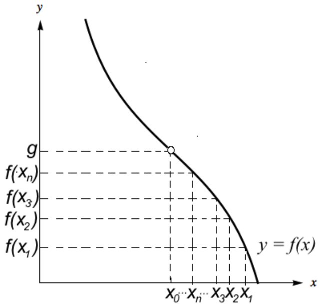
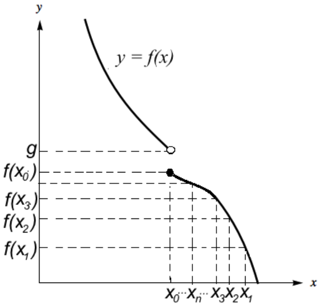
Limits of functions
Continuity of functions

Limit of a function at a point (by Heine)

The number g (or $\pm\infty$) is called **the limit of the function f at x_0** if and only if for every sequence of arguments $\{x_n\}$ near x_0 that converges to x_0, the sequence of values $\{f(x_n)\}$ converges to g (or $\pm\infty$).

$$\lim_{x\to x_0} f(x) = g, \qquad f(x) \xrightarrow{x\to x_0} g$$

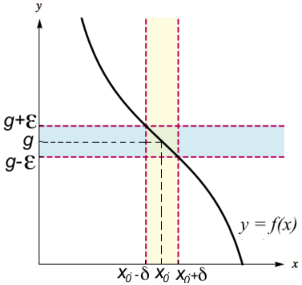


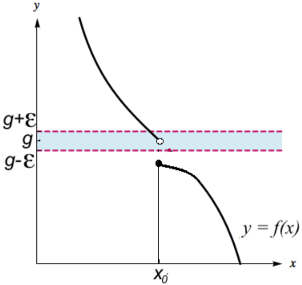


Limit of a function at a point (by Cauchy)

The number g is called **the limit of the function f at x_0** if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \neq x_0$ that satisfy $|x - x_0| < \delta$ it is true that $|f(x) - g| < \varepsilon$.

$$\lim_{x \to x_0} f(x) = g, \qquad f(x) \xrightarrow{x \to x_0} g$$





Evaluating limits

Arithmetic

If limits of $f(x)$ and $g(x)$ exist at x_0, then

- $\lim\limits_{x\to x_0}[f(x)\pm g(x)] = \lim\limits_{x\to x_0} f(x) \pm \lim\limits_{x\to x_0} g(x)$
- $\lim\limits_{x\to x_0}[c\cdot f(x)] = c\cdot \lim\limits_{x\to x_0} f(x)$, where $c\in\mathbb{R}$
- $\lim\limits_{x\to x_0}(f(x)g(x)) = \left(\lim\limits_{x\to x_0} f(x)\right)\cdot\left(\lim\limits_{x\to x_0} g(x)\right)$
- $\lim\limits_{x\to x_0}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}$, provided $\lim\limits_{x\to x_0} g(x)\neq 0$
- $\lim\limits_{x\to x_0}(f(x))^{g(x)} = \left(\lim\limits_{x\to x_0} f(x)\right)^{\lim_{x\to x_0} g(x)}$
- If $\lim\limits_{x\to x_0} f(x) = y_0$ and $\lim\limits_{y\to y_0} h(y) = q$ then $\lim\limits_{x\to x_0} h(f(x)) = q$

Evaluating limits in practice

1. Use the Heine definition, that is, first substitute x_0 for x and hope that it results in a number (or $\pm\infty$)
2. What if you get an undetermined symbol ?

$$[\infty - \infty], \quad [0 \cdot \infty], \quad \left[\frac{0}{0}\right], \quad \left[\frac{\infty}{\infty}\right], \quad [1^{\infty}], \quad [\infty^{0}], \quad [0^{0}]$$

- simplify the expression
- use one of the known limits

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$$

- use the squeeze theorem

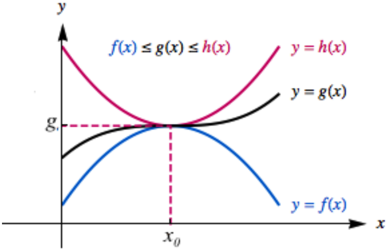
Squeeze (sandwich) theorem

If the functions f, g, and h are such that $f(x) \leq g(x) \leq h(x)$ for x near x_0 and

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = g$$

then

$$\lim_{x \to x_0} g(x) = g$$

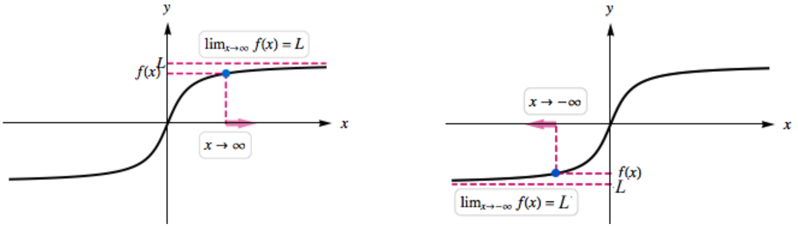


Limits at infinity

The essence of definitions *(details are in textbooks)*

- **finite limit** – for every sequence $x_n \to \infty$ $(x_n \to -\infty)$, $f(x_n) \to L$

$$\lim_{x\to\infty} f(x) = L \quad \left(\lim_{x\to-\infty} f(x) = L \right)$$

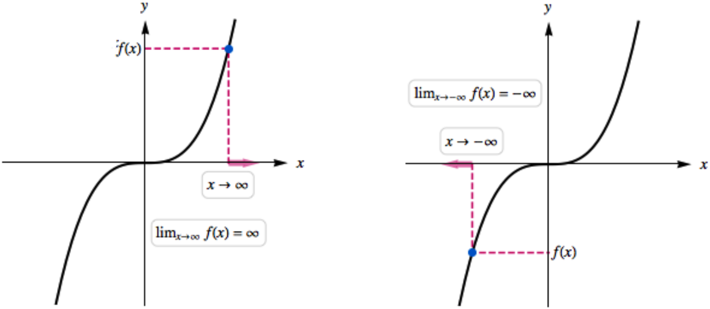


Limits at infinity

The essence of definitions *(details are in textbooks)*

- **infinite limit** – for every sequence $x_n \to \infty$ ($x_n \to -\infty$), $f(x_n) \to \pm\infty$

$$\lim_{x\to\infty} f(x) = \pm\infty \quad \left(\lim_{x\to-\infty} f(x) = \pm\infty\right)$$



Evaluating limits at infinity

- We proceed as we did with limits of sequences
- Be careful with limits at $x \to -\infty$
- We use the squeeze theorem
- $$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$
- Theorem: *The product of a function that converges to zero and a bounded function, converges to zero.*

One-sided limits

One-sided limits

Definitions by Heine (shortened versions)

- **Left-hand limit** at x_0 – for every sequence $x_n \to x_0^-$ the sequence $f(x_n) \to g$ (or ∞)

$$\lim_{x \to x_0^-} f(x) = g \text{ (or } \infty)$$

- **Right-handed limit** at x_0 – for every sequence $x_n \to x_0^+$ the sequence $f(x_n) \to g$ (or ∞)

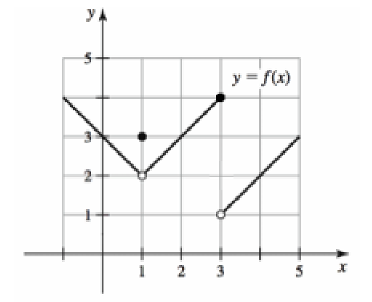
$$\lim_{x \to x_0^+} f(x) = g \text{ (or } \infty)$$

The connection between various limits

The necessary and sufficient condition

$$\lim_{x \to x_0} f(x) = g \quad \Longleftrightarrow \quad \lim_{x \to x_0^+} f(x) = g \quad \text{and} \quad \lim_{x \to x_0^-} f(x) = g$$

Example



$$\lim_{x \to 1^+} f(x) = \qquad \lim_{x \to 3^+} f(x) =$$

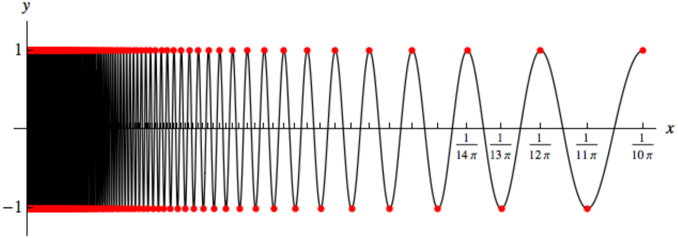
$$\lim_{x \to 1^-} f(x) = \qquad \lim_{x \to 3^-} f(x) =$$

$$\lim_{x \to 1} f(x) = \qquad \lim_{x \to 3} f(x) =$$

$$f(1) = \qquad f(3) =$$

Does one-sided limit always exist ?

$$\lim_{x\to 0^+} \cos\frac{1}{x} = ?$$

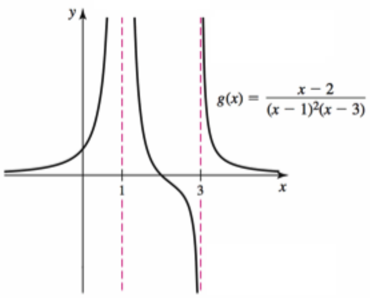


If one-sided limit is infinite ...

Vertical asymptote

We say that $x = x_0$ is the **vertical asymptote** of $f(x)$ if

$$\lim_{x \to x_0} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to x_0^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to x_0^-} f(x) = \pm\infty$$



Continuity

Continuity at a point

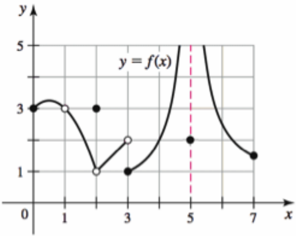
Definition

Let f be defined at and near x_0.
We say that the function is **continuous at** x_0 if and only if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

Otherwise we say that the function is discontinuous at x_0.

If the function is defined only on one side of x_0, then we say that it is continuous **from the left** or **from the right**.



Points of discontinuity

- $x = 2: \quad f(2) = 3 \neq \lim_{x\to 2} f(x) = 1$
- $x = 3: \quad \lim_{x\to 3^-} f(x) = 2 \neq \lim_{x\to 3^+} f(x) = 1$
- $x = 5: \quad \lim_{x\to 5} f(x) = \infty$

Types of discontinuities

$x = 1, x = 2$: removable, $\quad x = 3$: jump, $\quad x = 5$: infinite

If f and g are continuous at x_0, then the following functions are also continuous at x_0.

- $f \pm g$
- $c \cdot f$
- $f \cdot g$
- $\dfrac{f}{g}$, o ile $g(x_0) \neq 0$
- $f^{n/m}$, $f(x_0) \geq 0$ if $m -$ even

- If g is continuous at x_0 and f is continuous at $g(x_0)$, then the composite function $(f \circ g)(x) = f(g(x))$ is continuous at x_0.

Definition

A function is **continuous on an interval I** if it is continuous at every point of I. At the endpoints continuity means from the right or left.
A function is **continuous on its domain** if it is continuous at every point of the domain.

Functions continuous at all points of their domains

Polynomials, Rational, Radical, Trigonometric, Exponential, Logarithmic functions.

Continuity of an inverse function

If f is continuous and increasing (or decreasing) on an interval $A \subset \mathbb{R}$ then the invers function $f^{-1}(A)$ is also continuous and increasing (or decreasing) on the interval $f(A)$.