Discrete Numeric Function

Def^n A function whose domain is the set of natural nos (including zero) and whose range is the set of real nos is called 'discrete numeric function' or 'Numeric function'.

If $a: N \to R$ is a discrete numeric function then $a_0, a_1, a_2, \ldots, a_r, \ldots$ denote the values of the function a at $0, 1, 2, \ldots, r, \ldots$

i.e. $a(0) = a_0$, $a(1) = a_1$, $a(2) = a_2$, $\ldots$, $a(r) = a_r, \ldots$

The numeric function 'a' is written as,

$a = \{a_0, a_1, a_2, \ldots, a_r, \ldots\}$.

The numeric function often termed as 'sequence'.

Ex. 1

$$a_r = \begin{cases} 3r & 0 \le r \le 5 \\ r+1 & r \geq 6. \end{cases}$$

The above function can be written as,

$$a = \{0, 3, 6, 9, 12, 15, 7, 8, 9, \ldots\}$$

Ex. 2 $b_r = 2r^2 + 3$, $r \geq 0$.

For this numeric f^n, the expression is same for all the values of r, hence we can write

$$b = 2r^2 + 3, \quad r \geq 0.$$

Operations on Numeric Functions :-

1) **Addition** Let a & b be two numeric function. Then the sum of two numeric f^n a & b is a numeric f^n c whose value at r is equal to the sum of the values of two numeric functions a & b at r.

The sum of two numeric functions is denoted by $c = a + b$, where $c_r = a_r + b_r$ for given r

2) **Product | Multiplication**

The product of two numeric f^n a & b is a numeric f^n d whose value at r is equal to the product of the values of two numeric f^n at r.

The product is denoted by, $d = ab$, where $d_r = a_r b_r$ for given r.

3) **Scaling**

Let a be the numeric function and α be any real number, Then αa is a numeric function whose value at r is equal to α times a_r. The f^n αa is called a scaled version of a with scaling factor α.

4) **Convolution**

The convolution of the numeric function a & b is numeric function c denoted by $a * b$ where,

$$c_r = a_0 b_r + a_1 b_{r-1} + a_2 b_{r-2} + \cdots + a_i b_{r-i} + \cdots + a_{r-1} b_1 + a_r b_0.$$

$$c_r = \sum_{i=0}^{r} a_i b_{r-i}$$

Finite Differences of a Numeric Functions.

1) Shift of Numeric function

The shift of a numeric function ~~of~~ a_r is denoted by $E[a_r]$ and is defined as,

$$E[a_r] = a_{r-1}.$$

$$E^{-1}[a_r] = a_{r+1}.$$

$$E^{n}[a_r] = a_{r-n}$$

$$E^{-n}[a_r] = a_{r+n}.$$

It is also denoted by $S^n[a_r] = E^n[a_r]$

2) Forward difference of a numeric function.

The forward difference of a numeric function a is denoted by Δa, whose value at r is equal to $a_{r+1} - a_r$.

$$\Delta a_r = a_{r+1} - a_r \quad \cancel{at} \quad r \geq 0.$$

$$\cancel{\geq a_r}$$

3) Backward difference of numeric function.

The backword difference of a numeric function a is denoted by ∇a, whose value at r is equal to $a_r - a_{r-1}$

$$\nabla a_r = a_r - a_{r-1}$$

Generating Functions.

Generating functions are alternative representation of numeric function.

Defn For the numeric function,

$$a = \{a_0, a_1, a_2, \ldots, a_r, \ldots\}.$$

we define an infinite series,

$$a_0 + a_1 z + a_2 z^2 + \cdots + a_r z^r + \cdots$$

is called the generating function of the numeric function a.

It is denoted by $A(z)$.

$$\therefore \quad A(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_r z^r + \cdots$$

$$= \sum_{r=0}^{\infty} (a_r z^r).$$

Ex. The generating function of a is

$$a = \{4^0, 4^1, 4^2, 4^3, \ldots, 4^r, \ldots\} \text{ is,}$$

$$A(z) = 4^0 + 4z + 4^2 z^2 + 4^3 z^3 + \cdots + 4^r z^r + \cdots$$

The closed form of the generating f^n of some numeric function is,

Numeric Function	Generating Function
1) $a_r = k a^r$	$A(z) = \frac{k}{1-az}$
2) $a_r = r$	$A(z) = \frac{z}{(1-z)^2}$
3) $a_r = b r a^r$	$A(z) = \frac{abz}{(1-az)^2}$
4) $a_r = \frac{1}{r!}$	$A(z) = e^z$
5) $a_r = \begin{cases} {}^nC_r & 0 \le r \le n \\ 0 & r > n \end{cases}$	$A(z) = (1+z)^n$

Exs Determine the generating function of the numeric function a_r, where

1) $a_r = 3^r \quad r \geq 0$

→ Given numeric f^n is,

$a = a_r = 3^r$

$= \cancel{3^0 + 3^1 + 3^2 + \dots}$

The corresponding generating function A(Z) is given by,

$$A(Z) = 3^0 Z^0 + 3^1 Z^1 + 3^2 Z^2 + 3^3 Z^3 + \dots + 3^r Z^r + \dots$$

$$= 1 + 3Z + 3^2 Z^2 + 3^3 Z^3 + \dots + 3^r Z^r + \dots \quad r \geq 0$$

$$\boxed{A(Z) = \frac{1}{1-3Z}}$$

2) $a_r = 4^{r+1} \quad r \geq 0$

→ Given numeric function is,

$a = a_r = 4^{r+1} = 4 \cdot 4^r, \quad r \geq 0.$

The corresponding generating function A(Z) is given by,

$$A(Z) = 4Z^0 + 4^2 Z^1 + 4^3 Z^2 + \dots + 4^{r+1} Z^r + \dots$$

$$\boxed{A(Z) = \frac{4}{1-4Z}}$$

3) $a_r = 5$, $r \geq 0$.

→ Given numeric function is,

$a_r = 5$, $r \geq 0$.

∴ The corresponding generating function is,

$$A(z) = 5z^0 + 5z^1 + 5z^2 + \cdots + 5z^r + \cdots$$
$$= 5\{1^0 z^0 + 1z^1 + 1^2 z^2 + \cdots + 1^r z^r + \cdots\}$$
$$= \cancel{\frac{5}{1-8}} = \frac{5}{1-z} .$$

$$\boxed{\therefore A(z) = \frac{5}{1-z}}$$

4) $$a_r = \begin{cases} 0 & r \text{ odd} \\ 2^{r+1} & r \text{ even} . \end{cases}$$

→ The numeric function is,

$$a = \{2, 0, 2^3, 0, 2^4, 0, 2^5, \ldots\}$$

This numeric function can be written as,

$$a_r = 2^r + (-2)^r \qquad r \geq 0$$

∴ The corresponding generating function is,

$$A(z) = \frac{1}{1-2z} + \frac{1}{1+2z} .$$

$$\boxed{A(z) = \frac{2}{1-4z^2}}$$

Ex. Determine the discrete numeric functions corresponding to the following generating functions

1) $\frac{1}{1+z}$.

→ $A(z) = \frac{1}{1+z} = \frac{1}{1-(-z)}$

It is the sum of geometric progression whose first term is 1. & common ratio is $-z$

$A(z) = 1 - z + z^2 - z^3 + z^4 + \cdots$

∴ The corresponding numeric function is,

$$\boxed{a_r = (-1)^r}$$

2) $\frac{3-5z}{(1-2z-3z^2)}$

$3z^2 + 2z - 1$

→ $A(z) = \frac{3-5z}{(1-2z-3z^2)}$

$= \frac{3-5z}{(1-3z)(1+z)} = \frac{A}{1-3z} + \frac{B}{1+z}$.

$A(z) = \frac{1}{1-3z} + \frac{2}{1+z}$.

The corresponding ~~to~~ numeric function is,

$$\boxed{a_r = (3)^r + 2(-1)^r}, \quad r \geq 0$$

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3) $\dfrac{2+3z-6z^2}{(1-2z)}$

→

$$A(z) = \frac{2+3z-6z^2}{1-2z}$$

$$\begin{array}{r} 3z \\ 1-2z \,\overline{\big)\, -6z^2 + 3z + 2} \\ -6z^2 + 3z \\ + \quad\quad - \\ \hline 2 \end{array}$$

$$A(z) = 3z + \frac{2}{1-2z}$$

Here

$$A(z) = B(z) + C(z).$$

$$B(z) = 3z \quad \& \quad C(z) = \frac{2}{1-2z}$$

For $B(z)$, the numeric f^n is,

$$b = \{0, 3, 0, 0, 0, \ldots\}$$

For $C(z)$, the numeric f^n is,

$$c_r = 2(2^r) = 2^{r+1}$$

$$\therefore \quad c = \{2, 2^2, 2^3, 2^4, \ldots\}$$

Hence the numeric f^n corresponds to $A(z)$ is,

$$a = b + c$$

$$a = \{0, 3, 0, 0 \ldots\} + \{2, 2^2, 2^3, \ldots\}$$

$$a = \overset{0}{2} + \overset{1}{7} + \overset{2}{2^3} + 2^4 + \ldots$$

$$\therefore \quad a_r = \begin{cases} 2 & r = 0 \\ 7 & r = 1 \\ 2^{r+1} & r \geq 2 \end{cases}$$

* Recurrence Relations *

Def^n A relation in which previous terms are used to define next terms, is called the 'recurrence relation'. A recurrence relation is also called as difference equation.

That is the recursive formula for defining the numeric function (or sequence) is called recurrence relation.

eg i) $a_r = a_{r-1} + a_{r-2}$, $r \geq 2$

ii) $a_r = a_{r-1} + 5$

are recurrence relation.

iii) $a_r = r^2 + 5$

is not recurrence relation.

Def^n Linear Recurrence Relation with constant coefficients.

A recurrence relation of the form

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = f(r), \text{ for } r \geq k \quad \text{--- (1)}$$

where $C_0, C_1, C_2, \ldots, C_k$ are constants ~~(1)~~ is called a linear recurrence relation with constant coefficients of order k, provided C_0 & C_k are non zero.

Ex.

1) The relation,

$$a_r - 2a_{r-1} = 2r$$

is first order Linear recurrence relation with constant coeff.

2) The relation, $a_r + 2a_{r-3} = r^2$ is third order Linear recurrence relation.

3) The relation $a_r^2 + a_{r-1} = 5$ is not a linear recurrence relation.

Homogeneous recurrence relation.

A recurrence relation,

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = f(r), \quad r \geq k$$

is called homogeneous recurrence relation if $f(r) = 0$.

Homogeneous solution.

Each linear recurrence relation is associated with its homogeneous equation and the solution of homogeneous equation is called homogeneous solution of the given recurrence relation.

Consider k^{th} order linear recurrence relation with constant coeff.

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = f(r).$$

Homogeneous recurrence relation of above recurrence relation is,

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots + C_k a_{r-k} = 0.$$

To find the solution of homogeneous recurrence relation we consider characteristic eqn of homogeneous k^{th} order recurrence relation, as,

$$C_0 \alpha^k + C_1 \alpha^{k-1} + C_2 \alpha^{k-2} + \cdots + C_k = 0.$$

which is k^{th} degree polynomial in α.

The roots of this equation are called the characteristic roots.

The homogeneous solution depends upon the nature of characteristic roots.

The following table gives the details about homogeneous solution and roots of characteristic eqn.

Sr. No.	Nature of roots	Homogeneous soln $a_r^{(h)}$
1	Distinct real roots	
	i) α_1, α_2.	$a_r^{(h)} = A_1\alpha_1^r + A_2\alpha_2^r$
	ii) $\alpha_1, \alpha_2, \ldots, \alpha_n$	$a_r^{(h)} = A_1\alpha_1^r + A_2\alpha_2^r + \cdots + A_n\alpha_n^r$
2	Repeated real roots	
	i) α_1, α_1	$a_r^{(h)} = (A_1 + A_2 r)\alpha_1^r$
	ii) $\alpha_1, \alpha_1, \alpha_1$	$a_r^{(h)} = (A_1 + A_2 r + A_3 r^2)\alpha_1^r$
3	Distinct complex roots $\alpha + i\beta, \alpha - i\beta$	$a_r^{(h)} = r_1^r(A\cos r\theta + B\sin r\theta)$ where, $r_1 = \sqrt{\alpha^2+\beta^2}$, $\theta = \tan^{-1}\frac{\beta}{\alpha}$.

A solution of a recurrence relation which is obtained from homogeneous recurrence relation is called the homogeneous solution.

It is denoted by $a_r^{(h)}$.

Ex. Solve $a_n = 2a_{n-1} - a_{n-2}$ with $a_1 = 1.5$, $a_2 = 3$.

→ We have, homogeneous recurrence relation,

$$a_n - 2a_{n-1} + a_{n-2} = 0 \quad \text{—} \; (1)$$

~~the~~ which is of second order.

The characteristic eqn of (1) is,

$$\alpha^2 - 2\alpha + 1 = 0$$

$$(\alpha - 1)^2 = 0$$

$\alpha = 1, 1$ are real & repeated roots.

∴ The soln of given recurrence relation is,

$$a_r^{(h)} = (A_1 + A_2 r)1^r = A_1 + A_2 r.$$

given that, $a_1 = 1.5$

$$\therefore a_1 = A_1 + A_2. \Rightarrow A_1 + A_2 = 1.5 \quad \text{—} \; (a)$$

also, $a_2 = 3$

$$\therefore a_3 = A_1 + 3A_2 \Rightarrow A_1 + 3A_2 = 3 \quad \text{—} \; (b)$$

solving (a) & (b) we get,

$$A_1 = 0, \quad A_2 = 1.5$$

$$\therefore a_r^{(h)} = 0 + 1.5r$$

As $f(r) = 0$, $a_r^{(P)} = 0$.

$$\therefore \boxed{a_r = a_r^{(h)} + a_r^{(p)} = 1.5r}$$ is homog. soln.

Particular solution.

The solution which satisfies the linear recurrence relation with $f(r)$ on right side is called the particular solution. It is denoted by $a_r^{(P)}$

There is no general procedure for determining the particular solution of the given recurrence relation. It depends upon the nature of $f(r)$.

Consider the following table of particular solutions for given right hand side function $f(r)$.

Sr No.	f(r) Right hand side function	Form of particular solution.
1	$f(r) = d =$ constant	$a_r^{(P)} = P$ (a constant)
2	$d_0 + d_1 r$ (Linear f^n)	$a_r^{(P)} = P_0 + P_1 r$
3	$d_0 + d_1 r + d_2 r^2 + \cdots + d_n r^n$ (nth deg. polynomial)	$a_r^{(P)} = P_0 + P_1 r + P_2 r^2 + \cdots + P_n r^n$
4	$d b^r$ (an exponential f^n provided b is not the characteristic root)	$a_r^{(P)} = P b^r$
5	$d b^r$ (an exponential f^n where b is characteristic root with multiplicity m)	$a_r^{(P)} = P r^{m-1} b^r$

Total solution

The total solution or general solution of a recurrence relation is the sum of homogeneous solution & particular solution

$$a_r = a_r^{(h)} + a_r^{(p)}$$

Examples

Ex. Solve $a_r - 3a_{r-1} = 2$, $r \geq 1$ with $a_0 = 1$ using generating functions.

→ Given recurrence relation is,

$$a_r - 3a_{r-1} = 2 \quad \text{—(1)}$$

(1) Homogeneous solution

consider, characteristic eqn.

$$\alpha - 3 = 0$$

$$\alpha = 3$$

$$a_r^{(h)} = A(3)^r \quad \text{—(2)}$$

(2) Particular solution

$f(r) = 2$ constant

∴ particular solution is of const. type.

$$\therefore a_r^{(p)} = P, \quad a_{r-1} = P$$

$$\therefore P - 3P = 2$$

$$\boxed{P = -1}$$

$$\therefore a_r = A(3)^r - 1.$$

By using initial conditions,

$$a_0 = A - 1.$$

$$1 = A - 1$$

$$\boxed{A = 2}$$

$$\therefore \boxed{a_r = 2(3)^r - 1}$$ is Total solution.

Ex. Solve $a_r - a_{r-1} - 6a_{r-2} = -30$ given $a_0 = 20$ $a_1 = -5$.

→ Given, Linear recurrence relation is,

$$a_r - a_{r-1} - 6a_{r-2} = -30 \quad \text{--- (1)}$$

(1) Homogeneous solution

The corresponding Homogeneous recurrence relation is,

$$a_r - a_{r-1} - 6a_{r-2} = 0$$

The characteristic eq^n is,

$$\alpha^2 - \alpha - 6 = 0$$

$$\alpha = -2, 3$$

The homogeneous solution of the given recurrence relation is,

$$a_r^{(h)} = A_1(-2)^r + A_2(3)^r \quad \text{--- (2)}$$

(2) Particular solution

$f(r) = -30$ (constant)

∴ the particular solution will also be const. P contain i.e $a_r = P$ $\forall r$.

$\therefore a_{r-1} = a_{r-2} = P$

Put in (1)

$P - P - 6P = -30$ ∴ $\underline{P = 5}$

$$a_r^{(p)} = 5$$

∴ The total solution is,

$$a_r = a_r^{(h)} + a_r^{(p)}$$

$$a_r = A_1(-2)^r + A_2(3)^r + 5 \quad \text{--- (3)}$$

Put $r = 0$

$a_0 = A_1 + A_2 + 5$

$A_1 + A_2 + 5 = 20$

$A_1 + A_2 = 15$

Put ~~$a_1 = -5$~~ $r = 1$. in (3)

$a_1 = -2A_1 + 3A_2 + 5$

$-5 = -2A_1 + 3A_2 + 5$

$-2A_1 + 3A_2 = -10$

$\Rightarrow A_1 = 11, \quad A_2 = 4$

$\therefore$ complete solution is,

$$\boxed{a_r = 11(-2)^r + 4(3)^r + 5}$$

Ex Solve $a_r - 5a_{r-1} + 6a_{r-2} = 2^r + r, \quad r \geq 2$
with $a_0 = 1, \quad a_1 = 1.$

$\rightarrow$ Given recurrence relation is,

$$a_r - 5a_{r-1} + 6a_{r-2} = 2^r + r \quad \text{--- (1)}$$

(1) Homogeneous soln,
consider, characteristic eqn,

$$\alpha^2 - 5\alpha + 6 = 0$$

$$\alpha = 2, 3.$$

$\therefore$ Homogeneous solution is,

$$a_r^{(h)} = A_1(2)^r + A_2(3)^r \quad \text{--- (2)}$$

(2) For the particular solution.

$$f(r) = 2^r + r$$

$$a_r^{(p)} = P_1 r^{2-1} 2^r + (P_2 + P_3 r).$$

$$a_r^{(p)} = P_1 r 2^r + P_2 + P_3 r$$

Put in (1)

$$\{P_1 r 2^r + P_2 + P_3 r\} - 5\{P_1(r-1)2^{r-1} + P_2 + P_3(r-1)\}$$
$$+ 6\{P_1(r-2)2^{r-2} + P_2 + P_3(r-2)\} = 2^r + r$$

$+5P$
-12

$$2^r\left[P_1 r - \frac{5}{2}P_1(r-1) + \frac{6}{4}P_1(r-2)\right] + 2P_2 + [P_3 r - 5P_3(r-1)$$
$$+ 6P_3(r-2)] = 2^r + r$$

$$2^r\left[P_1 r - \frac{5}{2}P_1(r-1) + \frac{3}{2}P_1(r-2)\right] + 2P_2 + [2P_3 r - 7P_3] = 2^r + r$$

compare the coeff-s.

$P_1 r - \frac{5}{2} P_1 (r-1) + \frac{3}{2} P_1 (r-2) \Big] = 1$ & $2P_3 = 1.$

& $2P_2 - 7P_3 = 0$

$P_1 r \left(1 - \frac{5}{2} + \frac{3}{2}\right) + \frac{5}{2} P_1 - \frac{3}{2} P_1 = 1$ & $2P_3 = 1$, $2P_2 - 7P_3 = 0$

$0 - \frac{1}{2} P_1 = 1$ & $2P_3 = 1$, $2P_2 - 7P_3 = 0$

$\frac{5}{2} - 3$

$\therefore \boxed{P_1 = -2}$, $\boxed{P_3 = 1/2}$, $\boxed{P_2 = 7/4}$

$$\therefore a_r^{(P)} = A_1 2^r + A_2 3^r - 2r2^r + \frac{1}{2} + \frac{7}{4} r$$

- Using boundary conditions,

$a_0 = 1$ & $a_1 = 1.$

$-4 + \frac{9}{4}$

$\frac{-16+9}{4} = -\frac{7}{4}.$

$1 = A_1 + A_2 + \frac{1}{2}$ $\qquad 1 = 2A_1 + 3A_2 - 4 + \frac{1}{2} + \frac{7}{4}$ $\qquad 1 + \frac{7}{4}$

$A_1 + A_2 = -1/2$ $\qquad 2A_1 + 3A_2 = \frac{11}{4}$

$2A_1 + 2A_2 = -1$

$2A_1 + 3A_2 = 11/4$

$-A_2 = -1 - \frac{11}{4}$

$A_2 = \frac{15}{4}$

$\Rightarrow A_1 = -\frac{5}{4}$, $A_2 = \frac{7}{4}.$

- ~~complete~~ Total solution is,

$$a_r = -\frac{5}{4} 2^r + \frac{7}{4} 3^r - 2r2^r + \frac{1}{2} + \frac{7}{4} r$$