Topic: Linear Dependence and Independence of Vectors

Definitions:

- Linear Dependence of vectors: In the theory of vector spaces, a set of vectors is said to be linearly dependent if at least one of the vectors in the set can be defined as a linear combination of the others.
- Linear Independence of Vectors: In the theory of vector spaces, a set of vectors is said to be linearly independent if none of the vectors in the set can be defined as a linear combination of the others.

Concept:

A sequence of vectors **v1,v2,.....,vn** from a vector space V is said to be linearly dependent, if there exist scalars **a1,a2…,ak**, not all zero, such that :

a1v1 +a2v2+.....+akvk =0, …..1

where **0** denotes the zero vector.

A sequence of vectors **v1,v2,.....,vn** is said to be linearly independent if the equation :

a1v1 +a2v2+.....+akvk =0

can only be satisfied by ai=0 for i=1,....,n. This implies that no vector in the sequence can be represented as a linear combination of the remaining vectors in the sequence.

Steps: To examine if 2 vectors are dependent or independent :

1.Assume coefficients a1,a2,....an depending on the dimension of the matrix.

2.Multiply the coefficient matrix with the given matrix as per equation 1.

3.Using row transformations reduce the matrix to a simple form and find the value of a1,a2….an

4.If all a1=a2=....an=0 then the vectors are independent else the vectors are dependent.

Examples:

Q1 Examine wheather the vectors.

$V_1 - [3,1,1]$ $V_2 = [2,0,-1]$, $V_3 = [4,2,1]$ are linearly independant

Ans Consider the matrix equation

$$K_1V_1 + K_2V_2 + K_3V_3 = 0$$

$$K_1[3,1,1] + K_2[2,0,-1] + K_3[4,2,1] = [0,0,0]$$

i.e

$$\begin{bmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

By R_{13}

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - R_1$, $R_3 - 3R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - R_2$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 4 & 0 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore K_1 - K_2 + K_3 = 0$; $K_2 + K_3 = 0$; $4K_3 = 0$

$\therefore K_1 = K_2 = K_3 = 0$

$\because K_1, K_2$ & K_3 are 0 the vectors are linearly indepdant

Q2 Verify wheather (1,2), (2,3) are linearly dependent when placed with inital points at origin.

Ans: Let the vectors be $V_1 = (1,2)$ & $V_2 = (2,3)$ & consider

$$K_1V_1 + K_2V_2 = 0$$

$$K_1(1,2) + K_2(2,3) = 0$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

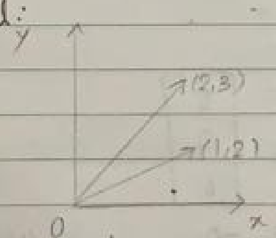
By $R_2 - 2R_1$

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore$ $K_1 + 2K_2 = 0$ & $-K_2 = 0$

$\therefore$ They are linearly independent.

Geometrical:



when we place the vectors with initial points on origin there is no line containg both the vectors.

$\therefore$ They are independent.

Q3 Use suitable trigonometric identities to determine wheather the following sets of functions are linearly independent & find the relation between them.

i) $8, 4\sin^2 x, 2\cos^2 x$ ii) $\cos 2x, \cos^2 x, \sin^2 x$.

Ans:

i) Let $f_1 = 8$, $f_2 = 4\sin^2 x$ $f_3 = 2(\cos^2 x)(2)$

consider :-

$$f_1 - 2f_2 - 4f_3 = 8 - 2(4\sin^2 x) - 4(2\cos^2 x)$$
$$= 8 - 8(\sin^2 x + \cos^2 x)$$
$$= 0$$

Since f_1 is expressed as a linear combination of f_2 & f_3 i.e $f_1 = 2f_2 + 4f_3$, the functions are linear dependent

ii) $f_1 = \cos 2x$, $f_2 = \cos^2 x$, $f_3 = \sin^2 x$

$$f_1 - f_2 - f_3 = \cos 2x - \cos^2 x - \sin^2 x$$
$$= \cos^2 x - \sin^2 x - \cos^2 x - \sin^2 x$$
$$= -2\sin^2 x$$

As the vectors cannot be expressed we take $f_1 - f_2 + f_3$

which gives $\cos 2x - \cos^2 x + \sin^2 x$

$$= \cos^2 x - \sin^2 x - \cos^2 x + \sin^2 x$$
$$= 0.$$

Thus the functions are linearly dependent.

Q4 Verify wheather the following vectors are linearly independent $V_1 = (4,5,1)$, $V_2 = (0,-1,-1)$, $V_3 = (3,9,4)$, $V_4 = (-4,4,4)$.

Ans Consider $K_1V_1 + K_2V_2 + K_3V_3 + V_4V_4 = 0$.

$K_1(4,5,1) + K_2(0,-1,-1) + K_3(3,9,4) + K_4(-4,4,4) = (0,0,0)$.

$$\begin{bmatrix} 4 & 0 & 3 & -4 \\ 5 & -1 & 9 & 4 \\ 1 & -1 & 4 & 4 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By R_{31}

$$\begin{bmatrix} 1 & -1 & 4 & 4 \\ 5 & -1 & 9 & 4 \\ 4 & 0 & 3 & -4 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_2 - 5R_1$ & $R_3 - 4R_1$

$$\begin{bmatrix} 1 & -1 & 4 & 4 \\ 0 & 4 & -11 & -16 \\ 0 & 4 & -13 & -20 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $R_3 - R_2$

$$\begin{bmatrix} 1 & -1 & 4 & 4 \\ 0 & 4 & -11 & -16 \\ 0 & 0 & -2 & -4 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \\ K_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad K_1 - K_2 + 4K_3 + 4K_4 = 0$

$4K_2 - 11K_3 - 16K_4 = 0$

$-2K_3 - 4K_4 = 0$

$\therefore K_3 = -2K_4$.

Putting $K_4 = t$, $K_3 = -2t$

$\therefore 4K_2 = 11K_3 + 16K_4$

$\therefore 4K_2 = -22t + 16t$

$= -6t$

$\therefore K_2 = -\frac{3}{2}t$.

$K_1 = K_2 - 4K_3 - 4K_4 = -\frac{3}{2}t + 8t - 4t$

$= \frac{5}{2}t$.

Putting these values:

$\frac{5}{2}tV_1 - \frac{3}{2}tV_2 - 2tV_3 + tV_4 = 0$

$\Rightarrow \frac{5}{2}V_1 - \frac{3}{2}V_2 - 2V_3 + V_4 = 0$

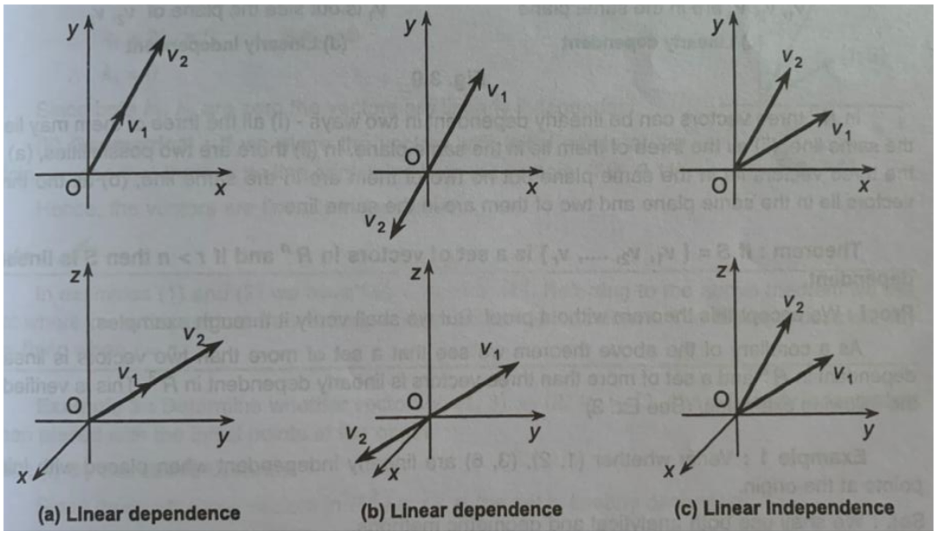
$\therefore \frac{5}{2}V_1 = \frac{3}{2}V_2 + 2V_3 - V_4$

As V_1 can be expressed in terms of V_2, V_3 & V_4 they are linearly dependent.

Application: It is used in the study of Movement of Rain drops as you move in a car . Movement of river water down a hill slope.Landslides, avalanches.Study of Tsunamis and Periodic winds.

Significance: The study of linear dependence and independence of vectors is important to study the relation between the vectors.

Geometrical Interpretation:
a.2 vectors are linearly independent if they do not lie on the same line. If 2 vectors are dependent they will lie in the same line and can be expressed in terms of one another.



b.In R3 3 vectors are linearly independent if and only if the vectors do not lie in the same plane when they are placed with their initial points at the origin.