Functions

Introductions

In this chapter we will discuss a special kind of relation called as function.
The knowledge of function is very useful both in Mathematics & computer science.
The terms such as mapping, transformation etc are also used for functions to depict a relation between two discrete objects.

Defⁿ Function

A relation f from a non empty set A to another non empty set B is said to be a function, if each element $x \in A$ corresponds to a unique element $y \in B$ and the y which corresponds in this way to a given x is denoted by $f(x)$ and is called the value of x under f.

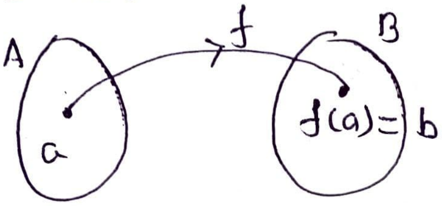
Note If f is a function from a set A to set B then we write it as $f: A \to B$.

Or Let A & B be any two non empty sets, not necessarily distinct. If there exists a rule or correspondence, denoted by f which relates (or associates) each element $x \in A$, to a unique (one & only one) element $y \in B$, then such a rule is called a function (or mapping) from A to B.

Symbolically the f^n f is expressed as,

$f: A \to B$ s.t. $y = f(x)$ for $x \in A$ & $y \in B$.

Thus A f^n takes an element from a set & maps it to a unique element in another set.



Graph

If f is a function from A to B then set of all ordered pairs $(x, f(x))$ is called the graph of f^n f.

Remark.

1) If $f: A \to B$ is a f^n from A to B & $(x, y) \in f$ the we write it as, $y = f(x)$. Here y is called image of x under f & x is called preimage of y under f.

2) If $f: A \to B$ be a function from the set A to the set B ie $f: A \to B$ then the set A is the domain of the function. And set B is the co-domain of the f^n.

Range: The set of all images in B is called the range of f

$$\text{Range} = \{ f(x) \mid x \in A \}$$

3) Every function is a relation but every relation need not be f^n.

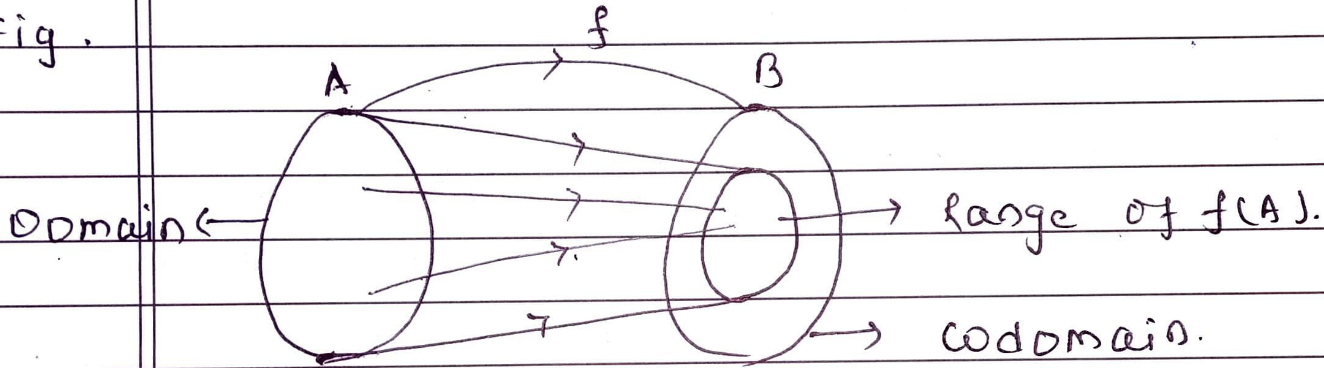
4) If the domain & codomain are not specified, they are assumed to be the set of real nos.

Domain, Co-domain And Range of a function :-

For a function f from A to B, the set A is called the Domain of f and the set of values of f is called the range of f & written as, $f(A)$, whereas B is called the co-domain of f.

Clearly $f(A) \subseteq B$.

All these sets can be seen in following Fig.



The range of f is denoted by,
i.e.

$$f(A) = \{ f(x) \mid x \in A \} = \{ y \mid y \in B,\ y = f(x) \text{ for some } x \in A \}.$$

Remark Each element of the first set A has unique image in B.

And each element of B need not appear as image of an element in A.

Also two or more elements of B should not be associated to any one element of A. There can be more than one element of A which have the same image in B.

Ex.

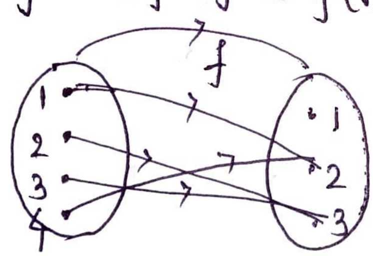
1) Let A be the set of books in a Library and N be the set of Natural no.s.
Let f be a rule which assigns to each book the no. of pages contained in it. Here some natural no. will be assigned to each book & no book can be assigned to more than one natural no.

$\therefore$ f is f^n from N to N.

2) Let $A = \{1, 2, 3, 4\}$ & $B = \{1, 2, 3\}$ then,
$f = \{(1,2)\ (2,3),\ (3,3),\ (4,2)\}$ is a f^n from A to B
Since each element in A we have assigned a unique element in B.

$\therefore$ $f(1) = 2$ $f(2) = 3$ $f(3) = 3$ $f(4) = 2$.

Range of $f = f(A) = \{2, 3\}$.



3) Let R be set of real no.s then the rule,

$f : R \to R$, $f(x) = x^2$ $\forall\ x \in R$.

is a f^n from R to R, because square of real no. is unique.

eg $f(1) = 1$ $f(2) = 4$ $f(-1) = 1$ $f(-2) = 4$ & soon

4) Let the rule defined by,

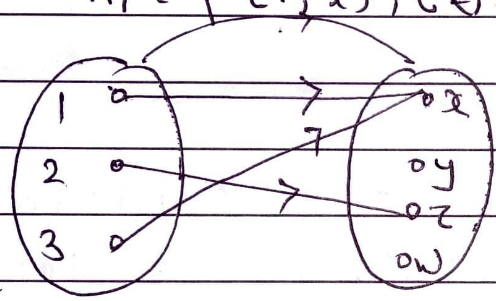
Function. $f: R_+ \rightarrow R$, $f(x) = \sqrt{x}$ $\forall x \in R$.

is not f^n, where R_+ is set of all real positive no. In this case for a single element say $1 \in R_+$ there are two ele.s -1 & 1 of R associated with it.

5) Let $A = \{1, 2, 3\}$ $B = \{x, y, z, w\}$.

Consider relation,

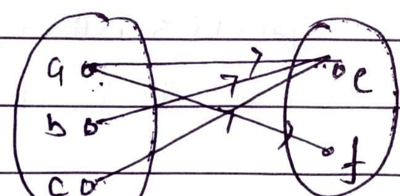
$R_1 = \{(1, x), (2, z), (2, z), (3, x)\}$



R is f^n, with Range of $f^n = \{x, z\}$.

6). $A = \{a, b, c\}$ $B = \{e, f\}$.

$R = \{(a, e), (b, e), (a, f), (c, e)\}$



is not f^n as, $f(a) = e$, $f(a) = f$ which violates the def^n of a f^n.

Counting of function

Let A & B be two finite sets having m & n elements resp.

Then each element of A can be associated to any one of n elements of set B.

So total no. of f^n from set A to B is equal to ways of having m job where each job can be done in n ways.

$\therefore$ total no. of f^n from A to B is n^m or $|B|^{|A|}$.

Ex. Let A be the set containing 10 ele.s & B be the set containing 5 elements.

Then total no. of f^n from A to B is 5^{10}

Function as a set of ordered pairs:-

Consider the f^n $f(x) = x^2+1$, where $A = \{1, 2, 3, 4\}$. $B = \{2, 5, 10, 17, 20\}$ we can form the set of ordered pairs of the corresponding elements as $\{(1,2)\ (2,5), (3,10), (4,17)\}$ In each of the ordered pairs here, first component is an element of set A & second component is an element of B. Thus we can consider the f^n f as a set of ordered pairs i.e a subset of $A \times B$. We can observe that no two pairs of this set have same first component for otherwise it will mean that one element of A is related to two different elements of set B.

Classification of function. (7)

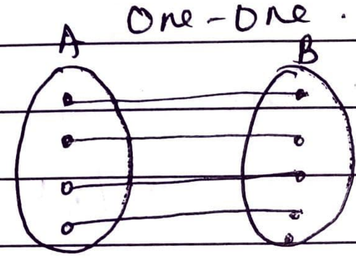
1) One - One (Injective) Function.

We say that a function $f: A \to B$ is one-one if $f(x_1) = f(x_2)$

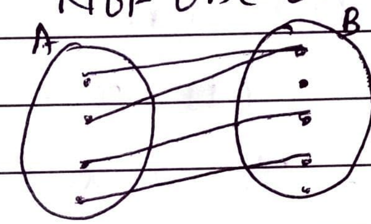
$$\Rightarrow x_1 = x_2 \quad \forall\, x_1, x_2 \in A.$$

i.e. no two elements in the domain corresponds to the same element in the range.

One-One.



Not One-One



Algorithm to check Injectivity of Function.

1) Take two arbitrary elements x_1 & x_2 in the domain

2) Put, $f(x_1) = f(x_2)$

3) Solve $f(x_1) = f(x_2)$.

If $f(x_1) = f(x_2)$ gives $x_1 = x_2$ only then $f: A \to B$ is one one function otherwise not.

or $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \quad \forall\, x_1, x_2 \in A.$

Ex. $f: [0,1] \to \mathbb{R}$ defined by, $f(x) = x^2$ is one-one function.

as $f(x) = f(y)$

$\Rightarrow x^2 = y^2$

$\Rightarrow x = y. \quad (x, y \geq 0) \quad \forall\, x, y \in [0,1]$

Remark

a) Injectivity of a function can also be checked from its graph. If any straight line parallel to x-axis intersects the curve $y = f(x)$ at most at one point, then the f^n $f(x)$ is one-one or an injection otherwise not

b) If $f: A \rightarrow B$ then $x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$ is true for all $x_1, x_2 \in A$. However $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \; \forall \; x_1, x_2 \in A$ is true only when f is an ~~injection map~~ injective map.

~~or $x_1 \neq x_2$~~

c) If A & B are finite sets having m & n elements respectively then, no. of one-one function from A to B

$$= \begin{cases} {}^{n}P_{m} & n \geq m. \\ 0 & n < m. \end{cases}$$

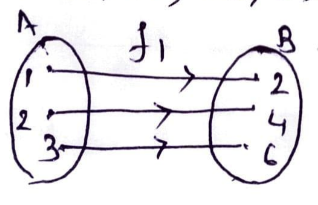
Alternate def^n of one-one function.

A function $f: A \rightarrow B$ is said to be one-one if distinct elements in A have distinct images in B.

ie $x_1, x_2 \in A$ s.t. $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

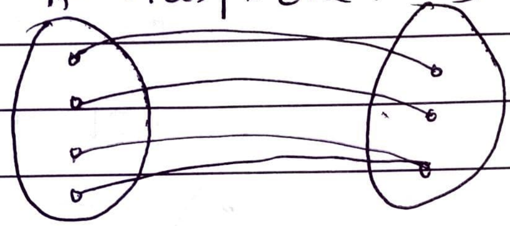
eg $A = \{1, 2, 3\}$ & $B = \{2, 4, 6\}$.

$f_1 = \{(1, 2)\ (2, 4), (3, 6)\}$. then f_1 is one-one f^n.



2) Many- One Function.

A function $f: A \to B$ is said to be many one function if two or more elements of set A (if exist) have the same image in B.



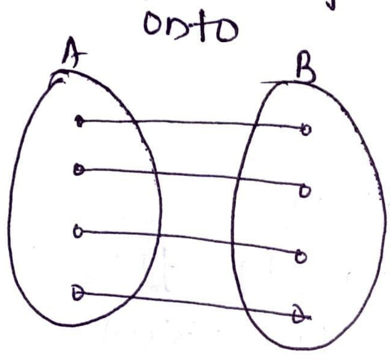
Ex. Consider the function,

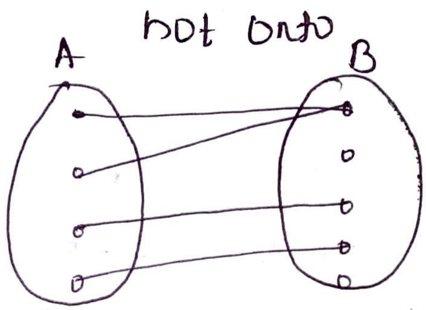
$f: [-1, 1] \to R$, defined by $f(x) = x^2$

then f is many to one function.

3) Onto (surjective) Function :-

A function $f: A \to B$ is said to be an onto function or a surjective function if every element of B is the f-image of some element of A i.e. $f(A) = B$ or range of f is the co-domain of f.





Ex. 1) The f^n $f: [0,1] \to \mathbb{R}$ defined by $f(x) = x^2$ is not an onto function.

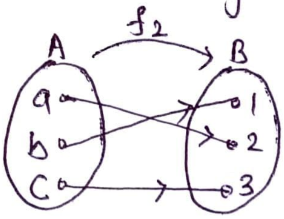
But $f: [0,1] \to [0,1]$ defined by $f(x) = x^2$ is an onto f^n.

2). Let $A = \{a, b, c\}$ $B = \{1, 2, 3\}$ &

$f_2 = \{(a,2), (b,1), (c,3)\}$.

Range of $f_2 = \{1, 2, 3\}$. And Range of f_2 = codomain of f_2 = Set B.

$\therefore$ The f^n f_2 is onto f^n.



Or def^n.

A function $f: A \to B$ is said to be onto if every element of B is the image of some element of A under the function f. Thus if f is onto then for every $y \in B$ there exist at least one element of A where $x \in A$ such that,

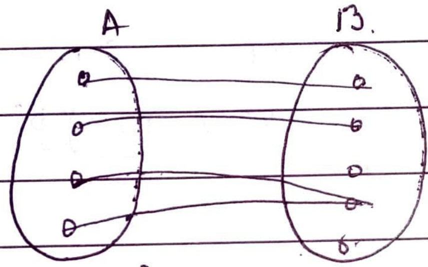
$y = f(x)$.

4) Into Function.

A function $f: A \to B$ is an into function if there exist at least one element in B having no pre image in A. But every element of A has f-image in B.

In other words,

$f: A \to B$ is an into function if $f(A) \subseteq B$ and $f(A) \neq B$.



into

clearly the range of f is a proper subset of its codomain ~~A~~ B.

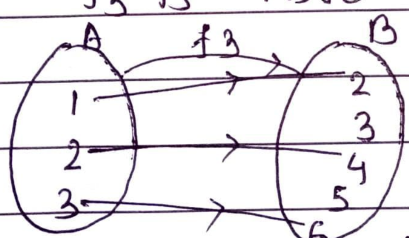
Ex. $A = \{1, 2, 3\}$ & $B = \{2, 3, 4, 5, 6\}$.

$f_3 = \{(1,2), (2,4), (3,6)\}$.

Range of $f_3 = \{2, 4, 6\}$.

$\therefore$ Range of $f_3 \neq$ co-domain of $f_3 = B$.

The f^n f_3 is into function.



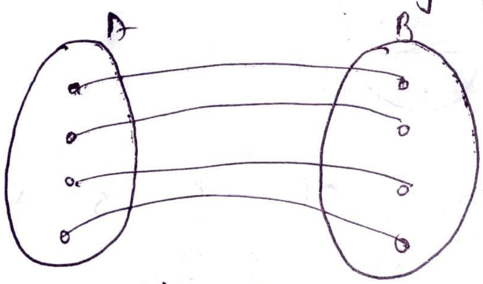
Thus, the function $f: A \to B$ is such that there exists at least one element in B which has no preimage in A, then f is said to be an into function.

5) One-One & Onto (Bijection) Function :-

The map $f: A \to B$ is bijective map if it is one-one as well as onto.

Or

A function $f: A \to B$ is said to be one-one onto function if to each element of A there corresponds one & only one element of B and every element of B have ~~one & only one~~ image in A. One-one & onto function are also called as bijective mapping.



Bijection.

Remark :-

Function f will be onto if for each $y \in B$ there exists an element $x \in A$ such that, $y = f(x)$.

Composition of functions:-

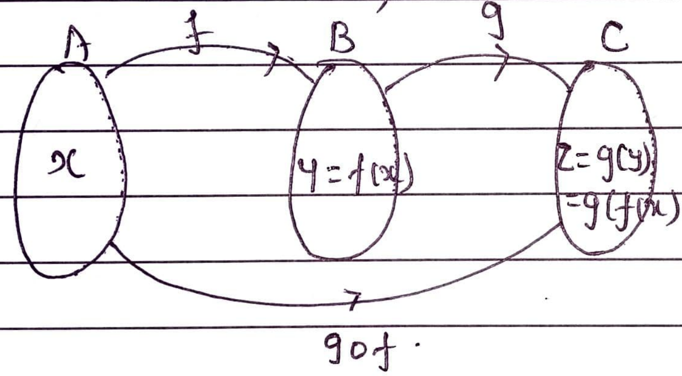
Let $f : A \to B$ & $g : B \to C$ be two functions. Then the composition of functions f & g denoted by $g \circ f$ is a function from A to C. and is written as,

$$(g \circ f) : A \to C.$$

Such that $(g \circ f)(x) = g(f(x))$ for each $x \in A$.

Obviously, domain of $(g \circ f)$ is set A & range of $(g \circ f)$ is subset of C.

The composition fn $(g \circ f)$ of f & g can be shown as,



Composition of Function.

Or Let $f : A \to B$ & $g : B \to C$ be two functions where domain of g is same as co-domain of f then we define the composite function of f & g from A to C denoted by $g \circ f$ & is given by,

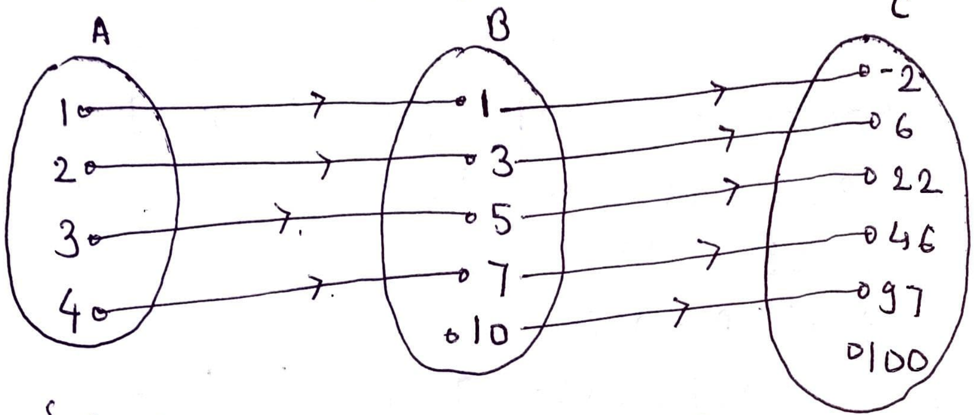
$$g \circ f(x) = g[f(x)]$$

$$\therefore g \circ f : A \to C \text{ where,}$$

$$g \circ f(x) = g[f(x)]$$

Ex. let $A = \{1, 2, 3, 4\}$. $B = \{1, 3, 5, 7, 10\}$.

$C = \{-2, 6, 22, 46, 97, 100\}$. where,



$f = \{(1,1), (2,3), (3,5), (4,7)\}$

$g = \{(1,-2), (3,6), (5,22), (7,46), (10,97)\}$

$g \circ f = \{(1,-2), (2,6)\ (3,22)\ (4,46)\}$.

since,

$f = \{(x, y) \mid x \in A, y \in B \ \& \ y = 2x - 1\}$.

$g = \{(y, z) \mid y \in B, z \in C \ \& \ z = y^2 - 3\}$

then,

$g \circ f(x) = \{(x, z) \mid x \in A, z \in C \ \& \ z = (2x-1)^2 - 3\}$.

Inverse Function :-

The domain & range of any function may be interchanged to form a new function. It is obtained by interchanging the position of elements in the ordered pair of original function.

Defn If $f: A \to B$ is one one, onto, $g: B \to A$ which is also one one & onto such that $g \circ f: A \to A$ & $f \circ g: B \to B$ are both identity functions (ie. $g \circ f(x) = x$, $f \circ g(y) = y$) then f & g are called as inverse functions of each other.

Function g is denoted by f^{-1} & is read as f inverse.

So we define a function f^{-1} as,

$f^{-1}: B \to A$ such that, if $f(x) = y$ then $f^{-1}(y) = x$.

Note:- Domain of f = Range of f^{-1} &

Range of f = Domain of f^{-1}.

Ex. Let $f: R \to R$, where $f(x) = 2x + 3$ &

$g: R \to R$ where $g(x) = \frac{x-3}{2}$

Consider,

$$g \circ f(x) = g[f(x)]$$

$$= g(2x+3)$$

$$= \frac{(2x+3)-3}{2} = x.$$

$\therefore g \circ f(x) = x$

Again,

$$fog(x) = f[g(x)]$$
$$= f\left(\frac{x-3}{2}\right)$$
$$= 2\left(\frac{x-3}{2}\right) + 3$$
$$= x$$

$\therefore$ gof & fog are identity functions.

$\therefore$ f & g are inverse of each other.

Note:- 1) Domain of f = Range of f^{-1} &
Range of f = Domain of f^{-1}.

2) When f & g are inverses of each other, both f & g are one-one & onto.

3). The inverse function of the given function f is unique

4). A function which possesses an inverse is said to be invertible.

5) Only one-one & onto functions are invertible.