momentum of a particle is known at any instant of time, it is possible to calculate its position and momentum at any later instant of time. The path of the particle could be traced. This concept breaks down in quantum mechanics leading to Heisenberg's Uncertainty Principle

Heisenberg's Uncertainty Principle

Statement:

It is impossible to measure simultaneously both the position and momentum of a particle accurately. If we make an effort to measure very accurately the position of a particle, it leads to large uncertainty in the measurement of momentum and vice versa.

If Δx and ΔP_x are the uncertainties in the measurement of position and momentum of the particle then the uncertainty can be written as

$$\Delta x \cdot \Delta P_x \geq \frac{h}{4\pi}$$

In any simultaneous determination of the position and momentum of the particle, the product of the corresponding uncertainties inherently present in the measurement is equal to or greater than h/4π.

Similarly

1) $\Delta E \cdot \Delta t \geq \frac{h}{4\pi}$ where ΔE and Δt are uncertainties in measurement of energy and time

2) $\Delta L \cdot \Delta \theta \geq \frac{h}{4\pi}$ where ΔL and $\Delta \theta$ are uncertainties in measurement of angular momentum and angular displacement

Application of Uncertainty Principle:

Impossibility of existence of electron in the atomic nucleus:

According to the theory of relativity, the energy E of a particle is: $E = mc^2 = \frac{m_o c^2}{\sqrt{1-\left(v^2/c^2\right)}}$

Where 'm_o' is the rest mass of the particle and 'm' is the mass when its velocity is 'v'.

i.e. $E^2 = \frac{m_o^2 c^4}{1-\left(v^2/c^2\right)} = \frac{m_o^2 c^6}{c^2 - v^2}$ 1

If 'p' is the momentum of the particle:

i.e. $p = mv = \frac{m_o v}{\sqrt{1-\left(v^2/c^2\right)}}$

$$p^2 = \frac{m_o^2 v^2 c^2}{c^2 - v^2} \quad \text{..................... 2}$$

Multiply by c² on both side of equation 2

$$p^2 c^2 = \frac{m_o^2 v^2 c^4}{c^2 - v^2} \quad \text{....... 3}$$

Subtracting equation 2 by equation 1 we have

$$E^2 - p^2c^2 = \frac{m_o^2 c^4 (c^2 - \mathrm{v}^2)}{c^2 - \mathrm{v}^2}$$

$$E^2 = p^2c^2 + m_o^2 c^4 \Rightarrow E^2 = c^2\left(p^2 + m_o^2 c^2\right) \qquad \text{...... 4}$$

Heisenberg's uncertainty principle states that

$$\Delta x \cdot \Delta p_x \geq \frac{h}{4\pi} \qquad \text{....... 5}$$

The diameter of the nucleus is of the order 10^{-14} m. If an electron is to exist inside the nucleus, the uncertainty in its position Δx must not exceed 10^{-14}m.

i. e. $\Delta x \leq 5\times10^{-15} m$

The minimum uncertainty in the momentum

$$\Delta p_x \geq \frac{h}{4\pi\ \Delta x} \geq \frac{6.63\times10^{-34}}{4\pi\times5\times10^{-15}} \geq 1.1\times10^{-20}\ \text{Ns} \qquad \text{-----} \quad 6$$

By considering minimum uncertainty in the momentum of the electron

i.e., $\Delta p_x \geq 1.1\times10^{-20}$ Ns $= p_x$ ----- 7

Now making use of inequality of equation 6 in 3 and Substituting the values of p_x, c and m_o in equation 3, we get

$$E^2 \geq c^2\left(p^2 + m_o^2 c^2\right)$$

$$E^2 \geq \left(3\times10^8\right)^2\left[\left(1.1\times10^{-20}\right)^2 + \left(9.1\times10^{-31}\right)^2\left(3\times10^8\right)^2\right]$$ Where $m_o = 9.1 \times 10^{-31}$ kg

If the electron exists in the nucleus its energy must be

$$E^2 \geq 1.09\times10^{-23} \Rightarrow E \geq 3.3\times10^{-12}\ J$$

$$E \geq 20.6\ MeV$$

If an electron exists in the nucleus its energy must be greater than or equal to 20.6 Mev. It is experimentally measured that the beta particles ejected from the nucleus during beta decay have energies of about 3 to 4 MeV. This shows that electrons cannot exist in the nucleus.