Problem 1 - 1D Transient Conduction with a non-homogeneous boundary condition:

A plane wall is initially at Ti = T∞ = 20°C when one side of the wall is exposed to a heat flux, q''= 1000 W/m2 and the other side is cooled by exposure to a fluid at T∞ = 20°C with heat transfer coefficient h = 100 W/m2-K. The wall is L = 0.1 m thick and has density ρ = 6000 kg/m3, conductivity k = 10 W/m-K, and specific heat capacity c = 700 J/kg-K.

- a) Develop a separation of variables solution for the temperature distribution in the wall.
- b) Plot the temperature as a function of position in the wall for time t = 100 s, 200 s, 500 s, 1000 s, 5000 s, and 10,000 s.

B.C.: $x=0: \dot{q}'' = 1000\,W/m^2 = -k\frac{\partial T}{\partial x}$

$x=L: -k\frac{\partial T}{\partial x} = h(T-T_\infty)$

I.C.: $t=0: T=T_i=T_\infty$

use $\theta = T(x,t) - T_\infty$

$\theta(x,t) = \psi(x,t) + \phi(x)$

GDE: $\frac{\partial^2\psi}{\partial x^2} + \frac{d^2\phi}{dx^2} = \frac{1}{\alpha}\frac{\partial\psi}{\partial t}$

BC: $\left(\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial x}\right)\Big|_{x=0} = \frac{\dot{q}''}{k}$

$\left(\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial x}\right)\Big|_{x=L} = h(\psi(L,t)+\phi(L))$

IC: $\psi(x,0)+\phi(x) = 0$

at steady state: $\psi(x,t)$ goes away

$\Rightarrow \frac{d\phi}{dx}\Big|_{x=0} = \frac{-\dot{q}''}{k} = \frac{d\phi}{dx}\Big|_{x=L} = -\frac{h}{k}(\phi(L)+\psi(L,t))$, $t\to\infty$

$\frac{d^2\phi}{dx^2} = 0 \Rightarrow \frac{d\phi}{dx} = C \Rightarrow \phi = C_1x + C_2$

$\frac{d\phi}{dx} = C_1 = \frac{-\dot{q}''}{k}$

$\phi = \frac{-\dot{q}''x}{k} + C_2$

finding C_2: at steady state $t\to\infty \Rightarrow \psi(x,t)=0$

$\dot{q}'' = h\phi(L)$

$\phi(L) = \dot{q}''/h = -\frac{\dot{q}''}{k}L + C_2$

$C_2 = \dot{q}''(1/h + L/k) = 20$

conditions for $\psi \rightarrow \frac{\partial^2\psi}{\partial x^2} = \frac{1}{\alpha}\frac{\partial\psi}{\partial t}$

$\frac{\partial\psi}{\partial x}(0,t) = 0$

$\psi(x,0) = -\phi(x)$

$\psi(x,t) = \sum_{i=1}^{\infty} C_i\cos(\lambda_i x)e^{-\lambda_i^2\alpha t}$

$\frac{\partial\psi}{\partial x} = \sum_{i=1}^{\infty} -C_i\lambda_i\sin(\lambda_i x)e^{-\lambda_i^2\alpha t}$

$-k\left(\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial x}\right) = h(\psi+\phi)$

$\frac{\partial\psi}{\partial x} + \frac{h}{k}\psi(L,0) = -\frac{\partial\phi}{\partial x} - \frac{h}{k}\phi(L) = \frac{\dot{q}''}{k} - \frac{h}{k}\left(\frac{\dot{q}''}{k}L + C_2\right) = \frac{\dot{q}''}{k}\left(1 - \frac{hL}{k} - \frac{hC_2}{\dot{q}''}\right)$

$\sum_{i=1}^{\infty} C_i\left[\cos(\lambda_i L)e^{-\lambda_i^2\alpha t} - \frac{h}{k}\lambda_i\sin(\lambda_i L)e^{-\lambda_i^2\alpha t}\right] = \frac{\dot{q}''}{k}\left(1 - \frac{hL}{k} - \frac{hC_2}{\dot{q}''}\right)$

at $(x,t) = (L,0)$: $\sum_{i=1}^{\infty} -C_i\frac{h}{k}\lambda_i\sin(\lambda_i L) - \frac{\dot{q}''}{k} = 0$

$\psi(x,0) = \sum_{i=1}^{\infty} C_i\cos(\lambda_i x) = \frac{\dot{q}''}{k}x - 20$

$C_i\int_0^L \cos^2(\lambda_i x)\,dx = \frac{\dot{q}''}{k}\int_0^L x\cos(\lambda_i x)\,dx - 20\int_0^L\cos(\lambda_i x)\,dx$

$$C_i = \frac{\frac{\dot{q}''}{k}\left(\frac{\cos(\lambda_i L)}{\lambda_i^2} + \frac{\sin(\lambda_i L)\cdot L}{\lambda_i} - \frac{1}{\lambda_i^2}\right) - 20\left(\frac{\sin(\lambda_i L)}{\lambda_i}\right)}{\left(\frac{\lambda_i\cdot L + \cos(\lambda_i L)\cdot\sin(\lambda_i L)}{2\lambda_i}\right)}$$

$-k\frac{\partial\psi}{\partial x}\Big|_{x=L} = \bar{h}\psi \rightarrow t=0$

$kC_i\lambda\sin(\lambda_i L) = \bar{h}C_i\cos(\lambda_i L)$

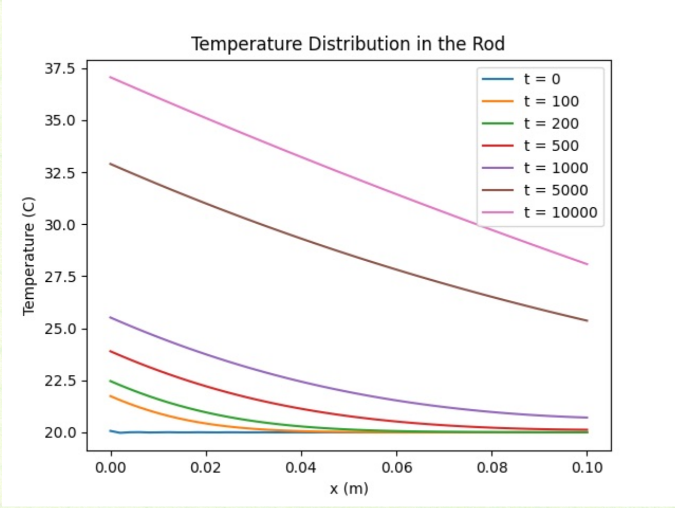
$\tan\lambda_i L = \frac{\bar{h}}{k\lambda_i} = \frac{B_i}{\lambda_i L}$

Find eigenvalues in ranges $\lambda L = \frac{(1+2i)\pi}{2} \rightarrow i = 0,1,2\ldots n$

Plug into C_i equation then plot

$$T(x,t) = \sum_{i=0}^{\infty} C_i\cos(\lambda_i x)e^{-(\lambda_i^2\alpha t)} + 20 - \frac{\dot{q}''}{k}x + T_\infty$$ ← SoV $T(x,t)$ solution

b)



Problem 2 - Textbook Problem 3-8

(Note: the boundary condition at the left surface (x=0) is different from what we are used to. It has convection in addition to the constant heat flux. Thus, the heat flux into the domain will be balanced by both convection and the given heat flux at the boundary).

3–8 Figure P3-8 shows a slab of material that is $L = 5$ cm thick and is heated from one side ($x = 0$) by a radiant heat flux $\dot{q}_s'' = 7500\ \mathrm{W/m^2}$. The material has conductivity $k = 2.4$ W/m-K and thermal diffusivity $\alpha = 2.2 \times 10^{-4}\ \mathrm{m^2/s}$. Both sides of the slab are exposed air at $T_\infty = 20°$C with heat transfer coefficient $\overline{h} = 15\ \mathrm{W/m^2}$-K. The initial temperature of the material is $T_{ini} = 20°$C.

a.) About how long do you expect it to take for the temperature of the material on the unheated side ($x = L$) to begin to rise?
b.) What do you expect the temperature of the material at the heated surface ($x = 0$) to be (approximately) at the time identified in (a)?
c.) Develop a simple and approximate model that can predict the temperature at the heated surface as a function of time for times that are less than the time calculated in (a). Plot the temperature as a function of time from $t = 0$ to the time identified in (a).
d.) Sketch the temperature as a function of position in the slab for several times less than the time identified in (a) and greater than the time identified in (a). Make sure that you get the qualitative features of the sketch correct. Also, sketch the temperature as a function of position in the slab at steady state; (make sure that you get the temperatures at either side correct).

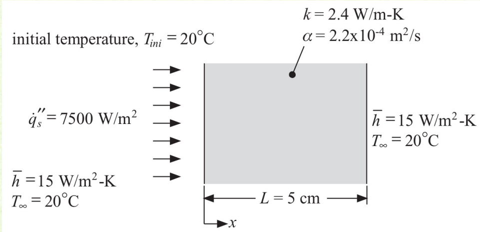


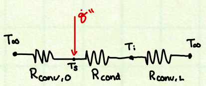
Figure P3-8: Slab of material heated at one surface.

a) The temperature at $x=L$ should begin to change at $t = \tau_{diff}$

$$\tau_{diff} = \frac{L^2}{4\alpha} = \frac{(0.05\,m)^2}{4 \cdot 2.2e\text{-}4\ m^2/s} = 2.84\ s$$

$$\underline{\underline{t = \tau_{diff} = 2.84\ s}}$$

b)
$$\dot{q}'' - h(T_s - T_\infty) = \frac{dU}{dt} = \frac{d}{dt}\left(\frac{(T_s - T_i)}{2}\rho c L\right) \quad , \quad \text{assumes } T_m = \frac{(T_s - T_i)}{2}$$



$$\dot{q}'' + hT_\infty = \frac{dT_s}{dt}\frac{\rho c L}{2} + hT_s$$

$$\frac{dT_s}{dt} + \frac{2h}{\rho c L}T_s = \frac{2\dot{q}''}{\rho c L} + \frac{2h}{\rho c L}T_\infty$$

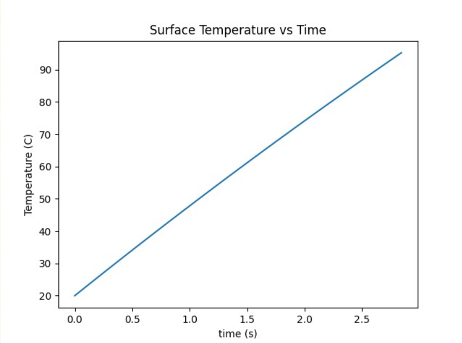
$$\frac{d}{dt}\left(e^{\frac{2h}{\rho c L}t}T_s\right) = e^{\frac{2h}{\rho c L}t}\left(\frac{2\dot{q}''}{\rho c L} + \frac{2h}{\rho c L}T_\infty\right)$$

$$T_s|_t - T_i = \int_0^t e^{\frac{2ht}{\rho c L}}dt\left(\frac{2\dot{q}''}{\rho c L} + \frac{2h}{\rho c L}T_\infty\right) / e^{\frac{2ht}{\rho c L}} = \frac{\rho c L}{2h}\left(e^{\frac{2ht}{\rho c L}} - 1\right)\left(\frac{2\dot{q}''}{\rho c L} + \frac{2h}{\rho c L}T_\infty\right)e^{\frac{-2ht}{\rho c L}}$$

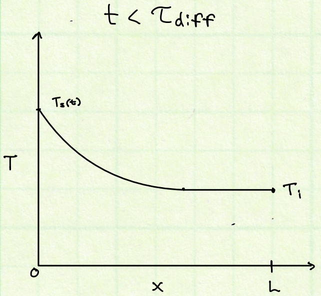
$$T_s = \left(\frac{\dot{q}''}{h} + T_\infty\right)\left(1 - e^{\frac{-2ht}{\rho c L}}\right) + T_i$$

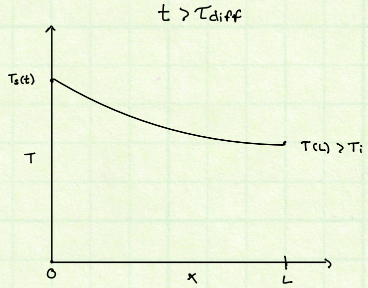
$$\underline{\underline{T_s(\tau_{diff}) = 95.22\ °C}}$$

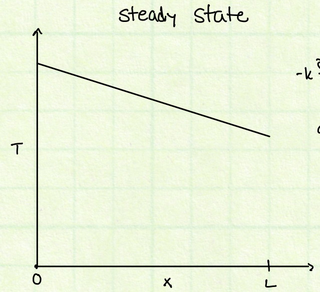
c) from previous problem $\quad T_s = \left(\frac{\dot{q}''}{h} + T_\infty\right)\left(1 - e^{\frac{-2ht}{\rho c L}}\right) + T_i$



d) T vs X Sketches







$$-k\frac{\partial T}{\partial x} = C = \dot{q} - h(T_s - T_\infty)$$

$$\dot{q}'' = h(T_s + T(L) - 2T_\infty)$$