MA 106 : Mathematics II Tutorial Sheet No.2

Autumn 2016 AR

Topics: $\mathbb{R}^n$, subspaces, linear independence, rank of a matrix, solvability of linear systems using rank.

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1. Suppose that the *state of land use* in a city area in 2003 was

   1 (Residential) 30percent

   2 (Commercial) 20percent

   3 (Industrial) 50percent

   Estimate the states of land use in 2008, 2013, 2018, assuming that the transitional probabilities for 5-year intervals are given by the stochastic matrix $\begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.0 & 0.1 & 0.9 \end{bmatrix}$,where $(i,j)^{th}$ entry is the probability for the i^{th} type to change to the j^{th} type. (For e.g. 0.2 is the probability for commercially used land to become industrial in a 5-year interval.)

   Solution. Let R, C and I denote the residential, the commercial and the industrial usage respectively in percentages. The given stochastic matrix represents schematically, the probabilities of the transitions: $\begin{bmatrix} R \mapsto R & R \mapsto C & R \mapsto I \\ C \mapsto R & C \mapsto C & C \mapsto I \\ I \mapsto R & I \mapsto C & I \mapsto I \end{bmatrix}$. Clearly, after 5 years $[R\ C\ I] = [R_0\ C_0\ I_0]S$.

   Hence

$$\begin{aligned} [R\ C\ I]_{2008} &= [30\ 20\ 50] \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.0 & 0.1 & 0.9 \end{bmatrix} \\ &= [26\ 22\ 52] \end{aligned}$$

$$\begin{aligned} [R\ C\ I]_{2013} &= [26\ 22\ 52] \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.0 & 0.1 & 0.9 \end{bmatrix} \\ &= [23.0\ 23.2\ 53.8] \end{aligned}$$

$$\begin{aligned}[R\ C\ I]_{2018} &= [23.0\ 23.2\ 53.8]\begin{bmatrix}0.8 & 0.1 & 0.1\\ 0.1 & 0.7 & 0.2\\ 0.0 & 0.1 & 0.9\end{bmatrix}\\ &= [20.72\ 23.92\ 55.36]\end{aligned}$$

Remark 0.0.1. *Note that* $[R\ C\ I] = [12.5\ 25.0\ 62.5]$ *gives the equilibrium solution.*

2. Find whether the following sets of vectors are linearly dependent or independent:

   (i) $[1,-1,1],[1,1,-1],[-1,1,1],[0,1,0]$.

   (ii) $[1,9,9,8],[2,0,0,3],[2,0,0,8]$.

   Solution. (i) $2[0,1,0] = [1,1,-1]+[-1,1,1]$ hence linearly dependent.

   Aliter:

$$\begin{bmatrix}1 & -1 & 1\\ 1 & 1 & -1\\ -1 & 1 & 1\\ 0 & 1 & 0\end{bmatrix} \mapsto \begin{bmatrix}\boxed{1} & -1 & 1\\ 0 & \boxed{1} & -1\\ 0 & 0 & \boxed{1}\\ 0 & 0 & 0\end{bmatrix}$$

   Hence dimension lin span $= 3 < 4 \implies$ lin DEPendent.

   Aliter-2:(changing to colulmns)

$$\begin{bmatrix}1 & 1 & -1 & 0\\ -1 & 1 & 1 & 1\\ 1 & -1 & 1 & 0\end{bmatrix} \mapsto \begin{bmatrix}\boxed{1} & 1 & -1 & 0\\ 0 & \boxed{2} & 0 & 1\\ 0 & 0 & \boxed{2} & 1\end{bmatrix}$$

   Hence dimension lin span $= 3 < 4 \implies$ lin DEPendent. Moreover, on dropping the fourth vector we get a linearly independent set, since the fourth column is pivot-free

   (ii) $a[1,9,9,8]+b[2,0,0,3]+c[2,0,0,8] = [0,0,0,0]$

   $\implies a+2b+2c = 9a = 8a+3b+8c = 0 \implies a = 0,\ b = 0,\ c = 0$. Hence linearly independent.

   Aliter:(changing to colulmns)

$$\begin{bmatrix}1 & 2 & 2\\ 9 & 0 & 0\\ 9 & 0 & 0\\ 8 & 3 & 8\end{bmatrix} \mapsto \begin{bmatrix}\boxed{1} & 2 & 2\\ 0 & \boxed{1} & 1\\ 0 & 0 & \boxed{5}\\ 0 & 0 & 0\end{bmatrix}.$$

No. of pivots $= 3 =$ no. of vectors, hence lin. INDependent.

3. Find the ranks of the following matrices:

(i) $\begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix}$, (ii) $\begin{bmatrix} m & n \\ n & m \\ p & p \end{bmatrix}$ $(m^2 \neq n^2)$, (iii) $\begin{bmatrix} 0 & 8 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 4 & 5 \end{bmatrix}$.

Solution. (i) By row operations, $\begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix} \mapsto \begin{bmatrix} 2 & -1 \\ -2 & 1 \\ 2 & -1 \end{bmatrix} \mapsto \begin{bmatrix} \boxed{2} & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. Rank is 1.

(ii) $\begin{bmatrix} m & n \\ n & m \\ p & p \end{bmatrix} \mapsto \begin{bmatrix} m+n & n+m \\ n & m \\ p & p \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 \\ n & m \\ 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} \boxed{1} & 1 \\ 0 & \boxed{m-n} \\ 0 & 0 \end{bmatrix}$. Rank is 2.

Remark 0.0.2. *We used the fact that $m \pm n \neq 0$.*

(iii) $\begin{bmatrix} 0 & 8 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 4 & 5 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & 0 \\ 0 & 8 & -1 \\ 0 & 4 & 5 \\ 0 & 0 & 3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -11 \\ 0 & 4 & 5 \\ 0 & 0 & 3 \end{bmatrix} \mapsto \begin{bmatrix} \boxed{1} & 2 & 0 \\ 0 & \boxed{4} & 5 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$. Rank is 3.

4. Solve the following system of linear equations in the unknowns $x_1, \ldots, x_5$ by GEM:

(i)
$$\begin{array}{rrrrl} & 2x_3 & -2x_4 & +x_5 & = 2 \\ 2x_2 & -8x_3 & +14x_4 & -5x_5 & = 2 \\ x_2 & +3x_3 & & +x_5 & = \alpha \end{array}$$

(ii)
$$\begin{array}{rrrrl} 2x_1 & -2x_2 & +x_3 & +x_4 & = 1 \\ & -2x_2 & +x_3 & +7x_4 & = 0 \\ 3x_1 & -x_2 & +4x_3 & -2x_4 & = -2 \end{array}$$

Solution. (i) Augmented matrix is $\left[\begin{array}{ccccc|c} 0 & 0 & 2 & -2 & 1 & 2 \\ 0 & 2 & -8 & 14 & -5 & 2 \\ 0 & 1 & 3 & 0 & 1 & \alpha \end{array}\right]$.

A REF is $\left[\begin{array}{ccccc|c} 0 & \boxed{1} & 3 & 0 & 1 & \alpha \\ 0 & 0 & \boxed{2} & -2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & \boxed{16-2\alpha} \end{array}\right]$. Hence no solutions if $\alpha \neq 8$.

For $\alpha = 8$, the REF is $\left[\begin{array}{ccccc|c} 0 & \boxed{1} & 3 & 0 & 1 & 8 \\ 0 & 0 & \boxed{2} & -2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ and the solution set is $x_3 = 1 + x_4 - x_5/2,\ x_2 = 5 - 3x_4 + x_5/2;\ x_1, x_4, x_5$ being arbitrary.

(ii) Augmented matrix $\left[\begin{array}{cccc|c} 2 & -2 & 1 & 1 & 1 \\ 0 & -2 & 1 & 7 & 0 \\ 3 & -1 & 4 & -2 & -2 \end{array}\right]$.

A REF is $\left[\begin{array}{cccc|c} \boxed{2} & -2 & 1 & 1 & 1 \\ 0 & \boxed{\text{-2}} & 1 & 7 & 0 \\ 0 & 0 & \boxed{1} & 1 & -1 \end{array}\right]$. Hence the solution set is $x_3 = -x_4 - 1,\ x_2 = 3x_4 - 1/2;\ x_1 = 3x_4 + 1/2;\ x_4$ being arbitrary.

5. Determine the equilibrium solution $(D_1 = S_1,\ D_2 = S_2)$ of the two-commodity market with linear model $(D, S, P) =$ (demand, supply, price):

$$D_1 = 40 - 2P_1 - P_2 \qquad S_1 = 4P_1 - P_2 + 4$$
$$D_2 = 16 + 5P_1 - 2P_2 \qquad S_2 = 3P_2 - 4.$$

Solution. Equilibrium condition implies

$$\begin{aligned} 40 - 2P_1 - P_2 &= 4P_1 - P_2 + 4 \\ 16 + 5P_1 - 2P_2 &= 3P_2 - 4 \\ \Longrightarrow \quad & P_1 = 6,\ P_2 = 5. \end{aligned}$$

6. For the following linear systems, find solvability by comparing the ranks of the coefficient matrix and the augmented matrix. Write down a basis for the solutions of the asociated homogeneous systems and hence describe the general solution of each of the systems.

(i)
$$\begin{array}{rrrrl} & & -2x_4 & +x_5 & = 2 \\ 2x_2 & -2x_3 & +14x_4 & -x_5 & = 2 \\ 2x_2 & +3x_3 & +13x_4 & +x_5 & = 3 \end{array}$$

(ii)
$$\begin{array}{rrrrl} 2x_1 & -2x_2 & +x_3 & +x_4 & = 1 \\ & -2x_2 & +x_3 & -x_4 & = 2 \\ x_1 & +x_2 & +2x_3 & -x_4 & = -2 \end{array}$$

Solution. (i) The augmented matrix $A^+ = \left[\begin{array}{ccccc|c} 0 & 0 & 0 & -2 & 1 & 2 \\ 0 & 2 & -2 & 14 & -1 & 2 \\ 0 & 2 & 3 & 13 & 1 & 3 \end{array}\right]$.

$$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & -2 & 1 & 2 \\ 0 & 2 & -2 & 14 & -1 & 2 \\ 0 & 2 & 3 & 13 & 1 & 3 \end{array}\right] \mapsto \left[\begin{array}{ccccc|c} 0 & 2 & 3 & 13 & 1 & 3 \\ 0 & 2 & -2 & 14 & -1 & 2 \\ 0 & 0 & 0 & -2 & 1 & 2 \end{array}\right]$$
$$\mapsto \left[\begin{array}{ccccc|c} 0 & \boxed{2} & 3 & 13 & 1 & 3 \\ 0 & 0 & \boxed{-5} & 1 & -2 & -1 \\ 0 & 0 & 0 & \boxed{-2} & 1 & 2 \end{array}\right].$$

Hence $\rho(A^+) = 3 = \rho(A)$ and the system is solvable. The homogeneous system has solutions $x_4 = 0.5x_5,\ x_3 = -0.3x_5,\ x_2 = -3.3x_5;\ x_1, x_5$ arbitrary. A basis of the homogeneous solutions is $\{[1,0,0,0,0]^T, [0,3.3,0.3,-0.5,-1]^T\}$. A particular solution of the given system is $[0,8,0,-1,0]^T$. The general solution is

$$[s, 8-33t, -3t, 5t-1, 10t]^T;\ s,t \text{ arbitrary.}$$

(ii) The augmented matrix $A^+ = \left[\begin{array}{cccc|c} 2 & -2 & 1 & 1 & 1 \\ 0 & -2 & 1 & -1 & 2 \\ 1 & 1 & 2 & -1 & -2 \end{array}\right]$.

$$\left[\begin{array}{cccc|c} 2 & -2 & 1 & 1 & 1 \\ 0 & -2 & 1 & -1 & 2 \\ 1 & 1 & 2 & -1 & -2 \end{array}\right] \mapsto \left[\begin{array}{cccc|c} 2 & 0 & 0 & 2 & -1 \\ 0 & -2 & 1 & -1 & 2 \\ 1 & 1 & 2 & -1 & -2 \end{array}\right]$$
$$\mapsto \left[\begin{array}{cccc|c} 2 & 0 & 0 & 2 & -1 \\ 0 & -2 & 1 & -1 & 2 \\ 0 & 2 & 4 & -4 & -1 \end{array}\right] \mapsto \left[\begin{array}{cccc|c} \boxed{2} & 0 & 0 & 2 & -1 \\ 0 & \boxed{-2} & 1 & -1 & 2 \\ 0 & 0 & \boxed{5} & -5 & 1 \end{array}\right].$$

Hence $\rho(A^+) = 3 = \rho(A)$ and the system is solvable. The homogeneous system has solutions $x_3 = x_4,\ x_2 = 0,\ x_1 = -x_4;\ x_4$ arbitrary. A basis of the homogeneous solutions is $\{[-1,0,1,1]^T\}$. A particular solution of the given system is $-[0.5,1.1,0.2,0]^T$. The general solution is

$$\left[-t-\frac{1}{2},-\frac{11}{10},t-\frac{2}{5},t\right]^T \text{ ; } t \text{ arbitrary.}$$

7. Is the given set of vectors a vector space?

   (i) All vectors $[v_1, v_2, v_3]^T$ in $\mathbb{R}^3$ such that $3v_1 - 2v_2 + v_3 = 0,\ 4v_1 + 5v_2 = 0$. (ii) All vectors in $\mathbb{R}^2$ with components less than 1 in absolute value.

   Solution. (i) YES. (ii) NO.

8. For $a < b$, consider the system of equations:

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 1 \\ ax & + & by & + & 2z & = & 3 \\ a^2x & + & b^2y & + & 4z & = & 9. \end{array}$$

   Find the pairs (a, b) for which the system has infinitely many solutions. Solution. For a square matrix A, $A\mathbf{x} = \mathbf{b}$ to have infinitely many solutions, a necessary (but not sufficient) condition is that $\det A = 0$.

   $A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & 2 \\ a^2 & b^2 & 4 \end{bmatrix} \implies \det A = (b-a)(2-a)(2-b) = 0$. Hence $a = 2$ or $b = 2$ since $b - a \neq 0$.

   <u>Case 1:</u>(a=2) The augmented matrix is

$$A^+ = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & b & 2 & 3 \\ 4 & b^2 & 4 & 9. \end{array}\right]$$

Elementary row operations give

$$A^+ \quad \mapsto \quad \begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & b-2 & 0 & | & 1 \\ 0 & b^2-4 & 0 & | & 5 \end{bmatrix}$$

$$\mapsto \quad \begin{bmatrix} \boxed{1} & 1 & 1 & | & 1 \\ 0 & \boxed{b-2} & 0 & | & 1 \\ 0 & 0 & 0 & | & \boxed{3-b} \end{bmatrix}$$

For existence $\rho(A^+) = \rho(A) \implies b = 3$. Thus $a = 2, b = 3$ will give infinitely many solutions.

Case 2:(b=2) Solving similarly, we find that $a = 3$ which is inadmissible since $a < b$.

9. Show that the row space of a matrix does not change by row operations. Show that the dimension of the column space is unchanged by row operations.

Proof:

(i)The new rows are linear combinations of previous rows and *vice versa.*

(ii) Suppose that $C_1, ..., C_n$ are the columns of a matrix. If an ERO is applied through a matrix E, then the new columns are $EC_1, ..., EC_n$. If $C_{j_1}, ..., C_{j_r}$ are lin. ind. then so are $EC_{j_1}, ..., EC_{j_r}$ and *vice versa* due to invertibility of E.