1.8-5 (1-18 in text) Figure P1.8-5 illustrates a fin that is to be used in the evaporator of a space conditioning system for a space-craft.

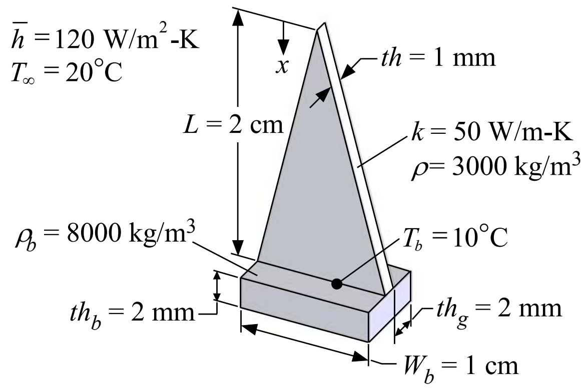

Figure P1.8-5: Fin on an evaporator.

The fin is a plate with a triangular shape. The thickness of the plate is $th = 1$ mm and the width of the fin at the base is $W_b = 1$ cm. The length of the fin is $L = 2$ cm. The fin material has conductivity $k = 50$ W/m-K. The average heat transfer coefficient between the fin surface and the air in the space-craft is $\bar{h} = 120$ W/m^2-K. The air is at $T_\infty = 20$°C and the base of the fin is at $T_b = 10$°C. Assume that the temperature distribution in the fin is 1-D in x. Neglect convection from the edges of the fin.

a.) Obtain an analytical solution for the temperature distribution in the fin. Plot the temperature as a function of position.

b.) Calculate the rate of heat transfer to the fin.

c.) Determine the fin efficiency.

The fin has density $\rho = 3000$ kg/m^3 and the fin is installed on a base material with thickness th_b = 2 mm and density $\rho_b = 8000$ kg/m^3. The half-width of the gap between adjacent fins is th_g = 2 mm. Therefore, the volume of the base material associated with each fin is $th_b\, W_b\, (th + 2\, th_g)$.

d.) Determine the ratio of the absolute value of the rate of heat transfer to the fin to the total mass of material (fin and base material associated with the fin).

e.) Prepare a contour plot that shows the ratio of the heat transfer to the fin to the total mass of material as a function of the length of the fin (L) and the fin thickness (th).

f.) What is the optimal value of L and th that maximizes the absolute value of the fin heat transfer rate to the mass of material?

a) $T(x)$

$$Bi = \frac{\bar{h} W_b}{2k} = \frac{120 \cdot 0.01}{2 \cdot 50} = 0.012 \quad \Rightarrow Bi << 1$$

$th = W_b = 1\,cm$, $W = th = 1\,mm$

$A_s(x) = 2\,th \frac{x}{L} dx$, convection surface area

$$\dot{q}_{conv} = 2\,th \frac{x}{L} dx\ \bar{h}(T - T_\infty)$$

$$\dot{q}_{cond} = -k\,th\,W \frac{x}{L} \frac{dT}{dx}$$

$$0 = \frac{d\dot{q}_{cond}}{dx} dx + \dot{q}_{conv}$$

$$0 = \frac{d}{dx}\left[-k\,th\,W \frac{x}{L}\frac{dT}{dx}\right] dx + 2\,th \frac{x}{L} dx\ \bar{h}(T - T_\infty)$$

$$\frac{d}{dx}\left[x \frac{dT}{dx}\right] dx - \frac{2\,th \frac{x}{L}\,\bar{h} L\,dx}{k\,th\,W} T = -\frac{2\,th\frac{x}{L}\,\bar{h}L\,dx}{k\,th\,W} T_\infty$$

$$\frac{d}{dx}\left[x\frac{dT}{dx}\right] - 2\frac{x\bar{h}}{kW} T = -2\frac{x\bar{h}}{kW} T_\infty$$

$$x\frac{d^2T}{dx^2} + \frac{dT}{dx} - 2\frac{x\bar{h}}{kW} T = -2\frac{x\bar{h}}{kW} T_\infty$$

$$x\frac{d^2\theta}{dx^2} + \frac{d\theta}{dx} - x\frac{2\bar{h}}{kW}\theta = 0 \quad , \theta = T - T_\infty \ , \ \frac{d\theta}{dx} = \frac{dT}{dx} \ , \ \frac{d^2\theta}{dx^2} = \frac{d^2T}{dx^2}$$

Plugging in to Maple:

$$\theta(x) = C_1 \,\text{BesselJ}(0, \sqrt{-\beta}\,x) + C_2\,\text{BesselY}(0, \sqrt{-\beta}\,x) \quad , \beta = \frac{2\bar{h}}{kW}$$

with β assumed positive

$$\theta(x) = C_1\,\text{BesselI}(0, \sqrt{\beta}\,x) + C_2\,\text{BesselK}(0, \sqrt{\beta}\,x)$$

at $x = 0$, $T < \infty \Rightarrow \theta < \infty \rightarrow C_1\,\text{BesselI}(0,0) + C_2\,\text{BesselK}(0,0) < \infty$

$\text{BesselK}(0,0) = \infty \Rightarrow C_2 = 0$

at $x = L$, $\theta = \theta_b = T_b - T_\infty$

$$C_1\,\text{BesselI}(0, \sqrt{\beta}\,L) = \theta_b$$

$$C_1 = \frac{\theta_b}{\text{BesselI}(0,\sqrt{\beta}\,L)}$$

$$\theta(x) = \theta_b \frac{\text{BesselI}(0,\sqrt{\beta}\,x)}{\text{BesselI}(0,\sqrt{\beta}\,L)} = \theta_b \frac{\text{BesselI}\left(0, \sqrt{\frac{2\bar{h}}{kW}}\,x\right)}{\text{BesselI}\left(0,\sqrt{\frac{2\bar{h}}{kW}}\,L\right)}$$

$$T(x) = \theta(x) + T_\infty \quad \leftarrow T(x)$$

b) $\dot{q}_{fin}$

$$\dot{q}_{fin} = kW\,th \left.\frac{dT}{dx}\right|_{x=L} = kW\,th \frac{d}{dx}\left[\theta_b \frac{\text{BesselI}(0,\sqrt{\beta}\,x)}{\text{BesselI}(0,\sqrt{\beta}\,L)}\right]_{x=L}$$

$$\dot{q}_{fin} = \frac{kW\,th\,\theta_b}{\text{BesselI}(0,\sqrt{\beta}\,L)} \frac{d}{dx}\left[\text{BesselI}(0,\sqrt{\beta}\,x)\right]_{x=L}$$

$$\frac{d}{dx}\left[\text{BesselI}(0,\sqrt{\beta}\,x)\right] = \text{BesselI}(1,\sqrt{\beta}\,x)\sqrt{\beta}/x$$

$$\dot{q}_{fin} = kW\,th\,\theta_b\sqrt{\beta}\frac{\text{BesselI}(1,\sqrt{\beta}\,L)}{\text{BesselI}(0,\sqrt{\beta}\,L)} \quad \leftarrow \dot{q}_{fin}$$

c) η_{fin}

$$\eta_{fin} = \frac{\dot{q}_{fin}}{\bar{h}\left(2\cdot\frac{th\cdot L}{2}\right)\theta_b} = \frac{kW\,th\,\theta_b\sqrt{\frac{2\bar{h}}{kW}}\frac{\text{BesselI}\left(1,\sqrt{\frac{2\bar{h}}{kW}}\,L\right)}{\text{BesselI}\left(0,\sqrt{\frac{2\bar{h}}{kW}}\,L\right)}}{\bar{h}\,th\,L\,\theta_b}$$

$$\eta_{fin} = \frac{\sqrt{2kW}}{L\sqrt{\bar{h}}}\frac{\text{BesselI}\left(1,\sqrt{\frac{2\bar{h}}{kW}}\,L\right)}{\text{BesselI}\left(0,\sqrt{\frac{2\bar{h}}{kW}}\,L\right)} \quad \leftarrow \eta_{fin}$$

equations determined here. Evaluated values and plots on next pages