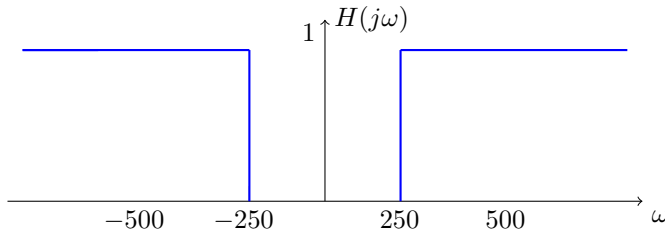


EE1101 Signals and Systems

Tutorial 6 Solutions

1)

$$H(j\omega) = \begin{cases} 1, & |\omega| \geq 250 \\ 0, & \text{otherwise} \end{cases}$$

Fig. 1: $H(j\omega)$ of Q1

The system $H(j\omega)$ passes only the frequency components greater than 250 rad/s. The characteristics are shown in Fig. 1.

Since the output is identical to input, this implies that the input contains only frequencies greater than 250.

Hence, for the input $x(t)$, Fourier coefficients, a_k (corresponding to the frequencies: $k\omega_0$) need to be 0 for:

$$\begin{aligned} |k\omega_0| &< 250 \\ |k| &< \frac{250}{14} = 17.85 \end{aligned}$$

Since k is integer, $a_k = 0$ for $|k| \leq 17$.

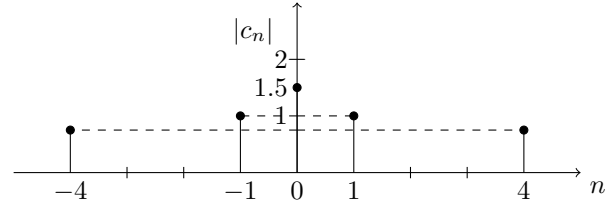
2)

$$\begin{aligned} x(t) &= 2 + \sum_{k=1}^3 3 \sin \frac{k\pi}{2} \cos 100k\pi t \\ &= 2 + 3 \left(\sin \frac{\pi}{2} \cos 100\pi t + \sin \pi \cos 200\pi t \right. \\ &\quad \left. + \sin \frac{3\pi}{2} \cos 300\pi t \right) \\ &= 2 + 3 \cos 100\pi t - 3 \cos 300\pi t \\ &= 2 + \frac{3}{2} (e^{j100\pi t} + e^{-j100\pi t}) \\ &\quad - \frac{3}{2} (e^{j300\pi t} + e^{-j300\pi t}). \end{aligned}$$

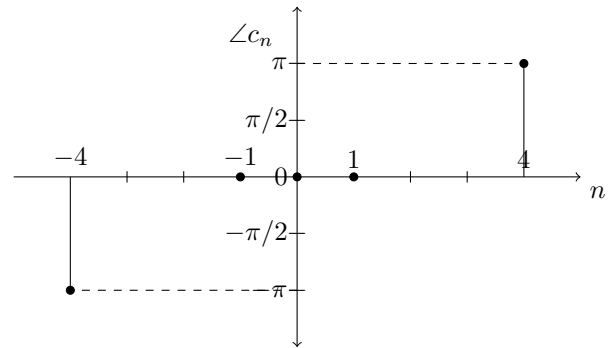
The fundamental frequency is $\omega_0 = 100\pi$, and the non-zero Fourier coefficients of $x(t)$ are

$$a_n = \begin{cases} 2, & \text{for } n = 0 \\ \frac{3}{2}, & \text{for } n = 1, -1 \\ -\frac{3}{2}, & \text{for } n = 3, -3 \end{cases}$$

The non-zero Fourier series of coefficients of $\cos(100\pi t)$ are $b_1 = b_{-1} = 1/2$. Using the



(a) Magnitude spectrum



(b) Phase spectrum

Fig. 2: Q2. Magnitude and phase spectra of the Fourier series coefficients of $y(t)$.

multiplication property of Fourier series we get

$$y(t) = x(t) \cos(100\pi t) \xrightarrow{\text{FS}} c_n = \sum_{l=-\infty}^{\infty} a_l b_{n-l}.$$

Therefore,

$$\begin{aligned} c_n &= a_n \star b_n \\ &= \left(-\frac{3}{2} \delta[n+3] + \frac{3}{2} \delta[n+1] + 2\delta[n] + \frac{3}{2} \delta[n-1] \right. \\ &\quad \left. - \frac{3}{2} \delta[n-3] \right) \star \left(\frac{1}{2} \delta[n+1] + \frac{1}{2} \delta[n-1] \right) \\ &= \frac{3}{2} \delta[n] + \delta[n-1] + \delta[n+1] \\ &\quad - \frac{3}{4} (\delta[n-4] + \delta[n+4]). \end{aligned}$$

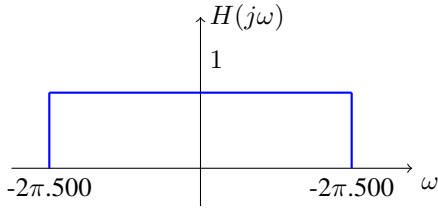
The magnitude and phase spectrum are plotted in Fig. 2a and Fig. 2b respectively

3) The frequency response:

$$H_l(j\omega) = \begin{cases} 1, & |\omega| < 2\pi \cdot 500 \text{ rad/s} \\ 0, & \text{otherwise} \end{cases}$$

The filter characteristics are shown in Fig. 3.

(a) $x(t) = \cos(2\pi \cdot 750t) + \sin(2\pi \cdot 1500t)$. Fourier

Fig. 3: $H(j\omega)$ of Q3

series expansion of $x(t)$:

$$x(t) = \frac{1}{2} \left(e^{j2\pi \cdot 750t} + e^{-j2\pi \cdot 750t} \right) + \frac{1}{2j} \left(e^{j2\pi \cdot 1500t} - e^{-j2\pi \cdot 1500t} \right).$$

The fundamental frequency $\omega_0 = 2\pi \cdot 750$. By inspection, the Fourier series coefficients of $x(t)$ are

$$a_{-1} = a_1 = \frac{1}{2}, \quad a_{-2}^* = a_2 = \frac{1}{2j}.$$

Using the synthesis equation, the output can be written as,

$$y(t) = \sum_{n=-\infty}^{\infty} a_n H(jn\omega_o) e^{jn\omega_o t}.$$

Since, $H(jn\omega_o)$ is 0 for all n , $y(t) = 0$.

- (b) $x(t) = \cos(2\pi \cdot 150t) + \sin(2\pi \cdot 1500t)$. Fourier series expansion of $x(t)$:

$$x(t) = \frac{1}{2} \left(e^{j2\pi \cdot 150t} + e^{-j2\pi \cdot 150t} \right) + \frac{1}{2j} \left(e^{j2\pi \cdot 1500t} - e^{-j2\pi \cdot 1500t} \right).$$

The fundamental frequency $\omega_0 = 2\pi \cdot 150$. By inspection, the Fourier series coefficients of $x(t)$ are

$$a_{-1} = a_1 = \frac{1}{2}, \quad a_{-10}^* = a_{10} = \frac{1}{2j}.$$

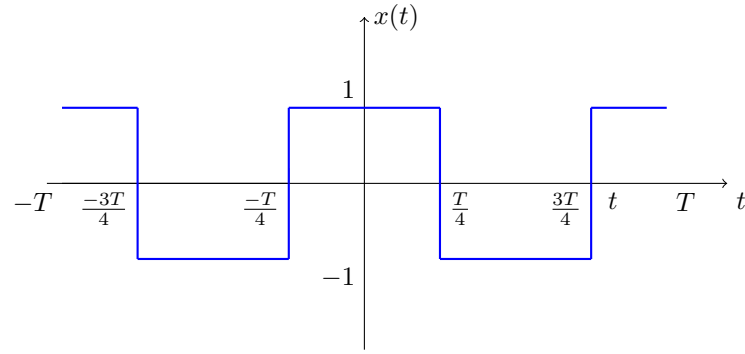
Using the synthesis equation, the output can be written as,

$$y(t) = \sum_{n=-\infty}^{\infty} a_n H(jn\omega_o) e^{jn\omega_o t}.$$

Since, $H(jn\omega_o)$ is 0 for all $n \neq \{1, -1\}$, $y(t) = \cos(2\pi \cdot 150t)$.

- (c) Periodic square wave with period 4.5 ms, oscillates between +1 V and -1 V with 50% duty cycle and is an even function of time.

$$x(t) = \begin{cases} 1, & 0 < t < T/4 \\ -1, & T/4 < t < 3T/4 \\ 1, & 3T/4 < t < T \end{cases} \quad (1)$$

Fig. 4: $x(t)$ of Q3 part c

$$x(t) = x(t + nT), T = 4.5 \text{ ms}$$

Fourier series expansion of $x(t)$:

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_o t}$$

$$\text{Here } a_o = \frac{1}{T} \int_0^T x(t) dt = 0$$

$$a_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_o t} dt = \frac{1}{T} \left(\int_0^{T/4} e^{-jn\omega_o t} dt - \int_{T/4}^{3T/4} e^{-jn\omega_o t} dt + \int_{3T/4}^T x(t) e^{-jn\omega_o t} dt \right)$$

simplification yields $a_n = \frac{2}{n\pi} \sin\left(n\frac{\pi}{2}\right)$.

Using the synthesis equation, the output can be written as,

$$y(t) = \sum_{n=-\infty}^{\infty} a_n H(jn\omega_o) e^{jn\omega_o t} = \sum_{n=-\infty, n \text{ odd}}^{\infty} \frac{2}{n\pi} \sin\left(n\frac{\pi}{2}\right) H(jn\omega_o) e^{jn\omega_o t}$$

Where, $\omega_o = \frac{2\pi}{4.5 \text{ ms}} = 2\pi(222.2) \text{ rad/s}$. Thus, $H(j\omega)$ is non-zero only for $n = -1, 1, -2, 2$

$$\begin{aligned} y(t) &= -\frac{2}{\pi} \sin\left(\frac{-\pi}{2}\right) e^{-j\omega_o t} + \frac{2}{\pi} \sin\left(\frac{\pi}{2}\right) e^{j\omega_o t} \\ &\quad - \frac{2}{2\pi} \sin\left(\frac{-2\pi}{2}\right) e^{-j2\omega_o t} + \frac{2}{2\pi} \sin\left(\frac{2\pi}{2}\right) e^{j2\omega_o t} \\ &= \frac{4}{\pi} \left(\frac{e^{-j\omega_o t} + e^{j\omega_o t}}{2} \right) + 0 \\ &= \frac{4}{\pi} \cos(\omega_o t). \end{aligned}$$

4) The impulse response for the LTI system is

$$h(t) = \delta(t) - e^{-t}u(t),$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)e^{-j\omega t} dt,$$

$$H(j\omega) = 1 - \frac{1}{1 + j\omega}$$

(a) $x(t) = \cos(3\pi t) + \frac{\pi}{3}$
Fourier series expansion of $x(t)$:

$$x(t) = \frac{\pi}{3} + \frac{1}{2} \left(e^{j3\pi t} + e^{-j3\pi t} \right).$$

The fundamental frequency $\omega_0 = 3\pi$ and period is $T_0 = 2/3$. By inspection, the Fourier series coefficients of $x(t)$ are

$$a_0 = \frac{\pi}{3}, \quad a_{-1} = \frac{1}{2}, \quad a_1 = \frac{1}{2}.$$

If $e^{j\omega t}$ is the input to a LTI system, then the output is $H(j\omega)t e^{j\omega t}$. Thus, the output of the given LTI system is,

$$y(t) = \sum_{n=-\infty}^{\infty} a_n H(jn\omega_0) e^{jn\omega_0 t}$$

$$= \frac{3\pi(3\pi \cos 3\pi t - \sin 3\pi t)}{1 + 9\pi^2}$$

(b) $x(t) = \sum_{n=-\infty}^{\infty} \delta(t - n)$

Fourier series expansion of $x(t)$:

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}$$

where, $a_0 = 1, a_n = 1, \omega_0 = 2\pi$

$$y(t) = \sum_{n=-\infty}^{\infty} a_n H(jn\omega_0) e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} \frac{nj2\pi}{1 + nj2\pi} e^{jn2\pi t}$$

(c) $x(t) = \sum_{n=-\infty}^{\infty} (-1)^n \delta(t - 2n)$

Fourier series expansion of $x(t)$:

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{jn\omega_0 t}$$

$$a_n = \frac{-1}{4} \int_0^4 [\delta(t) - \delta(t - 2)] e^{jn\omega_0 t} dt$$

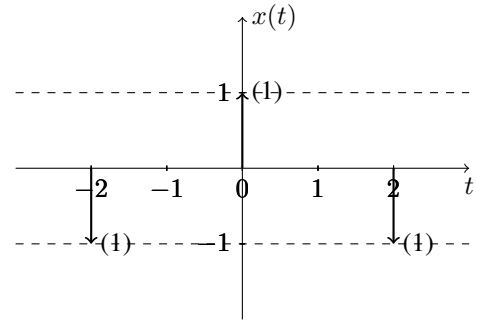


Fig. 5: Q4 (c)

where, $a_0 = 0, a_n = \frac{1 - e^{-jn\pi}}{4}, \omega_0 = \pi/2$

$$y(t) = \sum_{n=-\infty}^{\infty} a_n H(jn\omega_0) e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty, n \text{ odd}}^{\infty} \frac{nj\frac{\pi}{2}}{1 + nj\frac{\pi}{2}} \left(\frac{1}{2} e^{jn\frac{\pi}{2} t} \right).$$

5) Given the Fourier Series representation of $x(t)$, its Fourier Series coefficients, a_k are given by

$$a_k = \alpha^{|k|}$$

and its fundamental frequency $\omega_0 = \frac{\pi}{4}$. For inputs of the form $x(t) = e^{j\omega t}$, the output of an LTI system is given by

$$y(t) = H(j\omega)x(t).$$

Therefore we can write

$$y(t) = \sum_{k=-\infty}^{\infty} \alpha^{|k|} H(jk\omega_0) e^{jk\omega_0 t}.$$

Now $H(jk\omega_0)$ is non-zero only for

$$k\omega_0 \leq |W|,$$

i.e.,

$$k \leq \left| \frac{4W}{\pi} \right|.$$

Let

$$\frac{4W}{\pi} = N.$$

Then

$$y(t) = \sum_{k=-N}^N \alpha^{|k|} e^{jk\omega_0 t}.$$

The average energy of $x(t)$ over a period is given by

$$\begin{aligned}
 E_{avg}\{x(t)\} &= \frac{1}{T} \int_T |x(t)|^2 dt \\
 &= \sum_{k=-\infty}^{\infty} |a_k|^2 \quad (\text{Parseval's relation}) \\
 &= \sum_{k=-\infty}^{\infty} |\alpha^{k|}|^2 \\
 &= 1 + 2 \sum_{k=1}^{\infty} |\alpha^{k|}|^2 \\
 &= \frac{1 + \alpha^2}{1 - \alpha^2}.
 \end{aligned}$$

In last expression Geometric series formula is used. Similarly, the average energy of $y(t)$ is

$$\begin{aligned}
 E_{avg}\{y(t)\} &= \sum_{k=-N}^N |\alpha^{k|}|^2 \\
 &= 1 + 2 \sum_{k=1}^N |\alpha^{k|}|^2 \\
 &= \frac{1 - 2\alpha^{2N+2} + \alpha^2}{1 - \alpha^2}.
 \end{aligned}$$

Now, we have to find N for which

$$E_{avg}\{y(t)\} = 0.9 E_{avg}\{x(t)\}.$$

This implies

$$\frac{1 - 2\alpha^{2N+2} + \alpha^2}{1 - \alpha^2} = 0.9 \frac{1 + \alpha^2}{1 - \alpha^2}.$$

which gives the value of N as

$$N = \frac{\log(0.05) + \log(\frac{1+\alpha^2}{\alpha^2})}{2\log(\alpha)}.$$

Therefore $W = \frac{\pi}{4} N$ where N is as above.

- 6) We first evaluate the frequency response of the system. Consider an input $x(t)$ of the form $e^{j\omega t}$. To such an input, the output of the system will be

$$y(t) = H(j\omega) e^{j\omega t},$$

where $H(j\omega)$ is the frequency response of the system. Substituting the above input and output in the given differential equation, we get

$$H(j\omega) j\omega e^{j\omega t} + 4H(j\omega) e^{j\omega t} = e^{j\omega t}.$$

Therefore,

$$H(j\omega) = \frac{1}{j\omega + 4}.$$

Using the synthesis equation for the output, we have

$$y(t) = \sum_{n=-\infty}^{\infty} a_n H(jn\omega_0) e^{jn\omega_0 t},$$

where a_n are the Fourier series coefficients of input $x(t)$. Therefore, Fourier series coefficients of $y(t)$ are

$$a_n H(jn\omega_0).$$

We now apply this to the given input $x(t) = \cos(2\pi t)$. Here, the fundamental frequency is $\omega_0 = 2\pi$, and the non-zero Fourier series coefficients of $x(t)$ are $a_1, a_{-1} = 1/2$ (this is left for the reader to derive). Therefore, the non-zero Fourier series coefficients of $y(t)$ are as follows

$$\begin{aligned}
 b_1 &= a_1 H(j2\pi) = \frac{1}{2(4 + j2\pi)}, \\
 b_{-1} &= a_{-1} H(-j2\pi) = \frac{1}{2(4 - j2\pi)}.
 \end{aligned}$$

Using the synthesis equation, we get

$$y(t) = \frac{1}{2(4 + j2\pi)} e^{j2\pi t} + \frac{1}{2(4 - j2\pi)} e^{-j2\pi t},$$

which on simplification yields

$$y(t) = \frac{1}{4 + \pi^2} \cos(2\pi t) + \frac{\pi}{8 + 2\pi^2} \sin(2\pi t).$$

- 7) (a) The non-zero Fourier series coefficients of $x(t)$ are $a_1 = a_{-1} = \frac{1}{2}$ (left for the reader to derive).
 (b) The non-zero Fourier series coefficients of $y(t)$ are $b_1 = b_{-1}^* = \frac{1}{2j}$ (left for the reader to derive).
 (c) Using the multiplication property of Fourier series, we know that

$$z(t) = x(t)y(t) \xleftrightarrow{\text{FS}} c_n = \sum_{l=-\infty}^{\infty} a_l b_{n-l}.$$

Therefore,

$$c_n = a_n \star b_n = \frac{1}{4j} \delta[n-2] - \frac{1}{4j} \delta[n+2].$$

This implies that the nonzero Fourier series coefficients of $z(t)$ are $c_2 = c_{-2}^* = \frac{1}{4j}$.

(d)

$$z(t) = \sin(4\pi t) \cos(4\pi t) = \frac{1}{2} \sin(8\pi t).$$

The derivation of the Fourier series coefficients is left to the reader.

- 8) Properties of LTI systems and Fourier series result in convolution in time domain result in multiplication in Fourier domain. Which implies input Amplitude response is multiplied with System's amplitude response which results in amplitude scaling. Input phase response is multiplied with System's phase response which results in phase shift. So the output of LTI system is amplitude scaled and phase shifted version of the input. There won't be no new frequencies be generated which are not there in input.
 (a) Output is amplitude scaled and phase shifted version of input. So system is LTI.
 (b) Output frequency is different from input. So it is not LTI.