8-21. TEXSTAN *analysis of laminar thermal entry flow in a circular pipe with constant surface temperature*: Calculate the flow and construct a plot similar to Fig. 8-10 to show development of the Nusselt number with $x^+ = 2(x/D_h)/\text{Re Pr}$ over the range $x^+ = 0\text{–}0.3$. Let the Prandtl number be 0.7. Compare the results with Table 8-4. Feel free to evaluate the nondimensional temperature profiles at various x^+ locations to demonstrate the concept of how the profiles evolve from a flat profile into thermally fully developed profile, and to investigate any other attribute of the entry region or thermally fully developed region of the flow.

Let the Reynolds number be 1000, and pick fluid properties that are appropriate to the chosen Prandtl number. You can choose how to set up the TEXSTAN problem in terms of values for the thermal boundary and initial conditions, and for geometrical dimensions and mass flow rate for the pipe to provide the required Reynolds number and a pipe length equivalent to $x^+ = 0.3$. Use constant fluid properties and do not consider viscous dissipation. For initial conditions let the velocity profile be hydrodynamically fully developed and the temperature profile be flat at some value T_e.

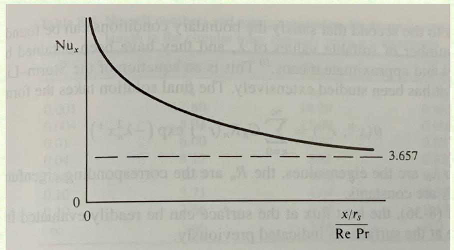
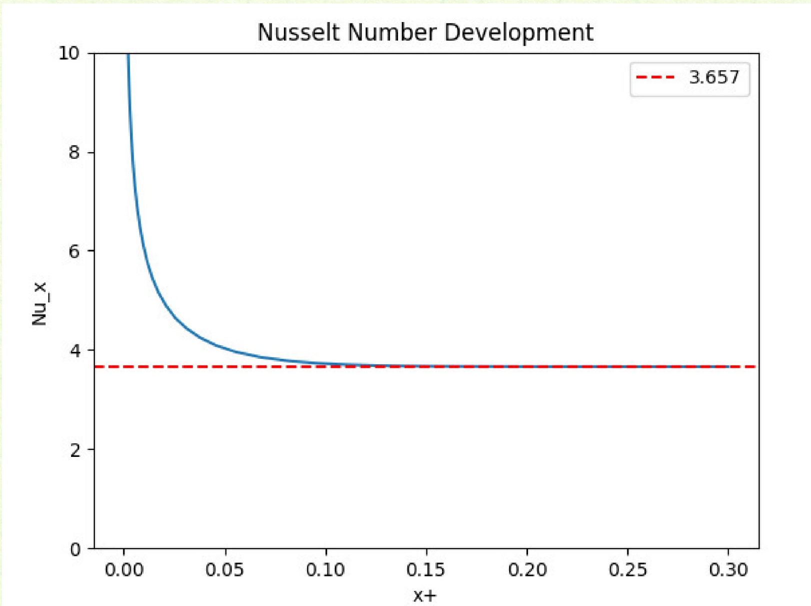


Figure 8-10 Variation of local Nusselt number in the thermal-entry region of a tube with constant surface temperature.

$x^+ = 2(x/D_h)/\text{RePr}$

$0 \le x^+ \le 0.3$, $P_r = 0.7$, $Re = 1000$

1) Calculate the flow and construct a plot similar to Fig. 8-10



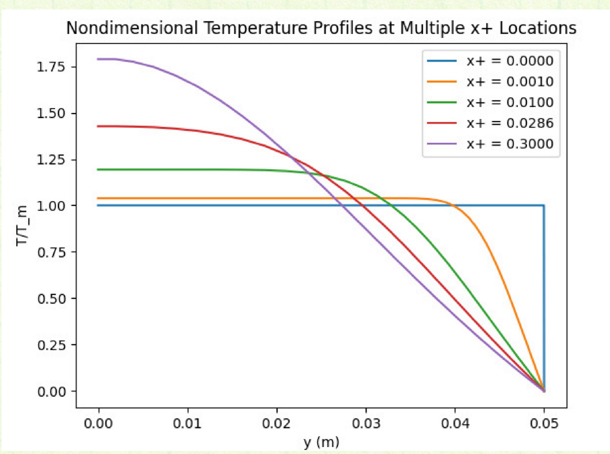
Nu develops like fig. 8-10 and table 8-4

table 8-4 :

x^+	table Nu_x	texstan* Nu_x
0.001	12.8	12.9
0.004	8.03	8.2
0.04	4.17	4.24
0.10	3.71	3.72
0.20	3.66	3.66

* points don't line up perfectly, using values from $x^+_{code} \approx x^+_{table}$

code and table values match up well



temperature profiles develop like velocity profiles

9-3. Solve the laminar boundary layer for constant free-stream velocity, using the momentum integral equation and an assumption that the velocity profile may be approximated by

$$\frac{u}{u_\infty} = \sin\frac{\pi y}{2\delta}$$

$\frac{\nu}{u_\infty^2}\left(\frac{\partial u}{\partial y}\right)_0 = \frac{d\delta_2}{dx}$, $\left.\frac{\partial u}{\partial y}\right|_{y=\delta} = 0$, $u|_{y=\delta} = u_\infty$

$u = u_\infty \sin\frac{\pi y}{2\delta}$

$\frac{\partial u}{\partial y} = \frac{u_\infty \pi}{2\delta}\cos\frac{\pi y}{2\delta}$

$\delta_2 = \int_0^\infty \frac{\rho u}{\rho_\infty u_\infty}\left(1-\frac{u}{u_\infty}\right)dy$

Check $y=\delta$ B.C.:

$u = u_\infty \sin\left(\frac{\pi\delta}{2\delta}\right) = u_\infty$ ✓

$\frac{\partial u}{\partial y} = \frac{u_\infty\pi}{2\delta}\cos\frac{\pi\delta}{2\delta} = 0$ ✓

$$\frac{\nu}{u_\infty^2}\left(\frac{\partial u}{\partial y}\right)_0 = \frac{\nu}{u_\infty^2}\left(\frac{u_\infty \pi}{2\delta}\right) = \frac{\pi\nu}{2u_\infty\delta}$$

$y \geq \delta$, $u = u_\infty \Rightarrow \frac{u}{u_\infty} = 1$

$$\delta_2 = \int_0^\infty \frac{u}{u_\infty}\left(1-\frac{u}{u_\infty}\right)dy = \int_0^\delta \frac{u}{u_\infty}\left(1-\frac{u}{u_\infty}\right) + \cancelto{0}{\int_\delta^\infty 1(1-1)\,dy}$$

$$\delta_2 = \int_0^\delta \sin\left(\frac{\pi y}{2\delta}\right)\left(1-\sin\left(\frac{\pi y}{2\delta}\right)\right)dy = \int_0^\delta \sin\left(\frac{\pi y}{2\delta}\right) - \sin^2\left(\frac{\pi y}{2\delta}\right)dy$$

$$\delta_2 = -\frac{2\delta}{\pi}\cos\frac{\pi y}{2\delta}\Big|_0^\delta - \frac{\delta}{2} = \frac{2\delta}{\pi} - \frac{\delta}{2} = \delta\left(\frac{2}{\pi}-\frac{1}{2}\right)$$

$$\frac{d\delta_2}{dx} = \left(\frac{2}{\pi}-\frac{1}{2}\right)\frac{d\delta}{dx}$$

$$\left(\frac{2}{\pi}-\frac{1}{2}\right)\frac{d\delta}{dx} = \frac{\pi\nu}{2u_\infty\delta}$$

$$\int \delta\, d\delta = \int \frac{\pi\nu}{u_\infty\left(\frac{4}{\pi}-1\right)}dx$$

$$\frac{\delta^2}{2} = \frac{\pi\nu x}{u_\infty\left(\frac{4}{\pi}-1\right)}$$

$$\underline{\underline{\delta = \sqrt{\frac{2\pi\nu x}{u_\infty\left(\frac{4}{\pi}-1\right)}} = 4.79\sqrt{\frac{\nu x}{u_\infty}}}} \longleftarrow \delta$$

$$\delta_1 = \int_0^\delta\left(1-\frac{u}{u_\infty}\right)dy = y + \frac{2\delta}{\pi}\cos\frac{\pi y}{2\delta}\Big|_0^\delta = \left(\delta + \frac{2\delta}{\pi}\cos\frac{\pi}{2}\right) - \left(\frac{2\delta}{\pi}\right) = \delta\left(1-\frac{2}{\pi}\right) = 4.79\sqrt{\frac{\nu x}{u_\infty}}\left(1-\frac{2}{\pi}\right) = \underline{\underline{1.74\sqrt{\frac{\nu x}{u}}}} \longleftarrow \delta_1$$

$$\delta_2 = \int_0^\delta \frac{u}{u_\infty}\left(1-\frac{u}{u_\infty}\right)dy = \delta\left(\frac{2}{\pi}-\frac{1}{2}\right) = 4.79\sqrt{\frac{\nu x}{u_\infty}}\left(\frac{2}{\pi}-\frac{1}{2}\right) = \underline{\underline{0.6544\sqrt{\frac{\nu x}{u_\infty}}}} \longleftarrow \delta_2$$

$$f_x = 2\frac{d\delta_2}{dx} = 2\frac{\nu}{u_\infty^2}\left(\frac{\partial u}{\partial y}\right)_0 = \frac{\nu}{u_\infty}\frac{\pi}{\delta} = \sqrt{\frac{\nu}{u_\infty}}\frac{\pi}{4.79\sqrt{x}} = \underline{\underline{0.66\,Re^{-1/2}}} \longleftarrow f_x$$

for the exact solution, $\delta_1 = 1.73\sqrt{\frac{\nu x}{u_\infty}}$

$\delta_2 = 0.664\sqrt{\frac{\nu x}{u_\infty}}$ — all of which are very close to the answers calculated above

$f_x = \frac{0.664}{Re^{1/2}}$ — δ_2 is the most different and is still within 1.5% of the Blasius solution

1) Derive the high and low Pr Nu correlations. Choose an Re value, and plot Nu vs Pr for the general solutions and determine what "High" and "Low" mean. Does it depend on Re?

2) Choose three Pr values (0.1, 1, 10) and plot Nu = f(Re) using equation 10-9. Interpret this as moving down the plate.

$\tau = \frac{T_s - T}{T_s - T_\infty}$, $\tau|_{y=0} = 0$, $\tau|_{y=\infty} = 1$, $\tau|_{x=0} = 1$, $u\frac{\partial\tau}{\partial x} + v\frac{\partial \tau}{\partial y} - \alpha\frac{\partial^2\tau}{\partial y^2} = 0$, $\eta = \frac{y}{\sqrt{\nu x/u_\infty}}$, $\tau = \tau(\eta)$

1. a) Low Pr : $Pr \ll 1 \Rightarrow \zeta' = \frac{u}{u_\infty} = 1$

$$\tau'' + \frac{Pr}{2}\zeta\tau' = 0$$

$$\frac{d}{d\eta}\left(\frac{\tau''}{\tau'}\right) + \frac{Pr}{2}\frac{d(\zeta)}{d\eta} = \frac{d}{d\eta}\left(\frac{\tau''}{\tau'}\right) + \frac{Pr}{2}\zeta' = \frac{d}{d\eta}\left(\frac{\tau''}{\tau'}\right) + \frac{Pr}{2} = 0$$

$$\frac{\tau''}{\tau'} = \int -\frac{Pr}{2}\,d\eta = -\frac{Pr\,\eta}{2} + C_3$$

$$\tau'' = -\frac{Pr\,\eta}{2}\tau' + C_3\tau' \quad \rightarrow \tau''(0) = 0$$

$$\tau''(0) = 0 = C_3\tau'(0) \Rightarrow C_3 = 0$$

$$\frac{1}{\tau'}\frac{d\tau'}{d\eta} = -\frac{Pr\,\eta}{2}$$

$$\ln(\tau') = -\frac{Pr}{2}\int \eta\, d\eta = -\frac{Pr\,\eta^2}{4} + C_4$$

$$\tau' = C_4 e^{-\frac{Pr\,\eta^2}{4}}$$

$$\tau' d\eta = C_4\int e^{-\frac{Pr\,\eta^2}{4}}\,d\eta$$

$$\tau = C_4\frac{\sqrt{\pi}}{2}\sqrt{\frac{4}{Pr}}\,\mathrm{erf}\left(\frac{\sqrt{Pr}\,\eta}{2}\right) + C_5$$

$$\tau(0) = 0 = C_5$$

$$\tau(\infty) = 1 = C_4\frac{\sqrt{\pi}}{2}\sqrt{\frac{4}{Pr}}\,\mathrm{erf}(\infty) = C_4\frac{\sqrt{\pi}}{2}\sqrt{\frac{2}{Pr}}$$

$$C_4 = \sqrt{\frac{Pr}{\pi}}$$

$$\eta = \frac{y}{\sqrt{\nu x/u_\infty}} = \frac{y}{\sqrt{x/Re}} = \frac{y\,Re^{1/2}}{x}$$

$$\tau = \mathrm{erf}\left(\frac{y\,Pr^{1/2}Re^{1/2}}{2x}\right)$$

$$h_x(T_s - T_\infty) = -k\left(\frac{\partial T}{\partial y}\right)_0 = k(T_s - T_\infty)\left(\frac{\partial\tau}{\partial\eta}\frac{\partial\eta}{\partial y}\right)_0 = k(T_s-T_\infty)\frac{\tau'(0)\sqrt{Re}}{x}$$

$$Nu_x = \frac{h_x x}{k} = \frac{x\sqrt{Re}}{x}\tau'(0) = Re^{1/2}\tau'(0)$$

$$\tau'(0) = \sqrt{\frac{Pr}{\pi}}\,e^{-\frac{Pr\,\eta^2}{4}}\Big|_{\eta=0} = \sqrt{\frac{Pr}{\pi}}$$

$$\underline{\underline{Nu_x = \frac{Re^{1/2}Pr^{1/2}}{\sqrt{\pi}} = 0.565\,Re^{1/2}Pr^{1/2}}} \longleftarrow \text{1.a) Low Pr}$$

1. b) High Pr : Pr > 15

use $\zeta'' = 0.3321 \Rightarrow \zeta = \frac{0.3321}{2}\eta^2$

$$\tau'' + Pr\frac{0.3321}{4}\eta^2\tau' = 0$$

$$\frac{1}{\tau'}d\tau' + Pr\frac{0.3321}{4}\eta^2\,d\eta + C_6 = 0$$

$$\ln\tau' = -\frac{0.3321\,Pr}{12}\eta^3 + C_6$$

$$\tau' = C_6 e^{-\frac{0.3321\,Pr}{12}\eta^3}$$

$$\tau = C_6\int e^{-\frac{0.3321\,Pr}{12}\eta^3}\,d\eta \quad \rightarrow k = \frac{0.3321\,Pr}{12}$$

$$\tau = C_6\left(-\frac{\eta\,\Gamma(\frac{1}{3}, k\eta^3)}{3\sqrt[3]{k\eta^3}}\right) + C_7 = -C_6\left(\frac{\Gamma(\frac{1}{3}, k\eta^3)}{3\sqrt[3]{k}}\right) + C_7$$

$$\tau(\infty) = 1 = -C_6\left(\frac{0}{3\sqrt[3]{k}}\right) + C_7 \quad \Rightarrow C_7 = 1$$

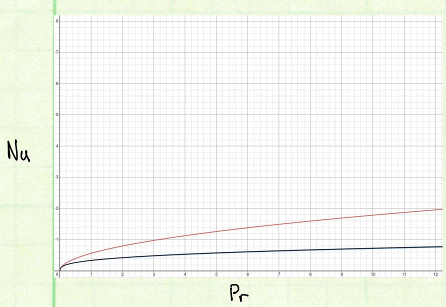
$$\tau(0) = 0 \Rightarrow C_6\left(\frac{\Gamma(\frac{1}{3}, 0)}{3\sqrt[3]{k}}\right) = 1$$

$$C_6 = \sqrt[3]{Pr}\,\frac{3\sqrt[3]{\frac{0.3321}{12}}}{\Gamma(\frac{1}{3}, 0)} = 0.339\,Pr^{1/3}$$

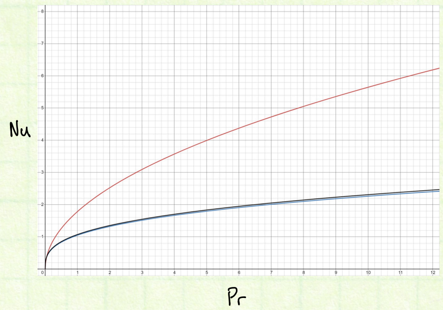
$$Nu_x = \frac{h_x x}{k} = \frac{k\,\tau'(0)\,x\sqrt{Re}}{k\,x}$$

$$\underline{\underline{Nu_x = 0.339\,Pr^{1/3}Re^{1/2}}} \longleftarrow \text{1.b) High Pr}$$

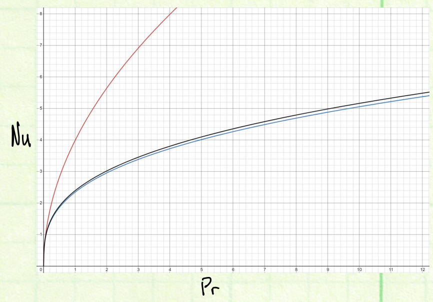
Re = 1



Re = 10



Re = 50



The red line is the solution for the low Pr case, the blue is the general solution, and black is for high Pr as the Re increases, the distance between solutions increases, but the general and high Pr solutions stay very close

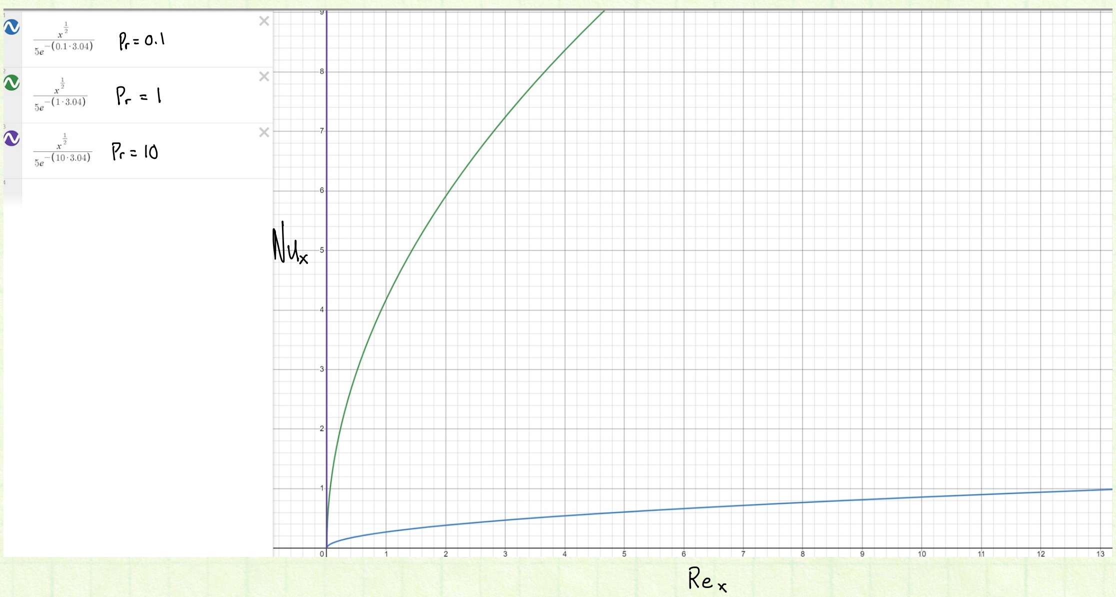
2) $$Nu_x = \frac{Re_x^{1/2}}{\int_0^\infty e^{-\left(\frac{Pr}{2}\int_0^\eta \zeta \, d\eta\right)} d\eta}$$

using numerical integration of table 8-4

$\int_0^\eta \zeta \, d\eta = 6.08$ for the range $\eta = 0 \to 5$

$$\therefore \int_0^\infty e^{-\left(\frac{Pr}{2}\int_0^\eta \zeta \, d\eta\right)} d\eta = 5\, e^{-\frac{Pr}{2}\cdot 6.08}$$

$$Nu_x = \frac{Re_x^{1/2}}{5e^{-3.04 \cdot Pr}}$$



this method gives a decent prediction for Pr = 0.1, but does poorly for larger Pr