Department of Mathematics
Indian Institute of Technology, Bombay

MA 106 : Mathematics II Tutorial Sheet No.1

Autumn 2016 AR

Topics: Matrix addition, Scalar multiplication, Transposition, Matrix multiplication, Elementary row operations,matrices as linear maps, GEM and GJEM.

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1. Show that every square matrix A can be written as $S+T$ in a unique way, where S is symmetric and T is skew-symmetric.
   Solution. $S = \dfrac{A+A^T}{2}$, $T = \dfrac{A-A^T}{2}$. Further, $A = S' + T' \Longrightarrow A^T = S' - T' \Longrightarrow S' = \dfrac{A+A^T}{2} = S$, $T' = \dfrac{A-A^T}{2} = T$.

2. A linear combination of matrices A and B of the same size is an expression of the form $P = aA + bB$, where a, b are arbitrary scalars. Show that for the square matrices A, B, the following is true: (i)If these are symmetric then so is P. (ii)If these are skewsymmetric, then so is P. (iii)If these are upper triangular, then so is P.

3. Let A and B be symmetric matrices of the same size. Show that AB is symmetric if and only if $AB = BA$.
   Solution. $(AB)^T = AB \Longleftrightarrow B^T A^T = AB \Longleftrightarrow BA = AB$.

4. Consider $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as a linear map $\mathbb{R} \longrightarrow \mathbb{R}^2$. Show that its range is a line through $\mathbf{0}$.
   Similarly, show that $\begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ as a linear map from $\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ has its range as a plane through $\mathbf{0}$. Find its equation.
   Solution. (i) $A(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} [t] = \begin{bmatrix} t \\ -t \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, say. Thus $x(t) = t$, $y(t) = -t$ which are parametric equations of the line $x + y = 0$ through $\mathbf{0}$.

(ii)$B\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}1&-1\\-1&2\\0&1\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}=\begin{bmatrix}u-v\\-u+2v\\v\end{bmatrix}=\begin{bmatrix}x\\y\\z\end{bmatrix}$, say. Thus $x=u-v,\ y=2v-u,\ z=v$ are the parametric equations of the plane $x+y-z=0$ through $\mathbf{0}$.

5. Consider the matrices:

$$\begin{bmatrix}1&1\\0&1\end{bmatrix},\begin{bmatrix}-1&0\\-1&1\end{bmatrix},\begin{bmatrix}2&0\\0&-3\end{bmatrix},\begin{bmatrix}1&1\\0&0\end{bmatrix},\begin{bmatrix}1&1\\1&1\end{bmatrix},\begin{bmatrix}0&1\\0&0\end{bmatrix}.$$

Determine the images of (i) Unit square $\{0\le x\le 1,\ 0\le y\le 1\}$, (ii) Unit circle $\{x^2+y^2=1\}$ and (iii) Unit disc $\{x^2+y^2\le 1\}$ under the above matrices viewed as linear maps $\mathbb{R}^2\longrightarrow\mathbb{R}^2$.

Solution.

- The matrix $\begin{bmatrix}1&1\\0&1\end{bmatrix}$:

  (i) The parallellogram with vertices at $\{(0,0),(-1,-1),(-1,0),(0,1)\}$.

  (ii)The ellipse $\{x^2-2xy+2y^2=1\}$.

  (iii) Elliptic disc enclosed by (ii).

- The matrix $\begin{bmatrix}-1&0\\-1&1\end{bmatrix}$:

  (i) The parallellogram with vertices at $\{(0,0),(1,0),(2,1),(1,1)\}$.

  (ii)The ellipse $\{2x^2-2xy+y^2=1\}$.

  (iii) Elliptic disc enclosed by (ii).

- The matrix $\begin{bmatrix}2&0\\0&-3\end{bmatrix}$:

  (i) The parallellogram (actually rectangle) with vertices at $\{(0,0),(2,0),(2,-3),(0,-3)\}$.

  (ii)The ellipse $\left\{\frac{x^2}{4}+\frac{y^2}{9}=1\right\}$.

  (iii) Elliptic disc enclosed by (ii).

- The matrix $\begin{bmatrix}1&1\\0&0\end{bmatrix}$:

  (i)The segment $[0,1]$ along the x-axis.

(ii) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} = \begin{bmatrix} \cos t + \sin t \\ 0 \end{bmatrix} \implies$ the segment $[-\sqrt{2}, \sqrt{2}]$ along the x-axis. (iii) Same as (ii).

- The matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$:

  (i)The segment from $(0,0)$ to $(1,1)$ along the line $\{x = y\}$.

  (ii) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} = \begin{bmatrix} \cos t + \sin t \\ \cos t + \sin t \end{bmatrix} \implies$ the segment between $\pm(\sqrt{2}, \sqrt{2})$ along the line $\{x = y\}$. (iii) Same as (ii).

- The matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$:

  (i)The segment $[0,1]$ along the x-axis.

  (ii) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} = \begin{bmatrix} \sin t \\ 0 \end{bmatrix} \implies$ the segment $[-1, 1]$ along the x-axis. (iii) Same as (ii).

6. Find the inverses of the following matrices using elementary row-operations:

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -7 \\ 0 & 1 & -4 \end{bmatrix}, \begin{bmatrix} 1 & -x & e^x \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}.$$

Solution.

- $$\left[\begin{array}{ccc|ccc}1 & 3 & -2 & 1 & 0 & 0\\ 2 & 5 & -7 & 0 & 1 & 0\\ 0 & 1 & -4 & 0 & 0 & 1\end{array}\right] \mapsto \left[\begin{array}{ccc|ccc}1 & 3 & -2 & 1 & 0 & 0\\ 0 & -1 & -3 & -2 & 1 & 0\\ 0 & 1 & -4 & 0 & 0 & 1\end{array}\right]$$
$$\mapsto \left[\begin{array}{ccc|ccc}1 & 3 & -2 & 1 & 0 & 0\\ 0 & -1 & -3 & -2 & 1 & 0\\ 0 & 0 & -7 & -2 & 1 & 1\end{array}\right] \mapsto \left[\begin{array}{ccc|ccc}1 & 3 & -2 & 1 & 0 & 0\\ 0 & 1 & 3 & 2 & -1 & 0\\ 0 & 0 & -7 & -2 & 1 & 1\end{array}\right]$$
$$\mapsto \left[\begin{array}{ccc|ccc}1 & 3 & -2 & 1 & 0 & 0\\ 0 & 1 & 3 & 2 & -1 & 0\\ 0 & 0 & -7 & -2 & 1 & 1\end{array}\right] \mapsto \left[\begin{array}{ccc|ccc}1 & 0 & -11 & -5 & 3 & 0\\ 0 & 1 & 3 & 2 & -1 & 0\\ 0 & 0 & 1 & 2/7 & -1/7 & -1/7\end{array}\right]$$
$$\mapsto \left[\begin{array}{ccc|ccc}1 & 0 & 0 & -13/7 & 10/7 & -11/7\\ 0 & 1 & 0 & 8/7 & -4/7 & 3/7\\ 0 & 0 & 1 & 2/7 & -1/7 & -1/7\end{array}\right].$$

Therefore, the inverse is $\frac{1}{7}\begin{bmatrix}-13 & 10 & -11\\ 8 & -4 & 3\\ 2 & -1 & -1\end{bmatrix}$

- $$\left[\begin{array}{ccc|ccc}1 & -x & e^x & 1 & 0 & 0\\ 0 & 1 & x & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 1\end{array}\right] \mapsto \left[\begin{array}{ccc|ccc}1 & 0 & x^2+e^x & 1 & x & 0\\ 0 & 1 & x & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 1\end{array}\right]$$
$$\mapsto \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & x & -x^2-e^x\\ 0 & 1 & 0 & 0 & 1 & -x\\ 0 & 0 & 1 & 0 & 0 & 1\end{array}\right].$$

Therefore, the inverse is $\begin{bmatrix}1 & x & -x^2-e^x\\ 0 & 1 & -x\\ 0 & 0 & 1\end{bmatrix}$.

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$$\left[\begin{array}{cc|cc} \cos x & -\sin x & 1 & 0 \\ \sin x & \cos x & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{cc|cc} \cos x & -\sin x & 1 & 0 \\ 0 & \sec x & -\tan x & 1 \end{array}\right]$$

$$\mapsto \left[\begin{array}{cc|cc} \cos x & -\sin x & 1 & 0 \\ 0 & 1 & -\sin x & \cos x \end{array}\right] \mapsto \left[\begin{array}{cc|cc} \cos x & 0 & \cos^2 x & \sin x \cos x \\ 0 & 1 & -\sin x & \cos x \end{array}\right]$$

$$\mapsto \left[\begin{array}{cc|cc} 1 & 0 & \cos x & \sin x \\ 0 & 1 & -\sin x & \cos x \end{array}\right].$$

Therefore, the inverse is $\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$.

7. Compute the last row of the inverse of the following matrices:

(i)
$$\begin{bmatrix} 1 & 0 & 1 \\ 8 & 1 & 0 \\ -7 & 3 & 1 \end{bmatrix}.$$

Solution.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 8 & 1 & 0 & 0 & 1 & 0 \\ -7 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 8 & 1 & 0 & 0 & 1 & 0 \\ -8 & 3 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\mapsto \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 8 & 1 & 0 & 0 & 1 & 0 \\ -32 & 0 & 0 & -1 & -3 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} -32 & 0 & 0 & -1 & -3 & 1 \\ 8 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\mapsto \left[\begin{array}{ccc|ccc} -32 & 0 & 0 & -1 & -3 & 1 \\ 8 & 1 & 0 & 0 & 1 & 0 \\ 32 & 0 & 32 & 32 & 0 & 0 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} -32 & 0 & 0 & -1 & -3 & 1 \\ 8 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 32 & 31 & -3 & 1 \end{array}\right]$$

Therefore, the last row of the inverse is $\begin{bmatrix} \frac{31}{32} & \frac{-3}{32} & \frac{1}{32} \end{bmatrix}$.

(ii)
$$\begin{bmatrix} 2 & 0 & -1 & 4 \\ 5 & 1 & 0 & 1 \\ 0 & 1 & 3 & -2 \\ -8 & -1 & 2 & 1 \end{bmatrix}.$$

Solution.

$$\left[\begin{array}{cccc|cccc} 2 & 0 & -1 & 4 & 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 & 0 & 1 & 0 \\ -8 & -1 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{cccc|cccc} 2 & 0 & -1 & 4 & 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 & 0 & 1 & 0 \\ -10 & -1 & 3 & -3 & -1 & 0 & 0 & 1 \end{array}\right]$$
$$\mapsto \left[\begin{array}{cccc|cccc} 2 & 0 & -1 & 4 & 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & -1 & -1 & 2 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{cccc|cccc} 2 & 0 & -1 & 4 & 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 2 & -1 & 1 \end{array}\right].$$

Therefore, the last row of the inverse is $[-1 \;\; 2 \;\; -1 \;\; 1]$.

8. A *Markov* or *stochastic* matrix is an $n \times n$ matrix $[a_{ij}]$ such that $a_{ij} \geq 0$ and $\sum_{j=1}^{n} a_{ij} = 1$. Prove that the product of two Markov matrices is again a Markov matrix.

Solution. Let $A = [a_{jk}]$ and $B = [b_{k\ell}]$ be Markov matrices and $C = AB = [c_{j\ell}]$.

$$\begin{aligned} \sum_{\ell=1}^{n} c_{j\ell} &= \sum_{\ell=1}^{n} \left[\sum_{k=1}^{n} a_{jk} b_{k\ell} \right] \\ &= \sum_{k=1}^{n} a_{jk} \underbrace{\left[\sum_{\ell=1}^{n} b_{k\ell} \right]}_{=1} \\ &= \sum_{k=1}^{n} a_{jk} = 1. \end{aligned}$$

9. Let $\Pi = \begin{bmatrix} 3 & -1 & 4 \\ -1 & 5 & -9 \\ 2 & -6 & 5 \end{bmatrix}$. Compute the products. (Note the patterns):

(i) $\Pi \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\Pi \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\Pi \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $[1\ 0\ 0]\Pi$, $[0\ 1\ 0]\Pi$, $[0\ 0\ 1]\Pi$.

(ii) $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix} \Pi$, $\Pi \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Pi$, $\Pi \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(iii) $\begin{bmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Pi$, $\Pi \begin{bmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mu & 0 & 1 \end{bmatrix} \Pi$, $\Pi \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mu & 0 & 1 \end{bmatrix}$.

10. Let A be a square matrix. Prove that there is a set of elementary matrices E_1 , E_2 , ..., E_N such that $E_N ... E_2 E_1 A$ is either the identity matrix or its bottom row is zero.

Solution. (i) Applying row operations is equivalent to multiplying by ERM's on the left.

(ii) Can achieve $E_N ... E_2 E_1 A$ to be the reduced row-echelon form. There are either n pivots or less than n.

(a) If less than n, then the last row is zero.

(b) If equal to n, then by the definition of reduced REF must be identity since the pivots are forced to occupy the diagonal entries.

11. List all possibilities for the reduced row echelon matrices of order 4×4 having exactly one pivot. Count the number of free parameters (degrees of freedom) in each case. For example one of the possibilty is $\begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ wherein there are 2 degrees of freedom. Repeat for 0, 2, 3 and 4 pivots.

Solution.

- For 0 pivots the only possibility is the zero-matrix.
- For 1 pivot there are 4 possibilities

$$\begin{bmatrix} \boxed{1} & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{3 degrees of freedom,} \begin{bmatrix} 0 & \boxed{1} & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{2 degrees of freedom,}$$

$$\begin{bmatrix} 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{1 degrees of freedom,} \begin{bmatrix} 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{no degrees of freedom (unique).}$$

- For 2 pivots there are 6 possibilities

$$\begin{bmatrix} \boxed{1} & 0 & * & * \\ 0 & \boxed{1} & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{4 degrees of freedom,} \begin{bmatrix} \boxed{1} & * & 0 & * \\ 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{3 degrees of freedom,}$$

$$\begin{bmatrix} \boxed{1} & * & * & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{2 degrees of freedom,} \begin{bmatrix} 0 & \boxed{1} & 0 & * \\ 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{2 degrees of freedom,}$$

$$\begin{bmatrix} 0 & \boxed{1} & * & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{1 degree of freedom,} \begin{bmatrix} 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{no degree of freedom.}$$

- For 3 pivots there are 4 possibilities

$$\begin{bmatrix} \boxed{1} & 0 & 0 & * \\ 0 & \boxed{1} & 0 & * \\ 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{3 degrees of freedom,} \begin{bmatrix} \boxed{1} & 0 & * & 0 \\ 0 & \boxed{1} & * & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{2 degrees of freedom,}$$

$$\begin{bmatrix} \boxed{1} & * & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ 1 degree of freedom, } \begin{bmatrix} 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix} \textit{ nil} \text{ degrees of freedom.}$$

- For 4 pivots there is only the identity matrix $\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$

12. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Verify that $(A - I)^3 = [0]$ and so the inverse is $A^2 - 3A + 3I$. Compute the same and verify by multiplying.

Solution.

$$\begin{aligned} (A-I)^3 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}^3 \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A^3 - 3A^2 + 3A - I. \end{aligned}$$

$$\therefore A^{-1} = A^2 - 3A + 3I = (A-I)^2 - (A-I) + I = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Verfication: $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

13. Let $X = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. (i) Show that $X^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}$, for all $\lambda \in \mathbb{R}, n \geq 1$.

(ii) If, as per the standard convention, we let $X^0 = \mathbf{I}$ (even when $X = [0]$), then show that $e^X = e^\lambda \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(iii) Show that (i) holds for integers $m = 0$ and also $m < 0$ if $\lambda \neq 0$.

Solution.

(i) Since the equation is true for $n = 1$, assume for n. Then

$$\begin{aligned} X^{n+1} &= \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix} \\ &= \begin{bmatrix} \lambda^{n+1} & (n+1)\lambda^n \\ 0 & \lambda^{n+1} \end{bmatrix}. \end{aligned}$$

(ii)

$$e^X \stackrel{def}{=} \mathbf{I} + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots = \begin{bmatrix} e^\lambda & e^\lambda \\ 0 & e^\lambda \end{bmatrix}.$$

(iii) For $n = 0$ we trivially get $\mathbf{I}$. For $n = -1$ we get the matrix $\begin{bmatrix} \lambda^{-1} & -\lambda^{-2} \\ 0 & \lambda^{-1} \end{bmatrix} = (-\lambda^{-2}) \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}$ which is actually the inverse X^{-1}.

Applying (i), we find $X^{-n} = \big(-\lambda^{-2} \big)^n \begin{bmatrix} (-\lambda)^n & n(-\lambda)^{n-1} \\ 0 & (-\lambda)^n \end{bmatrix} = \begin{bmatrix} \lambda^{-n} & -n\lambda^{-n-1} \\ 0 & \lambda^{-n} \end{bmatrix}$.

Therefore, $X^m = \begin{bmatrix} \lambda^m & m\lambda^{m-1} \\ 0 & \lambda^m \end{bmatrix}$, where $m = -n < 0$.