Problems for Wednesday Tutorial
March 23, 2022

Mainly Review.

(1) Let $\hat{a}, \hat{b}$ be unit vectors in $\mathbb{R}^3$
Discuss whether the equation $\hat{a} \times x = \hat{b}$ has solutions in $\mathbb{R}^3$; $\times$ is the cross product

(2) Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $p = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$ $(x_1^2 + x_2^2 = 1)$

What can you say about the set $\{p, Ap, A^2p, \dots\}$. Is it a finite set or an infinite set ?

(3) Consider the equation $x^2 + y^2 - z^2 + 7xy - 3yz + 6xz = 3$
write it in the form $[x \ y \ z] A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$
for some (3×3) symmetric matrix A.
Is such a matrix A unique ? What if we drop the symmetry requirement ??

(4) Recall the notion of an invertible matrix from Class 12.
How would ~~Can~~ you decide whether a 3×3 matrix is invertible or not. ?

If u is a unit vector in $\mathbb{R}^3$ (column vector)
Is $I - uu^T$ invertible ?
Can you discuss the map $f : \mathbb{R}^3 \to \mathbb{R}^3$
$f(x) = (I - 2uu^T)x$
geometrically ?

(5) Find two mutually $\perp$ unit vectors $\hat{u}, \hat{v}$ ~~set~~ s.t $\hat{u}, \hat{v}$ lie on the plane $x + y + z = 0$. Write out a parametrization for circle $x^2 + y^2 + z^2 = 1$, $x + y + z = 0$.

Department of Mathematics
Indian Institute of Technology, Bombay

MA 106 : Mathematics II Tutorial Sheet No.1

Autumn 2016 AR

Topics: Matrix addition, Scalar multiplication, Transposition, Matrix multiplication, Elementary row operations,matrices as linear maps, GEM and GJEM.

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1. Show that every square matrix A can be written as $S+T$ in a unique way, where S is symmetric and T is skew-symmetric.

2. A linear combination of matrices A and B of the same size is an expression of the form $P = aA + bB$, where $a,\ b$ are arbitrary scalars. Show that for the square matrices $A,\ B$, the following is true: (i)If these are symmetric then so is P. (ii)If these are skewsymmetric, then so is P. (iii)If these are upper triangular, then so is P.

3. Let A and B be symmetric matrices of the same size. Show that AB is symmetric if and only if $AB = BA$.

4. Consider $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as a linear map $\mathbb{R} \longrightarrow \mathbb{R}^2$. Show that its range is a line through $\mathbf{0}$.

   Similarly, show that $\begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ as a linear map from $\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ has its range as a plane through $\mathbf{0}$. Find its equation.

5. Consider the matrices:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Determine the images of (i) Unit square $\{0 \le x \le 1,\ 0 \le y \le 1\}$, (ii) Unit circle $\{x^2 + y^2 = 1\}$ and (iii) Unit disc $\{x^2 + y^2 \le 1\}$ under the above matrices viewed as linear maps $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

6. Find the inverses of the following matrices using elementary row-operations:

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -7 \\ 0 & 1 & -4 \end{bmatrix}, \begin{bmatrix} 1 & -x & e^x \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}.$$

7. Compute the last row of the inverse of the following matrices:

(i)
$$\begin{bmatrix} 1 & 0 & 1 \\ 8 & 1 & 0 \\ -7 & 3 & 1 \end{bmatrix}.$$

(ii)
$$\begin{bmatrix} 2 & 0 & -1 & 4 \\ 5 & 1 & 0 & 1 \\ 0 & 1 & 3 & -2 \\ -8 & -1 & 2 & 1 \end{bmatrix}.$$

8. A *Markov* or *stochastic* matrix is an $n \times n$ matrix $[a_{ij}]$ such that $a_{ij} \ge 0$ and $\sum_{j=1}^{n} a_{ij} = 1$. Prove that the product of two Markov matrices is again a Markov matrix.

9. Let $\Pi = \begin{bmatrix} 3 & -1 & 4 \\ -1 & 5 & -9 \\ 2 & -6 & 5 \end{bmatrix}$. Compute the products. (Note the patterns):

(i) $\Pi \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\Pi \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\Pi \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $[1\ 0\ 0]\Pi$, $[0\ 1\ 0]\Pi$, $[0\ 0\ 1]\Pi$.

(ii) $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix} \Pi$, $\Pi \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Pi$, $\Pi \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(iii) $\begin{bmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Pi,\ \Pi \begin{bmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mu & 0 & 1 \end{bmatrix} \Pi,\ \Pi \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mu & 0 & 1 \end{bmatrix}.$

10. Let A be a square matrix. Prove that there is a set of elementary matrices $E_1, E_2, ..., E_N$ such that $E_N ... E_2 E_1 A$ is either the identity matrix or its bottom row is zero.

11. List all possibilities for the reduced row echelon matrices of order 4×4 having exactly one pivot. Count the number of free parameters (degrees of freedom) in each case. For example one of the possibilty is $\begin{bmatrix} 0 & \boxed{1} & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ wherein there are 2 degrees of freedom. Repeat for 0, 2, 3 and 4 pivots.

12. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Verify that $(A - I)^3 = [0]$ and so the inverse is $A^2 - 3A + 3I$. Compute the same and verify by multiplying.

13. Let $X = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. (i) Show that $X^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}$, for all $\lambda \in \mathbb{R}, n \geq 1$.
(ii) If, as per the standard convention, we let $X^0 = \mathbf{I}$ (even when $X = [0]$), then show that $e^X = e^\lambda \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.
(iii) Show that (i) holds for integers $m = 0$ and also $m < 0$ if $\lambda \neq 0$.

MA 106 : Mathematics II Tutorial Sheet No.2

Autumn 2016 AR

Topics: $\mathbb{R}^n$, subspaces, linear independence, rank of a matrix, solvability of linear systems using rank.

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1. Suppose that the *state of land use* in a city area in 2003 was

1	(Residential)	30percent
2	(Commercial)	20percent
3	(Industrial)	50percent

Estimate the states of land use in 2008, 2013, 2018, assuming that the transitional probabilities for 5-year intervals are given by the stochastic matrix $\begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.0 & 0.1 & 0.9 \end{bmatrix}$,where $(i,j)^{th}$ entry is the probability for the i^{th} type to change to the j^{th} type. (For e.g. 0.2 is the probability for commercially used land to become industrial in a 5-year interval.)

2. Find whether the following sets of vectors are linearly dependent or independent:

(i) $[1,-1,1],[1,1,-1],[-1,1,1],[0,1,0]$.

(ii) $[1,9,9,8],[2,0,0,3],[2,0,0,8]$.

3. Find the ranks of the following matrices:

(i) $\begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix}$, (ii) $\begin{bmatrix} m & n \\ n & m \\ p & p \end{bmatrix}$ $(m^2 \neq n^2)$, (iii) $\begin{bmatrix} 0 & 8 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 4 & 5 \end{bmatrix}$.

4. Solve the following system of linear equations in the unknowns $x_1, \ldots, x_5$ by GEM:

(i)
$$\begin{array}{rrrrl} & 2x_3 & -2x_4 & +x_5 & = 2 \\ 2x_2 & -8x_3 & +14x_4 & -5x_5 & = 2 \\ x_2 & +3x_3 & & +x_5 & = \alpha \end{array}$$

(ii)
$$\begin{array}{rrrrl} 2x_1 & -2x_2 & +x_3 & +x_4 & = 1 \\ & -2x_2 & +x_3 & +7x_4 & = 0 \\ 3x_1 & -x_2 & +4x_3 & -2x_4 & = -2 \end{array}$$

5. Determine the equilibrium solution ($D_1 = S_1,\ D_2 = S_2$) of the two-commodity market with linear model $(D, S, P) =$ (demand, supply, price):

$$\begin{aligned} D_1 &= 40 - 2P_1 - P_2 & \qquad S_1 &= 4P_1 - P_2 + 4 \\ D_2 &= 16 + 5P_1 - 2P_2 & \qquad S_2 &= 3P_2 - 4. \end{aligned}$$

6. For the following linear systems, find solvability by comparing the ranks of the coefficient matrix and the augmented matrix. Write down a basis for the solutions of the asociated homogeneous systems and hence describe the general solution of each of the systems.

(i)
$$\begin{array}{cccccc} & & -2x_4 & +x_5 & = 2 \\ 2x_2 & -2x_3 & +14x_4 & -x_5 & = 2 \\ 2x_2 & +3x_3 & +13x_4 & +x_5 & = 3 \end{array}$$

(ii)
$$\begin{array}{ccccc} 2x_1 & -2x_2 & +x_3 & +x_4 & = 1 \\ & -2x_2 & +x_3 & -x_4 & = 2 \\ x_1 & +x_2 & +2x_3 & -x_4 & = -2 \end{array}$$

7. Is the given set of vectors a vector space?

(i) All vectors $[v_1, v_2, v_3]^T$ in $\mathbb{R}^3$ such that $3v_1 - 2v_2 + v_3 = 0,\ 4v_1 + 5v_2 = 0$. (ii) All vectors in $\mathbb{R}^2$ with components less than 1 in absolute value.

8. For $a < b$, consider the system of equations:

$$\begin{array}{ccccccc} x & + & y & + & z & = & 1 \\ ax & + & by & + & 2z & = & 3 \\ a^2x & + & b^2y & + & 4z & = & 9. \end{array}$$

Find the pairs (a, b) for which the system has infinitely many solutions.

9. Show that the row space of a matrix does not change by row operations. Show that the dimension of the column space is unchanged by row operations.

Proof:

(i)The new rows are linear combinations of previous rows and *vice versa.*

(ii) Suppose that $C_1, ..., C_n$ are the columns of a matrix. If an ERO is applied through a matrix E, then the new columns are $EC_1, ..., EC_n$. If $C_{j_1}, ..., C_{j_r}$ are lin. ind. then so are $EC_{j_1}, ..., EC_{j_r}$ and *vice versa* due to invertibility of E.

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MA 106 : Mathematics II Tutorial Sheet No.3

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Topics: Determinants, ranks by determinants, Adjoints, Inverses, Cramer's rule.

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1. Find the rank by determinants. Verify by row reduction.

$$\text{(i)} \begin{bmatrix} 0 & 2 & -3 \\ 2 & 0 & 5 \\ -3 & 5 & 0 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 4 & 3 \\ -8 & -6 \\ 16 & 12 \end{bmatrix} \quad \text{(iii)} \begin{bmatrix} -2 & -\sqrt{3} & -\sqrt{2} \\ -1 & 0 & 1 \\ \sqrt{2} & \sqrt{3} & 2 \end{bmatrix}$$

2. Find the values of β for which Cramer's rule is applicable. For the remaining value(s) of β, find the number of solutions.

$$\begin{aligned} x + 2y + 3z &= 20 \\ x + 3y + z &= 13 \\ x + 6y + \beta z &= \beta. \end{aligned}$$

3. Find whether the following set of vectors is linearly dependent or independent:

$$\{a\mathbf{i} + b\mathbf{j} + c\mathbf{k},\ b\mathbf{i} + c\mathbf{j} + a\mathbf{k},\ c\mathbf{i} + a\mathbf{j} + b\mathbf{k}\}.$$

4. Consider the system of equations

$$\begin{aligned} x + \lambda z &= \lambda - 1 \\ x + \lambda y &= \lambda + 1 \\ \lambda x + y + 3z &= 2\lambda - 1 (\text{ or } 1 - 2\lambda) \end{aligned}$$

Find the values of λ for which Cramer's rule can be used. For the remaining values of λ, discuss the solvability of the linear system.

5. Find the matrices of minors, cofactors and the adjoint of the following matrices:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \begin{bmatrix} 0 & 9 & 5 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & \sqrt{3} & \sqrt{2} \\ -1 & 0 & 1 \\ \sqrt{2} & \sqrt{3} & 2 \end{bmatrix}, \quad \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.0 & 0.1 & 0.9 \end{bmatrix}, \quad \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

6. In the previous problem verify that $A\mathrm{Adj}(A) = A\big(\mathrm{Adj}(A)\big) = \det A\mathbf{I}$. Hence compute the inverses in the valid cases.

7. Solve by Cramer's rule and verify by Gauss elimination.

$$\begin{aligned} 5x - 3y &= 37 \\ -2x + 7y &= -38. \end{aligned}$$

8. Solve by Cramer's rule and verify by Gauss elimination.

$$\begin{aligned} x + 2y + 3z &= 20 \\ 7x + 3y + z &= 13 \\ x + 6y + 2z &= 0. \end{aligned}$$

9. Invert the matrix $H = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}.$

10. (Vandermonde determinant)

    (a) Prove that $\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = (b-a)(c-a)(c-b).$

    (b) Prove an analogous formula for $n \times n$.

11. (Wronskian) Let $f_1, f_2, ..., f_n$ be functions over some interval (a, b). Their Wronskian is another function on (a, b) defined by a determinant involving the given functions and their derivatives upto the order $n-1$.

$$W_{f_1,f_2,...,f_n}(x) \stackrel{def}{=} \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

Prove that if $c_1f_1 + c_2f_2 + \cdots + c_nf_n = 0$ holds over the interval (a, b) for some constants $c_1, c_2, ..., c_n$ and $W_{f_1,f_2,...,f_n}(x_0) \neq 0$ at some x_0, then $c_1 = c_2 = \cdots = c_n = 0$. In other words, nonvanishing of $W_{f_1,f_2,...,f_n}$ at a single point establishes linear independence of $f_1, f_2, ..., f_n$ on (a, b).

Caution: The converse is false. $W \equiv 0 \not\Rightarrow$ $f_1, f_2, ..., f_n$ linearly dependent on (a, b). Though one can prove existence of a subinterval of (a, b) where linear dependence holds.

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MA 106 : Mathematics II Tutorial Sheet No.4

Autumn 2016 AR

Topics: Expansion in a basis, orthogonal sets, orthonormal basis, Gram-Schmidt process, Bessel's inequality.

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1. (Resolution into orthogonal components) Let $\mathbf{u}$ be a nonzero vector and $\mathbf{v}$ be any other vector. Let $\mathbf{w} = \mathbf{v} - \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\|\mathbf{u}\|^2}\mathbf{u}$ as in Gram-Schmidt process. Then show that
$$\mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\|\mathbf{u}\|^2}\mathbf{u} + \mathbf{w}$$
is resolution of $\mathbf{v}$ into two components-one parallel to $\mathbf{u}$ and the other orthogonal to $\mathbf{u}$.

2. Verify that the set of vectors $\left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right\}$ is an orthogonal set in $\mathbb{R}^3$. Is it a basis? If yes, express $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as a linear combination of these vectors and verify that Bessel's inequaliy is an equality.

3. Orthogonalize the following set of row-vectors in $\mathbb{R}^4$.
$$\{[1,1,1,1],\ [1,1,-1,-1],\ [1,1,0,0],\ [-1,1,-1,1]\}$$
Do you get an orthogonal basis?

4. Orthogonalize the following ordered set of row-vectors in $\mathbb{R}^4$.
$$\{[1,1,0,0],\ [1,0,1,0],\ [1,0,0,1],\ [0,1,1,0],\ [0,1,0,1],\ [0,0,1,1]\}$$

Do you get an orthogonal basis? Does $[-2,-1,1,2]$ belong to the linear span? Use Bessel's inequality.

5. For the following linear homogeneous systems, write down orthogonal bases for the solution spaces.

(i)
$$\begin{array}{rrrrl} & & -2x_4 & +x_5 & =0 \\ 2x_2 & -2x_3 & +14x_4 & -x_5 & =0 \\ 2x_2 & +3x_3 & +13x_4 & +x_5 & =0 \end{array}$$
(ii)
$$\begin{array}{rrrrl} 2x_1 & -2x_2 & +x_3 & +x_4 & =0 \\ & -2x_2 & +x_3 & -x_4 & =0 \\ x_1 & +x_2 & +2x_3 & -x_4 & =0 \end{array}$$
(This is a variant of the problem 6 in Sheet No.2)

6. For the following linear homogeneous systems, write down orthogonal bases for the solution spaces.

(i)
$$\begin{array}{rrrrl} & 2x_3 & -2x_4 & +x_5 & =0 \\ 2x_2 & -8x_3 & +14x_4 & -5x_5 & =0 \\ x_2 & +3x_3 & & +x_5 & =0 \end{array}$$
(ii)
$$\begin{array}{rrrrl} 2x_1 & -2x_2 & +x_3 & +x_4 & =0 \\ & -2x_2 & +x_3 & +7x_4 & =0 \\ 3x_1 & -x_2 & +4x_3 & -2x_4 & =0 \end{array}$$

7. Find whether the following sets of vectors are linearly dependent or independent by orthogonalizing them

(i) $[1,-1,1],\ [1,1,-1],\ [-1,1,1],\ [0,1,0]$.

(ii) $[2,0,0,3],\ [2,0,0,8],\ [2,0,1,3]$.

8. Orthonormalize the ordered set in $\mathbb{C}^5$:

$$\{[1,\imath,0,0,0],[0,1,\imath,0,0],[0,0,1,\imath,0],[0,0,0,1,\imath]\}$$

relative to the unitary inner product $\langle \mathbf{v},\mathbf{w}\rangle = \mathbf{w}^*\mathbf{v} = \sum_{j=1}^{5} v_j\bar{w}_j$.

Use Bessel's inequality to find whether $[1,\imath,1,\imath,1]$ is in the (complex) linear span of the above set or not.

9. Let $S=\{\mathbf{v}_1,\ \mathbf{v}_2,\dots,\mathbf{v}_k\}$ be any ordered set in $\mathbb{R}^n$ and $T=\{\mathbf{w}_1,\ \mathbf{w}_2,\dots,\mathbf{w}_k\}$ be the set resulting from the the Gram-Schmidt process applied to S. Prove that for $j=1,2,\dots,k$,

the linear span of $\{\mathbf{v}_1,\ \mathbf{v}_2, ..., \mathbf{v}_j\}$ equals that of $\{\mathbf{w}_1,\ \mathbf{w}_2, ..., \mathbf{w}_j\}$. Conclude that $\mathbf{w}_j$ is orthogonal to $\mathbf{v}_1, ..., \mathbf{v}_{j-1}$.

Hint: Use induction on j.

10. Let $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ be a linearly independent ordered set in $\mathbb{R}^3$. Let $\{\mathbf{x}(=\mathbf{p}),\ \mathbf{y},\ \mathbf{z}\}$ be the orthogonal set as a result of Gram-Schmidt process. Show that $\mathbf{z}$ must be a scalar multiple of $\mathbf{p} \times \mathbf{q}$.

11. For two <u>nonzero</u> vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, we define the angle between them by $\theta = \cos^{-1}\left[\dfrac{\langle \mathbf{v}, \mathbf{w}\rangle}{\|\mathbf{v}\|\|\mathbf{w}\|}\right]$.

    Note that by Cauchy-Schwartz inequality $\dfrac{\langle \mathbf{v}, \mathbf{w}\rangle}{\|\mathbf{v}\|\|\mathbf{w}\|} \in [-1, 1] \Longrightarrow \theta \in [0, \pi]$. Moreover, $\mathbf{v} \perp \mathbf{w}$ (orthogonal) if and only if $\theta = \dfrac{\pi}{2}$ as expected.

    Show that

    (i) $\theta = 0$ if and only if $\mathbf{w} = \dfrac{\|\mathbf{w}\|}{\|\mathbf{v}\|}\mathbf{v}$ is a positive scalar multiple of $\mathbf{v}$ (parallel).

    (ii) $\theta = \pi$ if and only if $\mathbf{w} = -\dfrac{\|\mathbf{w}\|}{\|\mathbf{v}\|}\mathbf{v}$ is a negative scalar multiple of $\mathbf{v}$ (anti-parallel).

    (iii) $$\|\mathbf{v} + \mathbf{w}\|^{\mathbf{2}} = \|\mathbf{v}\|^{\mathbf{2}} + \|\mathbf{w}\|^{\mathbf{2}} + 2\|\mathbf{v}\|\|\mathbf{w}\| \cos\theta.$$