TEXAS
The University of Texas at Austin

TOPIC 3 NONLINEAR PROGRAMMING

Optimization

- A general optimization problem is of the form
  - $\min_{x} f(x)$
  - s.t. $x \in S$
- So far, we have addressed a very specific class of optimization problems
  - $f(x) = c_1 x_1 + \cdots + c_n x_n$ (linear objective)
  - S is defined by $Ax \leq b$ (linear constraints)
  - S could also have integer constraints

Optimization

- What if $f(x)$ is some function that isn't linear
  - $f(x) = x_1^2 - x_2 + e^{x_3 x_2} - \sin(x_1 x_2^2)$
- What if S isn't defined by linear constraints
  - $x_1^2 + x_2^2 + x_3^2 \leq 1$
- If either of these occur, then we call the optimization problem a nonlinear program (NLP)

Nonlinear Programming

- Why can models become nonlinear?
  - Non-constant returns to scale (diminishing returns)
  - In supply/demand models when profit is maximized
  - In measuring goodness of fit with the sum of the squared differences
  - Models where risk is measured as the variance
  - Neural Networks

- Nonlinear models are often more realistic than linear models, but they are also more difficult to solve.

Nonlinear Programming

- With NLP we can’t rely on corner hopping anymore!
  - The feasible space might not have corners!
  - The optimal value might not be on the edge of the feasible space
- For example
  - $\max \cos(|x_1| + |x_2|)$
  - s.t. $x_1^2 + x_2^2 \leq \pi$

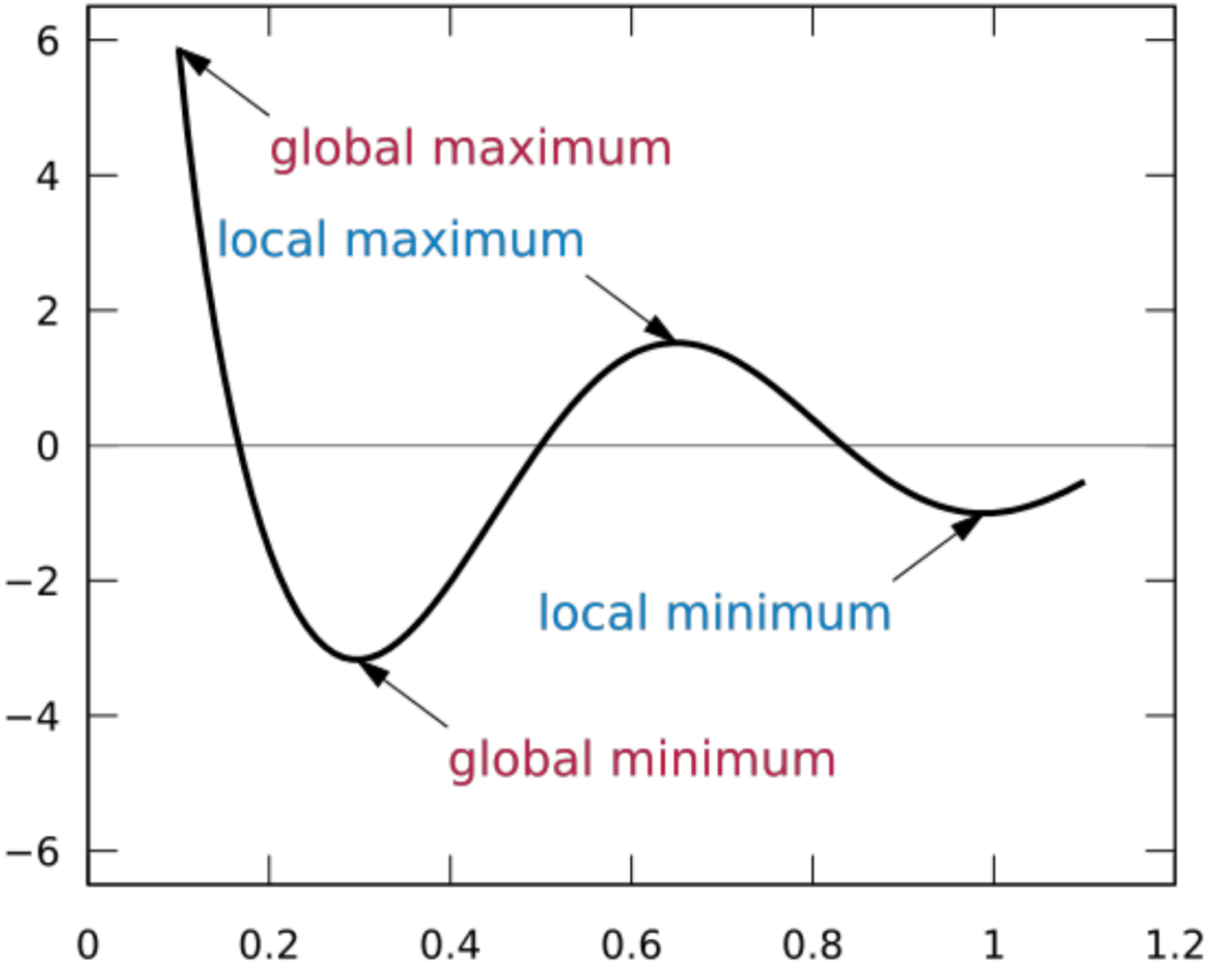
Calculus

- This brings us back to your first semester of calculus!
  - Find the slope of $f(x)$
  - Set the slope equal to zero
- $\min x^2 - 2x$

Limitations of Calculus

- What if the function is non-differentiable?
- What if the function is unknown?
- How do we handle constraints
  - Lagrange multipliers???
- Multiple places where slope is zero
- Min? Max? Saddle?
  - Second derivative test?

Global vs Local Optimum



Global vs Local Optimum

- Finding a local maximum or minimum is relatively much easier than finding the global, since it would be sufficient to check only the points around the local maximum or minimum.
- A local optimum is better than all nearby points.
- A global optimum is the best point in the entire feasible region.
- For NLP problems, some solving methods can get stuck at a local optimum and never find the global optimum.

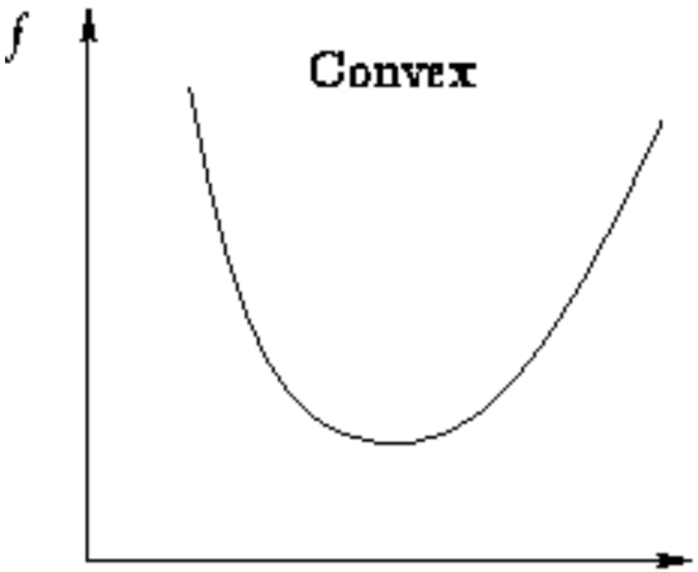
Global vs Local Optimum

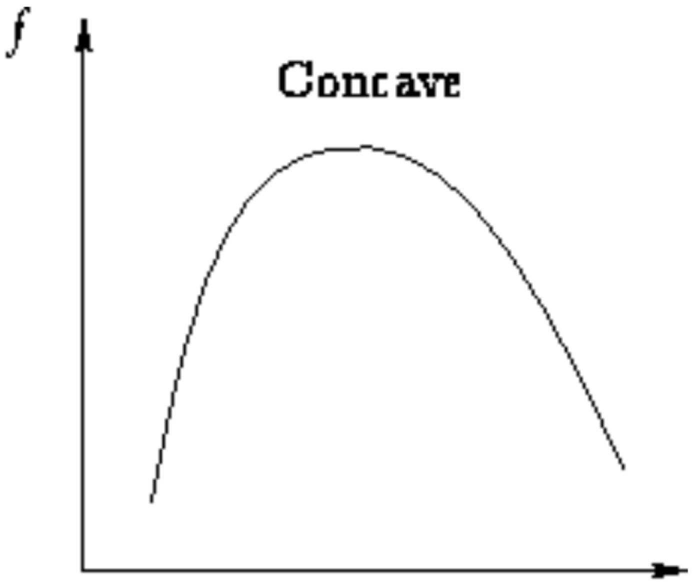
- When is a local optimum guaranteed to be global?

- Before we answer this question, we need to understand 2 concepts
  - Convex/Concave functions
  - Convex sets

Convex/Concave Functions

- A function, f, is convex if $\frac{f(x)+f(y)}{2} \geq f\left(\frac{x+y}{2}\right)\ \forall x, y$
- A function is concave if $\frac{f(x)+f(y)}{2} \leq f\left(\frac{x+y}{2}\right)\ \forall x, y$





Convex Set

- Convex sets are a little trickier
- A set, S, is convex if for any 2 points, x and y, that are in S, then the midpoint between them, $\frac{x+y}{2}$, is also in S

Global vs Local Optimum

- If
  - The objective function is convex (concave)
  - The feasible space is convex
- Then a local minimum (maximum) is guaranteed to be global!
- In general, we call both of these cases convex optimization (min or max…)

Class Participation

- Is the set defined by all points x, such that $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \leq b$ convex?

Global vs Local Optimum

- Why do we need the function to be convex?
- Why do we need the feasible space to be convex?

Convex Optimization

- When the optimization problem is convex it is WAY easier to find the optimal solution
  - Regression
- When the problem is non-convex it can be EXTREMELY difficult to solve!!!
  - Neural networks

Solution Algorithms

- There are LOTS of algorithms to solve NLPs
- Most of them are guaranteed to find the optimal solution if the problem is convex
- Very few algorithms exist that are guaranteed to find the solution to non-convex problems
  - Most are VERY slow
- Mostly people just modify convex algorithms to find local optimum of non-convex problems
  - Hopefully, the local optimum you find is good enough?