Ex Find the homogeneous solution for the recurrence relation,

$$a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3} \quad \text{with } a_0 = 2,\ a_1 = 5,\ a_2 = 15$$

→ Given recurrence relation,

$$a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0 \quad \text{—(1)}$$

The characteristic eqn is,

$$\alpha^3 - 6\alpha^2 + 11\alpha - 6 = 0.$$

$$\begin{array}{c|cccc} 1 & 1 & -6 & +11 & -6 \\ & & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$$(\alpha - 1)(\alpha^2 - 5\alpha + 6) = 0$$

$$(\alpha - 1)(\alpha - 2)(\alpha - 3) = 0$$

$$\alpha = 1, 2, 3.$$

Homogeneous solution is,

$$a_n = A_1(1)^n + A_2(2)^n + A_3(3)^n. \quad \text{—(2)}$$

Given $a_0 = 2,\ a_1 = 5,\ a_2 = 15$

$$a_0 = A_1 + A_2 + A_3 \Rightarrow A_1 + A_2 + A_3 = 2 \quad \text{—(3)}$$

$$a_1 = A_1 + 2A_2 + 3A_3 \Rightarrow A_1 + 2A_2 + 3A_3 = 5 \quad \text{—(4)}$$

$$a_2 = A_1 + 4A_2 + 9A_3 \Rightarrow A_1 + 4A_2 + 9A_3 = 15 \quad \text{—(5)}.$$

(4) − (3), (5) − (3) ⇒

$$A_2 + 2A_3 = 3$$

$$3A_2 + 8A_3 = 13$$

$$3\cancel{A_2} + 6A_3 = 9$$

$$3\cancel{A_2} + 8A_3 = 13$$

$$-2A_3 = -4$$

$$\boxed{A_3 = 2.}$$

$$\therefore \boxed{A_2 = -1} \qquad \boxed{A_1 = 1}$$

$$\therefore a_n = 1(1)^n - 1(2)^n + 2(3)^n$$

$$\boxed{a_n = 1 - 2^n + 2(3)^n}$$

ex. Solve recurrence relation

$$a_r + a_{r-1} = 3r\,2^r$$

→ Given recurrence relation is,

$$a_r + a_{r-1} = 3r\,2^r \quad \text{--- (1)}$$

(1) Homogeneous solution.

characteristic eqn of (1) is

$$\alpha + 1 = 0 \quad \Rightarrow \quad \alpha = -1$$

Pbl.

The homogeneous soln is,

$$a_r^{(h)} = A_1(-1)^r \quad \text{--- (2)}$$

(2) Particular solution.

$$f(r) = 3r\,2^r$$

Here the form of particular soln is,

$$a_r^{(P)} = (P_0 + P_1 r)\,2^r$$

Put in (1)

$$(P_0 + P_1 r)\,2^r + \left[P_0 + P_1(r-1)\right]2^{r-1} = 3r\,2^r$$

$$2^r\left\{(P_0 + P_1 r) + (P_0 + P_1 r - P_1)\tfrac{1}{2}\right\} = 3r\,2^r$$

$$2^r\left\{\left(P_0 + \tfrac{1}{2}P_0 - \tfrac{1}{2}P_1\right) + \left(P_1 + \tfrac{1}{2}P_1\right)r\right\} = 3r\,2^r$$

$$\left(\tfrac{3}{2}P_0 - \tfrac{1}{2}P_1\right)2^r + \left(\tfrac{3}{2}P_1\right)r\,2^r = 3r\,2^r$$

Equating

$$\tfrac{3}{2}P_1 = 3 \quad \& \quad \tfrac{3}{2}P_0 - \tfrac{1}{2}P_1 = 0$$

$$\boxed{P_1 = 2} \quad \& \quad \cancel{\tfrac{3}{2}P_0 - \tfrac{1}{2}P_1 = 0} \quad \tfrac{3}{2}P_0 - 1 = 0$$

$$\boxed{P_0 = \tfrac{2}{3}}$$

$$a_r^{(P)} = \left(\tfrac{2}{3} + 2r\right)2^r$$

Total soln, $a_r = a_r^{(h)} + a_r^{(P)}$

$$a_r = A_1(-1)^r + \left(\tfrac{2}{3} + 2r\right)2^r \quad \text{--- (Ans)}$$