One Dimensional Reaction-Diffusion equation with Holling Type II Functional Response

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The Model

Fisher – Kolmogorov
Non-dimensionalised equation

Logistic Growth Rate

$$\frac{\partial u}{\partial t} = \boxed{\frac{\partial u}{\partial x^2}} + \boxed{\lambda} \left[\boxed{u(1-u)} \boxed{- \frac{au}{1+bu}} \right], \quad t > 0, \quad x \in (0,1),$$

$$u(t,0) = u(t,1) = 0,$$

$$u(0,x) = f(x), \quad x \in (0,1).$$

Functional Response

The second order change of prey with spatial co-ordiantes (x in this case)

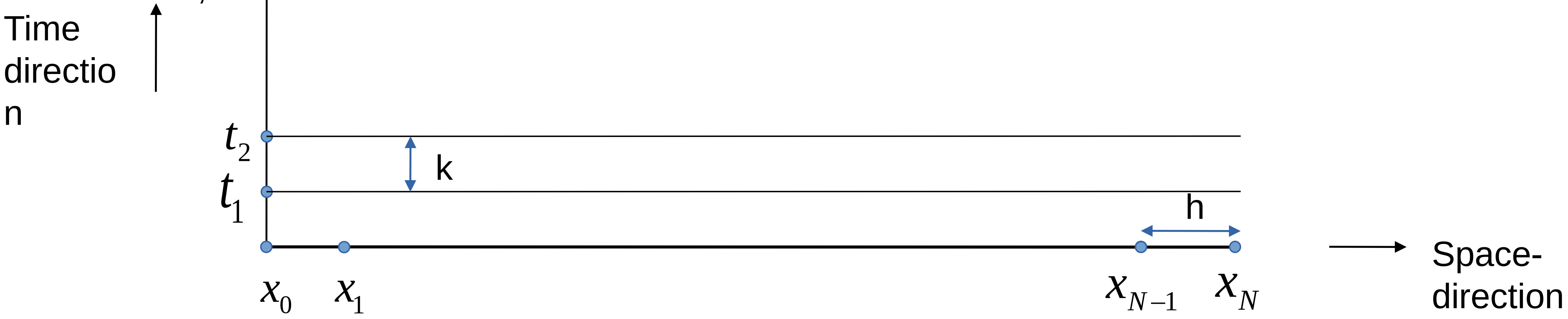
Interpretation of the model

The equation models a species which is harvested or predated, and the rate of predation has a saturation limit as $u \to \infty$

Homogeneous Dirichlet boundary condition signify that the Solution presumes the value zero for each point on the Boundary or we can say that there is no movement across the boundary. So in our case we can assume that any individual of the prey species cannot survive outside of Ω and even on the boundary. An example could be an isolated island or a high voltage electric fence around the Jurassic Park.

Discretization of the Problem

Using Backward in time and central in space we reduce our problem to the following discretization,



Number of unknowns are N – 1.

Iterative - Scheme

Let U_j^k Be an approximation to u_j^k. Then

Using Backward in time and central in space we reduce the problem to the following discretization,

$$u_t \approx \frac{U_j^{n+1} - U_j^n}{k} \quad \text{and} \quad u_{xx} \approx \frac{U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}}{h^2}, \quad j = 1, 2, \ldots\ldots, N-1.$$

For both the boundarypoints fulfilling Dirichlet boundary conditions,

$$U_0^{n+1} = U_N^{n+1} = 0.$$

Solution of the Problem

Substituting the iterative scheme in our equation, we get

$$\frac{(U_j^{n+1}-U_j^n)}{k}=\left(\frac{U_{j-1}^{n+1}-2U_j^{n+1}+U_j^{n+1}}{h^2}\right)+\lambda\left[U_j^{n+1}-(U_j^{n+1})^2-\frac{aU_j^{n+1}}{1+bU_j^{n+1}}\right]$$

On further simplification and letting $\mu=\frac{k}{h^2}$ We get,

For j = 1,

$$F(U_1^{n+1})=U_1^n+\mu\left[-2U_1^{n+1}+U_2^{n+1}\right]+k\lambda\left[U_1^{n+1}-\left(U_1^{n+1}\right)^2-\left(\frac{aU_1^{n+1}}{1+bU_1^{n+1}}\right)\right]-U_1^{n+1}=0$$

Solution of the Problem

For j = 2 to N-2,

$$F_j(U_1^{n+1},\dots,U_{N-1}^{n+1}) = U_j^n + \mu\left[U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}\right] + k\lambda\left[U_j^{n+1} - \left(U_j^{n+1}\right)^2 - \left(\frac{aU_j^{n+1}}{1+bU_j^{n+1}}\right)\right] - U_j^{n+1} = 0$$

For j = N-1,

$$\mathrm{F}_{N-1}(U_N^{n+1}, U_{N-1}^{n+1}, U_{N-2}^{n+1}) = U_{N-1}^n + \mu\left[U_{N-2}^{n+1} - 2U_{N-1}^{n+1}\right] + k\lambda\left[U_{N-1}^{n+1} - \left(U_{N-1}^{n+1}\right)^2 - \left(\frac{aU_{N-1}^{n+1}}{1+bU_{N-1}^{n+1}}\right)\right] - U_{N-1}^{n+1} = 0$$

Now we apply Newtons method to these system of algebraic equations.

Newton's Method

$$X_{i+1} = X_i - \frac{F(X_i)}{J\left(F(X_i)\right)} \longrightarrow$$ Iterative Scheme

Corresponding Jacobian forms a tri-diagonal system!

$$J(F(X_i)) = \begin{bmatrix} -(1+2\mu)+k\lambda\left[1-2U_1^{n+1}-\dfrac{a}{(1+bU_1^{n+1})^2}\right] & \mu & 0 & 0 \\ \mu & -(1+2\mu)+k\lambda\left[1-2U_2^{n+1}-\dfrac{a}{(1+bU_2^{n+1})^2}\right] & \mu & . \\ . & . & . & . \\ 0 & 0 & \mu & -(1+2\mu)+k\lambda\left[1-2U_{N-1}^{n+1}-\dfrac{a}{(1+bU_{N-1}^{n+1})^2}\right] \end{bmatrix}$$

Newton's Method

Let $\psi = -(J(F(X_j)))^{-1}(F(X_j))$,

then we can apply Gauss Elimination to find ψ and substitute it in newtons iterative scheme to find corresponding solutions.

Newton’s Method – Initial Guess

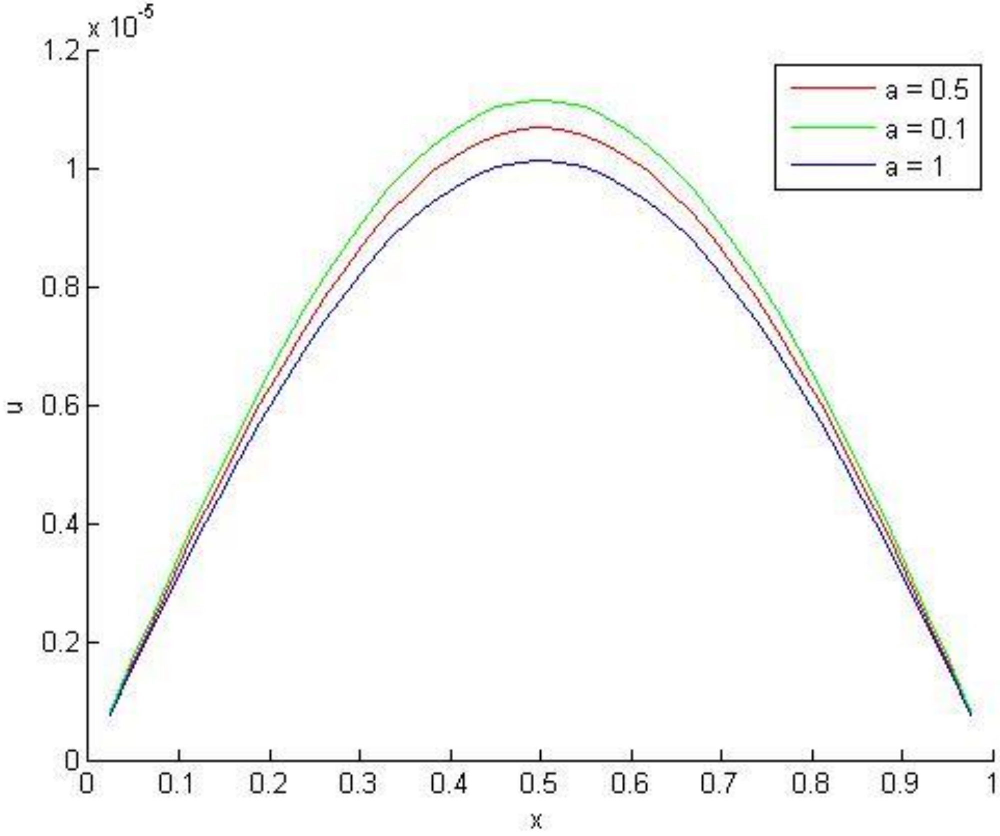
Three initial guesses are choosen and they are as follows,

1. U = zeros(N-1,1).
2. U = ones(N-1,1).
3. U = 0.5*ones(N-1,1).

Order of convergence corresponding to all three guesses have been found.

Dissipation

Graph of our approximate solution corresponding to parameters $N = 42, b = 1, \lambda = 1, a = 0.1, 0.5, 1.$



Refferences

[1] Lecture Notes.

[2] Math 490-01, Partial Differential Equations and Mathematical Biology Spring 2004, Prof. Junping Shi.

[3] Mathworks.