2.2-3 (2-3 in text) You are the engineer responsible for a simple device that is used to measure heat transfer coefficient as a function of position within a tank of liquid (Figure P2.2-3). The heat transfer coefficient can be correlated against vapor quality, fluid composition, and other useful quantities. The measurement device is composed of many thin plates of low conductivity material that are interspersed with large, copper interconnects. Heater bars run along both edges of the thin plates. The heater bars are insulated and can only transfer energy to the plate; the heater bars are conductive and can therefore be assumed to come to a uniform temperature as a current is applied. This uniform temperature is assumed to be applied to the top and bottom edges of the plates. The copper interconnects are thermally well-connected to the fluid; therefore, the temperature of the left and right edges of each plate are equal to the fluid temperature. This is convenient because it isolates the effect of adjacent plates from one another which allows each plate to measure the local heat transfer coefficient. Both surfaces of the plate are exposed to the fluid temperature via a heat transfer coefficient. It is possible to infer the heat transfer coefficient by measuring heat transfer required to elevate the heater bar temperature a specified temperature above the fluid temperature.

The nominal design of an individual heater plate utilizes metal with k = 20 W/m-K, th = 0.5 mm, a = 20 mm, and b = 15 mm (note that a and b are defined as the half-width and half-height of the heater plate, respectively, and th is the thickness as shown in Figure P2-3). The heater bar temperature is maintained at T_h = 40ºC and the fluid temperature is T_∞ = 20ºC. The nominal value of the average heat transfer coefficient is $\bar{h}$ = 50 W/m^2-K.

a.) Develop an analytical model that can predict the temperature distribution in the plate under these nominal conditions.

b.) The measured quantity is the rate of heat transfer to the plate from the heater ($\dot{q}_h$) and therefore the relationship between $\dot{q}_h$ and $\bar{h}$ (the quantity that is inferred from the heater power) determines how useful the instrument is. Determine the heater power.

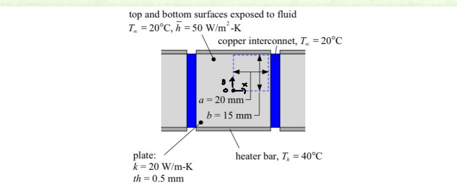


Figure P2.2-3: Device to measure heat transfer coefficient as a function of position.

a) $T(x,y)$

Step 1: Calculate Bi to justify ignoring z temperature gradient

$$B_i = \frac{hL}{k} = \frac{h\,th}{2k} = \frac{50\,W/m^2\cdot K \cdot 0.0005\,m}{2 \cdot 20\,W/m\cdot K} = 0.000625 \quad << 1$$

Step 2: differential CV energy balance

$$\frac{\partial \dot{q}_x}{\partial x}dx + \frac{\partial \dot{q}_y}{\partial y}dy = -\dot{q}_{conv}$$

$$\frac{\partial}{\partial x}\left[k\,th\,dy\frac{\partial T}{\partial x}\right]dx + \frac{\partial}{\partial y}\left[k\,th\,dx\frac{\partial T}{\partial y}\right]dy = 2\bar{h}(T-T_\infty)\,dx\,dy$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{2\bar{h}}{k\,th}(T-T_\infty)$$

Step 3: Boundary Conditions (use the middle of the plate as $(x,y) = (0,0)$)

1) $x=0$: $\frac{\partial T}{\partial x} = 0$
2) $x=a$: $T = T_\infty$
3) $y=0$: $\frac{\partial T}{\partial y} = 0$
4) $y=b$: $T = T_h$

→ use $\theta = T - T_\infty$ →

$\frac{\partial \theta}{\partial x} = 0$, $\theta = 0$ → x is homogeneous direction

$\frac{\partial \theta}{\partial y} = 0$, $\theta = T_h - T_\infty = \theta_h$ → y is nonhomogeneous direction

Step 4: Adjust equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{2\bar{h}}{k\,th}(T-T_\infty) \rightarrow \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = \frac{2\bar{h}}{k\,th}\theta \rightarrow G = \frac{2\bar{h}}{k\,th}$$

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} - G\theta = 0$$

Step 5: Solve boundary Conditions

$$\theta = X(x)\,Y(y)$$

$$\frac{\partial^2 (XY)}{\partial x^2} + \frac{\partial^2 (XY)}{\partial y^2} - G(XY) = Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} - GXY = 0$$

$$\underbrace{\frac{1}{X}\frac{\partial^2 X}{\partial x^2}}_{-\lambda^2} + \underbrace{\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} - G}_{\lambda^2} = 0$$

$$\frac{\partial^2 X}{\partial x^2} + \lambda^2 X = 0 \quad , \quad \frac{\partial^2 Y}{\partial y^2} - (\lambda^2 + G)Y = 0$$

$$X = C_1 \sin\lambda x + C_2 \cos\lambda x$$

$$Y = C_3 \cosh(\sqrt{\lambda^2+G}\,y) + C_4 \sinh(\sqrt{\lambda^2+G}\,y)$$

homogeneous direction

X: 1) $\left.\frac{\partial X}{\partial x}\right|_{x=0} = 0 = \lambda(C_1\cancelto{1}{\cos(0)} - C_2\cancelto{0}{\sin(0)}) \Rightarrow C_1 = 0$

2) $\left.\theta\right|_{x=a} = 0 = C_2\cos\lambda a\,(C_3\cosh(\sqrt{\lambda^2+G}\,y) + C_4\sinh(\sqrt{\lambda^2+G}\,y)) \Rightarrow \cos\lambda a = 0$

$$\lambda_n = \frac{(2i-1)\pi}{2a} \quad , \; i = 1,2,3,\ldots,\infty$$

nonhomogeneous direction

↓

3) $\frac{\partial \theta}{\partial y}\Big|_{y=0} = \frac{\partial Y}{\partial y} X$, $\frac{\partial Y}{\partial y}\Big|_{y=0} = \sqrt{\lambda^2+G}\left(C_3 \sinh(\cancelto{0}{\sqrt{\lambda^2+G}\, 0}) + C_4 \cosh(\cancelto{1}{\sqrt{\lambda^2+G}\, 0})\right) = 0$ $\Rightarrow C_4 = 0$

$$\theta = \sum \theta_i = \sum C_i \cos(\lambda x)\cosh(\sqrt{\lambda_i^2+G}\, y) \quad , \quad C_i = C_2 \cdot C_3$$

4) $\theta_{y=b} = \sum C_i \cos(\lambda_i x)\cosh(\sqrt{\lambda_i^2+G}\, b) = \theta_h$, $\lambda_i = \frac{(2i-1)\pi}{2a}$, $G = \frac{2\bar{h}}{k\, th}$

$$\sum_{i=1}^{\infty} C_i \int_0^a \cos(\lambda_i x)\cosh(\sqrt{\lambda_i^2+G}\, b)\cos(\lambda_j x)\cosh(\sqrt{\lambda_j^2+G}\, b)\, dx = \theta_h \int_0^a \cos(\lambda_j x)\cosh(\sqrt{\lambda_j^2+G}\, b)\, dx$$

$$C_i \cosh^2(\sqrt{\lambda_i^2+G}\, b)\int_0^a \cos^2(\lambda_i x)\, dx = \theta_h \cosh(\sqrt{\lambda_i^2+G}\, b)\int_0^a \cos(\lambda_i x)\, dx$$

$$C_i = \frac{\theta_h \int_0^a \cos(\lambda_i x)\, dx}{\cosh(\sqrt{\lambda_i^2+G}\, b)\int_0^a \cos^2(\lambda_i x)\, dx}$$

$\int_0^a \cos(\lambda_i x)\, dx = \frac{1}{\lambda_i}\left(\sin(\lambda_i a) - \sin(\lambda_i 0)\right)$, $\sin(\lambda_i a) = 1$ $\Rightarrow \int_0^a \sin(\lambda_i x)\, dx = \frac{1}{\lambda_i}$ → alternates between $\frac{1}{\lambda_i}$ and $-\frac{1}{\lambda_i}$ (+ if i is odd, - if i is even)

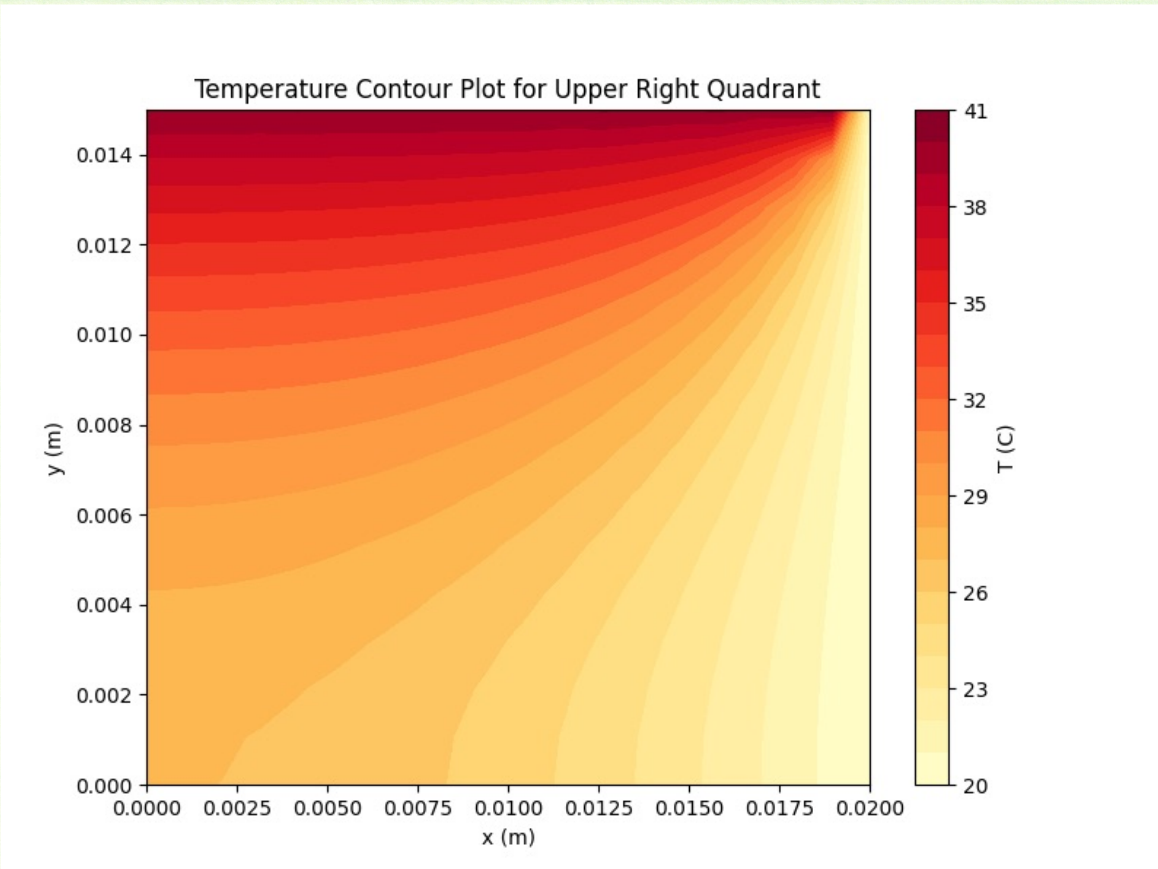
$\int_0^a \cos(\lambda_i x) = \frac{(-1+2(i\%2))}{\lambda_i}$, % is modulo operator (remainder after division)

$\int_0^a \cos^2(\lambda_i x)\, dx = \frac{1}{2}\left(a + \frac{1}{2\lambda_i}\sin(2\lambda_i a)\right) = \frac{a}{2}$

$$C_i = \frac{2\theta_h(-1+2(i\%2))}{a\lambda_i \cosh(\sqrt{\lambda_i^2+G}\, b)}$$

$$\theta = \sum_{i=1}^{\infty} \frac{2\theta_h(-1+2(i\%2))}{a\lambda_i \cosh(\sqrt{\lambda_i^2+G}\, b)}\cos(\lambda_i x)\cosh(\sqrt{\lambda_i^2+G}\, y) \quad , \quad \lambda_i = \frac{(2i-1)\pi}{2a} \quad , \quad G = \frac{2\bar{h}}{k\, th} \quad , \quad \theta_h = T_h - T_\infty$$

$$T(x,y) = \theta(x,y) + T_\infty$$



b) $\frac{\partial u}{\partial t} = q_{net} = \dot{q}_h - \dot{q}_{out} = 0$

$$\dot{q}_h = \dot{q}_{out} = 4\int_0^a\int_0^b 2\bar{h}(\theta(x,y))\, dy\, dx + 4k\, th\int_0^b -\frac{\partial \theta}{\partial x}\Big|_{x=a} dy$$

$$\dot{q}_{out} = 8\bar{h}\sum_{i=1}^{\infty} C_i \int_0^a\int_0^b \cos(\lambda_i x)\cosh(\sqrt{\lambda_i^2+G}\, y)\, dy\, dx + 4k\, th\sum_{i=1}^{\infty} C_i\left(\lambda_i \sin(\lambda_i a)\int_0^b \cosh(\sqrt{\lambda_i^2+G}\, y)\, dy\right)$$

$$\dot{q}_h = 8\bar{h}\sum_{i=1}^{\infty}\left[C_i\left(\frac{1}{\lambda_i}\sin(\lambda_i x)\right)\Big|_0^a \left(\frac{1}{\sqrt{\lambda_i^2+G}}\sinh(\sqrt{\lambda_i^2+G}\, y)\right)\Big|_0^b\right] + 4k\, th\sum_{i=1}^{\infty} C_i\left(\lambda_i \sin(\lambda_i a)\left(\frac{1}{\sqrt{\lambda_i^2+G}}\sinh(\sqrt{\lambda_i^2+G}\, y)\right)\right)\Big|_0^b$$

$$\dot{q}_h = 3.946\, W$$