VIRTUAL ELEMENT METHOD FOR A QUAD-CURL PROBLEM ON PLANAR DOMAINS

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Abstract. ...

1. Introduction

troduction⟩? Consider a bounded polygonal domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$. We are interested in the following quad-curl problem,

$$\begin{aligned} (\nabla\times)^4\boldsymbol{u} + \beta\,(\nabla\times)^2\boldsymbol{u} + \gamma\,\boldsymbol{u} = \boldsymbol{f} \quad &\text{in} \quad \Omega, \\ \nabla\times\boldsymbol{u} = 0 \quad &\text{on} \quad \partial\Omega, \\ \boldsymbol{n}\times\boldsymbol{u} = 0 \quad &\text{on} \quad \partial\Omega. \end{aligned} \tag{1.1}$$

eq:P.strong

Here $(\nabla\times\cdot)^4 = \nabla\times(\nabla\times(\nabla\times(\nabla\times\cdot)))$ and $(\nabla\times\cdot)^2 = \nabla\times(\nabla\times\cdot)$, are the quad-curl and curl-curl operators in two dimensions, $\boldsymbol{n}$ is the unit outward normal vector on $\partial\Omega$, and $\beta,\gamma \geqslant 0$ are given constants with $\gamma > 0$, if Ω is multiply connected. The forcing term $\boldsymbol{f} : \Omega \to \mathbb{R}^2$ is a given function.

The weak formulation of (1.1) reads: find $\boldsymbol{u} \in \tilde{\mathcal{E}}$ such that

$$(\nabla\times(\nabla\times\boldsymbol{u}), \nabla\times(\nabla\times\boldsymbol{v})) + \beta(\nabla\times\boldsymbol{u}, \nabla\times\boldsymbol{v}) + \gamma(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}) \quad \forall\boldsymbol{v}\in\tilde{\mathcal{E}}, \tag{1.2}$$

eq:P

where the energy space $\tilde{\mathcal{E}}$ is defined as

$$\tilde{\mathcal{E}} := \left\{\boldsymbol{v}\in[L^2(\Omega)]^2 : \nabla\times\boldsymbol{v}\in H_0^1(\Omega), \;\; \boldsymbol{n}\times\boldsymbol{v} = 0 \;\text{ on }\; \partial\Omega\right\}.$$

The notation $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ (or $[L^2(\Omega)]^2$) inner-product. This quad-curl problem is linked to the Maxwell's transmission eigenvalue problem (see (2.8) and consider for e.g., the case of homogeneous media in [12]). The authors here opt for a curl-conforming finite element method to discretize the problem. The quad-curl problem also has several applications including the resistive magnetohydrodynamic (MHD) ... [JT: To do - Add more literature] .

Another popular approach to numerically solve (1.1) is to use the Hodge decomposition of divergence-free vector fields (see (3.1) in Section 3). This reduces the problem into a sequence of second-order problems, which can then be discretized using H^1-conforming finite element methods. This approach has been studied, for e.g., in [5, 7] in two and three dimensions using simplicial and tetrahedral meshes, respectively. Since, we seek the solution in the divergence-free space (see Remark 3.1), we solve (1.2) using the reduced energy space $\mathcal{E} \subset \tilde{\mathcal{E}}$,

$$\mathcal{E} := \left\{\boldsymbol{v}\in[L^2(\Omega)]^2 : \nabla\times\boldsymbol{v}\in H_0^1(\Omega), \;\; \operatorname{div}\boldsymbol{v} = 0 \;\text{ and }\; \boldsymbol{n}\times\boldsymbol{v} = 0 \;\text{ on }\; \partial\Omega\right\}. \tag{1.3}$$

eq:E

Our goal in this paper is to extend the results to polygonal meshes using the conforming virtual element method (VEM). [JT: To do - Add literature on VEM for Maxwell and Hodge decomposition]

2. Preliminaries

liminaries⟩? In two dimensions, for a vector field $\boldsymbol{v} = (v_1, v_2)$, the curl operator is a scalar function,

$$\nabla\times\boldsymbol{v} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}.$$

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For a scalar function ϕ, the curl operator is a vector field,

$$\nabla \times \phi = \left[\frac{\partial \phi}{\partial y}, -\frac{\partial \phi}{\partial x}\right]^{\top} = \text{rot } \phi.$$

As a result, we have

$$(\nabla \times \phi, \nabla \times v) = (\nabla \phi, \nabla v) \quad \forall \phi, v \in H^1(\Omega),$$

and $\|\nabla \times v\|_{L^2(\Omega)} = |v|_{H^1(\Omega)}$ for all $v \in H^1(\Omega)$.

We collect regularity results for elliptic Boundary Value Problems (BVPs) on polygonal domains (see [11, Section 5.1] and [9, Section 2.5]), with ω denoting the largest interior angle at the corners of Ω.

g.reac-diff⟩ **Lemma 2.1** (Regularity of reaction-diffusion BVP)**.** *Given $g \in H^1(\Omega)$, let $\mu \in H^1_0(\Omega)$ satisfy*

$$(\nabla \mu, \nabla v) + \beta(\mu, v) = (g, v) \quad \forall v \in H^1_0(\Omega).$$

Then for any $\epsilon > 0, \mu \in H^{1+(\pi/\omega)-\epsilon}(\Omega)$. Also, we have the following continuous dependence on the data result, with positive constant C_ϵ, depending on ϵ and Ω,

$$\|\mu\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \leqslant C_\epsilon \|g\|_{H^1(\Omega)}.$$

ure-neumann⟩ **Lemma 2.2** (Regularity of pure-Neumann BVP)**.** *Given $g \in H^1(\Omega)$, let $\lambda \in H^1(\Omega)$ satisfy*

$$(\nabla \lambda, \nabla v) + (\lambda, 1)(v, 1) = (g, v) \quad \forall v \in H^1(\Omega).$$

Then for any $\epsilon > 0, \lambda \in H^{1+(\pi/\omega)-\epsilon}(\Omega)$. Also, we have the following continuous dependence on the data result, with positive constant C_ϵ, depending on ϵ and Ω,

$$\|\lambda\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \leqslant C_\epsilon \|g\|_{H^1(\Omega)}.$$

Using the above lemmas, we now state the regularity result for the energy space $\mathcal{E}$ defined in (1.3) (see [7, Section 2.1] for a proof).

⟨thm:reg.E⟩? **Theorem 2.3** (Regularity of the energy space)**.** *The quad-curl energy space $\mathbb{E}$ defined in* (1.3) *has regularity $[H^{(\pi/\omega)-\epsilon}(\Omega)]^2$ for any $\epsilon > 0$.*

As a consequence of this, the authors in [7, Section 2.2] established the well-posedness of the quad-curl problem (1.2) in the energy space $\mathcal{E}$.

We will use following Hilbert spaces in the rest of the article,

$$\begin{aligned} H(\text{div}^0; \Omega) &:= \left\{ \boldsymbol{v} \in [L^2(\Omega)]^2 : (\boldsymbol{v}, \nabla \eta) = 0 \quad \forall \eta \in H^1_0(\Omega) \right\}, \\ L^2_0(\Omega) &:= \left\{ v \in L^2(\Omega) : (1, v) = 0 \right\}. \end{aligned}$$

3. Hodge Decomposition Based Reduction

c:Reduction⟩ Any $\boldsymbol{v} \in H(\text{div}^0; \Omega)$ has a unique decomposition (see Chapter 3 in [10] and (1.2) in [3]):

$$\boldsymbol{v} = \nabla \times \phi + \sum_{j=1}^{m} d_j \nabla \varphi_j, \tag{3.1}$$ eq:HodgeDec

where $\phi \in H^1(\Omega) \cap L^2_0(\Omega)$, $m \in \mathbb{Z}^+_0$ is the Betti number, and $d_j (1 \leqslant j \leqslant m) \in \mathbb{R}$. The harmonic functions φ_j are defined by

eq:P.varphi⟩

$$(\nabla \varphi_j, \nabla v) = 0 \quad \forall v \in H^1_0(\Omega), \tag{3.2a}$$ eq:P.varphi

$$\varphi_j|_{\Gamma_0} = 0, \tag{3.2b}$$?eq:P.varphi

$$\varphi_j|_{\Gamma_l} = \delta_{jl} \quad \text{for} \quad 1 \leqslant l \leqslant m. \tag{3.2c}$$?eq:P.varphi

Here Γ_0 denotes the outer boundary, Γ_l denote the l components of the inner boundary when Betti number $m > 0$, and δ_{jl} is the Kronecker delta function.

⟨rem:kernel⟩ **Remark 3.1.** *The quad-curl energy has a large kernel consisting of gradient fields. Since the functions in $H(div^0; \Omega)$ are orthogonal to gradient fields, this kernel is fixed upto harmonic functions. Furthermore, in the case of simply connected domains, we infer from the boundary condition $\boldsymbol{u} \times \boldsymbol{n} = 0$ that the kernel reduces to the zero vector field. Therefore, we can put $\gamma = 0$. However, in the case of multiply connected domains, the harmonic functions in the kernel are non-zero vector fields. To account for this, we set $\gamma > 0$.*

3.1. **Ω is simply connected ($\gamma = 0$).** The Hodge decomposition (3.1) reduces to

$$\boldsymbol{u} = \nabla \times \phi. \tag{3.3}$$
u.simplyCon

It now remains to find ϕ. As shown in [7], this can be achieved by solving the following sequence of problems.

(1) First we find $\rho \in H^1(\Omega) \cap L_2^0(\Omega)$, or equivalently $\rho \in H^1(\Omega)$ such that

$$(\nabla \times \rho, \nabla \times \psi) + (\rho, 1)(\psi, 1) = (\boldsymbol{f}, \nabla \times \psi) \quad \forall \psi \in H^1(\Omega). \tag{3.4}$$
eq:P.rho

(2) Find ξ such that

$$(\nabla \times \xi, \nabla \times \eta) + \beta(\xi, \eta) = (\rho, \eta) \quad \forall \eta \in H_0^1(\Omega) \cap L_0^2(\Omega), \tag{3.5}$$
eq:xi

or equivalently, find $\xi \in H_0^1(\Omega) \cap L_0^2(\Omega)$ such that

$$\xi = \xi_0 - \frac{(1, \xi_0)}{(1, \xi_1)} \xi_1, \tag{3.6}$$
eq:P.xi

where $\xi_0, \xi_1 \in H_0^1(\Omega)$ satisfy

$$(\nabla \times \xi_0, \nabla \times \eta) + \beta(\xi_0, \eta) = (\rho, \eta) \quad \forall \eta \in H_0^1(\Omega), \tag{3.7}$$
eq:P.xi0

$$(\nabla \times \xi_1, \nabla \times \eta) + \beta(\xi_1, \eta) = (1, \eta) \quad \forall \eta \in H_0^1(\Omega). \tag{3.8}$$
eq:P.xi1

(3) Finally we find $\phi \in H^1(\Omega)$, or equivalently $\phi \in H^1(\Omega)$ such that

$$(\nabla \times \phi, \nabla \times \psi) + (\phi, 1)(\psi, 1) = (\xi, \psi) \quad \forall \psi \in H^1(\Omega). \tag{3.9}$$
eq:P.phi

3.2. **Ω is multiply connected ($\gamma > 0$).** Recalling the Hodge decomposition (3.1)

$$\boldsymbol{u} = \nabla \times \phi + \sum_{j=1}^{m} c_j \nabla \varphi_j.$$

It remains to find ϕ, c_j, and φ_j. Following [7], this is equivalent to solving the following sequence of problems.

(1) Find $(\zeta, \xi) \in H^1(\Omega) \cap L_2^0(\Omega) \times H_0^1(\Omega) \cap L_0^2(\Omega)$ such that

$$(\nabla \times \xi, \nabla \times \psi) + \gamma^{1/2}(\psi, \xi) = \gamma^{-1/2}(\boldsymbol{f}, \nabla \times \psi) \quad \forall \psi \in H^1(\Omega) \cap L_2^0(\Omega), \tag{3.10}$$
?eq:P.mc.zet

$$-\gamma^{1/2}(\zeta, \eta) + (\nabla \times \xi, \nabla \times \eta) + \beta(\xi, \eta) = 0 \quad \forall \eta \in H_0^1(\Omega) \cap L_0^2(\Omega). \tag{3.11}$$
eq:P.mc.xi

or equivalently, find (ζ, ξ) such that

$$(\zeta, \xi) = (\zeta_0, \xi_0) - \frac{(1, \xi_0)}{(1, \xi_1)} (\zeta_1, \xi_1), \tag{3.12}$$
eq:P.zeta.c

where $(\zeta_0, \xi_0), (\zeta_1, \xi_1) \in H^1(\Omega) \times H_0^1(\Omega)$ solve the following two coupled system:

$$\mathcal{A}((\zeta_0, \xi_0), (\psi, \eta)) + (\zeta_0, 1)(\psi, 1) = \gamma^{-\frac{1}{2}}(\boldsymbol{f}, \nabla \times \psi) \quad \forall (\psi, \eta) \in H^1(\Omega) \times H_0^1(\Omega), \tag{3.13}$$
eq:P.zeta0.

$$\mathcal{A}((\zeta_1, \xi_1), (\psi, \eta)) + (\zeta_1, 1)(\psi, 1) = (1, \eta) \quad \forall (\psi, \eta) \in H^1(\Omega) \times H_0^1(\Omega). \tag{3.14}$$
eq:P.zeta1.

Here, the bilinear form $\mathcal{A}(\cdot, \cdot)$ is defined by

$$\mathcal{A}((\zeta, \xi), (\psi, \eta)) = (\nabla \times \zeta, \nabla \times \psi) + \gamma^{\frac{1}{2}}(\psi, \xi) - \gamma^{\frac{1}{2}}(\zeta, \eta) + (\nabla \times \xi, \nabla \times \eta) + \beta(\xi, \eta). \tag{3.15}$$
?eq:A?

(2) Find $\phi \in H^1(\Omega)$ such that (3.9) holds.

(3) Finally $c_j (1 \leqslant j \leqslant m)$, are determined by solving the $m \times m$, SPD system

$$\sum_{j=1}^{m} (\nabla \varphi_i, \nabla \varphi_j) c_j = \gamma^{-1}(\boldsymbol{f}, \nabla \varphi_i) \quad \text{for} \quad 1 \leqslant i \leqslant m, \tag{3.16}$$
eq:P.cj

and the harmonic functions φ_j are defined by (3.2).

4. Conforming Virtual Element Discretization

cretization⟩ Let $\mathcal{T}_h$ be a triangulation of the polygonal domain $\Omega \subset \mathbb{R}^2$ into a finite collection of simple polygons D. The mesh parameter h is defined as $h := \max_{D\in\mathcal{T}_h} h_D$, where h_D denotes the diameter of D. Let $\mathcal{E}_D$ be the set of edges associated with D. We make the following shape-regularity assumptions for all $D \in \mathcal{T}_h$ (see [2, 6]). There exists a constant $\Theta \in (0,1)$ such that

hreg_assum⟩?

$$D \text{ is star-shaped with respect to a ball of radius } \Theta h_D, \text{ and} \tag{4.1a}$$

meshreg_ass

$$|e| \geqslant \Theta h_D \text{ for any edge } e \in \mathcal{E}_D. \tag{4.1b}$$

?meshreg_ass

The local enhanced virtual element space $V_h(D) \subset H^1(D)$ (see [1, 6]) is defined as follows:

$$V_h(D) := \left\{ v_h \in H^1(D) : v_h|_{\partial D} \in \mathbb{P}_k(\partial D),\ \ -\Delta v_h \in \mathbb{P}_k(D),\ \ \Pi^0_{k,D} v_h - \Pi^1_{k,D} v_h \in \mathbb{P}_{k-2}(D) \right\}. \tag{4.2}$$

?eq:VhK?

Here $\mathbb{P}_k(\partial D)$ (respectively, $\mathbb{P}_k(D)$) denote the space of continuous piecewise polynomials of degree at most k on the boundary ∂D (respectively, on D). The operators $\Pi^0_{k,D}$ and $\Pi^1_{k,D}$ are the standard L^2 and H^1-projections onto $\mathbb{P}_k(D)$, respectively (see [6, Section 2.2] for details). The global virtual element spaces V_h and V_h^0 are defined by concatenating the local spaces as follows:

$$V_h := \{ v_h \in H^1(\Omega) : v_h|_D \in V_h(D) \text{ for all } D \in \mathcal{T}_h \}, \tag{4.3}$$

?eq:Vh?

$$V_h^0 := \{ v_h \in H^1_0(\Omega) : v_h|_D \in V_h(D) \text{ for all } D \in \mathcal{T}_h \}. \tag{4.4}$$

?eq:Vh0?

4.1. **Ω is simply connected ($\gamma = 0$).** The $\mathbb{P}_k$ Virtual Element Method for approximating the sequence of problems (3.4)-(3.9) is as follows.

(1) Find $\rho_h \in V_h$ such that

$$a_h(\rho_h, \psi_h) + (\Pi^0_{k,h}\rho_h, 1)(\Pi^0_{k,h}\psi_h, 1) = (\boldsymbol{f}, \nabla \times \Pi^1_{k,h}\psi_h) \quad \forall \psi_h \in V_h. \tag{4.5}$$

eq:Ph.rhoh

(2) Find $\xi_h \in V_h$ given by

$$\xi_h = \xi_{0,h} - \frac{(1,\xi_{0,h})}{(1,\xi_{1,h})}\xi_{1,h}, \tag{4.6}$$

eq:Ph.xih

where $\xi_{0,h}, \xi_{1,h} \in V_h^0$ satisfy

$$a_h(\xi_{0,h}, \eta_h) + \beta(\Pi^0_{k,h}\xi_{0,h}, \Pi^0_{k,h}\eta_h) = (\Pi^0_{k,h}\rho_h, \Pi^0_{k,h}\eta_h) \quad \forall \eta_h \in V_h^0, \tag{4.7}$$

eq:Ph.xi0h

$$a_h(\xi_{1,h}, \eta_h) + \beta(\Pi^0_{k,h}\xi_{1,h}, \Pi^0_{k,h}\eta_h) = (1, \Pi^0_{k,h}\eta_h) \quad \forall \eta_h \in V_h^0. \tag{4.8}$$

eq:Ph.xi1h

(3) Find $\phi_h \in V_h$ such that

$$a_h(\phi_h, \psi_h) + (\Pi^0_{k,h}\phi_h, 1)(\Pi^0_{k,h}\psi_h, 1) = (\Pi^0_{k,h}\xi_h, \Pi^0_{k,h}\psi_h) \quad \forall \psi_h \in V_h. \tag{4.9}$$

eq:Ph.phih

Here, the global bilinear form $a_h(\cdot,\cdot)$ is given elementwise by:

$$\begin{aligned} a_h(w_h, v_h) &= \sum_{D\in\mathcal{T}_h} a_h^D(w_h, v_h) \quad \forall w_h, v_h \in V_h, \\ &= \sum_{D\in\mathcal{T}_h} (\nabla \times \Pi^1_{k,D} w_h, \nabla \times \Pi^1_{k,D} v_h)_D + S^D((I - \Pi^1_{k,D})w_h, (I - \Pi^1_{k,D})v_h). \end{aligned} \tag{4.10}$$

eq:ah

$\Pi^0_{k,h}$ and $\Pi^1_{k,h}$ are the global L^2 and H^1-projections onto $\mathbb{P}_k(\mathcal{T}_h)$ (discontinuous piecewise polynomials of degree $\leqslant k$), respectively, and are understood in terms of their local counterparts. The symmetric positive definite stabilization term is denoted by $S^D(\cdot,\cdot)$. Finally, we post-process $\boldsymbol{u}_h \in [\mathbb{P}_k(\mathcal{T}_h)]^2$ using the Hodge decomposition as follows:

$$\boldsymbol{u}_h = \nabla \times \Pi^1_{k,h}\phi_h. \tag{4.11}$$

uh.simplyCo

We choose the boundary version of the classical `Dofi-Dofi` definition of the stabilization term (see [4, Section 4.2] and [2]),

$$S^D(w_h, v_h) = \sum_{i=1}^{N^{\mathrm{dof}}_{\partial D}} \chi_i(w_h)\, \chi_i(v_h) \quad \forall w_h, v_h \in V_h(D), \tag{4.12}$$

def:bd.stab

where operator $\chi_i(\cdot)$ associates the function with its i-th degree of freedom, and $N^{\mathrm{dof}}_{\partial D}$ is the number of boundary degrees of freedom associated with the element in $D \in \mathcal{T}_h$. We have the following property associated with this choice of stabilization (see Remark 4.3 [4]),

$$\sum_{i=1}^{N^{\mathrm{dof}}_{\partial D}} \chi_i(w)^2 \lesssim \|w\|^2_{L^\infty(D)} \quad \forall w \in C(\bar{D}), \tag{4.13}$$
eq:stab.p2

where the hidden constants depend on Θ and k.

4.2. **Ω is multiply connected ($\gamma > 0$).** The $\mathbb{P}_k$ Virtual Element Method for approximating the sequence of problems (3.12)-(3.16) is as follows.

(1) Find $(\zeta_h, \xi_h) \in V_h \times V_h^0$ given by

$$(\zeta_h, \xi_h) = (\zeta_{0,h}, \xi_{0,h}) - \frac{(1, \xi_{0,h})}{(1, \xi_{1,h})}(\zeta_{1,h}, \xi_{1,h}), \tag{4.14}$$
eq:Ph.zeta.

where $(\zeta_{0,h}, \xi_{0,h}), (\zeta_{1,h}, \xi_{1,h}) \in V_h \times V_h^0$ solve the following two coupled systems:

$$\begin{aligned} \mathcal{A}_h((\zeta_{0,h}, \xi_{0,h}), (\psi_h, \eta_h)) &+ (\Pi^0_{k,h}\zeta_{0,h}, 1)(\Pi^0_{k,h}\psi_h, 1) \\ &= \gamma^{-\frac{1}{2}}(\boldsymbol{f}, \nabla \times \Pi^1_{k,h}\psi) \quad \forall (\psi_h, \eta_h) \in V_h \times V_h^0, \end{aligned} \tag{4.15}$$
eq:Ph.zeta0

$$\begin{aligned} \mathcal{A}_h((\zeta_{1,h}, \xi_{1,h}), (\psi_h, \eta_h)) &+ (\Pi^0_{k,h}\zeta_{1,h}, 1)(\Pi^0_{k,h}\psi_h, 1) \\ &= (1, \Pi^0_{k,h}\eta_h) \quad \forall (\psi_h, \eta_h) \in V_h \times V_h^0. \end{aligned} \tag{4.16}$$
eq:Ph.zeta1

The global coupled bilinear form $\mathcal{A}_h(\cdot,\cdot)$ is defined by

$$\begin{aligned} \mathcal{A}_h((\zeta_h, \xi_h), (\psi_h, \eta_h)) = a_h(\zeta_h, \psi_h) &+ \gamma^{\frac{1}{2}}(\Pi^0_{k,h}\psi_h, \Pi^0_{k,h}\xi_h) - \gamma^{\frac{1}{2}}(\Pi^0_{k,h}\zeta_h, \Pi^0_{k,h}\eta_h) \\ &+ a_h(\xi_h, \eta_h) + \beta(\Pi^0_{k,h}\xi_h, \Pi^0_{k,h}\eta_h). \end{aligned} \tag{4.17}$$
?eq:Ah?

(2) Find $\phi_h \in V_h$ such that (4.9) holds.

(3) The coefficients $c_{j,h}$ $(1 \leqslant j \leqslant m)$, are determined by solving

$$\sum_{j=1}^{m} a_h(\varphi_{i,h}, \varphi_{j,h}) c_{j,h} = \gamma^{-1}(\boldsymbol{f}, \nabla \Pi^1_{k,h}\varphi_{i,h}) \quad \text{for} \quad 1 \leqslant i \leqslant m. \tag{4.18}$$
eq:Ph.cj

Where the discrete harmonic functions $\varphi_{j,h}$ are determined by approximating (3.2) as follows:

q:Ph.varphi⟩

$$a_h(\varphi_{j,h}, v_h) = 0 \quad \forall v_h \in V_h^0, \tag{4.19a}$$
eq:Ph.varph

$$\varphi_{j,h}|_{\Gamma_0} = 0, \tag{4.19b}$$
?eq:Ph.varph

$$\varphi_{j,h}|_{\Gamma_l} = \delta_{jl} \quad \text{for} \quad 1 \leqslant l \leqslant m, \tag{4.19c}$$
eq:Ph.varph

with $a_h(\cdot,\cdot)$ given by (4.10).

Finally we post-process $\boldsymbol{u}_h \in [\mathbb{P}_k(\mathcal{T}_h)]^2$ using the Hodge decomposition as follows:

$$\boldsymbol{u}_h = \nabla \times \Pi^1_{k,h}\phi_h + \sum_{j=1}^{m} c_{j,h} \nabla \Pi^1_{k,h}\varphi_{j,h}. \tag{4.20}$$
uh.notSimpl

5. Convergence analysis

?⟨sec:CA⟩? We start by collecting some mathematical tools which will be helpful in the forthcoming analysis.

inequality⟩? **Lemma 5.1** (Sobolev inequality)**.** *Given any $\delta > 0$,*

$$\|v\|_{L^\infty(D)} \lesssim h_D^{-1}\|v\|_{L^2(D)} + |v|_{H^1(D)} + h_D^\delta |v|_{H^{1+\delta}(D)} \quad \forall v \in H^{1+\delta}(D). \tag{5.1}$$
eq:Sobolev.

Lemma 5.2 (Bramble-Hilbert estimates)**.** *Under the mesh regularity Assumption 4.1a, given any $\delta > 0$, there exists a positive constant independent of h_D such that*

$$\inf_{p \in \mathbb{P}_k(D)} |\lambda - \psi|_{H^1(D)} \lesssim h_D^{\min(\delta,k)} |\lambda|_{H^{1+\delta}(D)}, \quad \forall \lambda \in H^{1+\delta}(D).$$

inequality⟩? **Lemma 5.3** (Trace inequality)**.** *Let e be an edge of $D \subset \mathbb{R}^2$. Then, for all $v \in H^{1+\delta}(D)$ with given $\delta > 0$, we have*

$$h_D^{2\delta}|v|^2_{H^{1/2+\delta}(e)} \lesssim |v|^2_{H^1(D)} + h_D^{2\delta}|v|^2_{H^{1+\delta}(D)}.$$

Lemma 5.4 (H^1-projector stability and approximation)**.** *Given $\delta > 0$,*

$$|\Pi^1_{k,D} v|_{H^1(D)} \leqslant |v|_{H^1(D)} \quad \forall v \in H^1(D), \tag{5.2}$$

$$|v - \Pi^1_{k,D} v|_{H^1(D)} \lesssim h_D^{\min(\delta,k)} |v|_{H^{1+\delta}(D)} \quad \forall v \in H^{1+\delta}(D), \tag{5.3}$$

$$\|v - \Pi^1_{k,D} v\|_{L^2(D)} \lesssim h_D^{\min(\delta,k)+1} |v|_{H^{1+\delta}(D)} \quad \forall v \in H^{1+\delta}(D). \tag{5.4}$$

Lemma 5.5 (L^2-projector stability and approximation)**.**

$$\|\Pi^0_{k,D} v\|_{L^2(D)} \leqslant \|v\|_{L^2(D)} \quad \forall v \in L^2(D), \quad \|\Pi^0_{k,D} v\|_{H^1(D)} \leqslant \|v\|_{H^1(D)} \quad \forall v \in H^1(D), \tag{5.5}$$

$$\|v - \Pi^0_{k,D} v\|_{L^2(D)} \lesssim h_D^{l+1} |v|_{H^{l+1}(D)} \quad \forall v \in H^{l+1}(D), \quad 0 \leqslant l \leqslant k, \tag{5.6}$$

Now we recall the virtual element interpolation operator, which takes any sufficiently smooth function and maps it to the virtual element space. For $s > 1$, the global interpolation operator $I_{k,h} : H^s(\Omega) \to V_h$ is the global counterpart of the local interpolation operator $I_{k,D} : H^s(D) \to V_h(D)$ for all $D \in \mathcal{T}_h$ such that for any $v \in H^s(D)$,

$$I_{k,D} v(p) = v(p) \quad \forall p \in \mathcal{N}^{\partial D}, \tag{5.7}$$

$$\Pi^0_{k-2,D} I_{k,D} v = \Pi^0_{k-2,D} v. \tag{5.8}$$

Here $\mathcal{N}^{\partial D}$ is the set of boundary degrees of freedom associated with the local virtual element space.

Lemma 5.6 (Interpolation operator stability and approximation)**.** *Given $\delta > 0$, we have*

$$|I_{k,D} v|_{H^1(D)} \lesssim |v|_{H^1(D)} + h_D^{\delta} |v|_{H^{1+\delta}(D)} \quad \forall v \in H^{1+\delta}(D), \tag{5.9}$$

$$|v - I_{k,D} v|_{H^1(D)} + |v - \Pi^1_{k,D} I_{k,D} v|_{H^1(D)} \lesssim h_D^{\min(\delta,k)} |v|_{H^{1+\delta}(D)} \quad \forall v \in H^{1+\delta}(D), \tag{5.10}$$

$$\|v - I_{k,D} v\|_{L^2(D)} + \|v - \Pi^1_{k,D} I_{k,D} v\|_{L^2(D)} \lesssim h_D^{\min(\delta,k)+1} |v|_{H^{1+\delta}(D)} \quad \forall v \in H^{1+\delta}(D). \tag{5.11}$$

In the subsequent analysis, we work with the following assumption which is made concrete in Remark 5.9.

Assumption 5.7. *Let $\|\cdot\|_{h_\beta} = \sqrt{a_h(\cdot,\cdot) + \beta(\Pi^0_{k,h}\cdot, \Pi^0_{k,h}\cdot)}$ denote the mesh-dependent energy norm such that*

$$|v_h|_{H^1(\Omega)} \lesssim \|v_h\|_{h_\beta} \quad \forall v_h \in V_h^0, \tag{5.12}$$

$$|v_h|_{H^1(\Omega)} \lesssim \|v_h\|_{h_\beta} \quad \forall v_h \in V_h \quad \textit{with} \quad (v_h, 1) = 0. \tag{5.13}$$

We also define the piecewise-broken H^1-seminorm by

$$|v|_{1,h} := \left(\sum_{D \in \mathcal{T}_h} |v|^2_{H^1(D)} \right)^{\frac{1}{2}} \quad \forall v \in H^1(\mathcal{T}_h).$$

Some useful estimates in the mesh-dependent norm and piecewise broken semi-norm are in order.

Lemma 5.8. *The following estimates hold,*

$$\|I_{k,h} v - \Pi^1_{k,h} I_{k,h} v\|_{h_0} + |I_{k,h} v - \Pi^1_{k,h} I_{k,h} v|_{1,h} \lesssim h^{\min(\delta,k)} |v|_{H^{1+\delta}(\Omega)} \quad \forall v \in H^{1+\delta}(\Omega), \tag{5.14}$$

$$\|v - I_{k,h} v\|_{h_\beta} + \|v - \Pi^1_{k,h} v\|_{h_\beta} \lesssim h^{\min(\delta,k)} |v|_{H^{1+\delta}(\Omega)} \quad \forall v \in H^{1+\delta}(\Omega). \tag{5.15}$$

Remark 5.9. *[JT: To do - Making Assumption 5.7 concrete, for shape-regular meshes vs meshes with small edges]*

5.1. **Ω is simply connected** ($\gamma = 0$)**.** Our first aim is to estimate the error $\xi - \xi_h$. To this end, we first write down the error equation by testing (3.4) with $\psi_h \in V_h$ and subtracting it from (4.5),

$$(\nabla \times \rho, \nabla \times \psi_h) - a_h(\rho_h, \psi_h) = (\boldsymbol{f}, \nabla \times (\psi_h - \Pi^1_{k,h}(\psi_h))) \quad \forall \psi_h \in V_h. \tag{5.16}$$

We note that to obtain the above error equation we also used the definition of $\Pi^0_{k,h}$ and the fact that $\rho, \rho_h \in L^2_0(\Omega)$. Furthermore, in view of (3.4), (4.5), $\rho, \rho_h \in L^2_0(\Omega)$ and Assumption 5.7, we have the following relations:

$$\|\nabla \times \rho\|_{L^2(\Omega)} \leqslant \|\boldsymbol{f}\|_{L^2(\Omega)}, \quad \text{and} \quad \|\nabla \times \rho_h\|_{L^2(\Omega)} \lesssim \|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{5.17}$$

In the following lemma we estimate the error for ρ_h in the dual norm using a duality argument.

:rhoh.dual⟩? **Lemma 5.10.** *For any $\epsilon > 0$, there exists a positive constant dependent on ϵ and independent of h such that*

$$|(\rho - \rho_h, \chi)| \lesssim h^{\min((\pi/\omega)-\epsilon, k)} \|\chi\|_{H^1(\Omega)} \|\boldsymbol{f}\|_{L^2(\Omega)} \quad \forall \chi \in H^1(\Omega). \tag{5.18}$$
eq:err.est.

Proof. Given arbitary $\chi \in H^1(\Omega)$, let $\lambda \in H^1(\Omega)$ solve the following dual problem:

$$(\nabla \times \psi, \nabla \times \lambda) + (\psi, 1)(\lambda, 1) = (\psi, \chi) \quad \forall \psi \in H^1(\Omega). \tag{5.19}$$
eq:dual.rho

Testing (5.19) with $\rho - \rho_h \in H^1(\Omega)$ and exploiting $\rho, \rho_h \in L^2_0(\Omega)$, we get

$$\begin{aligned}(\rho - \rho_h, \chi) &= (\nabla \times (\rho - \rho_h), \nabla \times \lambda) + (\rho - \rho_h, 1)(\lambda, 1)\\ &= (\nabla \times (\rho - \rho_h), \nabla \times (\lambda - I_{k,h}\lambda)) + (\nabla \times (\rho - \rho_h), \nabla \times I_{k,h}\lambda).\end{aligned} \tag{5.20}$$
eq:err.rho.

The first term in (5.20) is bounded using the Cauchy-Schwarz inequality and (5.17), to get

$$\begin{aligned}(\nabla \times (\rho - \rho_h), \nabla \times (\lambda - I_{k,h}\lambda)) &\leqslant \|\nabla \times (\rho - \rho_h)\|_{L^2(\Omega)} \|\nabla \times (\lambda - I_{k,h}\lambda)\|_{L^2(\Omega)}.\\ &\lesssim \|\boldsymbol{f}\|_{L^2(\Omega)} |\lambda - I_{k,h}\lambda|_{H^1(\Omega)}.\end{aligned} \tag{5.21}$$
eq:err.rho.

The second term in (5.20) is bounded using the error equation (5.16) for $\psi_h = I_{k,h}\lambda$ as follows:

$$\begin{aligned}(\nabla \times (\rho - \rho_h), \nabla \times I_{k,h}\lambda) &= (\nabla \times \rho, \nabla \times I_{k,h}\lambda) - (\nabla \times \rho_h, \nabla \times I_{k,h}\lambda)\\ &= (\boldsymbol{f}, \nabla \times (I_{k,h}\lambda - \Pi^1_{k,h} I_{k,h}\lambda)) + a_h(\rho_h, I_{k,h}\lambda) - (\nabla \times \rho_h, \nabla \times I_{k,h}\lambda)\\ &\leqslant \|\boldsymbol{f}\|_{L^2(\Omega)} |(I - \Pi^1_{k,h}) I_{k,h}\lambda|_{1,h} + a_h(\rho_h, I_{k,h}\lambda) - (\nabla \times \rho_h, \nabla \times I_{k,h}\lambda).\end{aligned} \tag{5.22}$$
eq:err.rho.

The difference in (5.22) can be rewritten as follows:

$$\begin{aligned}&a_h(\rho_h, I_{k,h}\lambda) - (\nabla \times \rho_h, \nabla \times I_{k,h}\lambda)\\ &= \sum_{D \in \mathcal{T}_h} (\nabla \times \Pi^1_{k,D}\rho_h, \nabla \times \Pi^1_D I_{k,D}\lambda) + S^D((I - \Pi^1_{k,D})\rho_h, (I - \Pi^1_{k,D}) I_{k,D}\lambda) -\\ &\qquad (\nabla \times \rho_h, \nabla \times (I_{k,h}\lambda - \Pi^1_{k,D} I_{k,D}\lambda)) - (\nabla \times \rho_h, \nabla \times \Pi^1_{k,D} I_{k,D}\lambda)\\ &= \sum_{D \in \mathcal{T}_h} S^D((I - \Pi^1_{k,D})\rho_h, (I - \Pi^1_{k,D}) I_{k,D}\lambda) - (\nabla \times \rho_h, \nabla \times (I_{k,D}\lambda - \Pi^1_{k,D} I_{k,D}\lambda))\\ &\leqslant \Big(\sum_{D \in \mathcal{T}_h} S^D((I - \Pi^1_{k,D})\rho_h, (I - \Pi^1_{k,D})\rho_h)\Big)^{1/2} \Big(\sum_{D \in \mathcal{T}_h} S^D((I - \Pi^1_{k,D}) I_{k,D}\lambda, (I - \Pi^1_{k,D}) I_{k,D}\lambda)\Big)^{1/2}\\ &\qquad \Big(\sum_{D \in \mathcal{T}_h} |\rho_h|^2_{H^1(D)}\Big)^{1/2} \Big(\sum_{D \in \mathcal{T}_h} |I_{k,D}\lambda - \Pi^1_{k,D} I_{k,D}\lambda|^2_{H^1(D)}\Big)^{1/2}\\ &\lesssim \|(I - \Pi^1_{k,h})\rho_h\|_{h_0} \|(I - \Pi^1_{k,h}) I_{k,h}\lambda\|_{h_0} + |\rho_h|_{H^1(\Omega)} |(I - \Pi^1_{k,h}) I_{k,h}\lambda|_{1,h}.\end{aligned} \tag{5.23}$$
eq:err.rho.

Where in the passage to the second equality we used the definition of $\Pi^1_{k,D}$ projector followed by Cauchy-Schwarz inequality and finally the definition of $\|\cdot\|_h$ and $|\cdot|_{1,h}$. Using the definition of the stabilization (4.12), the approximation properties of $\Pi^1_{k,D}$ (5.3) and (5.4), and (5.17), we have

$$\begin{aligned}\|(I - \Pi^1_{k,h})\rho_h\|^2_{h_0} &= \sum_{D \in \mathcal{T}_h} S^D((I - \Pi^1_{k,h})\rho_h, (I - \Pi^1_{k,h})\rho_h)\\ &\overset{(4.13)}{\lesssim} \sum_{D \in \mathcal{T}_h} \|(I - \Pi^1_{k,h})\rho_h\|^2_{L^\infty(D)}\\ &\overset{(5.1)}{\lesssim} \sum_{D \in \mathcal{T}_h} h_D^{-1} \|(I - \Pi^1_{k,h})\rho_h\|^2_{L^2(D)} + |(I - \Pi^1_{k,h})\rho_h|^2_{H^1(D)}\\ &\lesssim \|\boldsymbol{f}\|_{L^2(\Omega)}.\end{aligned}$$

The regularity of the dual problem from Lemma 2.2, the above bound, the estimate (5.14) with $\delta = (\pi/\omega) - \epsilon$, and (5.17) lead to the following bound on (5.23),

$$a_h(\rho_h, I_{k,h}\lambda) - (\nabla \times \rho_h, \nabla \times I_{k,h}\lambda) \lesssim h^{\min((\pi/\omega)-\epsilon, k)} \|\boldsymbol{f}\|_{L^2(\Omega)} |\lambda|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)}. \tag{5.24}$$
eq:ref4coup

Substituting in (5.22), again using (5.14) and the data dependence relation of the dual problem in Lemma 2.2 yield

$$(\nabla \times (\rho - \rho_h), \nabla \times I_{k,h}\lambda) \lesssim h^{\min((\pi/\omega)-\epsilon,k)} \|\boldsymbol{f}\|_{L^2(\Omega)} \|\chi\|_{H^1(\Omega)}.$$

Substituting the above bound and (5.21) into (5.20), and using the definition of piecewise-broken H^1-seminorm and (5.10) yields the desired error estimate (5.18). □

It remains to estimate the error for ξ_h.

⟨lem:xi.est⟩ **Lemma 5.11.** *For any $\epsilon > 0$, there exists a positive constant dependent on ϵ and independent of h such that for $k = 1, 2$ and $\gamma = 0$,*

$$|\xi - \xi_h|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega)-\epsilon,k)} \|\boldsymbol{f}\|_{L^2(\Omega)}, \tag{5.25}$$
eq:err.est.

$$|\xi - \Pi^1_{k,h}\xi_h|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega)-\epsilon,k)} \|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{5.26}$$
eq:err.est.

Proof. Given $\rho_h \in V_h \cap L^2_0(\Omega)$, let $\tilde{\xi}_0 \in H^1_0(\Omega)$ solve the following auxiliary problem,

$$(\nabla \times \tilde{\xi}_0, \nabla \times \eta) + \beta(\tilde{\xi}_0, \eta) = (\rho_h, \eta) \quad \forall \eta \in H^1_0(\Omega). \tag{5.27}$$
eq:P.aux

Since $\rho_h \in V_h \subset H^1(\Omega)$, due to Lemma 2.1,

$$\tilde{\xi}_0 \in H^{1+(\pi/\omega)-\epsilon}(\Omega) \quad \text{and} \quad \|\tilde{\xi}_0\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \lesssim \|\rho_h\|_{H^1(\Omega)}. \tag{5.28}$$
tildexi0.re

It follows now from (5.64), (5.27), and the duality estimate (5.18) that

$$|\xi_0 - \tilde{\xi}_0|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega)-\epsilon,k)} \|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{5.29}$$
eq:xi0.a

It follows from (5.27) and (4.7) that $\xi_{0,h}$ is also a virtual element approximation of $\tilde{\xi}_0$ and hence we have the following analogous abstract error estimate following [8, Section 4.3],

$$\begin{aligned}\|\tilde{\xi}_0 - \xi_{0,h}\|_{h_\beta} \lesssim{}& \|\tilde{\xi}_0 - I_{k,h}\tilde{\xi}_0\|_{h_\beta} + |\tilde{\xi}_0 - \Pi^1_{k,h}\tilde{\xi}_0|_{1,h} + \|\tilde{\xi}_0 - \Pi^1_{k,h}\tilde{\xi}_0\|_{h_\beta} \\ &+ \beta h \|\tilde{\xi}_0 - \Pi^0_{k,h}\tilde{\xi}_0\|_{L^2(\Omega)} + \sup_{w_h \in V^0_h \setminus \{0\}} \frac{(\rho_h, w_h - \Pi^0_{k,h} w_h)}{|w_h|_{H^1(\Omega)}}.\end{aligned} \tag{5.30}$$
err.abs.xi0

The numerator in the last term of (5.30) is estimated using the definition of $\Pi^0_{k,h}$, the fact that it minimizes it's respective norm, the fact $\rho_h, w_h \in V_h \subset H^1(\Omega)$ and the approximation property of $\Pi^0_{k,h}$ follows:

$$\begin{aligned}(\rho_h, w_h - \Pi^0_{k,h} w_h) &= (\rho_h - \Pi^0_{0,h}\rho_h, w_h - \Pi^0_{k,h} w_h) \\ &\leqslant \|\rho_h - \Pi^0_{0,h}\rho_h\|_{L^2(\Omega)} \|w_h - \Pi^0_{0,h} w_h\|_{L^2(\Omega)} \\ &\lesssim h^2 |\rho_h|_{H^1(\Omega)} |w_h|_{H^1(\Omega)}.\end{aligned} \tag{5.31}$$
eq:err.abs.

Using the approximation properties stated in (5.15), the definition of $|\cdot|_{1,h}$, the estimates (5.3) and (5.6) for $\delta = (\pi/\omega) - \epsilon$, and (5.31), we can get the following concrete bound on (5.30).

$$\begin{aligned}\|\tilde{\xi}_0 - \xi_{0,h}\|_{h_\beta} &\lesssim h^{\min((\pi/\omega)-\epsilon,k)} (|\tilde{\xi}_0|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} + |\rho_h|_{H^1(\Omega)}) \\ &\overset{(5.28),(5.17)}{\lesssim} h^{\min((\pi/\omega)-\epsilon,k)} \|\boldsymbol{f}\|_{L^2(\Omega)}.\end{aligned}$$

Now, using Assumption 5.7 in the above estimate yields

$$|\tilde{\xi}_0 - \xi_{0,h}|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega)-\epsilon,k)} \|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{5.32}$$
eq:xi0h.b

Finally, a triangle inequality alongwith (5.29) and (5.32) leads to (5.33)

$$|\xi_0 - \xi_{0,h}|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega)-\epsilon,k)} \|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{5.33}$$
eq:err.xi0

Similarly, we can get the following control on $\xi_1 - \xi_{1,h}$ (see (3.8) and (4.8))

$$|\xi_1 - \xi_{1,h}|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega)-\epsilon,k)}. \tag{5.34}$$
eq:err.xi1

The estimate (5.25) now follows from (5.33), (5.34), (3.6), and (4.6). Since, $\xi \in H^{1+(\pi/\omega)-\epsilon}(\Omega)$ we can get the estimate (5.26) using Lemma 5.5 and (5.25) as follows,

$$\begin{aligned}|\xi - \Pi^1_{k,h}\xi_h|_{1,h} &\leqslant |\xi - \Pi^1_{k,h}\xi|_{1,h} + |\Pi^1_{k,h}(\xi - \xi_h)|_{1,h} \\ &\lesssim h^{\min((\pi/\omega)-\epsilon,k)}\|\xi\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} + |\xi - \xi_h|_{H^1(\Omega)} \\ &\lesssim h^{\min((\pi/\omega)-\epsilon,k)}\|\boldsymbol{f}\|_{L^2(\Omega)}.\end{aligned}$$

In the last step we used Lemma 2.1 on (3.5) followed by (5.17). This completes the proof. □

The error analysis for ϕ_h is analogous to that of ξ_h.

lem:est.phi⟩ **Lemma 5.12.** *For any $\epsilon > 0$, there esists a positive constant dependent on ϵ and independent of h such that for $k = 1, 2$,*

$$|\phi - \phi_h|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega)-\epsilon,k)}\|\boldsymbol{f}\|_{L^2(\Omega)}, \tag{5.35}$$ eq:err.est.

$$|\phi - \Pi^1_{k,h}\phi_h|_{1,h} \lesssim h^{\min((\pi/\omega)-\epsilon,k)}\|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{5.36}$$ eq:err.est.

Proof. Given $\xi_h \in V_h^0 \cap L_0^2(\Omega)$ (which is true by construction of ξ_h). Let $\tilde{\phi} \in H^1(\Omega)$ solve the following auxiliary problem,

$$(\nabla \times \tilde{\phi}, \nabla \times \psi) + (\tilde{\phi}, 1)(\tilde{\psi}, 1) = (\xi_h, \psi) \quad \forall \psi \in H^1(\Omega). \tag{5.37}$$ eq:P.aux.ph

Since $\xi_h \subset H^1(\Omega)$, due to Lemma 2.2,

$$\tilde{\phi} \in H^{1+(\pi/\omega)-\epsilon}(\Omega) \quad \text{and} \quad \|\tilde{\phi}\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \lesssim \|\xi_h\|_{H^1(\Omega)}. \tag{5.38}$$ tildephi.re

Comparing (3.9) and (5.37), using a standard Poincaré inequality for $\xi - \xi_h \in H_0^1(\Omega)$ alongwith (5.25) gives,

$$|\phi - \tilde{\phi}|_{H^1(\Omega)} \lesssim \|\xi - \xi_h\|_{L^2(\Omega)} \lesssim h^{\min((\pi/\omega)-\epsilon,k)}\|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{5.39}$$ eq:phi.a

It follows from (5.37) and (4.9) that ϕ_h is also a virtual element approximation of $\tilde{\phi}$ and hence using the fact the $\rho, \rho_h \in L_0^2(\Omega)$ we have the following analogous abstract error estimate following [8, Section 4.3],

$$\|\tilde{\phi} - \phi_h\|_{h_\beta} \lesssim \|\tilde{\phi} - I_{k,h}\tilde{\phi}\|_{h_0} + |\tilde{\phi} - \Pi^1_{k,h}\tilde{\phi}|_{1,h} + \|\tilde{\phi} - \Pi^1_{k,h}\tilde{\phi}\|_{h_0} + \sup_{w_h \in V_h^0\setminus\{0\}} \frac{(\xi_h, w_h - \Pi^0_{k,h}w_h)}{|w_h|_{H^1(\Omega)}}.$$

Following similar arguments as in the proof of Lemma 5.11 and (5.38), we can get the following concrete bound,

$$\begin{aligned}|\tilde{\phi} - \phi_h|_{H^1(\Omega)} &\lesssim h^{\min((\pi/\omega)-\epsilon,k)}(\|\tilde{\phi}\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} + |\xi_h|_{H^1(\Omega)}) \\ &\lesssim h^{\min((\pi/\omega)-\epsilon,k)}(\|\xi_h - \xi\|_{H^1(\Omega)} + \|\xi\|_{H^1(\Omega)} + |\xi|_{H^1(\Omega)}) \\ &\lesssim h^{\min((\pi/\omega)-\epsilon,k)}(|\xi_h - \xi|_{H^1(\Omega)} + \|\xi\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)}) \\ &\lesssim h^{\min((\pi/\omega)-\epsilon,k)}\|\boldsymbol{f}\|_{L^2(\Omega)}.\end{aligned} \tag{5.40}$$ eq:phi.b

Where in the second inequality we used (5.28) followed by a triangle inequality. In the third inequality we used Poincaré inequality and $H^{1+(\pi/\omega)-\epsilon}(\Omega) \hookrightarrow H^1(\Omega)$. Finally the last inequality is obtained by (5.25) and Lemma 2.1 on (3.5) followed by (5.17). Finally, a triangle inequality alongwith (5.39) and (5.40) leads to (5.35).

Moreover, it follows from (3.9) and Lemma 2.2 that

$$\phi \in H^{1+(\pi/\omega)-\epsilon}(\Omega) \quad \text{and} \quad \|\phi\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \lesssim \|\xi\|_{H^1(\Omega)}. \tag{5.41}$$ eq:reg.phi

Hence, using similar arguments as in the proof of (5.26) with (5.41) leads to (5.36). □

In view of Lemmas 5.11, 5.12, (3.3) and (4.11), we readily have the following result for simply connected domains.

lyconnected⟩ **Theorem 5.13.** *The approximations ξ_h and $\boldsymbol{u}_h$ obtained by $\mathbb{P}_k$-virtual finite element method satisfy*

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)} + |\nabla \times \boldsymbol{u} - \xi_h|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega)-\epsilon,k)}\|\boldsymbol{f}\|_{L^2(\Omega)}, \tag{5.42}$$?eq:err.est.

for any $\epsilon > 0$, and $k = 1, 2$, where ω is the largest angle at the corners of Ω.

Proof. The estimate for $\boldsymbol{u}-\boldsymbol{u}_h$ follows from (5.36) on observing the following,

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\Omega)} = \|\nabla\times\phi - \nabla\times\Pi^1_{k,h}\phi_h\|_{L^2(\Omega)} = |\phi-\Pi^1_{k,h}\phi_h|_{1,h}.$$

Recalling that $\xi = \nabla\times\boldsymbol{u}$, the estimate for $\nabla\times\boldsymbol{u}-\xi_h$ follows from (5.25). □

5.2. Ω **is multiply connected** ($\gamma > 0$)**.** The convergence analysis for the multiply connected case follows ideas similar to those in the simply connected case extended to the coupled setting. We first write down the error equation for $(\zeta_{0,h},\xi_{0,h})$ by testing (3.13) with $(\psi_h,\eta_h)\in V_h\times V_h^0$, subtracting it from (4.15) and using $\zeta_0,\zeta_{0,h}\in L^2_0(\Omega)$,

$$\mathcal{A}((\zeta_0,\xi_0),(\psi_h,\eta_h)) - \mathcal{A}_h((\zeta_{0,h},\xi_{0,h}),(\psi_h,\eta_h)) = \gamma^{-1/2}(\boldsymbol{f},\nabla\times(\psi_h-\Pi^1_{k,h}\psi_h)) \tag{5.43}$$ eq:err.zeta

Furthermore, testing (3.13) with (ζ_0,ξ_0) and applying Young's inequality gives

$$\begin{aligned}|\zeta_0|^2_{H^1(\Omega)} + (\zeta_0,1)^2 + |\xi_0|^2_{H^1(\Omega)} &\lesssim \|\boldsymbol{f}\|^2_{L^2(\Omega)},\\ \|\zeta_0\|^2_{H^1(\Omega)} + |\xi_0|^2_{H^1(\Omega)} &\lesssim \|\boldsymbol{f}\|^2_{L^2(\Omega)}.\end{aligned} \tag{5.44}$$ eq:data.zet

where the last inequality is a consequence of Poincaré-Wirtinger inequality. Similarly, testing (4.15) with $(\zeta_{0,h},\xi_{0,h})$, and in view of Assumption 5.7 and stability of $\Pi^1_{k,h}$, we obtain

$$\|\zeta_{0,h}\|^2_{H^1(\Omega)} + |\xi_{0,h}|^2_{H^1(\Omega)} \lesssim \|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{5.45}$$ eq:data.zet

Now analogous to the simply connected case, we estimate the error for $\zeta_{0,h}$ and $\zeta_{1,h}$ in $H^1(\Omega)'$.

zetah.dual⟩? **Lemma 5.14.** *For any $\epsilon > 0$, there exists a positive constant dependent on ϵ and independent of h such that for $k = 1,2$ and $\gamma > 0$,*

$$|(\zeta_0-\zeta_{0,h},\chi)| \lesssim h^{\min((\pi/\omega)-\epsilon,k)}\|\chi\|_{H^1(\Omega)}\|\boldsymbol{f}\|_{L^2(\Omega)} \quad \forall\chi\in H^1(\Omega) \tag{5.46}$$ eq:err.est.

$$|(\zeta_1-\zeta_{1,h},\chi)| \lesssim h^{\min((\pi/\omega)-\epsilon,k)}\|\chi\|_{H^1(\Omega)} \quad \forall\chi\in H^1(\Omega). \tag{5.47}$$ eq:err.est.

Proof. Consider the following dual problem. Given $\chi\in H^1(\Omega)$, let $(\lambda,\mu)\in H^1(\Omega)\times H^1_0(\Omega)$ solve

$$\mathcal{A}((\psi,\eta),(\lambda,\mu)) + (\psi,1)(\lambda,1) = (\psi,\chi) \quad \forall(\psi,\eta)\in H^1(\Omega)\times H^1_0(\Omega). \tag{5.48}$$ eq:DP.coupl

Observe that testing the above well-posed dual problem with (λ,μ) gives the following stability estimate,

$$\|\lambda\|^2_{H^1(\Omega)} + \|\mu\|^2_{H^1(\Omega)} \lesssim \|\chi\|^2_{L^2(\Omega)}. \tag{5.49}$$ eq:DP.stab.

Using the definition of $\mathcal{A}(\cdot,\cdot)$, we can rewrite (5.48) as the following coupled system:

$$(\nabla\times\psi,\nabla\times\lambda) + (\psi,1)(\lambda,1) = (\psi,\chi) + \gamma^{1/2}(\psi,\mu) \quad \forall\psi\in H^1(\Omega), \tag{5.50}$$ eq:DP.coupl

$$(\nabla\times\eta,\nabla\times\mu) + \beta(\eta,\mu) = -\gamma^{1/2}(\lambda,\eta) \quad \forall\eta\in H^1_0(\Omega). \tag{5.51}$$ eq:DP.coupl

Thus, using Lemma 2.2 on (5.50), and Lemma 2.1 on (5.51), followed by the stability estimate (5.49), we have the following regularity result for the dual problem,

$$\|\lambda\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} + \|\mu\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \lesssim \|\chi\|_{H^1(\Omega)}. \tag{5.52}$$ eq:DP.regul

Testing (5.48) with $(\psi,\eta) = (\zeta_0-\zeta_{0,h},\xi_0-\xi_{0,h})\in H^1(\Omega)\times H^1_0(\Omega)$ and using the fact that $\zeta_0,\zeta_{0,h}\in L^2_0(\Omega)$, we have

$$\begin{aligned}(\zeta_0-\zeta_{0,h},\chi) &= \mathcal{A}((\zeta_0-\zeta_{0,h},\xi_0-\xi_{0,h}),(\lambda,\mu))\\ &= \mathcal{A}((\zeta_0-\zeta_{0,h},\xi_0-\xi_{0,h}),(\lambda-I_{k,h}\lambda,\mu-I_{k,h}\mu)) + \mathcal{A}((\zeta_0-\zeta_{0,h},\xi_0-\xi_{0,h}),(I_{k,h}\lambda,I_{k,h}\mu)).\end{aligned} \tag{5.53}$$ eq:err.coup

The first term in (5.53) is bounded using the definition of $\mathcal{A}$, Cauchy-Schwarz inequality, (5.44), (5.45), Lemma 5.6 and (5.52) to get

$$\begin{aligned}&\mathcal{A}((\zeta_0-\zeta_{0,h},\xi_0-\xi_{0,h}),(\lambda-I_{k,h}\lambda,\mu-I_{k,h}\mu))\\ &\lesssim \left(\|\zeta_0-\zeta_{0,h}\|_{H^1(\Omega)} + |\xi_0-\xi_{0,h}|_{H^1(\Omega)}\right)\times\left(\|\lambda-I_{k,h}\lambda\|_{H^1(\Omega)} + |\mu-I_{k,h}\mu|_{H^1(\Omega)}\right)\\ &\lesssim \left(\|\lambda-I_{k,h}\lambda\|_{H^1(\Omega)} + |\mu-I_{k,h}\mu|_{H^1(\Omega)}\right)\|\boldsymbol{f}\|_{L^2(\Omega)}\\ &\lesssim h^{\min((\pi/\omega)-\epsilon,k)}\|\chi\|_{H^1(\Omega)}\|\boldsymbol{f}\|_{L^2(\Omega)}\end{aligned} \tag{5.54}$$ eq:err.coup

The second term in (5.53) is bounded using the error equation (5.43) for $(\psi_h, \eta_h) = (I_{k,h}\lambda, I_{k,h}\mu)$ as follows:

$$
\begin{aligned}
&\mathcal{A}((\zeta_0 - \zeta_{0,h}, \xi_0 - \xi_{0,h}), (I_{k,h}\lambda, I_{k,h}\mu)) \\
&= \gamma^{-1/2}(\boldsymbol{f}, \nabla \times (I_{k,h}\lambda - \Pi^1_{k,h} I_{k,h}\lambda)) + \mathcal{A}_h((\zeta_{0,h}, \xi_{0,h}), (I_{k,h}\lambda, I_{k,h}\mu)) - \mathcal{A}((\zeta_{0,h}, \xi_{0,h}), (I_{k,h}\lambda, I_{k,h}\mu)) \\
&\lesssim \|\boldsymbol{f}\|_{L^2(\Omega)} |(I - \Pi^1_{k,h}) I_{k,h}\lambda|_{1,h} + \mathcal{A}_h((\zeta_{0,h}, \xi_{0,h}), (I_{k,h}\lambda, I_{k,h}\mu)) - \mathcal{A}((\zeta_{0,h}, \xi_{0,h}), (I_{k,h}\lambda, I_{k,h}\mu)). \quad (5.55)
\end{aligned}
$$

The difference in (5.55) can be rewritten as follows:

$$
\begin{aligned}
&\mathcal{A}_h((\zeta_{0,h}, \xi_{0,h}), (I_{k,h}\lambda, I_{k,h}\mu)) - \mathcal{A}((\zeta_{0,h}, \xi_{0,h}), (I_{k,h}\lambda, I_{k,h}\mu)) \\
&\quad = (a_h(\zeta_{0,h}, I_{k,h}\lambda) - (\nabla \times \zeta_{0,h}, \nabla \times I_{k,h}\lambda)) + (a_h(\xi_{0,h}, I_{k,h}\mu) - (\nabla \times \xi_{0,h}, \nabla \times I_{k,h}\mu)) \\
&\qquad + \Big(\gamma^{1/2}(\Pi^0_{k,h} I_{k,h}\lambda, \Pi^0_{k,h}\xi_{0,h}) - \gamma^{1/2}(I_{k,h}\lambda, \xi_{0,h})\Big) + \Big(\gamma^{1/2}(\zeta_{0,h}, I_{k,h}\mu) - \gamma^{1/2}(\Pi^0_{k,h}\zeta_{0,h}, \Pi^0_{k,h} I_{k,h}\mu)\Big) \\
&\qquad + \big(\beta(\Pi^0_{k,h}\xi_{0,h}, \Pi^0_{k,h} I_{k,h}\mu) - \beta(\xi_{0,h}, I_{k,h}\mu)\big) \\
&\quad =: T_1 + T_2 + T_3 + T_4 + T_5.
\end{aligned}
$$

The terms T_1 and T_2 can be bounded similar to (5.24) to obtain

$$
\begin{aligned}
T_1 + T_2 &\lesssim h^{\min((\pi/\omega)-\epsilon, k)} \|\boldsymbol{f}\|_{L^2(\Omega)} \Big(\|\lambda\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} + \|\mu\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)}\Big) \\
&\overset{(5.52)}{\lesssim} h^{\min((\pi/\omega)-\epsilon, k)} \|\boldsymbol{f}\|_{L^2(\Omega)} \|\chi\|_{H^1(\Omega)}. \qquad (5.56)
\end{aligned}
$$

The term T_3 is estimated using the definition of $\Pi^0_{k,h}$ and Cauchy-Schwarz inequality as follows,

$$
\begin{aligned}
T_3 &= \gamma^{1/2}(\Pi^0_{k,h} I_{k,h}\lambda - I_{k,h}\lambda, \xi_{0,h}) \\
&\lesssim \|\Pi^0_{k,h} I_{k,h}\lambda - I_{k,h}\lambda\|_{L^2(\Omega)} \|\xi_{0,h}\|_{L^2(\Omega)} \\
&\lesssim h^{\min((\pi/\omega)-\epsilon, k)} |\lambda|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} |\xi_{0,h}|_{H^1(\Omega)} \\
&\overset{(5.52),(5.45)}{\lesssim} h^{\min((\pi/\omega)-\epsilon, k)} \|\boldsymbol{f}\|_{L^2(\Omega)} \|\chi\|_{H^1(\Omega)}. \qquad (5.57)
\end{aligned}
$$

Where the term $\|\Pi^0_{k,h} I_{k,h}\lambda - I_{k,h}\lambda\|_{L^2(\Omega)}$ is bounded analogous to [4, Lemma 3.16] using (5.9) and approximation property of $\Pi^0_{k,h}$. The terms T_4 and T_5 are bounded analogously.

$$
T_4 + T_5 \lesssim h^{\min((\pi/\omega)-\epsilon, k)} \|\boldsymbol{f}\|_{L^2(\Omega)} \|\chi\|_{H^1(\Omega)}. \qquad (5.58)
$$

Substituting the bounds (5.56), (5.57), and (5.58) into (5.55), and noting that the term $|(I - \Pi^1_{k,h}) I_{k,h}\lambda|_{1,h}$ can be bounded similar to the simply connected case gives

$$
\mathcal{A}((\zeta_0 - \zeta_{0,h}, \xi_0 - \xi_{0,h}), (I_{k,h}\lambda, I_{k,h}\mu)) \lesssim h^{\min((\pi/\omega)-\epsilon, k)} \|\boldsymbol{f}\|_{L^2(\Omega)} \|\chi\|_{H^1(\Omega)}. \qquad (5.59)
$$

Thus combining (5.54) and (5.59) in (5.53) leads to the following estimate (5.46). Furthermore, it is easy to see that one can readily obtain (5.47) following similar arguments. ☐

Lemma 5.15. *For any $\epsilon > 0$, there exists a positive constant dependent on ϵ and independent of h such that for $k = 1, 2$ and $\gamma > 0$,*

$$
|\xi - \xi_h|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega)-\epsilon, k)} \|\boldsymbol{f}\|_{L^2(\Omega)}, \qquad (5.60)
$$

$$
|\xi - \Pi^1_{k,h}\xi_h|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega)-\epsilon, k)} \|\boldsymbol{f}\|_{L^2(\Omega)}. \qquad (5.61)
$$

Proof. Consider the auxiliary problem. Given $\zeta_{0,h} \in V^0_h \cap L^2_0(\Omega)$, let $\tilde{\xi}_0 \in H^1_0(\Omega)$ solve

$$
(\nabla \times \tilde{\xi}_0, \nabla \times \eta) + \beta(\tilde{\xi}_0, \eta) = \gamma^{1/2}(\zeta_{0,h}, \eta) \quad \forall \eta \in H^1_0(\Omega). \qquad (5.62)
$$

This implies

$$
\tilde{\xi}_0 \in H^{1+(\pi/\omega)-\epsilon}(\Omega) \quad \text{and} \quad \|\tilde{\xi}_0\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \lesssim \|\zeta_{0,h}\|_{L^2(\Omega)}. \qquad (5.63)
$$

due to Lemma 2.1. Compaing the ξ_0 part of the coupled problem (3.13),

$$
(\nabla \times \xi_0, \nabla \times \eta) + \beta(\xi_0, \eta) = \gamma^{1/2}(\zeta_0, \eta) \quad \forall \eta \in H^1_0(\Omega), \qquad (5.64)
$$

with (5.62), and using the duality estimate (5.46) gives,

$$
|\xi_0 - \tilde{\xi}_0|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega)-\epsilon, k)} \|\boldsymbol{f}\|_{L^2(\Omega)}. \qquad (5.65)
$$

Now it follows from

$$a_h(\xi_{0,h},\eta_h)+\beta(\Pi^0_{k,h}\xi_{0,h},\Pi^0_{k,h}\eta_h)=\gamma^{1/2}(\Pi^0_{k,h}\zeta_{0,h},\Pi^0_{k,h}\eta_h)\quad\forall\eta_h\in V_h^0, \tag{5.66}$$
?eq:Ph.mc.xi

which is the $\xi_{0,h}$ part of the coupled discrete problem (4.15), and (5.62) that $\xi_{0,h}$ is also a virtual element approximation of $\tilde{\xi}_0$. Hence, using similar arguments as in the proof of Lemma 5.11, we can get the following estimate,

$$\begin{aligned}|\tilde{\xi}_0-\xi_{0,h}|_{H^1(\Omega)}&\lesssim h^{\min((\pi/\omega)-\epsilon,k)}(\|\tilde{\xi}_0\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)}+|\zeta_{0,h}|_{H^1(\Omega)})\\&\overset{(5.63)}{\lesssim} h^{\min((\pi/\omega)-\epsilon,k)}(\|\zeta_{0,h}\|_{L^2(\Omega)}+|\zeta_{0,h}|_{H^1(\Omega)})\\&\overset{(5.44)}{\lesssim} h^{\min((\pi/\omega)-\epsilon,k)}\|\boldsymbol{f}\|_{L^2(\Omega)}.\end{aligned} \tag{5.67}$$
eq:xi0h.b.m

Combining (5.67) with (5.65) via a triangle inequality leads to

$$|\xi_0-\xi_{0,h}|_{H^1(\Omega)}\lesssim h^{\min((\pi/\omega)-\epsilon,k)}\|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{5.68}$$
eq:xi0h.mc

Following similar arguments as above on the ξ_1 part of the coupled problem (3.14) and its discrete counterpart (4.16), and using (5.47) gives,

$$|\xi_1-\xi_{1,h}|_{H^1(\Omega)}\lesssim h^{\min((\pi/\omega)-\epsilon,k)}. \tag{5.69}$$
eq:xi1h.mc

Now in view of (5.69), (5.68) and (4.14), the estimate (5.60) readily follows. Finally, the estimate (5.61) follows from similar arguments as in the proof of (5.26) and using Lemma 2.1 on (3.11). ☐

It remains to estimate the error in the approximation of harmonic part φ_j and the real numbers c_j for $1\leqslant j\leqslant m$. The error estimate for φ_j can be estimated readily by using the abstract error estimate for second order elliptic problems approximated by virtual element methods [8, Section 4.3] and the regularity of $\varphi_j\in H^{1+(\pi/\omega)-\epsilon}(\Omega)$.

t.varphi.mc⟩ **Lemma 5.16.** *For any $\epsilon>0$, there exists a positive constant dependent on ϵ and independent of h such that for $k=1,2,\gamma>0$, and $1\leqslant j\leqslant m$,*

$$|\varphi_j-\varphi_{j,h}|_{H^1(\Omega)}\lesssim h^{\min((\pi/\omega)-\epsilon,k)}, \tag{5.70}$$
eq:err.est.

$$|\varphi_j-\Pi^1_{k,h}\varphi_{j,h}|_{H^1(\Omega)}\lesssim h^{\min((\pi/\omega)-\epsilon,k)}. \tag{5.71}$$
eq:err.est.

Proof. It follows from (3.2), (4.19) and the abstract error estimate [8, Section 4.3], we have

$$\|\varphi_j-\varphi_{j,h}\|_{h_0}\lesssim\|\varphi_j-I_{k,h}\varphi_j\|_{h_0}+|\varphi_j-\Pi^1_{k,h}\varphi_j|_{1,h}+\|\varphi_j-\Pi^1_{k,h}\varphi_j\|_{h_0}.$$

Using Lemma 5.8 and Assumption 5.7 leads to

$$\|\varphi_j-\varphi_{j,h}\|_{h_0}\lesssim h^{\min((\pi/\omega)-\epsilon,k)}\|\varphi_j\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \tag{5.72}$$
eq:varphi.a

In view of (3.2) there exists a lifting $g_j\in H^1(\Omega)$ such that $g_j|_{\Gamma_0}=0$ and $g_j|_{\Gamma_l}=\delta_{jl}$ for $1\leqslant l\leqslant m$. Moreover, since boundary data are just constants and finite ($j=\{1,\dots,m\}$), we can ensure $\|g_j\|_{H^1(\Omega)}\leqslant C_\Omega$. Thus, applying Lemma 2.1 to (5.72) gives

$$\|\varphi_j\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)}\lesssim\|g_j\|_{H^1(\Omega)}\lesssim C_\Omega, \tag{5.73}$$
eq:varphi.d

which results to (5.70). Finally, using similar arguments as in the proof of (5.26) with the regularity result for φ_j leads to (5.71). ☐

Before comparing c_j and $c_{j,h}$, we first note that (5.36) implies

$$|(\boldsymbol{f},\nabla\varphi_j)-(\boldsymbol{f},\nabla\Pi^1_{k,h}\varphi_{j,h})|\lesssim h^{\min((\pi/\omega)-\epsilon,k)}\|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{5.74}$$
eq:err.est.

Testing (3.2a) with $\varphi_j-I_{k,h}\varphi_j\in V_h^0\subset H_0^1(\Omega)$ and (4.19a) with $\varphi_{j,h}-I_{k,h}\varphi_j\in V_h^0$ results in

$$\begin{aligned}(\nabla\varphi_i,\nabla\varphi_j)-a_h(\varphi_{i,h},\varphi_{j,h})&=(\nabla\varphi_i,\nabla I_{k,h}\varphi_j)-a_h(\varphi_{i,h},I_{k,h}\varphi_j)\\&=\sum_{D\in\mathcal{T}_h}(\nabla\varphi_i,\nabla(I_{k,D}\varphi_j-\Pi^1_{k,D}I_{k,D}\varphi_j))+(\nabla(\varphi_i-\Pi^1_{k,D}\varphi_{i,h}),\nabla\Pi^1_{k,D}I_{k,D}\varphi_j)\\&\quad+S^D((I-\Pi^1_{k,D})\varphi_{i,h},(I-\Pi^1_{k,D})I_{k,D}\varphi_j).\end{aligned} \tag{5.75}$$
eq:err.est.

The first term on the right-hand side of (5.75) is bounded using Cauchy-Schwarz inequality, (5.10) and (5.73) to get

$$\sum_{D\in\mathcal{T}_h} (\nabla\varphi_i, \nabla(I_{k,D}\varphi_j - \Pi^1_{k,D} I_{k,D}\varphi_j)) \lesssim h^{\min((\pi/\omega)-\epsilon,k)}. \tag{5.76}$$

The second term on the right-hand side of (5.75) is bounded using Cauchy-Schwarz inequality, the error estimate (5.71), stability of projection and interpolation operators (5.2), (5.9), and (5.73) to get

$$\begin{aligned}
&\sum_{D\in\mathcal{T}_h} (\nabla(\varphi_i - \Pi^1_{k,D}\varphi_{i,h}), \nabla\Pi^1_{k,D} I_{k,D}\varphi_j)\\
&\lesssim \left(\sum_{D\in\mathcal{T}_h} \|\nabla(\varphi_i - \Pi^1_{k,D}\varphi_{i,h})\|^2_{L^2(D)}\right)^{1/2} \left(\sum_{D\in\mathcal{T}_h} \|\nabla\Pi^1_{k,D} I_{k,D}\varphi_j\|^2_{L^2(D)}\right)^{1/2}\\
&\lesssim h^{\min((\pi/\omega)-\epsilon,k)}.
\end{aligned} \tag{5.77}$$

The stabilization term on the right-hand side of (5.75) is bounded using the Cauchy-Schwarz inequality to obtain

$$\sum_{D\in\mathcal{T}_h} S^D((I-\Pi^1_{k,D})\varphi_{i,h}, (I-\Pi^1_{k,D})I_{k,D}\varphi_j) \lesssim \|\varphi_{i,h} - \Pi^1_{k,h}\varphi_{i,h}\|_{h_0} \|\varphi_j - I_{k,h}\varphi_j\|_{h_0}.$$

The second term in the above inequality is bounded using (5.15) and the first term is bounded using the definition of $S^D(\cdot,\cdot)$, (4.13), Sobolev inequality (5.1), Lemma 5.5 and (5.73) as follows,

$$\begin{aligned}
\|\varphi_{i,h} - \Pi^1_{k,h}\varphi_{i,h}\|^2_{h_0} &\lesssim \sum_{D\in\mathcal{T}_h} \|\varphi_{i,h} - \Pi^1_{k,h}\varphi_{i,h}\|^2_{L^\infty(D)}\\
&\lesssim \sum_{D\in\mathcal{T}_h} h_D^{-2}\|\varphi_{i,h} - \Pi^1_{k,D}\varphi_{i,h}\|^2_{L^2(D)} + |\varphi_{i,h} - \Pi^1_{k,D}\varphi_{i,h}|^2_{H^1(D)}\\
&\lesssim C_\Omega.
\end{aligned} \tag{5.78}$$

Substituting the bounds (5.76), (5.77), and (5.78) into (5.75) leads to

$$|(\nabla\varphi_i, \nabla\varphi_j) - a_h(\varphi_{i,h}, \varphi_{j,h})| \lesssim h^{\min((\pi/\omega)-\epsilon,k)} \quad \text{for} \quad 1 \leqslant i,j \leqslant m. \tag{5.79}$$

Lemma 5.17. *For $h < h_0$, and any $\epsilon > 0$, there exists a positive constant dependent on ϵ and independent of h such that for $k = 1, 2$, $\gamma > 0$, and $1 \leqslant j \leqslant m$,*

$$|c_j - c_{j,h}| \lesssim h^{\min((\pi/\omega)-\epsilon,k)} \|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{5.80}$$

Here $h_0 = \left(\frac{\lambda_{\min}(\boldsymbol{A})}{2}\right)^{1/\min((\pi/\omega)-\epsilon,k)}$, where $\lambda_{\min}(\boldsymbol{A})$ is the minimum eigenvalue of the $\boldsymbol{A} := [(\nabla\varphi_i, \nabla\varphi_j)]^m_{i,j=1}$.

Proof. The equations (3.16) and (4.18) can be rewritten in the following matrix form:

$$\boldsymbol{A}\boldsymbol{c} = \boldsymbol{b}, \quad \text{and} \quad \boldsymbol{A}_h\boldsymbol{c}_h = \boldsymbol{b}_h,$$

where $\boldsymbol{A} = [(\nabla\varphi_i, \nabla\varphi_j)]^m_{i,j=1}$, $\boldsymbol{A}_h = [a_h(\varphi_{i,h}, \varphi_{j,h})]^m_{i,j=1}$, $\boldsymbol{c} = [c_j]^m_{j=1}$, $\boldsymbol{c}_h = [c_{j,h}]^m_{j=1}$, $\boldsymbol{b} = [\gamma^{-1}(\boldsymbol{f}, \nabla\varphi_j)]^m_{j=1}$, and $\boldsymbol{b}_h = [\gamma^{-1}(\boldsymbol{f}, \nabla\Pi^1_{k,h}\varphi_{j,h})]^m_{j=1}$. Some algebra then gives,

$$\boldsymbol{c} - \boldsymbol{c}_h = \boldsymbol{A}^{-1}\boldsymbol{b} - \boldsymbol{A}_h^{-1}\boldsymbol{b}_h = \boldsymbol{A}^{-1}(\boldsymbol{b} - \boldsymbol{b}_h) + \boldsymbol{A}^{-1}(\boldsymbol{A}_h - \boldsymbol{A})\boldsymbol{A}_h^{-1}((\boldsymbol{b}_h - \boldsymbol{b}) + \boldsymbol{b}).$$

Hence, using the properties of matrix norm $\|\cdot\|_\infty$, we have

$$\|\boldsymbol{c} - \boldsymbol{c}_h\|_\infty \leqslant \|\boldsymbol{A}^{-1}\|_\infty \|\boldsymbol{b} - \boldsymbol{b}_h\|_\infty + \|\boldsymbol{A}^{-1}\|_\infty \|\boldsymbol{A}_h - \boldsymbol{A}\|_\infty \|\boldsymbol{A}_h^{-1}\|_\infty (\|\boldsymbol{b}_h - \boldsymbol{b}\|_\infty + \|\boldsymbol{b}\|_\infty). \tag{5.81}$$

Observe that

$$|\boldsymbol{b}| \leqslant |\gamma|^{-1} |(\boldsymbol{f}, \nabla\varphi_j)| \leqslant |\gamma|^{-1} \|\boldsymbol{f}\|_{L^2(\Omega)} \|\nabla\varphi_j\|_{L^2(\Omega)}.$$

Taking the max over j gives

$$\|\boldsymbol{b}\|_\infty \lesssim \|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{5.82}$$

Similarly, we have

$$\|\boldsymbol{b}_h\|_\infty \lesssim \|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{5.83}$$

The estimates (5.79) and (5.74) translate to

$$\|\boldsymbol{A}-\boldsymbol{A}_h\|_\infty \lesssim h^{\min((\pi/\omega)-\epsilon,k)} \quad \text{and} \quad \|\boldsymbol{b}-\boldsymbol{b}_h\|_\infty \lesssim h^{\min((\pi/\omega)-\epsilon,k)}\|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{5.84}$$

Furthermore, since $\boldsymbol{A}$ is symmetric positive definite matrix and due to equivalence of norms in finite dimensional spaces, we have

$$\|\boldsymbol{A}^{-1}\|_\infty \leqslant 2\|\boldsymbol{A}^{-1}\|_2 = \frac{2}{\lambda_{\min}(\boldsymbol{A})}, \tag{5.85}$$

where $\lambda_{\min}(\boldsymbol{A})$ denotes the minimum eigenvalue of $\boldsymbol{A}$. In order to control the $\|\boldsymbol{A}_h\|_\infty$, we write $\boldsymbol{A}_h^{-1} = (\boldsymbol{I}-\boldsymbol{A}^{-1}(\boldsymbol{A}_h-\boldsymbol{A}))^{-1}\boldsymbol{A}^{-1}$, and proceed using the Von Neumann lemma as follows,

$$\|\boldsymbol{A}_h^{-1}\|_\infty \leqslant \|(\boldsymbol{I}-\boldsymbol{A}^{-1}(\boldsymbol{A}_h-\boldsymbol{A}))^{-1}\|_\infty\|\boldsymbol{A}^{-1}\|_\infty \leqslant \frac{\|\boldsymbol{A}^{-1}\|_\infty}{1-\|\boldsymbol{A}^{-1}(\boldsymbol{A}_h-\boldsymbol{A})\|_\infty}. \tag{5.86}$$

The estimate (5.86) holds provided $\|\boldsymbol{A}^{-1}\|_\infty\|(\boldsymbol{A}_h-\boldsymbol{A})\|_\infty < 1$ which implies $\|\boldsymbol{A}^{-1}(\boldsymbol{A}_h-\boldsymbol{A})\|_\infty < 1$. On using (5.84) and further simplification this leads to the following condition on h:

$$h < h_0 := \left(\frac{\lambda_{\min}(\boldsymbol{A})}{2}\right)^{1/\min((\pi/\omega)-\epsilon,k)}.$$

Substituting the estimates (5.82), (5.83), (5.84), (5.85), and (5.86) into (5.81) leads to

$$\|\boldsymbol{c}-\boldsymbol{c}_h\|_\infty \lesssim h^{\min((\pi/\omega)-\epsilon,k)}\|\boldsymbol{f}\|_{L^2(\Omega)}.$$

This completes the proof of (5.80). □

In view of Lemmas 5.15, 5.12, 5.16, and 5.17, we can obtain the following estimate.

Theorem 5.18. *The approximations ξ_h and $\boldsymbol{u}_h$ obtained by $\mathbb{P}_k$-virtual finite element method satisfy*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\Omega)} + |\nabla\times\boldsymbol{u}-\xi_h|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega)-\epsilon,k)}\|\boldsymbol{f}\|_{L^2(\Omega)}, \tag{5.87}$$

for any $\epsilon > 0$, $h < h_0$ with h_0 defined in Lemma 5.17, and $k = 1, 2$, where ω is the largest angle at the corners of Ω.

Proof. The proof follows similarly to [3, Theorem 4.9]. □

6. Proof of some Lemmas

6.1. **Proof of Lemma 5.8.** Observing that $\Pi^1_{k,h}((I-\Pi^1_{k,h})I_{k,h}v) = 0$ due to linearity and idempotency of $\Pi^0_{k,h}$ we deduce that

$$\begin{aligned}
\|I_{k,h}v - \Pi^1_{k,h}I_{k,h}v\|^2_{h_0} &= \sum_{D\in\mathcal{T}_h} S^D((I-\Pi^1_{k,h})I_{k,h}v, (I-\Pi^1_{k,h})I_{k,h}v)\\
&= \sum_{D\in\mathcal{T}_h}\sum_{i=1}^{N^{\rm dof}_{\partial D}} \chi_i((I-\Pi^1_{k,h})I_{k,h}v)^2\\
&\overset{(4.13)}{\lesssim} \sum_{D\in\mathcal{T}_h} \|(I-\Pi^1_{k,h})I_{k,h}v\|^2_{L^\infty(D)}\\
&\overset{(5.1)}{\lesssim} \sum_{D\in\mathcal{T}_h} h_D^{-2}\|(I-\Pi^1_{k,D})I_{k,h}v\|^2_{L^2(D)} + |(I-\Pi^1_{k,D})I_{k,h}v|^2_{H^1(D)}\\
&\quad + h_D^{2\delta}|(I-\Pi^1_{k,D})I_{k,h}v|^2_{H^{1+\delta}(D)}
\end{aligned}$$

Now splitting $(I-\Pi^1_{k,D})I_{k,h}v = I_{k,h}v - v + v - \Pi^1_{k,D}I_{k,h}v$ and using the approximation properties of $I_{k,h}$ from (5.10) and (5.11), we get the bound on the first term stated in (5.14). Similarly, the second term in (5.14) follows from the definition of $|\cdot|_{1,h}$, the above mentioned split followed by a triangle inequality and the approximation properties of the interpolation operator stated in Lemma 5.6.

Using the linearity and idempotency of $\Pi^1_{k,h}$ and (4.13), we can get the following

$$\begin{aligned}\|v-\Pi^1_{k,h}v\|^2_{h_\beta} &\lesssim \sum_{D\in\mathcal{T}_h}\|v-\Pi^1_{k,D}v\|^2_{L^\infty(D)}+\beta\|v-\Pi^1_{k,D}v\|^2_{L^2(D)}\\ &\overset{(4.13),(5.1)}{\lesssim} \sum_{D\in\mathcal{T}_h} h_D^{-2}\|(I-\Pi^1_{k,D})v\|^2_{L^2(D)}+|(I-\Pi^1_{k,D})v|^2_{H^1(D)}\\ &\qquad + h_D^{2\delta}|(I-\Pi^1_{k,D})v|^2_{H^{1+\delta}(D)}+\beta\|v-\Pi^1_{k,D}v\|^2_{L^2(D)}.\end{aligned}$$

Now using the approximation property of $\Pi^1_{k,D}$ in Lemma 5.5 leads to the desired estimate on the first term in (5.15). Similarly, we have

$$\begin{aligned}\|v-I_{k,h}v\|^2_{h_\beta} &= \sum_{D\in\mathcal{T}_h}|\Pi^1_{k,D}(v-I_{k,D}v)|^2_{H^1(D)}+\sum_{i=1}^{N^{\text{dof}}_{\partial D}}\chi_i((I-\Pi^1_{k,D})(I-I_{k,D})v)^2+\beta\|\Pi^0_{k,D}(v-I_{k,D}v)\|^2_{L^2(D)}\\ &= \sum_{D\in\mathcal{T}_h}|\Pi^1_{k,D}(v-I_{k,D}v)|^2_{H^1(D)}+\sum_{i=1}^{N^{\text{dof}}_{\partial D}}\chi_i(\Pi^1_{k,D}(I-I_{k,D})v)^2+\beta\|\Pi^0_{k,D}(v-I_{k,D}v)\|^2_{L^2(D)}\\ &\lesssim \sum_{D\in\mathcal{T}_h}|v-I_{k,D}v|^2_{H^1(D)}+\|\Pi^1_{k,D}(I-\Pi^1_{k,D})v\|^2_{L^\infty(D)}+\beta\|v-I_{k,D}v\|^2_{L^2(D)},\end{aligned}$$

where in the passage to the second equality we use the fact that $\chi_i(v-I_{k,D}v)=0$ for boundary degrees of freedom by definition of interpolation operator (5.7). In the last inequality we used the stability of projectors (5.2), (5.5) and the property (4.13). The first and last terms are bounded using the approximation properties of $I_{k,D}$ from Lemma 5.6. The second term can be estimated as in [4, (4.24)]. Combining all the resulting bounds leads to the desired estimate on the second term in (5.15).

7. Numerical Results

sec:numexp⟩? We use the boundary version for the classical `Dofi-Dofi` stabilizer defined in (4.12) in our numerical experiments.

:smooth_sol⟩ 7.1. **Smooth solution on a convex simply connected domain.** We consider the problem (1.2) with $\gamma=\beta=0$ and $\Omega=(0,1)^2$. Given $\phi(x,y)=\sin^3(\pi x)\sin^3(\pi y)$, the manufactured solution is given by

$$\boldsymbol{u}=\nabla\times\phi,\quad \text{with}\quad \boldsymbol{f}=-\nabla\times(\Delta(\nabla\times\boldsymbol{u})).$$

We solve (1.2) using the scheme described in Section 4 for orders $k=1,2$ on a sequence of unstructured Voronoi meshes. Since the virtual element solution is not known explicitly inside the element, we compare the manufactured solution with suitable projection of the discrete solution. The errors we measure are as follows:

$$\begin{aligned} e_{\boldsymbol{u}_h} &:= \|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^2(\Omega)}=\sqrt{\sum_{D\in\mathcal{T}_h}\|\boldsymbol{u}-\boldsymbol{u}_h\|^2_{L^2(D)}},\\ e_{\xi_h} &:= |\xi-\Pi^1_{k,h}\xi_h|_{1,h}=\sqrt{\sum_{D\in\mathcal{T}_h}|\xi-\Pi^1_{k,D}\xi_h|^2_{H^1(D)}},\quad \text{with}\quad \xi=\nabla\times\boldsymbol{u}.\end{aligned}$$

and are reported in Tables 1 and 2 on a sequence of structured and unstructured voronoi meshes, respectively (see Figure 1). We recall that in this experiment $\boldsymbol{u}_h$ is computed using (4.11). The observed rates of convergence are in agreement with the theoretical rates predicted by Theorem 5.13 and (5.26) with $\omega=\pi/2$.

Table 1. Experimental errors and orders of convergence for Experiment 7.1 on structured voronoi meshes.

.structured⟩

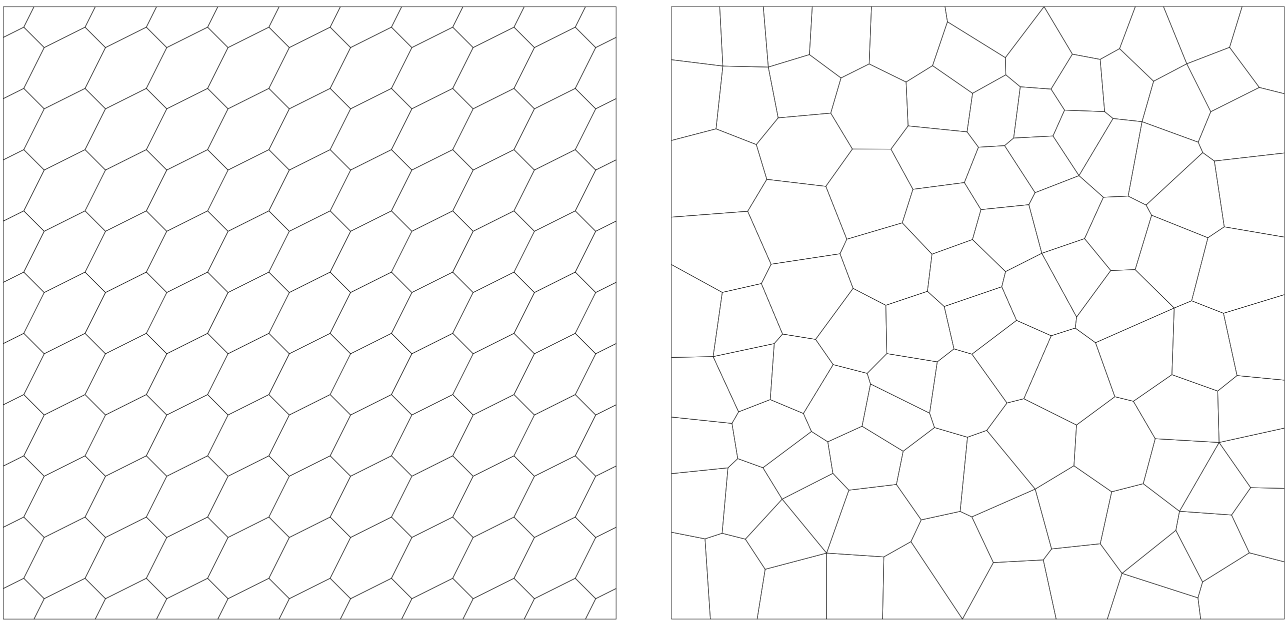
h	$N^{\texttt{Dofs}}$	$e_{\boldsymbol{u}_h}$	rate	e_{ξ_h}	rate
		$k=1$			

3.7637e-01	48	1.1963e+00	-	1.0194e+02	-
1.8360e-01	184	5.3872e-01	1.1114	5.1507e+01	0.9510
8.4283e-02	1234	1.8091e-01	1.4016	1.8902e+01	1.2876
4.1405e-02	5947	8.1748e-02	1.1176	8.6614e+00	1.0979
2.0988e-02	26477	3.8903e-02	1.0929	4.1493e+00	1.0831
1.0757e-02	96734	2.0202e-02	0.9804	2.1549e+00	0.9803
$k=2$					
3.7637e-01	145	3.0531e-01	-	3.4261e+01	-
1.8360e-01	567	7.2772e-02	1.9977	8.8858e+00	1.8800
8.4283e-02	3867	1.1100e-02	2.4152	1.3484e+00	2.4218
4.1405e-02	18693	2.3647e-03	2.1756	2.8656e-01	2.1789
2.0988e-02	82953	5.2424e-04	2.2171	6.3400e-02	2.2201
1.0757e-02	303467	1.4262e-04	1.9477	1.7275e-02	1.9453

TABLE 2. Experimental errors and orders of convergence for Experiment 7.1 on unstructured Voronoi meshes.

nstructured⟩

h	$N^{\mathtt{Dofs}}$	$e_{\boldsymbol{u}_h}$	`rate`	e_{ξ_h}	`rate`
$k=1$					
3.7637e-01	48	1.1963	-	1.0194e+02	-
1.8360e-01	184	5.3872e-01	1.1114	5.1507e+01	0.9510
8.4283e-02	1234	1.8091e-01	1.4016	1.8902e+01	1.2876
4.1405e-02	5947	8.1748e-02	1.1176	8.6614e+00	1.0979
2.0988e-02	26477	3.8903e-02	1.0929	4.1493e+00	1.0831
1.0757e-02	96734	2.0202e-02	0.9804	2.1549e+00	0.9803
$k=2$					
3.7637e-01	145	3.0531e-01	-	3.4261e+01	-
1.8360e-01	567	7.2772e-02	1.9977	8.8858e+00	1.8800
8.4283e-02	3867	1.1100e-02	2.4152	1.3484e+00	2.4218
4.1405e-02	18693	2.3647e-03	2.1756	2.8656e-01	2.1789
2.0988e-02	82953	5.2424e-04	2.2171	6.3400e-02	2.2201
1.0757e-02	303467	1.4262e-04	1.9477	1.7275e-02	1.9453



(A) Structured voronoi mesh (B) Unstructured voronoi mesh

:StVoro_sq⟩? ?⟨suB@fig:Voro_sq⟩?

FIGURE 1

Square_mesh⟩

:smoothdata⟩ 7.2. **Smooth data on a convex simply connected domain.** We keep the same domain and discretization introduced in Experiment 7.1. We take $\beta = \gamma = 0$, and choose

$$\boldsymbol{f} = ((x^2+1)\sin(x) + xy^3 + 2, (y^2+1)\cos(x) + x^3y^2 - 1).$$

Then we solve the sequence of discrete problems (4.5)-(4.11) for orders $k = 1, 2$. Since the exact solution is unknown, we compute relative errors

$$\texttt{rel}\ e^i_{\boldsymbol{u}_h} := \frac{\|\boldsymbol{u}^i_h - \boldsymbol{u}^{i+1}_h\|_{L^2(\Omega)}}{\|\boldsymbol{u}^{i+1}_h\|_{L^2(\Omega)}}, \quad \text{and} \quad \texttt{rel}\ e^i_{\xi_h} := \frac{|\Pi^{1,i}_{k,h}\xi^i_h - \Pi^{1,i+1}_{k,h}\xi^{i+1}_h|_{1,h_{i+1}}}{|\Pi^{1,i+1}_{k,h}\xi^{i+1}_h|_{1,h_{i+1}}},$$

where i denoetes the mesh level. Then we report them in Tables 3 and 4 on a sequence of structured and unstructured voronoi meshes, respectively (see Figures 1). The observed rates of convergence are in agreement with the theoretical rates predicted by Theorem 5.13 and (5.26) with $\omega = \pi/2$.

TABLE 3. Relative errors and convergence rates for Experiment 7.2 on structured voronoi meshes for orders $k = 1, 2$.

.structured⟩

h	$N^{\mathtt{Dofs}}$	$\texttt{rel}\ e_{\boldsymbol{u}_h}$	rate	$\texttt{rel}\ e_{\xi_h}$	rate
$k=1$					
2.9814e-01	90	-	-	-	-
1.4907e-01	280	2.3814e-01	-	4.6021e-01	-
7.4536e-02	960	1.2406e-01	0.9408	2.7185e-01	0.7595
3.7268e-02	3520	6.2627e-02	0.9862	1.4889e-01	0.8686
1.8634e-02	13440	3.1449e-02	0.9938	7.8161e-02	0.9297
9.3169e-03	52480	1.5765e-02	0.9963	4.0092e-02	0.9631
$k=2$					
2.9814e-01	251	-	-	-	-
1.4907e-01	801	3.2060e-02	-	8.4945e-02	-
7.4536e-02	2801	1.0155e-02	1.6586	2.8971e-02	1.5519
3.7268e-02	10401	2.9260e-03	1.7952	9.0584e-03	1.6773
1.8634e-02	40001	7.8719e-04	1.8941	2.6825e-03	1.7557
9.3169e-03	156801	2.0419e-04	1.9468	7.6646e-04	1.8073

TABLE 4. Relative errors and convergence rates for Experiment 7.2 on unstructured voronoi meshes for orders $k = 1, 2$.

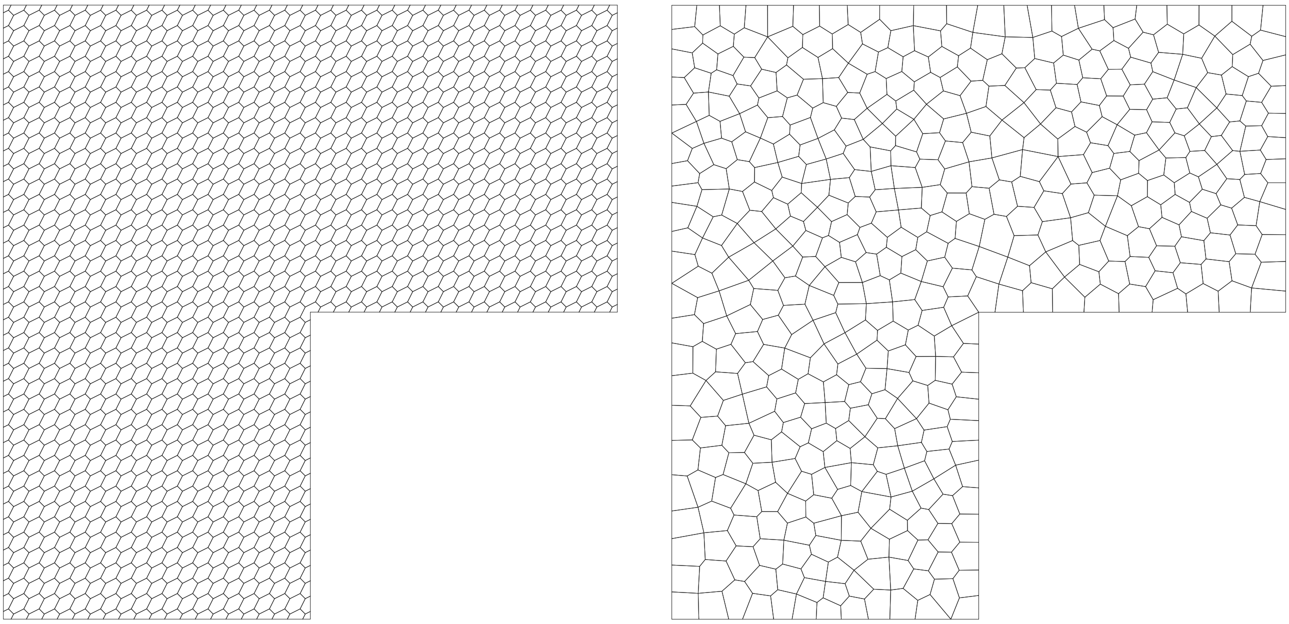
nstructured⟩

h	$N^{\mathtt{Dofs}}$	$\texttt{rel}\ e_{\boldsymbol{u}_h}$	rate	$\texttt{rel}\ e_{\xi_h}$	rate
$k=1$					
3.7637e-01	48	-	-	-	-
1.8360e-01	184	2.5848e-01	-	4.9213e-01	-
8.4283e-02	1234	1.2341e-01	0.9496	2.7482e-01	0.7483
4.1405e-02	5947	4.6503e-02	1.3731	1.1872e-01	1.1809
2.0988e-02	26477	2.1339e-02	1.1464	5.3990e-02	1.1597
1.0757e-02	96734	1.0318e-02	1.0873	2.6389e-02	1.0710
$k=2$					
3.7637e-01	145	-	-	-	-
1.8360e-01	567	3.8655e-02	-	1.1907e-01	-
8.4283e-02	3867	1.1807e-02	1.5233	3.7556e-02	1.4821
4.1405e-02	18693	2.0839e-03	2.4402	7.5986e-03	2.2481
2.0988e-02	82953	4.1737e-04	2.3666	1.9587e-03	1.9952
1.0757e-02	303467	9.6109e-05	2.1971	3.8104e-04	2.4494

nsmoothdata⟩ 7.3. **Piecewise smooth data on a non-convex simply connected domain.** We solve (1.2) on a L-shaped domain $\Omega = (-1,1)^2 \setminus (0,1) \times (-1,0)$ with $\beta = \gamma = 0$, and a piecewise constant vector field $\boldsymbol{f}$,

$$\boldsymbol{f} = \begin{cases} [1/4, 5/4]^\top, & \text{if} \quad |x| < 2^{-1/2}, \\ [1/2, 3/2]^\top, & \text{if} \quad 2^{-1/2} \leqslant |x| < 1, \\ [1, 2]^\top, & \text{if} \quad |x| \geqslant 1. \end{cases}$$

Since, the exact solution is unknown, we compute relative errors which are reported in Table 5 and 6 on a sequence of structured and unstructured voronoi meshes (see Figure 2). The observed rates of convergence are in agreement with the theoretical rates predicted by Theorem 5.13 for $\omega = 3\pi/2$, and the convergence rates for ξ_h are pre-asymptotic.



(A) Structured voronoi mesh (B) Unstructured voronoi mesh

g:stVoro_L⟩? ?⟨suB@fig:Voro_L⟩?

FIGURE 2

Lshape_mesh⟩

TABLE 5. Relative errors and convergence rates for Experiment 7.3 on structured voronoi meshes.

.structured⟩

h	$N^{\texttt{Dofs}}$	rel $e_{\boldsymbol{u}_h}$	rate	rel e_{ξ_h}	rate
		$k = 1$			
2.9814e-01	230	-	-	-	-
1.4907e-01	760	1.7658e-01	-	3.1600e-01	-
7.4536e-02	2720	1.0229e-01	0.7876	1.7874e-01	0.8221
3.7268e-02	10240	6.0474e-02	0.7583	9.5571e-02	0.9032
1.8634e-02	39680	3.6354e-02	0.7342	4.9551e-02	0.9476
9.3169e-03	156160	2.2161e-02	0.7141	2.5265e-02	0.9718
		$k = 2$			
2.9814e-01	651	-	-	-	-
1.4907e-01	2201	7.7042e-02	-	4.6187e-02	-
7.4536e-02	8001	4.8020e-02	0.6820	1.5768e-02	1.5505
3.7268e-02	30401	3.0151e-02	0.6714	5.4008e-03	1.5458
1.8634e-02	118401	1.8975e-02	0.6681	2.1096e-03	1.3562
9.3169e-03	467201	1.1949e-02	0.6672	1.0279e-03	1.0372

Table 6. Relative errors and convergence rates for Experiment 7.3 on unstructured voronoi meshes.

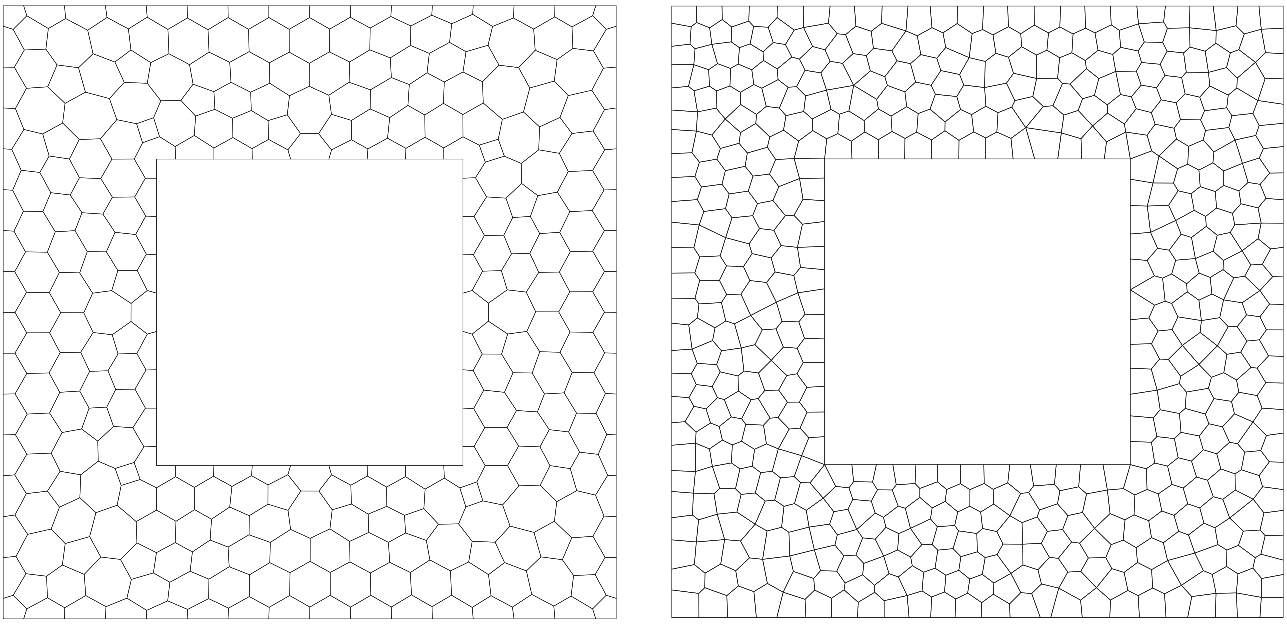
nstructured⟩

h	$N^{\mathtt{Dofs}}$	$\mathtt{rel}\ e_{\boldsymbol{u}_h}$	rate	$\mathtt{rel}\ e_{\xi_h}$	rate
$k=1$					
6.4474e-01	47	-	-	-	-
3.2497e-01	174	3.1455e-01	-	5.5756e-01	-
1.5600e-01	783	1.4861e-01	1.0218	2.9441e-01	0.8702
7.6972e-02	4279	7.6425e-02	0.9414	1.4527e-01	0.9999
3.8974e-02	16213	4.1667e-02	0.8913	6.6347e-02	1.1516
1.9933e-02	68184	2.4315e-02	0.8033	3.3254e-02	1.0301
1.0296e-02	255458	1.6116e-02	0.6226	1.6667e-02	1.0457
$k=2$					
6.4474e-01	143	-	-	-	-
3.2497e-01	547	8.2224e-02	-	1.5327e-01	-
1.5600e-01	2465	4.4343e-02	0.8414	4.9060e-02	1.5522
7.6972e-02	13557	2.7502e-02	0.6762	1.3178e-02	1.8607
3.8974e-02	51425	1.5440e-02	0.8483	3.0375e-03	2.1564
1.9933e-02	216367	9.2966e-03	0.7565	1.1150e-03	1.4946
1.0296e-02	810915	6.4954e-03	0.5428	6.6147e-04	0.7904

⟨exp:Betti1⟩ 7.4. **Piecewise smooth data on a multiply connected domain with Betti number 1.** In this experiment we consider the domain $\Omega = (0,1)^2 \setminus (1/4,3/4)^2$ with Betti number 1. We choose $\beta = \gamma = 1$ and the same piecewise smooth data $\boldsymbol{f}$ used in Experiment 7.3. We solve the sequence of discrete problems (4.14)-(4.19c) and post-process $\boldsymbol{u}_h$ using the relation (4.20) for orders $k = 1, 2$. The relative errors are reported in Table 7 and 8 on a sequence of structured and unstructured voronoi meshes, respectively (see Figure 3). We also report the relative error in the coefficient c_j,

$$\mathtt{rel}\ e^i_{c_j} = \frac{|c^i_j - c^{i+1}_j|}{|c^{i+1}_j|},$$

as well as the values of c_j for orders $k = 1, 2$



(a) Structured voronoi mesh (b) Unstructured voronoi mesh

oro_Betti1⟩? ?⟨sub@fig:Voro_Betti1⟩?

Figure 3

Betti1_mesh⟩

TABLE 7. Relative errors and convergence rates for Experiment 7.4 for orders $k = 1, 2$ on structured voronoi meshes.

etti1.struc⟩

h	$N^{\mathtt{Dofs}}$	rel $e_{\boldsymbol{u}_h}$	rate	rel e_{ξ_h}	rate	c_1	rate	rel e_{c_1}	rate
				$k = 1$					
2.64e-01	96	-	-	-	-	-0.15092	-	-	-
1.67e-01	200	5.55e-01	-	5.56e-01	-	-0.15258	-	1.09e-02	-
8.85e-02	644	3.96e-01	0.53	4.78e-01	0.24	-0.15201	1.68	3.74e-03	1.68
4.77e-02	1930	2.36e-01	0.84	2.65e-01	0.95	-0.15187	2.20	9.57e-04	2.20
2.40e-02	6584	1.48e-01	0.68	1.55e-01	0.78	-0.15180	1.08	4.57e-04	1.08
1.18e-02	24092	9.62e-02	0.61	8.80e-02	0.80	-0.15177	1.20	1.96e-04	1.20
6.11e-03	93334	6.09e-02	0.69	4.97e-02	0.86	-0.15175	1.24	8.66e-05	1.24
				$k = 2$					
2.64e-01	264	-	-	-	-	-0.15218	-	-	-
1.67e-01	552	2.30e-01	-	1.80e-01	-	-0.15214	-	3.20e-04	-
8.85e-02	1840	2.01e-01	0.22	1.33e-01	0.48	-0.15193	-2.25	1.33e-03	-2.25
4.77e-02	5614	1.14e-01	0.91	6.44e-02	1.17	-0.15182	0.89	7.66e-04	0.89
2.40e-02	19408	7.48e-02	0.62	3.88e-02	0.74	-0.15178	1.65	2.47e-04	1.65
1.18e-02	71592	4.98e-02	0.58	2.56e-02	0.59	-0.15176	0.85	1.35e-04	0.85
6.11e-03	278634	3.20e-02	0.67	1.63e-02	0.69	-0.15175	1.41	5.30e-05	1.41

TABLE 8. Relative errors and convergence rates for Experiment 7.4 for orders $k = 1, 2$ on unstructured voronoi meshes.

ti1.unstruc⟩

h	$N^{\mathtt{Dofs}}$	rel $e_{\boldsymbol{u}_h}$	rate	rel e_{ξ_h}	rate	c_1	rel e_{c_1}	rate
				$k = 1$				
3.01e-01	66	-	-	-	-	-0.14966	-	-
1.41e-01	197	5.59e-01	-	6.38e-01	-	-0.15085	7.93e-03	-
7.32e-02	908	2.99e-01	0.95	3.72e-01	0.83	-0.15136	3.35e-03	1.32
3.41e-02	4637	1.72e-01	0.72	1.92e-01	0.86	-0.15162	1.68e-03	0.90
1.57e-02	17139	9.62e-02	0.75	9.46e-02	0.92	-0.15169	5.14e-04	1.53
7.55e-03	92388	6.22e-02	0.60	5.32e-02	0.79	-0.15173	2.47e-04	1.00
				$k = 2$				
3.01e-01	204	-	-	-	-	-0.15137	-	-
1.41e-01	614	1.85e-01	-	1.59e-01	-	-0.15157	1.33e-03	-
7.32e-02	2816	1.16e-01	0.71	7.82e-02	1.08	-0.15168	7.04e-04	0.97
3.41e-02	14274	6.81e-02	0.70	4.08e-02	0.85	-0.15172	3.13e-04	1.06
1.57e-02	52278	3.88e-02	0.73	2.27e-02	0.76	-0.15174	9.70e-05	1.51
7.55e-03	284780	2.61e-02	0.54	1.50e-02	0.56	-0.15174	4.20e-05	1.14

⟨exp:Betti2⟩ 7.5. **Piecewise smooth data on a multiply connected domain with Betti number 2.** We solve (1.2) on a domain $\Omega = (-1,1)^2 \setminus (1/4, 3/4)^2 \cup (-1/4, -3/4)^2$ with Betti number 2. We choose $\beta = \gamma = 1$ and the same piecewise smooth data $\boldsymbol{f}$ used in Experiment 7.3. The relative errors are reported in Table 9 on a sequence of structured voronoi meshes (see Figure 4). We also report the relative error in the coefficients c_1, c_2 and their values.

TABLE 9. Relative errors and convergence rates for Experiment 7.5 for orders $k = 1, 2$ on structured meshes.

able:Betti2⟩

h	$N^{\mathtt{Dofs}}$	rel $e_{\boldsymbol{u}_h}$	rate	rel e_{ξ_h}	rate	c_1	rel e_{c_1}	rate	c_2	rel e_{c_2}	rate
						$k = 1$					
4.35e-01	178	-	-	-	-	-0.29668	-	-	-0.13608	-	-
2.20e-01	470	5.00e-01	-	3.92e-01	-	-0.29731	2.11e-03	-	-0.13590	1.37e-03	-
1.18e-01	1462	3.68e-01	0.49	2.51e-01	0.71	-0.29774	1.46e-03	0.58	-0.13602	9.36e-04	0.61
6.02e-02	5054	2.31e-01	0.70	1.44e-01	0.83	-0.29763	3.79e-04	2.02	-0.13586	1.18e-03	-0.35
3.06e-02	18014	1.46e-01	0.68	8.13e-02	0.84	-0.29752	3.87e-04	-0.03	-0.13579	5.05e-04	1.25

1.52e-02	67916	9.48e-02	0.62	4.53e-02	0.84	-0.29746	1.89e-04	1.03	-0.13577	1.99e-04	1.33
$k=2$											
4.35e-01	493	-	-	-	-	-0.29863	-	-	-0.13632	-	-
2.20e-01	1337	2.14e-01	-	9.76e-02	-	-0.29837	8.88e-04	-	-0.13615	1.27e-03	-
1.18e-01	4245	1.86e-01	0.22	6.10e-02	0.75	-0.29780	1.92e-03	-1.23	-0.13592	1.68e-03	-0.44
6.02e-02	14885	1.16e-01	0.70	3.06e-02	1.03	-0.29759	7.06e-04	1.50	-0.13582	7.35e-04	1.23
3.06e-02	53497	7.39e-02	0.67	1.93e-02	0.68	-0.29749	3.28e-04	1.13	-0.13578	3.18e-04	1.24
1.52e-02	202667	4.85e-02	0.60	1.23e-02	0.65	-0.29744	1.44e-04	1.18	-0.13576	1.44e-04	1.14

TABLE 10. Relative errors and convergence rates for Experiment 7.5 for orders $k = 1, 2$ on unstructured meshes.

i2_unstruc⟩?

h	$N^{\texttt{Dofs}}$	rel $e_{\boldsymbol{u}_h}$	rate	rel e_{ξ_h}	rate	c_1	rel e_{c_1}	rate	c_2	rel e_{c_2}	rate
$k=1$											
3.25e-01	181	-	-	-	-	-0.13483	-	-	-0.29259	-	-
1.70e-01	770	4.10e-01	-	3.37e-01	-	-0.13510	2.00e-03	-	-0.29611	1.19e-02	-
8.30e-02	3093	2.48e-01	0.70	1.76e-01	0.91	-0.13549	2.89e-03	-0.52	-0.29676	2.18e-03	2.37
4.08e-02	13033	1.49e-01	0.72	9.15e-02	0.92	-0.13562	9.33e-04	1.59	-0.29713	1.25e-03	0.78
2.00e-02	58140	9.48e-02	0.63	4.94e-02	0.87	-0.13570	6.31e-04	0.55	-0.29733	6.73e-04	0.87
$k=2$											
3.25e-01	563	-	-	-	-	-0.13555	-	-	-0.29716	-	-
1.70e-01	2341	1.70e-01	-	6.45e-02	-	-0.13565	7.63e-04	-	-0.29720	1.40e-04	-
8.30e-02	9387	1.03e-01	0.70	3.15e-02	1.00	-0.13572	4.81e-04	0.64	-0.29731	3.57e-04	-1.31
4.08e-02	39567	5.97e-02	0.77	1.61e-02	0.95	-0.13573	8.30e-05	2.47	-0.29737	2.10e-04	0.75
2.00e-02	176280	3.92e-02	0.59	1.09e-02	0.54	-0.13574	9.27e-05	-0.16	-0.29740	1.11e-04	0.89

ACKNOWLEDGMENTS

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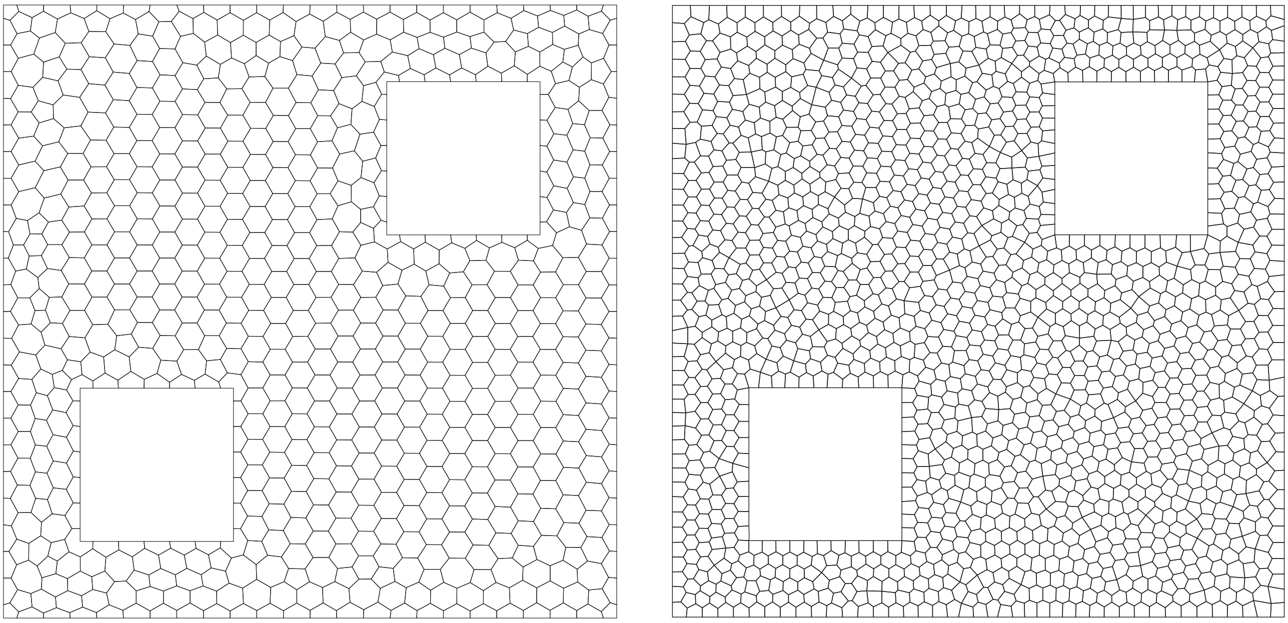

(A) Structured voronoi mesh

(B) Unstructured voronoi mesh

oro_Betti2⟩?

?⟨sub@fig:Voro_Betti2⟩?

FIGURE 4

Betti2_mesh⟩

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