Department of Mathematics
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MA 106 : Mathematics II Tutorial Sheet No.3

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Topics: Determinants, ranks by determinants, Adjoints, Inverses, Cramer's rule.

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1. Find the rank by determinants. Verify by row reduction.

$$\text{(i)} \begin{bmatrix} 0 & 2 & -3 \\ 2 & 0 & 5 \\ -3 & 5 & 0 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 4 & 3 \\ -8 & -6 \\ 16 & 12 \end{bmatrix} \quad \text{(iii)} \begin{bmatrix} -2 & -\sqrt{3} & -\sqrt{2} \\ -1 & 0 & 1 \\ \sqrt{2} & \sqrt{3} & 2 \end{bmatrix}$$

Solution. (i) The 3×3 determinant $\begin{vmatrix} 0 & 2 & -3 \\ 2 & 0 & 5 \\ -3 & 5 & 0 \end{vmatrix} = -60$ implies that the rank is 3.

Row reduction is

$$\begin{bmatrix} 0 & 2 & -3 \\ 2 & 0 & 5 \\ -3 & 5 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 0 & 5 \\ 0 & 2 & -3 \\ -3 & 5 & 0 \end{bmatrix} \mapsto \begin{bmatrix} 2 & 0 & 5 \\ 0 & 2 & -3 \\ 0 & 5 & 7.5 \end{bmatrix} \mapsto \begin{bmatrix} \boxed{2} & 0 & 5 \\ 0 & \boxed{2} & -3 \\ 0 & 0 & \boxed{15} \end{bmatrix}.$$

(ii) All 2×2 subdeterminants vanish and there are nonzero entries, hence the rank is 1.

Row reduction is $\begin{bmatrix} 4 & 3 \\ -8 & -6 \\ 16 & 12 \end{bmatrix} \mapsto \begin{bmatrix} \boxed{4} & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$

(iii) The 3×3 determinant vanishes, hence rank is < 3. The 2×2 subdeterminant $\begin{vmatrix} 0 & 1 \\ \sqrt{3} & 2 \end{vmatrix} = -\sqrt{3} \neq 0$, hence the rank is 2.

Row reduction is

$$\begin{bmatrix} -2 & -\sqrt{3} & -\sqrt{2} \\ -1 & 0 & 1 \\ \sqrt{2} & \sqrt{3} & 2 \end{bmatrix} \mapsto \begin{bmatrix} 0 & -\sqrt{3} & -\sqrt{2}-2 \\ -1 & 0 & 1 \\ 0 & \sqrt{3} & 2+\sqrt{2} \end{bmatrix} \mapsto \begin{bmatrix} \boxed{-1} & 0 & 1 \\ 0 & \boxed{\sqrt{3}} & 2+\sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

2. Find the values of β for which Cramer's rule is applicable. For the remaining value(s) of β, find the number of solutions.

$$\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 20 \\ x & + & 3y & + & z & = & 13 \\ x & + & 6y & + & \beta z & = & \beta. \end{array}$$

Solution. The coefficient matrix has determinant $\beta+5$, hence for $\beta \neq -5$ Cramer's Rule is applicable.

For $\beta = -5$, the augmented matrix $A^+ = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 1 & 3 & 1 & 13 \\ 1 & 6 & -5 & -5 \end{array}\right]$.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 1 & 3 & 1 & 13 \\ 1 & 6 & -5 & -5 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 0 & 1 & -2 & -7 \\ 0 & 4 & -8 & -25 \end{array}\right]$$

$$\mapsto \left[\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 20 \\ 0 & \boxed{1} & -2 & -7 \\ 0 & 0 & 0 & \boxed{3} \end{array}\right].$$

The pivot in the last column means that $\rho(A) = 2 < 3 = \rho(A^+)$. Hence the system is not solvable for $\beta = -5$.

3. Find whether the following set of vectors is linearly dependent or independent:

$$\{a\mathbf{i} + b\mathbf{j} + c\mathbf{k},\ b\mathbf{i} + c\mathbf{j} + a\mathbf{k},\ c\mathbf{i} + a\mathbf{j} + b\mathbf{k}\}.$$

Solution. Consider the matrix $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$. We have lin. ind. iff its rank is 3 i.e. its determinant is non zero. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - (a^3 + b^3 + c^3) = -\frac{(a+b+c)}{2}\Big[(a-b)^2 +$

$(b-c)^2+(c-a)^2\Big]$

Therefore, linear dependence if either $a+b+c=0$ or $a=b=c$. In the rest of the cases we have independence.

4. Consider the system of equations

$$\begin{aligned} x + \quad\ \lambda z &= \lambda - 1 \\ x + \lambda y \quad\ &= \lambda + 1 \\ \lambda x + \ y + 3z &= 2\lambda - 1(\text{ or } 1-2\lambda) \end{aligned}$$

Find the values of λ for which Cramer's rule can be used. For the remaining values of λ, discuss the solvability of the linear system.

Solution. The coefficient matrix is $A(\lambda) = \begin{bmatrix} 1 & 0 & \lambda \\ 1 & \lambda & 0 \\ \lambda & 1 & 3 \end{bmatrix}$.

$|A(\lambda)| = -\lambda(\lambda^2-4) \implies$ Cramer's Rule appplies if $\lambda \in \mathbb{R} \setminus \{0, \pm 2\}$. Hence we a a unique solution for these values of λ.

Case 1 ($\lambda = 0$): The augmented matrix is $A^+(0) = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & -1 \end{array}\right]$ has rank 3 since the determinant of the first, second and fourth columns is non zero. While $A(0)$ is of rank 2. Hence no solutions in this case.

Case 2 ($\lambda = 2$): The augmented matrix is $A^+(2) = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 3 \\ 2 & 1 & 3 & 3 \end{array}\right]$ has rank 2 since the determinant of any three columns vanishes. ($A(2)$ is of rank 2.) Hence ∞ly many solutions in this case.

Case 3 ($\lambda = -2$): The augmented matrix is $A^+(-2) = \left[\begin{array}{ccc|c} 1 & 0 & -2 & -3 \\ 1 & -2 & 0 & -1 \\ -2 & 1 & 3 & -5 \end{array}\right]$ has rank 3 since the determinant of the first, second and fourth columns is non zero. While $A(-2)$ is of rank 2. Hence no solutions in this case also.

If we replace $2\lambda+1$ by $1-2\lambda$ in the third equation, we find no solutions in case 1 and 2 and ∞ly many in case 3.

5. Find the matrices of minors, cofactors and the adjoint of the following matrices:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} 0 & 9 & 5 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & \sqrt{3} & \sqrt{2} \\ -1 & 0 & 1 \\ \sqrt{2} & \sqrt{3} & 2 \end{bmatrix}, \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.0 & 0.1 & 0.9 \end{bmatrix}, \begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solution.

- $\mathcal{M}(A) = \begin{bmatrix} d & c \\ b & a \end{bmatrix}$, $\mathcal{C}(A) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$, $\mathrm{Adj}(A) = \mathcal{C}(A)^T = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
- $\mathcal{M} = \begin{bmatrix} 0 & 0 & 4 \\ -10 & 0 & 0 \\ 0 & -10 & -18 \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} 0 & 0 & 4 \\ 10 & 0 & 0 \\ 0 & 10 & -18 \end{bmatrix}$, $\mathrm{Adj} = \begin{bmatrix} 0 & 10 & 0 \\ 0 & 0 & 10 \\ 4 & 0 & -18 \end{bmatrix}$.
- $\begin{bmatrix} -\sqrt{3} & -(2+\sqrt{2}) & -\sqrt{3} \\ (2-\sqrt{2})\sqrt{3} & 2 & (2-\sqrt{2})\sqrt{3} \\ \sqrt{3} & (2+\sqrt{2}) & \sqrt{3} \end{bmatrix}$, $\begin{bmatrix} -\sqrt{3} & (2+\sqrt{2}) & -\sqrt{3} \\ (\sqrt{2}-2)\sqrt{3} & 2 & (\sqrt{2}-2)\sqrt{3} \\ \sqrt{3} & -(2+\sqrt{2}) & \sqrt{3} \end{bmatrix}$, $\begin{bmatrix} -\sqrt{3} & (\sqrt{2}-2)\sqrt{3} & \sqrt{3} \\ (2+\sqrt{2}) & 2 & -(2+\sqrt{2}) \\ -\sqrt{3} & (\sqrt{2}-2)\sqrt{3} & \sqrt{3} \end{bmatrix}$.
- $\mathcal{M} = \frac{1}{100}\begin{bmatrix} 61 & 9 & 1 \\ 8 & 72 & 8 \\ -5 & 15 & 55 \end{bmatrix}$, $\mathcal{C} = \frac{1}{100}\begin{bmatrix} 61 & -9 & 1 \\ -8 & 72 & -8 \\ -5 & -15 & 55 \end{bmatrix}$, $\mathrm{Adj} = \frac{1}{100}\begin{bmatrix} 61 & -8 & -5 \\ -9 & 72 & -15 \\ 1 & -8 & 55 \end{bmatrix}$.
- $\begin{bmatrix} -1 & 0 & 0 & 0 \\ -4 & -3 & 6 & 0 \\ -1 & 0 & 3 & 0 \\ 4 & 3 & -6 & -3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 4 & -3 & -6 & 0 \\ -1 & 0 & 3 & 0 \\ -4 & 3 & 6 & -3 \end{bmatrix} \begin{bmatrix} -1 & 4 & -1 & -4 \\ 0 & -3 & 0 & 3 \\ 0 & -6 & 3 & 6 \\ 0 & 0 & 0 & -3 \end{bmatrix}$ respectively.

6. In the previous problem verify that $A\mathrm{Adj}(A) = A\big(\mathrm{Adj}(A)\big) = \det A\mathbf{I}$. Hence compute the inverses in the valid cases.

Solution.

- $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

  Threfore $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ if $\det A = ad-bc \neq 0$.

- $\begin{bmatrix} 0 & 9 & 5 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 10 & 0 \\ 0 & 0 & 10 \\ 4 & 0 & -18 \end{bmatrix} = 20\mathbf{I} = \begin{bmatrix} 0 & 10 & 0 \\ 0 & 0 & 10 \\ 4 & 0 & -18 \end{bmatrix} \begin{bmatrix} 0 & 9 & 5 \\ 2 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$

  Therefore, $A^{-1} = \frac{1}{20}\begin{bmatrix} 0 & 10 & 0 \\ 0 & 0 & 10 \\ 4 & 0 & -18 \end{bmatrix}$.

- $$\begin{bmatrix} 2 & \sqrt{3} & \sqrt{2} \\ -1 & 0 & 1 \\ \sqrt{2} & \sqrt{3} & 2 \end{bmatrix} \begin{bmatrix} -\sqrt{3} & (\sqrt{2}-2)\sqrt{3} & \sqrt{3} \\ (2+\sqrt{2}) & 2 & -(2+\sqrt{2}) \\ -\sqrt{3} & (\sqrt{2}-2)\sqrt{3} & \sqrt{3} \end{bmatrix}$$
  $$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} =$$
  $$\begin{bmatrix} -\sqrt{3} & (\sqrt{2}-2)\sqrt{3} & \sqrt{3} \\ (2+\sqrt{2}) & 2 & -(2+\sqrt{2}) \\ -\sqrt{3} & (\sqrt{2}-2)\sqrt{3} & \sqrt{3} \end{bmatrix} \begin{bmatrix} 2 & \sqrt{3} & \sqrt{2} \\ -1 & 0 & 1 \\ \sqrt{2} & \sqrt{3} & 2 \end{bmatrix}.$$

  $\det A = 0$ hence not invertible.

- $\begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.0 & 0.1 & 0.9 \end{bmatrix} \frac{1}{100}\begin{bmatrix} 61 & -8 & -5 \\ -9 & 72 & -15 \\ 1 & -8 & 55 \end{bmatrix} = 0.48\mathbf{I} = \frac{1}{100}\begin{bmatrix} 61 & -8 & -5 \\ -9 & 72 & -15 \\ 1 & -8 & 55 \end{bmatrix} \begin{bmatrix} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.0 & 0.1 & 0.9 \end{bmatrix}.$

Therefore, $A^{-1} = \dfrac{1}{48}\begin{bmatrix} 61 & -8 & -5 \\ -9 & 72 & -15 \\ 1 & -8 & 55 \end{bmatrix}$.

- $\begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 4 & -1 & -4 \\ 0 & -3 & 0 & 3 \\ 0 & -6 & 3 & 6 \\ 0 & 0 & 0 & -3 \end{bmatrix} = -3\mathbf{I} = \begin{bmatrix} -1 & 4 & -1 & -4 \\ 0 & -3 & 0 & 3 \\ 0 & -6 & 3 & 6 \\ 0 & 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

  $A^{-1} = \dfrac{1}{3}\begin{bmatrix} 1 & -4 & 1 & 4 \\ 0 & 3 & 0 & -3 \\ 0 & 6 & -3 & -6 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

7. Solve by Cramer's rule and verify by Gauss elimination.

$$\begin{array}{rcrcr} 5x & - & 3y & = & 37 \\ -2x & + & 7y & = & -38. \end{array}$$

Solution. $A^{+} = \left[\begin{array}{cc|c} 5 & -3 & 37 \\ -2 & 7 & -38 \end{array}\right]$. $D = 29,\ D_1 = \begin{vmatrix} 37 & -3 \\ -38 & 7 \end{vmatrix} = 145,$

$D_2 = \begin{vmatrix} 5 & 37 \\ -2 & -38 \end{vmatrix} = -116 \Longrightarrow x = 5,\ y = -4.$ OTOH row reduction gives

$\left[\begin{array}{cc|c} 5 & -3 & 37 \\ 0 & 5.8 & -23.2 \end{array}\right] \Longrightarrow y = -4,\ 5x + 12 = 37 \Longrightarrow x = 5.$ Verified!

8. Solve by Cramer's rule and verify by Gauss elimination.

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 20 \\ 7x & + & 3y & + & z & = & 13 \\ x & + & 6y & + & 2z & = & 0. \end{array}$$

Solution. $A^{+} = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 7 & 3 & 1 & 13 \\ 1 & 6 & 2 & 0 \end{array}\right]$. $D = \begin{vmatrix} 1 & 2 & 3 \\ 7 & 3 & 1 \\ 1 & 6 & 2 \end{vmatrix} = 91,$

$$D_1 = \begin{vmatrix} 20 & 2 & 3 \\ 13 & 3 & 1 \\ 0 & 6 & 2 \end{vmatrix} = 182,\ D_2 = \begin{vmatrix} 1 & 20 & 3 \\ 7 & 13 & 1 \\ 1 & 0 & 2 \end{vmatrix} = -273,\ D_3 = \begin{vmatrix} 1 & 2 & 20 \\ 7 & 3 & 13 \\ 1 & 6 & 0 \end{vmatrix} = 728.$$

Therefore, $x = 2,\ y = -3,\ z = 8$.

On the other hand row operations on A^+ yield $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 0 & 44 & 80 & 508 \\ 0 & 0 & 91 & 728 \end{array}\right]$. Consequently,

$z = 8,\ 44y + 640 = 508 \implies y = -3,\ x + (-6) + 24 = 20 \implies x = 2$. Verified!

Working:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 7 & 3 & 1 & 13 \\ 1 & 6 & 2 & 0 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 0 & -11 & -20 & -127 \\ 0 & 4 & -1 & -20 \end{array}\right]$$

$$\mapsto \left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 0 & 44 & 80 & 508 \\ 0 & -44 & 11 & 220 \end{array}\right] \mapsto \left[\begin{array}{ccc|c} 1 & 2 & 3 & 20 \\ 0 & 44 & 80 & 508 \\ 0 & 0 & 91 & 728 \end{array}\right].$$

9. Invert the matrix $H = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$.

Solution. $\mathcal{M} = \begin{bmatrix} 1/240 & 1/60 & 1/72 \\ 1/60 & 4/45 & 1/12 \\ 1/72 & 1/12 & 1/12 \end{bmatrix}$, $\mathcal{C} = \begin{bmatrix} 1/240 & -1/60 & 1/72 \\ -1/60 & 4/45 & -1/12 \\ 1/72 & -1/12 & 1/12 \end{bmatrix}$=Adj.

$$\det H = \frac{1}{240} - \frac{1}{120} + \frac{1}{216} = \frac{1}{2160} \implies H^{-1} = \begin{bmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{bmatrix}.$$

10. (Vandermonde determinant)

(a) Prove that $\det \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = (b-a)(c-a)(c-b)$.

Solution. Using column operations

$$\begin{aligned} \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} \\ &= (b-a)(c-a) \begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix} \\ &= (b-a)(c-a)(c-b). \end{aligned}$$

Aliter: The determinant is a polynomial of degree 3 in a, b, c and it vanishes whenever $a = b$ or $b = c$ or $c = a$. Hence equals $C(b-a)(c-a)(c-b)$. The coefficient of bc^2 is 1 by inspection of the diagonal entries, hence $C = 1$.

(b) Prove an analogous formula for $n \times n$.

Solution.

$$V(a_1, a_2, ..., a_n) \overset{def}{=} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ \vdots & \vdots & & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{vmatrix}.$$

This time V is a polynomial in $(a_1, a_2, ..., a_n)$ of degree $0+1+\cdots+(n-1) = \dfrac{n(n-1)}{2}$ and vanishes whenever $a_j = a_k,\ j \neq k$. Therefore, $\displaystyle\prod_{1 \le j < k \le n} (a_k - a_j)$ must divide V. The degree count makes $V(a_1, a_2, ..., a_n) = C \displaystyle\prod_{1 \le j < k \le n} (a_k - a_j)$ and a look at the diagonal terms forces $C = 1$.

11. (Wronskian) Let $f_1, f_2, ..., f_n$ be functions over some interval (a, b). Their Wronskian is another function on (a, b) defined by the determinant involving the given functions and

their derivatives upto the order $n-1$.

$$W_{f_1,f_2,\ldots,f_n}(x) \stackrel{def}{=} \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}.$$

Prove that if $c_1f_1 + c_2f_2 + \cdots + c_nf_n = 0$ holds over the interval (a,b) for some constants $c_1, c_2, ..., c_n$ and $W_{f_1,f_2,\ldots,f_n}(x_0) \neq 0$ at some x_0, then $c_1 = c_2 = \cdots = c_n = 0$. In other words, nonvanishing of $W_{f_1,f_2,\ldots,f_n}$ at a single point establishes linear independence of $f_1, f_2, ..., f_n$ on (a,b).

Caution: The converse is false. $W \equiv 0 \not\Rightarrow f_1, f_2, ..., f_n$ linearly dependent on (a,b). Though one can prove existence of a subinterval of (a,b) where linear dependence holds.

Solution. Starting with the equation $c_1f_1 + c_2f_2 + \cdots + c_nf_n = 0$ and its first $n-1$ derivatives we get a system of equations

$$\begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad x \in (a,b).$$

If $W_{f_1,f_2,\ldots,f_n}(x_0) \neq 0$ for some $x_0 \in (a,b)$, then $\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ as required.

Remark 0.0.3. *The converse is false in general. E.g. take*

$$f(x) = \begin{cases} 0, & x \leq 0 \\ e^{-1/x}, & x > 0 \end{cases} \qquad \text{and} \quad g(x) = \begin{cases} e^{1/x}, & x < 0 \\ 0, & x \geq 0 \end{cases}.$$

Then $f,\ g$ are C^∞, linearly independent but $W(f,g) \equiv 0$. However, can take $a = 1,\ b = 0$ in the subinterval $(-\infty, 0)$ or $a = 0,\ b = 1$ in $(0,\infty)$.