VIRTUAL ELEMENT METHOD FOR A QUAD-CURL PROBLEM ON PLANAR DOMAINS

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ABSTRACT. ...

1. INTRODUCTION

troduction⟩? Consider a bounded polygonal domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$. We are interested in the following quad-curl problem,

$$\begin{aligned} (\nabla\times)^4\boldsymbol{u} + \beta\,(\nabla\times)^2\boldsymbol{u} + \gamma\,\boldsymbol{u} &= \boldsymbol{f} \quad \text{in} \quad \Omega, \\ \nabla\times\boldsymbol{u} &= 0 \quad \text{on} \quad \partial\Omega, \\ \boldsymbol{n}\times\boldsymbol{u} &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{1.1}$$ eq:P.strong

Here $(\nabla\times\cdot)^4 = \nabla\times(\nabla\times(\nabla\times(\nabla\times\cdot)))$ and $(\nabla\times\cdot)^2 = \nabla\times(\nabla\times\cdot)$, are the quad-curl and curl-curl operators in two dimensions, $\boldsymbol{n}$ is the unit outward normal vector on $\partial\Omega$, and $\beta,\gamma \geqslant 0$ are given constants with $\gamma > 0$, if Ω is multiply connected. The forcing term $\boldsymbol{f}:\Omega\to\mathbb{R}^2$ is a given function.

The weak formulation of (1.1) reads: find $\boldsymbol{u}\in\tilde{\mathbb{E}}$ such that

$$(\nabla\times(\nabla\times\boldsymbol{u}),\nabla\times(\nabla\times\boldsymbol{v})) + \beta(\nabla\times\boldsymbol{u},\nabla\times\boldsymbol{v}) + \gamma(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}) \quad \forall\boldsymbol{v}\in\tilde{\mathbb{E}}, \tag{1.2}$$ eq:P

where the energy space $\tilde{\mathbb{E}}$ is defined as

$$\tilde{\mathbb{E}} := \left\{\boldsymbol{v}\in[L^2(\Omega)]^2 : \nabla\times\boldsymbol{v}\in H_0^1(\Omega),\ \ \boldsymbol{n}\times\boldsymbol{v} = 0 \ \text{ on } \ \partial\Omega\right\}.$$

The notation $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ (or $[L^2(\Omega)]^2$) inner-product. This quad-curl problem is linked to the Maxwell's transmission eigenvalue problem (see (2.8) and consider for e.g., the case of homogeneous media in [13]). The authors here opt for a curl-conforming finite element method to discretize the problem. The quad-curl problem also has several applications including the resistive magnetohydrodynamic (MHD) ... [JT: To do - Add more literature] .

Another popular approach to numerically solve (1.1) is to use the Hodge decomposition of divergence-free vector fields (see (3.1) in Section 3). This reduces the problem into a sequence of second-order problems, which can then be discretized using H^1-conforming finite element methods. This approach has been studied, for e.g., in [5, 8] in two and three dimensions using simplicial and tetrahedral meshes, respectively. Since, we seek the solution in the divergence-free space (see Remark 3.1), we solve (1.2) using the reduced energy space $\mathbb{E}\subset\tilde{\mathbb{E}}$,

$$\mathbb{E} := \left\{\boldsymbol{v}\in[L^2(\Omega)]^2 : \nabla\times\boldsymbol{v}\in H_0^1(\Omega),\ \ \operatorname{div}\boldsymbol{v} = 0 \ \text{ and } \ \boldsymbol{n}\times\boldsymbol{v} = 0 \ \text{ on } \ \partial\Omega\right\}. \tag{1.3}$$ eq:E

Our goal in this paper is to extend the results to polygonal meshes using the conforming virtual element method (VEM). [JT: To do - Add literature on VEM for Maxwell and Hodge decomposition]

2. PRELIMINARIES

liminaries⟩? In two dimensions, for a vector field $\boldsymbol{v} = (v_1, v_2)$, the curl operator is a scalar function,

$$\nabla\times\boldsymbol{v} = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}.$$

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For a scalar function ϕ, the curl operator is a vector field,

$$\nabla \times \phi = \left[\frac{\partial \phi}{\partial y}, -\frac{\partial \phi}{\partial x}\right]^{\top} = \text{rot } \phi.$$

As a result, we have

$$(\nabla \times \phi, \nabla \times v) = (\nabla \phi, \nabla v) \quad \forall \phi, v \in H^1(\Omega),$$

and $\|\nabla \times v\|_{L^2(\Omega)} = |v|_{H^1(\Omega)}$ for all $v \in H^1(\Omega)$.

We collect regularity results for elliptic Boundary Value Problems (BVPs) on polygonal domains (see [11, Section 5.1] and [9, Section 2.5]), with ω denoting the largest interior angle at the corners of Ω.

Lemma 2.1 (Regularity of reaction-diffusion BVP)**.** *Given $g \in H^1(\Omega)$, let $\mu \in H_0^1(\Omega)$ satisfy*

$$(\nabla \mu, \nabla v) + \beta(\mu, v) = (g, v) \quad \forall v \in H_0^1(\Omega).$$

Then for any $\epsilon > 0, \mu \in H^{1+(\pi/\omega)-\epsilon}(\Omega)$. Also, we have the following continuous dependence on the data result, with positive constant C_ϵ, depending on ϵ and Ω,

$$\|\mu\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \leqslant C_\epsilon \|g\|_{H^1(\Omega)}.$$

Lemma 2.2 (Regularity of pure-Neumann BVP)**.** *Given $g \in H^1(\Omega)$, let $\lambda \in H^1(\Omega)$ satisfy*

$$(\nabla \lambda, \nabla v) + (\lambda, 1)(v, 1) = (g, v) \quad \forall v \in H^1(\Omega).$$

Then for any $\epsilon > 0, \lambda \in H^{1+(\pi/\omega)-\epsilon}(\Omega)$. Also, we have the following continuous dependence on the data result, with positive constant C_ϵ, depending on ϵ and Ω,

$$\|\lambda\|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)} \leqslant C_\epsilon \|g\|_{H^1(\Omega)}.$$

Using the above lemmas, we now state the regularity result for the energy space $\mathbb{E}$ defined in (1.3) (see [8, Section 2.1] for a proof).

Theorem 2.3 (Regularity of the energy space)**.** *The quad-curl energy space $\mathbb{E}$ defined in* (1.3) *has regularity $[H^{(\pi/\omega)-\epsilon}(\Omega)]^2$ for any $\epsilon > 0$. Furthermore, if Ω is smooth, then the regularity of $\mathbb{E}$ increases to $[H^2(\Omega)]^2$.*

As a consequence of this, the authors in [8, Section 2.2] established the well-posedness of the quad-curl problem (1.2) in the energy space $\mathbb{E}$.

We will use following Hilbert spaces in the rest of the article,

$$H(\text{div}^0; \Omega) := \left\{\boldsymbol{v} \in [L^2(\Omega)]^2 : (\boldsymbol{v}, \nabla \eta) = 0 \quad \forall \eta \in H_0^1(\Omega)\right\},$$
$$L_0^2(\Omega) := \left\{v \in L^2(\Omega) : (1, v) = 0\right\}.$$

3. Hodge Decomposition Based Reduction

Any $\boldsymbol{v} \in H(\text{div}^0; \Omega)$ has a unique decomposition (see Chapter 3 in [10] and (1.2) in [3]):

$$\boldsymbol{v} = \nabla \times \phi + \sum_{j=1}^{m} d_j \nabla \varphi_j, \tag{3.1}$$

where $\phi \in H^1(\Omega) \cap L_0^2(\Omega)$, $m \in \mathbb{Z}_0^+$ is the Betti number, and $d_j (1 \leqslant j \leqslant m) \in \mathbb{R}$. The harmonic functions φ_j are defined by

$$(\nabla \varphi_j, \nabla v) = 0 \quad \forall v \in H_0^1(\Omega), \tag{3.2a}$$
$$\varphi_j|_{\Gamma_0} = 0, \tag{3.2b}$$
$$\varphi_j|_{\Gamma_l} = \delta_{jl} \quad \text{for} \quad 1 \leqslant l \leqslant m. \tag{3.2c}$$

Here Γ_0 denotes the outer boundary, Γ_l denote the l components of the inner boundary when Betti number $m > 0$, and δ_{jl} is the Kronecker delta function.

⟨rem:kernel⟩ **Remark 3.1.** *The quad-curl energy has a large kernel consisting of gradient fields. Since the functions in $H(div^0;\Omega)$ are orthogonal to gradient fields, this kernel is fixed upto harmonic functions. Furthermore, in the case of simply connected domains, we infer from the boundary condition $\boldsymbol{u} \times \boldsymbol{n} = 0$ that the kernel reduces to the zero vector field. Therefore, we can put $\gamma = 0$. However, in the case of multiply connected domains, the harmonic functions in the kernel are non-zero vector fields. To account for this, we set $\gamma > 0$.*

3.1. **Ω is simply connected ($\gamma = 0$).** The Hodge decomposition (3.1) reduces to

$$\boldsymbol{u} = \nabla \times \phi.$$

It now remains to find ϕ. As shown in [8], this can be achieved by solving the following sequence of problems.

(1) First we find $\rho \in H^1(\Omega) \cap L_2^0(\Omega)$, or equivalently $\rho \in H^1(\Omega)$ such that

$$(\nabla \times \rho, \nabla \times \psi) + (\rho, 1)(\psi, 1) = (\boldsymbol{f}, \nabla \times \psi) \quad \forall \psi \in H^1(\Omega). \tag{3.3}$$ eq:P.rho

(2) Find $\xi \in H^1(\Omega) \cap L_2^0(\Omega)$ given by

$$\xi = \xi_0 - \frac{(1,\xi_0)}{(1,\xi_1)}\xi_1, \tag{3.4}$$?eq:P.xi?

where $\xi_0, \xi_1 \in H_0^1(\Omega)$ satisfy

$$(\nabla \times \xi_0, \nabla \times \eta) + \beta(\xi_0, \eta) = (\rho, \eta) \quad \forall \eta \in H_0^1(\Omega), \tag{3.5}$$ eq:P.xi0

$$(\nabla \times \xi_1, \nabla \times \eta) + \beta(\xi_1, \eta) = (1, \eta) \quad \forall \eta \in H_0^1(\Omega). \tag{3.6}$$?eq:P.xi1?

(3) Finally we find $\phi \in H^1(\Omega) \cap L_2^0(\Omega)$, or equivalently $\phi \in H^1(\Omega)$ such that

$$(\nabla \times \phi, \nabla \times \psi) + (\phi, 1)(\psi, 1) = (\xi, \psi) \quad \forall \psi \in H^1(\Omega). \tag{3.7}$$ eq:P.phi

3.2. **Ω is multiply connected ($\gamma > 0$).** Recalling the Hodge decomposition (3.1)

$$\boldsymbol{u} = \nabla \times \phi + \sum_{j=1}^{m} c_j \nabla \varphi_j.$$

It remains to find ϕ, c_j, and φ_j. Following [8], this is equivalent to solving the following sequence of problems.

(1) Find $(\zeta, \xi) \in H^1(\Omega) \times H_0^1(\Omega)$ given by

$$(\zeta, \xi) = (\zeta_0, \xi_0) - \frac{(1,\xi_0)}{(1,\xi_1)}(\zeta_1, \xi_1), \tag{3.8}$$ eq:P.zeta.c

where $(\zeta_0, \xi_0), (\zeta_1, \xi_1) \in H^1(\Omega) \times H_0^1(\Omega)$ solve the following two coupled system:

$$\mathcal{A}((\zeta_0, \xi_0), (\psi, \eta)) + (\zeta_0, 1)(\psi, 1) = \gamma^{-\frac{1}{2}}(\boldsymbol{f}, \nabla \times \psi) \quad \forall (\psi, \eta) \in H^1(\Omega) \times H_0^1(\Omega), \tag{3.9}$$?eq:P.zeta0.

$$\mathcal{A}((\zeta_1, \xi_1), (\psi, \eta)) + (\zeta_1, 1)(\psi, 1) = (1, \eta) \quad \forall (\psi, \eta) \in H^1(\Omega) \times H_0^1(\Omega). \tag{3.10}$$?eq:P.zeta1.

Here, the bilinear form $\mathcal{A}(\cdot,\cdot)$ is defined by

$$\mathcal{A}((\zeta, \xi), (\psi, \eta)) = (\nabla \times \zeta, \nabla \times \psi) + \gamma^{\frac{1}{2}}(\psi, \xi) - \gamma^{\frac{1}{2}}(\zeta, \eta) + (\nabla \times \xi, \nabla \times \eta) + \beta(\xi, \eta). \tag{3.11}$$?eq:A?

(2) Find $\phi \in H^1(\Omega)$ such that (3.7) holds.

(3) Finally $c_j (1 \leqslant j \leqslant m)$, are determined by solving the $m \times m$, SPD system

$$\sum_{j=1}^{m} (\nabla \varphi_i, \nabla \varphi_j) c_j = \gamma^{-1}(\boldsymbol{f}, \nabla \varphi_i) \quad \text{for} \quad 1 \leqslant i \leqslant m, \tag{3.12}$$ eq:P.cj

and the harmonic functions φ_j are defined by (3.2).

4. Conforming Virtual Element Discretization

Let $\mathcal{T}_h$ be a triangulation of the polygonal domain $\Omega \subset \mathbb{R}^2$ into a finite collection of simple polygons D. The mesh parameter h is defined as $h := \max_{D \in \mathcal{T}_h} h_D$, where h_D denotes the diameter of D. Let $\mathcal{E}_D$ be the set of edges associated with D. We make the following shape-regularity assumptions for all $D \in \mathcal{T}_h$ (see [2, 6]). There exists a constant $\Theta \in (0,1)$ such that

$$D \text{ is star-shaped with respect to a ball of radius } \Theta h_D, \text{ and} \tag{4.1a}$$
$$|e| \geqslant \Theta h_D \text{ for any edge } e \in \mathcal{E}_D. \tag{4.1b}$$

The local enhanced virtual element space $V_h(D) \subset H^1(D)$ (see [1, 6]) is defined as follows:

$$V_h(D) := \left\{ v_h \in H^1(D) : v_h|_{\partial D} \in \mathbb{P}_k(\partial D),\ \ -\Delta v_h \in \mathbb{P}_k(D),\ \ \Pi^0_{k,D} v_h - \Pi^1_{k,D} v_h \in \mathbb{P}_{k-2}(D) \right\}. \tag{4.2}$$

Here $\mathbb{P}_k(\partial D)$ (respectively, $\mathbb{P}_k(D)$) denote the space of continuous piecewise polynomials of degree at most k on the boundary ∂D (respectively, on D). The operators $\Pi^0_{k,D}$ and $\Pi^1_{k,D}$ are the standard L^2 and H^1-projections onto $\mathbb{P}_k(D)$, respectively (see [6, Section 2.2] for details). The global virtual element spaces V_h and V_h^0 are defined by concatenating the local spaces as follows:

$$V_h := \{ v_h \in H^1(\Omega) : v_h|_D \in V_h(D) \text{ for all } D \in \mathcal{T}_h \}, \tag{4.3}$$
$$V_h^0 := \{ v_h \in H^1_0(\Omega) : v_h|_D \in V_h(D) \text{ for all } D \in \mathcal{T}_h \}. \tag{4.4}$$

4.1. **Ω is simply connected ($\gamma = 0$).** The $\mathbb{P}_k$ Virtual Element Method for approximating the sequence of problems (3.3)-(3.7) is as follows.

(1) Find $\rho_h \in V_h$ such that

$$a_h(\rho_h, \psi_h) + (\Pi^0_{k,h}\rho_h, 1)(\Pi^0_{k,h}\psi_h, 1) = (\boldsymbol{f}, \nabla \times \Pi^1_{k,h}\psi_h) \quad \forall \psi_h \in V_h. \tag{4.5}$$

(2) Find $\xi_h \in V_h$ given by

$$\xi_h = \xi_{0,h} - \frac{(1, \xi_{0,h})}{(1, \xi_{1,h})}\xi_{1,h}, \tag{4.6}$$

where $\xi_{0,h}, \xi_{1,h} \in V_h^0$ satisfy

$$a_h(\xi_{0,h}, \eta_h) + \beta(\Pi^0_{k,h}\xi_{0,h}, \Pi^0_{k,h}\eta_h) = (\Pi^0_{k,h}\rho_h, \Pi^0_{k,h}\eta_h) \quad \forall \eta_h \in V_h^0, \tag{4.7}$$
$$a_h(\xi_{1,h}, \eta_h) + \beta(\Pi^0_{k,h}\xi_{1,h}, \Pi^0_{k,h}\eta_h) = (1, \Pi^0_{k,h}\eta_h) \quad \forall \eta_h \in V_h^0. \tag{4.8}$$

(3) Find $\phi_h \in V_h$ such that

$$a_h(\phi_h, \psi_h) + (\Pi^0_{k,h}\phi_h, 1)(\Pi^0_{k,h}\psi_h, 1) = (\Pi^0_{k,h}\xi_h, \Pi^0_{k,h}\psi_h) \quad \forall \psi_h \in V_h. \tag{4.9}$$

Here, the global bilinear form $a_h(\cdot,\cdot)$ is given elementwise by:

$$a_h(w_h, v_h) = \sum_{D \in \mathcal{T}_h} a_h^D(w_h, v_h) \quad \forall w_h, v_h \in V_h, \tag{4.10}$$
$$= \sum_{D \in \mathcal{T}_h} (\nabla \times \Pi^1_{k,D} w_h, \nabla \times \Pi^1_{k,D} v_h)_D + S^D((I - \Pi^1_{k,D})w_h, (I - \Pi^1_{k,D})v_h). \tag{4.11}$$

$\Pi^0_{k,h}$ and $\Pi^1_{k,h}$ are the global L^2 and H^1-projections onto $\mathbb{P}_k(\mathcal{T}_h)$ (discontinuous piecewise polynomials of degree $\leqslant k$), respectively, and are understood in terms of their local counterparts. The symmetric positive definite stabilization term is denoted by $S^D(\cdot,\cdot)$. Finally, we post-process $\boldsymbol{u}_h \in [\mathbb{P}_k(\mathcal{T}_h)]^2$ using the Hodge decomposition as follows:

$$\boldsymbol{u}_h = \nabla \times \Pi^1_{k,h}\phi_h. \tag{4.12}$$

4.2. **Ω is multiply connected ($\gamma > 0$).** The $\mathbb{P}_k$ Virtual Element Method for approximating the sequence of problems (3.8)-(4.17) is as follows.

(1) Find $(\zeta_h, \xi_h) \in V_h \times V_h^0$ given by

$$(\zeta_h, \xi_h) = (\zeta_{0,h}, \xi_{0,h}) - \frac{(1, \xi_{0,h})}{(1, \xi_{1,h})}(\zeta_{1,h}, \xi_{1,h}), \tag{4.13}$$

where $(\zeta_{,h}, \xi_{0,h}), (\zeta_{1,h}, \xi_{1,h}) \in V_h \times V_h^0$ solve the following two coupled systems:

$$\begin{aligned}
&\mathcal{A}_h((\zeta_{0,h}, \xi_{0,h}), (\psi_h, \eta_h)) + (\Pi^0_{k,h}\zeta_{0,h}, 1)(\Pi^0_{k,h}\psi_h, 1) \\
&\qquad = \gamma^{-\frac{1}{2}}(\boldsymbol{f}, \nabla \times \Pi^1_{k,h}\psi) \quad \forall (\psi_h, \eta_h) \in V_h \times V_h^0, \qquad (4.14)\\
&\mathcal{A}_h((\zeta_{1,h}, \xi_{1,h}), (\psi_h, \eta_h)) + (\Pi^0_{k,h}\zeta_{1,h}, 1)(\Pi^0_{k,h}\psi_h, 1) \\
&\qquad = (1, \Pi^0_{k,h}\eta_h) \quad \forall (\psi_h, \eta_h) \in V_h \times V_h^0. \qquad (4.15)
\end{aligned}$$

The global coupled bilinear form $\mathcal{A}_h(\cdot,\cdot)$ is defined by

$$\begin{aligned}
\mathcal{A}_h((\zeta_h, \xi_h), (\psi_h, \eta_h)) &= a_h(\zeta_h, \xi_h) + \gamma^{\frac{1}{2}}(\Pi^0_{k,h}\psi_h, \Pi^0_{k,h}\xi_h) - \gamma^{\frac{1}{2}}(\Pi^0_{k,h}\zeta_h, \Pi^0_{k,h}\eta_h) \\
&\quad + a_h(\xi_h, \eta_h) + \beta(\Pi^0_{k,h}\xi_h, \Pi^0_{k,h}\eta_h). \qquad (4.16)
\end{aligned}$$

(2) Find $\phi_h \in V_h$ such that (4.9) holds.
(3) The coefficients $c_{j,h}$ $(1 \leqslant j \leqslant m)$, are determined by solving

$$\sum_{j=1}^{m} a_h(\varphi_{i,h}, \varphi_{j,h})c_{j,h} = \gamma^{-1}(\boldsymbol{f}, \nabla\Pi^1_{k,h}\varphi_{i,h}) \quad \text{for} \quad 1 \leqslant i \leqslant m. \tag{4.17}$$

Where the discrete harmonic functions $\varphi_{j,h}$ are determined by approximating (3.2) as follows:

$$\begin{aligned}
a_h(\varphi_{j,h}, v_h) &= 0 \quad \forall v_h \in V_h^0, && (4.18a)\\
\varphi_{j,h}|_{\Gamma_0} &= 0, && (4.18b)\\
\varphi_{j,h}|_{\Gamma_l} &= \delta_{jl} \quad \text{for} \quad 1 \leqslant l \leqslant m, && (4.18c)
\end{aligned}$$

with $a_h(\cdot,\cdot)$ given by (4.11).

Finally we post-process $\boldsymbol{u}_h \in [\mathbb{P}_k(\mathcal{T}_h)]^2$ using the Hodge decomposition as follows:

$$\boldsymbol{u}_h = \nabla \times \Pi^1_{k,h}\phi_h + \sum_{j=1}^{m} c_{j,h}\nabla\Pi^1_{k,h}\varphi_{j,h}. \tag{4.19}$$

5. Convergence analysis

We start by collecting some mathematical tools which will be helpful in the forthcoming analysis.

Lemma 5.1 (Sobolev inequality). *Given any $\delta > 0$,*

$$\|v\|_{L^\infty(D)} \lesssim h_D^{-1}\|v\|_{L^2(D)} + |v|_{H^1(D)} + h_D^{\delta}|v|_{H^{1+\delta}(D)} \quad \forall v \in H^{1+\delta}(D).$$

Lemma 5.2 (Bramble-Hilbert estimates). *Under the mesh regularity Assumption 4.1a, given any $\delta > 0$, there exists a positive constant independent of h_D such that*

$$\inf_{p \in \mathbb{P}_k(D)} |\lambda - \psi|_{H^1(D)} \lesssim h_D^{\min(\delta,k)}|\lambda|_{H^{1+\delta}(D)}, \quad \forall \lambda \in H^{1+\delta}(D).$$

Lemma 5.3 (Trace inequality). *Let e be an edge of $D \subset \mathbb{R}^2$. Then, for all $v \in H^{1+\delta}(D)$ with given $\delta > 0$, we have*

$$h_D^{2\delta}|v|^2_{H^{1/2+\delta}(e)} \lesssim |v|^2_{H^1(D)} + h_D^{2\delta}|v|^2_{H^{1+\delta}(D)}.$$

Lemma 5.4 (H^1-projector stability and approximation). *Given $\delta > 0$,*

$$|\Pi^1_{k,D}v|_{H^1(D)} \leqslant |v|_{H^1(D)} \quad \forall v \in H^1(D), \tag{5.1}$$

$$|v - \Pi^1_{k,D}v|_{H^1(D)} \lesssim h_D^{\min(\delta,k)}|v|_{H^{1+\delta}(D)} \quad \forall v \in H^{1+\delta}(D). \tag{5.2}$$

Lemma 5.5 (L^2-projector stability and approximation).

$$\|\Pi^0_{k,D}v\|_{L^2(D)} \leqslant \|v\|_{L^2(D)} \quad \forall v \in L^2(D), \quad \|\Pi^0_{k,D}v\|_{H^1(D)} \leqslant \|v\|_{H^1(D)} \quad \forall v \in H^1(D), \tag{5.3}$$

$$\|v - \Pi^0_{k,D}v\|_{L^2(D)} \lesssim h_D^{l+1}|v|_{H^{l+1}(D)} \quad \forall v \in H^{l+1}(D), \quad 0 \leqslant l \leqslant k, \tag{5.4}$$

The interpolation operator, which takes any sufficiently smooth function and maps it to the virtual element space. For $s > 1$, the global interpolation operator $I_{k,h} : H^s(\Omega) \to V_h$ is the global counterpart of the local interpolation operator $I_{k,D} : H^s(D) \to V_h(D)$ for all $D \in \mathcal{T}_h$ such that for any $v \in H^s(D)$,

$$I_{k,D}v(p) = v(p) \quad \forall p \in \mathcal{N}^{\partial D}, \tag{5.5}$$
eq:ID.bound

$$\Pi^0_{k-2,D} I_{k,D} v = \Pi^0_{k-2,D} v. \tag{5.6}$$
eq:ID.inter

Here $\mathcal{N}^{\partial D}$ is the set of boundary degrees of freedom associated with the local virtual element space. From (5.6), it follows that for $k \geqslant 2$, the interpolation $I_{k,D}v$ preserves the mean value of v on each element D, i.e., given $v \in H^s(D) \cap L^2_0(\Omega)$,

$$\int_D I_{k,D}v = \int_D \Pi^0_{k-2,D}(I_{k,D}v) = \int_D \Pi^0_{k-2,D}v = \int_D v = 0. \tag{5.7}$$
?rem:ID.mean

However, for $k = 1$, $I_{k,D}v$ is completely determined by boundary degrees of freedom (5.5) and does not necessarily preserve the mean value of v on D.

properties⟩? **Lemma 5.6** (Interpolation operator stability and approximation). *Given $\delta > 0$, we have*

$$|I_{k,D}v|_{H^1(D)} \lesssim |v|_{H^1(D)} + h_D^\delta |v|_{H^{1+\delta}(D)}, \qquad \forall v \in H^{1+\delta}(D) \tag{5.8}$$
?eq:ID.stabi

$$|v - I_{k,D}v|_{H^1(D)} + |I_{k,D}v - \Pi^1_{k,D} I_{k,D}v|_{H^1(D)} \lesssim h_D^{\min(\delta,k)} |v|_{H^{1+\delta}(D)}, \qquad \forall v \in H^{1+\delta}(D) \tag{5.9}$$
eq:ID.appro

We will also use the following inverse inequality for virtual element functions [4, Lemma 2.19].

.inequality⟩ **Lemma 5.7** (Inverse inequality). *Under the mesh regularity assumptions* (4.1a)-(4.1b) *we have for all $v_h \in V_h(D)$,*

$$|v_h|_{H^1(D)} \lesssim h_D^{-1} |||v_h|||_{k,D},$$

where, $|||\cdot|||_{k,D}$ plays the role of L^2-norm and is defined as

$$|||v_h|||^2_{k,D} = h_D \sum_{e\in\mathbb{E}_D} \|\Pi^0_{k,e} v_h\|^2_{L^2(e)} + \|\Pi^0_{k-2,D} v_h\|^2_{L^2(D)}.$$

In the subsequent analysis, we work with the following assumption:

:coercivity⟩ **Assumption 5.8.** *Let $\|\cdot\|_h = \sqrt{a_h(\cdot,\cdot)}$ denote the mesh-dependent energy norm such that*

$$|v_h|_{H^1(\Omega)} \lesssim \|v_h\|_h \quad \forall v_h \in V_h^0, \tag{5.10}$$
?eq:coercivi

$$|v_h|_{H^1(\Omega)} \lesssim \|v_h\|_h \quad \forall v_h \in V_h \quad \textit{with} \quad (v_h, 1) = 0. \tag{5.11}$$
?eq:coercivi

We also define the piecewise-broken H^1-seminorm by

$$|v|_{1,h} := \left(\sum_{D\in\mathcal{T}_h} |v|^2_{H^1(D)} \right)^{\frac{1}{2}} \qquad \forall v \in H^1(\mathcal{T}_h).$$

Some useful estimates in the mesh-dependent norm are in order.

.dependent⟩? **Lemma 5.9.** *The following estimates hold,*

$$\|I_{k,h}v - \Pi^1_{k,h} I_{k,h}v\|_h + |I_{k,h}v - \Pi^1_{k,h} I_{k,h}v|_{1,h} \lesssim h^{\min(\delta,k)} |v|_{H^{1+\delta}(\Omega)} \qquad \forall v \in H^{1+\delta}(\Omega), \tag{5.12}$$
eq:est.Ih.h

In order to prove optimal error estimates for ξ_h without assuming extra regularity on ρ, we define the following global conforming linear operator,

$$J_h : L^2(\Omega) \to V_h \quad \text{such that} \quad (J_h w, v_h) = (\Pi^0_{k,h} w, v_h) \quad \forall v_h \in V_h. \tag{5.13}$$
def:Jh

Since, $V_h \subset L^2(\Omega)$ is finite dimensional, the tuple $(V_h, (\cdot,\cdot))$ forms a Hilbert space. For any $w \in L^2(\Omega)$, the map $l_w : V_h \to \mathbb{R}$ defined by $l_w(v_h) := (\Pi^0_{k,h} w, v_h)$ is a bounded linear functional, with

$$|l_w(v_h)| \leqslant \|\Pi^0_{k,h} w\|_{L^2(\Omega)} \|v_h\|_{L^2(\Omega)} = \sum_{D\in\mathcal{T}_h} \|\Pi^0_{k,D} w\|_{L^2(D)} \|v_h\|_{L^2(\Omega)} \overset{(5.3)}{\lesssim} \|w\|_{L^2(\Omega)} \|v_h\|_{L^2(\Omega)}.$$

Hence, by the Riesz representation theorem, there exists a unique $J_h w \in V_h$ such that $(J_h w, v_h) = l_w(v_h)$ for all $v_h \in V_h$.

Lemma 5.10 (Approximation property of J_h). *For all $v_h \in V_h$, we have*

$$\|J_h v_h - v_h\|^2_{L^2(\Omega)} \lesssim \sum_{D \in \mathcal{T}_h} h_D^2 |v_h|^2_{H^1(D)}.$$

Proof. See Section 6.1. □

Lemma 5.11 (Stability of J_h). *The following hold,*

$$\|J_h w\|_{L^2(\Omega)} \lesssim \|w\|_{L^2(\Omega)} \quad \forall w \in L^2(\Omega), \tag{5.14}$$

$$|J_h v_h|_{H^1(\Omega)} \lesssim |v_h|_{H^1(\Omega)} \quad \forall v_h \in V_h^0. \tag{5.15}$$

Proof. see Section 6.2. □

5.1. **Ω is simply connected** ($\gamma = 0$). Our first aim is to estimate the error $\xi - \xi_h$. To this end, we first write down the error equation by testing (3.3) with $\psi_h \in V_h \subset H^1(\Omega)$ and subtracting it from (3.3),

$$(\nabla \times \rho, \nabla \times \psi_h) - a_h(\rho_h, \psi_h) = (\boldsymbol{f}, \nabla \times (\psi_h - \Pi^1_{k,h}(\psi_h))) \quad \forall \psi_h \in V_h. \tag{5.16}$$

We note that to obtain the above error equation we also used the definition of $\Pi^0_{k,h}$ and the fact that $\rho, \rho_h \in L^2_0(\Omega)$. Furthermore, in view of (3.3), (4.5), $\rho, \rho_h \in L^2_0(\Omega)$ and Assumption 5.8, we have the following relations:

$$\|\nabla \times \rho\|_{L^2(\Omega)} \leqslant \|\boldsymbol{f}\|_{L^2(\Omega)}, \quad \text{and} \quad \|\nabla \times \rho_h\|_{L^2(\Omega)} \lesssim \|\boldsymbol{f}\|_{L^2(\Omega)}. \tag{5.17}$$

In the following lemma we estimate the error for ρ_h in the dual norm using a duality argument.

Lemma 5.12. *For any $\epsilon > 0$, there exists a positive constant dependent on ϵ and independent of h such that*

$$|(\rho - \rho_h, \chi)| \lesssim h^{\min((\pi/\omega) - \epsilon, k)} \|\chi\|_{H^1(\Omega)} \|\boldsymbol{f}\|_{L^2(\Omega)} \quad \forall \chi \in H^1(\Omega). \tag{5.18}$$

Proof. Given arbitary $\chi \in H^1(\Omega)$, let $\lambda \in H^1(\Omega)$ solve the following dual problem:

$$(\nabla \times \psi, \nabla \times \lambda) + (\psi, 1)(\lambda, 1) = (\psi, \chi) \quad \forall \psi \in H^1(\Omega). \tag{5.19}$$

Testing (5.19) with $\rho - \rho_h \in H^1(\Omega)$ and exploiting $\rho, \rho_h \in L^2_0(\Omega)$, we get

$$\begin{aligned}(\rho - \rho_h, \chi) &= (\nabla \times (\rho - \rho_h), \nabla \times \lambda) + (\rho - \rho_h, 1)(\lambda, 1) \\ &= (\nabla \times (\rho - \rho_h), \nabla \times (\lambda - I_{k,h}\lambda)) + (\nabla \times (\rho - \rho_h), \nabla \times I_{k,h}\lambda).\end{aligned} \tag{5.20}$$

The first term in (5.20) is bounded using the Cauchy-Schwarz inequality and (5.17), to get

$$\begin{aligned}(\nabla \times (\rho - \rho_h), \nabla \times (\lambda - I_{k,h}\lambda)) &\leqslant \|\nabla \times (\rho - \rho_h)\|_{L^2(\Omega)} \|\nabla \times (\lambda - I_{k,h}\lambda)\|_{L^2(\Omega)}. \\ &\lesssim \|\boldsymbol{f}\|_{L^2(\Omega)} |\lambda - I_{k,h}\lambda|_{H^1(\Omega)}.\end{aligned} \tag{5.21}$$

The second term in (5.20) is bounded using the error equation (5.16) for $\psi_h = I_{k,h}\lambda$ as follows:

$$\begin{aligned}(\nabla \times (\rho - \rho_h), \nabla \times I_{k,h}\lambda) &= (\nabla \times \rho, \nabla \times I_{k,h}\lambda) - (\nabla \times \rho_h, \nabla \times I_{k,h}\lambda) \\ &= (\boldsymbol{f}, \nabla \times (I_{k,h}\lambda - \Pi^1_{k,h} I_{k,h}\lambda)) + a_h(\rho_h, I_{k,h}\lambda) - (\nabla \times \rho_h, \nabla \times I_{k,h}\lambda) \\ &\leqslant \|\boldsymbol{f}\|_{L^2(\Omega)} |(I - \Pi^1_{k,h}) I_{k,h}\lambda|_{1,h} + a_h(\rho_h, I_{k,h}\lambda) - (\nabla \times \rho_h, \nabla \times I_{k,h}\lambda).\end{aligned} \tag{5.22}$$

The difference in (5.22) can be rewritten as follows:

$$
\begin{aligned}
&a_h(\rho_h, I_{k,h}\lambda) - (\nabla\times\rho_h, \nabla\times I_{k,h}\lambda)\\
&\quad= \sum_{D\in\mathcal{T}_h} (\nabla\times\Pi^1_{k,D}\rho_h, \nabla\times\Pi^1_D I_{k,D}\lambda) + S^D((I-\Pi^1_{k,D})\rho_h, (I-\Pi^1_{k,D})I_{k,D}\lambda) -\\
&\qquad\quad (\nabla\times\rho_h, \nabla\times(I_{k,h}\lambda - \Pi^1_{k,D}I_{k,D}\lambda)) - (\nabla\times\rho_h, \nabla\times\Pi^1_{k,D}I_{k,D}\lambda)\\
&\quad= \sum_{D\in\mathcal{T}_h} S^D((I-\Pi^1_{k,D})\rho_h, (I-\Pi^1_{k,D})I_{k,D}\lambda) - (\nabla\times\rho_h, \nabla\times(I_{k,D}\lambda - \Pi^1_{k,D}I_{k,D}\lambda))\\
&\quad\leqslant \Big(\sum_{D\in\mathcal{T}_h} S^D((I-\Pi^1_{k,D})\rho_h, (I-\Pi^1_{k,D})\rho_h)\Big)^{1/2}\Big(\sum_{D\in\mathcal{T}_h} S^D((I-\Pi^1_{k,D})I_{k,D}\lambda, (I-\Pi^1_{k,D})I_{k,D}\lambda)\Big)^{1/2}\\
&\qquad\quad \Big(\sum_{D\in\mathcal{T}_h} |\rho_h|^2_{H^1(D)}\Big)^{1/2}\Big(\sum_{D\in\mathcal{T}_h} |I_{k,D}\lambda - \Pi^1_{k,D}I_{k,D}\lambda|^2_{H^1(D)}\Big)^{1/2}\\
&\quad\lesssim \|(I-\Pi^1_{k,h})\rho_h\|_h\|(I-\Pi^1_{k,h})I_{k,h}\lambda\|_h + |\rho_h|_{H^1(\Omega)}|(I-\Pi^1_{k,h})I_{k,h}\lambda|_{1,h}.
\end{aligned}
\tag{5.23}
$$

eq:err.rho.

Where in the passage to the second equality we used the definition of $\Pi^1_{k,D}$ projector followed by Cauchy-Schwarz inequality and finally the definition of $\|\cdot\|_h$ and $|\cdot|_{1,h}$. Using the linearity and idempotency of $\Pi^1_{k,D}$ and $\rho_h \in L^2_0(\Omega)$ we have that

$$
\begin{aligned}
\|(I-\Pi^1_{k,h}\rho_h)\|^2_h &= S((I-\Pi^1_{k,h})\rho_h, (I-\Pi^1_{k,h})\rho_h)\\
&\overset{(4.5)}{=} (\boldsymbol{f}, \nabla\times\Pi^1_{k,h}\rho_h) - (\nabla\times\Pi^1_{k,h}\rho_h, \nabla\times\Pi^1_{k,h}\rho_h)\\
&\leqslant \sum_{D\in\mathcal{T}_h} \|\boldsymbol{f}\|_{L^2(D)}\|\Pi^1_{k,D}\rho_h\|_{H^1(D)} + \|\Pi^1_{k,D}\rho_h\|^2_{H^1(D)}\\
&\overset{(5.1)}{\leqslant} \sum_{D\in\mathcal{T}_h} \|\boldsymbol{f}\|_{L^2(D)}\|\rho_h\|_{H^1(D)} + \|\rho_h\|^2_{H^1(D)} \overset{(5.17)}{\lesssim} \|\boldsymbol{f}\|^2_{L^2(\Omega)}.
\end{aligned}
$$

The regularity of the dual problem from Lemma 2.2, the above bound, the estimate (5.12) with $\delta = (\pi/\omega) - \epsilon$, and (5.17) lead to the following bound on (5.23),

$$
a_h(\rho_h, I_{k,h}\lambda) - (\nabla\times\rho_h, \nabla\times I_{k,h}\lambda) \lesssim h^{\min((\pi/\omega)-\epsilon,k)}\|\boldsymbol{f}\|_{L^2(\Omega)}|\lambda|_{H^{1+(\pi/\omega)-\epsilon}(\Omega)}.
$$

Substituting in (5.22), again using (5.12) and the data dependence relation of the dual problem in Lemma 2.2 yield

$$
(\nabla\times(\rho-\rho_h), \nabla\times I_{k,h}\lambda) \lesssim h^{\min((\pi/\omega)-\epsilon,k)}\|\boldsymbol{f}\|_{L^2(\Omega)}\|\chi\|_{H^1(\Omega)}.
$$

Substituting the above bound and (5.21) into (5.20), and using the definition of piecewise-broken H^1-seminorm and (5.9) yields the desired error estimate (5.18). $\square$

It remains to estimate the error for ξ_h.

Lemma 5.13. *For any $\epsilon > 0$, there exists a positive constant dependent on ϵ and independent of h such that*

$$
|\xi - \xi_h|_{H^1(\Omega)} \lesssim h^{\min((\pi/\omega)-\epsilon,k)}. \tag{5.24}
$$

?eq:err.est.

Proof. Given $\rho \in H^1(\Omega)$, let $\tilde{\xi}_{0,h} \in V^0_h$ solve the following discrete auxiliary problem,

$$
a_h(\tilde{\xi}_{0,h}, \eta_h) + \beta(\Pi^0_{k,h}\tilde{\xi}_{0,h}, \Pi^0_{k,h}\eta_h) = (\rho, J_h\eta_h) \quad \forall \eta_h \in V^0_h. \tag{5.25}
$$

eq:P.xi0h.a

On comparing with (4.7) after using the definition of $\Pi^0_{k,h}$ on $(\Pi^0_{k,h}\rho_h, \Pi^0_{k,h}\eta_h)$, we have

$$
a_h(\tilde{\xi}_{0,h} - \xi_{0,h}, \eta_h) + \beta(\Pi^0_{k,h}(\tilde{\xi}_{0,h} - \xi_{0,h}), \Pi^0_{k,h}\eta_h) = (\rho, J_h\eta_h) - (\rho_h, \Pi^0_{k,h}\eta_h) \quad \forall \eta_h \in V^0_h.
$$

The right-hand side can be rewritten using the definition of J_h in (5.13) as follows:

$$
\begin{aligned}
(\rho, J_h\eta_h) - (\rho_h, \Pi^0_{k,h}\eta_h) &= (\rho - \rho_h, J_h\eta_h) + (\rho_h, J_h\eta_h) - (\rho_h, \Pi^0_{k,h}\eta_h)\\
&= (\rho - \rho_h, J_h\eta_h).
\end{aligned}
$$

It follows that

$$
a_h(\tilde{\xi}_{0,h} - \xi_{0,h}, \eta_h) \leqslant (\rho - \rho_h, J_h\eta_h) \quad \forall \eta_h \in V^0_h. \tag{5.26}
$$

eq:aux.est.

Testing (5.26) with $\eta_h = \tilde{\xi}_{0,h} - \xi_{0,h} \in V_h^0$, using the dual norm estimate (5.18) with $\chi = J_h\eta_h$ and a Poicaré-Friedrichs inequality gives,

$$\|\tilde{\xi}_{0,h} - \xi_{0,h}\|_h^2 \lesssim h^{\min((\pi/\omega)-\epsilon,k)} |J_h\eta_h|_{H^1(\Omega)} \|\boldsymbol{f}\|_{L^2(\Omega)}.$$

Now the H^1 stability of the operator J_h given in (5.15) and Assumption 5.8 lead to

$$\|\tilde{\xi}_{0,h} - \xi_{0,h}\|_h \lesssim h^{\min((\pi/\omega)-\epsilon,k)} \|\boldsymbol{f}\|_{L^2(\Omega)} \tag{5.27}$$?eq:est.xi0h

Since $\tilde{\xi}_{0,h}$ is the virtual element approximation of ξ_0 (see (3.5) and (5.25)), we can obtain the following error equation by testing with $J_h\eta_h$ in (3.5) and subtracting it from (5.25),

$$a_h(\tilde{\xi}_{0,h}, \eta_h) - (\nabla\times\xi_0, \nabla\times J_h\eta_h) + \beta(\Pi_{k,h}^0\tilde{\xi}_{0,h}, \Pi_{k,h}^0\eta_h) - \beta(\xi_0, J_h\eta_h) = 0 \quad \forall \eta_h \in V_h^0. \tag{5.28}$$ eq:erreq.xi

The first difference in (5.28) can be rewritten in controllable quantities as follows:

$$\begin{aligned}
&a_h(\tilde{\xi}_{0,h} - \tilde{\xi}_0, \eta_h) + a_h(\tilde{\xi}_0, \eta_h) - (\nabla\times\xi_0, \nabla\times J_h\eta_h) \\
&\quad = a_h(\tilde{\xi}_{0,h} - \tilde{\xi}_0, \eta_h) + \sum_{D\in\mathcal{T}_h} (\nabla\times\Pi_{k,D}^1\xi_0, \nabla\times\Pi_{k,D}^1\eta_h) + S^D((I-\Pi_{k,D}^1)\xi_0, (I-\Pi_{k,D}^1)\eta_h) \\
&\qquad - (\nabla\times\xi_0, \nabla\times J_h\eta_h) \\
&\quad = a_h(\tilde{\xi}_{0,h} - \tilde{\xi}_0, \eta_h) + \sum_{D\in\mathcal{T}_h} (\nabla\times(\Pi_{k,D}^1\xi_0 - \xi_0), \nabla\times\eta_h) + S^D((I-\Pi_{k,D}^1)\xi_0, (I-\Pi_{k,D}^1)\eta_h) \\
&\qquad - (\nabla\times\xi_0, \nabla\times(\eta_h - J_h\eta_h)).
\end{aligned} \tag{5.29}$$ eq:err.xi0.

To estimate the last term in (5.29) we require that $\nabla\times(\eta_h - J_h\eta_h) \perp \nabla\times\mathbb{P}_k(\mathcal{T}_h)$,

$$(\nabla\times\xi_0, \nabla\times(\eta_h - J_h\eta_h)) = (\nabla\times(\xi_0 - \Pi_{k,h}^0\xi_0), \nabla\times(\eta_h - J_h\eta_h)).$$

The second difference in (5.28) can be rewritten in controllable quantities as follows:

$$\beta(\Pi_{k,h}^0\tilde{\xi}_{0,h}, \Pi_{k,h}^0\eta_h) - \beta(\xi_0, \eta_h) - \beta(\xi_0, J_h\eta_h - \eta_h) \tag{5.30}$$ eq:err.xi0.

The first difference in (5.30) can be readily estimated using the definition of the L^2-projector to get,

$$\beta(\Pi_{k,h}^0\tilde{\xi}_{0,h}, \Pi_{k,h}^0\eta_h) - \beta(\xi_0, \eta_h) = \beta(\Pi_{k,h}^0(\tilde{\xi}_{0,h} - \xi_0), \Pi_{k,h}^0\eta_h) + \beta(\Pi_{k,h}^0\xi_0 - \xi_0, \eta_h).$$

The last term in (5.30) requires that $J_h\eta_h - \eta_h \perp \mathbb{P}_k(\mathcal{T}_h)$,

$$\beta(\xi_0, J_h\eta_h - \eta_h) = \beta(\xi_0 - \Pi_{k,h}^0\xi_0, J_h\eta_h - \eta_h).$$

[JT: The current operator J_h is too strong to also satisfy the required orthogonality properties mentioned above. On going through the literature, we do have such a "companion" operator defined in [12, Theorem 2.5] with the help of a simplicial sub-mesh. However that operator is too weak to also satisfy the orthogonality property of J_h. I am currently stuck here in the analysis.] □

6. Proof of Lemmas

mCAhpappfsx?

6.1. Proof of Lemma 5.10. We split the defect $J_hv_h - v_h$ using $\Pi_{k,h}^0$ as

$$\|J_hv_h - v_h\|_{L^2(\Omega)} \leqslant \|J_hv_h - \Pi_{k,h}^0v_h\|_{L^2(\Omega)} + \|\Pi_{k,h}^0v_h - v_h\|_{L^2(\Omega)}. \tag{6.1}$$ eq:Jh.appro

The first term in (6.1) is bounded using the definition of J_h in (5.13) as follows:

$$\begin{aligned}
\|\Pi_{k,h}^0v_h - J_hv_h\|_{L^2(\Omega)}^2 &= (\Pi_{k,h}^0v_h - J_hv_h, \Pi_{k,h}^0v_h - J_hv_h) \\
&= (\Pi_{k,h}^0v_h - J_hv_h, \Pi_{k,h}^0v_h) \\
&= (\Pi_{k,h}^0v_h - J_hv_h, \Pi_{k,h}^0v_h - v_h).
\end{aligned}$$

Thus, upon using Causchy-Schwarz inequality and (5.4), we have

$$\|\Pi_{k,h}^0v_h - J_hv_h\|_{L^2(\Omega)} \lesssim \left(\sum_{D\in\mathcal{T}_h} \|\Pi_{k,D}^0v_h - v_h\|_{L^2(D)}^2\right)^{1/2} \lesssim \left(\sum_{D\in\mathcal{T}_h} h_D^2|v_h|_{H^1(D)}^2\right)^{1/2}.$$

h.stability⟩6.2. **Proof of Lemma 5.11.** Testing with $v_h = J_h w$ and using Cauchy-Schwarz yields the stability estimate for J_h,

$$\|J_h w\|_{L^2(\Omega)} \leqslant \|\Pi^0_{k,h} w\|_{L^2(\Omega)} \overset{(5.3)}{\lesssim} \|w\|_{L^2(\Omega)}.$$

To prove the H^1-stability of J_h, we proceed as follows,

$$|J_h v_h|^2_{H^1(\Omega)} \lesssim |J_h v_h - v_h|^2_{H^1(\Omega)} + |v_h|^2_{H^1(\Omega)} = \sum_{D\in\mathcal{T}_h} |J_h v_h - v_h|^2_{H^1(D)} + |v_h|^2_{H^1(\Omega)}. \tag{6.2}$$ eq:Jh.appro

Using the inverse estimate in Lemma 5.7 for $\delta_h := J_h v_h - v_h \in V_h(D)$, we have

$$\begin{aligned} |\delta_h|^2_{H^1(D)} &\lesssim h_D^{-2} |||\delta_h|||^2_{k,D} \\ &= h_D^{-1} \sum_{e\in\mathcal{E}_D} \|\Pi^0_{k,e}\delta_h\|^2_{L^2(e)} + h_D^{-2} \|\Pi^0_{k-2,D}\delta_h\|^2_{L^2(D)} \\ &\leqslant h_D^{-1}\|\delta_h\|^2_{L^2(\partial D)} + h_D^{-2}\|\delta_h\|^2_{L^2(D)}, \end{aligned} \tag{6.3}$$ eq:Jh.appro

where we used the stability of $\Pi^0_{k,e}$ and $\Pi^0_{k-2,D}$ from (5.3). Now using a multiplicative trace inequality [7, Theorem 1.6.6] followed by Young's inequality for $\epsilon > 0$, yields

$$h_D^{-1}\|\delta_h\|^2_{L^2(\partial D)} \lesssim h_D^{-1}\|\delta_h\|_{L^2(D)}\|\delta_h\|_{H^1(D)} \leqslant \frac{1}{2\epsilon} h_D^{-2}\|\delta_h\|^2_{L^2(D)} + \frac{\epsilon}{2}\|\delta_h\|^2_{H^1(D)}.$$

Substituting the above in (6.3), gives

$$|\delta_h|^2_{H^1(D)} \lesssim \frac{1}{2\epsilon} h_D^{-2}\|\delta_h\|^2_{L^2(D)} + \frac{\epsilon}{2}\|\delta_h\|^2_{H^1(D)}.$$

Hence, applying the approximation property of J_h from Lemma 5.10 alongwith quasi-uniformity of the mesh, and Poincaré-Friedrichs inequality yields

$$\begin{aligned} |\delta_h|^2_{H^1(\Omega)} = \sum_{D\in\mathcal{T}_h} |\delta_h|^2_{H^1(D)} &\lesssim \left(1 + \frac{1}{2\epsilon}\right) \sum_{D\in\mathcal{T}_h} h_D^{-2}\|\delta_h\|^2_{L^2(D)} + \frac{\epsilon}{2} \sum_{D\in\mathcal{T}_h} \|\delta_h\|^2_{H^1(D)} \\ &\approx \left(1 + \frac{1}{2\epsilon}\right) h^{-2} \sum_{D\in\mathcal{T}_h} \|\delta_h\|^2_{L^2(D)} + \frac{\epsilon}{2} \sum_{D\in\mathcal{T}_h} \|\delta_h\|^2_{H^1(D)} \\ &\lesssim \left(1 + \frac{1}{2\epsilon}\right) \sum_{D\in\mathcal{T}_h} |v_h|^2_{H^1(D)} + \frac{\epsilon}{2}\|\delta_h\|^2_{H^1(\Omega)} \\ &\lesssim \left(1 + \frac{1}{2\epsilon}\right) |v_h|^2_{H^1(\Omega)} + \frac{\epsilon}{2}|\delta_h|^2_{H^1(\Omega)}. \end{aligned}$$

Choose $\epsilon > 0$ sufficiently small to absorb the last term on the right-hand side into the left-hand side, and substitute the resulting bound in (6.2) to complete the proof.

7. Numerical Results

sec:numexp⟩? We choose the classical `Dofi-Dofi` definition of the stabilization term (see [2]),

$$S^D(w_h, v_h) = \sum_{i=1}^{N_D^{\mathtt{dof}}} \chi_i(w_h)\,\chi_i(v_h) \quad \forall w_h, v_h \in V_h(D), \tag{7.1}$$ {?}

where operator $\chi_i(\cdot)$ associates the function with its i-th degree of freedom, and $N_D^{\mathtt{dof}}$ is the number of degrees of freedom associated with an element in $D \in \mathcal{T}_h$.

:smooth_sol⟩7.1. **Smooth solution on a convex simply connected domain.** We consider the problem (1.2) with $\gamma = \beta = 0$ and $\Omega = (0,1)^2$. Given $\phi(x,y) = \sin^3(\pi x)\sin^3(\pi y)$, the manufactured solution is given by

$$\boldsymbol{u} = \nabla \times \phi, \quad \text{with} \quad \boldsymbol{f} = -\nabla \times (\Delta(\nabla \times \boldsymbol{u})).$$

We solve (1.2) using the scheme described in Section 4 for orders $k = 1, 2, 3, 4$ on a sequence of unstructured Voronoi meshes. Since the virtual element solution is not known explicitly inside the element, we compare the manufactured solution with suitable projection of the discrete solution. The errors we measure are as follows:

$$e_{\boldsymbol{u}_h} := \|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)} = \sqrt{\sum_{D \in \mathcal{T}_h} \|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{L^2(D)}},$$

$$e_{\xi_h} := |\nabla \times \boldsymbol{u} - \Pi^1_{k,h}\xi_h|_{1,h} = \sqrt{\sum_{D \in \mathcal{T}_h} |\nabla \times \boldsymbol{u} - \Pi^1_{k,D}\xi_h|^2_{H^1(D)}},$$

and are reported in Tables 1 and 2 for $k = 1, 2, 3$ on a sequence of structured and unstructured voronoi meshes, respectively (see Figure 1). We recall that in this experiment $\boldsymbol{u}_h$ is computed using (4.12).

TABLE 1. Experimental errors and orders of convergence for Experiment 7.1 on structured voronoi meshes for orders $k = 1, 2, 3$.

.structured⟩

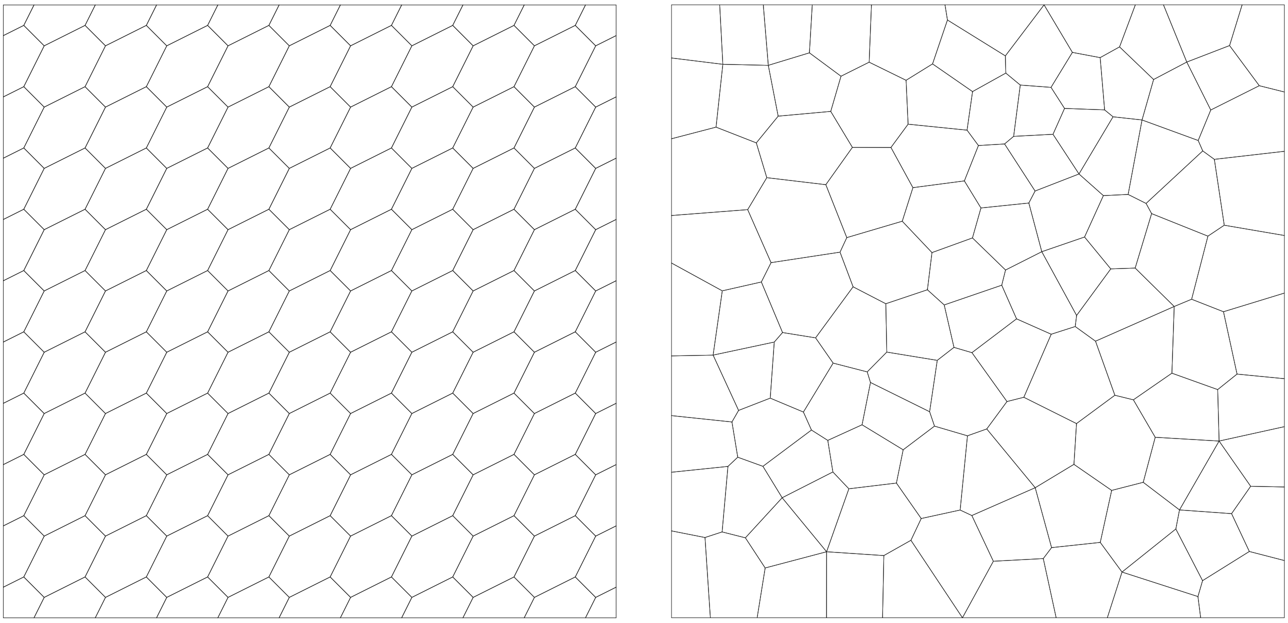
h	$N^{\mathtt{Dofs}}$	$e_{\boldsymbol{u}_h}$	rate	e_{ξ_h}	rate
$k = 1$					
2.9814e-01	90	1.1627	-	9.8919e+01	-
1.4907e-01	280	5.4896e-01	1.0827	5.1407e+01	0.9443
7.4536e-02	960	2.4669e-01	1.1540	2.5100e+01	1.0343
3.7268e-02	3520	1.1807e-01	1.0631	1.2425e+01	1.0145
1.8634e-02	13440	5.8296e-02	1.0181	6.1970e+00	1.0036
9.3169e-03	52480	2.9054e-02	1.0047	3.0971e+00	1.0006
$k = 2$					
2.9814e-01	251	2.7454e-01	-	3.0855e+01	-
1.4907e-01	801	7.1278e-02	1.9455	8.6421e+00	1.8361
7.4536e-02	2801	1.8366e-02	1.9564	2.2548e+00	1.9384
3.7268e-02	10401	4.6543e-03	1.9804	5.7300e-01	1.9764
1.8634e-02	40001	1.1703e-03	1.9916	1.4422e-01	1.9903
9.3169e-03	156801	2.9336e-04	1.9962	3.6163e-02	1.9957
$k = 3$					
2.9814e-01	448	6.3438e-02	-	8.2035e+00	-
1.4907e-01	1443	8.7780e-03	2.8534	1.1781e+00	2.7997
7.4536e-02	5083	1.1387e-03	2.9465	1.5452e-01	2.9306
3.7268e-02	18963	1.4425e-04	2.9807	1.9661e-02	2.9744
1.8634e-02	73123	1.8128e-05	2.9923	2.4755e-03	2.9895
9.3169e-03	287043	2.2713e-06	2.9966	3.1044e-04	2.9953

TABLE 2. Experimental errors and orders of convergence for Experiment 7.1 on unstructured Voronoi meshes for orders $k = 1, 2, 3$.

nstructured⟩

h	$N^{\mathtt{Dofs}}$	$e_{\boldsymbol{u}_h}$	rate	e_{ξ_h}	rate
$k = 1$					
3.7637e-01	48	1.1963	-	1.0194e+02	-
1.8360e-01	184	5.3872e-01	1.1114	5.1507e+01	0.9510
8.4283e-02	1234	1.8091e-01	1.4016	1.8902e+01	1.2876
4.1405e-02	5947	8.1748e-02	1.1176	8.6614e+00	1.0979
2.0988e-02	26477	3.8903e-02	1.0929	4.1493e+00	1.0831
1.0757e-02	96734	2.0202e-02	0.9804	2.1549e+00	0.9803
$k = 2$					
3.7637e-01	145	3.0531e-01	-	3.4261e+01	-
1.8360e-01	567	7.2772e-02	1.9977	8.8858e+00	1.8800
8.4283e-02	3867	1.1100e-02	2.4152	1.3484e+00	2.4218
4.1405e-02	18693	2.3647e-03	2.1756	2.8656e-01	2.1789
2.0988e-02	82953	5.2424e-04	2.2171	6.3400e-02	2.2201
1.0757e-02	303467	1.4262e-04	1.9477	1.7275e-02	1.9453

$k=3$					
3.7637e-01	267	6.7643e-02	-	9.2612e+00	-
1.8360e-01	1050	9.0071e-03	2.8088	1.2075e+00	2.8381
8.4283e-02	7200	5.2892e-04	3.6412	7.2091e-02	3.6199
4.1405e-02	34839	5.0638e-05	3.3008	6.7755e-03	3.3269
2.0988e-02	154429	5.4932e-06	3.2690	7.3215e-04	3.2747
1.0757e-02	565200	7.6463e-07	2.9503	1.0195e-04	2.9497



(A) Structured voronoi mesh (B) Unstructured voronoi mesh

:StVoro_sq⟩? ?⟨sub@fig:Voro_sq⟩?

FIGURE 1

Square_mesh⟩

:smoothdata⟩ 7.2. **Smooth data on a convex simply connected domain.** We keep the same domain and discretization introduced in Experiment 7.1. We take $\beta = \gamma = 0$, and choose

$$\boldsymbol{f} = ((x^2+1)\sin(x) + xy^3 + 2, (y^2+1)\cos(x) + x^3y^2 - 1).$$

Then we solve the sequence of discrete problems (4.5)-(4.12) for orders $k = 1, 2$. Since the exact solution is unknown, we compute relative errors

$$\texttt{rel } e^i_{\boldsymbol{u}_h} := \frac{\|\boldsymbol{u}^i_h - \boldsymbol{u}^{i+1}_h\|_{L^2(\Omega)}}{\|\boldsymbol{u}^{i+1}_h\|_{L^2(\Omega)}}, \quad \text{and} \quad \texttt{rel } e^i_{\xi_h} := \frac{|\Pi^{1,i}_{k,h}\xi^i_h - \Pi^{1,i+1}_{k,h}\xi^{i+1}_h|_{1,h_{i+1}}}{|\Pi^{1,i+1}_{k,h}\xi^{i+1}_h|_{1,h_{i+1}}},$$

where i denoetes the mesh level. Then we report them in Tables 3 and 4 on a sequence of structured and unstructured voronoi meshes, respectively (see Figures 1).

TABLE 3. Relative errors and convergence rates for Experiment 7.2 on structured voronoi meshes for orders $k = 1, 2$.

.structured⟩

h	$N^{\texttt{Dofs}}$	$\texttt{rel } e_{\boldsymbol{u}_h}$	rate	$\texttt{rel } e_{\xi_h}$	rate
$k=1$					
2.9814e-01	90	-	-	-	-
1.4907e-01	280	2.3814e-01	-	4.6021e-01	-
7.4536e-02	960	1.2406e-01	0.9408	2.7185e-01	0.7595
3.7268e-02	3520	6.2627e-02	0.9862	1.4889e-01	0.8686
1.8634e-02	13440	3.1449e-02	0.9938	7.8161e-02	0.9297
9.3169e-03	52480	1.5765e-02	0.9963	4.0092e-02	0.9631
$k=2$					

2.9814e-01	251	-	-	-	-
1.4907e-01	801	3.2060e-02	-	8.4945e-02	-
7.4536e-02	2801	1.0155e-02	1.6586	2.8971e-02	1.5519
3.7268e-02	10401	2.9260e-03	1.7952	9.0584e-03	1.6773
1.8634e-02	40001	7.8719e-04	1.8941	2.6825e-03	1.7557
9.3169e-03	156801	2.0419e-04	1.9468	7.6646e-04	1.8073

Table 4. Relative errors and convergence rates for Experiment 7.2 on unstructured voronoi meshes for orders $k = 1, 2$.

nstructured⟩

h	$N^{\mathtt{Dofs}}$	rel $e_{\boldsymbol{u}_h}$	rate	rel e_{ξ_h}	rate
		$k = 1$			
3.7637e-01	48	-	-	-	-
1.8360e-01	184	2.5848e-01	-	4.9213e-01	-
8.4283e-02	1234	1.2341e-01	0.9496	2.7482e-01	0.7483
4.1405e-02	5947	4.6503e-02	1.3731	1.1872e-01	1.1809
2.0988e-02	26477	2.1339e-02	1.1464	5.3990e-02	1.1597
1.0757e-02	96734	1.0318e-02	1.0873	2.6389e-02	1.0710
		$k = 2$			
3.7637e-01	145	-	-	-	-
1.8360e-01	567	3.8655e-02	-	1.1907e-01	-
8.4283e-02	3867	1.1807e-02	1.5233	3.7556e-02	1.4821
4.1405e-02	18693	2.0839e-03	2.4402	7.5986e-03	2.2481
2.0988e-02	82953	4.1737e-04	2.3666	1.9587e-03	1.9952
1.0757e-02	303467	9.6109e-05	2.1971	3.8104e-04	2.4494

nsmoothdata⟩ 7.3. **Piecewise smooth data on a non-convex simply connected domain.** We solve (1.2) on a L-shaped domain $\Omega = (-1,1)^2 \setminus [0,1) \times (-1,0]$ with $\beta = \gamma = 0$, and a piecewise constant vector field $\boldsymbol{f}$,

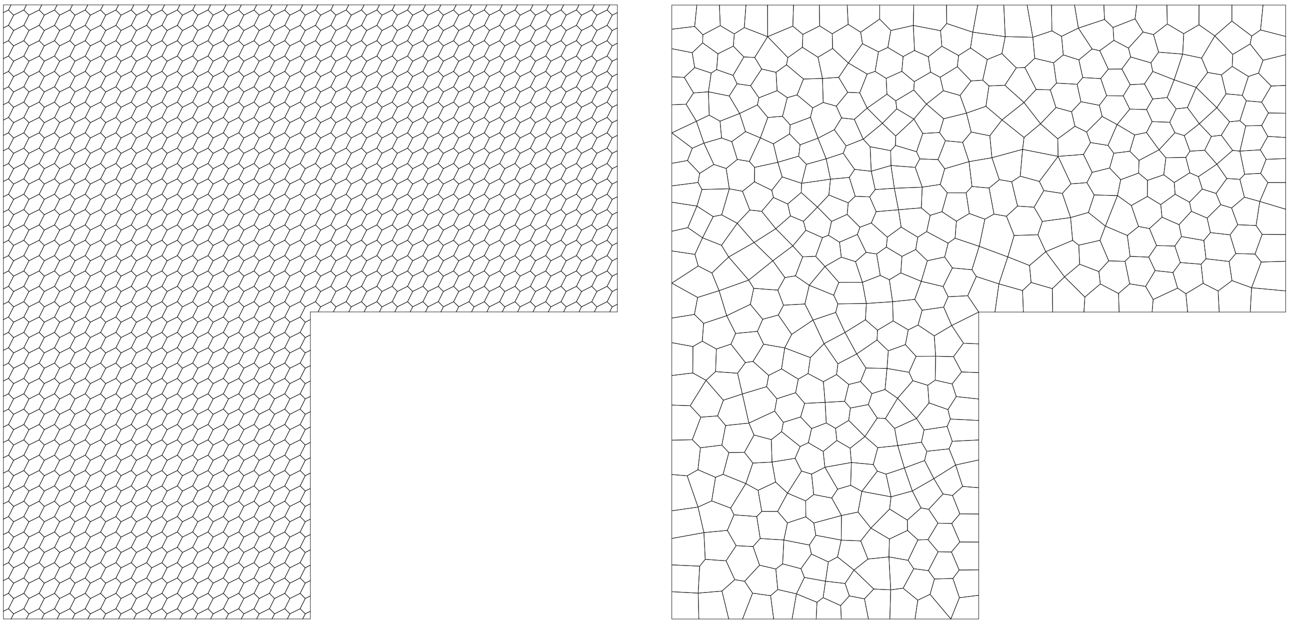
$$\boldsymbol{f} = \begin{cases} [1/4, 5/4]^\top, & \text{if} \quad |x| < 2^{-1/2}, \\ [1/2, 3/2]^\top, & \text{if} \quad 2^{-1/2} \leqslant |x| < 1, \\ [1, 2]^\top, & \text{if} \quad |x| \geqslant 1. \end{cases}$$

The relative errors are reported in Table 5 and 6 on a sequence of structured and unstructured voronoi meshes (see Figure 2) for orders $k = 1, 2$.

Table 5. Relative errors and convergence rates for Experiment 7.3 on structured voronoi meshes for orders $k = 1, 2$.

.structured⟩

h	$N^{\mathtt{Dofs}}$	rel $e_{\boldsymbol{u}_h}$	rate	rel e_{ξ_h}	rate
		$k = 1$			
2.9814e-01	230	-	-	-	-
1.4907e-01	760	1.7658e-01	-	3.1600e-01	-
7.4536e-02	2720	1.0229e-01	0.7876	1.7874e-01	0.8221
3.7268e-02	10240	6.0474e-02	0.7583	9.5571e-02	0.9032
1.8634e-02	39680	3.6354e-02	0.7342	4.9551e-02	0.9476
9.3169e-03	156160	2.2161e-02	0.7141	2.5265e-02	0.9718
		$k = 2$			
2.9814e-01	651	-	-	-	-
1.4907e-01	2201	7.7042e-02	-	4.6187e-02	-
7.4536e-02	8001	4.8020e-02	0.6820	1.5768e-02	1.5505
3.7268e-02	30401	3.0151e-02	0.6714	5.4008e-03	1.5458
1.8634e-02	118401	1.8975e-02	0.6681	2.1096e-03	1.3562
9.3169e-03	467201	1.1949e-02	0.6672	1.0279e-03	1.0372



(A) Structured voronoi mesh　　(B) Unstructured voronoi mesh

g:stVoro_L⟩?　　?⟨sub@fig:Voro_L⟩?

FIGURE 2

Lshape_mesh⟩

TABLE 6. Relative errors and convergence rates for Experiment 7.3 on unstructured voronoi meshes for orders $k = 1, 2$.

nstructured⟩

h	$N^{\texttt{Dofs}}$	$\texttt{rel}\ e_{\boldsymbol{u}_h}$	rate	$\texttt{rel}\ e_{\xi_h}$	rate
		$k = 1$			
6.4474e-01	47	-	-	-	-
3.2497e-01	174	3.1455e-01	-	5.5756e-01	-
1.5600e-01	783	1.4861e-01	1.0218	2.9441e-01	0.8702
7.6972e-02	4279	7.6425e-02	0.9414	1.4527e-01	0.9999
3.8974e-02	16213	4.1667e-02	0.8913	6.6347e-02	1.1516
1.9933e-02	68184	2.4315e-02	0.8033	3.3254e-02	1.0301
1.0296e-02	255458	1.6116e-02	0.6226	1.6667e-02	1.0457
		$k = 2$			
6.4474e-01	143	-	-	-	-
3.2497e-01	547	8.2224e-02	-	1.5327e-01	-
1.5600e-01	2465	4.4343e-02	0.8414	4.9060e-02	1.5522
7.6972e-02	13557	2.7502e-02	0.6762	1.3178e-02	1.8607
3.8974e-02	51425	1.5440e-02	0.8483	3.0375e-03	2.1564
1.9933e-02	216367	9.2966e-03	0.7565	1.1150e-03	1.4946
1.0296e-02	810915	6.4954e-03	0.5428	6.6147e-04	0.7904

⟨exp:Betti1⟩ 7.4. **Piecewise smooth data on a multiply connected domain with Betti number 1.** In this experiment we consider the domain $\Omega = (0,1)^2 \setminus [1/4, 3/4]^2$ with Betti number 1. We choose $\beta = \gamma = 1$ and the same piecewise smooth data $\boldsymbol{f}$ used in Experiment 7.3. We solve the sequence of discrete problems (4.13)-(4.18c) and post-process $\boldsymbol{u}_h$ using the relation (4.19) for orders $k = 1, 2$. The relative errors are reported in Table 7 and 8 on a sequence of structured and unstructured voronoi meshes, respectively (see Figure 3). We also report the relative error in the coefficient c_j,

$$\texttt{rel}\ e^i_{c_j} = \frac{|c^i_j - c^{i+1}_j|}{|c^{i+1}_j|},$$

as well as the values of c_j for orders $k = 1, 2$

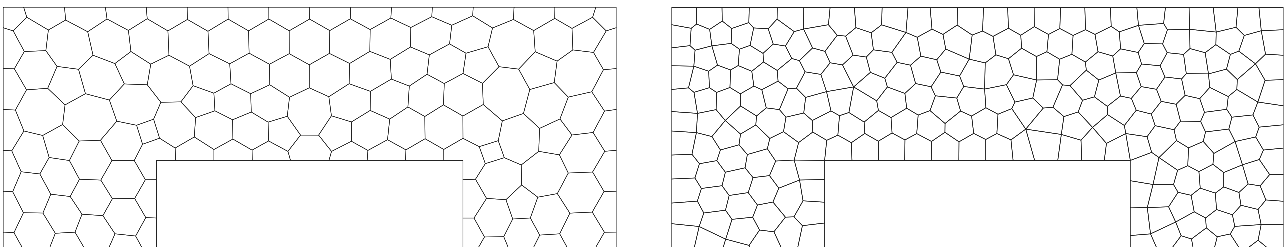


TABLE 7. Relative errors and convergence rates for Experiment 7.4 for orders $k = 1, 2$ on structured voronoi meshes.

etti1.struc⟩

h	$N^{\mathtt{Dofs}}$	rel $e_{\boldsymbol{u}_h}$	rate	rel e_{ξ_h}	rate	c_1	rate	rel e_{c_1}	rate
$k = 1$									
2.64e-01	96	-	-	-	-	-0.15092	-	-	-
1.67e-01	200	5.55e-01	-	5.56e-01	-	-0.15258	-	1.09e-02	-
8.85e-02	644	3.96e-01	0.53	4.78e-01	0.24	-0.15201	1.68	3.74e-03	1.68
4.77e-02	1930	2.36e-01	0.84	2.65e-01	0.95	-0.15187	2.20	9.57e-04	2.20
2.40e-02	6584	1.48e-01	0.68	1.55e-01	0.78	-0.15180	1.08	4.57e-04	1.08
1.18e-02	24092	9.62e-02	0.61	8.80e-02	0.80	-0.15177	1.20	1.96e-04	1.20
6.11e-03	93334	6.09e-02	0.69	4.97e-02	0.86	-0.15175	1.24	8.66e-05	1.24
$k = 2$									
2.64e-01	264	-	-	-	-	-0.15218	-	-	-
1.67e-01	552	2.30e-01	-	1.80e-01	-	-0.15214	-	3.20e-04	-
8.85e-02	1840	2.01e-01	0.22	1.33e-01	0.48	-0.15193	-2.25	1.33e-03	-2.25
4.77e-02	5614	1.14e-01	0.91	6.44e-02	1.17	-0.15182	0.89	7.66e-04	0.89
2.40e-02	19408	7.48e-02	0.62	3.88e-02	0.74	-0.15178	1.65	2.47e-04	1.65
1.18e-02	71592	4.98e-02	0.58	2.56e-02	0.59	-0.15176	0.85	1.35e-04	0.85
6.11e-03	278634	3.20e-02	0.67	1.63e-02	0.69	-0.15175	1.41	5.30e-05	1.41

TABLE 8. Relative errors and convergence rates for Experiment 7.4 for orders $k = 1, 2$ on unstructured voronoi meshes.

ti1.unstruc⟩

h	$N^{\mathtt{Dofs}}$	rel $e_{\boldsymbol{u}_h}$	rate	rel e_{ξ_h}	rate	c_1	rel e_{c_1}	rate
$k = 1$								
3.01e-01	66	-	-	-	-	-0.14966	-	-
1.41e-01	197	5.59e-01	-	6.38e-01	-	-0.15085	7.93e-03	-
7.32e-02	908	2.99e-01	0.95	3.72e-01	0.83	-0.15136	3.35e-03	1.32
3.41e-02	4637	1.72e-01	0.72	1.92e-01	0.86	-0.15162	1.68e-03	0.90
1.57e-02	17139	9.62e-02	0.75	9.46e-02	0.92	-0.15169	5.14e-04	1.53
7.55e-03	92388	6.22e-02	0.60	5.32e-02	0.79	-0.15173	2.47e-04	1.00
$k = 2$								
3.01e-01	204	-	-	-	-	-0.15137	-	-
1.41e-01	614	1.85e-01	-	1.59e-01	-	-0.15157	1.33e-03	-
7.32e-02	2816	1.16e-01	0.71	7.82e-02	1.08	-0.15168	7.04e-04	0.97
3.41e-02	14274	6.81e-02	0.70	4.08e-02	0.85	-0.15172	3.13e-04	1.06
1.57e-02	52278	3.88e-02	0.73	2.27e-02	0.76	-0.15174	9.70e-05	1.51
7.55e-03	284780	2.61e-02	0.54	1.50e-02	0.56	-0.15174	4.20e-05	1.14

⟨exp:Betti2⟩ 7.5. **Piecewise smooth data on a multiply connected domain with Betti number 2.** We solve (1.2) on a domain $\Omega = (-1, 1)^2 \setminus [1/4, 3/4]^2 \cup [-1/4, -3/4]^2$ with Betti number 2. We choose $\beta = \gamma = 1$ and the same piecewise smooth data $\boldsymbol{f}$ used in Experiment 7.3. The relative errors are reported in Table 9 on a sequence of structured voronoi meshes (see Figure 4). We also report the relative error in the coefficients c_1, c_2 and their values.

TABLE 9. Relative errors and convergence rates for Experiment 7.5 for orders $k = 1, 2$ on structured meshes.

able:Betti2⟩

h	$N^{\mathtt{Dofs}}$	rel $e_{\boldsymbol{u}_h}$	rate	rel e_{ξ_h}	rate	c_1	rel e_{c_1}	rate	c_2	rel e_{c_2}	rate
$k = 1$											
4.35e-01	178	-	-	-	-	-0.29668	-	-	-0.13608	-	-
2.20e-01	470	5.00e-01	-	3.92e-01	-	-0.29731	2.11e-03	-	-0.13590	1.37e-03	-
1.18e-01	1462	3.68e-01	0.49	2.51e-01	0.71	-0.29774	1.46e-03	0.58	-0.13602	9.36e-04	0.61
6.02e-02	5054	2.31e-01	0.70	1.44e-01	0.83	-0.29763	3.79e-04	2.02	-0.13586	1.18e-03	-0.35
3.06e-02	18014	1.46e-01	0.68	8.13e-02	0.84	-0.29752	3.87e-04	-0.03	-0.13579	5.05e-04	1.25

1.52e-02	67916	9.48e-02	0.62	4.53e-02	0.84	-0.29746	1.89e-04	1.03	-0.13577	1.99e-04	1.33
$k = 2$											
4.35e-01	493	-	-	-	-	-0.29863	-	-	-0.13632	-	-
2.20e-01	1337	2.14e-01	-	9.76e-02	-	-0.29837	8.88e-04	-	-0.13615	1.27e-03	-
1.18e-01	4245	1.86e-01	0.22	6.10e-02	0.75	-0.29780	1.92e-03	-1.23	-0.13592	1.68e-03	-0.44
6.02e-02	14885	1.16e-01	0.70	3.06e-02	1.03	-0.29759	7.06e-04	1.50	-0.13582	7.35e-04	1.23
3.06e-02	53497	7.39e-02	0.67	1.93e-02	0.68	-0.29749	3.28e-04	1.13	-0.13578	3.18e-04	1.24
1.52e-02	202667	4.85e-02	0.60	1.23e-02	0.65	-0.29744	1.44e-04	1.18	-0.13576	1.44e-04	1.14

TABLE 10. Relative errors and convergence rates for Experiment 7.5 for orders $k = 1, 2$ on unstructured meshes.

i2_unstruc⟩?

h	$N^{\mathtt{Dofs}}$	rel $e_{\boldsymbol{u}_h}$	rate	rel e_{ξ_h}	rate	c_1	rel e_{c_1}	rate	c_2	rel e_{c_2}	rate
$k = 1$											
3.25e-01	181	-	-	-	-	-0.13483	-	-	-0.29259	-	-
1.70e-01	770	4.10e-01	-	3.37e-01	-	-0.13510	2.00e-03	-	-0.29611	1.19e-02	-
8.30e-02	3093	2.48e-01	0.70	1.76e-01	0.91	-0.13549	2.89e-03	-0.52	-0.29676	2.18e-03	2.37
4.08e-02	13033	1.49e-01	0.72	9.15e-02	0.92	-0.13562	9.33e-04	1.59	-0.29713	1.25e-03	0.78
2.00e-02	58140	9.48e-02	0.63	4.94e-02	0.87	-0.13570	6.31e-04	0.55	-0.29733	6.73e-04	0.87
$k = 2$											
3.25e-01	563	-	-	-	-	-0.13555	-	-	-0.29716	-	-
1.70e-01	2341	1.70e-01	-	6.45e-02	-	-0.13565	7.63e-04	-	-0.29720	1.40e-04	-
8.30e-02	9387	1.03e-01	0.70	3.15e-02	1.00	-0.13572	4.81e-04	0.64	-0.29731	3.57e-04	-1.31
4.08e-02	39567	5.97e-02	0.77	1.61e-02	0.95	-0.13573	8.30e-05	2.47	-0.29737	2.10e-04	0.75
2.00e-02	176280	3.92e-02	0.59	1.09e-02	0.54	-0.13574	9.27e-05	-0.16	-0.29740	1.11e-04	0.89

ACKNOWLEDGMENTS

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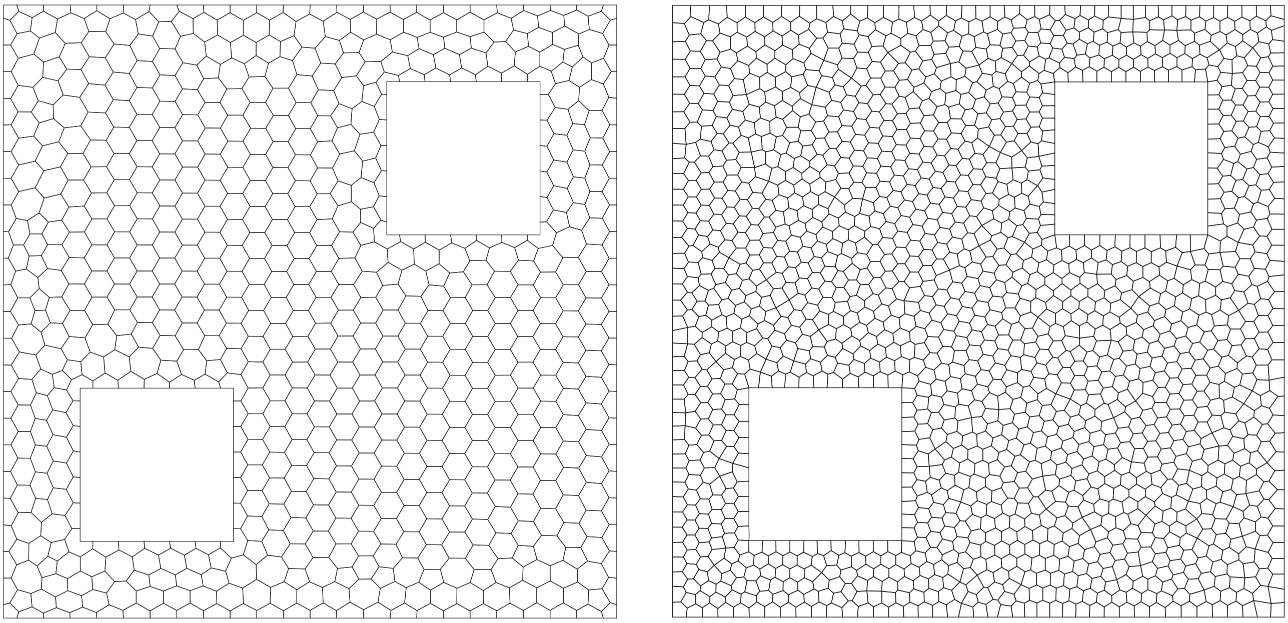

(A) Structured voronoi mesh (B) Unstructured voronoi mesh

oro_Betti2⟩? ?⟨sub@fig:Voro_Betti2⟩?

FIGURE 4

Betti2_mesh⟩

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