

# VIRTUAL ELEMENT METHOD FOR A QUAD-CURL PROBLEM ON PLANAR DOMAINS

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ABSTRACT. ...

## 1. INTRODUCTION

introduction>? Consider a bounded polygonal domain  $\Omega \subset \mathbb{R}^2$  with boundary  $\partial\Omega$ . We are interested in the following quad-curl problem,

$$\begin{aligned} (\nabla \times)^4 \mathbf{u} + \beta (\nabla \times)^2 \mathbf{u} + \gamma \mathbf{u} &= \mathbf{f} \quad \text{in } \Omega, \\ \nabla \times \mathbf{u} &= 0 \quad \text{on } \partial\Omega, \\ \mathbf{n} \times \mathbf{u} &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1} \boxed{\text{eq:P.strong}}$$

Here  $(\nabla \times \cdot)^4 = \nabla \times (\nabla \times (\nabla \times (\nabla \times \cdot)))$  and  $(\nabla \times \cdot)^2 = \nabla \times (\nabla \times \cdot)$ , are the quad-curl and curl-curl operators in two dimensions,  $\mathbf{n}$  is the unit outward normal vector on  $\partial\Omega$ , and  $\beta, \gamma \geq 0$  are given constants with  $\gamma > 0$ , if  $\Omega$  is multiply connected. The forcing term  $\mathbf{f} : \Omega \rightarrow \mathbb{R}^2$  is a given function.

The weak formulation of (1.1) reads: find  $\mathbf{u} \in \tilde{\mathcal{E}}$  such that

$$(\nabla \times (\nabla \times \mathbf{u}), \nabla \times (\nabla \times \mathbf{v})) + \beta(\nabla \times \mathbf{u}, \nabla \times \mathbf{v}) + \gamma(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \tilde{\mathcal{E}}, \tag{1.2} \boxed{\text{eq:P}}$$

where the energy space  $\tilde{\mathcal{E}}$  is defined as

$$\tilde{\mathcal{E}} := \{\mathbf{v} \in [L^2(\Omega)]^2 : \nabla \times \mathbf{v} \in H_0^1(\Omega), \mathbf{n} \times \mathbf{v} = 0 \text{ on } \partial\Omega\}.$$

The notation  $(\cdot, \cdot)$  denotes the  $L^2(\Omega)$  (or  $[L^2(\Omega)]^2$ ) inner-product. This quad-curl problem is linked to the Maxwell's transmission eigenvalue problem (see (2.8) and consider for e.g., the case of homogeneous media in [12]). The authors here opt for a curl-conforming finite element method to discretize the problem. The quad-curl problem also has several applications including the resistive magnetohydrodynamic (MHD) ... [JT: To do - Add more literature].

Another popular approach to numerically solve (1.1) is to use the Hodge decomposition of divergence-free vector fields (see (3.1) in Section 3). This reduces the problem into a sequence of second-order problems, which can then be discretized using  $H^1$ -conforming finite element methods. This approach has been studied, for e.g., in [5, 7] in two and three dimensions using simplicial and tetrahedral meshes, respectively. Since, we seek the solution in the divergence-free space (see Remark 3.1), we solve (1.2) using the reduced energy space  $\mathcal{E} \subset \tilde{\mathcal{E}}$ ,

$$\mathcal{E} := \{\mathbf{v} \in [L^2(\Omega)]^2 : \nabla \times \mathbf{v} \in H_0^1(\Omega), \operatorname{div} \mathbf{v} = 0 \text{ and } \mathbf{n} \times \mathbf{v} = 0 \text{ on } \partial\Omega\}. \tag{1.3} \boxed{\text{eq:E}}$$

Our goal in this paper is to extend the results to polygonal meshes using the conforming virtual element method (VEM). [JT: To do - Add literature on VEM for Maxwell and Hodge decomposition]

## 2. PRELIMINARIES

liminaries>? In two dimensions, for a vector field  $\mathbf{v} = (v_1, v_2)$ , the curl operator is a scalar function,

$$\nabla \times \mathbf{v} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}.$$

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For a scalar function  $\phi$ , the curl operator is a vector field,

$$\nabla \times \phi = \left[ \frac{\partial \phi}{\partial y}, -\frac{\partial \phi}{\partial x} \right]^\top = \text{rot } \phi.$$

As a result, we have

$$(\nabla \times \phi, \nabla \times v) = (\nabla \phi, \nabla v) \quad \forall \phi, v \in H^1(\Omega),$$

and  $\|\nabla \times v\|_{L^2(\Omega)} = |v|_{H^1(\Omega)}$  for all  $v \in H^1(\Omega)$ .

We collect regularity results for elliptic Boundary Value Problems (BVPs) on polygonal domains (see [11, Section 5.1] and [9, Section 2.5]), with  $\omega$  denoting the largest interior angle at the corners of  $\Omega$ .

`g.reac-diff` **Lemma 2.1** (Regularity of reaction-diffusion BVP). *Given  $g \in H^1(\Omega)$ , let  $\mu \in H_0^1(\Omega)$  satisfy*

$$(\nabla \mu, \nabla v) + \beta(\mu, v) = (g, v) \quad \forall v \in H_0^1(\Omega).$$

*Then for any  $\epsilon > 0$ ,  $\mu \in H^{1+(\pi/\omega)-\epsilon}(\Omega)$ . Also, we have the following continuous dependence on the data result, with positive constant  $C_\epsilon$ , depending on  $\epsilon$  and  $\Omega$ ,*

$$\|\mu\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \leq C_\epsilon \|g\|_{H^1(\Omega)}.$$

`ure-neumann` **Lemma 2.2** (Regularity of pure-Neumann BVP). *Given  $g \in H^1(\Omega)$ , let  $\lambda \in H^1(\Omega)$  satisfy*

$$(\nabla \lambda, \nabla v) + (\lambda, 1)(v, 1) = (g, v) \quad \forall v \in H^1(\Omega).$$

*Then for any  $\epsilon > 0$ ,  $\lambda \in H^{1+(\pi/\omega)-\epsilon}(\Omega)$ . Also, we have the following continuous dependence on the data result, with positive constant  $C_\epsilon$ , depending on  $\epsilon$  and  $\Omega$ ,*

$$\|\lambda\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \leq C_\epsilon \|g\|_{H^1(\Omega)}.$$

Using the above lemmas, we now state the regularity result for the energy space  $\mathcal{E}$  defined in (1.3) (see [7, Section 2.1] for a proof).

`thm:reg.E`? **Theorem 2.3** (Regularity of the energy space). *The quad-curl energy space  $\mathbb{E}$  defined in (1.3) has regularity  $[H^{(\pi/\omega)-\epsilon}(\Omega)]^2$  for any  $\epsilon > 0$ .*

As a consequence of this, the authors in [7, Section 2.2] established the well-posedness of the quad-curl problem (1.2) in the energy space  $\mathcal{E}$ .

We will use following Hilbert spaces in the rest of the article,

$$\begin{aligned} H(\text{div}^0; \Omega) &:= \left\{ \mathbf{v} \in [L^2(\Omega)]^2 : (\mathbf{v}, \nabla \eta) = 0 \quad \forall \eta \in H_0^1(\Omega) \right\}, \\ L_0^2(\Omega) &:= \left\{ v \in L^2(\Omega) : (1, v) = 0 \right\}. \end{aligned}$$

### 3. HODGE DECOMPOSITION BASED REDUCTION

`c:Reduction` Any  $\mathbf{v} \in H(\text{div}^0; \Omega)$  has a unique decomposition (see Chapter 3 in [10] and (1.2) in [3]):

$$\mathbf{v} = \nabla \times \phi + \sum_{j=1}^m d_j \nabla \varphi_j, \tag{3.1} \quad \boxed{\text{eq:HodgeDec}}$$

where  $\phi \in H^1(\Omega) \cap L_0^2(\Omega)$ ,  $m \in \mathbb{Z}_0^+$  is the Betti number, and  $d_j (1 \leq j \leq m) \in \mathbb{R}$ . The harmonic functions  $\varphi_j$  are defined by

$$(\nabla \varphi_j, \nabla v) = 0 \quad \forall v \in H_0^1(\Omega), \tag{3.2a} \quad \boxed{\text{eq:P.varphi}}$$

$$\varphi_j|_{\Gamma_0} = 0, \tag{3.2b} \quad ?\boxed{\text{eq:P.varphi}}$$

$$\varphi_j|_{\Gamma_l} = \delta_{jl} \quad \text{for } 1 \leq l \leq m. \tag{3.2c} \quad ?\boxed{\text{eq:P.varphi}}$$

Here  $\Gamma_0$  denotes the outer boundary,  $\Gamma_l$  denote the  $l$  components of the inner boundary when Betti number  $m > 0$ , and  $\delta_{jl}$  is the Kronecker delta function.

`rem:kernel`? **Remark 3.1.** *The quad-curl energy has a large kernel consisting of gradient fields. Since the functions in  $H(\text{div}^0; \Omega)$  are orthogonal to gradient fields, this kernel is fixed upto harmonic functions. Furthermore, in the case of simply connected domains, we infer from the boundary condition  $\mathbf{u} \times \mathbf{n} = 0$  that the kernel reduces to the zero vector field. Therefore, we can put  $\gamma = 0$ . However, in the case of multiply connected domains, the harmonic functions in the kernel are non-zero vector fields. To account for this, we set  $\gamma > 0$ .*

**3.1.  $\Omega$  is simply connected ( $\gamma = 0$ ).** The Hodge decomposition (3.1) reduces to

$$\mathbf{u} = \nabla \times \phi. \quad (3.3) \boxed{\text{u.simplyConn}}$$

It now remains to find  $\phi$ . As shown in [7], this can be achieved by solving the following sequence of problems.

(1) First we find  $\rho \in H^1(\Omega) \cap L_2^0(\Omega)$ , or equivalently  $\rho \in H^1(\Omega)$  such that

$$(\nabla \times \rho, \nabla \times \psi) + (\rho, 1)(\psi, 1) = (\mathbf{f}, \nabla \times \psi) \quad \forall \psi \in H^1(\Omega). \quad (3.4) \boxed{\text{eq:P.rho}}$$

(2) Find  $\xi$  such that

$$(\nabla \times \xi, \nabla \times \eta) + \beta(\xi, \eta) = (\rho, \eta) \quad \forall \eta \in H_0^1(\Omega) \cap L_0^2(\Omega), \quad (3.5) \boxed{\text{eq:xi}}$$

or equivalently, find  $\xi \in H_0^1(\Omega) \cap L_0^2(\Omega)$  such that

$$\xi = \xi_0 - \frac{(1, \xi_0)}{(1, \xi_1)} \xi_1, \quad (3.6) \boxed{\text{eq:P.xi}}$$

where  $\xi_0, \xi_1 \in H_0^1(\Omega)$  satisfy

$$(\nabla \times \xi_0, \nabla \times \eta) + \beta(\xi_0, \eta) = (\rho, \eta) \quad \forall \eta \in H_0^1(\Omega), \quad (3.7) \boxed{\text{eq:P.xi0}}$$

$$(\nabla \times \xi_1, \nabla \times \eta) + \beta(\xi_1, \eta) = (1, \eta) \quad \forall \eta \in H_0^1(\Omega). \quad (3.8) \boxed{\text{eq:P.xi1}}$$

(3) Finally we find  $\phi \in H^1(\Omega)$ , or equivalently  $\phi \in H^1(\Omega)$  such that

$$(\nabla \times \phi, \nabla \times \psi) + (\phi, 1)(\psi, 1) = (\xi, \psi) \quad \forall \psi \in H^1(\Omega). \quad (3.9) \boxed{\text{eq:P.phi}}$$

**3.2.  $\Omega$  is multiply connected ( $\gamma > 0$ ).** Recalling the Hodge decomposition (3.1)

$$\mathbf{u} = \nabla \times \phi + \sum_{j=1}^m c_j \nabla \varphi_j.$$

It remains to find  $\phi, c_j$ , and  $\varphi_j$ . Following [7], this is equivalent to solving the following sequence of problems.

(1) Find  $(\zeta, \xi) \in H^1(\Omega) \cap L_2^0(\Omega) \times H_0^1(\Omega) \cap L_0^2(\Omega)$  such that

$$(\nabla \times \xi, \nabla \times \psi) + \gamma^{1/2}(\psi, \xi) = \gamma^{-1/2}(\mathbf{f}, \nabla \times \psi) \quad \forall \psi \in H^1(\Omega) \cap L_2^0(\Omega), \quad (3.10) \boxed{\text{?eq:P.mc.zet}}$$

$$-\gamma^{1/2}(\zeta, \eta) + (\nabla \times \xi, \nabla \times \eta) + \beta(\xi, \eta) = 0 \quad \forall \eta \in H_0^1(\Omega) \cap L_0^2(\Omega). \quad (3.11) \boxed{\text{eq:P.mc.xi}}$$

or equivalently, find  $(\zeta, \xi)$  such that

$$(\zeta, \xi) = (\zeta_0, \xi_0) - \frac{(1, \xi_0)}{(1, \xi_1)} (\zeta_1, \xi_1), \quad (3.12) \boxed{\text{eq:P.zeta.c}}$$

where  $(\zeta_0, \xi_0), (\zeta_1, \xi_1) \in H^1(\Omega) \times H_0^1(\Omega)$  solve the following two coupled system:

$$\mathcal{A}((\zeta_0, \xi_0), (\psi, \eta)) + (\zeta_0, 1)(\psi, 1) = \gamma^{-\frac{1}{2}}(\mathbf{f}, \nabla \times \psi) \quad \forall (\psi, \eta) \in H^1(\Omega) \times H_0^1(\Omega), \quad (3.13) \boxed{\text{eq:P.zeta0..}}$$

$$\mathcal{A}((\zeta_1, \xi_1), (\psi, \eta)) + (\zeta_1, 1)(\psi, 1) = (1, \eta) \quad \forall (\psi, \eta) \in H^1(\Omega) \times H_0^1(\Omega). \quad (3.14) \boxed{\text{eq:P.zeta1..}}$$

Here, the bilinear form  $\mathcal{A}(\cdot, \cdot)$  is defined by

$$\mathcal{A}((\zeta, \xi), (\psi, \eta)) = (\nabla \times \zeta, \nabla \times \psi) + \gamma^{\frac{1}{2}}(\psi, \xi) - \gamma^{\frac{1}{2}}(\zeta, \eta) + (\nabla \times \xi, \nabla \times \eta) + \beta(\xi, \eta). \quad (3.15) \boxed{\text{?eq:A?}}$$

(2) Find  $\phi \in H^1(\Omega)$  such that (3.9) holds.

(3) Finally  $c_j (1 \leq j \leq m)$ , are determined by solving the  $m \times m$ , SPD system

$$\sum_{j=1}^m (\nabla \varphi_i, \nabla \varphi_j) c_j = \gamma^{-1}(\mathbf{f}, \nabla \varphi_i) \quad \text{for } 1 \leq i \leq m, \quad (3.16) \boxed{\text{eq:P.cj}}$$

and the harmonic functions  $\varphi_j$  are defined by (3.2).

#### 4. CONFORMING VIRTUAL ELEMENT DISCRETIZATION

cretization? Let  $\mathcal{T}_h$  be a triangulation of the polygonal domain  $\Omega \subset \mathbb{R}^2$  into a finite collection of simple polygons  $D$ . The mesh parameter  $h$  is defined as  $h := \max_{D \in \mathcal{T}_h} h_D$ , where  $h_D$  denotes the diameter of  $D$ . Let  $\mathcal{E}_D$  be the set of edges associated with  $D$ . We make the following shape-regularity assumptions for all  $D \in \mathcal{T}_h$  (see [2, 6]). There exists a constant  $\Theta \in (0, 1)$  such that

$$D \text{ is star-shaped with respect to a ball of radius } \Theta h_D, \text{ and} \quad (4.1a) \quad \boxed{\text{meshreg\_assum}}$$

$$|e| \geq \Theta h_D \text{ for any edge } e \in \mathcal{E}_D. \quad (4.1b) \quad \boxed{\text{?meshreg\_assum}}$$

The local enhanced virtual element space  $V_h(D) \subset H^1(D)$  (see [1, 6]) is defined as follows:

$$V_h(D) := \{v_h \in H^1(D) : v_h|_{\partial D} \in \mathbb{P}_k(\partial D), -\Delta v_h \in \mathbb{P}_k(D), \Pi_{k,D}^0 v_h - \Pi_{k,D}^1 v_h \in \mathbb{P}_{k-2}(D)\}. \quad (4.2) \quad \boxed{\text{?eq:vhK?}}$$

Here  $\mathbb{P}_k(\partial D)$  (respectively,  $\mathbb{P}_k(D)$ ) denote the space of continuous piecewise polynomials of degree at most  $k$  on the boundary  $\partial D$  (respectively, on  $D$ ). The operators  $\Pi_{k,D}^0$  and  $\Pi_{k,D}^1$  are the standard  $L^2$  and  $H^1$ -projections onto  $\mathbb{P}_k(D)$ , respectively (see [6, Section 2.2] for details). The global virtual element spaces  $V_h$  and  $V_h^0$  are defined by concatenating the local spaces as follows:

$$V_h := \{v_h \in H^1(\Omega) : v_h|_D \in V_h(D) \text{ for all } D \in \mathcal{T}_h\}, \quad (4.3) \quad \boxed{\text{?eq:vh?}}$$

$$V_h^0 := \{v_h \in H_0^1(\Omega) : v_h|_D \in V_h(D) \text{ for all } D \in \mathcal{T}_h\}. \quad (4.4) \quad \boxed{\text{?eq:vh0?}}$$

**4.1.  $\Omega$  is simply connected ( $\gamma = 0$ ).** The  $\mathbb{P}_k$  Virtual Element Method for approximating the sequence of problems (3.4)-(3.9) is as follows.

(1) Find  $\rho_h \in V_h$  such that

$$a_h(\rho_h, \psi_h) + (\Pi_{k,h}^0 \rho_h, 1)(\Pi_{k,h}^0 \psi_h, 1) = (\mathbf{f}, \nabla \times \Pi_{k,h}^1 \psi_h) \quad \forall \psi_h \in V_h. \quad (4.5) \quad \boxed{\text{eq:Ph.rhoh}}$$

(2) Find  $\xi_h \in V_h$  given by

$$\xi_h = \xi_{0,h} - \frac{(1, \xi_{0,h})}{(1, \xi_{1,h})} \xi_{1,h}, \quad (4.6) \quad \boxed{\text{eq:Ph.xih}}$$

where  $\xi_{0,h}, \xi_{1,h} \in V_h^0$  satisfy

$$a_h(\xi_{0,h}, \eta_h) + \beta(\Pi_{k,h}^0 \xi_{0,h}, \Pi_{k,h}^0 \eta_h) = (\Pi_{k,h}^0 \rho_h, \Pi_{k,h}^0 \eta_h) \quad \forall \eta_h \in V_h^0, \quad (4.7) \quad \boxed{\text{eq:Ph.xi0h}}$$

$$a_h(\xi_{1,h}, \eta_h) + \beta(\Pi_{k,h}^0 \xi_{1,h}, \Pi_{k,h}^0 \eta_h) = (1, \Pi_{k,h}^0 \eta_h) \quad \forall \eta_h \in V_h^0. \quad (4.8) \quad \boxed{\text{eq:Ph.xi1h}}$$

(3) Find  $\phi_h \in V_h$  such that

$$a_h(\phi_h, \psi_h) + (\Pi_{k,h}^0 \phi_h, 1)(\Pi_{k,h}^0 \psi_h, 1) = (\Pi_{k,h}^0 \xi_h, \Pi_{k,h}^0 \psi_h) \quad \forall \psi_h \in V_h. \quad (4.9) \quad \boxed{\text{eq:Ph.phih}}$$

Here, the global bilinear form  $a_h(\cdot, \cdot)$  is given elementwise by:

$$\begin{aligned} a_h(w_h, v_h) &= \sum_{D \in \mathcal{T}_h} a_h^D(w_h, v_h) \quad \forall w_h, v_h \in V_h, \\ &= \sum_{D \in \mathcal{T}_h} (\nabla \times \Pi_{k,D}^1 w_h, \nabla \times \Pi_{k,D}^1 v_h)_D + S^D((I - \Pi_{k,D}^1)w_h, (I - \Pi_{k,D}^1)v_h). \end{aligned} \quad (4.10) \quad \boxed{\text{eq:ah}}$$

$\Pi_{k,h}^0$  and  $\Pi_{k,h}^1$  are the global  $L^2$  and  $H^1$ -projections onto  $\mathbb{P}_k(\mathcal{T}_h)$  (discontinuous piecewise polynomials of degree  $\leq k$ ), respectively, and are understood in terms of their local counterparts. The symmetric positive definite stabilization term is denoted by  $S^D(\cdot, \cdot)$ . Finally, we post-process  $\mathbf{u}_h \in [\mathbb{P}_k(\mathcal{T}_h)]^2$  using the Hodge decomposition as follows:

$$\mathbf{u}_h = \nabla \times \Pi_{k,h}^1 \phi_h. \quad (4.11) \quad \boxed{\text{uh.simplyCon}}$$

We choose the boundary version of the classical **Dofi-Dofi** definition of the stabilization term (see [4, Section 4.2] and [2]),

$$S^D(w_h, v_h) = \sum_{i=1}^{N_{\partial D}^{\text{dof}}} \chi_i(w_h) \chi_i(v_h) \quad \forall w_h, v_h \in V_h(D), \quad (4.12) \quad \boxed{\text{def:bd.stab}}$$

where operator  $\chi_i(\cdot)$  associates the function with its  $i$ -th degree of freedom, and  $N_{\partial D}^{\text{dof}}$  is the number of boundary degrees of freedom associated with the element in  $D \in \mathcal{T}_h$ . We have the following property associated with this choice of stabilization (see Remark 4.3 [4]),

$$\sum_{i=1}^{N_{\partial D}^{\text{dof}}} \chi_i(w)^2 \lesssim \|w\|_{L^\infty(D)}^2 \quad \forall w \in C(\bar{D}), \quad (4.13) \quad [\text{eq:stab.p2}]$$

where the hidden constants depend on  $\Theta$  and  $k$ .

**4.2.  $\Omega$  is multiply connected ( $\gamma > 0$ ).** The  $\mathbb{P}_k$  Virtual Element Method for approximating the sequence of problems (3.12)-(4.18) is as follows.

(1) Find  $(\zeta_h, \xi_h) \in V_h \times V_h^0$  given by

$$(\zeta_h, \xi_h) = (\zeta_{0,h}, \xi_{0,h}) - \frac{(1, \xi_{0,h})}{(1, \xi_{1,h})} (\zeta_{1,h}, \xi_{1,h}), \quad (4.14) \quad [\text{eq:Ph.zeta.1}]$$

where  $(\zeta_{0,h}, \xi_{0,h}), (\zeta_{1,h}, \xi_{1,h}) \in V_h \times V_h^0$  solve the following two coupled systems:

$$\begin{aligned} \mathcal{A}_h((\zeta_{0,h}, \xi_{0,h}), (\psi_h, \eta_h)) + (\Pi_{k,h}^0 \zeta_{0,h}, 1)(\Pi_{k,h}^0 \psi_h, 1) \\ = \gamma^{-\frac{1}{2}}(\mathbf{f}, \nabla \times \Pi_{k,h}^1 \psi) \quad \forall (\psi_h, \eta_h) \in V_h \times V_h^0, \end{aligned} \quad (4.15) \quad [\text{eq:Ph.zeta0}]$$

$$\begin{aligned} \mathcal{A}_h((\zeta_{1,h}, \xi_{1,h}), (\psi_h, \eta_h)) + (\Pi_{k,h}^0 \zeta_{1,h}, 1)(\Pi_{k,h}^0 \psi_h, 1) \\ = (1, \Pi_{k,h}^0 \eta_h) \quad \forall (\psi_h, \eta_h) \in V_h \times V_h^0. \end{aligned} \quad (4.16) \quad [\text{eq:Ph.zeta1}]$$

The global coupled bilinear form  $\mathcal{A}_h(\cdot, \cdot)$  is defined by

$$\begin{aligned} \mathcal{A}_h((\zeta_h, \xi_h), (\psi_h, \eta_h)) = a_h(\zeta_h, \psi_h) + \gamma^{\frac{1}{2}}(\Pi_{k,h}^0 \psi_h, \Pi_{k,h}^0 \xi_h) - \gamma^{\frac{1}{2}}(\Pi_{k,h}^0 \zeta_h, \Pi_{k,h}^0 \eta_h) \\ + a_h(\xi_h, \eta_h) + \beta(\Pi_{k,h}^0 \xi_h, \Pi_{k,h}^0 \eta_h). \end{aligned} \quad (4.17) \quad [\text{eq:Ah}]$$

(2) Find  $\phi_h \in V_h$  such that (4.9) holds.

(3) The coefficients  $c_{j,h}$  ( $1 \leq j \leq m$ ), are determined by solving

$$\sum_{j=1}^m a_h(\varphi_{i,h}, \varphi_{j,h}) c_{j,h} = \gamma^{-1}(\mathbf{f}, \nabla \Pi_{k,h}^1 \varphi_{i,h}) \quad \text{for } 1 \leq i \leq m. \quad (4.18) \quad [\text{eq:P.cj}]$$

Where the discrete harmonic functions  $\varphi_{j,h}$  are determined by approximating (3.2) as follows:

q:Ph.varphi>

$$a_h(\varphi_{j,h}, v_h) = 0 \quad \forall v_h \in V_h^0, \quad (4.19a) \quad [\text{eq:Ph.varph}]$$

$$\varphi_{j,h}|_{\Gamma_0} = 0, \quad (4.19b) \quad [\text{eq:Ph.varph}]$$

$$\varphi_{j,h}|_{\Gamma_l} = \delta_{jl} \quad \text{for } 1 \leq l \leq m, \quad (4.19c) \quad [\text{eq:Ph.varph}]$$

with  $a_h(\cdot, \cdot)$  given by (4.10).

Finally we post-process  $\mathbf{u}_h \in [\mathbb{P}_k(\mathcal{T}_h)]^2$  using the Hodge decomposition as follows:

$$\mathbf{u}_h = \nabla \times \Pi_{k,h}^1 \phi_h + \sum_{j=1}^m c_{j,h} \nabla \Pi_{k,h}^1 \varphi_{j,h}. \quad (4.20) \quad [\text{uh.notSimpl}]$$

## 5. CONVERGENCE ANALYSIS

We start by collecting some mathematical tools which will be helpful in the forthcoming analysis.

inequality>? **Lemma 5.1** (Sobolev inequality). *Given any  $\delta > 0$ ,*

$$\|v\|_{L^\infty(D)} \lesssim h_D^{-1} \|v\|_{L^2(D)} + |v|_{H^1(D)} + h_D^\delta |v|_{H^{1+\delta}(D)} \quad \forall v \in H^{1+\delta}(D). \quad (5.1) \quad [\text{eq:Sobolev.1}]$$

**Lemma 5.2** (Bramble-Hilbert estimates). *Under the mesh regularity Assumption 4.1a, given any  $\delta > 0$ , there exists a positive constant independent of  $h_D$  such that*

$$\inf_{p \in \mathbb{P}_k(D)} |\lambda - \psi|_{H^1(D)} \lesssim h_D^{\min(\delta, k)} |\lambda|_{H^{1+\delta}(D)}, \quad \forall \lambda \in H^{1+\delta}(D).$$

inequality>? **Lemma 5.3** (Trace inequality). *Let  $e$  be an edge of  $D \subset \mathbb{R}^2$ . Then, for all  $v \in H^{1+\delta}(D)$  with given  $\delta > 0$ , we have*

$$h_D^{2\delta} |v|_{H^{1/2+\delta}(e)}^2 \lesssim |v|_{H^1(D)}^2 + h_D^{2\delta} |v|_{H^{1+\delta}(D)}^2.$$

properites)? **Lemma 5.4** ( $H^1$ -projector stability and approximation). Given  $\delta > 0$ ,

$$|\Pi_{k,D}^1 v|_{H^1(D)} \leq |v|_{H^1(D)} \quad \forall v \in H^1(D), \quad (5.2) \quad \text{[eq:stability]}$$

$$|v - \Pi_{k,D}^1 v|_{H^1(D)} \lesssim h_D^{\min(\delta,k)} |v|_{H^{1+\delta}(D)} \quad \forall v \in H^{1+\delta}(D), \quad (5.3) \quad \text{[eq:approxH1]}$$

$$\|v - \Pi_{k,D}^1 v\|_{L^2(D)} \lesssim h_D^{\min(\delta,k)+1} |v|_{H^{1+\delta}(D)} \quad \forall v \in H^{1+\delta}(D). \quad (5.4) \quad \text{[eq:approxL2]}$$

.properties) **Lemma 5.5** ( $L^2$ -projector stability and approximation).

$$\|\Pi_{k,D}^0 v\|_{L^2(D)} \leq \|v\|_{L^2(D)} \quad \forall v \in L^2(D), \quad \|\Pi_{k,D}^0 v\|_{H^1(D)} \leq \|v\|_{H^1(D)} \quad \forall v \in H^1(D), \quad (5.5) \quad \text{[eq:stability]}$$

$$\|v - \Pi_{k,D}^0 v\|_{L^2(D)} \lesssim h_D^{l+1} |v|_{H^{l+1}(D)} \quad \forall v \in H^{l+1}(D), \quad 0 \leq l \leq k, \quad (5.6) \quad \text{[eq:approxL2]}$$

Now we recall the virtual element interpolation operator, which takes any sufficiently smooth function and maps it to the virtual element space. For  $s > 1$ , the global interpolation operator  $I_{k,h} : H^s(\Omega) \rightarrow V_h$  is the global counterpart of the local interpolation operator  $I_{k,D} : H^s(D) \rightarrow V_h(D)$  for all  $D \in \mathcal{T}_h$  such that for any  $v \in H^s(D)$ ,

$$I_{k,D} v(p) = v(p) \quad \forall p \in \mathcal{N}^{\partial D}, \quad (5.7) \quad \text{[eq:ID.bound]}$$

$$\Pi_{k-2,D}^0 I_{k,D} v = \Pi_{k-2,D}^0 v. \quad (5.8) \quad \text{[eq:ID.inter]}$$

Here  $\mathcal{N}^{\partial D}$  is the set of boundary degrees of freedom associated with the local virtual element space.

.properties) **Lemma 5.6** (Interpolation operator stability and approximation). Given  $\delta > 0$ , we have

$$|I_{k,D} v|_{H^1(D)} \lesssim |v|_{H^1(D)} + h_D^\delta |v|_{H^{1+\delta}(D)} \quad \forall v \in H^{1+\delta}(D), \quad (5.9) \quad \text{[eq:ID.stab]}$$

$$|v - I_{k,D} v|_{H^1(D)} + |v - \Pi_{k,D}^1 I_{k,D} v|_{H^1(D)} \lesssim h_D^{\min(\delta,k)} |v|_{H^{1+\delta}(D)} \quad \forall v \in H^{1+\delta}(D), \quad (5.10) \quad \text{[eq:ID.approx]}$$

$$\|v - I_{k,D} v\|_{L^2(D)} + \|v - \Pi_{k,D}^1 I_{k,D} v\|_{L^2(D)} \lesssim h_D^{\min(\delta,k)+1} |v|_{H^{1+\delta}(D)} \quad \forall v \in H^{1+\delta}(D). \quad (5.11) \quad \text{[eq:ID.approx]}$$

In the subsequent analysis, we work with the following assumption which is made concrete in Remark 5.9.

:coercivity) **Assumption 5.7.** Let  $\|\cdot\|_{h_\beta} = \sqrt{a_h(\cdot, \cdot) + \beta(\Pi_{k,h}^0 \cdot, \Pi_{k,h}^0 \cdot)}$  denote the mesh-dependent energy norm such that

$$|v_h|_{H^1(\Omega)} \lesssim \|v_h\|_{h_\beta} \quad \forall v_h \in V_h^0, \quad (5.12) \quad \text{[eq:coercivity]}$$

$$|v_h|_{H^1(\Omega)} \lesssim \|v_h\|_{h_\beta} \quad \forall v_h \in V_h \quad \text{with} \quad (v_h, 1) = 0. \quad (5.13) \quad \text{[eq:coercivity]}$$

We also define the piecewise-broken  $H^1$ -seminorm by

$$|v|_{1,h} := \left( \sum_{D \in \mathcal{T}_h} |v|_{H^1(D)}^2 \right)^{\frac{1}{2}} \quad \forall v \in H^1(\mathcal{T}_h).$$

Some useful estimates in the mesh-dependent norm and piecewise broken semi-norm are in order.

h. dependent) **Lemma 5.8.** The following estimates hold,

$$\|I_{k,h} v - \Pi_{k,h}^1 I_{k,h} v\|_{h_0} + |I_{k,h} v - \Pi_{k,h}^1 I_{k,h} v|_{1,h} \lesssim h^{\min(\delta,k)} |v|_{H^{1+\delta}(\Omega)} \quad \forall v \in H^{1+\delta}(\Omega), \quad (5.14) \quad \text{[eq:est.Ih.h]}$$

$$\|v - I_{k,h} v\|_{h_\beta} + \|v - \Pi_{k,h}^1 v\|_{h_\beta} \lesssim h^{\min(\delta,k)} |v|_{H^{1+\delta}(\Omega)} \quad \forall v \in H^{1+\delta}(\Omega). \quad (5.15) \quad \text{[eq:est.hbet]}$$

:coercivity.assum) **Remark 5.9.** [JT: To do - Making Assumption 5.7 concrete, for shape-regular meshes vs meshes with small edges]

5.1.  **$\Omega$  is simply connected** ( $\gamma = 0$ ). Our first aim is to estimate the error  $\xi - \xi_h$ . To this end, we first write down the error equation by testing (3.4) with  $\psi_h \in V_h$  and subtracting it from (4.5),

$$(\nabla \times \rho, \nabla \times \psi_h) - a_h(\rho_h, \psi_h) = (\mathbf{f}, \nabla \times (\psi_h - \Pi_{k,h}^1(\psi_h))) \quad \forall \psi_h \in V_h. \quad (5.16) \quad \text{[eq:erreq.rh]}$$

We note that to obtain the above error equation we also used the definition of  $\Pi_{k,h}^0$  and the fact that  $\rho, \rho_h \in L_0^2(\Omega)$ . Furthermore, in view of (3.4), (4.5),  $\rho, \rho_h \in L_0^2(\Omega)$  and Assumption 5.7, we have the following relations:

$$\|\nabla \times \rho\|_{L^2(\Omega)} \leq \|\mathbf{f}\|_{L^2(\Omega)}, \quad \text{and} \quad \|\nabla \times \rho_h\|_{L^2(\Omega)} \lesssim \|\mathbf{f}\|_{L^2(\Omega)}. \quad (5.17) \quad \text{[eq:curl.dat]}$$

In the following lemma we estimate the error for  $\rho_h$  in the dual norm using a duality argument.

:rhoh.dual>? **Lemma 5.10.** For any  $\epsilon > 0$ , there exists a positive constant dependent on  $\epsilon$  and independent of  $h$  such that

$$|(\rho - \rho_h, \chi)| \lesssim h^{\min((\pi/\omega) - \epsilon, k)} \|\chi\|_{H^1(\Omega)} \|\mathbf{f}\|_{L^2(\Omega)} \quad \forall \chi \in H^1(\Omega). \quad (5.18) \quad [\text{eq:err.est}]$$

*Proof.* Given arbitrary  $\chi \in H^1(\Omega)$ , let  $\lambda \in H^1(\Omega)$  solve the following dual problem:

$$(\nabla \times \psi, \nabla \times \lambda) + (\psi, 1)(\lambda, 1) = (\psi, \chi) \quad \forall \psi \in H^1(\Omega). \quad (5.19) \quad [\text{eq:dual.rho}]$$

Testing (5.19) with  $\rho - \rho_h \in H^1(\Omega)$  and exploiting  $\rho, \rho_h \in L_0^2(\Omega)$ , we get

$$\begin{aligned} (\rho - \rho_h, \chi) &= (\nabla \times (\rho - \rho_h), \nabla \times \lambda) + (\rho - \rho_h, 1)(\lambda, 1) \\ &= (\nabla \times (\rho - \rho_h), \nabla \times (\lambda - I_{k,h}\lambda)) + (\nabla \times (\rho - \rho_h), \nabla \times I_{k,h}\lambda). \end{aligned} \quad (5.20) \quad [\text{eq:err.rho}]$$

The first term in (5.20) is bounded using the Cauchy-Schwarz inequality and (5.17), to get

$$\begin{aligned} (\nabla \times (\rho - \rho_h), \nabla \times (\lambda - I_{k,h}\lambda)) &\leq \|\nabla \times (\rho - \rho_h)\|_{L^2(\Omega)} \|\nabla \times (\lambda - I_{k,h}\lambda)\|_{L^2(\Omega)} \\ &\lesssim \|\mathbf{f}\|_{L^2(\Omega)} |\lambda - I_{k,h}\lambda|_{H^1(\Omega)}. \end{aligned} \quad (5.21) \quad [\text{eq:err.rho}]$$

The second term in (5.20) is bounded using the error equation (5.16) for  $\psi_h = I_{k,h}\lambda$  as follows:

$$\begin{aligned} (\nabla \times (\rho - \rho_h), \nabla \times I_{k,h}\lambda) &= (\nabla \times \rho, \nabla \times I_{k,h}\lambda) - (\nabla \times \rho_h, \nabla \times I_{k,h}\lambda) \\ &= (\mathbf{f}, \nabla \times (I_{k,h}\lambda - \Pi_{k,h}^1 I_{k,h}\lambda)) + a_h(\rho_h, I_{k,h}\lambda) - (\nabla \times \rho_h, \nabla \times I_{k,h}\lambda) \\ &\leq \|\mathbf{f}\|_{L^2(\Omega)} |(I - \Pi_{k,h}^1)I_{k,h}\lambda|_{1,h} + a_h(\rho_h, I_{k,h}\lambda) - (\nabla \times \rho_h, \nabla \times I_{k,h}\lambda). \end{aligned} \quad (5.22) \quad [\text{eq:err.rho}]$$

The difference in (5.22) can be rewritten as follows:

$$\begin{aligned} &a_h(\rho_h, I_{k,h}\lambda) - (\nabla \times \rho_h, \nabla \times I_{k,h}\lambda) \\ &= \sum_{D \in \mathcal{T}_h} (\nabla \times \Pi_{k,D}^1 \rho_h, \nabla \times \Pi_{k,D}^1 I_{k,D}\lambda) + S^D((I - \Pi_{k,D}^1) \rho_h, (I - \Pi_{k,D}^1) I_{k,D}\lambda) - \\ &\quad (\nabla \times \rho_h, \nabla \times (I_{k,h}\lambda - \Pi_{k,D}^1 I_{k,D}\lambda)) - (\nabla \times \rho_h, \nabla \times \Pi_{k,D}^1 I_{k,D}\lambda) \\ &= \sum_{D \in \mathcal{T}_h} S^D((I - \Pi_{k,D}^1) \rho_h, (I - \Pi_{k,D}^1) I_{k,D}\lambda) - (\nabla \times \rho_h, \nabla \times (I_{k,D}\lambda - \Pi_{k,D}^1 I_{k,D}\lambda)) \\ &\leq \left( \sum_{D \in \mathcal{T}_h} S^D((I - \Pi_{k,D}^1) \rho_h, (I - \Pi_{k,D}^1) \rho_h) \right)^{1/2} \left( \sum_{D \in \mathcal{T}_h} S^D((I - \Pi_{k,D}^1) I_{k,D}\lambda, (I - \Pi_{k,D}^1) I_{k,D}\lambda) \right)^{1/2} \\ &\quad \left( \sum_{D \in \mathcal{T}_h} |\rho_h|_{H^1(D)}^2 \right)^{1/2} \left( \sum_{D \in \mathcal{T}_h} |I_{k,D}\lambda - \Pi_{k,D}^1 I_{k,D}\lambda|_{H^1(D)}^2 \right)^{1/2} \\ &\lesssim \|(I - \Pi_{k,h}^1) \rho_h\|_{h_0} \|(I - \Pi_{k,h}^1) I_{k,h}\lambda\|_{h_0} + |\rho_h|_{H^1(\Omega)} |(I - \Pi_{k,h}^1) I_{k,h}\lambda|_{1,h}. \end{aligned} \quad (5.23) \quad [\text{eq:err.rho}]$$

Where in the passage to the second equality we used the definition of  $\Pi_{k,D}^1$  projector followed by Cauchy-Schwarz inequality and finally the definition of  $\|\cdot\|_h$  and  $|\cdot|_{1,h}$ . Using the definition of the stabilization (4.12), the approximation properties of  $\Pi_{k,D}^1$  (5.3) and (5.4), and (5.17), we have

$$\begin{aligned} \|(I - \Pi_{k,h}^1) \rho_h\|_{h_0}^2 &= \sum_{D \in \mathcal{T}_h} S^D((I - \Pi_{k,h}^1) \rho_h, (I - \Pi_{k,h}^1) \rho_h) \\ &\stackrel{(4.13)}{\lesssim} \sum_{D \in \mathcal{T}_h} \|(I - \Pi_{k,h}^1) \rho_h\|_{L^\infty(D)}^2 \\ &\stackrel{(5.1)}{\lesssim} \sum_{D \in \mathcal{T}_h} h_D^{-1} \|(I - \Pi_{k,h}^1) \rho_h\|_{L^2(D)}^2 + |(I - \Pi_{k,h}^1) \rho_h|_{H^1(D)}^2 \\ &\lesssim \|\mathbf{f}\|_{L^2(\Omega)}. \end{aligned}$$

The regularity of the dual problem from Lemma 2.2, the above bound, the estimate (5.14) with  $\delta = (\pi/\omega) - \epsilon$ , and (5.17) lead to the following bound on (5.23),

$$a_h(\rho_h, I_{k,h}\lambda) - (\nabla \times \rho_h, \nabla \times I_{k,h}\lambda) \lesssim h^{\min((\pi/\omega) - \epsilon, k)} \|\mathbf{f}\|_{L^2(\Omega)} |\lambda|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)}. \quad (5.24) \quad [\text{eq:ref4coup}]$$

Substituting in (5.22), again using (5.14) and the data dependence relation of the dual problem in Lemma 2.2 yield

$$(\nabla \times (\rho - \rho_h), \nabla \times I_{k,h} \lambda) \lesssim h^{\min((\pi/\omega) - \epsilon, k)} \|f\|_{L^2(\Omega)} \|\chi\|_{H^1(\Omega)}.$$

Substituting the above bound and (5.21) into (5.20), and using the definition of piecewise-broken  $H^1$ -seminorm and (5.10) yields the desired error estimate (5.18).  $\square$

It remains to estimate the error for  $\xi_h$ .

`<lem:xi.est>` **Lemma 5.11.** *For any  $\epsilon > 0$ , there exists a positive constant dependent on  $\epsilon$  and independent of  $h$  such that for  $k = 1, 2$  and  $\gamma = 0$ ,*

$$|\xi - \xi_h|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega) - \epsilon, k)} \|f\|_{L^2(\Omega)}, \quad (5.25) \quad \text{[eq:err.est..]}$$

$$|\xi - \Pi_{k,h}^1 \xi_h|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega) - \epsilon, k)} \|f\|_{L^2(\Omega)}. \quad (5.26) \quad \text{[eq:err.est..]}$$

*Proof.* Given  $\rho_h \in V_h \cap L_0^2(\Omega)$ , let  $\tilde{\xi}_0 \in H_0^1(\Omega)$  solve the following auxiliary problem,

$$(\nabla \times \tilde{\xi}_0, \nabla \times \eta) + \beta(\tilde{\xi}_0, \eta) = (\rho_h, \eta) \quad \forall \eta \in H_0^1(\Omega). \quad (5.27) \quad \text{[eq:P.aux]}$$

Since  $\rho_h \in V_h \subset H^1(\Omega)$ , due to Lemma 2.1,

$$\tilde{\xi}_0 \in H^{1+(\pi/\omega)-\epsilon}(\Omega) \quad \text{and} \quad \|\tilde{\xi}_0\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \lesssim \|\rho_h\|_{H^1(\Omega)}. \quad (5.28) \quad \text{[tildexi0.re..]}$$

It follows now from (5.64), (5.27), and the duality estimate (5.18) that

$$|\xi_0 - \tilde{\xi}_0|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega) - \epsilon, k)} \|f\|_{L^2(\Omega)}. \quad (5.29) \quad \text{[eq:xi0.a]}$$

It follows from (5.27) and (4.7) that  $\xi_{0,h}$  is also a virtual element approximation of  $\xi_0$  and hence we have the following analogous abstract error estimate following [8, Section 4.3],

$$\begin{aligned} \|\tilde{\xi}_0 - \xi_{0,h}\|_{h_\beta} &\lesssim \|\tilde{\xi}_0 - I_{k,h} \tilde{\xi}_0\|_{h_\beta} + |\tilde{\xi}_0 - \Pi_{k,h}^1 \tilde{\xi}_0|_{1,h} + \|\tilde{\xi}_0 - \Pi_{k,h}^1 \tilde{\xi}_0\|_{h_\beta} \\ &\quad + \beta h \|\tilde{\xi}_0 - \Pi_{k,h}^0 \tilde{\xi}_0\|_{L^2(\Omega)} + \sup_{w_h \in V_h^0 \setminus \{0\}} \frac{(\rho_h, w_h - \Pi_{k,h}^0 w_h)}{|w_h|_{H^1(\Omega)}}. \end{aligned} \quad (5.30) \quad \text{[err.abs.xi0..]}$$

The numerator in the last term of (5.30) is estimated using the definition of  $\Pi_{k,h}^0$ , the fact that it minimizes its respective norm, the fact  $\rho_h, w_h \in V_h \subset H^1(\Omega)$  and the approximation property of  $\Pi_{k,h}^0$  follows:

$$\begin{aligned} (\rho_h, w_h - \Pi_{k,h}^0 w_h) &= (\rho_h - \Pi_{0,h}^0 \rho_h, w_h - \Pi_{k,h}^0 w_h) \\ &\leq \|\rho_h - \Pi_{0,h}^0 \rho_h\|_{L^2(\Omega)} \|w_h - \Pi_{0,h}^0 w_h\|_{L^2(\Omega)} \\ &\lesssim h^2 |\rho_h|_{H^1(\Omega)} |w_h|_{H^1(\Omega)}. \end{aligned} \quad (5.31) \quad \text{[eq:err.abs..]}$$

Using the approximation properties stated in (5.15), the definition of  $|\cdot|_{1,h}$ , the estimates (5.3) and (5.6) for  $\delta = (\pi/\omega) - \epsilon$ , and (5.31), we can get the following concrete bound on (5.30).

$$\begin{aligned} \|\tilde{\xi}_0 - \xi_{0,h}\|_{h_\beta} &\lesssim h^{\min((\pi/\omega) - \epsilon, k)} (|\tilde{\xi}_0|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} + |\rho_h|_{H^1(\Omega)}) \\ &\stackrel{(5.28),(5.17)}{\lesssim} h^{\min((\pi/\omega) - \epsilon, k)} \|f\|_{L^2(\Omega)}. \end{aligned}$$

Now, using Assumption 5.7 in the above estimate yields

$$|\tilde{\xi}_0 - \xi_{0,h}|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega) - \epsilon, k)} \|f\|_{L^2(\Omega)}. \quad (5.32) \quad \text{[eq:xi0h.b]}$$

Finally, a triangle inequality alongwith (5.29) and (5.32) leads to (5.33)

$$|\xi_0 - \xi_{0,h}|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega) - \epsilon, k)} \|f\|_{L^2(\Omega)}. \quad (5.33) \quad \text{[eq:err.xi0]}$$

Similarly, we can get the following control on  $\xi_1 - \xi_{1,h}$  (see (3.8) and (4.8))

$$|\xi_1 - \xi_{1,h}|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega) - \epsilon, k)}. \quad (5.34) \quad \text{[eq:err.xi1]}$$

The estimate (5.25) now follows from (5.33), (5.34), (3.6), and (4.6). Since,  $\xi \in H^{1+(\pi/\omega)-\epsilon}(\Omega)$  we can get the estimate (5.26) using Lemma 5.5 and (5.25) as follows,

$$\begin{aligned} |\xi - \Pi_{k,h}^1 \xi_h|_{1,h} &\leq |\xi - \Pi_{k,h}^1 \xi|_{1,h} + |\Pi_{k,h}^1 (\xi - \xi_h)|_{1,h} \\ &\lesssim h^{\min((\pi/\omega)-\epsilon,k)} \|\xi\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} + |\xi - \xi_h|_{H^1(\Omega)} \\ &\lesssim h^{\min((\pi/\omega)-\epsilon,k)} \|\mathbf{f}\|_{L^2(\Omega)}. \end{aligned}$$

In the last step we used Lemma 2.1 on (3.5) followed by (5.17). This completes the proof.  $\square$

The error analysis for  $\phi_h$  is analogous to that of  $\xi_h$ .

lem:est.phi **Lemma 5.12.** *For any  $\epsilon > 0$ , there exists a positive constant dependent on  $\epsilon$  and independent of  $h$  such that for  $k = 1, 2$ ,*

$$|\phi - \phi_h|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega)-\epsilon,k)} \|\mathbf{f}\|_{L^2(\Omega)}, \quad (5.35) \quad [\text{eq:err.est.}]$$

$$|\phi - \Pi_{k,h}^1 \phi_h|_{1,h} \lesssim h^{\min((\pi/\omega)-\epsilon,k)} \|\mathbf{f}\|_{L^2(\Omega)}. \quad (5.36) \quad [\text{eq:err.est.}]$$

*Proof.* Given  $\xi_h \in V_h^0 \cap L_0^2(\Omega)$  (which is true by construction of  $\xi_h$ ). Let  $\tilde{\phi} \in H^1(\Omega)$  solve the following auxiliary problem,

$$(\nabla \times \tilde{\phi}, \nabla \times \psi) + (\tilde{\phi}, 1)(\tilde{\psi}, 1) = (\xi_h, \psi) \quad \forall \psi \in H^1(\Omega). \quad (5.37) \quad [\text{eq:P.aux.phi}]$$

Since  $\xi_h \subset H^1(\Omega)$ , due to Lemma 2.2,

$$\tilde{\phi} \in H^{1+(\pi/\omega)-\epsilon}(\Omega) \quad \text{and} \quad \|\tilde{\phi}\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \lesssim \|\xi_h\|_{H^1(\Omega)}. \quad (5.38) \quad [\text{tildephi.re}]$$

Comparing (3.9) and (5.37), using a standard Poincaré inequality for  $\xi - \xi_h \in H_0^1(\Omega)$  alongwith (5.25) gives,

$$|\phi - \tilde{\phi}|_{H^1(\Omega)} \lesssim \|\xi - \xi_h\|_{L^2(\Omega)} \lesssim h^{\min((\pi/\omega)-\epsilon,k)} \|\mathbf{f}\|_{L^2(\Omega)}. \quad (5.39) \quad [\text{eq:phi.a}]$$

It follows from (5.37) and (4.9) that  $\phi_h$  is also a virtual element approximation of  $\tilde{\phi}$  and hence using the fact the  $\rho, \rho_h \in L_0^2(\Omega)$  we have the following analogous abstract error estimate following [8, Section 4.3],

$$\|\tilde{\phi} - \phi_h\|_{h_\beta} \lesssim \|\tilde{\phi} - I_{k,h} \tilde{\phi}\|_{h_0} + |\tilde{\phi} - \Pi_{k,h}^1 \tilde{\phi}|_{1,h} + \|\tilde{\phi} - \Pi_{k,h}^1 \tilde{\phi}\|_{h_0} + \sup_{w_h \in V_h^0 \setminus \{0\}} \frac{(\xi_h, w_h - \Pi_{k,h}^0 w_h)}{|w_h|_{H^1(\Omega)}}.$$

Following similar arguments as in the proof of Lemma 5.11 and (5.38), we can get the following concrete bound,

$$\begin{aligned} |\tilde{\phi} - \phi_h|_{H^1(\Omega)} &\lesssim h^{\min((\pi/\omega)-\epsilon,k)} (\|\tilde{\phi}\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} + |\xi_h|_{H^1(\Omega)}) \\ &\lesssim h^{\min((\pi/\omega)-\epsilon,k)} (\|\xi_h - \xi\|_{H^1(\Omega)} + \|\xi\|_{H^1(\Omega)} + |\xi|_{H^1(\Omega)}) \\ &\lesssim h^{\min((\pi/\omega)-\epsilon,k)} (|\xi_h - \xi|_{H^1(\Omega)} + \|\xi\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)}) \\ &\lesssim h^{\min((\pi/\omega)-\epsilon,k)} \|\mathbf{f}\|_{L^2(\Omega)}. \end{aligned} \quad (5.40) \quad [\text{eq:phi.b}]$$

Where in the second inequality we used (5.28) followed by a triangle inequality. In the third inequality we used Poincaré inequality and  $H^{1+(\pi/\omega)-\epsilon}(\Omega) \hookrightarrow H^1(\Omega)$ . Finally the last inequality is obtained by (5.25) and Lemma 2.1 on (3.5) followed by (5.17). Finally, a triangle inequality alongwith (5.39) and (5.40) leads to (5.35).

Moreover, it follows from (3.9) and Lemma 2.2 that

$$\phi \in H^{1+(\pi/\omega)-\epsilon}(\Omega) \quad \text{and} \quad \|\phi\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \lesssim \|\xi\|_{H^1(\Omega)}. \quad (5.41) \quad [\text{eq:reg.phi}]$$

Hence, using similar arguments as in the proof of (5.26) with (5.41) leads to (5.36).  $\square$

In view of Lemmas 5.11, 5.12, (3.3) and (4.11), we readily have the following result for simply connected domains.

lyconnected **Theorem 5.13.** *The approximations  $\xi_h$  and  $\mathbf{u}_h$  obtained by  $\mathbb{P}_k$ -virtual finite element method satisfy*

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} + |\nabla \times \mathbf{u} - \xi_h|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega)-\epsilon,k)} \|\mathbf{f}\|_{L^2(\Omega)}. \quad (5.42) \quad [\text{?eq:err.est.}]$$

for any  $\epsilon > 0$ , and  $k = 1, 2$ , where  $\omega$  is the largest angle at the corners of  $\Omega$ .

*Proof.* The estimate for  $\mathbf{u} - \mathbf{u}_h$  follows from (5.36) on observing the following,

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} = \|\nabla \times \phi - \nabla \times \Pi_{k,h}^1 \phi_h\|_{L^2(\Omega)} = |\phi - \Pi_{k,h}^1 \phi_h|_{1,h}.$$

Recalling that  $\xi = \nabla \times \mathbf{u}$ , the estimate for  $\nabla \times \mathbf{u} - \xi_h$  follows from (5.25).  $\square$

**5.2.  $\Omega$  is multiply connected ( $\gamma > 0$ ).** The convergence analysis for the multiply connected case follows ideas similar to those in the simply connected case extended to the coupled setting. We first write down the error equation for  $(\zeta_{0,h}, \xi_{0,h})$  by testing (3.13) with  $(\psi_h, \eta_h) \in V_h \times V_h^0$ , subtracting it from (4.15) and using  $\zeta_0, \zeta_{0,h} \in L_0^2(\Omega)$ ,

$$\mathcal{A}((\zeta_0, \xi_0), (\psi_h, \eta_h)) - \mathcal{A}_h((\zeta_{0,h}, \xi_{0,h}), (\psi_h, \eta_h)) = \gamma^{-1/2}(\mathbf{f}, \nabla \times (\psi_h - \Pi_{k,h}^1 \psi_h)) \quad (5.43) \quad [\text{eq:err.zeta}]$$

Furthermore, testing (3.13) with  $(\zeta_0, \xi_0)$  and applying Young's inequality gives

$$\begin{aligned} |\zeta_0|_{H^1(\Omega)}^2 + (\zeta_0, 1)^2 + |\xi_0|_{H^1(\Omega)}^2 &\lesssim \|\mathbf{f}\|_{L^2(\Omega)}^2, \\ \|\zeta_0\|_{H^1(\Omega)}^2 + |\xi_0|_{H^1(\Omega)}^2 &\lesssim \|\mathbf{f}\|_{L^2(\Omega)}^2. \end{aligned} \quad (5.44) \quad [\text{eq:err.zeta}]$$

where the last inequality is a consequence of Poincaré-Wirtinger inequality. Similarly, testing (4.15) with  $(\zeta_{0,h}, \xi_{0,h})$ , and in view of Assumption 5.7 and stability of  $\Pi_{k,h}^1$ , we obtain

$$\|\zeta_{0,h}\|_{H^1(\Omega)}^2 + |\xi_{0,h}|_{H^1(\Omega)}^2 \lesssim \|\mathbf{f}\|_{L^2(\Omega)}. \quad (5.45) \quad [\text{eq:err.zeta}]$$

Now analogous to the simply connected case, we estimate the error for  $\zeta_{0,h}$  and  $\zeta_{1,h}$  in  $H^1(\Omega)'$ .

**Lemma 5.14.** *For any  $\epsilon > 0$ , there exists a positive constant dependent on  $\epsilon$  and independent of  $h$  such that for  $k = 1, 2$  and  $\gamma > 0$ ,*

$$|(\zeta_0 - \zeta_{0,h}, \chi)| \lesssim h^{\min((\pi/\omega) - \epsilon, k)} \|\chi\|_{H^1(\Omega)} \|\mathbf{f}\|_{L^2(\Omega)} \quad \forall \chi \in H^1(\Omega) \quad (5.46) \quad [\text{eq:err.est.}]$$

$$|(\zeta_1 - \zeta_{1,h}, \chi)| \lesssim h^{\min((\pi/\omega) - \epsilon, k)} \|\chi\|_{H^1(\Omega)} \quad \forall \chi \in H^1(\Omega). \quad (5.47) \quad [\text{eq:err.est.}]$$

*Proof.* Consider the following dual problem. Given  $\chi \in H^1(\Omega)$ , let  $(\lambda, \mu) \in H^1(\Omega) \times H_0^1(\Omega)$  solve

$$\mathcal{A}((\psi, \eta), (\lambda, \mu)) + (\psi, 1)(\lambda, 1) = (\psi, \chi) \quad \forall (\psi, \eta) \in H^1(\Omega) \times H_0^1(\Omega). \quad (5.48) \quad [\text{eq:DP.coupl.}]$$

Observe that testing the above well-posed dual problem with  $(\lambda, \mu)$  gives the following stability estimate,

$$\|\lambda\|_{H^1(\Omega)}^2 + \|\mu\|_{H^1(\Omega)}^2 \lesssim \|\chi\|_{L^2(\Omega)}^2. \quad (5.49) \quad [\text{eq:DP.stab.}]$$

Using the definition of  $\mathcal{A}(\cdot, \cdot)$ , we can rewrite (5.48) as the following coupled system:

$$(\nabla \times \psi, \nabla \times \lambda) + (\psi, 1)(\lambda, 1) = (\psi, \chi) + \gamma^{1/2}(\psi, \mu) \quad \forall \psi \in H^1(\Omega), \quad (5.50) \quad [\text{eq:DP.coupl.}]$$

$$(\nabla \times \eta, \nabla \times \mu) + \beta(\eta, \mu) = -\gamma^{1/2}(\lambda, \eta) \quad \forall \eta \in H_0^1(\Omega). \quad (5.51) \quad [\text{eq:DP.coupl.}]$$

Thus, using Lemma 2.2 on (5.50), and Lemma 2.1 on (5.51), followed by the stability estimate (5.49), we have the following regularity result for the dual problem,

$$\|\lambda\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} + \|\mu\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \lesssim \|\chi\|_{H^1(\Omega)}. \quad (5.52) \quad [\text{eq:DP.regul.}]$$

Testing (5.48) with  $(\psi, \eta) = (\zeta_0 - \zeta_{0,h}, \xi_0 - \xi_{0,h}) \in H^1(\Omega) \times H_0^1(\Omega)$  and using the fact that  $\zeta_0, \zeta_{0,h} \in L_0^2(\Omega)$ , we have

$$\begin{aligned} (\zeta_0 - \zeta_{0,h}, \chi) &= \mathcal{A}((\zeta_0 - \zeta_{0,h}, \xi_0 - \xi_{0,h}), (\lambda, \mu)) \\ &= \mathcal{A}((\zeta_0 - \zeta_{0,h}, \xi_0 - \xi_{0,h}), (\lambda - I_{k,h}\lambda, \mu - I_{k,h}\mu)) + \mathcal{A}((\zeta_0 - \zeta_{0,h}, \xi_0 - \xi_{0,h}), (I_{k,h}\lambda, I_{k,h}\mu)). \end{aligned} \quad (5.53) \quad [\text{eq:err.coupl.}]$$

The first term in (5.53) is bounded using the definition of  $\mathcal{A}$ , Cauchy-Schwarz inequality, (5.44), (5.45), Lemma 5.6 and (5.52) to get

$$\begin{aligned} &\mathcal{A}((\zeta_0 - \zeta_{0,h}, \xi_0 - \xi_{0,h}), (\lambda - I_{k,h}\lambda, \mu - I_{k,h}\mu)) \\ &\lesssim (\|\zeta_0 - \zeta_{0,h}\|_{H^1(\Omega)} + |\xi_0 - \xi_{0,h}|_{H^1(\Omega)}) \times (\|\lambda - I_{k,h}\lambda\|_{H^1(\Omega)} + |\mu - I_{k,h}\mu|_{H^1(\Omega)}) \\ &\lesssim (\|\lambda - I_{k,h}\lambda\|_{H^1(\Omega)} + |\mu - I_{k,h}\mu|_{H^1(\Omega)}) \|\mathbf{f}\|_{L^2(\Omega)} \\ &\lesssim h^{\min((\pi/\omega) - \epsilon, k)} \|\chi\|_{H^1(\Omega)} \|\mathbf{f}\|_{L^2(\Omega)} \end{aligned} \quad (5.54) \quad [\text{eq:err.coupl.}]$$

The second term in (5.53) is bounded using the error equation (5.43) for  $(\psi_h, \eta_h) = (I_{k,h}\lambda, I_{k,h}\mu)$  as follows:

$$\begin{aligned} & \mathcal{A}((\zeta_0 - \zeta_{0,h}, \xi_0 - \xi_{0,h}), (I_{k,h}\lambda, I_{k,h}\mu)) \\ &= \gamma^{-1/2}(\mathbf{f}, \nabla \times (I_{k,h}\lambda - \Pi_{k,h}^1 I_{k,h}\lambda)) + \mathcal{A}_h((\zeta_{0,h}, \xi_{0,h}), (I_{k,h}\lambda, I_{k,h}\mu)) - \mathcal{A}((\zeta_{0,h}, \xi_{0,h}), (I_{k,h}\lambda, I_{k,h}\mu)) \\ &\lesssim \|\mathbf{f}\|_{L^2(\Omega)} |(I - \Pi_{k,h}^1) I_{k,h}\lambda|_{1,h} + \mathcal{A}_h((\zeta_{0,h}, \xi_{0,h}), (I_{k,h}\lambda, I_{k,h}\mu)) - \mathcal{A}((\zeta_{0,h}, \xi_{0,h}), (I_{k,h}\lambda, I_{k,h}\mu)). \end{aligned} \quad (5.55) \quad [\text{eq:err.coup}]$$

The difference in (5.55) can be rewritten as follows:

$$\begin{aligned} & \mathcal{A}_h((\zeta_{0,h}, \xi_{0,h}), (I_{k,h}\lambda, I_{k,h}\mu)) - \mathcal{A}((\zeta_{0,h}, \xi_{0,h}), (I_{k,h}\lambda, I_{k,h}\mu)) \\ &= (a_h(\zeta_{0,h}, I_{k,h}\lambda) - (\nabla \times \zeta_{0,h}, \nabla \times I_{k,h}\lambda)) + (a_h(\xi_{0,h}, I_{k,h}\mu) - (\nabla \times \xi_{0,h}, \nabla \times I_{k,h}\mu)) \\ &\quad + \left( \gamma^{1/2}(\Pi_{k,h}^0 I_{k,h}\lambda, \Pi_{k,h}^0 \xi_{0,h}) - \gamma^{1/2}(I_{k,h}\lambda, \xi_{0,h}) \right) + \left( \gamma^{1/2}(\zeta_{0,h}, I_{k,h}\mu) - \gamma^{1/2}(\Pi_{k,h}^0 \zeta_{0,h}, \Pi_{k,h}^0 I_{k,h}\mu) \right) \\ &\quad + (\beta(\Pi_{k,h}^0 \xi_{0,h}, \Pi_{k,h}^0 I_{k,h}\mu) - \beta(\xi_{0,h}, I_{k,h}\mu)) \\ &=: T_1 + T_2 + T_3 + T_4 + T_5. \end{aligned}$$

The terms  $T_1$  and  $T_2$  can be bounded similar to (5.24) to obtain

$$\begin{aligned} T_1 + T_2 &\lesssim h^{\min((\pi/\omega)-\epsilon, k)} \|\mathbf{f}\|_{L^2(\Omega)} \left( \|\lambda\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} + \|\mu\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \right) \\ &\stackrel{(5.52)}{\lesssim} h^{\min((\pi/\omega)-\epsilon, k)} \|\mathbf{f}\|_{L^2(\Omega)} \|\chi\|_{H^1(\Omega)}. \end{aligned} \quad (5.56) \quad [\text{eq:err.coup}]$$

The term  $T_3$  is estimated using the definition of  $\Pi_{k,h}^0$  and Cauchy-Schwarz inequality as follows,

$$\begin{aligned} T_3 &= \gamma^{1/2}(\Pi_{k,h}^0 I_{k,h}\lambda - I_{k,h}\lambda, \xi_{0,h}) \\ &\lesssim \|\Pi_{k,h}^0 I_{k,h}\lambda - I_{k,h}\lambda\|_{L^2(\Omega)} \|\xi_{0,h}\|_{L^2(\Omega)} \\ &\lesssim h^{\min((\pi/\omega)-\epsilon, k)} |\lambda|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} |\xi_{0,h}|_{H^1(\Omega)} \\ &\stackrel{(5.52), (5.45)}{\lesssim} h^{\min((\pi/\omega)-\epsilon, k)} \|\mathbf{f}\|_{L^2(\Omega)} \|\chi\|_{H^1(\Omega)}. \end{aligned} \quad (5.57) \quad [\text{eq:err.coup}]$$

Where the term  $\|\Pi_{k,h}^0 I_{k,h}\lambda - I_{k,h}\lambda\|_{L^2(\Omega)}$  is bounded analogous to [4, Lemma 3.16] using (5.9) and approximation property of  $\Pi_{k,h}^0$ . The terms  $T_4$  and  $T_5$  are bounded analogously.

$$T_4 + T_5 \lesssim h^{\min((\pi/\omega)-\epsilon, k)} \|\mathbf{f}\|_{L^2(\Omega)} \|\chi\|_{H^1(\Omega)}. \quad (5.58) \quad [\text{eq:err.coup}]$$

Substituting the bounds (5.56), (5.57), and (5.58) into (5.55), and noting that the term  $|(I - \Pi_{k,h}^1) I_{k,h}\lambda|_{1,h}$  can be bounded similar to the simply connected case gives

$$\mathcal{A}((\zeta_0 - \zeta_{0,h}, \xi_0 - \xi_{0,h}), (I_{k,h}\lambda, I_{k,h}\mu)) \lesssim h^{\min((\pi/\omega)-\epsilon, k)} \|\mathbf{f}\|_{L^2(\Omega)} \|\chi\|_{H^1(\Omega)}. \quad (5.59) \quad [\text{eq:err.coup}]$$

Thus combining (5.54) and (5.59) in (5.53) leads to the following estimate (5.46). Furthermore, it is easy to see that one can readily obtain (5.47) following similar arguments.  $\square$

`:xi.est.mc?` **Lemma 5.15.** *For any  $\epsilon > 0$ , there exists a positive constant dependent on  $\epsilon$  and independent of  $h$  such that for  $k = 1, 2$  and  $\gamma > 0$ ,*

$$|\xi - \xi_h|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega)-\epsilon, k)} \|\mathbf{f}\|_{L^2(\Omega)}, \quad (5.60) \quad [\text{eq:err.est..}]$$

$$|\xi - \Pi_{k,h}^1 \xi_h|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega)-\epsilon, k)} \|\mathbf{f}\|_{L^2(\Omega)}. \quad (5.61) \quad [\text{eq:err.est..}]$$

*Proof.* Consider the auxiliary problem. Given  $\zeta_{0,h} \in V_h^0 \cap L_0^2(\Omega)$ , let  $\tilde{\xi}_0 \in H_0^1(\Omega)$  solve

$$(\nabla \times \tilde{\xi}_0, \nabla \times \eta) + \beta(\tilde{\xi}_0, \eta) = \gamma^{1/2}(\zeta_{0,h}, \eta) \quad \forall \eta \in H_0^1(\Omega). \quad (5.62) \quad [\text{eq:P.aux.xi}]$$

This implies

$$\tilde{\xi}_0 \in H^{1+(\pi/\omega)-\epsilon}(\Omega) \quad \text{and} \quad \|\tilde{\xi}_0\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \lesssim \|\zeta_{0,h}\|_{L^2(\Omega)}. \quad (5.63) \quad [\text{tildexi0.re}]$$

due to Lemma 2.1. Compaing the  $\xi_0$  part of the coupled problem (3.13),

$$(\nabla \times \xi_0, \nabla \times \eta) + \beta(\xi_0, \eta) = \gamma^{1/2}(\zeta_0, \eta) \quad \forall \eta \in H_0^1(\Omega), \quad (5.64) \quad [\text{eq:P.xi0}]$$

with (5.62), and using the duality estimate (5.46) gives,

$$|\xi_0 - \tilde{\xi}_0|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega)-\epsilon, k)} \|\mathbf{f}\|_{L^2(\Omega)}. \quad (5.65) \quad [\text{eq:xi0.a.mc}]$$

Now it follows from

$$a_h(\xi_{0,h}, \eta_h) + \beta(\Pi_{k,h}^0 \xi_{0,h}, \Pi_{k,h}^0 \eta_h) = \gamma^{1/2}(\Pi_{k,h}^0 \zeta_{0,h}, \Pi_{k,h}^0 \eta_h) \quad \forall \eta_h \in V_h^0, \quad (5.66) \boxed{\text{eq:Ph.mc.xi}}$$

which is the  $\xi_{0,h}$  part of the coupled discrete problem (4.15), and (5.62) that  $\xi_{0,h}$  is also a virtual element approximation of  $\tilde{\xi}_0$ . Hence, using similar arguments as in the proof of Lemma 5.11, we can get the following estimate,

$$\begin{aligned} |\tilde{\xi}_0 - \xi_{0,h}|_{H^1(\Omega)} &\lesssim h^{\min((\pi/\omega)-\epsilon,k)} (\|\tilde{\xi}_0\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} + |\zeta_{0,h}|_{H^1(\Omega)}) \\ &\stackrel{(5.63)}{\lesssim} h^{\min((\pi/\omega)-\epsilon,k)} (\|\zeta_{0,h}\|_{L^2(\Omega)} + |\zeta_{0,h}|_{H^1(\Omega)}) \\ &\stackrel{(5.44)}{\lesssim} h^{\min((\pi/\omega)-\epsilon,k)} \|\mathbf{f}\|_{L^2(\Omega)}. \end{aligned} \quad (5.67) \boxed{\text{eq:xi0h.b.m}}$$

Combining (5.67) with (5.65) via a triangle inequality leads to

$$|\xi_0 - \xi_{0,h}|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega)-\epsilon,k)} \|\mathbf{f}\|_{L^2(\Omega)}. \quad (5.68) \boxed{\text{eq:xi0h.mc}}$$

Following similar arguments as above on the  $\xi_1$  part of the coupled problem (3.14) and its discrete counterpart (4.16), and using (5.47) gives,

$$|\xi_1 - \xi_{1,h}|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega)-\epsilon,k)}. \quad (5.69) \boxed{\text{eq:xi1h.mc}}$$

Now in view of (5.69), (5.68) and (4.14), the estimate (5.60) readily follows. Finally, the estimate (5.61) follows from similar arguments as in the proof of (5.26) and using Lemma 2.1 on (3.11).  $\square$

It remains to estimate the error in the approximation of harmonic part  $\varphi_j$  and the real numbers  $c_j$  for  $1 \leq j \leq m$ . The error estimate for  $\varphi_j$  can be estimated readily by using the abstract error estimate for second order elliptic problems approximated by virtual element methods [8, Section 4.3] and the regularity of  $\varphi_j \in H^{1+(\pi/\omega)-\epsilon}(\Omega)$ .

*est.phi.mc?* **Lemma 5.16.** *For any  $\epsilon > 0$ , there exists a positive constant dependent on  $\epsilon$  and independent of  $h$  such that for  $k = 1, 2, \gamma > 0$ , and  $1 \leq j \leq m$ ,*

$$|\varphi_j - \varphi_{j,h}|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega)-\epsilon,k)}, \quad (5.70) \boxed{\text{eq:err.est.}}$$

$$|\varphi_j - \Pi_{k,h}^1 \varphi_{j,h}|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega)-\epsilon,k)}. \quad (5.71) \boxed{\text{eq:err.est.}}$$

*Proof.* It follows from (3.2), (4.19) and the abstract error estimate [8, Section 4.3], we have

$$\|\varphi_j - \varphi_{j,h}\|_{h_0} \lesssim \|\varphi_j - I_{k,h} \varphi_j\|_{h_0} + |\varphi_j - \Pi_{k,h}^1 \varphi_j|_{1,h} + \|\varphi_j - \Pi_{k,h}^1 \varphi_j\|_{h_0}.$$

Using Lemma 5.8 and Assumption 5.7 leads to

$$\|\varphi_j - \varphi_{j,h}\|_{h_0} \lesssim h^{\min((\pi/\omega)-\epsilon,k)} \|\varphi_j\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \quad (5.72) \boxed{\text{eq:varphi.a}}$$

In view of (3.2) there exists a lifting  $g_j \in H^1(\Omega)$  such that  $g_j|_{\Gamma_0} = 0$  and  $g_j|_{\Gamma_l} = \delta_{jl}$  for  $1 \leq l \leq m$ . Moreover, since boundary data are just constants and finite ( $j = \{1, \dots, m\}$ ), we can ensure  $\|g_j\|_{H^1(\Omega)} \leq C_\Omega$ . Thus, applying Lemma 2.1 to (5.72) gives

$$\|\varphi_j\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \lesssim \|g_j\|_{H^1(\Omega)} \lesssim C_\Omega, \quad (5.73) \boxed{\text{eq:varphi.d}}$$

which results to (5.70). Finally, using similar arguments as in the proof of (5.26) with the regularity result for  $\varphi_j$  leads to (5.71).  $\square$

Before comparing  $c_j$  and  $c_{j,h}$ , we first note that (5.36) implies

$$|(\mathbf{f}, \nabla \varphi_j) - (\mathbf{f}, \nabla \Pi_{k,h}^1 \varphi_{j,h})| \lesssim h^{\min((\pi/\omega)-\epsilon,k)} \|\mathbf{f}\|_{L^2(\Omega)}. \quad (5.74) \boxed{\text{eq:err.est.}}$$

Testing (3.2a) with  $\varphi_j - I_{k,h} \varphi_j \in V_h^0 \subset H_0^1(\Omega)$  and (4.19a) with  $\varphi_{j,h} - I_{k,h} \varphi_j \in V_h^0$  results in

$$\begin{aligned} (\nabla \varphi_i, \nabla \varphi_j) - a_h(\varphi_{i,h}, \varphi_{j,h}) &= (\nabla \varphi_i, \nabla I_{k,h} \varphi_j) - a_h(\varphi_{i,h}, I_{k,h} \varphi_j) \\ &= \sum_{D \in \mathcal{T}_h} (\nabla \varphi_i, \nabla (I_{k,D} \varphi_j - \Pi_{k,D}^1 I_{k,D} \varphi_j)) + (\nabla (\varphi_i - \Pi_{k,D}^1 \varphi_{i,h}), \nabla \Pi_{k,D}^1 I_{k,D} \varphi_j) \\ &\quad + S^D((I - \Pi_{k,D}^1) \varphi_{i,h}, (I - \Pi_{k,D}^1) I_{k,D} \varphi_j). \end{aligned} \quad (5.75) \boxed{\text{eq:err.est.}}$$

The first term on the right-hand side of (5.75) is bounded using Cauchy-Schwarz inequality, (5.10) and (5.73) to get

$$\sum_{D \in \mathcal{T}_h} (\nabla \varphi_i, \nabla (I_{k,D} \varphi_j - \Pi_{k,D}^1 I_{k,D} \varphi_j)) \lesssim h^{\min((\pi/\omega) - \epsilon, k)}. \quad (5.76) \quad [\text{eq:err.est.}]$$

The second term on the right-hand side of (5.75) is bounded using Cauchy-Schwarz inequality, the error estimate (5.71), stability of projection and interpolation operators (5.2), (5.9), and (5.73) to get

$$\begin{aligned} & \sum_{D \in \mathcal{T}_h} (\nabla (\varphi_i - \Pi_{k,D}^1 \varphi_{i,h}), \nabla \Pi_{k,D}^1 I_{k,D} \varphi_j) \\ & \lesssim \left( \sum_{D \in \mathcal{T}_h} \|\nabla (\varphi_i - \Pi_{k,D}^1 \varphi_{i,h})\|_{L^2(D)}^2 \right)^{1/2} \left( \sum_{D \in \mathcal{T}_h} \|\nabla \Pi_{k,D}^1 I_{k,D} \varphi_j\|_{L^2(D)}^2 \right)^{1/2} \\ & \lesssim h^{\min((\pi/\omega) - \epsilon, k)}. \end{aligned} \quad (5.77) \quad [\text{eq:err.est.}]$$

The stabilization term on the right-hand side of (5.75) is bounded using the Cauchy-Schwarz inequality to obtain

$$\sum_{D \in \mathcal{T}_h} S^D ((I - \Pi_{k,D}^1) \varphi_{i,h}, (I - \Pi_{k,D}^1) I_{k,D} \varphi_j) \lesssim \|\varphi_{i,h} - \Pi_{k,h}^1 \varphi_{i,h}\|_{h_0} \|\varphi_j - I_{k,h} \varphi_j\|_{h_0}.$$

The second term in the above inequality is bounded using (5.15) and the first term is bounded using the definition of  $S^D(\cdot, \cdot)$ , (4.13), Sobolev inequality (5.1), Lemma 5.5 and (5.73) as follows,

$$\begin{aligned} \|\varphi_{i,h} - \Pi_{k,h}^1 \varphi_{i,h}\|_{h_0}^2 & \lesssim \sum_{D \in \mathcal{T}_h} \|\varphi_{i,h} - \Pi_{k,h}^1 \varphi_{i,h}\|_{L^\infty(D)}^2 \\ & \lesssim \sum_{D \in \mathcal{T}_h} h_D^{-2} \|\varphi_{i,h} - \Pi_{k,D}^1 \varphi_{i,h}\|_{L^2(D)}^2 + |\varphi_{i,h} - \Pi_{k,D}^1 \varphi_{i,h}|_{H^1(D)}^2 \\ & \lesssim C_\Omega. \end{aligned} \quad (5.78) \quad [\text{eq:err.est.}]$$

Substituting the bounds (5.76), (5.77), and (5.78) into (5.75) leads to

$$|(\nabla \varphi_i, \nabla \varphi_j) - a_h(\varphi_{i,h}, \varphi_{j,h})| \lesssim h^{\min((\pi/\omega) - \epsilon, k)} \quad \text{for } 1 \leq i, j \leq m. \quad (5.79) \quad [\text{eq:err.est.}]$$

**Lemma 5.17.** For  $h < h_0$ , and any  $\epsilon > 0$ , there exists a positive constant dependent on  $\epsilon$  and independent of  $h$  such that for  $k = 1, 2$ ,  $\gamma > 0$ ,  $1 \leq j \leq m$ , and  $h_0 =$ ,

$$|c_j - c_{j,h}| \lesssim h^{\min((\pi/\omega) - \epsilon, k)} \|\mathbf{f}\|_{L^2(\Omega)}. \quad (5.80) \quad [\text{eq:err.est.}]$$

## 6. PROOF OF SOME LEMMAS

**6.1. Proof of Lemma 5.8.** Observing that  $\Pi_{k,h}^1((I - \Pi_{k,h}^1)I_{k,h}v) = 0$  due to linearity and idempotency of  $\Pi_{k,h}^0$  we deduce that

$$\begin{aligned} \|I_{k,h}v - \Pi_{k,h}^1 I_{k,h}v\|_{h_0}^2 &= \sum_{D \in \mathcal{T}_h} S^D ((I - \Pi_{k,h}^1)I_{k,h}v, (I - \Pi_{k,h}^1)I_{k,h}v) \\ &= \sum_{D \in \mathcal{T}_h} \sum_{i=1}^{N_{\partial D}^{\text{dof}}} \chi_i ((I - \Pi_{k,h}^1)I_{k,h}v)^2 \\ &\stackrel{(4.13)}{\lesssim} \sum_{D \in \mathcal{T}_h} \|(I - \Pi_{k,h}^1)I_{k,h}v\|_{L^\infty(D)}^2 \\ &\stackrel{(5.1)}{\lesssim} \sum_{D \in \mathcal{T}_h} h_D^{-2} \|(I - \Pi_{k,D}^1)I_{k,D}v\|_{L^2(D)}^2 + |(I - \Pi_{k,D}^1)I_{k,D}v|_{H^1(D)}^2 \\ &\quad + h_D^{2\delta} |(I - \Pi_{k,D}^1)I_{k,D}v|_{H^{1+\delta}(D)}^2 \end{aligned}$$

Now splitting  $(I - \Pi_{k,D}^1)I_{k,D}v = I_{k,D}v - v + v - \Pi_{k,D}^1 I_{k,D}v$  and using the approximation properties of  $I_{k,D}$  from (5.10) and (5.11), we get the bound on the first term stated in (5.14). Similarly, the second

term in (5.14) follows from the definition of  $|\cdot|_{1,h}$ , the above mentioned split followed by a triangle inequality and the approximation properties of the interpolation operator stated in Lemma 5.6.

Using the linearity and idempotency of  $\Pi_{k,h}^1$  and (4.13), we can get the following

$$\begin{aligned} \|v - \Pi_{k,h}^1 v\|_{h_\beta}^2 &\lesssim \sum_{D \in \mathcal{T}_h} \|v - \Pi_{k,D}^1 v\|_{L^\infty(D)}^2 + \beta \|v - \Pi_{k,D}^1 v\|_{L^2(D)}^2 \\ &\stackrel{(4.13),(5.1)}{\lesssim} \sum_{D \in \mathcal{T}_h} h_D^{-2} \|(I - \Pi_{k,D}^1)v\|_{L^2(D)}^2 + |(I - \Pi_{k,D}^1)v|_{H^1(D)}^2 \\ &\quad + h_D^{2\delta} |(I - \Pi_{k,D}^1)v|_{H^{1+\delta}(D)}^2 + \beta \|v - \Pi_{k,D}^1 v\|_{L^2(D)}^2. \end{aligned}$$

Now using the approximation property of  $\Pi_{k,D}^1$  in Lemma 5.5 leads to the desired estimate on the first term in (5.15). Similarly, we have

$$\begin{aligned} \|v - I_{k,h} v\|_{h_\beta}^2 &= \sum_{D \in \mathcal{T}_h} |\Pi_{k,D}^1(v - I_{k,D} v)|_{H^1(D)}^2 + \sum_{i=1}^{N_{\partial D}^{\text{dof}}} \chi_i((I - \Pi_{k,D}^1)(I - I_{k,D})v)^2 + \beta \|\Pi_{k,D}^0(v - I_{k,D} v)\|_{L^2(D)}^2 \\ &= \sum_{D \in \mathcal{T}_h} |\Pi_{k,D}^1(v - I_{k,D} v)|_{H^1(D)}^2 + \sum_{i=1}^{N_{\partial D}^{\text{dof}}} \chi_i(\Pi_{k,D}^1(I - I_{k,D})v)^2 + \beta \|\Pi_{k,D}^0(v - I_{k,D} v)\|_{L^2(D)}^2 \\ &\lesssim \sum_{D \in \mathcal{T}_h} |v - I_{k,D} v|_{H^1(D)}^2 + \|\Pi_{k,D}^1(I - \Pi_{k,D}^1)v\|_{L^\infty(D)}^2 + \beta \|v - I_{k,D} v\|_{L^2(D)}^2, \end{aligned}$$

where in the passage to the second equality we use the fact that  $\chi_i(v - I_{k,D} v) = 0$  for boundary degrees of freedom by definition of interpolation operator (5.7). In the last inequality we used the stability of projectors (5.2), (5.5) and the property (4.13). The first and last terms are bounded using the approximation properties of  $I_{k,D}$  from Lemma 5.6. The second term can be estimated as in [4, (4.24)]. Combining all the resulting bounds leads to the desired estimate on the second term in (5.15).

## 7. NUMERICAL RESULTS

`sec:numexp` We use the boundary version for the classical **Dofi-Dofi** stabilizer defined in (4.12) in our numerical experiments.

`:smooth_sol` 7.1. **Smooth solution on a convex simply connected domain.** We consider the problem (1.2) with  $\gamma = \beta = 0$  and  $\Omega = (0, 1)^2$ . Given  $\phi(x, y) = \sin^3(\pi x) \sin^3(\pi y)$ , the manufactured solution is given by

$$\mathbf{u} = \nabla \times \phi, \quad \text{with } \mathbf{f} = -\nabla \times (\Delta(\nabla \times \mathbf{u})).$$

We solve (1.2) using the scheme described in Section 4 for orders  $k = 1, 2$  on a sequence of unstructured Voronoi meshes. Since the virtual element solution is not known explicitly inside the element, we compare the manufactured solution with suitable projection of the discrete solution. The errors we measure are as follows:

$$\begin{aligned} e_{\mathbf{u}_h} &:= \|\mathbf{u} - \mathbf{u}_h\|_{L^2(\Omega)} = \sqrt{\sum_{D \in \mathcal{T}_h} \|\mathbf{u} - \mathbf{u}_h\|_{L^2(D)}^2}, \\ e_{\xi_h} &:= |\xi - \Pi_{k,h}^1 \xi_h|_{1,h} = \sqrt{\sum_{D \in \mathcal{T}_h} |\xi - \Pi_{k,D}^1 \xi_h|_{H^1(D)}^2}, \quad \text{with } \xi = \nabla \times \mathbf{u}. \end{aligned}$$

and are reported in Tables 1 and 2 on a sequence of structured and unstructured voronoi meshes, respectively (see Figure 1). We recall that in this experiment  $\mathbf{u}_h$  is computed using (4.11). The observed rates of convergence are in agreement with the theoretical rates predicted by Theorem 5.13 and (5.26) with  $\omega = \pi/2$ .

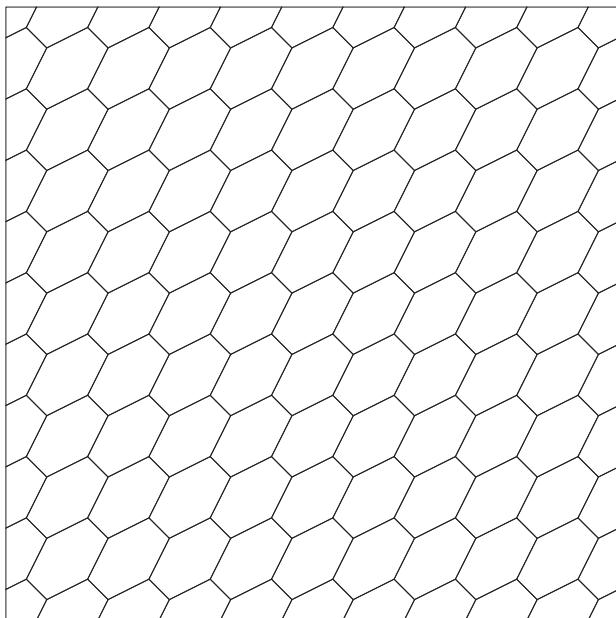
TABLE 1. Experimental errors and orders of convergence for Experiment 7.1 on structured voronoi meshes.

$h$	$N^{\text{dofs}}$	$e_{\mathbf{u}_h}$	rate	$e_{\xi_h}$	rate
		$k = 1$			

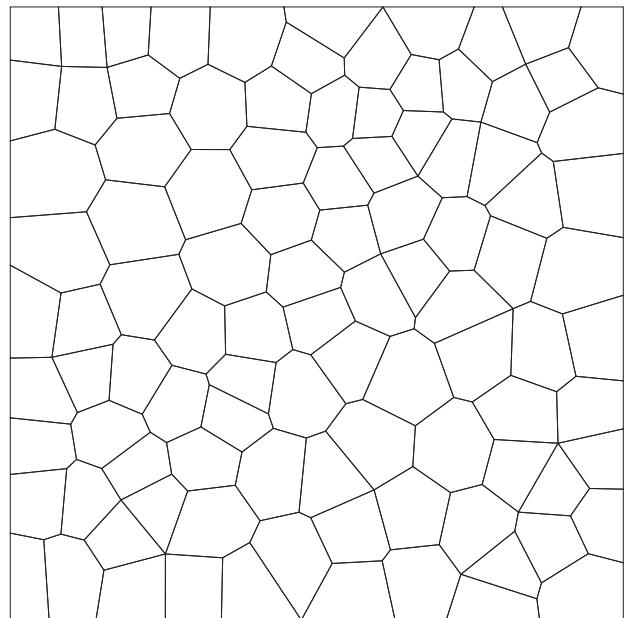
3.7637e-01	48	1.1963e+00	-	1.0194e+02	-
1.8360e-01	184	5.3872e-01	1.1114	5.1507e+01	0.9510
8.4283e-02	1234	1.8091e-01	1.4016	1.8902e+01	1.2876
4.1405e-02	5947	8.1748e-02	1.1176	8.6614e+00	1.0979
2.0988e-02	26477	3.8903e-02	1.0929	4.1493e+00	1.0831
1.0757e-02	96734	2.0202e-02	0.9804	2.1549e+00	0.9803
$k = 2$					
3.7637e-01	145	3.0531e-01	-	3.4261e+01	-
1.8360e-01	567	7.2772e-02	1.9977	8.8858e+00	1.8800
8.4283e-02	3867	1.1100e-02	2.4152	1.3484e+00	2.4218
4.1405e-02	18693	2.3647e-03	2.1756	2.8656e-01	2.1789
2.0988e-02	82953	5.2424e-04	2.2171	6.3400e-02	2.2201
1.0757e-02	303467	1.4262e-04	1.9477	1.7275e-02	1.9453

TABLE 2. Experimental errors and orders of convergence for Experiment 7.1 on unstructured Voronoi meshes.

$h$	$N^{\text{Dofs}}$	$e_{\mathbf{u}_h}$	rate	$e_{\xi_h}$	rate
$k = 1$					
3.7637e-01	48	1.1963	-	1.0194e+02	-
1.8360e-01	184	5.3872e-01	1.1114	5.1507e+01	0.9510
8.4283e-02	1234	1.8091e-01	1.4016	1.8902e+01	1.2876
4.1405e-02	5947	8.1748e-02	1.1176	8.6614e+00	1.0979
2.0988e-02	26477	3.8903e-02	1.0929	4.1493e+00	1.0831
1.0757e-02	96734	2.0202e-02	0.9804	2.1549e+00	0.9803
$k = 2$					
3.7637e-01	145	3.0531e-01	-	3.4261e+01	-
1.8360e-01	567	7.2772e-02	1.9977	8.8858e+00	1.8800
8.4283e-02	3867	1.1100e-02	2.4152	1.3484e+00	2.4218
4.1405e-02	18693	2.3647e-03	2.1756	2.8656e-01	2.1789
2.0988e-02	82953	5.2424e-04	2.2171	6.3400e-02	2.2201
1.0757e-02	303467	1.4262e-04	1.9477	1.7275e-02	1.9453



(A) Structured voronoi mesh



(B) Unstructured voronoi mesh

FIGURE 1

:smoothdata> 7.2. Smooth data on a convex simply connected domain. We keep the same domain and discretization introduced in Experiment 7.1. We take  $\beta = \gamma = 0$ , and choose

$$\mathbf{f} = ((x^2 + 1)\sin(x) + xy^3 + 2, (y^2 + 1)\cos(x) + x^3y^2 - 1).$$

Then we solve the sequence of discrete problems (4.5)-(4.11) for orders  $k = 1, 2$ . Since the exact solution is unknown, we compute relative errors

$$\text{rel } e_{\mathbf{u}_h}^i := \frac{\|\mathbf{u}_h^i - \mathbf{u}_h^{i+1}\|_{L^2(\Omega)}}{\|\mathbf{u}_h^{i+1}\|_{L^2(\Omega)}}, \quad \text{and} \quad \text{rel } e_{\xi_h}^i := \frac{|\Pi_{k,h}^{1,i} \xi_h^i - \Pi_{k,h}^{1,i+1} \xi_h^{i+1}|_{1,h_{i+1}}}{|\Pi_{k,h}^{1,i+1} \xi_h^{i+1}|_{1,h_{i+1}}},$$

where  $i$  denotes the mesh level. Then we report them in Tables 3 and 4 on a sequence of structured and unstructured voronoi meshes, respectively (see Figures 1). The observed rates of convergence are in agreement with the theoretical rates predicted by Theorem 5.13 and (5.26) with  $\omega = \pi/2$ .

TABLE 3. Relative errors and convergence rates for Experiment 7.2 on structured voronoi meshes for orders  $k = 1, 2$ .

$h$	$N^{\text{Dofs}}$	$\text{rel } e_{\mathbf{u}_h}$	rate	$\text{rel } e_{\xi_h}$	rate
$k = 1$					
2.9814e-01	90	-	-	-	-
1.4907e-01	280	2.3814e-01	-	4.6021e-01	-
7.4536e-02	960	1.2406e-01	0.9408	2.7185e-01	0.7595
3.7268e-02	3520	6.2627e-02	0.9862	1.4889e-01	0.8686
1.8634e-02	13440	3.1449e-02	0.9938	7.8161e-02	0.9297
9.3169e-03	52480	1.5765e-02	0.9963	4.0092e-02	0.9631
$k = 2$					
2.9814e-01	251	-	-	-	-
1.4907e-01	801	3.2060e-02	-	8.4945e-02	-
7.4536e-02	2801	1.0155e-02	1.6586	2.8971e-02	1.5519
3.7268e-02	10401	2.9260e-03	1.7952	9.0584e-03	1.6773
1.8634e-02	40001	7.8719e-04	1.8941	2.6825e-03	1.7557
9.3169e-03	156801	2.0419e-04	1.9468	7.6646e-04	1.8073

TABLE 4. Relative errors and convergence rates for Experiment 7.2 on unstructured voronoi meshes for orders  $k = 1, 2$ .

$h$	$N^{\text{Dofs}}$	$\text{rel } e_{\mathbf{u}_h}$	rate	$\text{rel } e_{\xi_h}$	rate
$k = 1$					
3.7637e-01	48	-	-	-	-
1.8360e-01	184	2.5848e-01	-	4.9213e-01	-
8.4283e-02	1234	1.2341e-01	0.9496	2.7482e-01	0.7483
4.1405e-02	5947	4.6503e-02	1.3731	1.1872e-01	1.1809
2.0988e-02	26477	2.1339e-02	1.1464	5.3990e-02	1.1597
1.0757e-02	96734	1.0318e-02	1.0873	2.6389e-02	1.0710
$k = 2$					
3.7637e-01	145	-	-	-	-
1.8360e-01	567	3.8655e-02	-	1.1907e-01	-
8.4283e-02	3867	1.1807e-02	1.5233	3.7556e-02	1.4821
4.1405e-02	18693	2.0839e-03	2.4402	7.5986e-03	2.2481
2.0988e-02	82953	4.1737e-04	2.3666	1.9587e-03	1.9952
1.0757e-02	303467	9.6109e-05	2.1971	3.8104e-04	2.4494

`nsmoothdata` 7.3. **Piecewise smooth data on a non-convex simply connected domain.** We solve (1.2) on a L-shaped domain  $\Omega = (-1, 1)^2 \setminus (0, 1) \times (-1, 0)$  with  $\beta = \gamma = 0$ , and a piecewise constant vector field  $\mathbf{f}$ ,

$$\mathbf{f} = \begin{cases} [1/4, 5/4]^\top, & \text{if } |x| < 2^{-1/2}, \\ [1/2, 3/2]^\top, & \text{if } 2^{-1/2} \leq |x| < 1, \\ [1, 2]^\top, & \text{if } |x| \geq 1. \end{cases}$$

Since, the exact solution is unknown, we compute relative errors which are reported in Table 5 and 6 on a sequence of structured and unstructured voronoi meshes (see Figure 2). The observed rates of convergence are in agreement with the theoretical rates predicted by Theorem 5.13 for  $\omega = 3\pi/2$ , and the convergence rates for  $\xi_h$  are pre-asymptotic.

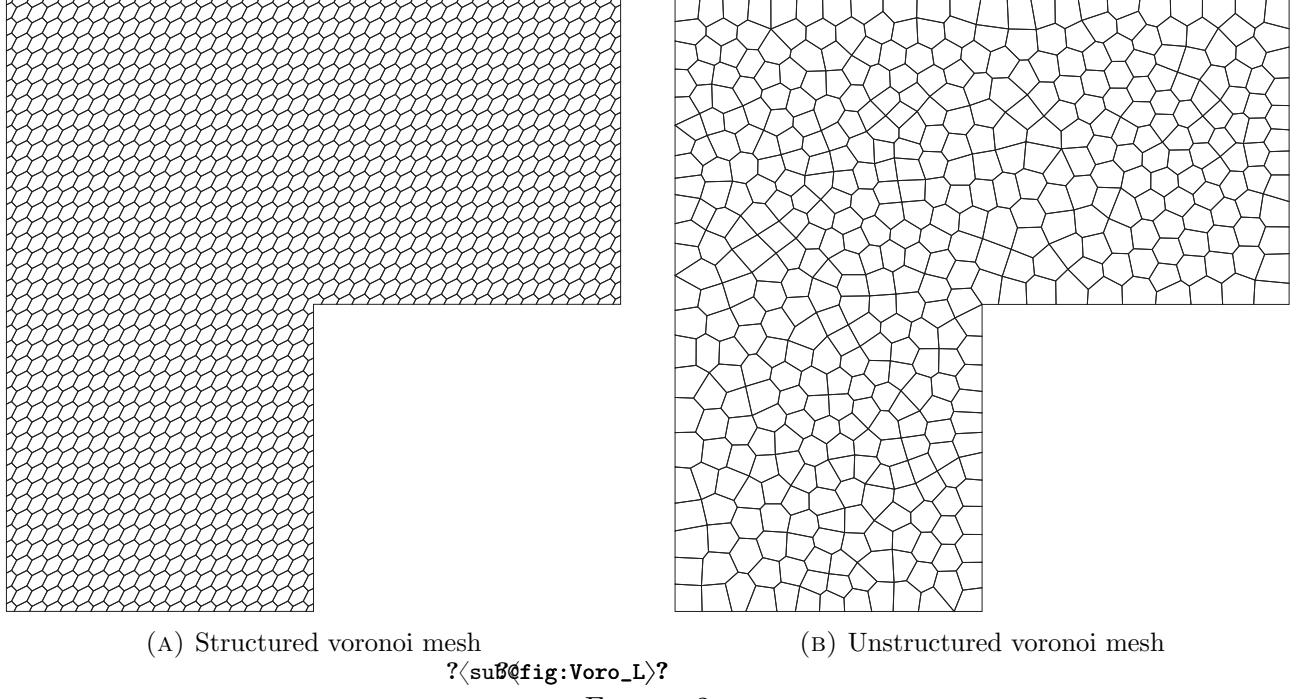


FIGURE 2

TABLE 5. Relative errors and convergence rates for Experiment 7.3 on structured voronoi meshes.

$h$	$N^{\text{Dofs}}$	$\text{rel } e_{\mathbf{u}_h}$	rate	$\text{rel } e_{\xi_h}$	rate
$k = 1$					
2.9814e-01	230	-	-	-	-
1.4907e-01	760	1.7658e-01	-	3.1600e-01	-
7.4536e-02	2720	1.0229e-01	0.7876	1.7874e-01	0.8221
3.7268e-02	10240	6.0474e-02	0.7583	9.5571e-02	0.9032
1.8634e-02	39680	3.6354e-02	0.7342	4.9551e-02	0.9476
9.3169e-03	156160	2.2161e-02	0.7141	2.5265e-02	0.9718
$k = 2$					
2.9814e-01	651	-	-	-	-
1.4907e-01	2201	7.7042e-02	-	4.6187e-02	-
7.4536e-02	8001	4.8020e-02	0.6820	1.5768e-02	1.5505
3.7268e-02	30401	3.0151e-02	0.6714	5.4008e-03	1.5458
1.8634e-02	118401	1.8975e-02	0.6681	2.1096e-03	1.3562
9.3169e-03	467201	1.1949e-02	0.6672	1.0279e-03	1.0372

TABLE 6. Relative errors and convergence rates for Experiment 7.3 on unstructured voronoi meshes.

$h$	$N^{\text{Dofs}}$	$\text{rel } e_{\mathbf{u}_h}$	rate	$\text{rel } e_{\xi_h}$	rate
$k = 1$					
6.4474e-01	47	-	-	-	-
3.2497e-01	174	3.1455e-01	-	5.5756e-01	-
1.5600e-01	783	1.4861e-01	1.0218	2.9441e-01	0.8702
7.6972e-02	4279	7.6425e-02	0.9414	1.4527e-01	0.9999
3.8974e-02	16213	4.1667e-02	0.8913	6.6347e-02	1.1516
1.9933e-02	68184	2.4315e-02	0.8033	3.3254e-02	1.0301
1.0296e-02	255458	1.6116e-02	0.6226	1.6667e-02	1.0457
$k = 2$					
6.4474e-01	143	-	-	-	-
3.2497e-01	547	8.2224e-02	-	1.5327e-01	-
1.5600e-01	2465	4.4343e-02	0.8414	4.9060e-02	1.5522
7.6972e-02	13557	2.7502e-02	0.6762	1.3178e-02	1.8607
3.8974e-02	51425	1.5440e-02	0.8483	3.0375e-03	2.1564
1.9933e-02	216367	9.2966e-03	0.7565	1.1150e-03	1.4946
1.0296e-02	810915	6.4954e-03	0.5428	6.6147e-04	0.7904

7.4. **Piecewise smooth data on a multiply connected domain with Betti number 1.** In this experiment we consider the domain  $\Omega = (0, 1)^2 \setminus (1/4, 3/4)^2$  with Betti number 1. We choose  $\beta = \gamma = 1$  and the same piecewise smooth data  $\mathbf{f}$  used in Experiment 7.3. We solve the sequence of discrete problems (4.14)-(4.19c) and post-process  $\mathbf{u}_h$  using the relation (4.20) for orders  $k = 1, 2$ . The relative errors are reported in Table 7 and 8 on a sequence of structured and unstructured voronoi meshes, respectively (see Figure 3). We also report the relative error in the coefficient  $c_j$ ,

$$\text{rel } e_{c_j}^i = \frac{|c_j^i - c_j^{i+1}|}{|c_j^{i+1}|},$$

as well as the values of  $c_j$  for orders  $k = 1, 2$

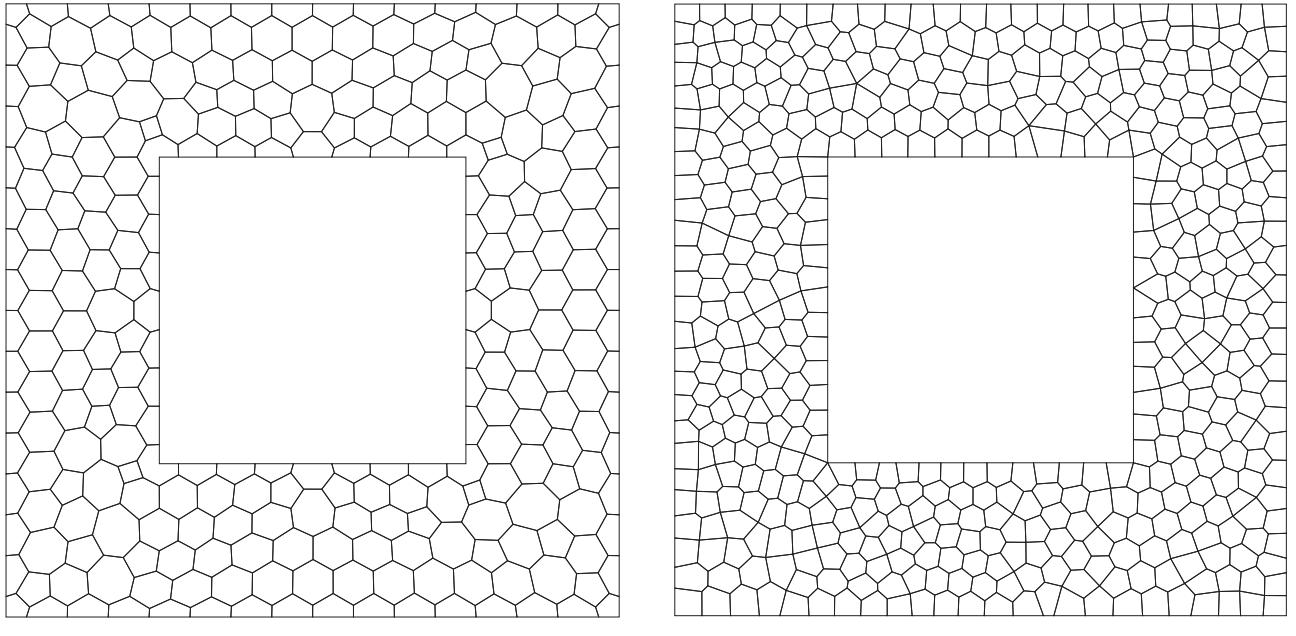


FIGURE 3

TABLE 7. Relative errors and convergence rates for Experiment 7.4 for orders  $k = 1, 2$  on structured voronoi meshes.

$h$	$N^{\text{Dofs}}$	$\text{rel } e_{\mathbf{u}_h}$	rate	$\text{rel } e_{\xi_h}$	rate	$c_1$	rate	$\text{rel } e_{c_1}$	rate
$k = 1$									
2.64e-01	96	-	-	-	-	-0.15092	-	-	-
1.67e-01	200	5.55e-01	-	5.56e-01	-	-0.15258	-	1.09e-02	-
8.85e-02	644	3.96e-01	0.53	4.78e-01	0.24	-0.15201	1.68	3.74e-03	1.68
4.77e-02	1930	2.36e-01	0.84	2.65e-01	0.95	-0.15187	2.20	9.57e-04	2.20
2.40e-02	6584	1.48e-01	0.68	1.55e-01	0.78	-0.15180	1.08	4.57e-04	1.08
1.18e-02	24092	9.62e-02	0.61	8.80e-02	0.80	-0.15177	1.20	1.96e-04	1.20
6.11e-03	93334	6.09e-02	0.69	4.97e-02	0.86	-0.15175	1.24	8.66e-05	1.24
$k = 2$									
2.64e-01	264	-	-	-	-	-0.15218	-	-	-
1.67e-01	552	2.30e-01	-	1.80e-01	-	-0.15214	-	3.20e-04	-
8.85e-02	1840	2.01e-01	0.22	1.33e-01	0.48	-0.15193	-2.25	1.33e-03	-2.25
4.77e-02	5614	1.14e-01	0.91	6.44e-02	1.17	-0.15182	0.89	7.66e-04	0.89
2.40e-02	19408	7.48e-02	0.62	3.88e-02	0.74	-0.15178	1.65	2.47e-04	1.65
1.18e-02	71592	4.98e-02	0.58	2.56e-02	0.59	-0.15176	0.85	1.35e-04	0.85
6.11e-03	278634	3.20e-02	0.67	1.63e-02	0.69	-0.15175	1.41	5.30e-05	1.41

TABLE 8. Relative errors and convergence rates for Experiment 7.4 for orders  $k = 1, 2$  on unstructured voronoi meshes.

$h$	$N^{\text{Dofs}}$	$\text{rel } e_{\mathbf{u}_h}$	rate	$\text{rel } e_{\xi_h}$	rate	$c_1$	$\text{rel } e_{c_1}$	rate
$k = 1$								
3.01e-01	66	-	-	-	-	-0.14966	-	-
1.41e-01	197	5.59e-01	-	6.38e-01	-	-0.15085	7.93e-03	-
7.32e-02	908	2.99e-01	0.95	3.72e-01	0.83	-0.15136	3.35e-03	1.32
3.41e-02	4637	1.72e-01	0.72	1.92e-01	0.86	-0.15162	1.68e-03	0.90
1.57e-02	17139	9.62e-02	0.75	9.46e-02	0.92	-0.15169	5.14e-04	1.53
7.55e-03	92388	6.22e-02	0.60	5.32e-02	0.79	-0.15173	2.47e-04	1.00
$k = 2$								
3.01e-01	204	-	-	-	-	-0.15137	-	-
1.41e-01	614	1.85e-01	-	1.59e-01	-	-0.15157	1.33e-03	-
7.32e-02	2816	1.16e-01	0.71	7.82e-02	1.08	-0.15168	7.04e-04	0.97
3.41e-02	14274	6.81e-02	0.70	4.08e-02	0.85	-0.15172	3.13e-04	1.06
1.57e-02	52278	3.88e-02	0.73	2.27e-02	0.76	-0.15174	9.70e-05	1.51
7.55e-03	284780	2.61e-02	0.54	1.50e-02	0.56	-0.15174	4.20e-05	1.14

7.5. Piecewise smooth data on a multiply connected domain with Betti number 2. We solve (1.2) on a domain  $\Omega = (-1, 1)^2 \setminus (1/4, 3/4)^2 \cup (-1/4, -3/4)^2$  with Betti number 2. We choose  $\beta = \gamma = 1$  and the same piecewise smooth data  $\mathbf{f}$  used in Experiment 7.3. The relative errors are reported in Table 9 on a sequence of structured voronoi meshes (see Figure 4). We also report the relative error in the coefficients  $c_1, c_2$  and their values.

TABLE 9. Relative errors and convergence rates for Experiment 7.5 for orders  $k = 1, 2$  on structured meshes.

$h$	$N^{\text{Dofs}}$	$\text{rel } e_{\mathbf{u}_h}$	rate	$\text{rel } e_{\xi_h}$	rate	$c_1$	$\text{rel } e_{c_1}$	rate	$c_2$	$\text{rel } e_{c_2}$	rate
$k = 1$											
4.35e-01	178	-	-	-	-	-0.29668	-	-	-0.13608	-	-
2.20e-01	470	5.00e-01	-	3.92e-01	-	-0.29731	2.11e-03	-	-0.13590	1.37e-03	-
1.18e-01	1462	3.68e-01	0.49	2.51e-01	0.71	-0.29774	1.46e-03	0.58	-0.13602	9.36e-04	0.61
6.02e-02	5054	2.31e-01	0.70	1.44e-01	0.83	-0.29763	3.79e-04	2.02	-0.13586	1.18e-03	-0.35
3.06e-02	18014	1.46e-01	0.68	8.13e-02	0.84	-0.29752	3.87e-04	-0.03	-0.13579	5.05e-04	1.25

1.52e-02	67916	9.48e-02	0.62	4.53e-02	0.84	-0.29746	1.89e-04	1.03	-0.13577	1.99e-04	1.33
$k = 2$											
4.35e-01	493	-	-	-	-	-0.29863	-	-	-0.13632	-	-
2.20e-01	1337	2.14e-01	-	9.76e-02	-	-0.29837	8.88e-04	-	-0.13615	1.27e-03	-
1.18e-01	4245	1.86e-01	0.22	6.10e-02	0.75	-0.29780	1.92e-03	-1.23	-0.13592	1.68e-03	-0.44
6.02e-02	14885	1.16e-01	0.70	3.06e-02	1.03	-0.29759	7.06e-04	1.50	-0.13582	7.35e-04	1.23
3.06e-02	53497	7.39e-02	0.67	1.93e-02	0.68	-0.29749	3.28e-04	1.13	-0.13578	3.18e-04	1.24
1.52e-02	202667	4.85e-02	0.60	1.23e-02	0.65	-0.29744	1.44e-04	1.18	-0.13576	1.44e-04	1.14

TABLE 10. Relative errors and convergence rates for Experiment 7.5 for orders  $k = 1, 2$  on unstructured meshes.

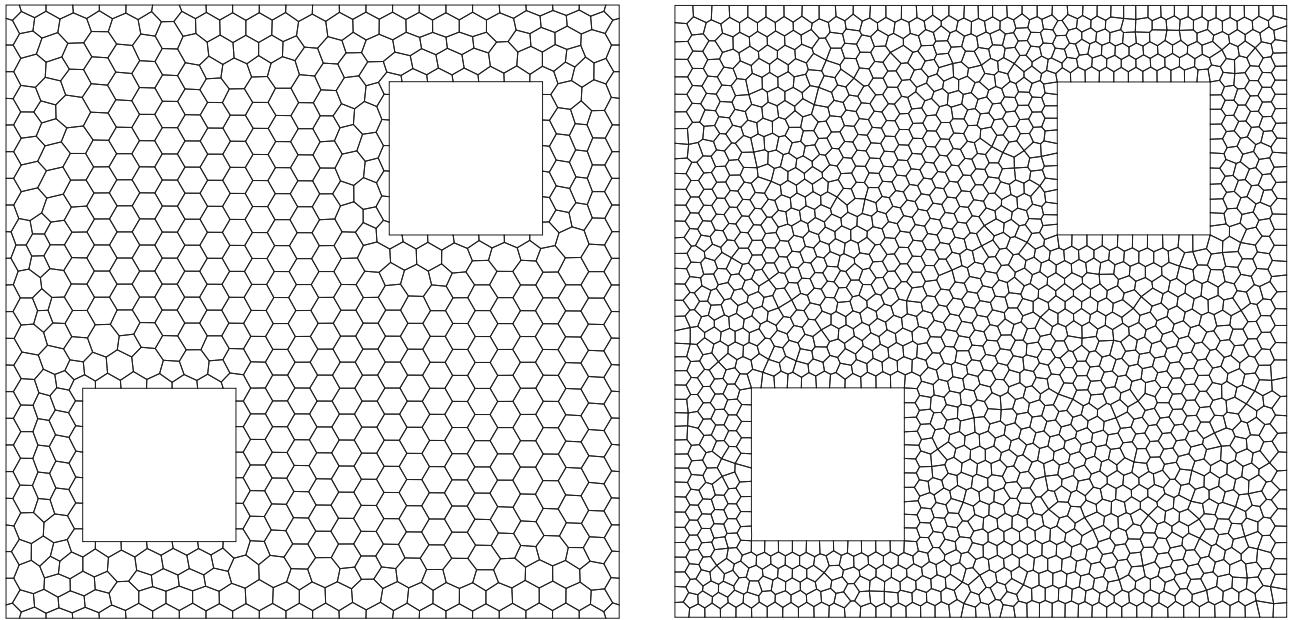
$h$	$N^{\text{Dofs}}$	$\text{rel } e_{\mathbf{u}_h}$	rate	$\text{rel } e_{\xi_h}$	rate	$c_1$	$\text{rel } e_{c_1}$	rate	$c_2$	$\text{rel } e_{c_2}$	rate
$k = 1$											
3.25e-01	181	-	-	-	-	-0.13483	-	-	-0.29259	-	-
1.70e-01	770	4.10e-01	-	3.37e-01	-	-0.13510	2.00e-03	-	-0.29611	1.19e-02	-
8.30e-02	3093	2.48e-01	0.70	1.76e-01	0.91	-0.13549	2.89e-03	-0.52	-0.29676	2.18e-03	2.37
4.08e-02	13033	1.49e-01	0.72	9.15e-02	0.92	-0.13562	9.33e-04	1.59	-0.29713	1.25e-03	0.78
2.00e-02	58140	9.48e-02	0.63	4.94e-02	0.87	-0.13570	6.31e-04	0.55	-0.29733	6.73e-04	0.87
$k = 2$											
3.25e-01	563	-	-	-	-	-0.13555	-	-	-0.29716	-	-
1.70e-01	2341	1.70e-01	-	6.45e-02	-	-0.13565	7.63e-04	-	-0.29720	1.40e-04	-
8.30e-02	9387	1.03e-01	0.70	3.15e-02	1.00	-0.13572	4.81e-04	0.64	-0.29731	3.57e-04	-1.31
4.08e-02	39567	5.97e-02	0.77	1.61e-02	0.95	-0.13573	8.30e-05	2.47	-0.29737	2.10e-04	0.75
2.00e-02	176280	3.92e-02	0.59	1.09e-02	0.54	-0.13574	9.27e-05	-0.16	-0.29740	1.11e-04	0.89

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(A) Structured voronoi mesh

?&lt;sub&gt;fig:Voro\_Betti2&lt;/sub&gt;?

(B) Unstructured voronoi mesh

FIGURE 4

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