1. You are building an instrument for measuring the heat transfer coefficient (h) between a sphere and a flowing fluid. The instrument is a spherical temperature sensor with diameter D = 3 mm that is initially in equilibrium with the fluid at T_∞. The sensor has density $\rho = 7500\ kg/m^3$, specific heat capacity $c = 820\ J/kg K$, and conductivity $k = 75\ W/m\cdot k$. The sensor is heated with a constant rate of thermal energy generation of $\dot{g} = 0.1\ W$ and the time required for the sensor temperature to increase by $\Delta T = 10\ K$ is recorded. The range of the instrument is expected to be from $h = 30\ W/m^2 K$ to $h = 300\ W/m^2 K$

a) can the sensor be treated as a lumped capacitance? Justify your answer

b) Develop an equation that relates the measured time to the heat transfer coefficient

c) Plot the heat transfer coefficient as a function of measured time

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a) $L_{cond} = \frac{V}{A_s} = \frac{\frac{4}{3}\pi r^3}{4\pi r^2} = \frac{r}{3}$

$Bi = \frac{L_{cond}\bar{h}}{k} = \frac{Dh}{6k}$

$Bi_{min} = \frac{0.003m \cdot 30 W/m^2\cdot K}{6 \cdot 75 W/m\cdot K} = 0.0002$, $Bi_{max} = \frac{0.003m \cdot 300 W/m^2\cdot K}{6 \cdot 75 W/m\cdot K} = 0.002$

because $Bi_{max} < 0.1$ the lumped capacitance assumption is satisfied so the LCM method can be used

b) $T_0 = T_\infty$

$\rho V c \frac{dT}{dt} = \dot{g} - hA_s(T - T_\infty)$, $\theta = T - T_\infty$, $\theta' = \frac{d(T-T_\infty)}{dt} = \frac{dT}{dt}$

$\theta' + \frac{hA_s}{\rho V c}\theta = \frac{\dot{g}}{\rho V c}$, $\frac{hA_s}{\rho V c} = \frac{1}{\tau}$, $\frac{\dot{g}}{\rho V c} = f$

$\theta' + \frac{\theta}{\tau} = f$

$e^{t/\tau}\theta' + \frac{e^{t/\tau}\theta}{\tau} = e^{t/\tau} f$

$\frac{d}{dt}(\theta e^{t/\tau}) = e^{t/\tau} f$

$\theta e^{t/\tau} = \int_0^t e^{t/\tau} f\, dt = \tau f(e^{t/\tau} - 1)$

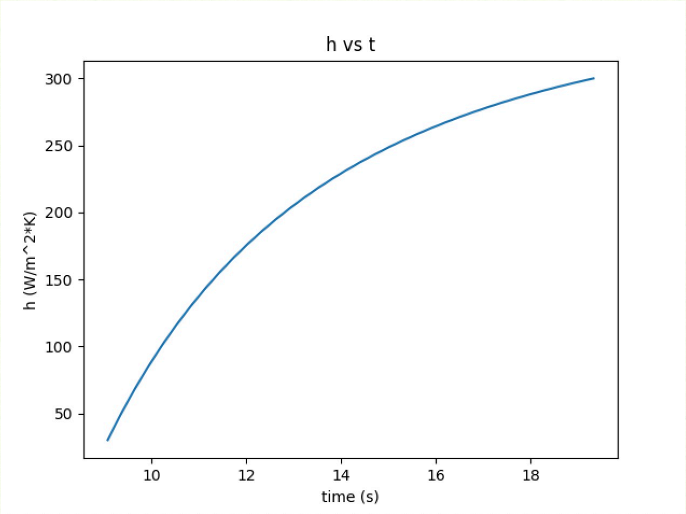
$\theta = \tau f(1 - e^{-t/\tau})$

$h = \frac{\dot{g}}{\Delta T A_s}\left(1 - e^{\frac{-thA_s}{\rho V c}}\right)$, $\dot{g}/\Delta T A_s = 353.68\ W/m^2 K$, $\frac{A_s}{\rho V c} = 0.000325\ \frac{K\cdot m^2}{J}$

$e^{-\frac{thA_s}{\rho V c}} = 1 - \frac{h\Delta T A_s}{\dot{g}}$

$$\underline{\underline{t = -\frac{\rho V c}{hA_s}\ln\left(1 - \frac{h\Delta T A_s}{\dot{g}}\right)}}$$

c)



2. Reconsider Problem 3.10 using a separation of variables solution.

3.10: A rod with uniform cross sectional area, $A_c = 0.1\,m^2$ and perimeter $P = 0.05\,m$ is placed in a vacuum environment. The length of the rod is $L = 0.09\,m$ and the external surfaces of the rod can be assumed to be adiabatic. For a long time, a heat transfer rate of $\dot{q}_{in} = 100\,W$ is provided to the end of the rod at $x=0$. The tip of the rod at $x=L$ is always maintained at $T_t = 20°C$. The rod material has density $\rho = 5000\,kg/m^3$, specific heat capacity $c = 500\,J/kgK$, and conductivity $k = 5\,W/m{\cdot}K$. The rod is at a steady state operating condition when, at a time $t=0$, the heat transfer rate at $x=0$ becomes zero.

a) About how long does it take for the rod to respond to the change in heat transfer?

b) Sketch the temperature distribution at $t=0$ and $t\to\infty$. Make sure the temperatures at either end and the temperature distribution is correct.

c) overlay the profile from (b) with a profile at the time in (a) as well as half and double that time

d) Sketch the heat transfer rate from the tip as a function of time. Make sure the sketch shows the behavior before and after the time in (a). Make sure that you get the rate at $t=0$ and $t\to\infty$ are correct

a) Derive the governing differential equation, the boundary conditions, and the initial conditions for the problem

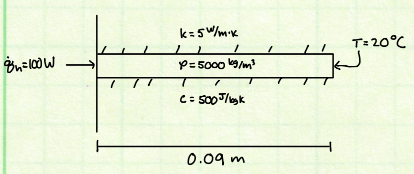
b) Does the mathematical problem statement derived in (a) satisfy all of the requirements for a separation of variables solution? If not, provide a simple transformation that can be applied so that the problem can be solved using separation of variables.

c) Prepare a separation of variables solution to the transformed problem from (b) and implement your solution in EES.

d) Prepare a plot of the temperature as a function of position for $t=0$ and $t\to\infty$ as well as the times requested in 3-10 (c)

Plot temperature profiles at $t = 0.5\,\tau_{diff}$ and $t = 2.0\,\tau_{diff}$, assume $\tau_{diff} = L^2/4\alpha$ (Fourier Number definition uses $\tau_{diff} = L^2/\alpha$)

$A_c = 0.1\,m^2$, $P = 0.05\,m$



$$\alpha = \frac{k}{\rho c_p} = \frac{5\,W/m{\cdot}K}{5000\,kg/m^3 \cdot 500\,J/kg{\cdot}K} = 2\times10^{-6}\,m^2/s$$

$$\tau_{diff} = L^2/4\alpha = 1012.5\ s$$

a) Because the outer edges are effectively insulated, the problem can be treated as 1D $\left(\frac{\partial T}{\partial y}=0\right)$ using the 1D heat equation:

$$k\frac{\partial^2 T}{\partial x^2} + \dot{q} = \rho c_p \frac{\partial T}{\partial t} \quad , \dot{q}=0$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\rho c_p}{k}\frac{\partial T}{\partial t} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$\frac{\partial^2 T}{\partial x^2} - \frac{1}{\alpha}\frac{\partial T}{\partial t} = 0$$

B.C. $x=0: \frac{\partial T}{\partial x} = 0$

$x=L: T = 20°C$

I.C. $\substack{x=L \\ t=0}: -k\frac{\partial T}{\partial x} = \dot{q}_{in}/A_c$ → assumes that the tip of the rod is not adiabatic due to steady state conditions

b) the GDE is 1st order wrt time and 2nd order wrt x, so it needs 1 initial condition and 2 boundary conditions

the GDE is linear and homogeneous and the boundary / initial conditions are linear, but there is no direction where all B.C.s are homogeneous

so SoV is not valid

use $\theta = T - T_L = T - 20°C$

$$\frac{\partial^2\theta}{\partial x^2} - \frac{1}{\alpha}\frac{\partial\theta}{\partial t} = 0$$

B.C. $\frac{\partial\theta}{\partial x}\Big|_{x=0} = 0$

$\theta(L,t) = 0$

I.C. $t=0: -k\frac{\partial\theta}{\partial x}\Big|_{x=L} = \dot{q}_{in}/A_c$

c) SoV solution

$$\theta(x,t) = X(x)\,t(t)$$

$$t\frac{d^2X}{dx} - \frac{X}{\alpha}\frac{dt}{dt} = 0$$

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx}}_{-\lambda^2} - \underbrace{\frac{1}{t\alpha}\frac{dt}{dt}}_{\lambda^2} = 0$$

$$\frac{d^2X}{dx} + \lambda^2 X = 0 \quad , \quad \frac{dt}{dt} + \lambda^2 \alpha t = 0$$

$$X = C_1 \cos(\lambda x) + C_2 \sin(\lambda x) \quad \rightarrow \frac{dX}{dx} = \lambda C_2 \cos(\lambda x)$$

Check B.C.

$$x=0: \ \frac{\partial \theta}{\partial x} = 0 = t(t)\frac{\partial X}{\partial x} = t(t)\lambda C_2$$

$$C_2 = 0$$

$$x = L: \ \theta = 0 = t(t) C_1 \cos(\lambda L) \Rightarrow \cos(\lambda L) = 0$$

$$\lambda L = \frac{i\pi}{2} \Rightarrow \lambda_i = \frac{i\pi}{2L} \quad , \ i = 1, 2, \dots, \infty$$

$$t(t) = C_{3,i}\, e^{(-\lambda_i^2 \alpha t)}$$

$$\theta(x,t) = a_i \cos(\lambda_i x) e^{-\lambda_i^2 \alpha t} \quad , \ a_i = C_{1,i} C_{3,i}$$

$$\frac{\partial^2 \theta}{\partial x^2} - \frac{1}{\alpha}\frac{\partial \theta}{\partial t} = 0$$

Check GDE:

$$a_i\left(-\lambda^2 \cos(\lambda_i x) e^{-\lambda_i^2 \alpha t} + \lambda_i^2 \cos(\lambda_i x) e^{-\lambda_i^2 \alpha t}\right) = 0 \quad \checkmark$$

Series solution and I.C.:

$$\theta = \sum_{i=1}^{\infty} a_i \cos(\lambda_i x) e^{-\lambda_i^2 \alpha t} \xrightarrow{\text{satisfies}} -k\frac{\partial \theta}{\partial x}\bigg|_{x=L,t=0} = q_h / A_c$$

$$\frac{\partial \theta}{\partial x}\bigg|_{x=L,t=0} = -a_i \frac{i\pi}{2L} \sin\left(i\frac{\pi}{2}\right) e^{-\left(\frac{i\pi}{2L}\right)^2 \alpha \cdot 0} = -a_i \frac{i\pi}{2L}$$

$$\sum_{i=1}^{\infty} k a_i \frac{i\pi}{2L} = q_h / A_c$$

$$k a_i \frac{i\pi}{2L}$$

$$a_i = \frac{2L q_h}{i\pi k A_c}$$

$$\theta(x,t) = \sum_{i=1}^{\infty} \frac{2L q_h}{i\pi k A_c} \cos\left(\frac{i\pi}{2L}x\right) e^{-\left(\frac{i\pi}{2L}\right)^2 \alpha t}$$

d) T profiles at $t = 0, \infty, 0.5\,\tau_{diff}, 2.0\,\tau_{diff}$. Tested at different number of terms in summation