Time Value of Money

READ TIME VALUE OF MONEY

Time Value of Money – FV, PV, interest or discount rate, time

time value of money The return investors can earn over a period of time without taking any risk of loss of their money.

future value The value at a specific point in the future of a stipulated amount of cash today. Future value includes the stipulated amount as well as interest earned and compounded over the appropriate period.

present value The value today of a cash flow to be paid or received in the future; the discounted value of a series of future cash flows.

discount rate The rate at which future cash flows are discounted to determine their present value.

Time Value: $1 today is worth more than $1 tomorrow

2.1 Present and Future Value Over a Single Period

The notion that money has ***time value*** is a fundamental concept in finance. Someone who has $1 may consume $1 worth of goods today, or save and invest it for consumption at a later date. If the dollar is saved at the beginning of the period, it can earn interest at the ***risk-free interest rate***, r, so its value at the end of the period will be greater than $1. This extra value—the return on risk-free investing—is time value.

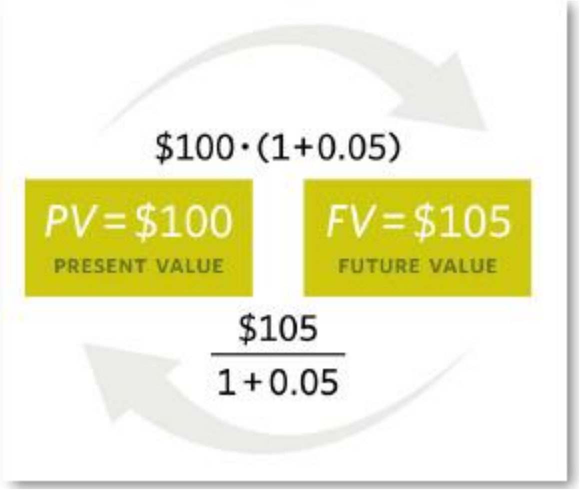
Present Value & Future Value

$$FV = PV \times (1 + r)$$

$$or \quad PV = \frac{FV}{1+r}$$

where $FV =$ future value, at the end of the period
$PV =$ present value, at the beginning of the period
$r =$ the risk-free rate of return per period

Invest $100 today at 5% risk-free rate for 1 year
Present Value = _____
Future Value = ________



An Example: N = 1

One year from now you will need a $1,000 deposit to rent a new apartment. You now have $906. If you invest your $906 for one year, what is the lowest annual interest rate that will enable you to meet your goal?

$906 * (1 + r) = $1,000

Annual interest rate = r = __________

$906 * (1 + r) = $1,000 Annual interest rate = r = <u>(1000/906) – 1 = 10.38%</u>

(FV = 1000, PV = -906, N = 1, <u>I=10.38</u>)

Compounding & Discounting over multiple periods

2.2 Compounding and Discounting Over Multiple Periods

If we invest \$100 for two periods instead of one, at the end of the second period we will have $[\$100 \cdot (1+r)] \cdot (1+r)$ or $\$100 \cdot (1+r)^2$. Not only will we earn interest on \$100 for two periods instead of one, but we will also earn *interest on interest* during the second period. If the risk-free interest rate is 5% for both periods, we will have $\$100 \cdot (1.05)^2 = \$110.25$ at the end of the second period. After a third period, we will have $\$110.25 \cdot 1.05 = \$100 \cdot (1.05)^3 = \$115.76$, and so forth. Earning interest on interest, known as ***compounding***, reflects the accumulation of time value over multiple periods.

Interactive Illustration 1 computes the value of \$100 at various points in time—one, two, or three years into the future (the FV of \$100) or one, two, or three years in the past (the PV of \$100) at various rates of interest. At points in the future, the value is greater than the initial \$100 investment owing to the effects of compounding. At points in the past, the value is less than \$100 owing to the effects of ***discounting***, the calculation of the (lesser) present value associated with a given (greater) future value. In discounting calculations the interest rate is often referred to as the ***discount rate.***

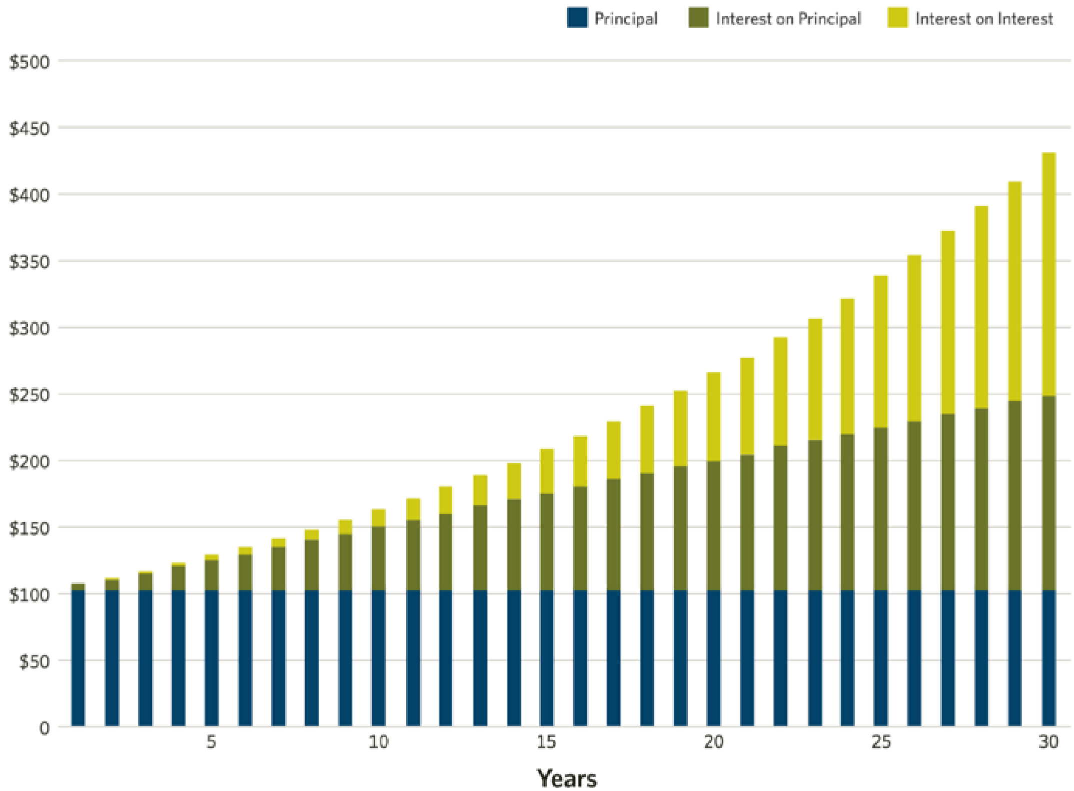
In general, over T period of time:

$$FV = PV \times (1+r)^T$$

$$or \quad PV = \frac{FV}{(1+r)^T}$$

where $(1+\mathrm{r})^{\mathrm{T}}$ is referred to as the compounding or discounting factor

The Power of Compounding



*Note: Your wealth/ investment increases exponentially instead of linearly because you not only earn interest over principal but also interest over interest.

Besides, your wealth/ investment increases by a larger percentage with a higher interest rate or a longer investment period.

However, this is also why your wealth/ investment has a much lower present value.

Try it out:
https://s3.amazonaws.com/he-assets-prod/interactives/149_time_value_of_100_annual/Launch.html

Examples

You invest \$162 today. What will be the value of your investment 4 years from now if the annual risk-free interest rate is 6%?

FV = 162 * 1.06^4 = \$204.52 (PV = -162, I = 6, N = 4, <u>FV=204.52</u>)

You invest \$162 today. What will be the value of your investment **30 years** from now if the annual risk-free interest rate is 6%?

FV = 162 * 1.06^30 = \$930.45 (PV = -162, I = 6, N = 30, <u>FV=930.45</u>)

You invest \$162 today. What will be the value of your investment 30 years from now if the annual risk-free interest rate is **12%**?

FV = 162 * 1.12^30 = \$4,853.51 (PV = -162, I = 12, N = 30, <u>FV=4,853.51</u>)

Multiple Cash Flows and Multiple Periods

2.3 Multiple Cash Flows and Multiple Periods

The same principle of time value and tools for compounding and discounting apply to multiple cash flows. We simply apply the appropriate discount or compounding factor to each cash flow, according to its location in time, and then sum the results.

For multiple periods and multiple cash flows, the accumulated future value at time T of a series of periodic cash flows is given by:

$$FV_T = Cf_0(1+r)^T + Cf_1(1+r)^{T-1} + \ldots + Cf_{T-1}(1+r) = \sum_{t=0}^{T-1} Cf_t(1+r)^{T-t}$$

where $T =$ the total number of periods (the endpoint) and $t =$ the date at which a cash flow occurs.

More generally, the formula for the calculation of the present value (at $t = 0$) of a finite series of periodic cash flows is given by:

$$PV = \frac{Cf_1}{1+r} + \frac{Cf_2}{(1+r)^2} + \frac{Cf_3}{(1+r)^3} + \ldots + \frac{Cf_T}{(1+r)^T}$$

where $T =$ the total number of periods (the endpoint) and $t =$ the date at which a cash flow occurs.

$$PV = \sum_{t=1}^{T} \frac{Cf_t}{(1+r)^t}$$

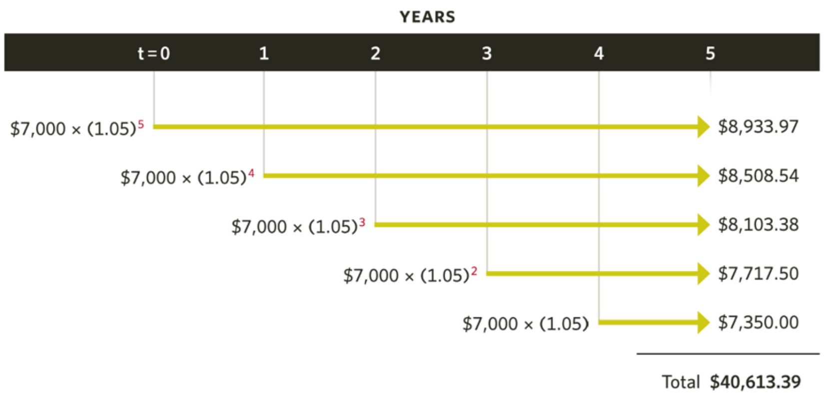
Guided Example

You would like to save $40,000 as the down payment for buying a house in 5 years. The current risk-free rate is 5%. What is the **least** amount of money *(rounded to the nearest thousands)* you need to save per year *(saved at the beginning of each year),* so you have just enough for the down payment in 5 years?

- You know for sure saving $8,000 per year works/ doesn't work because $8,000 x 5 = $40,000. This means, even without interest, you will have enough/ not enough money in 5 years.

- Therefore, the question is: will saving $7,000 under a 5% risk-free rate per year do the same thing compared to saving $8,000 without interest.

Guided Example (Con't)



Point in Time (yearly)	New Investment During the Year	Number of Years in Which Interest Is Earned	Compounding (FV) Factor	Future Value at the End of the Program
t	Cf_t	$T-t$	$(1+r)^{T-t}$	FV
0 (today)	$7,000	5	1.2763	$8,933.97
1	$7,000	4	1.2155	$8,508.54
2	$7,000	3	1.1576	$8,103.38
3	$7,000	2	1.1025	$7,717.50
4	$7,000	1	1.0500	$7,350.00
5 (total)				$40,613.39

Suppose you saved $7,000 per year at a risk-free rate of 5%:

By the end of year 5, you would have $40,613.39 sitting in your bank account.

This means you will/ will not have enough money for the down payments.

(calculator 2nd row all the way to right, 2nd BEG; then PMT = -7000 I = 5, N = 5, FV = 40,613.39)

Quick Thought: will it work if you save at the end of each year?

(2nd END; then PMT = -7000 I = 5, N = 5, FV = 36,679.42)

Examples

Given an 11% rate of return, what is the amount that must be put into an investment account at the end of each of the next ten years in order to accumulate $60,000?

N = ______; I/YR = ______; PV = _____; FV = ______

Annual Investment = PMT = $___________

N = 10; I/YR = 11; PV = 0; FV = 60,000

Annual Investment = PMT = $3,588.09

Examples

If 10,000 is invested today in an account that earns interest at a rate of 9.5%, what is the value of the equal withdrawals that can be taken out of he account at the end of each of the next 5 years if the investor plans to deplete the account at the end of the time period (i.e. account balance reaches 0)?

N = ____; I/YR = ____; PV = ____; FV = ____

Annual withdrawals = PMT = ____

N = 5; I/YR = 9.5; PV = -10,000; FV = 0

Annual withdrawals = PMT = $2,604.36

Examples

(6) An investment you made has the following cashflows:

Year 1	Year 2	Year 3	Year 4	Year 5
-200	-100	100	150	350

Risk-free rate = 4%

(i) Assume these cashflows happen at the end of each year; for example, you make a payment of $200 one year from now. What is the present value of this investment?

CF0 = ___; CF1 = ___; CF2 = ___; CF3 = ___; CF4 = ___; CF5 = ___ I/YR =___

PV = NPV = _____

CF0 = 0; CF1 = -200; CF2 = -100; CF3 = 100; CF4 = 150; CF5 = 350; I/YR = 4%

PV = NPV = $220.03

The Frequency of Compounding and Discounting

2.4 The Frequency of Compounding and Discounting

So far, all of our examples have utilized annual compounding or discounting. What if we performed the compounding or discounting more (or less) frequently?

For compounding periods less than one year, the formula for the effective annual interest rate is

$$\text{Effective interest rate} = \left(1 + \frac{r}{m}\right)^{m} - 1$$

where r is the simple annual rate, and m is the number of compounding periods per year.

*Note: Effective annual interest rate is often referred to as **Effect Annual Rate (EAR).** Besides, simple annual rate is assumed in most cases and simple annual rate is typically not equal to the effective interest rate unless it is compounded annually. This is because $[(1 + \frac{r}{1})^{1} - 1] = r$

APR and EAR – We Quote at APR but effective might be higher

simple interest rate An interest rate for which the compounding interval and the payment period are the same (most commonly annual). For example, a simple annual interest rate of 5% denotes 5% interest per year, compounded annually.

annual percentage rate (APR) The simple rate of interest on a loan (before compounding). Also referred to as the *nominal rate of return*.

effective interest rate The annual rate of interest implied when compounding takes place more than once per year, calculated as

$$\left(1+\frac{r}{m}\right)^{m}-1$$

where r is the simple annual rate and m is the number of compounding periods per year.

Effects of various compounding frequencies

Fill in the empty blue boxes:

Simple Interest Rate r	Compounding Frequency	Number of Compounding Periods/Year m	$\left(1+\frac{r}{m}\right)^m$	Effective Annual Interest Rate	Value of \$1,000,000 at $T=1$
10%	Annual	1	1.1	10%	1,100,000
10%	Semiannual	2	1.1025	10.25%	1,102,500
10%	Quarterly	4	1.103813	10.3813%	1,103,813
10%	Monthly	12	1.104713	10.4713%	1,104,713
10%	Daily	365	1.105156	10.5156%	1,105,156

Which compounding frequency would you prefer to receive interest on?
Annual/ Semiannual/ Quarterly/ Monthly/ Daily

Examples

A friend tells you that his credit card is charging an annual percentage rate of 18% compounded monthly. You tell him you will compute the effective rate or EAR. What is that rate?

EAR or Effective Annual Rate = _______

EAR = (1+18%/12)^(1*12) – 1 = (1 + 1.5%)^12 – 1 = 19.56%

Examples

A bond has an 8% coupon rate (with interest paid semi-annually), a face value of $1,000, and matures in 10 years. If the bond is priced to yield 6% (you can treat this as the annual discount rate), what is the bond's current price (i.e. Present Value)?

* Note: (i) A 8% semi-annual bond pay two interests per year with each worth 8%/2 = 4% of face value

(ii) The face value of a bond is what you receive by the time the bond matures (i.e. in 10 years)

N = ___; I/YR = ___PMT = ___; FV = ___

Price = PV = ___

FV = 1000

PMT = (.08*1000/2) = 40

I = (6/2) = 3

N = (10*2) = 20

<u>PV = -1,148.77   or bond price = 1,148.77</u>

Perpetuities and Annuities

2.5 Perpetuities and Annuities

Some cash flow streams have special properties that give rise to simple mathematical formulas for their present values. We will examine three of them in this section.

Perpetuities

You have already computed the value of an *annuity*, a fixed amount of cash paid or received every period for a finite number of periods. Now suppose we received a periodic cash flow, say $100 per year, *forever*. Such a stream of payments is called a ***perpetuity***: a perpetual stream of periodic cash flows. How much would such a stream of cash flows be worth today?

*Note: Perpetuity is basically an unending annuity. You might think the PV of a perpetuity is indefinite since you receive indefinite numbers of payments in the future. However, perpetuities are in fact priceable since their amount per payment is definite and with the help of some math tricks – summation of geometric series.

Perpetuities and Annuities – Level-Type Cash Flows

perpetuity Usually a stream of periodic, constant cash flows paid or received forever. *See also* growing (shrinking) perpetuity.

growing (shrinking) perpetuity A perpetuity in which the periodic payment increases (or decreases) by a constant rate of growth (or decay).

annuity A stream of constant periodic payments paid or received every period for a finite number of periods.

mortgage A collateralized loan, typically secured by real estate, which also exhibits the characteristics of an *annuity* (periodic payments of an equal amount) and an *amortizing loan*.

amortizing loan A type of loan in which the principal is repaid over the life of the loan, via periodic payments containing both interest and repayment of a portion of the outstanding principal balance.

Annuity Application to Mortgage

To buy a house, which costs $250,000 you make a down payment of $50,000, and take out a fixed-rate amortizing mortgage of $200,000 for 15 years, with an annual percentage rate (APR) of 4.25%, compounded monthly. What will be your required monthly payment?

* Note: APR is basically a simple annual rate in mortgage terms

Interest rate per month = 4.25% / 12 = 0.35417%

Solve for PMT in Calculator: PV = ___ N = ___ I/YR = ___

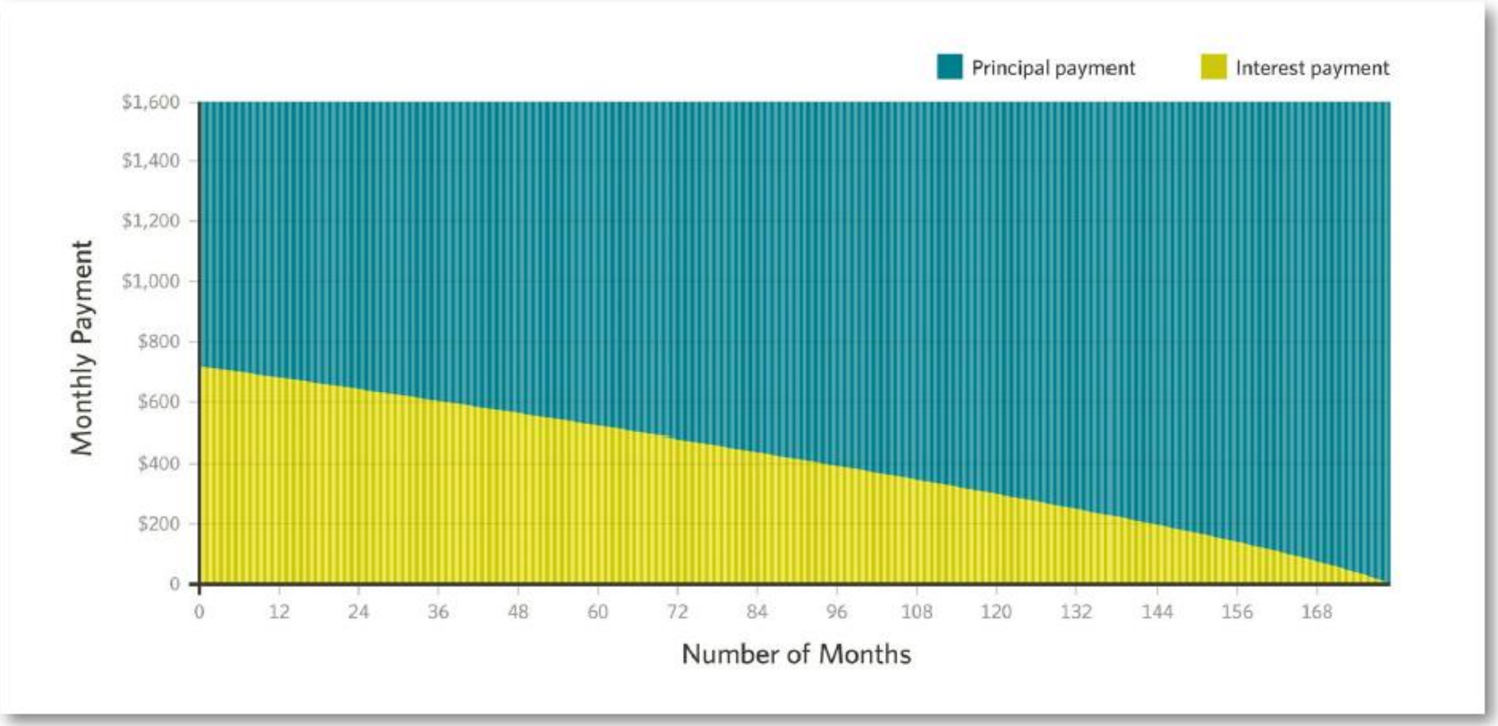
PV = 200,000 N = (15*12) = 180 I/YR = (4.25/12) = .35417

PMT = -$1504.56 (or just $1504.56)

Payment Split over time between Interest and Principal

EXHIBIT 7

Constant Mortgage Payment Split Between Interest and Principal Payments



Growing Perpetuity

Now let's return to perpetuities and consider one for which the periodic payment, initially equal to Cf, increases by a growth rate g per period. As long as g is known and constant and $g < r$, this ***growing perpetuity*** also has a convenient compact mathematical formula for its present value:

$$PV = \frac{Cf}{r - g}$$

Note the similarity between this formula and the one presented above with constant periodic payments.[8] When $g = 0$ the two formulas are identical. Another convenient property of this formula is that it also works when $g < 0$; that is, when the periodic cash flow is *shrinking* at a constant rate. Note that the formula makes no economic sense when $g \geq r$. When $g = r$, the denominator equals zero and the present value is undefined; when $g > r$, it gives a negative present value, which is clearly illogical. Fortunately, this latter circumstance can't actually occur in the real world—it is not economically possible for any cash flow to grow faster than the risk-free rate *forever*. It would take over the world!

$$PV = \frac{Cf}{r} \qquad where\ Cf = expected\ or\ \ Cf1$$

Application of Growing Perpetuity – solving for T

Example: Credit Card Debt

As a final application, let's explore credit card debt, a form of consumer borrowing that many personal finance experts warn can be difficult to escape when overused. Like the home mortgage we examined above, credit card debt requires monthly payments. But credit card balances aren't secured by a parcel of real estate, and the interest rates are generally much higher than mortgage rates. In the United States, average APRs for all types of consumer credit cards have been about 15% for a long time (though rates for specific cards vary widely), so we'll assume an APR of 15% per year. If you pay off your entire balance each month, the credit card issuer (typically a bank) charges you no interest. But if you don't pay it off entirely, you owe interest on the unpaid balance of $15\% \div 12 = 1.25\%$ per month.[h]

Since you would quickly be overwhelmed by a debt growing at 15% per year (and compounding monthly!), most credit card agreements require you to make some minimum payment each month, part of which covers the interest you owe and the remainder of which goes toward reducing the principal. The minimum monthly payment is often a percentage—say, 2% of the outstanding balance. Let's suppose you have a balance of $500 that you would like to pay off. Your required monthly payment is $10 (2% of $500). If you pay $10 every month, how long will it take you to pay off the entire $500?

Example of Minimum Payment to Prevent Growing Perpetuity

A friend of yours has a credit card balance of $500 with an APR of 15% compounded monthly. Because it is not secured by a house like a mortgage loan is, the credit card company requires a minimum monthly payment, usually 2% of the loan balance. Otherwise, if your peer paid nothing, the debt would turn into growing perpetuity. The debt would grow and never be paid off.

Your peer will not charge anymore and thinks it is ok to go with the minimum payment of .02*500 = 10. You are concerned and say, let me tell you how many months / years that will take to pay off.

PV = ___ I = ___ PMT = ___ now solve: N = ___ months /12 = ______ years

(PV = -500, I = 15/12 = 1.25, PMT = 10, <u>N=78.96 or about 79 months; /12 = 6.58 or about 7 years</u>)

Examples: Annuity and Perpetuity

You are planning to donate money to fund a community sporting event every year indefinitely, beginning one year from now. Assume the event costs $9,800 in the first year, and this amount will grow each year by 1.3%. If the annual interest rate is 1.9%, how much money should you donate today?

Money to donate = PV = _______

Money to donate = PV = 9800/(1.9% - 1.3%) = $1,633,333

PRACTICE (what you don't need to do)

Come to class to do these problems, many more and analyze results. These are some of your practice problems. Do the problems several times to make sure you understand them and that you can pick up speed.

The following are in the reading but totally *optional and only recommended if you want to do more. You really do not need to do these things unless you feel you already know what we are learning and need a bigger challenge.*

- Excel functions are simply FYI
- Key terms starting on pages 40-42
- practice questions w/ QR code p. 43 w/ solution provided