Faculty of Natural Sciences

Department	**: Mathematics and Statistics**
Programme	**: B. Tech.**
Semester/Batch	**: 3rd /2023**
Course Code	**:**
Course Title	**: Engineering Mathematics – 3**

Tutorial-2

Gradient, Directional derivative of scalar field, Divergence and Curl of vector field

1. Obtain the grad ϕ for $\phi(x, y, z) = x^2yz^2$ at the point $(1,2,3)$
2. If $\phi(x, y, z) = xy^2z + 3x^2 - z^3$, determine the grad$\phi$ at the point $(1,3,4)$
3. Compute the gradient for the function $\phi(x, y) = x^2 - x^3y^2 + y^4$
4. Obtain the directional derivative of $\phi(x, y, z) = 2x^2y^3 + 6xy$ at $(1,1)$ in the direction of a unit vector whose angle with positive $x - axis$ is $\pi/6$.
5. Determine the gradient of scalar field $\phi = x^3y^3z^3$
6. Determine the directional derivative of $\phi(x, y, z) = x^2yz + 4xz^2$ at the point $(1, -2, -1)$in the direction of $2\boldsymbol{i} - \boldsymbol{j} - 2\boldsymbol{k}$
7. Obtain the directional derivative of $\phi(x, y, z) = xy^2 - 4x^2y + z^2$ at $(1, -1,2)$ in the direction of $6i + 2j + 3k$.
8. Find the directional derivative of the function $\boldsymbol{\phi} = xe^y + ye^z + ze^x$ at the point $(0,0,0)$ in the direction of the vector $\vec{v} = -\hat{\imath} - 2\hat{\jmath} + 2\hat{k}$. Also find the maximal directional derivative.
9. Find the directional derivative of $\phi = x^2 - y^2 + 2z^2$ at $P(1,2,3)$ in the direction from P to $Q(5,0,4)$.
10. Suppose that the temperature at a point $x, y, z)$ in space is given by $\boldsymbol{T(x, y, z)} = \frac{80}{1+x^2+2y^2+3z^2}$, where T is measured in degrees Celsius and x, y, z in meters. In which direction does the temperature increase fastest at the point $(1,1, -2)$? What is the maximum rate of increase?
11. The temperature in a rectangular box is approximated by $\boldsymbol{T(x, y, z)} = xyz(1 - x)(2 - y)(3 - z)$, $0 \leq x \leq 1, 0 \leq y \leq 2, 0 \leq y \leq 3$. If a mosquito is located at $\left(\frac{1}{2}, 1,1\right)$, in which direction should it fly to cool off as rapidly as possible.
12. If $F = xy^2i + 2x^2yz^2j - 3yz^2k$, Find div$F$ at the point $(1,1, -1)$.
13. If $F = 2xy^2i + 3x^2zj - 4xyzk$, Find div$F$.
14. If the vector field $\vec{F} = (Ax + 3y + 4z)i + (x - 2y + 3z)j + (3x + 2y - z)k$ is solenoidal then find A.
15. Given the vector field $F = (bx + 4y^2z)\mathrm{i} + (x^2 sin\, z - 3y)\mathrm{j} + (e^x + 4\cos^2 yx)\mathrm{k}$. For what value of b the vector field is solenoidal.
16. If $F = 2xy^2\mathrm{i} + 3x^2z\mathrm{j} - 4xyz\mathrm{k}$, Find curl of F at $P(1,1 - ,2)$.
17. Show that fluid motion given by the vector field $F = (y + z)i + (z + x)j + (x + y)k$ is irrotational. Hence find its scalar field.
18. Find the value of the constant 'a' such that:
$F = (axy - z^3)i + (a - 2)x^2\, j + (1 - a)\, xz^2$ is irrotational.
19. Show that $F = (2xy^2 + yz)i + (2x^2y + xz + 2yz^2)\, j + (2y^2z + xy)k$ is a conservative force field. Find its scalar potential.

20. Obtain the curl and divergence of the vector field $F = xyz\, i + x^2y^2z\, j + yz^3\boldsymbol{k}$ and hence find scalar potential ϕ if $curl\,\vec{F} = 0$.
21. Show that the vector field
$\boldsymbol{F} = (x^2 - yz)\mathbf{i} + (y^2 - xz)\mathbf{j} + (z^2 - xy)\mathbf{k}$ Is irrotational. Obtain the scalar potential ϕ such that $\vec{\mathbf{F}} = \nabla\phi$

Answer:

1. $\nabla\phi(1,2,3) = 36\,\boldsymbol{i} + 9\,\boldsymbol{j} + 12\,\boldsymbol{k}$
2. $\nabla\phi(1,3,4) = 24\,\boldsymbol{i} + 24\boldsymbol{j} - 39\,\boldsymbol{k}$
3. $\nabla\phi = (2x - 3x^2\,y^2)\boldsymbol{i} + (-2x^3y + 4y^3)\boldsymbol{j}$
4. $D_{\vec{v}}\phi(1,1) = \frac{4\sqrt{3}}{2} + 3$
5. $\nabla\phi = (3x^2\,y^3z^3)\boldsymbol{i} + (3x^3y^2z^3)\boldsymbol{j} + 3x^3y^3z^2\boldsymbol{k}$
6. $D_{\vec{v}}\phi(1,-2,-1) = (8i - j - 10k)\cdot\left(\frac{2}{3}i - \frac{1}{3}j - \frac{2}{3}k\right) = \frac{37}{3}$
7. $D_{\vec{v}}\phi(1,-1,2) = (9i - 6j + 4k)\cdot(76i + 72j + 73k) = \frac{54}{7}$
8. $D_{\vec{v}}\phi(1,-2,-1) = (i + j + k)\cdot\left(-\frac{1}{3}i - \frac{2}{3}j + \frac{2}{3}k\right) = -\frac{1}{3};$

The maximal directional derivative occurs in the direction of the gradient$\nabla\phi$. The magnitude of the gradient $\nabla\phi(0,0,0) = (1,1,1)$ is:

$$|\ \nabla\phi\ | = \sqrt{1^2 + 1^2 + 1^2} = 3$$

So, the maximal directional derivative is:

$$|\ \nabla\phi\ | = 3$$

9. $D_{\vec{v}}\phi(1,2,3) = (2i - 4j + 12k)\cdot\left(\frac{4}{\sqrt{21}}i - \frac{2}{\sqrt{21}}j + \frac{1}{\sqrt{21}}k\right)$

The directional derivative of ϕ at $P(1,2,3)$ in the direction from P to $Q(5,0,4)$ is:

$$D_{\vec{v}}\phi(1,2,3) = \frac{28}{\sqrt{21}} = \frac{4\sqrt{21}}{3}$$

10. $\nabla T = \frac{5}{8}\,(i - 2j + 6k);|\ \nabla T\ | = \frac{5}{8}\sqrt{41} = 4^0\ c/m.$
11. $\nabla T\left(\frac{1}{2},1,1\right) = \frac{1}{4}$

The mosquito should fly in the direction $-\frac{1}{4}k$ to cool off rapidly.

12. $\nabla\cdot\vec{\mathrm{F}} = y^2 + 2x^2z^2 - 6yz;\ \nabla\cdot\vec{\mathrm{F}}(1,1,-1) = 1^2 + 21^2(-1)^2 - 61(-1) = 9$
13. $\nabla\cdot\vec{\mathrm{F}} = 2y^2 - 4xy$
14. $A = 3$
15. For $b = 3$ the given vector field is incompressible.
16. $\nabla\times\vec{\mathrm{F}} = 5i - 8j - 6k$
17. $\nabla\times\vec{\mathrm{F}} = 0$, the vector filed is irrotational and The scalar potential function $\phi = xy + zx + yz$
18. $\nabla\times\vec{\mathrm{F}} = (az - 4z^2)j + k(ax - 4x) = 0$. Hence $a = 4$.
19. $\nabla\times\vec{\mathrm{F}} = 0$, the vector filed is irrotational and The scalar potential function $\phi = x^2y^2 + yzx + y^2z^2$.
20. $\nabla\times\vec{\mathrm{F}} = (z^3 - x^2y^2)j + (xy)j + (2xyz - xz)k$; Since this curl is not zero, F is not a conservative field, meaning it does not have a scalar potential ϕ such that $\vec{\mathbf{F}} = \nabla\phi$

$\nabla\cdot\vec{\mathrm{F}} = \mathrm{yz} + 2\mathrm{x}^2\mathrm{yz} + 3\mathrm{yz}^2.$

21. The vector field $\vec{\mathrm{F}}$ is irrotational. The scalar potential is given by: $\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz + C$.