Department of Mathematics
Indian Institute of Technology, Bombay

MA 106 : Mathematics II — Tutorial Sheet No.4

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Topics: Expansion in a basis, orthogonal sets, orthonormal basis, Gram-Schmidt process, Bessel's inequality.

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1. (Resolution into orthogonal components) Let $\mathbf{u}$ be a nonzero vector and $\mathbf{v}$ be any other vector. Let $\mathbf{w} = \mathbf{v} - \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\|\mathbf{u}\|^2}\mathbf{u}$ as in Gram-Schmidt process. Then show that

$$\mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\|\mathbf{u}\|^2}\mathbf{u} + \mathbf{w}$$

is resolution of $\mathbf{v}$ into two components-one parallel to $\mathbf{u}$ and the other orthogonal to $\mathbf{u}$.

Solution. Clearly, the first term (summand) is parallel to $\mathbf{u}$. Hence must verify that $\langle \mathbf{w}, \mathbf{u} \rangle = 0$.

$$\begin{aligned} \langle \mathbf{w}, \mathbf{u} \rangle &= \langle \mathbf{v}, \mathbf{u} \rangle - \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\cancel{\|\mathbf{u}\|^2}}\cancel{\langle \mathbf{u}, \mathbf{u} \rangle} \\ &= 0. \end{aligned}$$

2. Verify that the set of vectors $\left\{ \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right\}$ is an orthogonal set in $\mathbb{R}^3$. Is it a basis? If yes, express $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as a linear combination of these vectors and verify that Bessel's inequaliy is an equality.

Solution. Orthogonality is easily verified. Now linear independence makes it a basis. The coefficients in the expansion are:

$$a = \frac{\mathbf{v} \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2} = \frac{1}{9},\ b = \frac{\mathbf{v} \cdot \mathbf{w}_2}{\|\mathbf{w}_2\|^2} = \frac{5}{9},\ \frac{\mathbf{v} \cdot \mathbf{w}_3}{\|\mathbf{w}_3\|^2} = \frac{1}{9}.$$

Bessel's: $\|\mathbf{v}\|^2 = 3$, while $(\mathbf{v}\cdot\mathbf{u}_1)^2 = \frac{1}{9}$, $(\mathbf{v}\cdot\mathbf{u}_2)^2 = \frac{25}{9}$, $(\mathbf{v}\cdot\mathbf{u}_3)^2 = \frac{1}{9}$ and $3 = \frac{1}{9}+\frac{25}{9}+\frac{1}{9}$.

3. Orthogonalize the following set of row-vectors in $\mathbb{R}^4$.

$$\{[1,1,1,1],\ [1,1,-1,-1],\ [1,1,0,0],\ [-1,1,-1,1]\}$$

Do you get an orthogonal basis?

Solution.

$$\begin{aligned}
\mathbf{w}_1 = \mathbf{v}_1 &= [1,1,1,1] \\
\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2\cdot\mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 &= [1,1,-1,-1] \\
\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3\cdot\mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\mathbf{v}_3\cdot\mathbf{w}_2}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 &= [0,0,0,0] \\
\mathbf{w}_4 = \mathbf{v}_4 - \frac{\mathbf{v}_4\cdot\mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\mathbf{v}_4\cdot\mathbf{w}_2}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 &= [-1,1,-1,1].
\end{aligned}$$

Since only 3 nonzero orthogonal elements appear, it is NOT a basis of $\mathbb{R}^4$. (The linear span is 3-dimensional.)

4. Orthogonalize the following ordered set of row-vectors in $\mathbb{R}^4$.

$$\{[1,1,0,0],\ [1,0,1,0],\ [1,0,0,1],\ [0,1,1,0],\ [0,1,0,1],\ [0,0,1,1]\}$$

Do you get an orthogonal basis? Does $[-2,-1,1,2]$ belong to the linear span? Use Bessel's inequality.

Solution. After the Gram-Schmidt process we get

$$\{[1,1,0,0],\ \frac{1}{2}[1,-1,2,0],\ \frac{1}{3}[1,-1,-1,3],\ \frac{1}{2}[-1,1,1,1],\ \mathbf{0},\ \mathbf{0}\}.$$

The 4 nonzero elements give an orthogonal basis. $[-2,-1,1,2]$ must be in linear span. To verify Bessel's equality, we first note that

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}[1,1,0,0],\ \mathbf{u}_2 = \frac{1}{\sqrt{6}}[1,-1,2,0],\ \mathbf{u}_3 = \frac{1}{2\sqrt{3}}[1,-1,-1,3],\ \mathbf{u}_4 = \frac{1}{2}[-1,1,1,1].$$

Now $\sum(\mathbf{v}\cdot\mathbf{u}_j)^2 = \frac{9}{2}+\frac{1}{6}+\frac{4}{3}+4 = 10 = \|\mathbf{v}\|^2$. (Verified!)

5. For the following linear homogeneous systems, write down orthogonal bases for the solution spaces.

(i)
$$\begin{array}{rrrrl} & & -2x_4 & +x_5 & =0\\ 2x_2 & -2x_3 & +14x_4 & -x_5 & =0\\ 2x_2 & +3x_3 & +13x_4 & +x_5 & =0\end{array}$$

(ii)
$$\begin{array}{rrrrl} 2x_1 & -2x_2 & +x_3 & +x_4 & =0\\ & -2x_2 & +x_3 & -x_4 & =0\\ x_1 & +x_2 & +2x_3 & -x_4 & =0\end{array}$$

(This is a variant of the problem 6 in Sheet No.2)

Solution. (i) The augmented matrix $A^+ = \left[\begin{array}{ccccc|c} 0 & 0 & 0 & -2 & 1 & 0\\ 0 & 2 & -2 & 14 & -1 & 0\\ 0 & 2 & 3 & 13 & 1 & 0\end{array}\right]$.

$$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & -2 & 1 & 0\\ 0 & 2 & -2 & 14 & -1 & 0\\ 0 & 2 & 3 & 13 & 1 & 0\end{array}\right] \mapsto \left[\begin{array}{ccccc|c} 0 & 2 & 3 & 13 & 1 & 0\\ 0 & 2 & -2 & 14 & -1 & 0\\ 0 & 0 & 0 & -2 & 1 & 0\end{array}\right]$$
$$\mapsto \left[\begin{array}{ccccc|c} 0 & \boxed{2} & 3 & 13 & 1 & 0\\ 0 & 0 & \boxed{-5} & 1 & -2 & 0\\ 0 & 0 & 0 & \boxed{-2} & 1 & 0\end{array}\right].$$

The homogeneous system has solutions $x_4 = 0.5x_5,\ x_3 = -0.3x_5,\ x_2 = -3.3x_5;\ x_1, x_5$ arbitrary. A basis of the homogeneous solutions is $\{[1,0,0,0,0]^T, [0,3.3,0.3,-0.5,-1]^T\}$, which is already orthogonal.

(ii) The augmented matrix $A^+ = \left[\begin{array}{cccc|c} 2 & -2 & 1 & 1 & 0\\ 0 & -2 & 1 & -1 & 0\\ 1 & 1 & 2 & -1 & 0\end{array}\right]$.

$$\left[\begin{array}{cccc|c} 2 & -2 & 1 & 1 & 0\\ 0 & -2 & 1 & -1 & 0\\ 1 & 1 & 2 & -1 & 0\end{array}\right] \mapsto \left[\begin{array}{cccc|c} 2 & 0 & 0 & 2 & 0\\ 0 & -2 & 1 & -1 & 0\\ 1 & 1 & 2 & -1 & 0\end{array}\right]$$
$$\mapsto \left[\begin{array}{cccc|c} 2 & 0 & 0 & 2 & 0\\ 0 & -2 & 1 & -1 & 0\\ 0 & 2 & 4 & -4 & 0\end{array}\right] \mapsto \left[\begin{array}{cccc|c} \boxed{2} & 0 & 0 & 2 & 0\\ 0 & \boxed{-2} & 1 & -1 & 0\\ 0 & 0 & \boxed{5} & -5 & 0\end{array}\right].$$

The homogeneous system has solutions $x_3 = x_4,\ x_2 = 0,\ x_1 = -x_4;\ x_4$ arbitrary.

A basis of the homogeneous solutions is $\{[-1,0,1,1]^T\}$. A singleton set is vacuously orthogonal.

6. For the following linear homogeneous systems, write down orthogonal bases for the solution spaces.

$$\text{(i)}\quad \begin{array}{rrrrl} & 2x_3 & -2x_4 & +x_5 & =0 \\ 2x_2 & -8x_3 & +14x_4 & -5x_5 & =0 \\ x_2 & +3x_3 & & +x_5 & =0 \end{array} \qquad \text{(ii)}\quad \begin{array}{rrrrl} 2x_1 & -2x_2 & +x_3 & +x_4 & =0 \\ & -2x_2 & +x_3 & +7x_4 & =0 \\ 3x_1 & -x_2 & +4x_3 & -2x_4 & =0 \end{array}$$

Solution. (i)

Augmented matrix is $\left[\begin{array}{ccccc|c} 0 & 0 & 2 & -2 & 1 & 0 \\ 0 & 2 & -8 & 14 & -5 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \end{array}\right]$.

A REF is $\left[\begin{array}{ccccc|c} 0 & \boxed{1} & 3 & 0 & 1 & 0 \\ 0 & 0 & \boxed{2} & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$. The solution set is

$x_3 = x_4 - x_5/2,\ x_2 = -3x_4 + x_5/2;\ x_1, x_4, x_5$ being arbitrary.

A basis is $\{[1,0,0,0,0]^T,\ [0,-3,1,1,0]^T,\ [0,1,-1,0,2]^T\}$. An orthogonal basis is

$$\{[1,0,0,0,0]^T,\ [0,-3,1,1,0]^T,\ [0,-1,-7,4,22]^T\}.$$

(ii) Augmented matrix $\left[\begin{array}{cccc|c} 2 & -2 & 1 & 1 & 0 \\ 0 & -2 & 1 & 7 & 0 \\ 3 & -1 & 4 & -2 & 0 \end{array}\right]$.

A REF is $\left[\begin{array}{cccc|c} \boxed{2} & -2 & 1 & 1 & 0 \\ 0 & \boxed{-2} & 1 & 7 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \end{array}\right]$. Hence the solution set is

$x_3 = -x_4,\ x_2 = 3x_4;\ x_1 = 3x_4;\ x_4$ being arbitrary. A basis is $\{[3,3,-1,1]\}$ which is vacuously orthogonal.

7. Find whether the following sets of vectors are linearly dependent or independent by orthogonalizing them

(i) $[1,-1,1],\ [1,1,-1],\ [-1,1,1],\ [0,1,0]$.

(ii) $[2,0,0,3],\ [2,0,0,8],\ [2,0,1,3]$.

Solution. (i) Orthogonalization yields: $\{[1,-1,1],\ [2,1,-1],\ [0,1,1],\ [0,0,0]\}$. Hence l.d. as $\mathbf{0}$ occurs.

(ii) Orthogonalization yields: $\{[2,0,0,3],\ [-3,0,0,2],\ [0,0,1,0]\}$. Hence linearly independent. (We scaled $\mathbf{w}_2$ for covvenience by $13/10$.)

8. Orthonormalize the ordered set in $\mathbb{C}^5$:

$$\{[1,\imath,0,0,0],[0,1,\imath,0,0],[0,0,1,\imath,0],[0,0,0,1,\imath]\}$$

relative to the unitary inner product $\langle \mathbf{v},\mathbf{w}\rangle = \mathbf{w}^*\mathbf{v} = \sum_{j=1}^{5} v_j \bar{w}_j$.

Use Bessel's inequality to find whether $[1,\imath,1,\imath,1]$ is in the (complex) linear span of the above set or not.

Solution.

$$\begin{aligned}
\mathbf{w}_1 = \mathbf{v}_1 &= [1,\imath,0,0,0] \\
\mathbf{w}_2 = [0,1,\imath,0,0] - \frac{-\imath}{2}[1,\imath,0,0,0] &= \frac{1}{2}[\imath,1,2\imath,0,0] \\
\text{Similarly, } \mathbf{w}_3 &= \frac{1}{3}[-1,\imath,1,3\imath,0] \\
\text{and } \mathbf{w}_4 &= \frac{1}{4}[-\imath,-1,\imath,1,4\imath].
\end{aligned}$$

The corresponding orthonormal (unitary) set is

$$\left\{\frac{1}{\sqrt{2}}[1,\imath,0,0,0],\ \frac{1}{\sqrt{6}}[\imath,1,2\imath,0,0],\ \frac{1}{2\sqrt{3}}[-1,\imath,1,3\imath,0],\ \frac{1}{2\sqrt{5}}[-\imath,-1,\imath,1,4\imath]\right\}.$$

$\|[1,\imath,1,\imath,1]\|^2 = 5$. OTOH,

$$\begin{aligned}
|\langle \mathbf{v},\mathbf{u}_1\rangle|^2 &= 2, & |\langle \mathbf{v},\mathbf{u}_2\rangle|^2 &= 2/3, \\
|\langle \mathbf{v},\mathbf{u}_3\rangle|^2 &= 4/3, & |\langle \mathbf{v},\mathbf{u}_4\rangle|^2 &= 4/5.
\end{aligned}$$

The sum is $24/5 < 5$, hence can not be in the linear span.

9. Let $S = \{\mathbf{v}_1,\ \mathbf{v}_2, ..., \mathbf{v}_k\}$ be any ordered set in $\mathbb{R}^n$ and $T = \{\mathbf{w}_1,\ \mathbf{w}_2, ..., \mathbf{w}_k\}$ be the set resulting from the the Gram-Schmidt process applied to S. Prove that for $j = 1, 2, ..., k$, the linear span of $\{\mathbf{v}_1,\ \mathbf{v}_2, ..., \mathbf{v}_j\}$ equals that of $\{\mathbf{w}_1,\ \mathbf{w}_2, ..., \mathbf{w}_j\}$. Conclude that $\mathbf{w}_j$ is orthogonal to $\mathbf{v}_1, ..., \mathbf{v}_{j-1}$.

Hint: Use induction on j.

10. Let $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ be a linearly independent ordered set in $\mathbb{R}^3$. Let $\{\mathbf{x}(=\mathbf{p}),\ \mathbf{y},\ \mathbf{z}\}$ be the orthogonal set as a result of Gram-Schmidt process. Show that $\mathbf{z}$ must be a scalar multiple of $\mathbf{p} \times \mathbf{q}$.

Solution. By the previous theorem, plane of $\{\mathbf{x}, \mathbf{y}\}$ is the plane of $\{\mathbf{p}, \mathbf{q}\}$. Hence $\mathbf{z}$ being orthogonal to this plane must be a scalar multiple of $\mathbf{p} \times \mathbf{q}$.

11. For two nonzero vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, we define the angle between them by $\theta = \cos^{-1}\left[\dfrac{\langle \mathbf{v}, \mathbf{w}\rangle}{\|\mathbf{v}\|\|\mathbf{w}\|}\right]$.

Note that by Cauchy-Schwartz inequality $\dfrac{\langle \mathbf{v}, \mathbf{w}\rangle}{\|\mathbf{v}\|\|\mathbf{w}\|} \in [-1, 1] \Longrightarrow \theta \in [0, \pi]$. Moreover, $\mathbf{v} \perp \mathbf{w}$ (orthogonal) if and only if $\theta = \dfrac{\pi}{2}$ as expected.

Show that

(i) $\theta = 0$ if and only if $\mathbf{w} = \dfrac{\|\mathbf{w}\|}{\|\mathbf{v}\|}\mathbf{v}$ is a positive scalar multiple of $\mathbf{v}$ (parallel).

(ii) $\theta = \pi$ if and only if $\mathbf{w} = -\dfrac{\|\mathbf{w}\|}{\|\mathbf{v}\|}\mathbf{v}$ is a negative scalar multiple of $\mathbf{v}$ (anti-parallel).

(iii) $$\|\mathbf{v} + \mathbf{w}\|^{\mathbf{2}} = \|\mathbf{v}\|^{\mathbf{2}} + \|\mathbf{w}\|^{\mathbf{2}} + 2\|\mathbf{v}\|\|\mathbf{w}\| \cos\theta.$$

Solution.

(i) $(\Longrightarrow)$: $\theta = 0 \Longrightarrow \langle \mathbf{v}, \mathbf{w}\rangle = \|\mathbf{v}\|\|\mathbf{w}\| \Longrightarrow \lambda = \dfrac{\|\mathbf{w}\|}{\|\mathbf{v}\|} > 0$ in $\mathbf{w} = \lambda\mathbf{v}$.

$(\Longleftarrow)$: $\mathbf{w} = \dfrac{\|\mathbf{w}\|}{\|\mathbf{v}\|}\mathbf{v} \Longrightarrow \langle \mathbf{v}, \mathbf{w}\rangle = \|\mathbf{w}\|\|\mathbf{v}\| \Longrightarrow \cos\theta = 1 \Longrightarrow \theta = 0.$

(ii) Similar to (i).

(iii)

$$\begin{aligned}
\|\mathbf{v} + \mathbf{w}\|^2 &= \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w}\rangle \\
&= \langle \mathbf{v}, \mathbf{v}\rangle + \langle \mathbf{w}, \mathbf{w}\rangle + 2\langle \mathbf{v}, \mathbf{w}\rangle \\
&= \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 + 2\|\mathbf{v}\|\|\mathbf{w}\| \cos\theta.
\end{aligned}$$