Line Integral

A line integral is a type of integral where a function is evaluated along a curve or path, rather than over an interval (as in a standard integral) or a region (as in a double integral). In mathematics, it is often used in fields like vector calculus, physics, and engineering to calculate quantities like work, flux, or circulation along a path in a vector field.

Types of Line Integrals

1. **Scalar Line Integral**: Integrates a scalar field (a function of position that assigns a scalar value to each point) along a curve. If a scalar field $f(x,y,z)$ is defined along a path C with parameterization $\vec{r}(t) = (x(t), y(t), z(t))$, the line integral is:

$$\int_c f(x,y,z)ds$$

where ds is the infinitesimal arc length element along C, given by

$$ds = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2}\ dt$$

2. **Vector Line Integral**: Integrates a vector field $\vec{F} = (P, Q, R)$ along a curve, often used to compute work done by a force field along a path. For a curve C, the line integral is:

$$\int_c \vec{F} \cdot \overrightarrow{dr} = \int_c Pdx + Qdy + Rdz$$

3. where $\overrightarrow{dr} = (dx, dy, dz)$ represents the differential displacement along C, and $\vec{F} \cdot \overrightarrow{dr}$is the dot product of the vector field and the tangent vector to the curve.

Applications of Line Integrals

- **Physics**: Calculating work done by a force along a path, such as moving an object in a gravitational or electric field.
- **Engineering**: Analysing fluid flow along a path in fluid dynamics.
- **Electromagnetism**: Computing circulation of a magnetic field around a closed loop.

Method of evaluation

I. Curve defined by parametrically:

If C is a smooth curve parameterized by $x = f(t), y = g(t),\ a \le t \le b$, then we replace $x\ by\ f(t)\ and\ y\ by\ g(t)$ and the approximate differentials $dx\ by\ f'(t)$ and $dy\ by\ g'(t)$.

1. $\int_C G(x,y)dx = \int_a^b G\big(f(t), g(t)\big)f'(t)dt.$
2. $\int_C G(x,y)dy = \int_a^b G\big(f(t), g(t)\big)g'(t)dt.$
3. $\int_C G(x,y)ds = \int_a^b G\big(f(t), g(t)\big)|r'(t)|dt. =$
$\int_a^b G\big(f(t), g(t)\big).\sqrt{\left(f'(t)\right)^2 + \left(g'(t)\right)^2}\ dt$

II. Curve defined by an explicit function:

If the curve C is defined by an explicit function $y = f(x), a \leq x \leq b$, we can use x as parameter with $dy = f'(x)dx \; and \; ds = \sqrt{1 + (f'(x))^2} \; dx$.

1. $\int_C G(x,y)dx = \int_a^b G(x, f(x))dx$.
2. $\int_C G(x,y)dy = \int_a^b G(x, f(x))f'(x)dx$.
3. $\int_C G(x,y)ds = \int_a^b G(x, f(x))\sqrt{1 + (f'(x))^2}dx$.

Examples:

1. Evaluate
   a) $\int_C xy^2 dx$ b) $\int_C xy^2 dy$ c) $\int_C xy^2 ds$ on the quadrature circle defined by $x = 4\cos t, y = 4\sin t$ in the interval $0 \leq t \leq \frac{\pi}{2}$.

Solutions:

Given: $x = 4\cos t, y = 4\sin t, dx = -4\sin t\, dt, dy = 4\cos t dt$

a) $\int_C xy^2 dx = \int_0^{\frac{\pi}{2}} 4\cos t\,(4\sin t)^2(-4\sin t)dt =$

$-4^4 \int_0^{\frac{\pi}{2}} \sin^3 t \cos t\; dt = (-4)^4 \left[\frac{\sin^4 t}{4}\right]_0^{\frac{\pi}{2}} = -64 \qquad \therefore \left\{\int f^n f'(x)dx = \frac{f^{n+1}}{n+1}\right.$

$\therefore \int_C xy^2 dx = -64$

b) $\int_C xy^2 dy = \int_0^{\frac{\pi}{2}} 4\cos t\,(4\sin t)^2(4\cos t)dt$

$= 4^4 \int_0^{\frac{\pi}{2}} \sin^2 t \cos^2 t\; dt \qquad \therefore \left\{\begin{array}{l} \sin(2t) = 2\sin t\cos t \\ \frac{1-cos2t}{2} = \sin^2 t \end{array}\right.$

$$= \frac{4^4}{4}\int_0^{\frac{\pi}{2}} \sin^2(2t)dt$$

$$= 4^3 \int_0^{\frac{\pi}{2}} \frac{1-\cos(4t)}{2}dt$$

$$= \frac{4^3}{2}\left(t - \frac{\sin 4t}{4}\right)_0^{\frac{\pi}{2}} = 32\left\{\frac{\pi}{2} - \frac{\sin(2\pi)}{4} - 0 + 0\right\} = \frac{32\pi}{2}$$

$$\therefore \int_C xy^2 dy = 16\pi$$

c) $\int_C xy^2 ds = \int_0^{\frac{\pi}{2}} 4\cos t\,(4\sin t)^2\sqrt{4^2\sin^2 t + 4^2\cos^2 t}\,dt$

$$= 4^4 \int_0^{\frac{\pi}{2}} \sin^2 t\cos t\; dt$$

$$= (4)^4 \left[\frac{\sin^3 t}{3}\right]_0^{\frac{\pi}{2}} = \frac{256}{3}$$

$$\therefore \int_C xy^2 ds = \frac{256}{3}$$

2. Evaluate

   $\int_C (xydx + x^2dy)$,where c is given by $y = x^3, -1 \le x \le 2$

   Solution: Given $y = x^3, dy = 3x^2dx$

   $$\int_C (xydx + x^2dy) = \int_{-1}^{2} x\, x^3dx + x^2(3x^2)dx$$

   $$\int_{-1}^{2} 4\, x^4dx = 4\left[\frac{x^5}{5}\right]_{-1}^{2} = \frac{4 \times 33}{5} = \frac{132}{5}$$

   $$\therefore \int_C (xydx + x^2dy) = \frac{132}{5}$$

3. Evaluate

   $\int_C (4xdx + 2ydy)$,where c is given by $x = y^3 + 1$ from the point $(0, -1)$to$(9,2)$

Solution:

Given $x = y^3 + 1, dx = 3y^2dy$

$$\int_{-1}^{2} (4xdx + 2ydy) = \int_{-1}^{2} 4y^3 + 4)3y^2dy + 2ydy$$

$$= \int_{-1}^{2} (12y^5 + 12y^2 + 2y)dy$$

$$= \left[\frac{12y^6}{6} + \frac{12y^3}{3} + \frac{2y^2}{2}\right]_{-1}^{2} = [2y^6 + 4y^3 + y^2]_{-1}^{2} = 165$$

$$\therefore \int_{-1}^{2} (4xdx + 2ydy) = 165$$

4. If $\vec{F} = xy\boldsymbol{i} + yz\boldsymbol{j} + zx\boldsymbol{k}$, then evaluate $\int_C \vec{F} \cdot d\vec{r}$, where C is the curve represented by $x = t, y = t^2, z = t^3, -1 \le t \le 1$.

   Solution:

   Given: $x = t, y = t^2, z = t^3$

   $\vec{r} = x\, \boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$;

   $\vec{r} = t\, \boldsymbol{i} + t^2\boldsymbol{j} + t^3\boldsymbol{k}$

   $d\vec{r} = dt\, \boldsymbol{i} + 2tdt\boldsymbol{j} + 3t^2dt\boldsymbol{k}$

   $\vec{F} = xy\boldsymbol{i} + yz\boldsymbol{j} + zx\boldsymbol{k} = \vec{F} = t^3\boldsymbol{i} + t^5\boldsymbol{j} + t^4\boldsymbol{k}$

   $\vec{F} \cdot d\vec{r} = (t^3\boldsymbol{i} + t^5\boldsymbol{j} + t^4\boldsymbol{k}) \cdot (\, \boldsymbol{i} + 2t\boldsymbol{j} + 3t^2\boldsymbol{k})dt = t^3 + 2t^6 + 3t^6$

   $$\int_C \vec{F} \cdot d\vec{r} = \int_{-1}^{1} t^3 + 5t^6\, dt$$

   $$\left[\frac{t^4}{4} + \frac{5t^7}{7}\right]_{-1}^{1} = \frac{10}{7}$$

   $$\therefore \int_C \vec{F} \cdot d\vec{r} = \frac{10}{7}$$

5. If $\vec{F} = (3x^2 + 6y)\boldsymbol{i} - 14yz\boldsymbol{j} + 20z^2x\boldsymbol{k}$, , over the curve C is given by $x = t, y = t^2, z = t^3$, from the point $(0,0,0)$to the point $(1,1,1)$

Solution:
Given: $x = t, y = t^2, z = t^3$
$\vec{r} = x\,\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$;
$\vec{r} = t\,\boldsymbol{i} + t^2\boldsymbol{j} + t^3\boldsymbol{k}$
$d\vec{r} = (\boldsymbol{i} + 2t\boldsymbol{j} + 3t^2\boldsymbol{k})dt$
$\vec{F} = (3x^2 + 6y)\boldsymbol{i} - 14yz\boldsymbol{j} + 20z^2x\boldsymbol{k} = \vec{F} = (3t^2 + 6t^2)\boldsymbol{i} - 14t^5\boldsymbol{j} + 20t^7\boldsymbol{k}$
$\vec{F}\cdot d\vec{r} = (3t^2 + 6t^2)\boldsymbol{i} - 14t^5\boldsymbol{j} + 20t^7\boldsymbol{k}\cdot(\boldsymbol{i} + 2t\boldsymbol{j} + 3t^2\boldsymbol{k})dt = 9t^2 - 28t^6 + 60t^4$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (9t^2 - 28t^6 + 60t^4)\,dt$$

$$= \left[\frac{9t^3}{3} - \frac{28t^7}{7} + \frac{60t^5}{5}\right]_0^1 = 3 - 4 + 5 = 5$$

$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = 5$

6. Evaluate the integral I along the circular arc C given by $x = \cos t$, $y = \sin t$, $0 \le t \le \frac{\pi}{2}$

$$I = \int_C 2xydx + (x^2 + y^2)dy.$$

Solution: Given $x = \cos t\,, y = \sin t$

$$dx = -\sin t\,dt, dy = costdt$$

Substitute $x, y, dx, and\ dy$ into the integral: We can rewrite III as:

$$I = \int_C 2\cos t\sin t\,(-\sin t)dt + (\cos^2 t + \sin^2 t)costdt.$$
$$I = \int_0^{\frac{\pi}{2}}(-2\sin^2 t\cos t + cost\,)dt$$
$$I = -2\int_0^{\frac{\pi}{2}}\sin^2 t\cos t\,)dt + \int_0^{\frac{\pi}{2}} costdt$$
$$I = -2\left[\frac{\sin^3 t}{3}\right]_0^{\frac{\pi}{2}} + [\sin t]_0^{\frac{\pi}{2}} = -\frac{2}{3} + 1 = \frac{1}{3}$$

7. Evaluate $\int_C 4x^3 ds$ where C is the line segment from $(-2,-1)$ to $(1,2)$.

Solution: Express $4x^3$ $in\ terms\ of\ t$**:**

$Since\ \ x = -2 + 3t$ we have:
$4x^3 = 4(-2 + 3t)^3$; $ds = \sqrt{3^2 + 3^2} = \sqrt{18}\,dt = 3\sqrt{2}\,dt$

$I = \int_C 4x^3 ds = \int_0^1 4(2 + 3t)^3 3\sqrt{2}\,dt$
$I = 12\sqrt{2}\int_0^1(-2 + 3t)^3\,dt$
$I = 12\sqrt{2}\int_0^1(-8 + 36t - 54t^2 + 27t^3)\,dt$
Integrate and apply the limit,

$$I = 12\sqrt{2}\left[-8t + 18t^2 - 18t^3 + \frac{27t^4}{4}\right]_0^1 = -15\sqrt{2}$$

8. Define work done along the curve C. Find the work done in moving particle in the force field

$\vec{\mathrm{F}} = 3x^2\hat{\imath} + (2xz - y)\hat{\jmath} + z\hat{\mathrm{k}}$, over the curve defined by $x = t, y = \frac{t^2}{4}, z = \frac{3t^3}{8}$ in the interval $0 \leq t \leq 1$.

Solution: Find

$\vec{F} \cdot d\vec{r} = 3t^2 + \frac{3t^5 - t^3}{8} + \frac{27t^5}{64}$

$I = \int_C \vec{F} \cdot d\vec{r} = \int_0^1 (3t^2 + \frac{3t^5 - t^3}{8} + \frac{27t^5}{64})\, dt$

$I = \int_0^1 3t^2 dt + \int_0^1 \frac{3t^5}{8} dt - \int_0^1 \frac{t^3}{8} dt + \int_0^1 \frac{27t^5}{64} dt$

Integrate and apply the limit, we get

$I = \frac{141}{128}$