3-14: Solve problem 3-8 using the Laplace transform technique for the period of time where the material can be treated as a semi-infinite body.

Problem 2 - Textbook Problem 3-8

(Note: the boundary condition at the left surface (x=0) is different from what we are used to. It has convection in addition to the constant heat flux. Thus, the heat flux into the domain will be balanced by both convection and the given heat flux at the boundary).

3–8 Figure P3-8 shows a slab of material that is $L = 5$ cm thick and is heated from one side ($x = 0$) by a radiant heat flux $\dot{q}_s'' = 7500\,\text{W/m}^2$. The material has conductivity $k = 2.4$ W/m-K and thermal diffusivity $\alpha = 2.2 \times 10^{-4}\,\text{m}^2/\text{s}$. Both sides of the slab are exposed air at $T_\infty = 20°\text{C}$ with heat transfer coefficient $\bar{h} = 15\,\text{W/m}^2\text{-K}$. The initial temperature of the material is $T_{ini} = 20°\text{C}$.

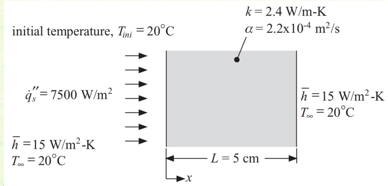


Figure P3-8: Slab of material heated at one surface.

a) Prepare and implement a solution and plot the temperature as a function of position for several times.

Governing equation and Boundary / Initial conditions

GDE : $\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$

B.C. : $x=0: -k\frac{\partial T}{\partial x} = \dot{q}_s'' - h(T(0,t) - T_\infty)$

$x=L: \; T(L,t) = T_i$

I.C. : $t=0: \; T(x,0) = T_i$

Laplace Transform of equation and boundary / initial conditions

$$\left.\begin{aligned}
&\mathcal{L}\left\langle \frac{\partial T}{\partial t}\right\rangle = s\hat{T}(x,s) - T(x,0)\\
&\mathcal{L}\left\langle \alpha\frac{\partial^2 T}{\partial x^2}\right\rangle = \alpha\frac{\partial^2 \hat{T}(x,s)}{\partial x^2}\\
&\mathcal{L}\left\langle -k\frac{\partial T}{\partial x}\right\rangle = -k\frac{\partial \hat{T}(x,s)}{\partial x}\\
&\mathcal{L}\left\langle \dot{q}_s'' - h(T(0,t) - T_\infty)\right\rangle = \frac{\dot{q}_s''}{s} + \frac{hT_\infty}{s} - h\hat{T}(0,t)
\end{aligned}\right\}$$

$$s\hat{T}(x,s) - T(x,0) = \alpha\frac{\partial^2 \hat{T}}{\partial x^2}$$

B.C. :

$$-k\frac{\partial \hat{T}(0,s)}{\partial x} = \frac{\dot{q}_s''}{s} - h\left(\hat{T}(0,t) - \frac{T_\infty}{s}\right)$$

$$\hat{T}(L,s) = \frac{T_i}{s}$$

Solve

$$s\hat{T}(x,s) - T_i = \alpha\frac{\partial^2 \hat{T}}{\partial x^2}$$

$$\frac{\partial^2 \hat{T}}{\partial x^2} - \frac{s}{\alpha}\hat{T} = -\frac{T_i}{\alpha}$$

$$\left.\begin{aligned}
&\text{use } \hat{\theta} = \hat{T} - \frac{T_i}{s}\\
&\frac{\partial^2\hat{\theta}}{\partial x^2} - \frac{s}{\alpha}\hat{\theta} = 0
\end{aligned}\right\}
\quad
\begin{aligned}
&\text{B.C. } \hat{T} = \hat{\theta} + \frac{T_i}{s}\\
&-k\frac{\partial \hat{\theta}(0,s)}{\partial x} = \frac{\dot{q}''}{s} - h\left(\left(\hat{\theta}(0,s) + \frac{T_i}{s}\right) - \frac{T_\infty}{s}\right)\\
&\hat{\theta}(L,s) = 0
\end{aligned}$$

$$\hat{\theta} = C_1 e^{\sqrt{\frac{s}{\alpha}}x} + C_2 e^{-\sqrt{\frac{s}{\alpha}}x}$$

$$\frac{\partial\hat{\theta}}{\partial x} = C_1\sqrt{\frac{s}{\alpha}}e^{\sqrt{\frac{s}{\alpha}}x} - C_2\sqrt{\frac{s}{\alpha}}e^{\sqrt{\frac{s}{\alpha}}x}$$

B.C. 1: $-k\sqrt{\frac{s}{\alpha}}(C_1 - C_2) = \frac{\dot{q}''}{s} - h\left(\left(C_1 + C_2 + \frac{T_i}{s}\right) - \frac{T_\infty}{s}\right)$

2: $\hat{\theta}(L,s) = 0 \;\rightarrow$ treating as semi-infinite, $L \rightarrow \infty \Rightarrow \hat{\theta}(\infty,s) = C_1 e^{\infty} + C_2 e^{-\infty} = 0$

$\therefore \; C_1 = 0$

$$\Rightarrow k\sqrt{\frac{s}{\alpha}}C_2 = \frac{\dot{q}''}{s} - h\left(\frac{T_\infty}{s} - \left(C_2 + \frac{T_i}{s}\right)\right)$$

$$C_2\left(k\sqrt{\frac{s}{\alpha}} + h\right) = \frac{\dot{q}''}{s} - h\frac{T_\infty}{s} + h\frac{T_i}{s}$$

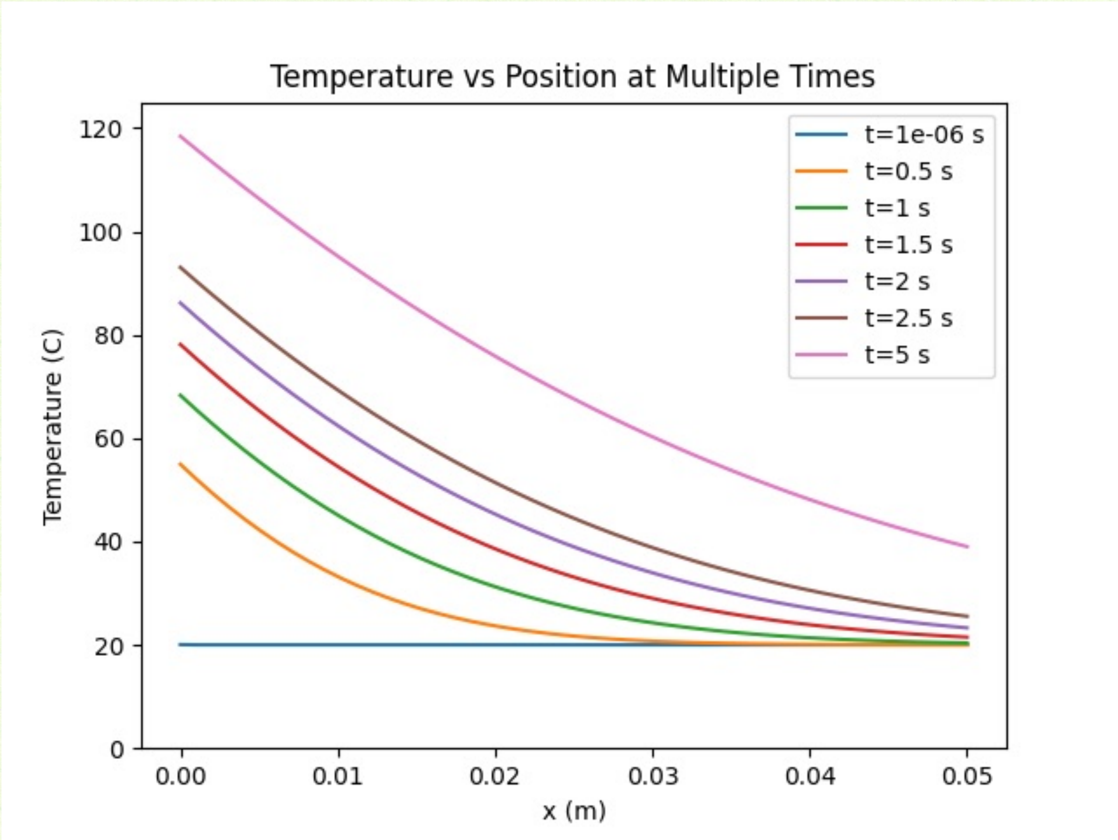
$$C_2 = \frac{\dot{q}'' + h(T_i - T_\infty)}{s\left(k\sqrt{\frac{s}{\alpha}} + h\right)} \quad \rightarrow T_i = 20°C = T_\infty \Rightarrow C_2 = \frac{\dot{q}''}{s\left(k\sqrt{\frac{s}{\alpha}} + h\right)}$$

$$\hat{\theta} = \frac{\dot{q}''e^{-\sqrt{\frac{s}{\alpha}}x}}{s\left(k\sqrt{\frac{s}{\alpha}} + h\right)}, \quad \hat{T} = \hat{\theta} + \frac{T_i}{s}$$

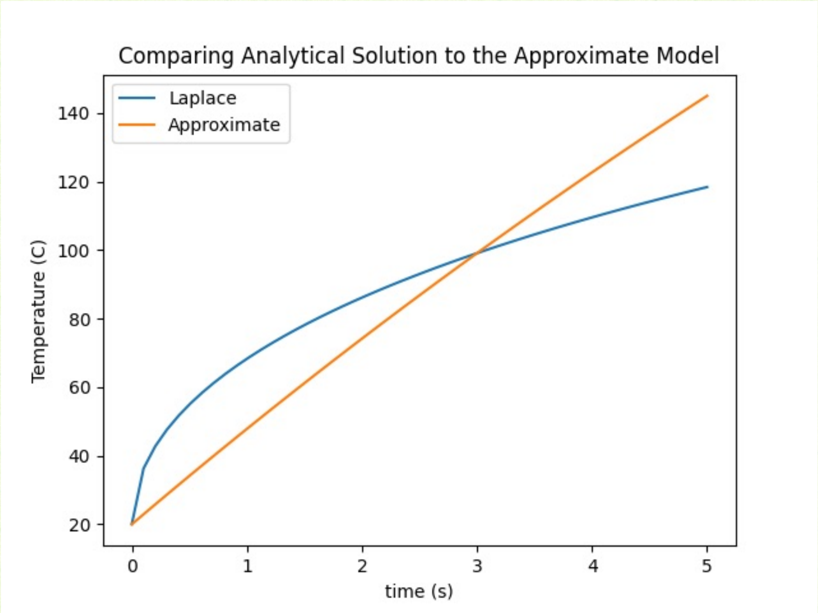
$$\hat{T} = \frac{\dot{q}''e^{-\sqrt{s/\alpha}\,x}}{s\left(k\sqrt{s/\alpha} + h\right)} + \frac{T_i}{s} \quad \xrightarrow[\text{to match table}]{\text{fix form}} \quad \frac{\frac{\sqrt{\alpha}}{k}}{\frac{\sqrt{\alpha}}{k}} \quad \hat{T} = \frac{\dot{q}''}{h}\frac{\frac{\sqrt{\alpha}h}{k}e^{-\sqrt{s/\alpha}\,x}}{s\left(\sqrt{s} + \frac{\sqrt{\alpha}h}{k}\right)} + \frac{T_i}{s}, \quad \text{matches} \quad \frac{a\exp(-C\sqrt{s})}{s(a+\sqrt{s})}$$

$$T(x,t) = \frac{\dot{q}''}{h}\left(\text{erfc}\left(\frac{C}{2\sqrt{t}}\right) - \exp(aC)\exp(a^2t)\,\text{erfc}\left(a\sqrt{t} + \frac{C}{2\sqrt{t}}\right)\right), \quad a = \frac{\sqrt{\alpha}h}{k}, \quad C = \frac{x}{\sqrt{\alpha}}$$

$$\underline{\underline{T(x,t) = \frac{\dot{q}''}{h}\left(\text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) - \exp\left(\frac{hx}{k}\right)\exp\left(\frac{\alpha h^2 t}{k^2}\right)\text{erfc}\left(\frac{\sqrt{\alpha t}\,h}{k} + \frac{x}{2\sqrt{\alpha t}}\right)\right) + T_i}} \longleftarrow \text{Analytical Solution}$$



b) Compare the analytical solution in a) to the approximate model that you derived in c) of problem 3-8



The approximate solution acts more linear for longer times. This results in higher surface temperatures for the approximate solution.

* Uses $T_s = \left(\frac{q''}{h} + T_\infty\right)\left(1 - e^{\frac{-2ht}{\rho c L}}\right) + T_i$, which is my solution for T_s from problem 3-8