Mathematical Induction

One of the most important technique to prove many mathematical statements or formulae which can not be easily derived by direct methods is sometimes derived by using the principle of mathematical induction.

In this principle, the word induction is associated with the inductive reasoning by which a conclusion is drawn from a large no. special cases.

Actually mathematical induction is deductive in nature, for it leads to a definite conclusion.

It is usually employed in proving the validity of a statement involving all positive integer values of the no. n

Alternately,

In mathematics, many times we need to generalise a particular solution. This process is known as mathematical induction.

In this case, we first establish a particular solution of the problem and then generalise it for rest all values of the variable.

Principle of Mathematical induction

Let $P(n)$ be a statement or proposition defined by the set of positive integers, I_+ such that it is either true or false for all $n \in I_+$. For the given statement $P(n)$, if we can prove that,

a) $P(n)$ is true for $n = n_0$ (the smallest integer)

b) $P(n)$ is true for $n = k+1$, assuming that it is true for $n = k$ ($k \geq n_0$)

then we can conclude that $P(n)$ is true for all natural numbers $n \geq n_0$.

Assuming the validity of the statement for the smallest integer n (ie $n = 1, 2, 3$) for which it is true, is called as basis of induction.

If the statement is true for $n = k$, where k denotes any value of n, then it is also true for $n = k+1$, is called as the induction step

Here the assumption $P(n)$ is true for $n = k$, is called as induction hypothesis.

Ex. Using principle of mathematical induction Prove that,

$$P(n) = 1 + 3 + 5 + \cdots + (2n-1) = n^2.$$

→ Let, $P(n) = 1 + 3 + 5 + \cdots + (2n-1) = n^2$

Basis of induction

For $n = 1$

$LHS = 1$ & $RHS = n^2 = 1^2 = 1$.

Hence $P(n)$ is true for $n = 1$.

Induction step

Assuming that $P(n)$ is true for $n = k$.

$$P(k) = 1 + 3 + 5 + \cdots (2k-1) = k^2$$

(Induction Hypothesis)

Adding $(2k+1)$ to both sides of $P(k)$

$$1 + 3 + 5 + \cdots + (2k-1) + (2k+1) = k^2 + (2k+1)$$

$$\therefore 1 + 3 + 5 + \cdots + (2k-1) + (2k+1) = (k+1)^2$$

This shows that, it is also true for $n = k+1$.

Hence by mathematical induction $P(n)$ is true for every integral value of n.

Ex Prove by the principle of mathematical induction,

$$P(n) = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

→ Let,

$$P(n) = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Basis of Induction

If $n = 1$. then,

$$LHS = \frac{1}{n(n+1)} = \frac{1}{1(1+1)} = \frac{1}{1\cdot 2} = \frac{1}{2}$$

$$RHS = \frac{n}{n+1} = \frac{1}{1+1} = \frac{1}{2}$$

Hence $P(n)$ is true for $n=1$.

Induction step

Assume that $P(n)$ is true for $n=k$.

$$P(k) = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

(*is Induction Hypothesis)

To prove it is true for $n = k+1$.

Adding the term $\frac{1}{(k+1)(k+2)}$ to both sides of $P(k)$

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}=\frac{k^2+2k+1}{(k+1)(k+2)}$$

$$=\frac{(k+1)^2}{(k+1)(k+2)}$$

$$=\frac{k+1}{(k+1)+1}$$

$\Rightarrow$ It is also true for

Ex Prove by mathematical induction that

$P(n) = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$.

→ Let,

$$P(n) = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

Basis of Induction

For $n=1$

$LHS = 1^2 = 1$ $\quad$ $RHS = \frac{n(n+1)(2n+1)}{6}$

$$= \frac{1(1+1)(2+1)}{6} = 1$$

$\therefore$ $P(n)$ is true for $n=1$.

Induction step

Assuming that $P(k)$ is true for $n=k$

then,

$$1^2+2^2+3^2+\cdots+k^2 = \frac{k(k+1)(2k+1)}{6}$$

To prove $P(k+1)$ is true. (induction hypothesis)

Adding $(k+1)^2$ to both sides,

$$1^2 + 2^2 + 3^2 + \cdots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{(k+1)}{6}\left\{k(2k+1) + 6(k+1)\right\}$$

$$= \frac{(k+1)}{6}\left\{2k^2 + 7k + 6\right\}$$

$$= \frac{(k+1)}{6}\left\{(k+2)(2k+3)\right\}$$

$$= \frac{(k+1)\left[(k+1)+1\right]\left[2(k+1)+1\right]}{6}$$

This shows that if $P(n)$ is true for $n=k$, then it is also true for $n=k+1$.

Hence by mathematical induction $P(n)$ is true for every integral value of n.

1) Prove by Principle of mathematical induction,

$$P(n) = 1 + 3 + 6 + \cdots + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

→ Let, $P(n) = 1 + 3 + 6 + \cdots + \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$

Basis of induction,

For $n = 1$

$$LHS = P(1) = \frac{1(1+1)}{2} = 1$$

$$RHS = \frac{1(1+1)(1+2)}{6} = 1$$

∴ $P(n)$ is true for $n = 1$.

Induction step

Assume $P(n)$ is true for $n = k$.

Thus we get,

$$P(k) = 1 + 3 + 6 + \cdots + \frac{k(k+1)}{2} = \frac{k(k+1)(k+2)}{6}$$

To prove $P(k+1)$ is also true.

Adding the term $\frac{(k+1)(k+2)}{2}$ to both sides of $P(k)$

$$1 + 3 + 6 + \cdots + \frac{k(k+1)}{2} + \frac{(k+1)(k+2)}{2}$$

$$\text{—''—} = \frac{k(k+1)(k+2)}{6} + \frac{(k+1)(k+2)}{2}$$

$$\text{—''—} = \frac{k(k+1)(k+2)}{6} + \frac{3(k+1)(k+2)}{6}$$

$$1+3+6+ \cdots + \frac{K(K+1)}{2} + \frac{(K+1)(K+2)}{2} = \frac{(K+1)(K+2)(K+3)}{6}$$

This shows that if $P(n)$ is true for $n=K$ then it is also true for $n=K+1$.

Hence by mathematical induction it is true for every value of n.

Ex Prove by the principle of mathematical induction,

$$P(n) = 1\cdot 2 + 2\cdot 2^2 + \cdots + n\cdot 2^n = (n-1)\,2^{n+1} + 2$$

→ Let, $P(n) = 1\cdot 2 + 2\cdot 2^2 + \cdots + n\cdot 2^n = (n-1)\,2^{n+1} + 2$

Basis of induction

Assuming for $n=1$

$LHS = 1\cdot 2^1 = 2$, $RHS = (1-1)\,2^{1+1} + 2 = 2$

Hence $P(n)$ is true for $n=1$.

Induction step

Assuming that $P(n)$ is true for $n=K$.

thus,

$$P(K) = 1\cdot 2 + 2\cdot 2^2 + \cdots + K\cdot 2^K = (K-1)\,2^{K+1} + 2$$

(induction Hypothesis)

Now adding $(K+1)\,2^{K+1}$ to both sides of $P(K)$

$$1\cdot 2 + 2\cdot 2^2 + \cdots + K\cdot 2^K + (K+1)\,2^{K+1} = (K-1)\,2^{K+1} + 2 + (K+1)\,2^{K+1}$$

$$= 2^{K+1}[K-1+K+1] + 2$$

$$= 2^{K+1}\, 2K + 2$$

$$= K\, 2^{K+2} + 2$$

$$= [(K+1)-1]\,2^{(K+1)+1} + 2$$

⇒ $P(n)$ is true for every value of n