4-1. Consider steady flow of a constant-property fluid in a long duct formed by two parallel planes. Consider a point sufficiently far removed from the duct entrance that the y component of velocity is zero and the flow is entirely in the x direction. Write the Navier–Stokes equations for both the x and y directions. What can you deduce about the pressure gradients?

Navier Stokes Equation: $\rho \frac{D\vec{V}}{Dt} = -\vec{\nabla}P + \rho\vec{g} + \mu \nabla^2 \vec{V}$, use 2D

assume $\vec{g} = \langle g_x, g_y \rangle = \langle 0, -g \rangle$, $\vec{V} = \langle u, v \rangle = \langle u, 0 \rangle$, $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = 0$

continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

x: $\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial P}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

$\rho u \frac{\partial u}{\partial x} = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$

$\frac{\partial P}{\partial x} = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \rho u \frac{\partial u}{\partial x}$

Pressure gradient depends on y and x , viscosity and density

y: $\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial P}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$

because flow is entirely in x direction, $\frac{\partial v}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial y^2} = 0$

$\rho(0) = -\frac{\partial P}{\partial y} - \rho g + \mu(0)$

$\frac{\partial P}{\partial y} = -\rho g$ $\rightarrow$ if gravitational / body forces were ignored, the RHS goes to 0 and $\frac{\partial P}{\partial y} = 0$

Pressure gradient only depends on gravity, density, and y (will not change along x)

Same formula as hydrostatics

4-4. Starting with the general viscous energy equation, show by a succession of steps how and why it reduces to the classic heat-conduction equation for a solid, and finally to the Laplace equation.

Viscous Energy Equation: $\rho\left(u\frac{\partial i}{\partial x}+v\frac{\partial i}{\partial y}+w\frac{\partial i}{\partial z}+\frac{\partial i}{\partial \theta}\right)-\left[\frac{\partial}{\partial x}\left(k\frac{\partial t}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial t}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial t}{\partial z}\right)\right]-\mu\left[2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]+\left[\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2\right]-\frac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)^2\right]$ ...

$-\left(u\frac{dP}{dx}+v\frac{dP}{dy}+w\frac{dP}{dz}+\frac{dP}{d\theta}\right)-S=0$

a) Heat Equation: $\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}+\frac{\partial^2 t}{\partial z^2}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial t}{\partial \theta}$, $\alpha=\frac{k}{\rho c_p}$

in a solid, the particles don't flow like they do for a fluid. So we can set $u=v=w=\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=\frac{\partial w}{\partial z}=\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}=\frac{\partial v}{\partial z}=\frac{\partial w}{\partial y}=\frac{\partial w}{\partial x}=\frac{\partial u}{\partial z}=0$ because there is no motion

this makes the viscous energy equation become: $\rho\frac{\partial i}{\partial \theta}-\left[\frac{\partial}{\partial x}\left(k\frac{\partial t}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial t}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial t}{\partial z}\right)\right]-\frac{\partial P}{\partial \theta}-S=0$

assuming the pressure on the solid doesn't change, $\frac{\partial P}{\partial \theta}=0$, and assuming constant density, ρ, and conductivity, k

then the viscous energy equation becomes $\rho\frac{\partial i}{\partial \theta}-k\frac{\partial^2 t}{\partial x^2}-k\frac{\partial^2 t}{\partial y^2}-k\frac{\partial^2 t}{\partial z^2}-S=0$

use $\frac{\partial i}{\partial \theta}=c_p\frac{\partial t}{\partial \theta}$ and rearrange

$$\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}+\frac{\partial^2 t}{\partial z^2}+\frac{S}{k}=\frac{\rho c_p}{k}\frac{\partial t}{\partial \theta}$$

set $S=\dot{q}$ and substitute $\frac{\rho c_p}{k}=\frac{1}{\alpha}$

$$\underline{\underline{\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}+\frac{\partial^2 t}{\partial z^2}+\frac{\dot{q}}{k}=\frac{1}{\alpha}\frac{\partial t}{\partial \theta}}} \longleftarrow \text{heat equation}$$

b) Laplace Equation: $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=0$

starting from the heat equation above, we assume no generation , $\dot{q}=0$

the equation becomes $\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}+\frac{\partial^2 t}{\partial z^2}=\frac{1}{\alpha}\frac{\partial t}{\partial \theta}$

then we assume steady state, $\frac{\partial t}{\partial \theta}=0$

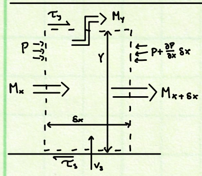
this gets $\frac{\partial^2 t}{\partial x^2}+\frac{\partial^2 t}{\partial y^2}+\frac{\partial^2 t}{\partial z^2}=0$

set $f(x,y,z)=t(x,y,z)$

Substituting, we get

$$\underline{\underline{\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=0}} \longleftarrow \text{Laplace Equation}$$

3: Derive the integral momentum for a flat plate with no flow through the wall and constant density fluid



control volume

$$* \quad -\tau_s = \frac{1}{R}\frac{d}{dx}\left(R\int_0^Y \rho u^2 dy\right) - \frac{u_\infty}{R}\frac{d}{dx}\left(R\int_0^Y \rho u dy\right) + \rho_s v_s u_\infty + \int_0^Y \left(-\rho_\infty u_\infty \frac{du_\infty}{dx}\right) dy$$

Conservation of mass: $\delta_x \delta_z v_y \rho_y + \delta_z \int_0^Y u\rho\, dy + \delta_z \frac{d}{dx}\left(\int_0^Y u\rho\, dy\right)\delta x - \delta_z \delta x v_s \rho_s - \delta_z \int_0^Y u\rho\, dy = 0$

use $v_s = 0$ and combine terms

$$v_y \rho_y + \frac{d}{dx}\left(\int_0^Y u\rho\, dy\right) = 0$$

$$v_Y \rho_Y = -\frac{d}{dx}\left(\int_0^Y u\rho\, dy\right)$$

Momentum flux: $M_x = \delta_z \int_0^Y \rho u^2 dy$

$$M_{x+\delta x} = \delta_z \int_0^Y \rho u^2 dy + \delta_z \frac{d}{dx}\left(\int_0^Y \rho u^2 dy\right)\delta x$$

$$M_Y = \delta_z \delta x\, v_Y \rho_Y u_Y$$

combine with shear and pressure terms

$$\tau_Y \delta x \delta_z + PY\delta_z + \delta_z \int_0^Y \rho u^2 dy = \tau_s \delta x \delta_z + PY\delta_z + \frac{dP}{dx}\delta x \delta_z Y + \delta_z \delta x\, v_Y \rho_Y u_Y + \delta_z \int_0^Y \rho u^2 dy + \frac{d}{dx}\left(\int_0^Y \rho u^2 dy\right)\delta x \delta_z$$

simplify and combine terms

$$\tau_Y = \tau_s + \frac{dP}{dx}Y + v_Y \rho_Y u_Y + \frac{d}{dx}\int_0^Y \rho u^2 dy$$

set Y to be greater than the boundary layer thickness $\Rightarrow u_Y = u_\infty$ and $\tau_y = 0$, use $v_Y \rho_Y = -\frac{d}{dx}\int_0^Y u\rho\, dy$ from CoM

$$0 = \tau_s + \int_0^Y \frac{dP}{dx} - u_\infty \frac{d}{dx}\int_0^Y \rho u dy + \frac{d}{dx}\int_0^Y \rho u^2 dy$$

from Bernoulli, $\frac{dP}{dx} = -\rho_\infty u_\infty \frac{du_\infty}{dx}$, and ρ is constant

$$\tau_s = \rho u_\infty \frac{d}{dx}\int_0^Y u dy + \rho_\infty u_\infty \int_0^Y \left(\frac{du_\infty}{dx}\right) dy - \rho \frac{d}{dx}\int_0^Y u^2 dy$$ ← Momentum Integral for flat plate