Unit II Functions And Relations

Surjective, Injective, Bijective & Inverse functions, composition of function, Reflexivity, symmetry, Transitivity & equivalence relations.

Introduction

The relation such as less than, perpendicular to, not equal to, subset of etc are often used in mathematics & computer science problems concerning discrete objects.

These relations help establishing a relation between pair of objects taken in a particular order.

A relation between two sets can be defined by listing their elements as an ordered pairs. An ordered pair consists of two elements say a & b, in which one of them is designated as the first element and the other as second element.

An ordered pair is usually denoted by (a,b).

There can be ordered pairs which have the same first and second element such as $(1,1)$, $(2,2)$, (a,a) etc.

Two ordered pair (a,b) & (c,d) are said to be equal if & only if $a=c$ & $b=d$.

In other words, $(a,b) = (c,d) \Leftrightarrow a=c$ & $b=d$.

Thus, the ordered pair $(2,3)$ & $(2,3)$ are equal while the ordered pairs $(2,3)$ & $(3,2)$ are different.

Cartesian Product of sets

If A & B are any two non-empty finite sets then the set of all distinct ordered pairs whose first member (or coordinate) belongs to A & second member (or coordinate) belongs to B is called the cartesian product of A & B (in that order) and is denoted by $A \times B$ (read as $A \times B$).

Or

Let A & B be non empty sets. The product set or the cartesian product $A \times B$ is defined as,

$$A \times B = \{(a, b) \mid a \in A \ \& \ b \in B\}.$$

If $A = \phi$ & $B = \phi$ then $A \times B = \phi$.

Ex.s. 1) $A = \{1, 2, 3\}$, $B = \{2, 3\}$. then,

$A \times B = \{(1, 2) (1, 3), (2, 2), (2, 3), (3, 2), (3, 3)\}$.

$n(A \times B)$ = No. of elements in $A \times B = 6$.

2) Let $A = \{a, b, c\}$, $B = \{1, 2\}$.

$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$.

3) If R denotes the set of all real nos then $R \times R$ denotes the set of all points in the co-ordinate plane.

4) we know that a complex no. $x + iy$ can be considered as an ordered pair (x, y) hence if c denotes the set of all complex nos then C is the cartesian product $R \times R$.

Remarks

1) Cartesian product of two sets is not commutative

$$A \times B \neq B \times A.$$

2) If either A or B is infinite and the other is empty, then $A \times B = \phi$.

3) If either A or B is infinite & the other is not empty, then $A \times B$ is infinite.

Thm If A & B are finite sets with cardinalities m, n respectively then

$$|A \times B| = m \cdot n.$$

i.e. If set A has m elements & the set B has n elements then the product set $A \times B$ has mn elements.

Ex. If $A = \{n \in N \mid 1 \leq n \leq 100\}$.

$B = \{n \in N \mid 1 \leq n \leq 50\}$.

then,

$$|A \times B| = 100 \times 50 = 5000.$$

Ex. If $A = \{1\}$, $B = \{a, b\}$, $C = \{2, 3\}$ Find $A \times B \times C$, A^2, $B^2 \times A$, ~~C³~~

→ 1) $A \times B \times C = \{(1, a, 2), (1, b, 2), (1, a, 3), (1, b, 3)\}$.

2) $A^2 = A \times A = \{(1, 1)\}$.

3) $B^2 = B \times B = \{(a, a) (a, b), (b, a), (b, b)\}$.

4) $B^2 \times A = \{((a, a), 1), ((a, b), 1), ((b, a), 1), ((b, b), 1)\}$

~~5) $C^3 = C \times C \times C = \{(2, 2), (2, 3), (3, 2), (3, 3)$~~

Def^n n-ary relation :-

Let $A_1, A_2, \ldots, A_n$ be a finite collection of sets.

A subset R of $A_1 \times A_2 \times \cdots \times A_n$ is called an n-ary relation on $A_1, A_2, \ldots, A_n$.

If $R = \phi$, then R is called void or empty relation.

If $R = A_1 \times A_2 \times \cdots \times A_n$ then R is called the universal relation.

If $A_i = A$ for all i, then R is called ~~the~~ an n-ary relation on A.

If $n = 1$, then R is called unary relation

If $n = 2$ then R is called Binary relation.

If $n = 3$ then R is called ternary relation.

Def^n Binary Relation

Let A & B be two non empty sets. Then the Binary relation R from the set A to B is defined to be the subset of $A \times B$.

symbolically,

$$R : A \to B \text{ iff } R \subseteq A \times B \text{ \& } (a, b) \in R.$$

where $a \in A$ & $b \in B$.

If this relation holds then we say that a is related to b by R & we write aRb.

If a is not related to b by R, we write $a \not R b$.

Note If $A = B$, then R is said to be the relation on A.

Ex. 1) If $A = \{1, 2, 3, ---, 20\}$. and

~~then~~ $R = \{(x, y) \mid x = 3y\}$

then ~~$R = \{(1,3), (2,6)$~~

$R = \{(3,1), (6,2), (9,3), (12,4), (15,5), (18,6)\}$

2) $A = \{2, 3, 5, 6\}$

and, $R = \{(x, y) \mid x \text{ divides } y \text{ ie } x \mid y\}$.

then

$R = \{(2,2), (2,6), (3,3), (3,6), (5,5), (6,6)\}$

Defn Domain And Range of a Relation

If R is the relation from a set A to another set B, then the set of elements in A that are related to some element or elements in B is called the Domain of R.

In other words, domain of R, a subset of A is the set of all first elements in the ordered pairs which belongs to R. ~~is given by~~ symbolically,

$D(R) = \{x \mid (x, y) \in R,\ x \in A \text{ for some } y \in B\}$

The range of the relation R, is the set of elements in B that appear as second element in the ordered pair which belongs to R ie all elements in B that are related to some element in A.

Symbolically,

$Rn(R) = \{y \mid (x, y) \in R \text{ for some } x \in A\}$

Defⁿ Complement of a Relation.

Let A & B be two non empty sets. A relation R from A to B is defined as,

$$R = \{(a,b) \mid a \in A \text{ \& } b \in B\}.$$

then complement of relation R is denoted by R' or $\overline{R}$ & is defined as,

$$R' = \overline{R} = \{(a,b) \mid (a,b) \notin R\}.$$

ie $a \overline{R} b$ iff $a \not R b$.

De Morgans Laws

1) $\overline{R \cup S} = \overline{R} \cap \overline{S}$

2) $\overline{R \cap S} = \overline{R} \cup \overline{S}$.

Defⁿ Inverse Relation (or Converse Relation)

Let R be any relation from a set A to set B. The inverse relation denoted by R^{-1} or R^{C} is a relation from B to A and is ~~denoted~~ defined by,

$$R^{C} = R^{-1} = \{(y,x) \mid x \in A,\ y \in B,\ (x,y) \in R\}.$$

In other words, the inverse relation is obtained by reversing each of the ordered pairs belongs to R. Thus,

$$(x,y) \in R \leftrightarrow (y,x) \in R^{-1} \quad \text{or} \quad xRy \leftrightarrow yR^{-1}x.$$

Clearly $R^{-1} \subseteq B \times A$.

Imp Results

1) $(R^{-1})^{-1} = R$.

2) $(R_1 \cup R_2)^{-1} = R_2^{-1} \cup R_1^{-1} = R_1^{-1} \cup R_2^{-1}$

3) $(R_1 \cap R_2)^{-1} = R_2^{-1} \cap R_1^{-1} = R_1^{-1} \cap R_2^{-1}$.

Def^n Composition of Binary Relation.

Let R_1 be a relation from A to B & R_2 is a relation from B to C, the composite relation from A to C denoted by $R_1 \circ R_2$ or $R_1 R_2$ is defined as,

$$R_1 \circ R_2 = \{(a,c) \mid a \in A \ \& \ c \in C \ \& \ b \in B \text{ s.t. } (a,b) \in R_1 \ \& \ (b,c) \in R_2\}.$$

Def^n Identity Relation in a Set.

Let A be any set. Then the relation R in a set A denoted by I_A is said to be identity relation if,

$$I_A = \{(x,y) \mid x \in A,\ y \in A,\ x = y\}.$$

In other words, the identity relation I_A in a set A is the set of all ordered pairs (x,y) of $A \times B$ for which $x = y$.

The domain & Range of I_A are both A.

Ex. If $A = \{a,b,c,d\}$ then Identity relation in A is,

$$I_A = \{(a,a),(b,b),(c,c),(d,d)\}.$$

Examples

1) Let $A = \{2,3,4,5\}$ and R be a relation on A defined as aRb iff ~~at~~ $a<b$. Find $D(R)$ & $Rn(R)$.

→ $A = \{2,3,4,5\}$.

$R = \{(2,3), (2,4), (2,5), (3,4), (3,5), (4,5)\}$.

$D(R) = \{2,3,4\}$

$Rn(R) = \{3,4,5\}$.

2) For $A = \{1,2,3,4\}$, $B = \{a,b,c\}$ R is,
$R = \{(2,a), (4,a), (4,c)\}$. Find domain & range of f

→ $D(R) = \{~~c~~\ 2,4\}$, $Rn(R) = \{a,c\}$..

3). For $A = \{1,2,3,4\}$ & $B = \{a,b,c\}$.

$R = \{(1,a), (1,b), (2,c), (3,a), (4,b)\}$.

$S = \{(1,b), (1,c), (2,a), (3,b), (4,b)\}$.

Find $\overline{R}$ & $\overline{S}$ Also verify De morgans law.

→ Given $A = \{1,2,3,4\}$, $B = \{a,b,c\}$

$A \times B = \{(1,a), (1,b), (1,c), (2,a), (2,b), (2,c), (3,a)$
$(3,b), (3,c), (4,a), (4,b), (4,c)\}$.

$\overline{R} = \{(1,c), (2,a), (2,b), (3,b), (3,c), (4,a), (4,c)\}$

$\overline{S} = \{(1,a), (2,b), (2,c), (3,a), (3,c), (4,a), (4,c)\}$.

To verify De Morgans Law.

(10)

1) $R \cup S = \{(1,a), (1,b), (1,c), (2,a), (2,c), (3,a), (3,b), (4,b)\}$

$\overline{R \cup S} = \{(2,b), (3,c), (4,a), (4,c)\}$.

$\bar{R} \cap \bar{S} = \{(2,b), (3,c), (4,a), (4,c)\}$

$\Rightarrow \quad \overline{R \cup S} = \bar{R} \cap \bar{S}$.

2) $R \cap S = \{(1,b), (4,b)\}$.

$\overline{R \cap S} = \{(1,a), (1,c), (2,a), (2,b), (2,c), (3,a), (3,b)$
$(3,c), (4,a), (4,c)\}$.

$\bar{R} \cup \bar{S} = \{(1,a), (1,c), (2,a), (2,b), (2,c), (3,a), (3,b)$
$(3,c), (4,a), (4,c)\}$.

$\Rightarrow \quad \overline{R \cap S} = \bar{R} \cup \bar{S}$.

Ex. For $A = \{1,2,3,4\}$ & $B = \{a,b,c\}$.
$R = \{(1,a), (3,a), (3,c)\}$
Find R^{-1}, $D(R^{-1})$, $R_D(R^{-1})$

→ Given, $R = \{(1,a), (3,a), (3,c)\}$.

$\therefore R^{-1} = \{(a,1), (a,3), (c,3)\}$.

$D(R^{-1}) = \{a, c\}$ $\qquad$ $R_D(R^{-1}) = \{1, 3\}$.

Ex Let $A = \{a, b, c, d\}$ where,

$R_1 = \{(a,a), (a,b), (b,d)\}$.

$R_2 = \{(a,d), (b,c), (b,d), (c,b)\}$.

Find R_1R_2, R_2R_1, R_1^2, R_2^3.

$\rightarrow$ $R_1 = \{(a,a), (a,b), (b,d)\}$

$R_2 = \{(a,d), (b,c), (b,d), (c,b)\}$

$R_1R_2 = \{(a,d), (a,c)\}$.

$R_2R_1 = \{(c,d)\}$.

$R_1^2 = \{(a,a), (a,b), (a,d)\}$

$R_2^2 = \{(b,b), (c,c), (c,d)\}$.

$R_2^3 = \{(b,c), (c,b), (b,d)\}$.

Ex. Let $A = \{2,3,4,6,5\}$. & Let R_1 & R_2 be relation on A such that,

$R_1 = \{(a,b) \mid a-b=2\}$, $R_2 = \{(a,b) \mid a+1=b$ or $a=2b\}$.

Find i) R_1R_2 ii) R_2R_1 iii) $R_1.R_2R_1$..

$\rightarrow$ $R_1 = \{(4,2), (5,3), (6,4)\}$.

$R_2 = \{(2,3), (3,4), (4,5), (5,6), (4,2), (6,3)\}$

$R_1R_2 = \{(4,3), (5,4), (6,5)\ (6,2).\}$ $(5,$

$R_2R_1 = \{(3,2), (4,3)\ (5,4)\}$.

$R_1R_2R_1 = R_1(R_2R_1) = \{(5,2), (6,3)\}$.

Properties of Relation

1) Reflexive Relation

A relation R on a set A is reflexive iff each element in A is related to itself, ie aRa for all $a \in A$.
In other words, for all $a \in A$, $(a,a) \in R$.

Ex 1) Let $A = \{a, b\}$ & $R = \{(a,a), (a,b), (b,b)\}$.
Then R is Reflexive.

2) If A is set of all straight lines in 2-D plane & R is a relation,
$R = \{(a,b) \mid a \text{ is parallel to } b\}$
then R is reflexive relation as every straight line is parallel to itself.

3) $A = \{1,2,3,4\}$ then the relation,
$R_1 = \{(1,1), (2,4), (3,3), (4,1), (4,4)\}$ on A
is not reflexive because $2 \in A$ but $(2,2) \notin R_1$

2) Irreflexive Relation.

A relation R on a set R is said to be irreflexive if for every element $a \in A$, $a \not R a$ i.e $(a,a) \notin R$.

Ex. 1) Let $A = \{1,2\}$. & $R = \{(1,2), (2,1)\}$
then R is irreflexive since, $(1,1), (2,2) \notin R$.

2) Let $A = \{1,2\}$, & $R_1 = \{(1,2), (2,2)\}$. then
R_1 is not irreflexive since $(2,2) \in R_1$

Note R_1 is not reflexive as, $(1,1) \notin R$.

3) Symmetric Relation :-

A relation R on a set A is said to be symmetric iff for all $a, b \in R$, $aRb \Rightarrow bRa$
i.e. $(a,b) \in R \Rightarrow (b,a) \in R$.

The necessary & sufficient condition for the relation R to be symmetric is $R = R^{-1}$.

Ex 1) The relation in N defined by $x+y=8$ is symmetric because if $a+b=8$ then $b+a=8$.

The relation in N defined by 'x is a divisor of y' is not symmetric because 2 divides 4 but 4 does not divide 2.

2) Let A is the set of people and let aRb if a is brother of b. Then this R is not a symmetric relation since b can be sister of a. This relation will be symmetric only if A is the set of males.

4) Asymmetric Relation :-

A relation R on a ~~as~~ set A is said to be asymmetric if whenever aRb then $b \not R a$.

Hence R is not asymmetric if for some $a \& b \in A$ we have both aRb & bRa.

Ex. 1) Let $A = R =$ set of real nos & R is relation '<' then $a < b \rightarrow b \not< a$ hence < is asymmetric.

2) $A = \{2, 4, 5\}$ & R is "is a divisor of"
$R = \{(2,2), (2,4), (4,4), (5,5)\}$ is not asymmetric as $(2,2), (4,4), (5,5) \in R$.