13.1 Functions of Two or More Variables

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Lecture continued

In previous courses we studied real-valued functions of a real variable and vector-valued functions of a real variable. Here we will consider real-valued functions of two or more real variables.

NOTATION AND TERMINOLOGY

There are many familiar formulas in which a given variable depends on two or more other variables. For example, the area A of a triangle depends on the base length b and height h by the formula $A = \frac{1}{2}bh$; the volume V of a rectangular box depends on the length l, the width w, and the height h by the formula $V = lwh$; and the arithmetic average $\bar{x}$ of n real numbers, $x_1, x_2, \ldots, x_n$, depends on those numbers by the formula

$$\bar{x} = \frac{1}{n}(x_1 + x_2 + \cdots + x_n)$$

Lecture continued

Thus, we say that

A is a function of the two variables b and h;

V is a function of the three variables l, w, and h;

$\bar{x}$ is a function of the n variables $x_1, x_2, \ldots, x_n$.

The terminology and notation for functions of two or more variables is similar to that for functions of one variable. For example, the expression

$$z = f(x, y)$$

means that z is a function of x and y in the sense that a unique value of the dependent variable z is determined by specifying values for the independent variables x and y. Similarly,

$$w = f(x, y, z)$$

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expresses w as a function of x, y, and z, and

$$u = f(x_1, x_2, \ldots, x_n)$$

expresses u as a function of $x_1, x_2, \ldots, x_n$.

As with functions of one variable, the independent variables of a function of two or more variables may be restricted to lie in some set D, which we call the ***domain*** of f. Sometimes the domain will be determined by physical restrictions on the variables. If the function is defined by a formula and if there are no physical restrictions or other restrictions stated explicitly, then it is understood that the domain consists of all points for which the formula yields a real value for the dependent variable. We call this the ***natural domain*** of the function. The following definitions summarize this discussion.

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13.1.1 DEFINITION A ***function f of two variables***, x and y, is a rule that assigns a unique real number $f(x, y)$ to each point (x, y) in some set D in the xy-plane.

13.1.2 DEFINITION A ***function f of three variables***, x, y, and z, is a rule that assigns a unique real number $f(x, y, z)$ to each point (x, y, z) in some set D in three-dimensional space.

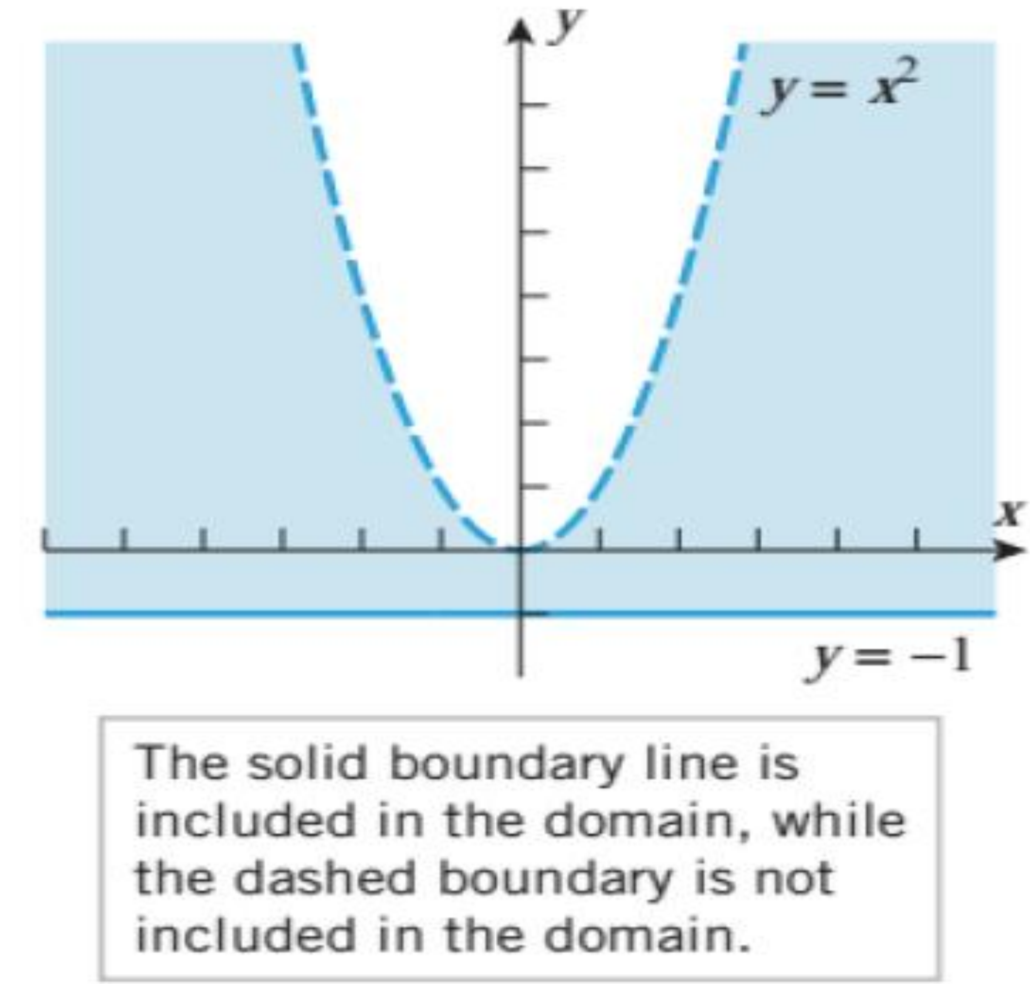
► **Example 1** Let $f(x, y) = \sqrt{y+1} + \ln(x^2 - y)$. Find $f(e, 0)$ and sketch the natural domain of f.

Solution. By substitution,

$$f(e, 0) = \sqrt{0+1} + \ln(e^2 - 0) = \sqrt{1} + \ln(e^2) = 1 + 2 = 3$$

To find the natural domain of f, we note that $\sqrt{y+1}$ is defined only when $y \geq -1$, while $\ln(x^2 - y)$ is defined only when $0 < x^2 - y$ or $y < x^2$. Thus, the natural domain of f consists of all points in the xy-plane for which $-1 \leq y < x^2$. To sketch the natural domain, we first sketch the parabola $y = x^2$ as a "dashed" curve and the line $y = -1$ as a solid curve. The natural domain of f is then the region lying above or on the line $y = -1$ and below the parabola $y = x^2$ (Figure 13.1.1). ◄

Lecture continued



▲ Figure 13.1.1

Lecture continued

▶ **Example 2** Let

$$f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$$

Find $f\left(0, \frac{1}{2}, -\frac{1}{2}\right)$ and the natural domain of f.

Solution. By substitution,

$$f\left(0, \tfrac{1}{2}, -\tfrac{1}{2}\right) = \sqrt{1 - (0)^2 - \left(\tfrac{1}{2}\right)^2 - \left(-\tfrac{1}{2}\right)^2} = \sqrt{\tfrac{1}{2}}$$

Because of the square root sign, we must have $0 \le 1 - x^2 - y^2 - z^2$ in order to have a real value for $f(x, y, z)$. Rewriting this inequality in the form

$$x^2 + y^2 + z^2 \le 1$$

we see that the natural domain of f consists of all points on or within the sphere

$$x^2 + y^2 + z^2 = 1 \blacktriangleleft$$

Lecture continued

FUNCTIONS DESCRIBED BY TABLES

Sometimes it is either desirable or necessary to represent a function of two variables in table form, rather than as an explicit formula. For example, the U.S. National Weather Service uses the formula

$$W = 35.74 + 0.6215T + (0.4275T - 35.75)v^{0.16} \quad (1)$$

Lecture continued

to model the wind chill index W (in $^\circ$F) as a function of the temperature T (in $^\circ$F) and the wind speed v (in mi/h) for wind speeds greater than 3 mi/h. This formula is sufficiently complex that it is difficult to get an intuitive feel for the relationship between the variables. One can get a clearer sense of the relationship by selecting sample values of T and v and constructing a table, such as Table 13.1.1, in which we have rounded the values of W to the nearest integer. For example, if the temperature is 30°F and the wind speed is 5 mi/h, it feels as if the temperature is 25°F. If the wind speed increases to 15 mi/h, the temperature then feels as if it has dropped to 19°F. Note that in this case, an increase in wind speed of 10 mi/h causes a 6°F decrease in the wind chill index. To estimate wind chill values not displayed in the table, we can use ***linear interpolation***. For example, suppose that the temperature is 30°F and the wind speed is 7 mi/h. A reasonable estimate for the drop in the wind chill index from its value when the wind speed is 5 mi/h would be $\frac{2}{10} \cdot 6^\circ\text{F} = 1.2^\circ\text{F}$. (Why?) The resulting estimate in wind chill would then be $25^\circ - 1.2^\circ = 23.8^\circ\text{F}$.

In some cases, tables for functions of two variables arise directly from experimental data, in which case one must either work directly with the table or else use some technique to construct a formula that models the data in the table. Such modeling techniques are developed in statistics and numerical analysis texts.

Lecture continued

Table 13.1.1

WIND SPEED v (mi/h) \ TEMPERATURE T(°F)	20	25	30	35
5	13	19	25	31
15	6	13	19	25
25	3	9	16	23
35	0	7	14	21
45	−2	5	12	19

Lecture continued

GRAPHS OF FUNCTIONS OF TWO VARIABLES

Recall that for a function f of one variable, the graph of $f(x)$ in the xy-plane was defined to be the graph of the equation $y = f(x)$. Similarly, if f is a function of two variables, we define the ***graph*** of $f(x, y)$ in xyz-space to be the graph of the equation $z = f(x, y)$. In general, such a graph will be a surface in 3-space.

▶ **Example 3** In each part, describe the graph of the function in an xyz-coordinate system.

(a) $f(x, y) = 1 - x - \frac{1}{2}y$ (b) $f(x, y) = \sqrt{1 - x^2 - y^2}$

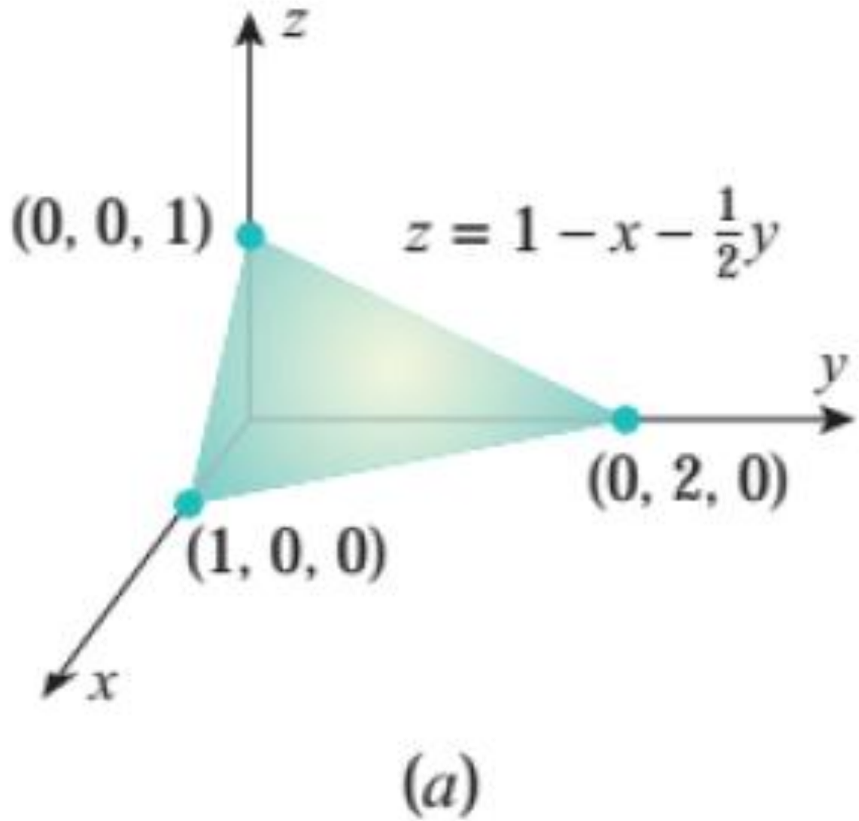
(c) $f(x, y) = -\sqrt{x^2 + y^2}$

Solution (a). By definition, the graph of the given function is the graph of the equation

$$z = 1 - x - \tfrac{1}{2}y$$

which is a plane. A triangular portion of the plane can be sketched by plotting the intersections with the coordinate axes and joining them with line segments (Figure 13.1.2a).

Lecture continued



▲ Figure 13.1.2

Lecture continued

Solution (b). By definition, the graph of the given function is the graph of the equation

$$z = \sqrt{1 - x^2 - y^2} \tag{2}$$

After squaring both sides, this can be rewritten as

$$x^2 + y^2 + z^2 = 1$$

which represents a sphere of radius 1, centered at the origin. Since (2) imposes the added condition that $z \geq 0$, the graph is just the upper hemisphere (Figure 13.1.2*b*).

Solution (c). The graph of the given function is the graph of the equation

$$z = -\sqrt{x^2 + y^2} \tag{3}$$

After squaring, we obtain

$$z^2 = x^2 + y^2$$

which is the equation of a circular cone (see Table 11.7.1). Since (3) imposes the condition that $z \leq 0$, the graph is just the lower nappe of the cone (Figure 13.1.2*c*). ◀

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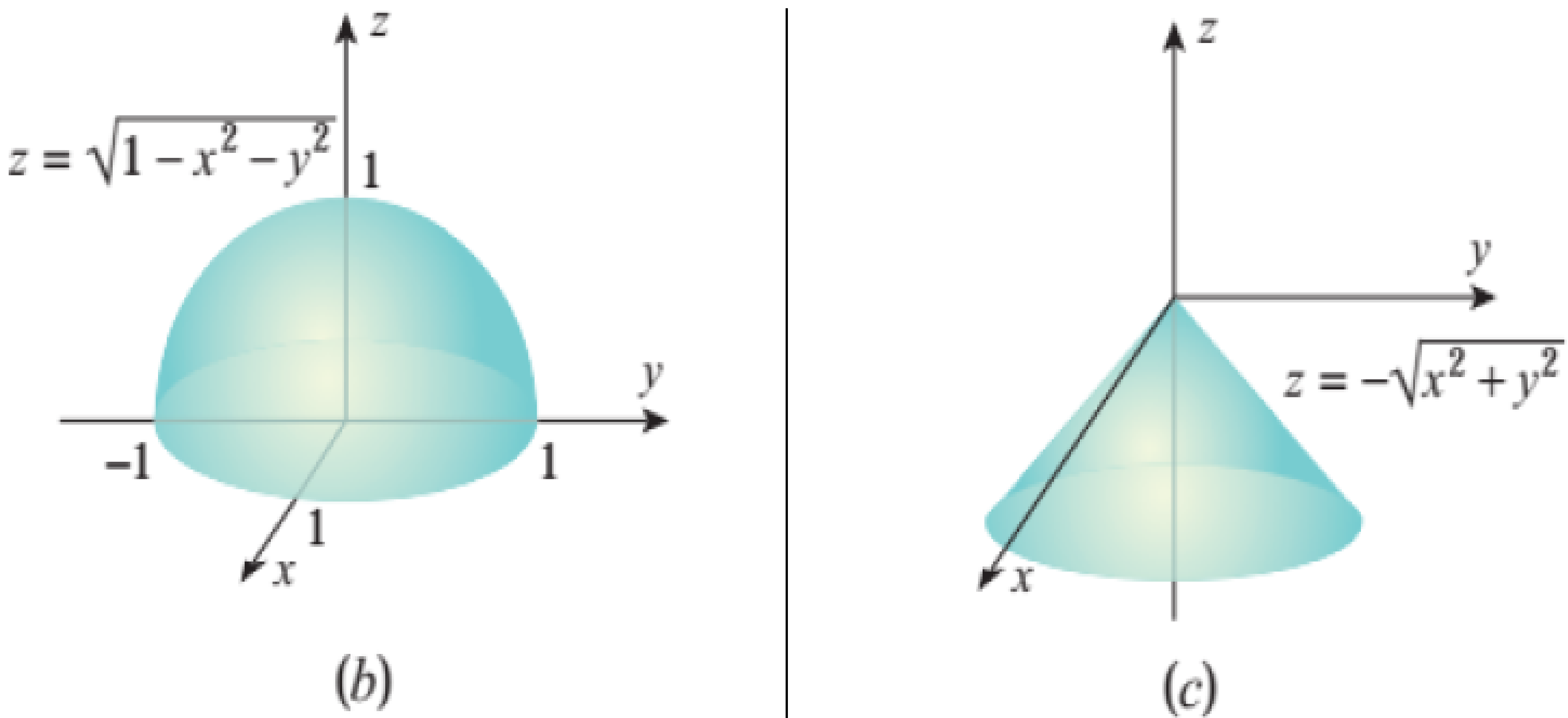


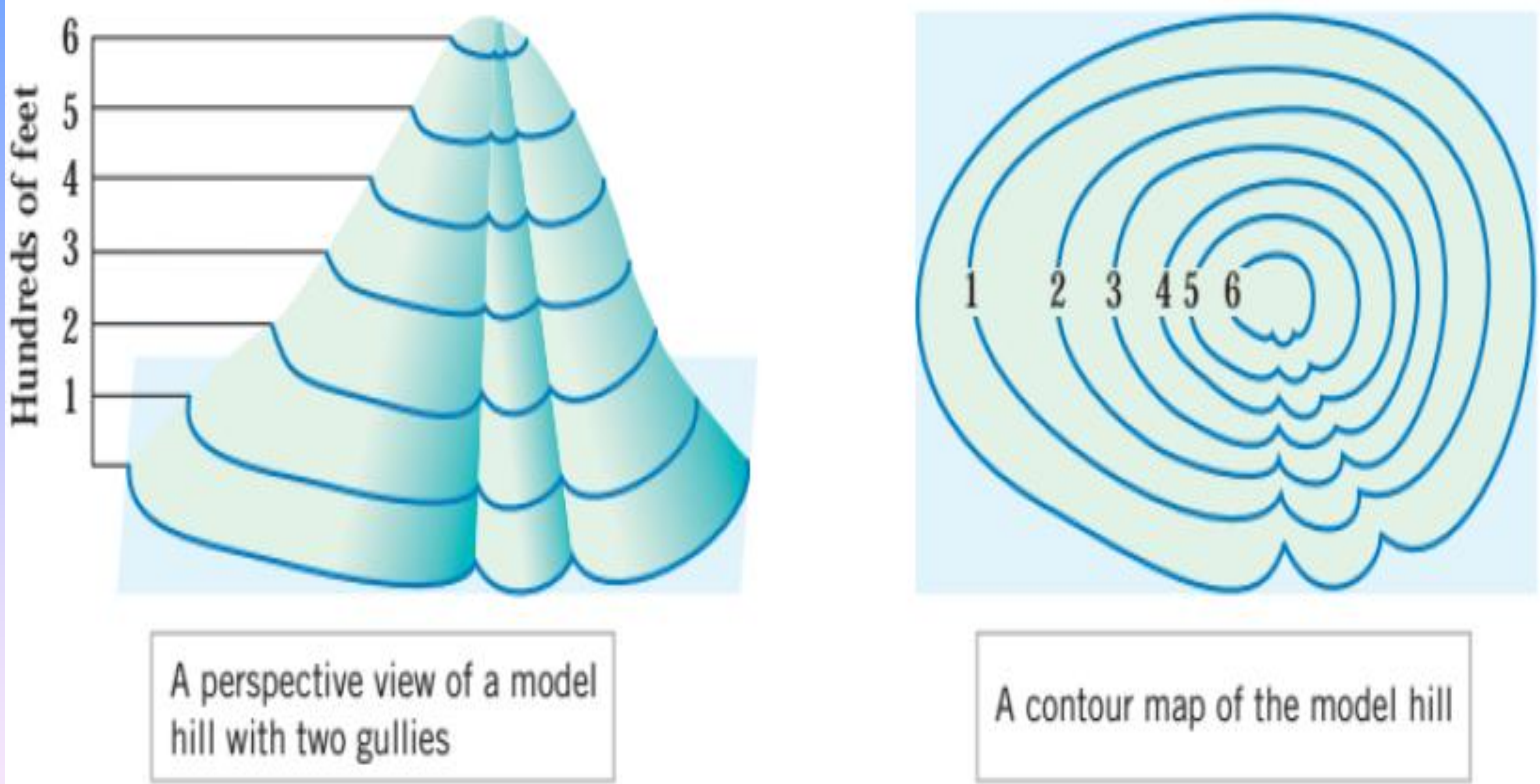
▲ Figure 13.1.2

Lecture continued

LEVEL CURVES

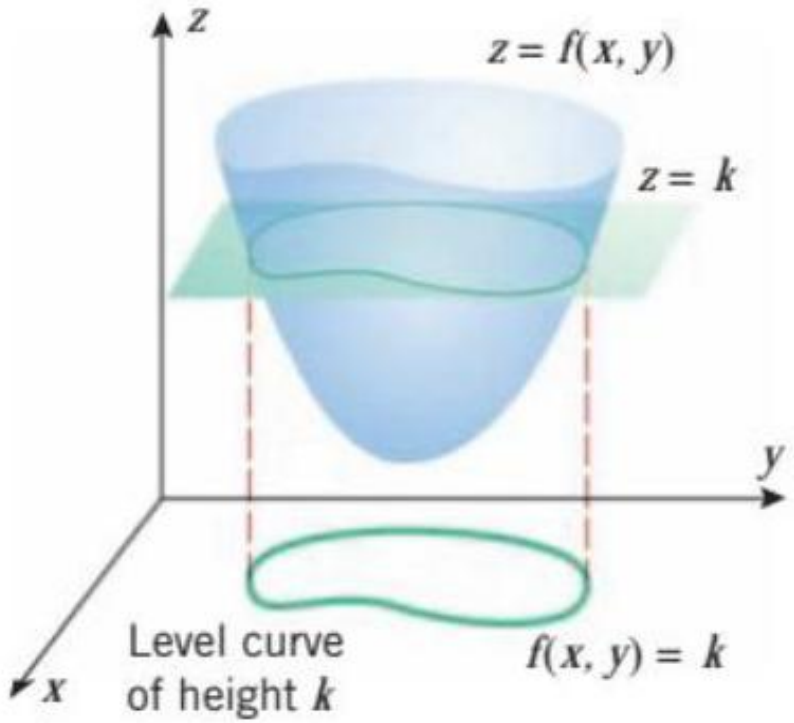
We are all familiar with the topographic (or contour) maps in which a three-dimensional landscape, such as a mountain range, is represented by two-dimensional contour lines or curves of constant elevation. Consider, for example, the model hill and its contour map shown in Figure 13.1.3. The contour map is constructed by passing planes of constant elevation through the hill, projecting the resulting contours onto a flat surface, and labeling the contours with their elevations. In Figure 13.1.3, note how the two gullies appear as indentations in the contour lines and how the curves are close together on the contour map where the hill has a steep slope and become more widely spaced where the slope is gradual.

Lecture continued



▲ Figure 13.1.3

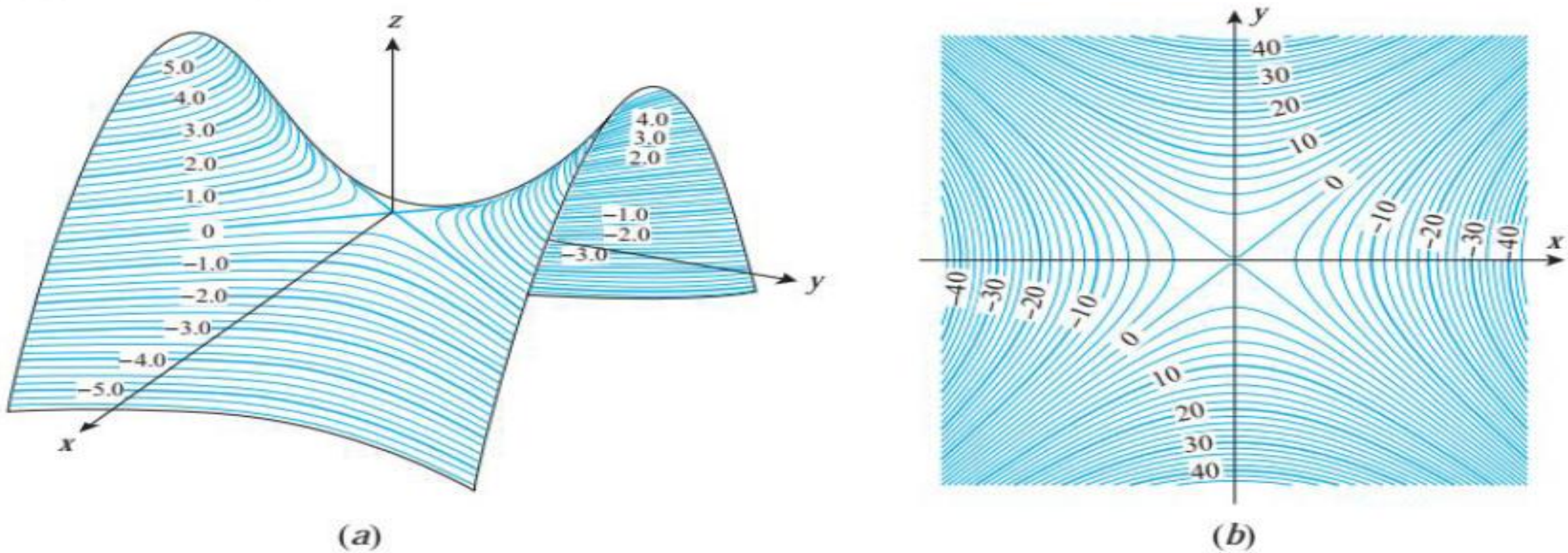
Contour maps are also useful for studying functions of two variables. If the surface $z = f(x, y)$ is cut by the horizontal plane $z = k$, then at all points on the intersection we have $f(x, y) = k$. The projection of this intersection onto the xy-plane is called the ***level curve of height k*** or the ***level curve with constant k*** (Figure 13.1.4). A set of level curves for $z = f(x, y)$ is called a ***contour plot*** or ***contour map*** of f.



▲ **Figure 13.1.4**

Lecture continued

▶ **Example 4** The graph of the function $f(x, y) = y^2 - x^2$ in xyz-space is the hyperbolic paraboloid (saddle surface) shown in Figure 13.1.5a. The level curves have equations of the form $y^2 - x^2 = k$. For $k > 0$ these curves are hyperbolas opening along lines parallel to the y-axis; for $k < 0$ they are hyperbolas opening along lines parallel to the x-axis; and for $k = 0$ the level curve consists of the intersecting lines $y + x = 0$ and $y - x = 0$ (Figure 13.1.5b). ◀



▲ **Figure 13.1.5**

Lecture continued

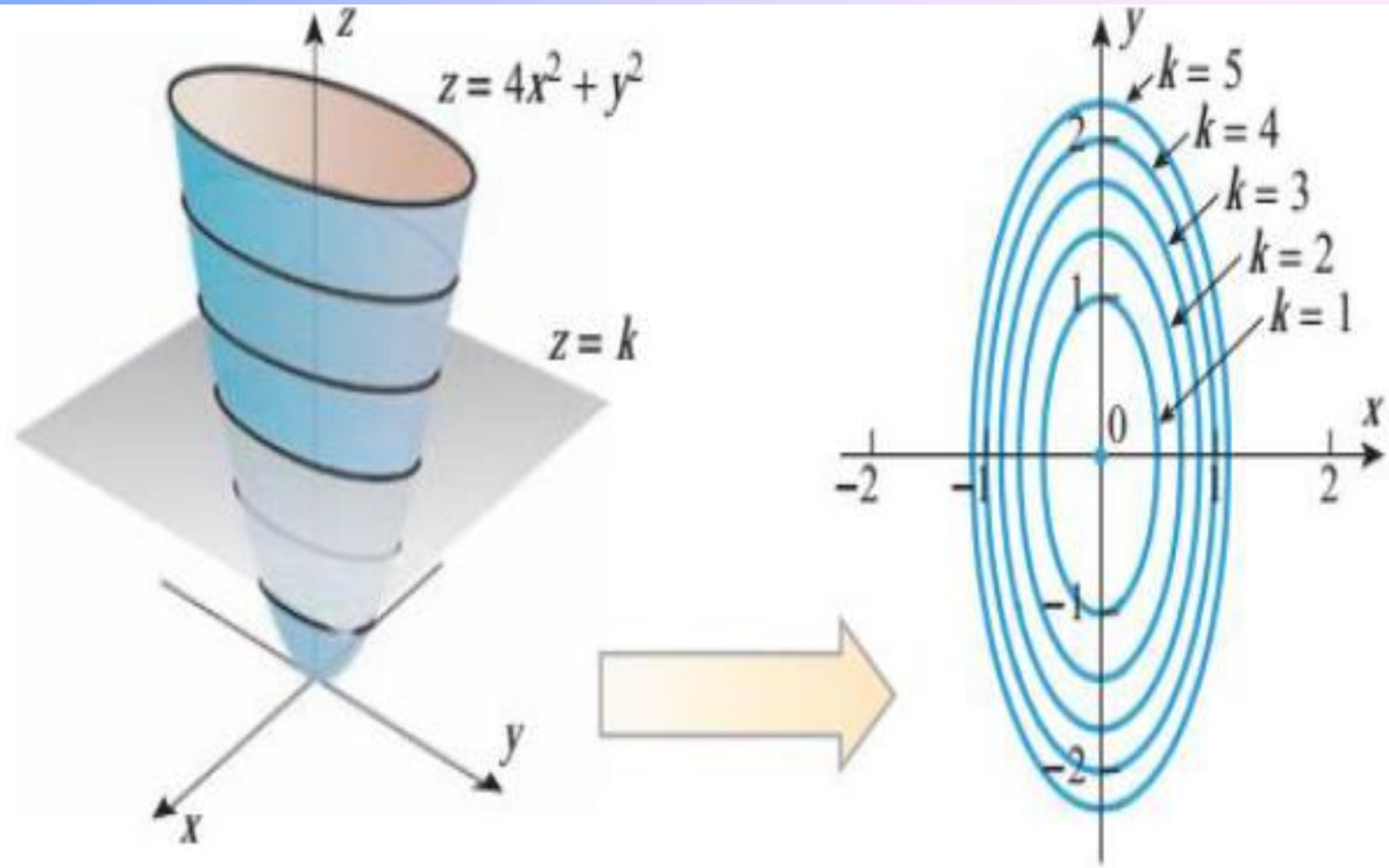
Example 5 Sketch the contour plot of $f(x, y) = 4x^2 + y^2$ using level curves of height $k = 0, 1, 2, 3, 4, 5$.

Solution. The graph of the surface $z = 4x^2 + y^2$ is the paraboloid shown in the left part of Figure 13.1.6, so we can reasonably expect the contour plot to be a family of ellipses centered at the origin. The level curve of height k has the equation $4x^2 + y^2 = k$. If $k = 0$, then the graph is the single point $(0, 0)$. For $k > 0$ we can rewrite the equation as

$$\frac{x^2}{k/4} + \frac{y^2}{k} = 1$$

which represents a family of ellipses with x-intercepts $\pm\sqrt{k}/2$ and y-intercepts $\pm\sqrt{k}$. The contour plot for the specified values of k is shown in the right part of Figure 13.1.6. ◀

Lecture continued



▶ Figure 13.1.6

Lecture continued

LEVEL SURFACES

Observe that the graph of $y = f(x)$ is a curve in 2-space, and the graph of $z = f(x, y)$ is a surface in 3-space, so the number of dimensions required for these graphs is one greater than the number of independent variables. Accordingly, there is no "direct" way to graph a function of three variables since four dimensions are required. However, if k is a constant, then the graph of the equation $f(x, y, z) = k$ will generally be a surface in 3-space (e.g., the graph of $x^2 + y^2 + z^2 = 1$ is a sphere), which we call the ***level surface with constant k***. Some geometric insight into the behavior of the function f can sometimes be obtained by graphing these level surfaces for various values of k.

> The term "level surface" is standard but confusing, since a level surface need not be level in the sense of being horizontal—it is simply a surface on which all values of f are the same.

Lecture continued

Example 7 Describe the level surfaces of

(a) $f(x, y, z) = x^2 + y^2 + z^2$ (b) $f(x, y, z) = z^2 - x^2 - y^2$

Solution (a). The level surfaces have equations of the form

$$x^2 + y^2 + z^2 = k$$

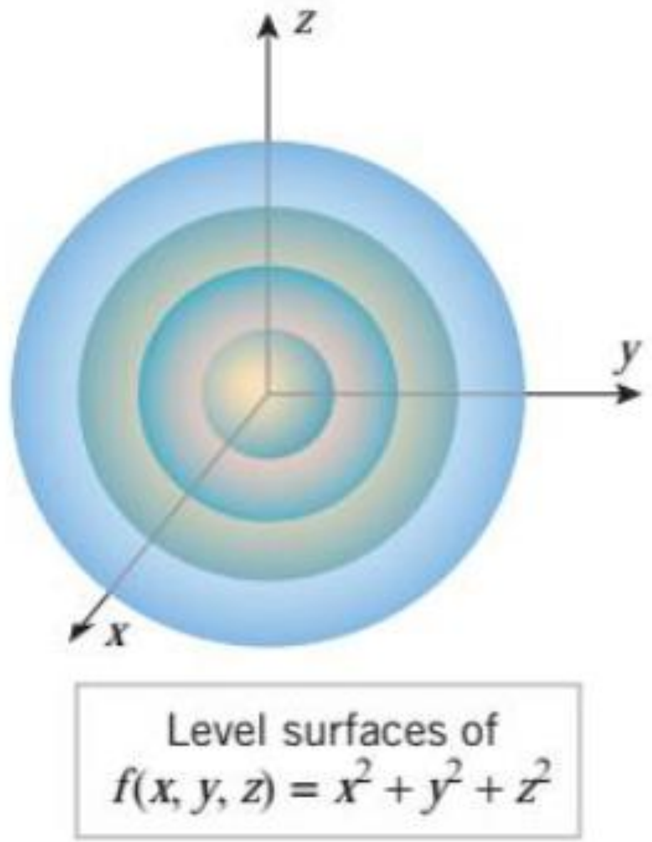
For $k > 0$ the graph of this equation is a sphere of radius $\sqrt{k}$, centered at the origin; for $k = 0$ the graph is the single point (0, 0, 0); and for $k < 0$ there is no level surface (Figure 13.1.9).

Solution (b). The level surfaces have equations of the form

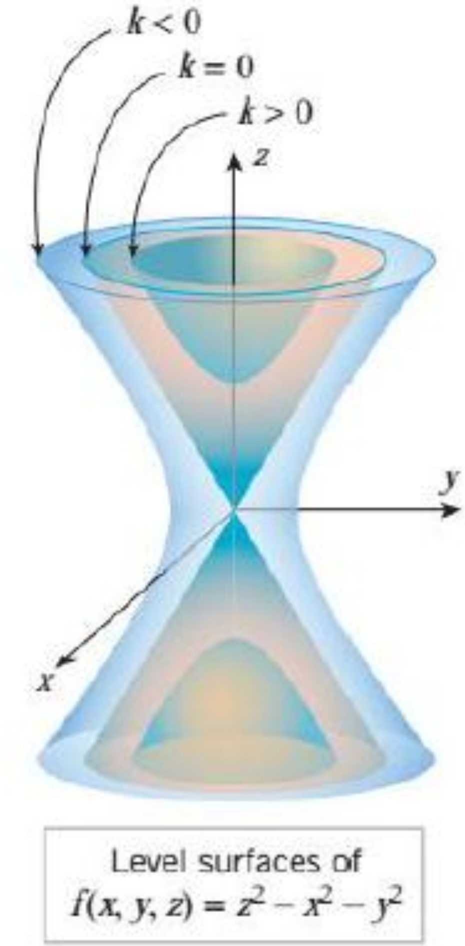
$$z^2 - x^2 - y^2 = k$$

As discussed in Section 11.7, this equation represents a cone if $k = 0$, a hyperboloid of two sheets if $k > 0$, and a hyperboloid of one sheet if $k < 0$ (Figure 13.1.10).

Lecture continued



▶ Figure 13.1.9



▶ Figure 13.1.10

Lecture continued

1–8 These exercises are concerned with functions of two variables. ■

1. Let $f(x, y) = x^2y + 1$. Find
(a) $f(2, 1)$ (b) $f(1, 2)$ (c) $f(0, 0)$
(d) $f(1, -3)$ (e) $f(3a, a)$ (f) $f(ab, a - b)$.

2. Let $f(x, y) = x + \sqrt[3]{xy}$. Find
(a) $f(t, t^2)$ (b) $f(x, x^2)$ (c) $f(2y^2, 4y)$.

3. Let $f(x, y) = xy + 3$. Find
(a) $f(x + y, x - y)$ (b) $f(xy, 3x^2y^3)$.

4. Let $g(x) = x \sin x$. Find
(a) $g(x/y)$ (b) $g(xy)$ (c) $g(x - y)$.

Lecture continued

5. Find $F(g(x), h(y))$ if $F(x, y) = xe^{xy}$, $g(x) = x^3$, and $h(y) = 3y + 1$.

6. Find $g(u(x, y), v(x, y))$ if $g(x, y) = y\sin(x^2y)$, $u(x, y) = x^2y^3$, and $v(x, y) = \pi xy$.

7. Let $f(x, y) = x + 3x^2y^2$, $x(t) = t^2$, and $y(t) = t^3$. Find

(a) $f(x(t), y(t))$ (b) $f(x(0), y(0))$

(c) $f(x(2), y(2))$.

8. Let $g(x, y) = ye^{-3x}$, $x(t) = \ln(t^2 + 1)$, and $y(t) = \sqrt{t}$. Find $g(x(t), y(t))$.

Lecture continued

23–26 Sketch the domain of f. Use solid lines for portions of the boundary included in the domain and dashed lines for portions not included. ■

23. $f(x, y) = \ln(1 - x^2 - y^2)$

24. $f(x, y) = \sqrt{x^2 + y^2 - 4}$

25. $f(x, y) = \dfrac{1}{x - y^2}$

26. $f(x, y) = \ln xy$

33–42 Sketch the graph of f. ■

33. $f(x, y) = 3$

34. $f(x, y) = \sqrt{9 - x^2 - y^2}$

35. $f(x, y) = \sqrt{x^2 + y^2}$

36. $f(x, y) = x^2 + y^2$

41. $f(x, y) = y + 1$

42. $f(x, y) = x^2$

Lecture continued

51–56 Sketch the level curve $z = k$ for the specified values of k.

51. $z = x^2 + y^2;\ k = 0, 1, 2, 3, 4$

52. $z = y/x;\ k = -2, -1, 0, 1, 2$

53. $z = x^2 + y;\ k = -2, -1, 0, 1, 2$

54. $z = x^2 + 9y^2;\ k = 0, 1, 2, 3, 4$

55. $z = x^2 - y^2;\ k = -2, -1, 0, 1, 2$

Lecture continued

57–60 Sketch the level surface $f(x, y, z) = k$.

57. $f(x, y, z) = 4x^2 + y^2 + 4z^2;\ k = 16$

58. $f(x, y, z) = x^2 + y^2 - z^2;\ k = 0$

59. $f(x, y, z) = z - x^2 - y^2 + 4;\ k = 7$

Thanks a lot …