Department of Mathematics
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MA 106 : Mathematics II Tutorial Sheet No.1

Autumn 2016 AR

Topics: Matrix addition, Scalar multiplication, Transposition, Matrix multiplication, Elementary row operations,matrices as linear maps, GEM and GJEM.

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1. Show that every square matrix A can be written as $S + T$ in a unique way, where S is symmetric and T is skew-symmetric.
   Solution. $S = \dfrac{A + A^T}{2}$, $T = \dfrac{A - A^T}{2}$. Further, $A = S' + T' \Longrightarrow A^T = S' - T' \Longrightarrow S' = \dfrac{A + A^T}{2} = S,\ T' = \dfrac{A - A^T}{2} = T.$

2. A linear combination of matrices A and B of the same size is an expression of the form $P = aA + bB$, where $a,\ b$ are arbitrary scalars. Show that for the square matrices $A,\ B$, the following is true: (i)If these are symmetric then so is P. (ii)If these are skewsymmetric, then so is P. (iii)If these are upper triangular, then so is P.

3. Let A and B be symmetric matrices of the same size. Show that AB is symmetric if and only if $AB = BA$.
   Solution. $(AB)^T = AB \Longleftrightarrow B^T A^T = AB \Longleftrightarrow BA = AB.$

4. Consider $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ as a linear map $\mathbb{R} \longrightarrow \mathbb{R}^2$. Show that its range is a line through $\mathbf{0}$. Similarly, show that $\begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ as a linear map from $\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ has its range as a plane through $\mathbf{0}$. Find its equation.
   Solution. (i) $A(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} [t] = \begin{bmatrix} t \\ -t \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, say. Thus $x(t) = t,\ y(t) = -t$ which are parametric equations of the line $x + y = 0$ through $\mathbf{0}$.

(ii)$B\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u-v \\ -u+2v \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, say. Thus $x = u-v,\ y = 2v-u,\ z = v$ are the parametric equations of the plane $x+y-z=0$ through $\mathbf{0}$.

5. Consider the matrices:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Determine the images of (i) Unit square $\{0 \le x \le 1,\ 0 \le y \le 1\}$, (ii) Unit circle $\{x^2+y^2=1\}$ and (iii) Unit disc $\{x^2+y^2 \le 1\}$ under the above matrices viewed as linear maps $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

Solution.

- The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$:

  (i) The parallellogram with vertices at $\{(0,0),(-1,-1),(-1,0),(0,1)\}$.

  (ii)The ellipse $\{x^2-2xy+2y^2=1\}$.

  (iii) Elliptic disc enclosed by (ii).

- The matrix $\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$:

  (i) The parallellogram with vertices at $\{(0,0),(1,0),(2,1),(1,1)\}$.

  (ii)The ellipse $\{2x^2-2xy+y^2=1\}$.

  (iii) Elliptic disc enclosed by (ii).

- The matrix $\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$:

  (i) The parallellogram (actually rectangle) with vertices at $\{(0,0),(2,0),(2,-3),(0,-3)\}$.

  (ii)The ellipse $\left\{\dfrac{x^2}{4}+\dfrac{y^2}{9}=1\right\}$.

  (iii) Elliptic disc enclosed by (ii).

- The matrix $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$:

  (i)The segment $[0,1]$ along the x-axis.

(ii) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} = \begin{bmatrix} \cos t + \sin t \\ 0 \end{bmatrix} \Longrightarrow$ the segment $[-\sqrt{2}, \sqrt{2}]$ along the x-axis. (iii) Same as (ii).

- The matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$:

  (i)The segment from $(0, 0)$ to $(1, 1)$ along the line $\{x = y\}$.

  (ii) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} = \begin{bmatrix} \cos t + \sin t \\ \cos t + \sin t \end{bmatrix} \Longrightarrow$ the segment between $\pm(\sqrt{2}, \sqrt{2})$ along the line $\{x = y\}$. (iii) Same as (ii).

- The matrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$:

  (i)The segment $[0, 1]$ along the x-axis.

  (ii) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} = \begin{bmatrix} \sin t \\ 0 \end{bmatrix} \Longrightarrow$ the segment $[-1, 1]$ along the x-axis. (iii) Same as (ii).

6. Find the inverses of the following matrices using elementary row-operations:

$$\begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -7 \\ 0 & 1 & -4 \end{bmatrix}, \begin{bmatrix} 1 & -x & e^x \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}.$$

Solution.

- $$\left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 2 & 5 & -7 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -1 & -3 & -2 & 1 & 0 \\ 0 & 1 & -4 & 0 & 0 & 1 \end{array}\right]$$
$$\mapsto \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -7 & -2 & 1 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & -1 & 0 \\ 0 & 0 & -7 & -2 & 1 & 1 \end{array}\right]$$
$$\mapsto \left[\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & 1 & 3 & 2 & -1 & 0 \\ 0 & 0 & -7 & -2 & 1 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 1 & 0 & -11 & -5 & 3 & 0 \\ 0 & 1 & 3 & 2 & -1 & 0 \\ 0 & 0 & 1 & 2/7 & -1/7 & -1/7 \end{array}\right]$$
$$\mapsto \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -13/7 & 10/7 & -11/7 \\ 0 & 1 & 0 & 8/7 & -4/7 & 3/7 \\ 0 & 0 & 1 & 2/7 & -1/7 & -1/7 \end{array}\right].$$
Therefore, the inverse is $\frac{1}{7}\begin{bmatrix} -13 & 10 & -11 \\ 8 & -4 & 3 \\ 2 & -1 & -1 \end{bmatrix}$

- $$\left[\begin{array}{ccc|ccc} 1 & -x & e^x & 1 & 0 & 0 \\ 0 & 1 & x & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 1 & 0 & x^2+e^x & 1 & x & 0 \\ 0 & 1 & x & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$
$$\mapsto \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & x & -x^2-e^x \\ 0 & 1 & 0 & 0 & 1 & -x \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].$$
Therefore, the inverse is $\begin{bmatrix} 1 & x & -x^2-e^x \\ 0 & 1 & -x \\ 0 & 0 & 1 \end{bmatrix}$.

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$$\left[\begin{array}{cc|cc} \cos x & -\sin x & 1 & 0 \\ \sin x & \cos x & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{cc|cc} \cos x & -\sin x & 1 & 0 \\ 0 & \sec x & -\tan x & 1 \end{array}\right]$$

$$\mapsto \left[\begin{array}{cc|cc} \cos x & -\sin x & 1 & 0 \\ 0 & 1 & -\sin x & \cos x \end{array}\right] \mapsto \left[\begin{array}{cc|cc} \cos x & 0 & \cos^2 x & \sin x \cos x \\ 0 & 1 & -\sin x & \cos x \end{array}\right]$$

$$\mapsto \left[\begin{array}{cc|cc} 1 & 0 & \cos x & \sin x \\ 0 & 1 & -\sin x & \cos x \end{array}\right].$$

Therefore, the inverse is $\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$.

7. Compute the last row of the inverse of the following matrices:

(i) $$\begin{bmatrix} 1 & 0 & 1 \\ 8 & 1 & 0 \\ -7 & 3 & 1 \end{bmatrix}.$$

Solution.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 8 & 1 & 0 & 0 & 1 & 0 \\ -7 & 3 & 1 & 0 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 8 & 1 & 0 & 0 & 1 & 0 \\ -8 & 3 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\mapsto \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 8 & 1 & 0 & 0 & 1 & 0 \\ -32 & 0 & 0 & -1 & -3 & 1 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} -32 & 0 & 0 & -1 & -3 & 1 \\ 8 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\mapsto \left[\begin{array}{ccc|ccc} -32 & 0 & 0 & -1 & -3 & 1 \\ 8 & 1 & 0 & 0 & 1 & 0 \\ 32 & 0 & 32 & 32 & 0 & 0 \end{array}\right] \mapsto \left[\begin{array}{ccc|ccc} -32 & 0 & 0 & -1 & -3 & 1 \\ 8 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 32 & 31 & -3 & 1 \end{array}\right]$$

Therefore, the last row of the inverse is $\begin{bmatrix} \frac{31}{32} & \frac{-3}{32} & \frac{1}{32} \end{bmatrix}$.

(ii)
$$\begin{bmatrix} 2 & 0 & -1 & 4 \\ 5 & 1 & 0 & 1 \\ 0 & 1 & 3 & -2 \\ -8 & -1 & 2 & 1 \end{bmatrix}.$$

Solution.

$$\left[\begin{array}{cccc|cccc} 2 & 0 & -1 & 4 & 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 & 0 & 1 & 0 \\ -8 & -1 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{cccc|cccc} 2 & 0 & -1 & 4 & 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 & 0 & 1 & 0 \\ -10 & -1 & 3 & -3 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\mapsto \left[\begin{array}{cccc|cccc} 2 & 0 & -1 & 4 & 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & -1 & -1 & 2 & 0 & 1 \end{array}\right] \mapsto \left[\begin{array}{cccc|cccc} 2 & 0 & -1 & 4 & 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 2 & -1 & 1 \end{array}\right].$$

Therefore, the last row of the inverse is $[-1 \;\; 2 \;\; -1 \;\; 1]$.

8. A *Markov* or *stochastic* matrix is an $n \times n$ matrix $[a_{ij}]$ such that $a_{ij} \geq 0$ and $\sum_{j=1}^{n} a_{ij} = 1$. Prove that the product of two Markov matrices is again a Markov matrix.

Solution. Let $A = [a_{jk}]$ and $B = [b_{k\ell}]$ be Markov matrices and $C = AB = [c_{j\ell}]$.

$$\begin{aligned} \sum_{\ell=1}^{n} c_{j\ell} &= \sum_{\ell=1}^{n} \left[\sum_{k=1}^{n} a_{jk} b_{k\ell} \right] \\ &= \sum_{k=1}^{n} a_{jk} \underbrace{\left[\sum_{\ell=1}^{n} b_{k\ell} \right]}_{=1} \\ &= \sum_{k=1}^{n} a_{jk} = 1. \end{aligned}$$

9. Let $\Pi = \begin{bmatrix} 3 & -1 & 4 \\ -1 & 5 & -9 \\ 2 & -6 & 5 \end{bmatrix}$. Compute the products. (Note the patterns):

(i) $\Pi \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\Pi \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\Pi \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $[1\ 0\ 0]\Pi$, $[0\ 1\ 0]\Pi$, $[0\ 0\ 1]\Pi$.

(ii) $\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix} \Pi,\ \Pi \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{bmatrix},\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Pi,\ \Pi \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

(iii) $\begin{bmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Pi,\ \Pi \begin{bmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mu & 0 & 1 \end{bmatrix} \Pi,\ \Pi \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mu & 0 & 1 \end{bmatrix}.$

10. Let A be a square matrix. Prove that there is a set of elementary matrices E_1 , E_2 , ..., E_N such that $E_N ... E_2 E_1 A$ is either the identity matrix or its bottom row is zero.

Solution. (i) Applying row operations is equivalent to multiplying by ERM's on the left.

(ii) Can achieve $E_N ... E_2 E_1 A$ to be the reduced row-echelon form. There are either n pivots or less than n.

(a) If less than n, then the last row is zero.

(b) If equal to n, then by the definition of reduced REF must be identity since the pivots are forced to occupy the diagonal entries.

11. List all possibilities for the reduced row echelon matrices of order 4×4 having exactly one pivot. Count the number of free parameters (degrees of freedom) in each case. For example one of the possibilty is $\begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ wherein there are 2 degrees of freedom. Repeat for 0, 2, 3 and 4 pivots.

Solution.

- For 0 pivots the only possibility is the zero-matrix.
- For 1 pivot there are 4 possibilities

$$\begin{bmatrix} \boxed{1} & * & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{3 degrees of freedom,} \begin{bmatrix} 0 & \boxed{1} & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{2 degrees of freedom,}$$

$$\begin{bmatrix} 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{1 degrees of freedom,} \begin{bmatrix} 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{no degrees of freedom (unique).}$$

- For 2 pivots there are 6 possibilities

$$\begin{bmatrix} \boxed{1} & 0 & * & * \\ 0 & \boxed{1} & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{4 degrees of freedom,} \begin{bmatrix} \boxed{1} & * & 0 & * \\ 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{3 degrees of freedom,}$$

$$\begin{bmatrix} \boxed{1} & * & * & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{2 degrees of freedom,} \begin{bmatrix} 0 & \boxed{1} & 0 & * \\ 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{2 degrees of freedom,}$$

$$\begin{bmatrix} 0 & \boxed{1} & * & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{1 degree of freedom,} \begin{bmatrix} 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{no degree of freedom.}$$

- For 3 pivots there are 4 possibilities

$$\begin{bmatrix} \boxed{1} & 0 & 0 & * \\ 0 & \boxed{1} & 0 & * \\ 0 & 0 & \boxed{1} & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{3 degrees of freedom,} \begin{bmatrix} \boxed{1} & 0 & * & 0 \\ 0 & \boxed{1} & * & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{2 degrees of freedom,}$$

$$\begin{bmatrix} \boxed{1} & * & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ 1 degree of freedom, } \begin{bmatrix} 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix} \textit{ nil} \text{ degrees of freedom.}$$

- For 4 pivots there is only the identity matrix $\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$

12. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$. Verify that $(A - I)^3 = [0]$ and so the inverse is $A^2 - 3A + 3I$. Compute the same and verify by multiplying.

Solution.

$$\begin{aligned} (A - I)^3 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}^3 \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = A^3 - 3A^2 + 3A - I. \end{aligned}$$

$$\therefore A^{-1} = A^2 - 3A + 3I = (A - I)^2 - (A - I) + I = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Verfication: $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$

13. Let $X = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$. (i) Show that $X^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}$, for all $\lambda \in \mathbb{R}, n \geq 1$.

(ii) If, as per the standard convention, we let $X^0 = \mathbf{I}$ (even when $X = [0]$), then show that $e^X = e^\lambda \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(iii) Show that (i) holds for integers $m = 0$ and also $m < 0$ if $\lambda \neq 0$.

Solution.

(i) Since the equation is true for $n = 1$, assume for n. Then

$$\begin{aligned} X^{n+1} &= \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix} \\ &= \begin{bmatrix} \lambda^{n+1} & (n+1)\lambda^n \\ 0 & \lambda^{n+1} \end{bmatrix}. \end{aligned}$$

(ii)

$$e^X \overset{def}{=} \mathbf{I} + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots = \begin{bmatrix} e^\lambda & e^\lambda \\ 0 & e^\lambda \end{bmatrix}.$$

(iii) For $n = 0$ we trivially get $\mathbf{I}$. For $n = -1$ we get the matrix $\begin{bmatrix} \lambda^{-1} & -\lambda^{-2} \\ 0 & \lambda^{-1} \end{bmatrix} = (-\lambda^{-2}) \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}$ which is actually the inverse X^{-1}.

Applying (i), we find $X^{-n} = \big(-\lambda^{-2} \big)^n \begin{bmatrix} (-\lambda)^n & n(-\lambda)^{n-1} \\ 0 & (-\lambda)^n \end{bmatrix} = \begin{bmatrix} \lambda^{-n} & -n\lambda^{-n-1} \\ 0 & \lambda^{-n} \end{bmatrix}$.

Therefore, $X^m = \begin{bmatrix} \lambda^m & m\lambda^{m-1} \\ 0 & \lambda^m \end{bmatrix}$, where $m = -n < 0$.

MA 106 : Mathematics II Tutorial Sheet No.2

Autumn 2016 AR

Topics: $\mathbb{R}^n$, subspaces, linear independence, rank of a matrix, solvability of linear systems using rank.

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1. Suppose that the *state of land use* in a city area in 2003 was

   1 (Residential) 30percent

   2 (Commercial) 20percent

   3 (Industrial) 50percent

   Estimate the states of land use in 2008, 2013, 2018, assuming that the transitional probabilities for 5-year intervals are given by the stochastic matrix $\begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.0 & 0.1 & 0.9 \end{bmatrix}$,where $(i, j)^{th}$ entry is the probability for the i^{th} type to change to the j^{th} type. (For e.g. 0.2 is the probability for commercially used land to become industrial in a 5-year interval.)

   Solution. Let R, C and I denote the residential, the commercial and the industrial usage respectively in percentages. The given stochastic matrix represents schematically, the probabilities of the transitions: $\begin{bmatrix} R \mapsto R & R \mapsto C & R \mapsto I \\ C \mapsto R & C \mapsto C & C \mapsto I \\ I \mapsto R & I \mapsto C & I \mapsto I \end{bmatrix}$. Clearly, after 5 years $[R\ C\ I] = [R_0\ C_0\ I_0]S$.

   Hence

$$\begin{aligned} [R\ C\ I]_{2008} &= [30\ 20\ 50] \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.0 & 0.1 & 0.9 \end{bmatrix} \\ &= [26\ 22\ 52] \end{aligned}$$

$$\begin{aligned} [R\ C\ I]_{2013} &= [26\ 22\ 52] \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.0 & 0.1 & 0.9 \end{bmatrix} \\ &= [23.0\ 23.2\ 53.8] \end{aligned}$$

$$\begin{aligned}[R\ C\ I]_{2018} &= [23.0\ 23.2\ 53.8]\begin{bmatrix}0.8 & 0.1 & 0.1\\ 0.1 & 0.7 & 0.2\\ 0.0 & 0.1 & 0.9\end{bmatrix}\\ &= [20.72\ 23.92\ 55.36]\end{aligned}$$

Remark 0.0.1. *Note that* $[R\ C\ I] = [12.5\ 25.0\ 62.5]$ *gives the equilibrium solution.*

2. Find whether the following sets of vectors are linearly dependent or independent:
(i) $[1,-1,1],[1,1,-1],[-1,1,1],[0,1,0]$.
(ii) $[1,9,9,8],[2,0,0,3],[2,0,0,8]$.

Solution. (i) $2[0,1,0] = [1,1,-1]+[-1,1,1]$ hence linearly dependent.

Aliter:

$$\begin{bmatrix}1 & -1 & 1\\ 1 & 1 & -1\\ -1 & 1 & 1\\ 0 & 1 & 0\end{bmatrix} \mapsto \begin{bmatrix}\boxed{1} & -1 & 1\\ 0 & \boxed{1} & -1\\ 0 & 0 & \boxed{1}\\ 0 & 0 & 0\end{bmatrix}$$

Hence dimension lin span $= 3 < 4 \Longrightarrow$ lin DEPendent.

Aliter-2:(changing to colulmns)

$$\begin{bmatrix}1 & 1 & -1 & 0\\ -1 & 1 & 1 & 1\\ 1 & -1 & 1 & 0\end{bmatrix} \mapsto \begin{bmatrix}\boxed{1} & 1 & -1 & 0\\ 0 & \boxed{2} & 0 & 1\\ 0 & 0 & \boxed{2} & 1\end{bmatrix}$$

Hence dimension lin span $= 3 < 4 \Longrightarrow$ lin DEPendent. Moreover, on dropping the fourth vector we get a linearly independent set, since the fourth column is pivot-free

(ii) $a[1,9,9,8]+b[2,0,0,3]+c[2,0,0,8] = [0,0,0,0]$

$\Longrightarrow a+2b+2c = 9a = 8a+3b+8c = 0 \Longrightarrow a = 0,\ b = 0,\ c = 0$. Hence linearly independent.

Aliter:(changing to colulmns)

$$\begin{bmatrix}1 & 2 & 2\\ 9 & 0 & 0\\ 9 & 0 & 0\\ 8 & 3 & 8\end{bmatrix} \mapsto \begin{bmatrix}\boxed{1} & 2 & 2\\ 0 & \boxed{1} & 1\\ 0 & 0 & \boxed{5}\\ 0 & 0 & 0\end{bmatrix}.$$

No. of pivots $= 3 =$ no. of vectors, hence lin. INDependent.

3. Find the ranks of the following matrices:

$$\text{(i)} \begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix}, \qquad \text{(ii)} \begin{bmatrix} m & n \\ n & m \\ p & p \end{bmatrix} (m^2 \neq n^2), \qquad \text{(iii)} \begin{bmatrix} 0 & 8 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 4 & 5 \end{bmatrix}.$$

Solution. (i) By row operations, $\begin{bmatrix} 8 & -4 \\ -2 & 1 \\ 6 & -3 \end{bmatrix} \mapsto \begin{bmatrix} 2 & -1 \\ -2 & 1 \\ 2 & -1 \end{bmatrix} \mapsto \begin{bmatrix} \boxed{2} & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. Rank is 1.

(ii) $\begin{bmatrix} m & n \\ n & m \\ p & p \end{bmatrix} \mapsto \begin{bmatrix} m+n & n+m \\ n & m \\ p & p \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 \\ n & m \\ 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} \boxed{1} & 1 \\ 0 & \boxed{m-n} \\ 0 & 0 \end{bmatrix}$. Rank is 2.

Remark 0.0.2. *We used the fact that $m \pm n \neq 0$.*

(iii) $\begin{bmatrix} 0 & 8 & -1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \\ 0 & 4 & 5 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & 0 \\ 0 & 8 & -1 \\ 0 & 4 & 5 \\ 0 & 0 & 3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -11 \\ 0 & 4 & 5 \\ 0 & 0 & 3 \end{bmatrix} \mapsto \begin{bmatrix} \boxed{1} & 2 & 0 \\ 0 & \boxed{4} & 5 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}$. Rank is 3.

4. Solve the following system of linear equations in the unknowns $x_1, \ldots, x_5$ by GEM:

$$\text{(i)} \quad \begin{array}{rrrrl} & 2x_3 & -2x_4 & +x_5 & = 2 \\ 2x_2 & -8x_3 & +14x_4 & -5x_5 & = 2 \\ x_2 & +3x_3 & & +x_5 & = \alpha \end{array} \qquad \text{(ii)} \quad \begin{array}{rrrrl} 2x_1 & -2x_2 & +x_3 & +x_4 & = 1 \\ & -2x_2 & +x_3 & +7x_4 & = 0 \\ 3x_1 & -x_2 & +4x_3 & -2x_4 & = -2 \end{array}$$

Solution. (i) Augmented matrix is $\left[\begin{array}{ccccc|c} 0 & 0 & 2 & -2 & 1 & 2 \\ 0 & 2 & -8 & 14 & -5 & 2 \\ 0 & 1 & 3 & 0 & 1 & \alpha \end{array}\right]$.

A REF is $\left[\begin{array}{ccccc|c} 0 & \boxed{1} & 3 & 0 & 1 & \alpha \\ 0 & 0 & \boxed{2} & -2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & \boxed{16-2\alpha} \end{array}\right]$. Hence no solutions if $\alpha \neq 8$.

For $\alpha = 8$, the REF is $\left[\begin{array}{ccccc|c} 0 & \boxed{1} & 3 & 0 & 1 & 8 \\ 0 & 0 & \boxed{2} & -2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$ and the solution set is $x_3 = 1 + x_4 - x_5/2,\ x_2 = 5 - 3x_4 + x_5/2;\ x_1, x_4, x_5$ being arbitrary.

(ii) Augmented matrix $\left[\begin{array}{cccc|c} 2 & -2 & 1 & 1 & 1 \\ 0 & -2 & 1 & 7 & 0 \\ 3 & -1 & 4 & -2 & -2 \end{array}\right]$.

A REF is $\left[\begin{array}{cccc|c} \boxed{2} & -2 & 1 & 1 & 1 \\ 0 & \boxed{-2} & 1 & 7 & 0 \\ 0 & 0 & \boxed{1} & 1 & -1 \end{array}\right]$. Hence the solution set is $x_3 = -x_4 - 1,\ x_2 = 3x_4 - 1/2;\ x_1 = 3x_4 + 1/2;\ x_4$ being arbitrary.

5. Determine the equilibrium solution $(D_1 = S_1,\ D_2 = S_2)$ of the two-commodity market with linear model $(D, S, P) =$ (demand, supply, price):

$$\begin{aligned} D_1 &= 40 - 2P_1 - P_2 & \qquad S_1 &= 4P_1 - P_2 + 4 \\ D_2 &= 16 + 5P_1 - 2P_2 & \qquad S_2 &= 3P_2 - 4. \end{aligned}$$

Solution. Equilibrium condition implies

$$\begin{aligned} 40 - 2P_1 - P_2 &= 4P_1 - P_2 + 4 \\ 16 + 5P_1 - 2P_2 &= 3P_2 - 4 \\ \implies & \quad P_1 = 6,\ P_2 = 5. \end{aligned}$$

6. For the following linear systems, find solvability by comparing the ranks of the coefficient matrix and the augmented matrix. Write down a basis for the solutions of the asociated homogeneous systems and hence describe the general solution of each of the systems.

(i)
$$\begin{array}{rrrrl} & & -2x_4 & +x_5 & = 2 \\ 2x_2 & -2x_3 & +14x_4 & -x_5 & = 2 \\ 2x_2 & +3x_3 & +13x_4 & +x_5 & = 3 \end{array}$$

(ii)
$$\begin{array}{rrrrl} 2x_1 & -2x_2 & +x_3 & +x_4 & = 1 \\ & -2x_2 & +x_3 & -x_4 & = 2 \\ x_1 & +x_2 & +2x_3 & -x_4 & = -2 \end{array}$$

Solution. (i) The augmented matrix $A^+ = \left[\begin{array}{ccccc|c} 0 & 0 & 0 & -2 & 1 & 2 \\ 0 & 2 & -2 & 14 & -1 & 2 \\ 0 & 2 & 3 & 13 & 1 & 3 \end{array}\right]$.

$$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & -2 & 1 & 2 \\ 0 & 2 & -2 & 14 & -1 & 2 \\ 0 & 2 & 3 & 13 & 1 & 3 \end{array}\right] \mapsto \left[\begin{array}{ccccc|c} 0 & 2 & 3 & 13 & 1 & 3 \\ 0 & 2 & -2 & 14 & -1 & 2 \\ 0 & 0 & 0 & -2 & 1 & 2 \end{array}\right]$$

$$\mapsto \left[\begin{array}{ccccc|c} 0 & \boxed{2} & 3 & 13 & 1 & 3 \\ 0 & 0 & \boxed{-5} & 1 & -2 & -1 \\ 0 & 0 & 0 & \boxed{-2} & 1 & 2 \end{array}\right].$$

Hence $\rho(A^+) = 3 = \rho(A)$ and the system is solvable. The homogeneous system has solutions $x_4 = 0.5x_5,\ x_3 = -0.3x_5,\ x_2 = -3.3x_5;\ x_1, x_5$ arbitrary. A basis of the homogeneous solutions is $\{[1,0,0,0,0]^T, [0,3.3,0.3,-0.5,-1]^T\}$. A particular solution of the given system is $[0,8,0,-1,0]^T$. The general solution is

$$[s, 8-33t, -3t, 5t-1, 10t]^T;\ s,t \text{ arbitrary.}$$

(ii) The augmented matrix $A^+ = \left[\begin{array}{cccc|c} 2 & -2 & 1 & 1 & 1 \\ 0 & -2 & 1 & -1 & 2 \\ 1 & 1 & 2 & -1 & -2 \end{array}\right]$.

$$\left[\begin{array}{cccc|c} 2 & -2 & 1 & 1 & 1 \\ 0 & -2 & 1 & -1 & 2 \\ 1 & 1 & 2 & -1 & -2 \end{array}\right] \mapsto \left[\begin{array}{cccc|c} 2 & 0 & 0 & 2 & -1 \\ 0 & -2 & 1 & -1 & 2 \\ 1 & 1 & 2 & -1 & -2 \end{array}\right]$$

$$\mapsto \left[\begin{array}{cccc|c} 2 & 0 & 0 & 2 & -1 \\ 0 & -2 & 1 & -1 & 2 \\ 0 & 2 & 4 & -4 & -1 \end{array}\right] \mapsto \left[\begin{array}{cccc|c} \boxed{2} & 0 & 0 & 2 & -1 \\ 0 & \boxed{-2} & 1 & -1 & 2 \\ 0 & 0 & \boxed{5} & -5 & 1 \end{array}\right].$$

Hence $\rho(A^+) = 3 = \rho(A)$ and the system is solvable. The homogeneous system has solutions $x_3 = x_4,\ x_2 = 0,\ x_1 = -x_4;\ x_4$ arbitrary. A basis of the homogeneous solutions is $\{[-1,0,1,1]^T\}$. A particular solution of the given system is $-[0.5, 1.1, 0.2, 0]^T$. The general solution is

$$\left[-t-\frac{1}{2},-\frac{11}{10},t-\frac{2}{5},t\right]^T \;;\; t \text{ arbitrary.}$$

7. Is the given set of vectors a vector space?

(i) All vectors $[v_1, v_2, v_3]^T$ in $\mathbb{R}^3$ such that $3v_1 - 2v_2 + v_3 = 0,\ 4v_1 + 5v_2 = 0$. (ii) All vectors in $\mathbb{R}^2$ with components less than 1 in absolute value.

Solution. (i) YES. (ii) NO.

8. For $a < b$, consider the system of equations:

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 1 \\ ax & + & by & + & 2z & = & 3 \\ a^2x & + & b^2y & + & 4z & = & 9. \end{array}$$

Find the pairs (a, b) for which the system has infinitely many solutions. Solution. For a square matrix A, $A\mathbf{x} = \mathbf{b}$ to have infinitely many solutions, a necessary (but not sufficient) condition is that $\det A = 0$.

$A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & 2 \\ a^2 & b^2 & 4 \end{bmatrix} \implies \det A = (b-a)(2-a)(2-b) = 0$. Hence $a = 2$ or $b = 2$ since $b - a \neq 0$.

Case 1:(a=2) The augmented matrix is

$$A^+ = \begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 2 & b & 2 & | & 3 \\ 4 & b^2 & 4 & | & 9. \end{bmatrix}$$

Elementary row operations give

$$A^+ \quad \mapsto \quad \begin{bmatrix} 1 & 1 & 1 & | & 1 \\ 0 & b-2 & 0 & | & 1 \\ 0 & b^2-4 & 0 & | & 5 \end{bmatrix}$$

$$\mapsto \quad \begin{bmatrix} \boxed{1} & 1 & 1 & | & 1 \\ 0 & \boxed{b-2} & 0 & | & 1 \\ 0 & 0 & 0 & | & \boxed{3-b} \end{bmatrix}$$

For existence $\rho(A^+) = \rho(A) \implies b = 3$. Thus $a = 2, b = 3$ will give infinitely many solutions.

<u>Case 2:</u>(b=2) Solving similarly, we find that $a = 3$ which is inadmissible since $a < b$.

9. Show that the row space of a matrix does not change by row operations. Show that the dimension of the column space is unchanged by row operations.

Proof:

(i)The new rows are linear combinations of previous rows and *vice versa.*

(ii) Suppose that $C_1, ..., C_n$ are the columns of a matrix. If an ERO is applied through a matrix E, then the new columns are $EC_1, ..., EC_n$. If $C_{j_1}, ..., C_{j_r}$ are lin. ind. then so are $EC_{j_1}, ..., EC_{j_r}$ and *vice versa* due to invertibility of E.

Department of Mathematics
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MA 106 : Mathematics II Tutorial Sheet No.3

Autumn 2016 AR

Topics: Determinants, ranks by determinants, Adjoints, Inverses, Cramer's rule.

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1. Find the rank by determinants. Verify by row reduction.

$$\text{(i)} \begin{bmatrix} 0 & 2 & -3 \\ 2 & 0 & 5 \\ -3 & 5 & 0 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 4 & 3 \\ -8 & -6 \\ 16 & 12 \end{bmatrix} \quad \text{(iii)} \begin{bmatrix} -2 & -\sqrt{3} & -\sqrt{2} \\ -1 & 0 & 1 \\ \sqrt{2} & \sqrt{3} & 2 \end{bmatrix}$$

Solution. (i) The 3×3 determinant $\begin{vmatrix} 0 & 2 & -3 \\ 2 & 0 & 5 \\ -3 & 5 & 0 \end{vmatrix} = -60$ implies that the rank is 3.

Row reduction is

$$\begin{bmatrix} 0 & 2 & -3 \\ 2 & 0 & 5 \\ -3 & 5 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 0 & 5 \\ 0 & 2 & -3 \\ -3 & 5 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 0 & 5 \\ 0 & 2 & -3 \\ 0 & 5 & 7.5 \end{bmatrix} \mapsto \begin{bmatrix} \boxed{2} & 0 & 5 \\ 0 & \boxed{2} & -3 \\ 0 & 0 & \boxed{15} \end{bmatrix}.$$

(ii) All 2×2 subdeterminants vanish and there are nonzero entries, hence the rank is 1.

Row reduction is $\begin{bmatrix} 4 & 3 \\ -8 & -6 \\ 16 & 12 \end{bmatrix} \mapsto \begin{bmatrix} \boxed{4} & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$

(iii) The 3×3 determinant vanishes, hence rank is < 3. The 2×2 subdeterminant $\begin{vmatrix} 0 & 1 \\ \sqrt{3} & 2 \end{vmatrix} = -\sqrt{3} \neq 0$, hence the rank is 2.

Row reduction is

$$\begin{bmatrix} -2 & -\sqrt{3} & -\sqrt{2} \\ -1 & 0 & 1 \\ \sqrt{2} & \sqrt{3} & 2 \end{bmatrix} \mapsto \begin{bmatrix} 0 & -\sqrt{3} & -\sqrt{2}-2 \\ -1 & 0 & 1 \\ 0 & \sqrt{3} & 2+\sqrt{2} \end{bmatrix} \mapsto \begin{bmatrix} \boxed{\text{-1}} & 0 & 1 \\ 0 & \boxed{\sqrt{3}} & 2+\sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

2. Find the values of β for which Cramer's rule is applicable. For the remaining value(s) of β, find the number of solutions.

$$\begin{array}{ccccccc} \text{x} & + & \text{2y} & + & \text{3z} & = & 20 \\ \text{x} & + & \text{3y} & + & \text{z} & = & 13 \\ \text{x} & + & \text{6y} & + & \beta\text{z} & = & \beta. \end{array}$$

Solution. The coefficient matrix has determinant $\beta+5$, hence for $\beta \neq -5$ Cramer's Rule is applicable.

For $\beta = -5$, the augmented matrix $A^{+} = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 1 & 3 & 1 & 13 \\ 1 & 6 & -5 & -5 \end{array}\right]$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 1 & 3 & 1 & 13 \\ 1 & 6 & -5 & -5 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 0 & 1 & -2 & -7 \\ 0 & 4 & -8 & -25 \end{array}\right]$$

$$\mapsto \left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 20 \\ 0 & \boxed{1} & -2 & -7 \\ 0 & 0 & 0 & \boxed{3} \end{array}\right].$$

The pivot in the last column means that $\rho(A) = 2 < 3 = \rho(A^{+})$. Hence the system is not solvable for $\beta = -5$.

3. Find whether the following set of vectors is linearly dependent or independent:

$$\{a\mathbf{i} + b\mathbf{j} + c\mathbf{k},\ b\mathbf{i} + c\mathbf{j} + a\mathbf{k},\ c\mathbf{i} + a\mathbf{j} + b\mathbf{k}\}.$$

Solution. Consider the matrix $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$. We have lin. ind. iff its rank is 3 i.e. its determinant is non zero. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - (a^3 + b^3 + c^3) = -\dfrac{(a+b+c)}{2}\Big[(a-b)^2 +$

$(b-c)^2+(c-a)^2\Big]$

Therefore, linear dependence if either $a+b+c=0$ or $a=b=c$. In the rest of the cases we have independence.

4. Consider the system of equations

$$\begin{aligned} x + \quad \lambda z &= \lambda - 1 \\ x + \lambda y \quad &= \lambda + 1 \\ \lambda x + \ y + 3z &= 2\lambda - 1(\text{ or } 1 - 2\lambda) \end{aligned}$$

Find the values of λ for which Cramer's rule can be used. For the remaining values of λ, discuss the solvability of the linear system.

Solution. The coefficient matrix is $A(\lambda) = \begin{bmatrix} 1 & 0 & \lambda \\ 1 & \lambda & 0 \\ \lambda & 1 & 3 \end{bmatrix}$.

$|A(\lambda)| = -\lambda(\lambda^2-4) \implies$ Cramer's Rule appplies if $\lambda \in \mathbb{R} \setminus \{0, \pm 2\}$. Hence we a a unique solution for these values of λ.

Case 1 ($\lambda = 0$): The augmented matrix is $A^+(0) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & -1 \end{array}\right]$ has rank 3 since the determinant of the first, second and fourth columns is non zero. While $A(0)$ is of rank 2. Hence no solutions in this case.

Case 2 ($\lambda = 2$): The augmented matrix is $A^+(2) = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 3 \\ 2 & 1 & 3 & 3 \end{array}\right]$ has rank 2 since the determinant of any three columns vanishes. ($A(2)$ is of rank 2.) Hence ∞ly many solutions in this case.

Case 3 ($\lambda = -2$): The augmented matrix is $A^+(-2) = \left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 1 & -2 & 0 & -1 \\ -2 & 1 & 3 & -5 \end{array}\right]$ has rank 3 since the determinant of the first, second and fourth columns is non zero. While $A(-2)$ is of rank 2. Hence no solutions in this case also.

If we replace $2\lambda+1$ by $1-2\lambda$ in the third equation, we find no solutions in case 1 and 2 and ∞ly many in case 3.

5. Find the matrices of minors, cofactors and the adjoint of the following matrices:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} 0 & 9 & 5 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & \sqrt{3} & \sqrt{2} \\ -1 & 0 & 1 \\ \sqrt{2} & \sqrt{3} & 2 \end{bmatrix}, \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.0 & 0.1 & 0.9 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solution.

- $\mathcal{M}(A) = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$, $\mathcal{C}(A) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$, $\mathrm{Adj}(A) = \mathcal{C}(A)^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
- $\mathcal{M} = \begin{bmatrix} 0 & 0 & 4 \\ -10 & 0 & 0 \\ 0 & -10 & -18 \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} 0 & 0 & 4 \\ 10 & 0 & 0 \\ 0 & 10 & -18 \end{bmatrix}$, $\mathrm{Adj} = \begin{bmatrix} 0 & 10 & 0 \\ 0 & 0 & 10 \\ 4 & 0 & -18 \end{bmatrix}$.
- $\begin{bmatrix} -\sqrt{3} & -(2+\sqrt{2}) & -\sqrt{3} \\ (2-\sqrt{2})\sqrt{3} & 2 & (2-\sqrt{2})\sqrt{3} \\ \sqrt{3} & (2+\sqrt{2}) & \sqrt{3} \end{bmatrix}$, $\begin{bmatrix} -\sqrt{3} & (2+\sqrt{2}) & -\sqrt{3} \\ (\sqrt{2}-2)\sqrt{3} & 2 & (\sqrt{2}-2)\sqrt{3} \\ \sqrt{3} & -(2+\sqrt{2}) & \sqrt{3} \end{bmatrix}$, $\begin{bmatrix} -\sqrt{3} & (\sqrt{2}-2)\sqrt{3} & \sqrt{3} \\ (2+\sqrt{2}) & 2 & -(2+\sqrt{2}) \\ -\sqrt{3} & (\sqrt{2}-2)\sqrt{3} & \sqrt{3} \end{bmatrix}$.
- $\mathcal{M} = \frac{1}{100}\begin{bmatrix} 61 & 9 & 1 \\ 8 & 72 & 8 \\ -5 & 15 & 55 \end{bmatrix}$, $\mathcal{C} = \frac{1}{100}\begin{bmatrix} 61 & -9 & 1 \\ -8 & 72 & -8 \\ -5 & -15 & 55 \end{bmatrix}$, $\mathrm{Adj} = \frac{1}{100}\begin{bmatrix} 61 & -8 & -5 \\ -9 & 72 & -15 \\ 1 & -8 & 55 \end{bmatrix}$.
- $\begin{bmatrix} -1 & 0 & 0 & 0 \\ -4 & -3 & 6 & 0 \\ -1 & 0 & 3 & 0 \\ 4 & 3 & -6 & -3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 4 & -3 & -6 & 0 \\ -1 & 0 & 3 & 0 \\ -4 & 3 & 6 & -3 \end{bmatrix} \begin{bmatrix} -1 & 4 & -1 & -4 \\ 0 & -3 & 0 & 3 \\ 0 & -6 & 3 & 6 \\ 0 & 0 & 0 & -3 \end{bmatrix}$ respectively.

6. In the previous problem verify that $A\mathrm{Adj}(A) = A\big(\mathrm{Adj}(A)\big) = \det A\mathbf{I}$. Hence compute the inverses in the valid cases.

Solution.

- $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
  Threfore $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ if $\det A = ad - bc \neq 0$.

- $$\begin{bmatrix} 0 & 9 & 5 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 10 & 0 \\ 0 & 0 & 10 \\ 4 & 0 & -18 \end{bmatrix} = 20\mathbf{I} = \begin{bmatrix} 0 & 10 & 0 \\ 0 & 0 & 10 \\ 4 & 0 & -18 \end{bmatrix}\begin{bmatrix} 0 & 9 & 5 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$
  Therefore, $A^{-1} = \dfrac{1}{20}\begin{bmatrix} 0 & 10 & 0 \\ 0 & 0 & 10 \\ 4 & 0 & -18 \end{bmatrix}$.

- $$\begin{bmatrix} 2 & \sqrt{3} & \sqrt{2} \\ -1 & 0 & 1 \\ \sqrt{2} & \sqrt{3} & 2 \end{bmatrix}\begin{bmatrix} -\sqrt{3} & (\sqrt{2}-2)\sqrt{3} & \sqrt{3} \\ (2+\sqrt{2}) & 2 & -(2+\sqrt{2}) \\ -\sqrt{3} & (\sqrt{2}-2)\sqrt{3} & \sqrt{3} \end{bmatrix}$$
  $$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} =$$
  $$\begin{bmatrix} -\sqrt{3} & (\sqrt{2}-2)\sqrt{3} & \sqrt{3} \\ (2+\sqrt{2}) & 2 & -(2+\sqrt{2}) \\ -\sqrt{3} & (\sqrt{2}-2)\sqrt{3} & \sqrt{3} \end{bmatrix}\begin{bmatrix} 2 & \sqrt{3} & \sqrt{2} \\ -1 & 0 & 1 \\ \sqrt{2} & \sqrt{3} & 2 \end{bmatrix}.$$
  $\det A = 0$ hence not invertible.

- $$\begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.0 & 0.1 & 0.9 \end{bmatrix}\frac{1}{100}\begin{bmatrix} 61 & -8 & -5 \\ -9 & 72 & -15 \\ 1 & -8 & 55 \end{bmatrix} = 0.48\mathbf{I} = \frac{1}{100}\begin{bmatrix} 61 & -8 & -5 \\ -9 & 72 & -15 \\ 1 & -8 & 55 \end{bmatrix}\begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.0 & 0.1 & 0.9 \end{bmatrix}.$$

Therefore, $A^{-1} = \dfrac{1}{48}\begin{bmatrix} 61 & -8 & -5 \\ -9 & 72 & -15 \\ 1 & -8 & 55 \end{bmatrix}$.

- $\begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 4 & -1 & -4 \\ 0 & -3 & 0 & 3 \\ 0 & -6 & 3 & 6 \\ 0 & 0 & 0 & -3 \end{bmatrix} = -3\mathbf{I} = \begin{bmatrix} -1 & 4 & -1 & -4 \\ 0 & -3 & 0 & 3 \\ 0 & -6 & 3 & 6 \\ 0 & 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

  $A^{-1} = \dfrac{1}{3}\begin{bmatrix} 1 & -4 & 1 & 4 \\ 0 & 3 & 0 & -3 \\ 0 & 6 & -3 & -6 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

7. Solve by Cramer's rule and verify by Gauss elimination.

$$\begin{array}{rcrcr} 5x & - & 3y & = & 37 \\ -2x & + & 7y & = & -38. \end{array}$$

Solution. $A^+ = \left[\begin{array}{cc|c} 5 & -3 & 37 \\ -2 & 7 & -38 \end{array}\right]$. $D = 29$, $D_1 = \begin{vmatrix} 37 & -3 \\ -38 & 7 \end{vmatrix} = 145$,

$D_2 = \begin{vmatrix} 5 & 37 \\ -2 & -38 \end{vmatrix} = -116 \Longrightarrow x = 5,\ y = -4$. OTOH row reduction gives

$\left[\begin{array}{cc|c} 5 & -3 & 37 \\ 0 & 5.8 & -23.2 \end{array}\right] \Longrightarrow y = -4,\ 5x + 12 = 37 \Longrightarrow x = 5$. Verified!

8. Solve by Cramer's rule and verify by Gauss elimination.

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 20 \\ 7x & + & 3y & + & z & = & 13 \\ x & + & 6y & + & 2z & = & 0. \end{array}$$

Solution. $A^+ = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 7 & 3 & 1 & 13 \\ 1 & 6 & 2 & 0 \end{array}\right]$. $D = \begin{vmatrix} 1 & 2 & 3 \\ 7 & 3 & 1 \\ 1 & 6 & 2 \end{vmatrix} = 91$,

$$D_1 = \begin{vmatrix} 20 & 2 & 3 \\ 13 & 3 & 1 \\ 0 & 6 & 2 \end{vmatrix} = 182,\ D_2 = \begin{vmatrix} 1 & 20 & 3 \\ 7 & 13 & 1 \\ 1 & 0 & 2 \end{vmatrix} = -273,\ D_3 = \begin{vmatrix} 1 & 2 & 20 \\ 7 & 3 & 13 \\ 1 & 6 & 0 \end{vmatrix} = 728.$$

Therefore, $x = 2,\ y = -3,\ z = 8.$

On the other hand row operations on A^+ yield $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 0 & 44 & 80 & 508 \\ 0 & 0 & 91 & 728 \end{array}\right]$. Consequently,

$z = 8,\ 44y + 640 = 508 \Longrightarrow y = -3,\ x + (-6) + 24 = 20 \Longrightarrow x = 2.$ Verified!

Working:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 7 & 3 & 1 & 13 \\ 1 & 6 & 2 & 0 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 0 & -11 & -20 & -127 \\ 0 & 4 & -1 & -20 \end{array}\right]$$

$$\mapsto \left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 0 & 44 & 80 & 508 \\ 0 & -44 & 11 & 220 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 0 & 44 & 80 & 508 \\ 0 & 0 & 91 & 728 \end{array}\right].$$

9. Invert the matrix $H = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$.

Solution. $\mathcal{M} = \begin{bmatrix} 1/240 & 1/60 & 1/72 \\ 1/60 & 4/45 & 1/12 \\ 1/72 & 1/12 & 1/12 \end{bmatrix},\ \mathcal{C} = \begin{bmatrix} 1/240 & -1/60 & 1/72 \\ -1/60 & 4/45 & -1/12 \\ 1/72 & -1/12 & 1/12 \end{bmatrix}$=Adj.

$$\det H = \frac{1}{240} - \frac{1}{120} + \frac{1}{216} = \frac{1}{2160} \Longrightarrow H^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}.$$

10. (Vandermonde determinant)

(a) Prove that $\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = (b-a)(c-a)(c-b)$.

Solution. Using column operations

$$\begin{aligned} \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} \\ &= (b-a)(c-a) \begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix} \\ &= (b-a)(c-a)(c-b). \end{aligned}$$

Aliter: The determinant is a polynomial of degree 3 in a, b, c and it vanishes whenever $a = b$ or $b = c$ or $c = a$. Hence equals $C(b-a)(c-a)(c-b)$. The coefficient of bc^2 is 1 by inspection of the diagonal entries, hence $C = 1$.

(b) Prove an analogous formula for $n \times n$.

Solution.

$$V(a_1, a_2, ..., a_n) \overset{def}{=} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{vmatrix}.$$

This time V is a polynomial in $(a_1, a_2, ..., a_n)$ of degree $0+1+\cdots+(n-1) = \dfrac{n(n-1)}{2}$ and vanishes whenever $a_j = a_k,\ j \neq k$. Therefore, $\prod_{1 \le j < k \le n} (a_k - a_j)$ must divide V. The degree count makes $V(a_1, a_2, ..., a_n) = C \prod_{1 \le j < k \le n} (a_k - a_j)$ and a look at the diagonal terms forces $C = 1$.

11. (Wronskian) Let $f_1, f_2, ..., f_n$ be functions over some interval (a, b). Their Wronskian is another function on (a, b) defined by the determinant involving the given functions and

their derivatives upto the order $n-1$.

$$W_{f_1,f_2,\ldots,f_n}(x) \overset{def}{=} \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

Prove that if $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$ holds over the interval (a,b) for some constants $c_1, c_2, \ldots, c_n$ and $W_{f_1,f_2,\ldots,f_n}(x_0) \neq 0$ at some x_0, then $c_1 = c_2 = \cdots = c_n = 0$. In other words, nonvanishing of $W_{f_1,f_2,\ldots,f_n}$ at a single point establishes linear independence of $f_1, f_2, \ldots, f_n$ on (a,b).

Caution: The converse is false. $W \equiv 0 \;\not\Rightarrow\; f_1, f_2, \ldots, f_n$ linearly dependent on (a,b). Though one can prove existence of a subinterval of (a,b) where linear dependence holds.

Solution. Starting with the equation $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$ and its first $n-1$ derivatives we get a system of equations

$$\begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad x \in (a,b).$$

If $W_{f_1,f_2,\ldots,f_n}(x_0) \neq 0$ for some $x_0 \in (a,b)$, then $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ as required.

Remark 0.0.3. *The converse is false in general. E.g. take*

$$f(x) = \begin{cases} 0, & x \leq 0 \\ e^{-1/x}, & x > 0 \end{cases} \qquad \textit{and} \quad g(x) = \begin{cases} e^{1/x}, & x < 0 \\ 0, & x \geq 0 \end{cases}.$$

Then $f,\ g$ are C^∞, linearly independent but $W(f,g) \equiv 0$. However, can take $a = 1,\ b = 0$ in the subinterval $(-\infty, 0)$ or $a = 0,\ b = 1$ in $(0, \infty)$.

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MA 106 : Mathematics II Tutorial Sheet No.4

Autumn 2016 AR

Topics: Expansion in a basis, orthogonal sets, orthonormal basis, Gram-Schmidt process, Bessel's inequality.

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1. (Resolution into orthogonal components) Let $\mathbf{u}$ be a nonzero vector and $\mathbf{v}$ be any other vector. Let $\mathbf{w} = \mathbf{v} - \frac{\langle \mathbf{v}, \mathbf{u}\rangle}{\|\mathbf{u}\|^2}\mathbf{u}$ as in Gram-Schmidt process. Then show that

$$\mathbf{v} = \frac{\langle \mathbf{v}, \mathbf{u}\rangle}{\|\mathbf{u}\|^2}\mathbf{u} + \mathbf{w}$$

is resolution of $\mathbf{v}$ into two components-one parallel to $\mathbf{u}$ and the other orthogonal to $\mathbf{u}$.

Solution. Clearly, the first term (summand) is parallel to $\mathbf{u}$. Hence must verify that $\langle \mathbf{w}, \mathbf{u}\rangle = 0$.

$$\begin{aligned}\langle \mathbf{w}, \mathbf{u}\rangle &= \langle \mathbf{v}, \mathbf{u}\rangle - \frac{\langle \mathbf{v}, \mathbf{u}\rangle}{\cancel{\|\mathbf{u}\|^2}}\cancel{\langle \mathbf{u}, \mathbf{u}\rangle} \\ &= 0.\end{aligned}$$

2. Verify that the set of vectors $\left\{\begin{bmatrix}2\\-2\\1\end{bmatrix}, \begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}2\\1\\-2\end{bmatrix}\right\}$ is an orthogonal set in $\mathbb{R}^3$. Is it a basis? If yes, express $\begin{bmatrix}1\\1\\1\end{bmatrix}$ as a linear combination of these vectors and verify that Bessel's inequaliy is an equality.

Solution. Orthogonality is easily verified. Now linear independence makes it a basis. The coefficients in the expansion are:

$$a = \frac{\mathbf{v}\cdot\mathbf{w}_1}{\|\mathbf{w}_1\|^2} = \frac{1}{9},\ b = \frac{\mathbf{v}\cdot\mathbf{w}_2}{\|\mathbf{w}_2\|^2} = \frac{5}{9},\ \frac{\mathbf{v}\cdot\mathbf{w}_3}{\|\mathbf{w}_3\|^2} = \frac{1}{9}.$$

Bessel's: $\|\mathbf{v}\|^2 = 3$, while $(\mathbf{v}\cdot\mathbf{u}_1)^2 = \frac{1}{9}$, $(\mathbf{v}\cdot\mathbf{u}_2)^2 = \frac{25}{9}$, $(\mathbf{v}\cdot\mathbf{u}_3)^2 = \frac{1}{9}$ and $3 = \frac{1}{9} + \frac{25}{9} + \frac{1}{9}$.

3. Orthogonalize the following set of row-vectors in $\mathbb{R}^4$.

$$\{[1,1,1,1],\ [1,1,-1,-1],\ [1,1,0,0],\ [-1,1,-1,1]\}$$

Do you get an orthogonal basis?

Solution.

$$\begin{aligned}
\mathbf{w}_1 = \mathbf{v}_1 &= [1,1,1,1] \\
\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2\cdot\mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 &= [1,1,-1,-1] \\
\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3\cdot\mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\mathbf{v}_3\cdot\mathbf{w}_2}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 &= [0,0,0,0] \\
\mathbf{w}_4 = \mathbf{v}_4 - \frac{\mathbf{v}_4\cdot\mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\mathbf{v}_4\cdot\mathbf{w}_2}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 &= [-1,1,-1,1].
\end{aligned}$$

Since only 3 nonzero orthogonal elements appear, it is NOT a basis of $\mathbb{R}^4$. (The linear span is 3-dimensional.)

4. Orthogonalize the following ordered set of row-vectors in $\mathbb{R}^4$.

$$\{[1,1,0,0],\ [1,0,1,0],\ [1,0,0,1],\ [0,1,1,0],\ [0,1,0,1],\ [0,0,1,1]\}$$

Do you get an orthogonal basis? Does $[-2,-1,1,2]$ belong to the linear span? Use Bessel's inequality.

Solution. After the Gram-Schmidt process we get

$$\{[1,1,0,0],\ \frac{1}{2}[1,-1,2,0],\ \frac{1}{3}[1,-1,-1,3],\ \frac{1}{2}[-1,1,1,1],\ \mathbf{0},\ \mathbf{0}\}.$$

The 4 nonzero elements give an orthogonal basis. $[-2,-1,1,2]$ must be in linear span. To verify Bessel's equality, we first note that

$$\mathbf{u}_1 = \frac{1}{\sqrt{2}}[1,1,0,0],\ \mathbf{u}_2 = \frac{1}{\sqrt{6}}[1,-1,2,0],\ \mathbf{u}_3 = \frac{1}{2\sqrt{3}}[1,-1,-1,3],\ \mathbf{u}_4 = \frac{1}{2}[-1,1,1,1].$$

Now $\sum(\mathbf{v}\cdot\mathbf{u}_j)^2 = \frac{9}{2} + \frac{1}{6} + \frac{4}{3} + 4 = 10 = \|\mathbf{v}\|^2$. (Verified!)

5. For the following linear homogeneous systems, write down orthogonal bases for the solution spaces.

(i)
$$\begin{array}{rrrrl} & & -2x_4 & +x_5 & =0 \\ 2x_2 & -2x_3 & +14x_4 & -x_5 & =0 \\ 2x_2 & +3x_3 & +13x_4 & +x_5 & =0 \end{array}$$

(ii)
$$\begin{array}{rrrrl} 2x_1 & -2x_2 & +x_3 & +x_4 & =0 \\ & -2x_2 & +x_3 & -x_4 & =0 \\ x_1 & +x_2 & +2x_3 & -x_4 & =0 \end{array}$$

(This is a variant of the problem 6 in Sheet No.2)

Solution. (i) The augmented matrix $A^+ = \left[\begin{array}{ccccc|c} 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 2 & -2 & 14 & -1 & 0 \\ 0 & 2 & 3 & 13 & 1 & 0 \end{array}\right]$.

$$\left[\begin{array}{ccccc|c} 0 & 0 & 0 & -2 & 1 & 0 \\ 0 & 2 & -2 & 14 & -1 & 0 \\ 0 & 2 & 3 & 13 & 1 & 0 \end{array}\right] \mapsto \left[\begin{array}{ccccc|c} 0 & 2 & 3 & 13 & 1 & 0 \\ 0 & 2 & -2 & 14 & -1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 \end{array}\right]$$
$$\mapsto \left[\begin{array}{ccccc|c} 0 & \boxed{2} & 3 & 13 & 1 & 0 \\ 0 & 0 & \boxed{-5} & 1 & -2 & 0 \\ 0 & 0 & 0 & \boxed{-2} & 1 & 0 \end{array}\right].$$

The homogeneous system has solutions $x_4 = 0.5x_5$, $x_3 = -0.3x_5$, $x_2 = -3.3x_5$; x_1, x_5 arbitrary. A basis of the homogeneous solutions is $\{[1,0,0,0,0]^T, [0,3.3,0.3,-0.5,-1]^T\}$, which is already orthogonal.

(ii) The augmented matrix $A^+ = \left[\begin{array}{cccc|c} 2 & -2 & 1 & 1 & 0 \\ 0 & -2 & 1 & -1 & 0 \\ 1 & 1 & 2 & -1 & 0 \end{array}\right]$.

$$\left[\begin{array}{cccc|c} 2 & -2 & 1 & 1 & 0 \\ 0 & -2 & 1 & -1 & 0 \\ 1 & 1 & 2 & -1 & 0 \end{array}\right] \mapsto \left[\begin{array}{cccc|c} 2 & 0 & 0 & 2 & 0 \\ 0 & -2 & 1 & -1 & 0 \\ 1 & 1 & 2 & -1 & 0 \end{array}\right]$$
$$\mapsto \left[\begin{array}{cccc|c} 2 & 0 & 0 & 2 & 0 \\ 0 & -2 & 1 & -1 & 0 \\ 0 & 2 & 4 & -4 & 0 \end{array}\right] \mapsto \left[\begin{array}{cccc|c} \boxed{2} & 0 & 0 & 2 & 0 \\ 0 & \boxed{-2} & 1 & -1 & 0 \\ 0 & 0 & \boxed{5} & -5 & 0 \end{array}\right].$$

The homogeneous system has solutions $x_3 = x_4$, $x_2 = 0$, $x_1 = -x_4$; x_4 arbitrary.

A basis of the homogeneous solutions is $\{[-1,0,1,1]^T\}$. A singleton set is vacuously orthogonal.

6. For the following linear homogeneous systems, write down orthogonal bases for the solution spaces.

(i)
$$\begin{array}{rrrrl} & 2x_3 & -2x_4 & +x_5 & =0 \\ 2x_2 & -8x_3 & +14x_4 & -5x_5 & =0 \\ x_2 & +3x_3 & & +x_5 & =0 \end{array}$$

(ii)
$$\begin{array}{rrrrl} 2x_1 & -2x_2 & +x_3 & +x_4 & =0 \\ & -2x_2 & +x_3 & +7x_4 & =0 \\ 3x_1 & -x_2 & +4x_3 & -2x_4 & =0 \end{array}$$

Solution. (i)

Augmented matrix is $\left[\begin{array}{rrrrr|r} 0 & 0 & 2 & -2 & 1 & 0 \\ 0 & 2 & -8 & 14 & -5 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \end{array}\right]$.

A REF is $\left[\begin{array}{rrrrr|r} 0 & \boxed{1} & 3 & 0 & 1 & 0 \\ 0 & 0 & \boxed{2} & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$. The solution set is

$x_3 = x_4 - x_5/2,\ x_2 = -3x_4 + x_5/2;\ x_1, x_4, x_5$ being arbitrary.

A basis is $\{[1,0,0,0,0]^T,\ [0,-3,1,1,0]^T,\ [0,1,-1,0,2]^T\}$. An orthogonal basis is

$$\{[1,0,0,0,0]^T,\ [0,-3,1,1,0]^T,\ [0,-1,-7,4,22]^T\}.$$

(ii) Augmented matrix $\left[\begin{array}{rrrr|r} 2 & -2 & 1 & 1 & 0 \\ 0 & -2 & 1 & 7 & 0 \\ 3 & -1 & 4 & -2 & 0 \end{array}\right]$.

A REF is $\left[\begin{array}{rrrr|r} \boxed{2} & -2 & 1 & 1 & 0 \\ 0 & \boxed{-2} & 1 & 7 & 0 \\ 0 & 0 & \boxed{1} & 1 & 0 \end{array}\right]$. Hence the solution set is

$x_3 = -x_4,\ x_2 = 3x_4;\ x_1 = 3x_4;\ x_4$ being arbitrary. A basis is $\{[3,3,-1,1]\}$ which is vacuously orthogonal.

7. Find whether the following sets of vectors are linearly dependent or independent by orthogonalizing them

(i) $[1,-1,1],\ [1,1,-1],\ [-1,1,1],\ [0,1,0]$.

(ii) $[2,0,0,3],\ [2,0,0,8],\ [2,0,1,3]$.

Solution. (i) Orthogonalization yields: $\{[1,-1,1],\ [2,1,-1],\ [0,1,1],\ [0,0,0]\}$. Hence l.d. as $\mathbf{0}$ occurs.

(ii) Orthogonalization yields: $\{[2,0,0,3],\ [-3,0,0,2],\ [0,0,1,0]\}$. Hence linearly independent. (We scaled $\mathbf{w}_2$ for covvenience by $13/10$.)

8. Orthonormalize the ordered set in $\mathbb{C}^5$:

$$\{[1,\imath,0,0,0],[0,1,\imath,0,0],[0,0,1,\imath,0],[0,0,0,1,\imath]\}$$

relative to the unitary inner product $\langle \mathbf{v},\mathbf{w}\rangle = \mathbf{w}^*\mathbf{v} = \sum_{j=1}^{5} v_j \bar{w}_j$.

Use Bessel's inequality to find whether $[1,\imath,1,\imath,1]$ is in the (complex) linear span of the above set or not.

Solution.

$$\begin{aligned}
\mathbf{w}_1 = \mathbf{v}_1 &= [1,\imath,0,0,0] \\
\mathbf{w}_2 = [0,1,\imath,0,0] - \frac{-\imath}{2}[1,\imath,0,0,0] &= \frac{1}{2}[\imath,1,2\imath,0,0] \\
\text{Similarly, } \mathbf{w}_3 &= \frac{1}{3}[-1,\imath,1,3\imath,0] \\
\text{and } \mathbf{w}_4 &= \frac{1}{4}[-\imath,-1,\imath,1,4\imath].
\end{aligned}$$

The corresponding orthonormal (unitary) set is

$$\left\{\frac{1}{\sqrt{2}}[1,\imath,0,0,0],\ \frac{1}{\sqrt{6}}[\imath,1,2\imath,0,0],\ \frac{1}{2\sqrt{3}}[-1,\imath,1,3\imath,0],\ \frac{1}{2\sqrt{5}}[-\imath,-1,\imath,1,4\imath]\right\}.$$

$\|[1,\imath,1,\imath,1]\|^2 = 5$. OTOH,

$$\begin{aligned}
|\langle \mathbf{v},\mathbf{u}_1\rangle|^2 &= 2, & |\langle \mathbf{v},\mathbf{u}_2\rangle|^2 &= 2/3, \\
|\langle \mathbf{v},\mathbf{u}_3\rangle|^2 &= 4/3, & |\langle \mathbf{v},\mathbf{u}_4\rangle|^2 &= 4/5.
\end{aligned}$$

The sum is $24/5 < 5$, hence can not be in the linear span.

9. Let $S = \{\mathbf{v}_1,\ \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be any ordered set in $\mathbb{R}^n$ and $T = \{\mathbf{w}_1,\ \mathbf{w}_2, \ldots, \mathbf{w}_k\}$ be the set resulting from the the Gram-Schmidt process applied to S. Prove that for $j = 1,2,\ldots,k$, the linear span of $\{\mathbf{v}_1,\ \mathbf{v}_2,\ldots,\mathbf{v}_j\}$ equals that of $\{\mathbf{w}_1,\ \mathbf{w}_2,\ldots,\mathbf{w}_j\}$. Conclude that $\mathbf{w}_j$ is orthogonal to $\mathbf{v}_1,\ldots,\mathbf{v}_{j-1}$.

Hint: Use induction on j.

10. Let $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ be a linearly independent ordered set in $\mathbb{R}^3$. Let $\{\mathbf{x}(=\mathbf{p}),\ \mathbf{y},\ \mathbf{z}\}$ be the orthogonal set as a result of Gram-Schmidt process. Show that $\mathbf{z}$ must be a scalar multiple of $\mathbf{p} \times \mathbf{q}$.

    Solution. By the previous theorem, plane of $\{\mathbf{x}, \mathbf{y}\}$ is the plane of $\{\mathbf{p}, \mathbf{q}\}$. Hence $\mathbf{z}$ being orthogonal to this plane must be a scalar multiple of $\mathbf{p} \times \mathbf{q}$.

11. For two <u>nonzero</u> vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, we define the angle between them by $\theta = \cos^{-1}\left[\frac{\langle \mathbf{v}, \mathbf{w}\rangle}{\|\mathbf{v}\|\|\mathbf{w}\|}\right]$.

    Note that by Cauchy-Schwartz inequality $\frac{\langle \mathbf{v}, \mathbf{w}\rangle}{\|\mathbf{v}\|\|\mathbf{w}\|} \in [-1, 1] \Longrightarrow \theta \in [0, \pi]$. Moreover, $\mathbf{v} \perp \mathbf{w}$ (orthogonal) if and only if $\theta = \frac{\pi}{2}$ as expected.

    Show that

    (i) $\theta = 0$ if and only if $\mathbf{w} = \frac{\|\mathbf{w}\|}{\|\mathbf{v}\|}\mathbf{v}$ is a positive scalar multiple of $\mathbf{v}$ (parallel).

    (ii) $\theta = \pi$ if and only if $\mathbf{w} = -\frac{\|\mathbf{w}\|}{\|\mathbf{v}\|}\mathbf{v}$ is a negative scalar multiple of $\mathbf{v}$ (anti-parallel).

    (iii) $$\|\mathbf{v} + \mathbf{w}\|^{\mathbf{2}} = \|\mathbf{v}\|^{\mathbf{2}} + \|\mathbf{w}\|^{\mathbf{2}} + 2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta.$$

    Solution.

    (i) $(\Longrightarrow)$: $\theta = 0 \Longrightarrow \langle \mathbf{v}, \mathbf{w}\rangle = \|\mathbf{v}\|\|\mathbf{w}\| \Longrightarrow \lambda = \frac{\|\mathbf{w}\|}{\|\mathbf{v}\|} > 0$ in $\mathbf{w} = \lambda\mathbf{v}$.

    $(\Longleftarrow)$: $\mathbf{w} = \frac{\|\mathbf{w}\|}{\|\mathbf{v}\|}\mathbf{v} \Longrightarrow \langle \mathbf{v}, \mathbf{w}\rangle = \|\mathbf{w}\|\|\mathbf{v}\| \Longrightarrow \cos\theta = 1 \Longrightarrow \theta = 0$.

    (ii) Similar to (i).

    (iii)

$$\begin{aligned}
\|\mathbf{v} + \mathbf{w}\|^2 &= \langle \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w}\rangle \\
&= \langle \mathbf{v}, \mathbf{v}\rangle + \langle \mathbf{w}, \mathbf{w}\rangle + 2\langle \mathbf{v}, \mathbf{w}\rangle \\
&= \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 + 2\|\mathbf{v}\|\|\mathbf{w}\|\cos\theta.
\end{aligned}$$