5 Inner Product Spaces

1. Introduction

We have already defined Euclidean inner product of spaces in R^2 and R^3 in § 11, page 2-10. In this chapter we shall extend the concept of inner product still further and define Euclidean inner product on R^n.

2. General Inner Product

Instead of denoting the inner product (or the dot product) of two vectors u, v as $u \cdot v$, we shall now denote the inner product by $\langle u, v\rangle$. Thus, we have

$$\boxed{\langle u, v\rangle = u \cdot v}$$

Definition : An **inner product** on a real vector space V is a function that associates a **real number** $\langle u, v\rangle$ with each pair of u, v in V such that the following axioms are satisfied for all vectors u, v, z in V and for all scalars k.

(i) $\langle u, v\rangle = \langle v, u\rangle$ **[Axiom of Symmetry]**

(ii) $\langle u + v, z\rangle = \langle u, z\rangle + \langle v, z\rangle$ **[Axiom of Additivity]**

(iii) $\langle ku, v\rangle = k\langle u, v\rangle$ **[Axiom of Homogeneity]**

(iv) $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0$ if and only if $v = 0$. **[Positivity Axiom]**

3. Matrix Form of Inner Product

If $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ then the inner product

$$\langle u, v\rangle = (u_1, u_2, \ldots, u_n) \cdot (v_1, v_2, \ldots, v_n)$$
$$= u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

can be written as a matrix product

$$\langle u, v\rangle = [v_1, v_2, \ldots, v_n] \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = [v'u]$$

4. Weighted Euclidean Inner product

Definition : If $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$ are vectors in R^n and if $w_1, w_2, ..., w_n$ are positive real numbers called **weights** then

$$\langle u, v\rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + ... + w_n u_n v_n$$

define an inner product on R^n called **weighted Euclidean inner product with weights** $w_1, w_2, ..., w_n$.

Example 1 : Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$.

Prove that $\langle u, v\rangle = 3\, u_1 v_1 + 2\, u_2 v_2$

is an inner product on R^2.

Sol. : (i) Since $\langle u, v\rangle = 3\, u_1 v_1 + 2\, u_2 v_2$

and $\langle v, u\rangle = 3\, v_1 u_1 + 2\, v_2 u_2$

$\therefore\ \langle u, v\rangle = \langle v, u\rangle$

(ii) If $z = (z_1, z_2)$ then

$$\begin{aligned}
\langle u+v, z\rangle &= 3\,(u_1 + v_1)\, z_1 + 2\,(u_2 + v_2)\, z_2 \\
&= (3u_1 z_1 + 3v_1 z_1) + (2u_2 v_2 + 2v_2 z_2) \\
&= (3u_1 z_1 + 2u_2 z_2) + (3v_1 z_1 + 2v_2 z_2) \\
\therefore\ \langle u+v, z\rangle &= \langle u, z\rangle + \langle v, z\rangle
\end{aligned}$$

(iii) Now, $\langle ku, v\rangle = 3\,(ku_1)\, v_1 + 2\,(ku_2)\, v_2$

$$= k\,(3u_1 v_1 + 2u_2 v_2)$$

$$= k\,\langle u, v\rangle$$

(iv) Lastly, $\langle v, v\rangle = 3\, v_1 v_1 + 2\, v_2 v_2$

$$= 3v_1^2 + 2v_2^2$$

Clearly $3v_1^2 + 2v_2^2 \geq 0$ and $(v, v) = 0$ if and only $v_1 = 0$ and $v_2 = 0$.

Thus, all the four axioms of the inner product are satisfied.

Hence, $\langle u, v\rangle$ is an inner product in R^2.

5. Norm (or Length) and Distance

Recall that we defined the norm (or length) of the vector u in Euclidean space as

$$\| u \| = (u \cdot u)^{1/2}$$

and the distance between two vectors u, v as

$$d\,(u, v) = \| u - v \| = [(u - v) \cdot (u - v)]^{1/2}$$

We generalise this concept and define these terms for R^n as follows.

Definition : If V is the inner product space then the **norm (or length)** of a vector u in V denoted by $\| u \|$ is defined by

$$\boxed{\| u \| = \langle u, u \rangle^{1/2}}$$

The distance between two points (vectors) u, v denoted by $d(u, v)$ is defined by

$$\boxed{d(u, v) = \| u - v \|}$$

If a vector has norm 1, then the vector is called a **unit** vector.

If $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ are any two vectors in R^n with inner product then

$$\| u \| = \langle u, u \rangle^{1/2}$$

$$\boxed{\|u\| = (u \cdot u)^{1/2} = \sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}}$$

and
$$\begin{aligned} d(u, v) &= \| u - v \| \\ &= \langle u - v, u - v \rangle^{1/2} \\ &= [(u - v) \cdot (u - v)]^{1/2} \end{aligned}$$

$$\boxed{d(u, v) = [(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots + (u_n - v_n)]^{1/2}}$$

Example 1 : If $u = (u_1, u_2, u_3)$ and $\langle u_1, u_2 \rangle = 1$, $\langle u_2, u_3 \rangle = 2$, $\langle u_3, u_1 \rangle = -3$ and $\| u_2 \| = 1$, calculate

(i) $\langle u_2, 2u_1 - 5u_3 \rangle$ (ii) $\langle u_1 + 2u_2, u_2 + u_3 \rangle$

Sol. : (i)
$$\begin{aligned} \langle u_2, 2u_1 - 5u_3 \rangle &= \langle u_2, 2u_1 \rangle + \langle u_2, -5u_3 \rangle \\ &= \langle 2u_1, u_2 \rangle + \langle -5u_3, u_2 \rangle && \text{[By (ii), page 5-1]} \\ &= 2\langle u_1, u_2 \rangle - 5\langle u_3, u_2 \rangle && \text{[By (iii), page 5-1]} \\ &= 2 - 5(2) = -8 \end{aligned}$$

(ii)
$$\begin{aligned} \langle u_1 + 2u_2, u_2 + u_3 \rangle &= \langle u_1 + 2u_2, u_2 \rangle + \langle u_1 + 2u_2, u_3 \rangle && \text{[By (ii)]} \\ &= \langle u_1, u_2 \rangle + \langle 2u_2, u_2 \rangle + \langle u_1, u_3 \rangle + \langle 2u_2, u_3 \rangle && \text{[By (ii)]} \\ &= \langle u_1, u_2 \rangle + 2\langle u_2, u_2 \rangle + \langle u_1, u_3 \rangle + 2\langle u_2, u_3 \rangle && \text{[By (iii)]} \\ &= 1 + 2 \cdot 1 + (-3) + 2(2) = 4 \end{aligned}$$

Example 2 : If $u = (3, 2)$, $v = (4, 5)$, find $\| u \|$, $d(u, v)$ using

(a) Euclidean inner product,

(b) Weighted Euclidean inner product $\langle u, v \rangle = 3u_1u_2 + 2v_1v_2$.

Sol. : (a) Using Euclidean inner product

(i) $\|u\| = \langle u, u \rangle^{1/2} = (u \cdot u)^{1/2}$

$$\therefore \ \|u\| = \sqrt{(u_1^2 + u_2^2)} = \sqrt{(9+4)} = \sqrt{13}$$

(ii) $$d(u, v) = \|u - v\|$$
$$= \sqrt{(u_1 - u_2)^2 + (v_1 - v_2)^2}$$
$$= \sqrt{1+9} = \sqrt{10}$$

(b) Using weighted Euclidean inner product,

(i) $$\|u\| = \langle u, u\rangle^{1/2} = (u \cdot u)^{1/2}$$
$$= \sqrt{3u_1^2 + 2u_1^2} = \sqrt{3 \cdot (3)^2 + 2(2)^2}$$
$$= \sqrt{27+8} = \sqrt{35}$$

(ii) $$d(u, v) = \|u - v\|$$
$$= \sqrt{3(u_1 - u_2)^2 + 2(v_1 - v_2)^2}$$
$$= \sqrt{3(-1)^2 + 2(-3)^2} = \sqrt{21}.$$

Example 3 : Show that
$$\| u + v \|^2 + \| u - v \|^2 = 2 \| u \|^2 + 2 \| v \|^2.$$

Sol. : We have by definition

$$\|u + v\| = \sqrt{(u+v) \cdot (u+v)} = \sqrt{u^2 + 2uv + v^2}$$
$$\|u - v\| = \sqrt{(u-v) \cdot (u-v)} = \sqrt{u^2 - 2uv + v^2}$$
$$\therefore \ \| u + v \|^2 + \| u - v \|^2 = (u^2 + 2uv + v^2) + (u^2 - 2uv + v^2)$$
$$= 2u^2 + 2v^2$$
$$= 2 \| u \|^2 + 2 \| v \|^2.$$

6. Inner Product Generated by Matrices

If $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and A is an $n \times n$ invertible matrix (*i.e.* $|A| \neq 0$)

then the inner product given by

$$\boxed{\langle u, v\rangle = Au \cdot Av}$$

is called the **inner product generated by *A***.

As we have seen above the inner product

$\langle u, v\rangle = u \cdot v = v'$ $u$in matrix form.

Using the matrix form, we can write

$$\langle u, v\rangle = Au \cdot Av = (Av)' Au$$

$$\boxed{\langle u, v\rangle = v' A' Au}$$

Example 1 : Find the inner product $\langle u, v\rangle$ generated by $\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$ where $u = (-3, 2)$, $v = (2, 4)$.

Sol. : By definition, the inner product generated by the matrix A is given by

$$\langle u, v\rangle = v' A' Au$$

$$= [2, 4] \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$= [2, 4] \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ 9 \end{bmatrix} = [2, 4] \begin{bmatrix} -17 \\ 23 \end{bmatrix}$$

$$= (-34 + 92) = 58.$$

Example 2 : Show that $\langle u, v\rangle = 9\, u_1 v_1 + 4\, u_2 v_2$ is the inner product of $u = (u_1, v_1)$ and $v = (u_2, v_2)$ generated by $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.

Sol. : By definition, the inner product generated by the matrix A is given by

$$\langle u, v\rangle = v' A' Au$$

Since $|A| = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6 \neq 0$

A is invertible. Hence, we can write

$$\langle u, v\rangle = v' A' Au$$

$$= [v_1, v_2] \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad (\because A' = A)$$

$$= [v_1, v_2] \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3u_1 \\ 2u_2 \end{bmatrix}$$

$$= [v_1, v_2] \begin{bmatrix} 9u_1 \\ 4u_2 \end{bmatrix}$$

$$= 9 u_1 v_1 + 4 u_2 v_2$$

Hence, the result.

Example 3 : Find a matrix that generates the inner product

$$\langle u, v\rangle = 3\, u_1 v_1 + 5\, u_2 v_2.$$

Sol. : By definition, the inner product generated by the matrix A is given by

$$\langle u, v\rangle = v' A' Au$$

We want $A'A = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$. Looking carefully the above examples, we see that if we take $A = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$ and $A' = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$ and we get $A'A = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$.

$$\langle u, v\rangle = [v_1, v_2] \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{5} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= [v_1, v_2] \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$= 3u_1 v_1 + 5u_2 v_2$$

Hence, the required matrix is $\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$.

In general, the matrix that generates the inner product,

$$\langle u, v\rangle = w_1 u_1 v_1 + w_2 u_2 v_2 + \ldots\ldots + w_n u_n v_n$$

is $$A = \begin{bmatrix} \sqrt{w_1} & 0 & 0 & \ldots & 0 \\ 0 & \sqrt{w_2} & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & \sqrt{w_n} \end{bmatrix}$$

Example 4 : Find the inner product $\langle u, v\rangle$ generated by $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ where $u = (2u_1, u_2)$ and $v = (v_1, 3v_2)$.

Sol. : By definition

$$\langle u, v\rangle = v' A' Au$$

$$= [v_1, 3v_2] \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2u_1 \\ u_2 \end{bmatrix}$$

$$= [v_1, 3v_2] \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2u_1 + 3u_2 \\ 4u_1 + 4u_2 \end{bmatrix}$$

$$= [v_1, 3v_2] \begin{bmatrix} 2u_1 + 3u_2 + 8u_1 + 8u_2 \\ 6u_1 + 9u_2 + 16u_1 + 16u_2 \end{bmatrix}$$

$$= [v_1, 3v_2] \begin{bmatrix} 10u_1 + 11u_2 \\ 22u_1 + 25u_2 \end{bmatrix}$$

$$= 10u_1 v_1 + 11u_2 v_1 + 66u_1 v_2 + 75u_2 v_2$$

$$= 10u_1 v_1 + 66u_1 v_2 + 11u_2 v_1 + 75u_2 v_2$$

Example 5 : Find $\| w \|$ where $w = (3, 4)$ using

(i) Euclidean inner product

(ii) Weighted Euclidean inner product $3\, u_1 v_1 + 2\, u_2 v_2$ where $u = (u_1, u_2)$, $v = (v_1, v_2)$

(iii) the inner product generated by $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$.

Sol. : (i) By using Euclidean inner product

$$\|w\| = \langle w, w\rangle^{1/2} = (w \cdot w)^{1/2}$$
$$= \sqrt{w_1^2 + w_2^2} = \sqrt{9+16} = 5$$

(ii) By using weighted Euclidean inner product

$$\|w\| = \langle w, w\rangle^{1/2} = (w \cdot w)^{1/2}$$
$$= \sqrt{3\, w_1^2 + 2\, w_2^2} = \sqrt{3\,(3)\,(3) + 2\,(4)\,(4)}$$
$$= \sqrt{27+32} = \sqrt{59}.$$

(iii) Using the inner product generated by A

$$\|w\| = \sqrt{w'\, A'\, A\, w}$$

Now, $$w'\, A'\, A\, w = [3,\ 4] \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$
$$= [3,\ 4] \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 11 \\ 9 \end{bmatrix} = [3,\ 4] \begin{bmatrix} 2 \\ 49 \end{bmatrix}$$
$$= 6 + 196 = 212$$

$$\therefore \quad \|w\| = \sqrt{212} = 2\sqrt{53}.$$

Example 6 : Find $d(u, v)$ for $u = (-1, 2)$, $v = (2, 5)$ using

(i) Euclidean inner product

(ii) Weighted Euclidean inner product $\langle u, v\rangle = 3\, u_1 v_1 + 2\, u_2 v_2$.

Sol. : (i) Using Euclidean inner product

$$d(u, v) = \| u - v \| = \| -3, -3 \|$$
$$= \langle -3, -3\rangle^{1/2}$$
$$= \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}.$$

(ii) Using weighted Euclidean in product

$$d(u, v) = \| u - v \| = \langle -3, -3\rangle^{1/2}$$
$$= [\,3\,(-3)^2 + 2\,(-3)^2\,]^{1/2}$$
$$= \sqrt{27+18} = \sqrt{45} = 3\sqrt{5}.$$

Example 7 : Find $d(u, v)$ for $u = (-1, 2)$, $v = (2, 5)$ using the inner product generated by $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$.

Sol. : We have $u - v = (-3, -3)$.

Now, $d(u, v) = \| u - v \| = \| w \|$ say.

By definition, $\langle w, w \rangle = w' A' A w$

$$\therefore \quad \| -3, -3 \| = [-3, -3] \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

$$= [-3, -3] \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -9 \\ -6 \end{bmatrix}$$

$$= [-3, -3] \begin{bmatrix} -3 \\ -36 \end{bmatrix}$$

$$= 9 + 108 = 117$$

$$\therefore \quad d(u, v) = \| u - v \| = \| w \|$$

$$= \sqrt{\langle w, w \rangle} = \sqrt{117} = 3\sqrt{13}.$$

7. Inner Product on M_{22}

Definition : If $U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$ and $V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$, then the inner product of U, V is **defined** as the trace of $U' V$ or trace of $V' U$.

(The sum of the diagonal elements of a square matrix is called its trace.)

$$\therefore \quad \langle U, V \rangle = \text{tr}\,(U'V) = \text{tr}\,(V'U)$$

$$= \text{tr} \begin{bmatrix} u_1 & u_3 \\ u_2 & u_4 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$$

$$= \text{tr} \begin{bmatrix} u_1 v_1 + u_3 v_3 & u_1 v_2 + u_3 v_4 \\ u_2 v_1 + u_4 v_3 & u_2 v_2 + u_4 v_4 \end{bmatrix}$$

$$= u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

Similarly, we can see that trace of $[V' U]$ gives the same result.

Thus,
$$\boxed{\begin{aligned} \langle U, V \rangle &= u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4 \\ &= \text{tr}\, U' V = \text{tr}\, V' U \end{aligned}}$$

The norm of U is given by

$$\| U \| = \langle U, U \rangle^{1/2} = (u \cdot u)^{1/2}$$

$$\boxed{||U|| = \sqrt{u_1^{\,2} + u_2^{\,2} + u_3^{\,2} + u_4^{\,2}}}$$

A unit sphere in this space is given by $u_1^{\,2} + u_2^{\,2} + u_3^{\,2} + u_4^{\,2} = 1$.

(It can be seen that $\langle u, v\rangle$ is obtained by taking row $\times$ row product of u and v).

Example 1 : Find the inner product of $U = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ and $V = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}$ also find the norm of U.

Sol. : As seen above,

$$\begin{aligned} \langle u, v\rangle &= u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4 \\ &= 3 \times 1 + 4 \times -2 + 1 \times 0 + 2 \times (-3) \\ &= 3 - 8 - 6 = -11 \end{aligned}$$

$$\begin{aligned} \text{And} \quad ||U|| &= \sqrt{u_1^{\,2} + u_2^{\,2} + u_3^{\,2} + u_4^{\,2}} \\ &= \sqrt{3^2 + 4^2 + 1^2 + 2^2} \\ &= \sqrt{30} \end{aligned}$$

Example 2 : If $u = \begin{bmatrix} 2 & 6 \\ 9 & 4 \end{bmatrix}$ and $v = \begin{bmatrix} -4 & 7 \\ 1 & 6 \end{bmatrix}$, find $d(u, v)$.

Sol. : By definition,

$$d(u, v) = ||u - v|| = \sqrt{\langle (u - v) \cdot (u - v) \rangle}$$

$$\text{Now,} \quad u - v = \begin{bmatrix} 2 & 6 \\ 9 & 4 \end{bmatrix} - \begin{bmatrix} -4 & 7 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ 8 & -2 \end{bmatrix}$$

$$\begin{aligned} d(u, v) &= \sqrt{u_1^{\,2} + u_2^{\,2} + u_3^{\,2} + u_4^{\,2}} \\ &= \sqrt{36 + 1 + 64 + 4} \\ &= \sqrt{105}. \end{aligned}$$

8. Inner Product On P_2

If $p = a_0 + a_1 x + a_2 x^2$ and $q = b_0 + b_1 x + b_2 x^2$, then the inner product on P_2 is given by

$$\boxed{\langle p, q\rangle = a_0 b_0 + a_1 b_1 + a_2 b_2}$$

Further, we have $\boxed{||p|| = \sqrt{a_0^{\,2} + a_1^{\,2} + a_2^{\,2}}}$

Unit sphere in this space is given by $a_0^{\,2} + a_1^{\,2} + a_2^{\,2} = 1$.

Example 1 : If $p = -2 + x + 3x^2$ and $q = 5 - 7x^2$, find $\langle p, q\rangle$ and $\| p \|$ on P_2.

Sol. : Since $p = -2 + x + 3x^2$ and $q = 5 + 0 - 7x^2$, as above

$$\langle p, q\rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$$

$$\langle p, q\rangle = (-2)(5) + (1)(0) + (3)(-7)$$

$$= -10 - 21 = -31$$

Further, $\|p\| = \sqrt{4+1+9} = \sqrt{14}$.

Example 2 : If $p = -1 + 2x - x^2$ and $q = 2x - 4x^2$, then find $\langle p, q\rangle$ and $\| p \|$ on P_2.

Sol. : By definition, since $p = -1 + 2x - x^2$ and $q = 0 + 2x - 4x^2$.

$$\langle p, q\rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$$

$$= (-1)(0) + (2)(2) + (-1)(-4)$$

$$= 4 + 4 = 8$$

Also $\|p\| = \sqrt{a_0^2 + a_1^2 + a_2^2} = \sqrt{1+4+1} = \sqrt{6}$.

9. Inner Product On $C(a, b)$

If $f(x)$ and $g(x)$ are any two functions defined on $C[a, b]$ and if we define

$$\boxed{\langle f, g\rangle = \int_a^b f(x)\, g(x)\, dx}$$

then $\langle f, g\rangle$ defines an inner product on $C[a, b]$.

If $C[a, b]$ has the inner product defined as above then the norm of a function $f = f(x)$ is given by

$$\boxed{\|f\| = \langle f, f\rangle = (f \cdot f) = \sqrt{\int_a^b f^2(x)\, dx}}$$

Example 1 : If $p = x$, find $\| p \|$ if $\langle p, q\rangle = \int_{-1}^{1} p(x)\, q(x)\, dx$ on $C(-1, 1)$.

Sol. : As defined above

$$\|p\| = \sqrt{\int_{-1}^{1} p(x)\, p(x) \cdot dx} = \sqrt{\int_{-1}^{1} x \cdot x \cdot dx}$$

$$= \int_{-1}^{1} x^2\, dx = 2\int_0^1 x^2\, dx = 2\left[\frac{x^3}{3}\right]_0^1 \qquad [\because f \text{ is even}]$$

$$= \sqrt{\frac{2}{3}} = \frac{1}{3}\sqrt{6}.$$

Example 2 : Find $d(p, q)$ if $p = 1$ and $q = x$; $\langle p, q\rangle$ defined as above in Ex. 1.

Sol. : By definition

$$d(p,q) = \sqrt{(u-v)\cdot(u-v)} = \sqrt{\int_{-1}^{1}(1-x)\cdot(1-x)\,dx}$$

$$= \sqrt{\int_{-1}^{1}(1-2x+x^2)\,dx}$$

$$= \sqrt{2\cdot\int_{0}^{1}(1+x^2)\,dx} \qquad \left[\because \int_{-1}^{1}-2x\,dx = 0,\ f \text{ is odd}\right]$$

$$= \sqrt{2\left[x+\frac{x^3}{3}\right]_0^1} = \sqrt{2\left[1+\frac{1}{3}\right]} = \sqrt{\frac{2\cdot 4}{3}} \qquad [\because f \text{ is even}]$$

$$= \frac{2}{\sqrt{3}}\cdot\sqrt{2} = \frac{2}{3}\sqrt{6}.$$

Example 3 : If $f(x) = \cos 2\pi x$, $g(x) = \sin 2\pi x$, compute $d(f, g)$ using the inner product $\langle f, g\rangle = \int_0^1 f(x)\,g(x)\,dx$.

Sol. : By definition,

$$d(f,g) = \sqrt{(f-g)\cdot(f-g)}$$

$$= \sqrt{\int_0^1(\cos 2\pi x - \sin 2\pi x)\cdot(\cos 2\pi x - \sin 2\pi x)\,dx}$$

$$= \sqrt{\int_0^1(\cos^2 2\pi x - 2\cos 2\pi x \sin 2\pi x + \sin^2 2\pi x)\,dx}$$

$$= \sqrt{\int_0^1(1-2\cos 2\pi x\sin 2\pi x)\,dx}$$

$$= \sqrt{\int_0^1(1-\sin 4\pi x)\,dx} = \left[x+\frac{\cos 4\pi x}{4}\right]_0^1$$

$$= \sqrt{\left(1+\frac{1}{4}\right)-\left(0+\frac{1}{4}\right)} = \sqrt{1} = 1.$$

Example 4 : Using the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)\cdot g(x)\,dx$ in $C(-1, 1)$, find $\langle f, g\rangle$ where

(i) $f(x) = 2 + x$, $g(x) = 1 + x^2$

(ii) $f(x) = x$, $g(x) = 1 + 2x + 3x^2$.

Sol. : (i) By definition,

$$\langle f, g \rangle = \int_{-1}^{1} (2+x)(1+x^2)\, dx$$

$$= \int_{-1}^{1} (2+2x^2+x+x^3)\, dx$$

$$= 2\int_{0}^{1} 2(1+x^2)\, dx \qquad \left[\because \int_{-1}^{1} \text{Odd function} = 0\right]$$

$$= 4\left[x+\frac{x^3}{3}\right]_0^1 = 4\left[1+\frac{1}{3}\right] = \frac{16}{3}.$$

(ii) $$\langle f, g \rangle = \int_{-1}^{1} x(1+2x+3x^2)\, dx$$

$$= \int_{-1}^{1} (x+2x^2+3x^3)\, dx$$

$$= 2\int_{0}^{1} 2x^2\, dx \qquad \left[\int_{-1}^{1} \text{Odd function} = 0\right]$$

$$= 4\left[\frac{x^3}{3}\right]_0^1 = 4\left[\frac{1}{3}\right] = \frac{4}{3}.$$

Example 4 : Use the inner product $\langle p, q \rangle = \int_{-1}^{1} p(x)\, q(x)\, dx$ to compute $\langle p, q \rangle$ for vectors $p(x) = 1 - x + x^2 + 5x^3$ and $q(x) = x - 3x^2$ in P_3.

Sol. : $$\langle p, q \rangle = \int_{-1}^{1} p(x) \cdot q(x)\, dx$$

$$= \int_{-1}^{1} (1-x+x^2+5x^3)(x-3x^2)\, dx$$

$$= \int_{-1}^{1} (x-3x^2-x^2+3x^3+x^3-3x^4+5x^4-15x^5)\, dx$$

$$= \int_{-1}^{1} (x-4x^2+4x^3+2x^4-15x^5)\, dx$$

$$= 2\int_{-1}^{1} (-4x^2+2x^4)\, dx \qquad \left[\because \int_{-1}^{1} (\text{Odd function})\, dx = 0\right]$$

$$= 2\left[-\frac{4x^3}{3}+\frac{2x^5}{5}\right]_0^1 = 2\left[-\frac{4}{3}+\frac{2}{5}\right]$$

$$= 2\left[\frac{14}{15}\right] = \frac{28}{15}.$$

EXERCISE - I

1. Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be any two vectors in R^2. Let $\langle u, v \rangle$ be defined as given below. Show that it is an inner product.

(i) $\langle u, v \rangle = u_1 v_1 + u_2 v_2$

(ii) $\langle u, v \rangle = u_1 v_1 - u_1 v_2 - u_2 v_1 + u_2 v_2$

2. Let $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ be any two vectors in R^3. Let $\langle u, v \rangle$ be defined as follows. Check whether $\langle u, v \rangle$ is an inner product

(i) $\langle u, v \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3$ [**Ans. :** Yes]

(ii) $\langle u, v \rangle = 2\, u_1 v_1 + u_2 v_2 + 4\, u_3 v_3$ [**Ans. :** Yes]

(iii) $\langle u, v \rangle = u_1 v_1 + u_3 v_3$ [**Ans. :** No]

3. If (i) $u = \begin{bmatrix} 3 & -2 \\ 4 & 8 \end{bmatrix}$, $v = \begin{bmatrix} -1 & 3 \\ 4 & 3 \end{bmatrix}$

(ii) $u = \begin{bmatrix} 2 & -4 \\ 3 & 1 \end{bmatrix}$, $v = \begin{bmatrix} 6 & 2 \\ -1 & 3 \end{bmatrix}$

find $\langle u, v \rangle = \text{tr}\,(v'\, u)$. [**Ans. : (i)** 31, **(ii)** 4]

4. If U is an inner product space and $u = (u_1, u_2, u_3)$, such that $\langle u_1, u_2 \rangle = 2$, $\langle u_2, u_3 \rangle = 1$, $\langle u_1, u_3 \rangle = -3$, $\| u_1 \| = 1$, calculate,

(i) $\langle u_2, 2u_1 - 3u_3 \rangle$ (ii) $\langle 2u_1 + u_2, -u_1 - u_3 \rangle$ [**Ans. : (i)** 1, **(ii)** 9]

5. If $\langle u_1, u_2 \rangle = 25$, $\langle u_2, u_1 - u_2 \rangle = 11$, find $\| u_2 \|$. [**Ans. :** 6]

6. If $u = (2, -3, 4)$, $v = (1, 0, 2)$ are vectors in R^3 and $w_1 = 1$, $w_2 = 2$, $w_3 = 3$ are the weights, find (i) weighted inner product of u, v, (ii) $\| u \|$, $\| v \|$, (iii) $d\,(u, v)$. [**Ans. : (i)** 26, **(ii)** 70, **(iii)** $\sqrt{101}$, $\sqrt{14}$, **(iv)** $\sqrt{31}$]

7. Let $u = (u_1, u_2)$, $v = (v_1, v_2)$. Prove that

(i) $\langle u, v \rangle = 4\, u_1 v_1 + 9\, u_2 v_2$

(ii) $\langle u, v \rangle = 3\, u_1 v_1 + 6\, u_2 v_2$

is an inner product space on R^2. Find a matrix that generates it.

[**Ans. : (i)** $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, **(ii)** $\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{6} \end{bmatrix}$

8. Find the inner product of

(i) $u = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$, $v = \begin{bmatrix} 4 & -2 \\ 1 & 3 \end{bmatrix}$ [**Ans. :** 20]

(ii) $u = \begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}$, $v = \begin{bmatrix} 4 & 6 \\ 0 & 8 \end{bmatrix}$ [**Ans. :** 56]

9. If (i) $p = -5 + 2x + x^2$ and $q = 3 + 4x^2$

(ii) $p = 3 - x + x^2$ and $q = 1 - 4x$.

Compute the inner product $\langle p, q \rangle$. [**Ans. :** (i) -11, (ii) 7]

10. Find the inner product $\langle u, v \rangle$ generated by A if $u = (-3, 2)$, $v = (1, 7)$ and $A = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. [**Ans. :** 29]

11. Find the inner product of

(i) $u = \begin{bmatrix} 2 & 6 \\ 9 & -4 \end{bmatrix}$ and $v = \begin{bmatrix} -4 & 7 \\ 1 & 6 \end{bmatrix}$ [**Ans. :** 19]

(ii) $u = \begin{bmatrix} 2 & 5 \\ -3 & 2 \end{bmatrix}$ and $v = \begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix}$ [**Ans. :** 10]

12. If (i) $p = 1$, (ii) $p = x^2$, find $\| p \|$ where

$\langle p, q \rangle = \int_{-1}^{1} p(x)\, q(x)\, dx.$ [**Ans. :** (i) $\sqrt{2}$, (ii) $\frac{1}{5}\sqrt{10}$]

13. If $p = 1$, $q = x^2$ and $\langle p, q \rangle = \int_{-1}^{1} p(x)\, q(x)\, dx$, find (p, q).

[**Ans. :** 16 / 3]

14. If $u = (-1, 2)$, $v = (2, 7)$, find $d(u, v)$ using

(i) Euclidean inner product of u, v.

(ii) Weighted Euclidean inner product $\langle u, v \rangle = 3\, u_1 v_1 + 2\, u_2 v_2$.

(iii) Inner product generated by the matrix $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$.

[**Ans. :** (i) $\sqrt{34}$, (ii) $\sqrt{77}$, (iii) $\sqrt{313}$.]

15. Using the inner product $\langle f, g \rangle = \int_{0}^{1} f(x)\, g(x)\, dx$, find $\langle f, g \rangle$ if

(i) $f = x$, $g = e^x$,

(ii) $f = \tan\left(\frac{\pi}{4}\right) x$, $g = 1$. [**Ans. :** (i) 1, (ii) $\frac{2}{\pi} \log 2$]

16. Using the inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$ in $C(-1, 1)$, find $\langle f, g \rangle$ where

(i) $f = 2x - 5x^3$, $g = 5 + 6x^2$

(ii) $f = 3 - x + x^2 + 6x^3$, $g = x - 6x^2$. [**Ans. :** (i) 0, (ii) $-11/2$]

10. Angle and Orthogonality

We have seen that the angle θ between any two vectors u, v in R^2 and R^3 is given by

$$\boxed{\cos\theta = \frac{u \cdot v}{||u||\,||v||}}$$

Since $\cos\theta \leq 1$ to apply this definition to R^n, we must have

$$|u \cdot v| \leq ||u||\,||v||$$

This is assured by Cauchy-Schwartz inequality.

Definition : We define angle θ between two vectors u and v as

$$\boxed{\cos\theta = \frac{\langle u, v\rangle}{||u||\,||v||}}$$

11. Condition of Orthogonality

Two vectors are said to be orthogonal if $\theta = \pi/2$. From the above relation, we get the following condition of orthogonality. u and v are orthogonal if

$$\boxed{\langle u, v\rangle = 0}$$

12. Cauchy-Schewartz Inequality

If u, v are vectors in a real inner product space then

$$\boxed{|\langle u, v\rangle| \leq ||u||\,||v||}$$

Sol. : We accept this result and shall verify it in Ex. 3, 4, pages 5-19, 5-20.

Example 1 : Let R^4 have Euclidean inner product. Find the cosine of the angle between vectors $u = (4, 3, 1, -2)$ and $v = (-2, 1, 2, 3)$.

Sol. : We have

$$\begin{aligned}\langle u, v\rangle &= (4, 3, 1, -2) \cdot (-2, 1, 2, 3)\\ &= (4)(-2) + (3)(1) + (1)(2) + (-2)(3)\\ &= -8 + 3 + 2 - 6 = -9\end{aligned}$$

$$||u|| = \sqrt{16+9+1+4} = \sqrt{30}$$

$$||v|| = \sqrt{4+1+4+9} = \sqrt{18}$$

$$\therefore \quad \cos\theta = \frac{\langle u, v\rangle}{||u||\,||v||} = \frac{-9}{\sqrt{30}\,\sqrt{18}}$$

$$= -\frac{9}{\sqrt{3\cdot 5\cdot 2}\,\sqrt{9\cdot 2}} = -\frac{9}{3\times 2\sqrt{15}} = -\frac{3}{2\sqrt{15}}.$$

Example 2 : Check whether the following vectors are orthogonal

(i) $u = (-2, 3, 4), \; v = (2, 4, -2)$

(ii) $u = (-4, 6, -10, 1), \; v = (2, 1, -2, 9)$

Sol. : (i) We have

$$\langle u, v \rangle = u \cdot v = (-2, 3, 4) \cdot (2, 4, -2)$$
$$= -4 + 12 - 8 = 0$$

Hence, u, v are orthogonal.

(ii) $$\langle u, v \rangle = u \cdot v = (-4, 6, -10, 1) \cdot (2, 1, -2, 9)$$
$$= -8 + 6 + 20 + 9 = 27 \neq 0$$

Hence, u, v are not orthogonal.

Example 3 : If R^3 has the Euclidean inner product, find k such that u, v are orthogonal where

(a) $u = (k, k, 1), \; v = (k, 5, 6)$

(b) $u = (2, 1, 3), \; v = (1, 7, k)$

Sol. : Since u, v are orthogonal $\langle u, v \rangle = 0$.

(a) $$\langle u, v \rangle = (k, k, 1) \cdot (k, 5, 6)$$
$$= k(k) + k(5) + (1)\,6 = 0$$
$$\therefore \quad k^2 + 5k + 6 = 0$$
$$\therefore \quad (k+3)(k+2) = 0 \qquad \therefore \quad k = -3, -2$$

(b) $$\langle u, v \rangle = (2, 1, 3) \cdot (1, 7, k)$$
$$= 2(1) + 1(7) + 3(k) = 0$$
$$\therefore \quad 3k + 9 = 0 \qquad \therefore \quad k = -3.$$

Example 4 : If $u = (1, 1, -1)$ and $v = (6, 7, -15)$ in R^3 having Euclidean product and if $\| ku + v \| = 13$, find k.

Sol. : We have $$(ku + v) = (k(1, 1, -1) + (6, 7, -15))$$
$$= ((k, k, -k) + (6, 7, -15))$$
$$= (k+6, k+7, -(k+15))$$
$$\| ku + v \| = \sqrt{(k+6)^2 + (k+7)^2 + (k+15)^2}$$
$$= \sqrt{k^2 + 12k + 36 + k^2 + 14k + 49 + k^2 + 30k + 225}$$
$$= \sqrt{3k^2 + 56k + 310}$$

But this is equal to 13.

$$\therefore \quad 3k^2 + 56k + 310 = 13^2 = 169$$
$$\therefore \quad 3k^2 + 56k + 141 = 0$$

$\therefore\quad 3k^2+9k+47k+141=0$

$\therefore\quad 3k(k+3)+47(k+3)=0$

$\therefore\quad (3k+47)(k+3)=0 \quad \therefore\quad k=-3 \text{ or } -47/3.$

Example 5 : Find two unit vectors which are orthogonal to $(2, 1, -4, 0)$, $(-1, -1, 2, 2)$, $(3, 2, 5, 4)$ in the Euclidean space in R^4.

Sol. : Let (u_1, u_2, u_3, u_4) be the required unit vector which is orthogonal to each of the above vectors. Then, we have

$$\begin{aligned} 2u_1 + u_2 - 4u_3 + 0u_4 &= 0 \\ -u_1 - u_2 + 2u_3 + 2u_4 &= 0 \\ 3u_1 + 2u_2 + 5u_3 + 4u_4 &= 0 \end{aligned}$$

We solve these equations,

$$\begin{bmatrix} 2 & 1 & -4 & 0 \\ -1 & -1 & 2 & 2 \\ 3 & 2 & 5 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $\begin{matrix} -R_2 \\ R_{1,2} \end{matrix}$ $\begin{bmatrix} 1 & 1 & -2 & -2 \\ 2 & 1 & -4 & 0 \\ 3 & 2 & 5 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

By $\begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$ $\begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & -1 & 0 & 4 \\ 0 & -1 & 11 & 10 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

By $\begin{matrix} R_3 - R_2 \\ -R_2 \end{matrix}$ $\begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 11 & 6 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$\therefore\quad \begin{aligned} u_1 + u_2 - 2u_3 - 2u_4 &= 0 \\ u_2 \qquad - 4u_4 &= 0 \\ 11u_3 + 6u_4 &= 0 \end{aligned}$$

Let $u_4 = t \quad \therefore\quad u_3 = -\dfrac{6}{11}t \quad \therefore\quad u_2 = 4u_4 = 4t.$

$$\therefore\quad u_1 = -u_2 + 2u_3 + 2u_4$$

$$= -4t - \frac{12}{11}t + 2t = -\frac{34}{11}t.$$

To remove the fraction, we may take $t = 11$.

$$\therefore \quad u_1 = -34, \ u_2 = 44, \ u_3 = -6, \ u_4 = 11$$

$$\therefore \quad \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2} = \sqrt{34^2 + 44^2 + 36 + 121}$$

$$= \sqrt{1156 + 1936 + 36 + 121}$$

$$= \sqrt{3249} = \sqrt{57^2} = \pm 57$$

Hence, the two required vectors are $\dfrac{1}{\pm 57}(-34, 44, -6, 11)$.

13. Orthogonal Vectors in M_{22}

We have defined [See § 7, page 5-8] the inner product of two matrices $\begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}, \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$ as $\langle u, v \rangle = \text{tr}\,[\,v'\,u\,]$

$$\langle u, v \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

Then the cosine of the angle θ between these matrices as vectors is given by

$$\boxed{\cos\theta = \frac{\langle u, v \rangle}{||u||\,||v||}}$$

The matrices (as vectors) are orthogonal if $\langle u, v \rangle = 0$ *i.e.* if

$$\boxed{u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4 = 0}$$

Example 1 : Check whether $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ is orthogonal to

(i) $\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Sol. : We first calculate inner product $\langle A, B \rangle$ of the given matrices

(i) $\langle A, B \rangle = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$

$$= u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

$$= (3)(-2) + (1)(0) + (-1)(0) + (2)(3) = 0$$

Hence, A, B are orthogonal.

(ii) $\langle A, B \rangle = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$= (-3)(1) + (1)(-1) + (-1)(1) + (2)(1)$$

$$= -3 \neq 0$$

Hence, A, B are not orthogonal.

(iii) Left to you.

Clearly Orthogonal.

Example 2 : Let M_{22} have the inner product $\langle A, B \rangle = \text{tr}(A^T B)$. Find the cosine of the angle between A, B when $A = \begin{bmatrix} 2 & 6 \\ 1 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$.

Sol. : We have

$$\begin{aligned} \langle A, B \rangle &= \text{tr}(A' B) \\ &= u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4 \\ &= (2)(3) + (6)(2) + (1)(1) + (-3)(0) \\ &= 19 \end{aligned}$$

$$\begin{aligned} ||A|| &= \langle A', A \rangle^{1/2} = \sqrt{\text{tr}(A' A)} \\ &= \sqrt{u_1^2 + u_2^2 + u_3^2 + u_4^2} \\ &= \sqrt{4 + 36 + 1 + 9} = \sqrt{50} \end{aligned}$$

$$\begin{aligned} ||B|| &= \langle B', B \rangle^{1/2} = \sqrt{\text{tr}(B' B)} \\ &= \sqrt{v_1^2 + v_2^2 + v_3^2 + v_4^2} \\ &= \sqrt{9 + 4 + 1 + 0} = \sqrt{14} \end{aligned}$$

Now, $\cos\theta = \dfrac{\langle A, B \rangle}{||u||\,||v||} = \dfrac{19}{\sqrt{50}\sqrt{14}} = \dfrac{19}{\sqrt{25 \cdot 2}\sqrt{7 \cdot 2}} = \dfrac{19}{10\sqrt{7}}$.

Example 3 : For $U = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$, $V = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$ in M_{22} define

$$\langle U, V \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$$

for matrices A and B. Verify Cauchy-Schwartz inequality and find the cosine of the angle between them where $A = \begin{bmatrix} 2 & 6 \\ 1 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}$.

Sol. : We have, proved in the above example that

$$\begin{aligned} \therefore \quad \cos\theta &= \frac{\langle u, v \rangle}{||u||\,||v||} = \frac{19}{\sqrt{50}\,\sqrt{14}} \\ &= \frac{19}{5\sqrt{2}\sqrt{2}\sqrt{7}} = \frac{19}{10\sqrt{7}}. \end{aligned}$$

Further, $|\langle u, v \rangle| = 19$ and $||u|| = \sqrt{50}$, $||v|| = \sqrt{14}$.

$\therefore \quad |\langle u, v \rangle| \leq ||\,u\,||\,||\,v\,|| \qquad [\because\ 19^2 < 50 \times 14]$

Example 4 : Verify the Cauchy-Schwartz inequality for the following vectors $p = 1 + 2x + x^2$, $q = 3 + 4x^2$ using the inner product

$$\langle p, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2.$$

Sol. : We have $\langle p, q \rangle = (1)(3) + (2)(0) + (1)(4) = 7$

$$||u|| = \sqrt{a_0^2 + a_1^2 + a_2^2} = \sqrt{1+4+1} = \sqrt{6}$$

$$||v|| = \sqrt{b_0^2 + b_1^2 + b_2^2} = \sqrt{9+0+16} = \sqrt{25} = 5$$

$$\therefore \quad |\langle u, v \rangle| = 7 < \sqrt{6} \cdot 5 \qquad [49 < 6 \times 25]$$

$$\therefore \quad |\langle u, v \rangle| \le ||u|| \cdot ||v||.$$

14. Angle Between the Vectors in $C(-1, 1)$

Let P_2 have the inner product $\langle p, q \rangle = \int_{-1}^{1} p(x)\, q(x)\, dx$, then the angle between two (vectors) p, q is $\cos\theta = \dfrac{\langle p, q \rangle}{||p||\, ||q||}$.

The vectors are perpendicular if $\cos\theta = 0$ *i.e.* if $\boxed{\langle p, q \rangle = 0}$

Example 1 : Let $C(-1, 1)$ have an inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x) \cdot g(x)\, dx.$$

Prove that $f_1(x) = 1$, $f_2(x) = x$ and $f_3(x) = \dfrac{3x^2 - 1}{2}$ are orthogonal with respect to the given inner product.

Sol. : We have

(i) $\displaystyle \int_{-1}^{1} f_1(x) \cdot f_2(x)\, dx = \int_{-1}^{1} 1 \cdot x\, dx = 0$ $\qquad [\because$ Odd function]

(ii) $\displaystyle \int_{-1}^{1} f_1(x) \cdot f_3(x)\, dx = \frac{1}{2} \int_{-1}^{1} 1 \cdot (3x^2 - 1)\, dx$

$$= \int_{0}^{1} (3x^2 - 1)\, dx \qquad [\because \text{ Even function }]$$

$$= \left[x^3 - x\right]_0^1 = 0$$

(iii) $\displaystyle \int_{-1}^{1} f_2(x) \cdot f_3(x) = \frac{1}{2} \int_{-1}^{1} x(3x^2 - 1)\, dx = 0$ $\qquad [\because$ Odd function]

$\therefore$ f_1, f_2, f_3 are orthogonal.

Example 2 : Let $C(-1, 1)$ have an inner product

$$\langle f, g \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx.$$

Check whether, (i) $f_1(x) = 1$, (ii) $f_2(x) = x^2$, (iii) $f_3(x) = 5x^4 - 1$ are orthogonal with respect to the above inner product.

Sol. : We have to check whether $\langle p, q \rangle = 0$ for f_1, f_2, f_3 taken two at a time.

(i) $$\int_{-1}^{1} f_1(x) \cdot f_2(x)\, dx = \int_{-1}^{1} 1 \cdot x^2\, dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3} \neq 0$$

$\therefore$ $f_1(x)$, $f_2(x)$ are not orthogonal.

(ii) $$\int_{-1}^{1} f_1(x) \cdot f_3(x)\, dx = \int_{-1}^{1} 1 \cdot (5x^4 - 1)\, dx$$

$$= 2\int_{0}^{1} (5x^4 - 1)\, dx = 2\left[x^5 - x\right]_0^1 = 0$$

$\therefore$ $f_1(x)$, $f_3(x)$ are orthgonal.

(iii) $$\int_{-1}^{1} f_2(x)\, f_3(x)\, dx = \int_{-1}^{1} x^2 (5x^4 - 1)\, dx = \int_{-1}^{1} (5x^6 - x^2)\, dx$$

$$= 2\left[\frac{5x^7}{7} - \frac{x^3}{3}\right]_0^1 = 2\left(\frac{5}{7} - \frac{1}{3}\right) \neq 0$$

$\therefore$ $f_2(x)$, $f_3(x)$ are not orthgonal.

15. Orthonormal Bases

Definition : A set of vectors in an inner product space is called an **orthogonal set** if each vector in the set is orthogonal to every other vector in the set. If the norm of each vector in this set is unit then the set is called **orthonormal**.

Example 1 : An orthogonal set in R^3.

Let $u_1 = (1, 0, 1)$, $u_2 = (0, 1, 0)$, $u_3 = (1, 0, -1)$ have the Euclidean inner product. Prove that the vectors are orthogonal.

Sol. : Since $\langle u_1, u_2 \rangle = \langle u_1, u_3 \rangle = \langle u_2, u_3 \rangle = 0$, vectors are orthogonal.

Example 2 : Normalise the above vectors.

Sol. : The norms of the above vectors are $\| u \| = \sqrt{2}$, $\| u_2 \| = 1$, $\| u_3 \| = \sqrt{2}$.

If we divide a vector by its norm, we get a unit vector thus, $\dfrac{v}{\|v\|}$ is a unit vector.

Hence, dividing u_1, u_2, u_3 by their norms we get the normalised vectors

$$\frac{u_1}{\|u_1\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \quad \frac{u_2}{\|u_2\|} = (0, 1, 0), \quad \frac{u_3}{\|u_3\|} = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

16. Generalised Theorem of Pythagoras

If u, v are orthogonal vectors in an inner product space then the Pythagoras Theorem states that

$$\boxed{\| u+v \|^2 = \| u \|^2 + \| v \|^2}$$

Proof : Since u and v are orthogonal $\langle u \cdot v \rangle = 0$.

$$\begin{aligned} \text{Now, } \| u+v \|^2 &= \langle (u+v), (u+v) \rangle \\ &= (u+v) \cdot (u+v) \\ &= u \cdot u + v \cdot v + v \cdot u + u \cdot v \\ &= \| u \|^2 + 2 \langle u, v \rangle + \| v \|^2 \\ &= \| u \|^2 + \| v \|^2 \qquad [\because \langle u, v \rangle = 0] \end{aligned}$$

Example 1 : Verify the theorem of Pythagoras, for $p = -1 + 2x + 4x^2$, $q = 4 + x^2$ using the inner product $\langle p, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$ where $p = a_0 + a_1 x + a_2 x^2$ and $q = b_0 + b_1 x + b_2 x^2$.

Sol. : We first see that p and q are orthogonal.

$$\begin{aligned} \text{Since, } \langle p, q \rangle &= (-1)(4) + (2)(0) + 4(1) \\ &= -4 + 4 = 0 \end{aligned}$$

$\therefore$ p, q are orthogonal.

$$\begin{aligned} \text{Now, } \| p \|^2 &= \langle p \cdot p \rangle = a_0^2 + a_1^2 + a_2^2 \\ &= 1 + 4 + 16 = 21 \\ \| q \|^2 &= b_0^2 + b_1^2 + b_2^2 \\ &= 16 + 1 = 17 \\ p + q &= -3 + 2x + 5x^2 = c_0 + c_1 x + c_2 x^2 \\ \| p+q \|^2 &= c_0^2 + c_1^2 + c_2^2 \\ &= 9 + 4 + 25 = 38 \end{aligned}$$

But $\| p \|^2 + \| q \|^2 = 21 + 17 = 38$

Hence, $\| p+q \|^2 = \| p \|^2 + \| q \|^2$.

Example 2 : Check whether $p = x$, $q = x^2$ are orthogonal relative to the inner product $\langle p, q \rangle = \int_{-1}^{1} p(x)\, q(x)\, dx$ on $C(-1, 1)$ and if so verify the generalised Pythagoras theorem.

Sol. : Since $\langle p, q \rangle = \int_{-1}^{1} x \cdot x^2 \, dx = 0$; p, q are orthogonal.

We have $\qquad [\because$ Odd function $]$

$$\| p \|^2 = \langle p, p \rangle = \int_{-1}^{1} x \cdot x \, dx \qquad [\because \text{Even}]$$

$$\therefore \quad ||p||^2 = 2\int_0^1 x^2\,dx = 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3}.$$

$$||q||^2 = \langle q \cdot q \rangle = \int_{-1}^{1} x^2 \cdot x^2\,dx \qquad [\because \text{Even}]$$

$$= 2\int_0^1 x^4\,dx = 2\left[\frac{x^5}{5}\right]_0^1 = \frac{2}{5}.$$

$$p + q = x + x^2$$

$$\therefore \quad ||p+q||^2 = \langle (p+q), (p+q) \rangle$$

$$= \int_{-1}^{1} (x + x^2)(x + x^2)\,dx$$

$$= \int_{-1}^{1} (x^2 + 2x^3 + x^4)\,dx$$

$$= 2\int_0^1 (x^2 + x^4)\,dx \qquad \left[\because \int_{-1}^{1} x^3\,dx = 0\right]$$

$$= 2\left[\frac{x^3}{3} + \frac{x^5}{5}\right]_0^1 = 2\left[\frac{1}{3} + \frac{1}{5}\right] = \frac{16}{25}.$$

But $||p||^2 + ||q||^2 = \frac{2}{3} + \frac{2}{5} = \frac{16}{25}$.

$$\therefore \quad ||p+q||^2 = ||p||^2 + ||q||^2$$

$\therefore$ The generalised Pythagorus theorem is true.

17. Orthogonal Complements

Definition : Let W be a subspace of an inner product space in V. A vector u in V is said to be orthogonal to W if it is orthogonal to every vector in W. The set of all vectors in V that are orthogonal to W is called the **orthogonal complement of W**. The orthogonal complement of W is denoted by $W^{\perp}$ (read "W perp").

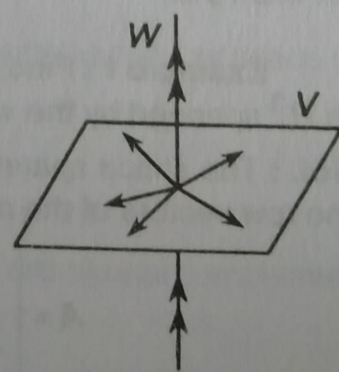


Fig. 5.1

Example : (i) Let W be the line $y = 3x$ in R^2, find an equation for $W^{\perp}$.

(ii) Let W be the plane in R^3 with the equation $x - 2y + 3z = 0$. Find the parametric equations of $W^{\perp}$.

Sol. : (i) The equation of the line is $y = 3x$.

The line perpendicular to it in R^2 is $y = -\frac{1}{3}x$.

$\therefore W^{\perp}$ is $y = -\frac{1}{3}x$. [Fig. 9.2 (a)]

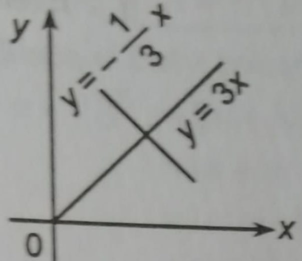


Fig. 5.2 (a)

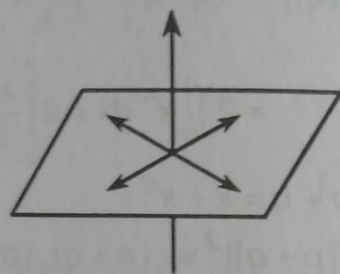


Fig. 5.2 (b)

(ii) The plane $x - 2y + 3z = 0$ in R^3 is perpendicular to the line

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{3} = t$$

$\therefore \quad x = t, \; y = -2\,t, \; z = 3\,t.$

$\therefore W^{\perp}$ is $x = t, \; y = -2\,t, \; z = 3\,t$. [Fig. 9.2 (b)]

Theorem : If A is an $m \times n$ matrix, then

(a) the null-space of A and the row space of A are orthogonal complements in R^n.

(b) the null-space of A' and the column space of A are orthogonal complements in R^n with respect to the Euclidean inner product.

Proof : We shall accept the theorem without proof. But, we shall verify it through an example.

Example 1 : Find a basis for the orthogonal complement of the subspace in R^3 spanned by the vectors $v_1 = (1, -1, 3)$, $v_2 = (5, -4, -4)$, $v_3 = (7, -6, 2)$.

Sol. : The space spanned by v_1, v_2, v_3 is the same as the space spanned by the row vectors of the matrix

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix}$$

We obtain the basis for the null-space of A. [See § 10, page 4-38]

We first reduce A to row-echelon form

By $\begin{matrix} R_2 - 5R_1 \\ R_3 - 7R_1 \end{matrix} \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 1 & -19 \end{bmatrix}$ By $R_3 - R_2 \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}$

The corresponding homogeneous system of equations is

$$x_1 - x_2 + 3x_3 = 0$$
$$x_2 - 19\,x_3 = 0$$

Putting $x_3 = t,\ x_2 = 19\,t,\ x_1 = 19\,t - 3\,t = 16\,t.$

$$\therefore \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16\,t \\ 19\,t \\ t \end{bmatrix} = t\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$$

The basis for the null-space is $u = \begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$.

This is the orthogonal complement of the row space of A and hence, of the space spanned by the given vectors.

It can be easily verified that u_1 is orthogonal to v_1, v_2, v_3.

[For $\langle u, v_1\rangle = (16, 19, 1)\cdot(1, -1, 3)$
$= (16 - 19 + 3) = 0$
$\langle u, v_2\rangle = (16, 19, 1)\cdot(5, -4, -4)$
$= (80 - 76 - 4) = 0$
$\langle u, v_3\rangle = (16, 19, 1)\cdot(7, -6, 2)$
$= (112 - 114 + 2) = 0$]

Example 2 : Find a basis for the orthogonal complement of the sub-space of R^n spanned by the vectors $W_1 = (1, 4, 5, 2)$, $W_2 = (2, 1, 3, 0)$, $W_3 = (-1, 3, 2, 2)$.

Sol. : The space W spanned by w_1, w_2, w_3, w_4 is the same as the row space of

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$$

By the above theorem the null-space of A is the orthogonal complement of W.

We first reduce A to row echelon form.

$$\text{By } \begin{matrix} R_2 - 2R_1 \\ R_3 + R_1 \end{matrix} \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 4 \end{bmatrix} \quad \text{By } \begin{matrix} R_3 + R_2 \\ -R_2/7 \end{matrix} \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & 4/7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad x_1 + 4x_2 + 5x_3 + 2x_4 = 0$$
$$x_2 + x_3 + \frac{4}{7}x_4 = 0$$

Putting $x_3 = s,\ x_4 = t,\ x_2 = -s - \frac{4}{7}t,$

But $x_1 = -4x_2 - 5x_3 - 2x_4$

$\therefore \quad x_1 = 4s + \frac{16}{7}t - 5s - 2t = -s + \frac{2}{7}t$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s+(2/7)t \\ -s-(4/7)t \\ s+0 \\ 0+t \end{bmatrix} = s\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2/7 \\ -4/7 \\ 0 \\ 1 \end{bmatrix}$$

The bases for the null-space are

$$u_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 2/7 \\ -4/7 \\ 0 \\ 1 \end{bmatrix}$$

Since this is the null-space of A, it forms the orthogonal complement of W.

[You can verify that u_1, u_2 are orthogonal to w_1, w_2, w_3, w_4.

For $\langle u_1, w_1 \rangle = (-1, -1, 1, 0) \cdot (1, 4, 5, 2)$
$= (-1 - 4 + 5 + 0) = 0$

$\langle u_1, w_2 \rangle = (-1, -1, 1, 0) \cdot (2, 1, 3, 0)$
$= (-2 - 1 + 3 + 0) = 0$

$\langle u_1, w_3 \rangle = (-1, -1, 1, 0) \cdot (-1, 3, 2, 2)$
$= (1 - 3 + 2 + 0) = 0$

We leave to you to verify that u_2 is orthogonal to w_1, w_2, w_3, w_4.]

EXERCISE - II

1. Determine whether the following vectors are orthogonal

   (i) $u = (1, 2, 3)$, $\quad v = (4, 1, -2)$

   (ii) $u = (-2, -2, -2)$, $\quad v = (3, 3, -3)$

   (iii) $u = (u_1, 0, u_3)$, $\quad v = (0, v_2, 0)$

   (iv) $u = (0, 3, -2, 4)$, $\quad v = (1, -2, 3, 0)$

   [**Ans.** : (i) Yes, (ii) No, (iii) Yes, (iv) Yes.]

2. Determine whether the following matrices are orthogonal to

$$A = \begin{bmatrix} 3 & 2 \\ -2 & 4 \end{bmatrix}$$

(i) $\begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 2 \\ 5 & 0 \end{bmatrix}$ [**Ans. :** (i) Yes, (ii) Yes.]

3. Find the cosine of the angle between $A = \begin{bmatrix} 2 & 4 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 \\ 4 & 2 \end{bmatrix}$.

[**Ans. :** 2 / 15]

4. Verify Cauchy-Schwartz inequality for

(i) $u = \begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$, $v = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$ using inner product

$\langle u, v \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$

(ii) $p = 1 - 2x + x^2$ and $q = 2 - 4x^2$ using in inner product

$\langle p, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$.

5. Check whether p, q are orthogonal and then verify generalised theorem of Pythagoras for $p = 1 - x + 3x^2$, $q = 3 + x^2$ with respect tc the inner product of $\langle p, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$.

6. Verify Cauchy-Schwartz inequality for $u = \begin{bmatrix} 2 & 3 \\ 6 & 1 \end{bmatrix}$, $v = \begin{bmatrix} 1 & 0 \\ 4 & 5 \end{bmatrix}$ with respect to the inner product $\langle u, v \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3 + u_4 v_4$.

7. Find a basis for the orthogonal complement of the sub-space of R^n spanned by the vectors

(i) $v_1 = (2, 0, -1)$, $v_2 = (4, 0, -2)$

(ii) $v_1 = (2, 2, -1, 0, 1)$, $v_2 = (-1, -1, 2, -3, 1)$,
$v_3 = (1, 1, -2, 0, -1)$, $v_4 = (0, 0, 1, 1, 1)$.

[**Ans. : (i)** (1, 0, 1), (1/2, 0, 1), **(ii)** (–1, 1, 0, 0, 0), (–1, 0, –1, 0, 1)]

8. Let R^4 have the Euclidean inner product and let $u = (-1, 1, 0, 2)$. Determine whether the vector u is orthogonal to the subspace spanned by the vectors $w_1 = (0, 0, 0, 0)$, $w_2 = (2, -2, 4, 0)$, $w_3 = (6, 0, 0, -3)$. [**Ans. :** Yes]

9. Find the cosine of the angle between the following vectors.

(i) (–2, 5, 11), (4, 6, – 2) (ii) (1, 1, 0, 0), (3, 3, 3, 3)

(iii) (2, 1, 7, –1), (4, 0, 0, 0) [**Ans. : (i)** 0, **(ii)** $\frac{1}{\sqrt{2}}$, **(iii)** $\frac{2}{\sqrt{55}}$.]

10. Show that $p = 1 - x + 3x^2$ and $q = 3x - x^2$ are orthogonal with respect to the inner product defined by $\langle p, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$ where $p = a_0 + a_1 x + a_2 x^2$ and $q = b_0 + b_1 x + b_2 x^2$.

11. Show that the following matrices are orthogonal to $A = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$.

(i) $\begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$ (ii) $\begin{bmatrix} 3 & 1 \\ -1 & -2 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 4 \\ -4 & -3 \end{bmatrix}$

if the inner product is defined by $\langle u, v\rangle = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$

for $u = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$, $v = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$.

12. If $\langle f, g\rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$,

show that $f_1(x) = 1$, $f_2(x) = x$ are orthogonal with respect to the above inner product. Find a, b such that $f_3(x) = -1 + ax + bx^2$ is orthogonal to both $f_1(x)$ and $f_2(x)$ with respect to the inner product. **[Ans. :** $f_3(x) = 3x^2 - 1$ **]**

13. Let $A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 3 & 5 & 0 & 4 \\ 1 & 1 & 2 & 0 \end{bmatrix}$

Find the bases for the row space and null-space of A. Verify that every vector in row space is orthogonal to every vector in null-space.

18. Orthonormal Basis

In this section we shall learn orthonormal vectors, orthonormal basis. Then we shall learn how to construct orthogonal basis from an ordinary basis of an inner product space by using Gram-Schmidt process.

Definition : A set of vectors in an inner product space is called an **orthogonal set** if all the distinct vectors are orthogonal. If the norm of each vector in the orthogonal set is 1, then it is called **orthonormal set**.

Example 1 : Check whether the following sets of vectors in R^3 are orthogonal with respect to the Euclidean inner product.

(a) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$

(b) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

Sol. : (a) Taking the inner products

$$\langle u_1, u_2\rangle = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$= \left(\frac{1}{\sqrt{6}} + 0 + \frac{1}{\sqrt{6}}\right) \neq 0$$

$$\langle u_1, u_3 \rangle = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) \cdot \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{2} + 0 - \frac{1}{2} = 0$$

$$\langle u_2, u_3 \rangle = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \cdot \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}} = 0$$

Since $\langle u_1, u_2 \rangle \neq 0$, the vectors are not orthogonal.

(b) $$\langle u_1, u_2 \quad \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) \cdot \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$$

$$= \frac{4}{3} - \frac{2}{3} - \frac{2}{3} = 0$$

$$\langle u_1, u_3 \rangle = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) \cdot \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$= \frac{2}{3} - \frac{4}{3} + \frac{2}{3} = 0$$

$$\langle u_2, u_3 \rangle = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) \cdot \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$= \frac{2}{3} + \frac{2}{3} - \frac{4}{3} = 0$$

Since all inner products are zero, the vectors are orthogonal.

Example 2 : Check whether the above vectors are orthonormal.

Sol. : (a) Since the vectors are not orthogonal, they are not orthonormal.

(b) $$||u_1|| = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} = 1$$

$$||u_2|| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}} = 1$$

$$||u_3|| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = 1$$

Since the vectors u_1, u_2, u_3 are orthogonal and their norms are 1, they are orthonormal.

Example 3 : Which of the following sets are orthonormal with respect to the inner product P_2 defined by

$$\langle p, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$$

where $p = a_0 + a_1 x + a_2 x^2$, $q = b_0 + b_1 x + b_2 x^2$?

(a) $x, \quad \frac{1}{2}+x^2, \quad 1-x-\frac{1}{2}x^2$

(b) $\frac{2}{3}-\frac{2}{3}x+\frac{1}{3}x^2, \quad \frac{2}{3}+\frac{1}{3}x-\frac{2}{3}x^2, \quad \frac{1}{3}+\frac{2}{3}x+\frac{2}{3}x^2$

Sol. : (a) We have

$$\langle u_1, u_2\rangle = (0+1+0)\cdot\left(\frac{1}{2}+0+1\right) = 0$$

$$\langle u_1, u_3\rangle = (0+1+0)\cdot\left(1-1-\frac{1}{2}\right) = -1 \neq 0$$

$$\langle u_2, u_3\rangle = \left(\frac{1}{2}+0+1\right)\cdot\left(1-1-\frac{1}{2}\right) = \frac{1}{2}+0-\frac{1}{2} = 0$$

Since all inner products are not zero, the vectors are not orthogonal and hence not orthonormal.

(b) $$\langle u_1, u_2\rangle = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)\cdot\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$$

$$= \frac{4}{9}-\frac{2}{9}-\frac{2}{9} = 0$$

$$\langle u_1, u_3\rangle = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)\cdot\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$= \frac{2}{9}-\frac{4}{9}+\frac{2}{9} = 0$$

$$\langle u_2, u_3\rangle = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)\cdot\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

$$= \frac{2}{9}+\frac{2}{9}-\frac{4}{9} = 0$$

$\therefore$ Vectors are orthogonal.

Further, $$||u_1|| = \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}} = 1$$

$$||u_2|| = \sqrt{\frac{4}{9}+\frac{1}{9}+\frac{4}{9}} = 1$$

$$||u_3|| = \sqrt{\frac{1}{9}+\frac{4}{9}+\frac{4}{9}} = 1$$

Since the vectors are orthogonal and have norms 1, they are orthonormal. [Note that this i the same example as Ex. 1 (b).]

Example 4 : Check whether the following vectors are orthogonal with respect to the inner product of two matrices $u = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$, $v = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$ defined by $\langle u, v \rangle = \text{tr}(v' u)$.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2/3 \\ 2/3 & -2/3 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 2/3 \\ -2/3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2/3 \\ 2/3 & 2/3 \end{bmatrix}$

Sol. : (a) We know that

$$\langle u, v \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4$$

$$\therefore \quad \langle u, v \rangle = 0 + 0 + 0 + 0 = 0$$

$\therefore$ u, v are orthogonal.

(b) $$\langle u, v \rangle = 0 + \frac{4}{9} - \frac{4}{9} + 0 = 0$$

$\therefore$ u, v are orthogonal.

Example 5 : Verify that

$$v_1 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right), \quad v_2 = \left(\frac{4}{5}, \frac{3}{5}, 0\right), \quad v_3 = (0, 0, 1)$$

are orthonormal and if $u = (1, -1, 2)$ then express u as a linear combination of v_1, v_2, v_3.

Sol. : We have

$$\langle v_1, v_2 \rangle = -\frac{12}{5} + \frac{12}{5} + 0 = 0$$

$$\langle v_1, v_3 \rangle = 0 + 0 + 0 = 0$$

$$\langle v_2, v_3 \rangle = 0 + 0 + 0 = 0$$

Hence, v_1, v_2, v_3 are orthogonal.

Further, $$||u_1|| = \sqrt{\frac{9}{25} + \frac{16}{25} + 0} = 1$$

$$||u_2|| = \sqrt{\frac{16}{25} + \frac{9}{25} + 0} = 1$$

$$||u_3|| = \sqrt{0 + 0 + 1} = 1$$

Since, the vectors are orthogonal and of norms one, they are orthonormal.

Now, $$u = \langle u, v_1 \rangle v_1 + \langle u, v_2 \rangle v_2 + \langle u, v_3 \rangle v_3$$

$$= \left[(1, -1, 2) \cdot \left(-\frac{3}{5}, \frac{4}{5}, 0\right)\right] v_1$$

$$+ \left[(1, -1, 2) \cdot \left(\frac{4}{5}, \frac{3}{5}, 0\right)\right] v_2$$

$$+ [(1, -1, 2) \cdot (0, 0, 1)] v_3$$

$$\therefore \quad u = \left(-\frac{3}{5}-\frac{4}{5}+0\right)\left(-\frac{3}{5}, \frac{4}{5}, 0\right)+\left(\frac{4}{5}-\frac{3}{5}+0\right)\left(\frac{4}{5}, \frac{3}{5}, 0\right)$$
$$+(0+0+2)(0, 0, 1)$$
$$= -\frac{7}{5}v_1+\frac{1}{5}v_2+2v_3.$$

(You can verify the equality.)

Example 6 : Find the coordinate vector of u with respect to the orthonormal basis $\{v_1, v_2, v_3\}$ where

$$u = (-1, 0, 2), \; v_1 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \; v_2 = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \; v_3 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

Sol. : We have already checked that the set $\{v_1, v_2, v_3\}$ is an orthonormal set [Ex. 1 (b), page 5-28]

Now, $\quad u = \langle u, v_1\rangle v_1 + \langle u, v_2\rangle v_2 + \langle u, v_3\rangle v_3$

$$= (-1, 0, 2)\cdot\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)v_1 + (-1, 0, 2)\cdot\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)v_2$$
$$+(-1, 0, 2)\cdot\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)v_3$$
$$= 0\,v_1 + \left(-\frac{2}{3}-\frac{4}{3}\right)v_2 + \left(-\frac{1}{3}+\frac{4}{3}\right)v_3$$
$$= 0\,v_1 - 2\,v_2 + v_3$$
$$\therefore \quad (u)_S = (0, -2, 1).$$

EXERCISE - III

1. Check whether the following sets of vectors in R^3 are orthogonal with respect to the Euclidean inner product.

(a) $(0, 1), (2, 0)$ (b) $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}\right)$

(c) $(1, 0, 0), (0, 1, 0), (0, 0, 1)$

(d) $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right)$ **[Ans. : All]**

2. Check whether the following set of polynomials is orthonormal with respect to the inner product in P_2.

$$1, \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}x^2, x^2$$

[Ans. : No]

3. Check whether the following sets of matrices are orthogonal with respect to the inner product on M_{22}.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2/3 \\ 1/3 & -2/3 \end{bmatrix}, \begin{bmatrix} 0 & 2/3 \\ -2/3 & 1/3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ [**Ans. :** Both]

4. Verify that the vectors

$$v_1 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right),\ v_2 = \left(\frac{4}{5}, \frac{3}{5}, 0\right),\ v_3 = (0, 0, 1)$$

form an orthonormal basis in R^3 with respect to the Euclidean inner product. Express the vector $(3, -7, 4)$ as a linear combination of v_1, v_2, v_3.

[**Ans. :** $\left(-\frac{37}{5} v_1 - \frac{9}{5} v_2 + 4 v_3\right)$

5. Check whether the following vectors are orthonormal and then find the coordinates of w with respect to u_1, u_2.

$$u_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right),\ v_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),\ w = (3, 7)$$

[**Ans. :** $(w)_s = \left(-2\sqrt{2}, 5\sqrt{2}\right)$]

6. If $v_1 = (0, 1, 0)$, $v_2 = \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$, $v_3 = \left(\frac{3}{5}, 0, \frac{4}{5}\right)$, show that the vectors are orthonormal. Express $u = (1, 1, 1)$ as a linear combination of v_1, v_2, v_3 and find the coordinate vectors of u with respect to $S = \{v_1, v_2, v_3\}$.

[**Ans. :** $u = 1 \cdot u_1 - \frac{1}{5} u_2 + \frac{7}{5} u_3$, $(u)_s = \left(1, -\frac{1}{5}, \frac{7}{5}\right)$

19. Projection Theorem

We now state the following theorem known as projection theorem.

Projection Theorem

If W is a finite dimensional subspace of an inner product space V, then every vector u in V can be expressed uniquely as

$$\boxed{u = w_1 + w_2}$$

where w_1 is in W and w_2 is in $W^{\perp}$.

(For $W^{\perp}$ see § 17, page 5-23)

Proof : We shall accept this theorem without proof.

Example 1 : Let $W = \text{span}\left\{(0, 1, 0), \left(-\frac{4}{5}, 0, \frac{3}{5}\right)\right\}$.

Express, $w = (1, 1, 1)$ in the form $w = w_1 + w_2$ where $w_1 \in W$ and $w_2 \in W^{\perp}$.

Sol. : Let $v_1 = (0, 1, 0)$, $v_2 = \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$.

Then projection of w on W

$$\text{proj}_W w = \langle w, v_1 \rangle v_1 + \langle w, v_2 \rangle v_2$$

$$= [(1, 1, 1) \cdot (0, 1, 0)] v_1 + (1, 1, 1) \cdot \left(-\frac{4}{5}, 0, \frac{3}{5}\right) v_2$$

$$= 1 \cdot v_1 - \frac{1}{5} v_2$$

$$= 1(0, 1, 0) - \frac{1}{5}\left(-\frac{4}{5}, 0, \frac{3}{5}\right)$$

$$w_1 = \left(\frac{4}{25}, 1, -\frac{3}{25}\right)$$

This is the component of w which is in W.

The component of w orthogonal to W is

$$\text{proj}_{W^{\perp}} w = w - \text{proj}_W w$$

$$= (1, 1, 1) - \left(\frac{4}{25}, 1, -\frac{3}{25}\right)$$

$$w_2 = \left(\frac{21}{25}, 0, \frac{28}{25}\right)$$

This is the component of w which is in $W^{\perp}$.

Hence, $w = w_1 + w_2$ (w_1 is in W and w_2 is in $W^{\perp}$)

$$= \left(\frac{4}{25}, 1, -\frac{3}{25}\right) + \left(\frac{21}{25}, 0, \frac{28}{25}\right)$$

(Verify that $w_1 + w_2 = (1, 0, 1)$.)

Example 2 : Let $W = \text{span}\left\{(0, 1, 0), \left(-\frac{4}{5}, 0, \frac{3}{5}\right)\right\}$.

Express $w = (1, 2, 3)$ in the form of $w = w_1 + w_2$, where $w_1 \in W$ and $w_2 \in W^{\perp}$.

Sol. : Let $v_1 = (0, 1, 0)$, $v_2 = \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$.

Then the projection of w on W.

$$\text{proj}_W w = \langle w, v_1 \rangle v_1 + \langle w, v_2 \rangle v_2$$

$$= [(1, 2, 3) \cdot (0, 1, 0)]\, v_1 + \left[(1, 2, 3) \cdot \left(-\frac{4}{5}, 0, \frac{3}{5}\right)\right] v_2$$

$$= 2\, v_1 + 1 \cdot v_2$$

$$= 2(0, 1, 0) + 1 \cdot \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$$

$$w_1 = \left(-\frac{4}{5}, 2, \frac{3}{5}\right)$$

This is the component of w in W.

The component of w orthogonal to W is

$$\text{proj}_{W^\perp} w = w - \text{proj}_W w$$

$$\therefore \quad \text{proj}_{W^\perp} w = (1, 2, 3) - \left(-\frac{4}{5}, 2, \frac{3}{5}\right)$$

$$w_2 = \left(\frac{9}{5}, 0, \frac{12}{5}\right)$$

This is the component of w in $W^\perp$.

Hence, $w = w_1 + w_2$ (where $w_1 \in W$ and $w_2 \in W^\perp$)

$$= \left(-\frac{4}{2}, 2, \frac{3}{5}\right) + \left(\frac{9}{5}, 0, \frac{12}{5}\right)$$

Example 3 : Let R^4 have the Euclidean inner product. Express $w = (-1, 2, 6, 0)$ in the form $w = w_1 + w_2$ where $w_1 \in W$ and $w_2 \in W^\perp$ where w is spanned by $v_1 = (-1, 0, 1, 2)$ and $v_2 = (0, 1, 0, 1)$.

Sol. : The project of u on W

$$\text{proj}_W w = \langle w, v_1 \rangle\, v_1 + \langle w, v_2 \rangle\, v_2$$

$$= [(-1, 2, 6, 0) \cdot (-1, 0, 1, 2)]\, v_1$$

$$+ [(-1, 2, 6, 0) \cdot (0, 1, 0, 1)]\, v_2$$

$$= 7v_1 + 2v_2$$

$$= 7\,(-1, 0, 1, 2) + 2\,(0, 1, 0, 1)$$

$$\therefore \quad w_1 = (-7, 2, 7, 16)$$

The component of w orthogonal to W is

$$\text{proj}_{W^\perp} w = w - \text{proj}_W w$$

$$= (-1, 2, 6, 0) - (-7, 2, 7, 16)$$

$$\therefore \quad w_2 = (6, 0, -1, -16)$$

$\therefore \quad w = w_1 + w_2 \quad$ where $w_1 \in W$ and $w_2 \in W^\perp$

$$= (-7, 2, 7, 16) + (6, 0, -1, -16).$$

20. Gram-Schmidt Process

Gram-Schmidt process gives us a method of finding orthonormal vectors.

Let for convenience V be a non-zero inner product space in R^3 and let $u = \{u_1, u_2, u_3\}$ be any base for V. The following steps lead to an orthogonal basis $\{v_1, v_2, v_3\}$ for V.

Step 1 : Let $v_1 = u_1$.

Step 2 : As shown in the Fig. 5.3 (a), we obtain a vector v_2, orthogonal to v_1.

$$v_2 = u_2 - \text{proj } u_2 \text{ which is given by}$$

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{||v_1||^2} v_1$$

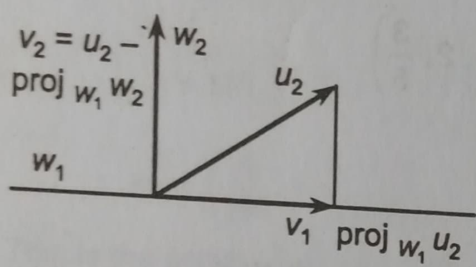


Fig. 5.3 (a)

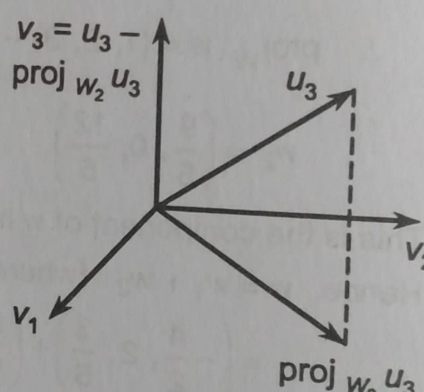


Fig. 5.3 (b)

Step 3 : To find v_3 that is orthogonal to both v_1 and v_2, we find u_3 orthogonal to the space w_2 spanned by v_1 and v_2 which is given by

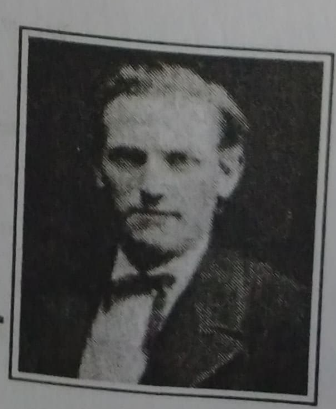
$$v_3 = v_3 - \text{proj } u_3$$

$$= u_3 - \frac{\langle u_3, v_1\rangle}{||v_1||^2} v_1 - \frac{\langle u_2, v_2\rangle}{||v_2||^2} v_2$$

Thus, we get the orthogonal set $\{v_1, v_2, v_3\}$.

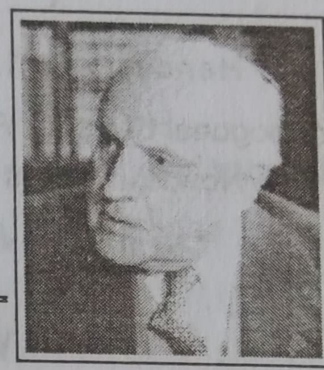
Jörgen Pederson Gram (1850-1916)

Jörgen Pederson Gram was an insurance expert in Denmark. He got his master's degree in mathematics with specialisation in then the new developing subject modern algebra. He then joined the Hafnia Life Insurance Company where he developed mathematical foundations of accident insurance. During this period he got his Ph.D. on "On Series Development Utilising the Least Square Methods". On this thesis Gram-Schmidt process is based. In 1910 he became



director of the Danish (Denmark) Insurance Board. Gram eventually developed interest in number theory and won a gold medal from "The Royal Danish Society of Sciences And Letters". He had keen interest in the interplay between applied mathematics and pure mathematics, which led to four treatises on Danish forest management. Unfortunately he died in a an accident while going to attend a meeting of the Royal Danish Society.

Erhardt Schmidt (1876-1959)



Erhardt Schmidt was a reputed German mathematician. He got his Ph.D. degree from Göttingen University, Germany in 1905. He was a student of another giant German mathematician David Hilbert. He then went to teach at Berlin University in 1917 and stayed there for the rest of his life. He made important contributions to various mathematical fields. Schmidt first described 'Gram-Schmidt process' in a paper on integral equations published in 1907. He is credited to have developed the general concept "Hilbert Space" which is fundamental in the study of infinite-dimensional vector spaces.

Example 1 : Let R^3 have the Euclidean inner product. Use Gram-Schmidt process to transform the basis $\{u_1, u_2, u_3\}$ into orthonormal bases where $u_1 = (1, 1, 1)$, $u_2 = (0, 1, 1)$, $u_3 = (0, 0, 1)$.

Sol. : Step 1 : $v_1 = u_1 = (1, 1, 1)$.

Step 2 : $v_2 = u_2 - \text{proj } u_2$

$$= u_2 - \frac{\langle u_2, v_1\rangle}{||v_1||^2} v_1$$

Now, $\langle u_2, v_1\rangle = 0 + 1 + 1 = 2$ and $|| v_1 ||^2 = 1 + 1 + 1 = 3$

$$\therefore \quad v_2 = (0, 1, 1) - \frac{2}{3}(1, 1, 1) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

Step 3 : $v_3 = u_3 - \text{proj } u_3$

$$= u_3 - \frac{\langle u_3, v_1\rangle}{||v_1||^2} v_1 - \frac{\langle u_3, v_2\rangle}{||v_2||^2} v_2$$

Now, $\langle u_3, v_1\rangle = 0 + 0 + 1 = 1$

$$\langle u_3, v_2\rangle = 0 + 0 + \frac{1}{3} = \frac{1}{3}$$

$|| v_1 ||^2 = 3$ as before.

$$|| v_2 ||^2 = \frac{4}{9} + \frac{1}{9} + \frac{1}{9} = \frac{6}{9} = \frac{2}{3}$$

$$\therefore \quad v_3 = (0, 0, 1) - \frac{1}{3}(1, 1, 1) - \frac{1/3}{2/3}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= (0, 0, 1) - \frac{1}{3}(1, 1, 1) - \frac{1}{2}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

Hence, $v_1 = (1, 1, 1)$, $v_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $v_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$ form the orthogonal basis for R^3.

Now, the norms of these vectors are

$$\|v_1\| = \sqrt{1+1+1} = \sqrt{3}$$

$$\|v_2\| = \sqrt{\frac{4}{9}+\frac{1}{9}+\frac{1}{9}} = \frac{\sqrt{6}}{3}$$

$$\|v_3\| = \sqrt{0+\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{2}{4}} = \frac{1}{\sqrt{2}}$$

And hence orthonormal basis for R^3 is

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$q_3 = \frac{v_3}{\|v_3\|} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

Example 2 : Let R^3 have the Euclidean inner product. Use Gram-Schmidt process to transform the basis $\{u_1, u_2, u_3\}$ into an orthonormal basis where $u_1 = (1, 1, 1)$, $u_2 = (-1, 1, 0)$, $u_3 = (1, 2, 1)$.

Sol. : Step 1 : $v_1 = u_1 = (1, 1, 1)$

Step 2 : $v_2 = u_2 - \text{proj } u_2$

$$= u_2 - \frac{\langle u_2, u_1\rangle}{\|v_1\|^2} \cdot v_1$$

Now, $\langle u_2, v_1\rangle = -1 + 1 - 0 = 0$

and $\| v_1 \|^2 = 1 + 1 + 1 = 3$

$$\therefore \quad v_2 = (-1, 1, 0) - \frac{0}{3}(1, 1, 1) = (-1, 1, 0)$$

Step 3 : $v_3 = u_3 - \text{proj } u_3$

$$= u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} \cdot v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2} \cdot v_2$$

Now, $\langle u_3, v_1\rangle = 1 + 2 + 1 = 4$

$\langle u_3, v_2\rangle = -1 + 2 + 0 = 1$

$\| v_1 \|^2 = 3$ as before.

$\| v_2 \|^2 = 1 + 1 + 0 = 2$

$$\therefore \quad v_3 = (1, 2, 1) - \frac{4}{3}(1, 1, 1) - \frac{1}{2}(-1, 1, 0)$$

$$= \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right)$$

Hence, $v_1 = (1, 1, 1)$, $v_2 = (-1, 1, 0)$, $v_3 = \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right)$ form orthogonal basis for R^3.

Now, the norms of these vectors are

$$\|v_1\|^2 = \sqrt{1+1+1} = \sqrt{3}$$

$$\|v_2\|^2 = \sqrt{1+1+0} = \sqrt{2}$$

$$\|v_3\|^2 = \sqrt{\frac{1}{36}+\frac{1}{36}+\frac{1}{9}} = \sqrt{\frac{6}{36}} = \frac{1}{\sqrt{6}}$$

Hence, the orthonormal basis for R^3 is

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$$

$$q_3 = \frac{v_3}{\|v_3\|} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right) \quad \left[\because \frac{\sqrt{6}}{3} = \frac{\sqrt{6}\cdot\sqrt{6}}{3\sqrt{6}} = \frac{6}{3\sqrt{6}} = \frac{2}{\sqrt{6}}\right]$$

Example 3 : Let R^3 have the Euclidean inner product. Use the Gram-Schmidt process to transform the basis $\{u_1, u_2, u_3\}$ into orthonormal basis where $u_1 = (1, 0, 0)$, $u_2 = (3, 7, -2)$, $u_3 = (0, 4, 1)$.

Sol. : Step 1 : $v_1 = u_1 = (1, 0, 0)$

Step 2 : $v_2 = u_2 - \text{proj}\ u_2 = u_2 - \dfrac{\langle u_2, u_1\rangle}{\|v_1\|^2}\bullet v_1$

Now, $\langle u_2, v_1\rangle = 3 + 0 + 0 = 3$

and $\| v_1 \|^2 = 1 + 0 + 0 = 1$

$$\therefore \quad v_2 = (3, 7, -2) - \frac{3}{1}(1, 0, 0) = (0, 7, -2)$$

Step 3 : $v_3 = u_3 - \text{proj}\ u_3$

$$= u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2}\bullet v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2}\bullet v_2$$

Now, $\langle u_3, v_1 \rangle = 0 + 0 + 0 = 0$

$\langle u_3, v_2 \rangle = 0 + 28 - 2 = 26$

$\| v_1 \|^2 = 1$ as before.

$\| v_2 \|^2 = 0 + 49 + 4 = 53$

$$\therefore \quad v_3 = (0, 4, 1) - 0 - \frac{26}{53}(0, 7, -2)$$

$$= \left(0, \frac{30}{53}, \frac{105}{53}\right)$$

Hence, $v_1 = (1, 0, 0)$, $v_2 = (0, 7, -2)$, $v_3 = \left(0, \frac{30}{53}, \frac{105}{53}\right)$ form orthogonal basis for R^3.

Now, the norms of these vectors are

$$\|v_1\|^2 = \sqrt{1 + 0 + 0} = 1$$

$$\|v_2\|^2 = \sqrt{0 + 49 + 4} = \sqrt{53}$$

$$\|v_3\|^2 = \sqrt{0 + \frac{900}{53^2} + \frac{11025}{53^2}} = \frac{15}{\sqrt{53}}$$

And hence, the orthonormal basis for R^3 is

$$q_1 = \frac{v_1}{\|v_1\|} = (1, 0, 0)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \left(0, \frac{7}{\sqrt{53}}, -\frac{2}{\sqrt{53}}\right)$$

$$q_3 = \frac{v_3}{\|v_3\|} = \left(0, \frac{2}{\sqrt{53}}, \frac{7}{\sqrt{53}}\right)$$

Example 4 : Let R^4 have the Euclidean inner product. Use Gram-Schmidt Process to transform the basis $\{u_1, u_2, u_3\}$ into orthogonal basis, where $u_1 = \{1, 0, 1, 1\}$, $u_2 = \{-1, 0, -1, 1\}$, $u_3 = \{0, -1, 1, 1\}$.

Sol. : Step 1 : $v_1 = u_1 = (1, 0, 1, 1)$

Step 2 : $v_2 = u_2 - \text{proj } u_2$

$$= u_2 - \frac{\langle u_2, u_1 \rangle}{\|v_1\|^2} \cdot v_1$$

Now, $\langle u_2, v_1 \rangle = -1 + 0 - 1 + 1 = -1$

and $\| v_1 \|^2 = 1 + 0 + 1 + 1 = 3$

$$\therefore \quad v_2 = (-1, 0, -1, 1) - \frac{-1}{3}(1, 0, 1, 1)$$

$$= \left(-\frac{2}{3}, 0, -\frac{2}{3}, \frac{4}{3}\right)$$

Step 3 : $v_3 = u_3 - \text{proj } u_3$

$$= u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} \cdot v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} \cdot v_2$$

Now, $\langle u_3, v_1 \rangle = 0 + 0 + 1 + 1 = 2$

$$\langle u_3, v_2 \rangle = (0, -1, 1, 1) \cdot \left(-\frac{2}{3}, 0, -\frac{2}{3}, \frac{4}{3}\right)$$

$$= 0 + 0 - \frac{2}{3} + \frac{4}{3} = \frac{2}{3}$$

$\| v_1 \|^2 = 3$ as before.

$$\|v_2\|^2 = \frac{4}{9} + 0 + \frac{4}{9} + \frac{16}{9} = \frac{24}{9} = \frac{8}{3}$$

$$\therefore \quad v_3 = (0, -1, 1, 1) - \frac{2}{3}(1, 0, 1, 1) - \frac{2/3}{8/3}\left(-\frac{2}{3}, 0, -\frac{2}{3}, \frac{4}{3}\right)$$

$$= (0, -1, 1, 1) - \frac{2}{3}(1, 0, 1, 1) - \frac{1}{4}\left(-\frac{2}{3}, 0, -\frac{2}{3}, \frac{4}{3}\right)$$

$$= (0, -1, 1, 1) + \left(-\frac{2}{3}, 0, -\frac{2}{3}, -\frac{2}{3}\right) + \left(\frac{1}{6}, 0, \frac{1}{6}, -\frac{1}{3}\right)$$

$$= \left(\frac{1}{2}, -1, \frac{1}{2}, 0\right)$$

Hence, the required orthogonal set is

$$v_1 = (1, 0, 1, 1), \quad v_2 = \left(-\frac{2}{3}, 0, -\frac{2}{3}, \frac{4}{3}\right),$$

$$v_3 = \left(\frac{1}{2}, -1, \frac{1}{2}, 0\right).$$

Example 5 : Let the vector space P_2 have the inner product defined by

$$\langle p, q \rangle = \int_{-1}^{1} p(x) \cdot q(x)\, dx$$

Use Gram-Schmidt process to transform the standard basis $S = \{1, x, x^2\}$ into an orthonormal basis.

Sol. : We have $u_1 = 1, \ u_2 = x, \ u_3 = x^2$.

Step 1 : $v_1 = u_1 = 1$

Step 2 : $v_2 = u_2 - \text{proj } u_2$

$$= u_2 - \frac{\langle u_2, u_1 \rangle}{\|v_1\|^2} \cdot v_1$$

Now, $\langle u_2, v_1 \rangle = \int_{-1}^{1} x \cdot 1 \cdot dx = 0$ [$\because$ Odd function] and $||v_1||^2 = \int_{-1}^{1} [v_1(x)]^2 \, dx = \int_{-1}^{1} 1^2 \cdot dx$

$$= 2\int_0^1 dx = 2[x]_0^1 = 2. \qquad [\because \text{ Even function }]$$

$$\therefore \quad v_2 = x - \frac{0}{2}(1) = x$$

Step 3 : $v_3 = u_3 - \text{proj } u_3$

$$= u_3 - \frac{\langle u_3, v_1\rangle}{||v_1||^2} \cdot v_1 - \frac{\langle u_3, v_2\rangle}{||v_2||^2} \cdot v_2$$

Now, $\langle u_3, v_1\rangle = \int_{-1}^{1} x^2 \cdot 1 \cdot dx = \int_{-1}^{1} x^2 \, dx$

$$= 2\int_0^1 x^2 \, dx = 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3} \qquad [\because \text{ Even function }]$$

$$\therefore \quad \langle u_3, v_2\rangle = \int_{-1}^{1} x^2 \cdot x \cdot dx = 0 \qquad [\because \text{ Odd function }]$$

$|| v_1 ||^2 = 2$ as before.

$$||v_2||^2 = \int_{-1}^{1} [v_2(x)]^2 \, dx = \int_{-1}^{1} x^2 \, dx$$

$$\therefore \quad ||v_2||^2 = 2\int_0^1 x^2 \, dx = 2\left[\frac{x^3}{3}\right]_0^1 = \frac{2}{3} \qquad [\because \text{ Even function }]$$

$$\therefore \quad v_3 = x^2 - \frac{2/3}{2} \cdot 1 - 0 = x^2 - \frac{1}{3}$$

Hence, $v_1 = 1, \; v_2 = x, \; v_3 = x^2 - \frac{1}{3}$.

Now, $||v_1|| = \sqrt{2}$

$||v_2|| = \sqrt{\frac{2}{3}}$ as above.

and $||v_3||^2 = \int_{-1}^{1} [v_3(x)]^2 \, dx = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx$

$$= \int_{-1}^{1} \left(x^4 - \frac{2x^2}{3} + \frac{1}{9}\right) dx = 2\int_0^1 \left(x^4 - \frac{2x^2}{3} + \frac{1}{9}\right) dx$$

$$= 2\left[\frac{x^5}{5} - \frac{2x^3}{9} + \frac{x}{9}\right]_0^1 = 2\left[\frac{1}{5} - \frac{2}{9} + \frac{1}{9}\right]$$

$$= 2\left[\frac{1}{5} - \frac{1}{9}\right] = \frac{8}{45}.$$

$$\therefore \quad \|v_3\| = \frac{2}{3}\sqrt{\frac{2}{5}}.$$

Hence, the orthonormal basis is

$$q_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}$$

$$q_2 = \frac{v_2}{\|v_2\|} = \frac{x}{\sqrt{2/3}} = \sqrt{\frac{3}{2}} \cdot x$$

$$q_3 = \frac{v_3}{\|v_3\|} = \frac{3}{2} \cdot \sqrt{\frac{5}{2}} \cdot \left(x^2 - \frac{1}{3}\right) = \frac{\sqrt{5}}{2\sqrt{2}}(3x^2 - 1)$$

21. Singular Value Decomposition (S.V.D.)

Singular Value Decomposition (SVD) is based on the following theorem.

Theorem : A rectangular matrix A_{mn} can be decomposed into the product of three matrices, an orthogonal matrix U_{mm}, a diagonal matrix D_{mn} and the transpose of another orthogonal matrix V_{nn}' i.e., we can have

$$\boxed{A_{mn} = U_{mm}\, D_{mn}\, V_{nn}'}$$

Since U and V are orthogonal matrices, we have $UU' = I$ and $VV' = I$. Further, the columns of U are orthonormal eigen vectors of AA' and the columns of V are orthonormal eigen vector of $A'A$ and D is a diagonal matrix whose elements are square roots of eigen values of U or V in descending order.

We accept this theorem without proof.

We shall learn the method of obtaining singular value decomposition of a given matrix through the following example.

Example : Find the singular value decomposition of the following matrix.

$$A = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

Sol. : Step 1 : We first find the eigen values of AA'.

$$\text{Now,} \quad A' = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$\therefore \quad B = AA' = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix}$$

$$\therefore \quad B = \begin{bmatrix} 11 & 1 \\ 1 & 11 \end{bmatrix}$$

The characteristic equation of B is

$$\begin{vmatrix} 11-\lambda & 1 \\ 1 & 11-\lambda \end{vmatrix} = 0$$

$\therefore\ (11-\lambda)^2 - 1 = 0 \qquad \therefore\ 121 - 22\lambda + \lambda^2 - 1 = 0$

$\therefore\ \lambda^2 - 22\lambda + 120 = 0 \qquad \therefore\ (\lambda - 10)(\lambda - 12) = 0$

$\therefore\ \lambda = 10, 12.$

Step 2 : Now, we find eigen vectors corresponding these eigen values.

For $\lambda = 10$, $[B - \lambda_1 I]X = 0$ gives

$$\begin{bmatrix} 11-10 & 1 \\ 1 & 11-10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By $\quad R_2 - R_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore\ x_1 + x_2 = 0$

Putting $x_2 = -t,\ x_1 = -x_2 = t.$

$\therefore$ The eigen vector is

$$X = \begin{bmatrix} t \\ -t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \therefore\ X = [1, -1]'.$$

For $\lambda = 12$, $[B - \lambda_2 I]X = 0$ gives

$$\begin{bmatrix} 11-12 & 1 \\ 1 & 11-12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By $\quad R_2 + R_1 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore\ -x_1 + x_2 = 0$

Putting $x_2 = t,\ x_1 = x_2 = t.$

$\therefore$ The eigen vector is

$$X = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \therefore\ X = [1, 1]'.$$

Step 3 : Now, we find orthogonal matrix U, which is obtained from eigen vectors of B arranged in descending order of the corresponding eigen values. The eigen vector corresponding to highest value is the first column, the eigen vector corresponding to the next immediate smaller eigen value is the second column and so on.

By this rule, the eigen vector corresponding to the highest eigen value $\lambda = 12$ is the first column and the eigen vector corresponding to the next eigen vector $\lambda = 10$ is the second column.

Thus, we have $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

We now convert this matrix into an orthonormal matrix by using Gram-Schmidt orthonormalisation process.

Here, we have $u_1 = (1, 1)$ and $u_2 = (1, -1)$.

Now, $v_1 = u_1 = (1, 1)$

$$v_2 = u_2 - \frac{\langle u_2, u_1 \rangle}{\|v_1\|^2} \cdot v_1$$

Now, $\langle u_2, u_1 \rangle = 1 - 1 = 0$

and $\| v_1 \|^2 = 1 + 1 = 2$

$$\therefore \quad v_2 = (1, -1) - \frac{0}{2}(1, 1) = (1, -1)$$

Now, norms of these vectors are

$$\|v_1\| = \sqrt{1+1} = \sqrt{2}$$

$$\|v_2\| = \sqrt{1+1} = \sqrt{2}$$

$\therefore$ The normalised vectors are

$$v_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$v_2 = \frac{v_2}{\|v_2\|} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

$$\therefore \quad U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

Step 4 : Now, we find orthonormal matrix V' which is obtained from $A'A$.

We have, $C = A'A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 2 \\ 0 & 10 & 4 \\ 2 & 4 & 2 \end{bmatrix}$

The characteristic equation is

$$\begin{vmatrix} 10-\lambda & 0 & 2 \\ 0 & 10-\lambda & 4 \\ 2 & 4 & 2-\lambda \end{vmatrix} = 0$$

$\therefore\ (10-\lambda)[(10-\lambda)(2-\lambda)-16]-0+2[0-2(10-\lambda)]=0$

$\therefore\ (10-\lambda)[20-12\lambda+\lambda^2-16]-4[10-\lambda]=0$

$\therefore\ (10-\lambda)(\lambda^2-12\lambda+4)-4(10-\lambda)=0$

$\therefore\ (10-\lambda)(\lambda^2-12\lambda+4-4)=0$

$\therefore\ (10-\lambda)(\lambda^2-12\lambda)=0$

$\therefore\ (10-\lambda)\lambda(\lambda-12)=0$

$\therefore\ \lambda=0, 10, 12.$

Now, we find the eigen vectors corresponding to these eigen values.

For $\lambda=0$, we have

$$\begin{bmatrix}10 & 0 & 2\\ 0 & 10 & 4\\ 2 & 4 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

By $\begin{array}{c}(1/2)R_1\\(1/2)R_2\\(1/2)R_3\end{array}\begin{bmatrix}5 & 0 & 1\\ 0 & 5 & 2\\ 1 & 2 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

By $R_{13}\begin{bmatrix}1 & 2 & 1\\ 0 & 5 & 2\\ 5 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

By $R_3-5R_1\begin{bmatrix}1 & 2 & 1\\ 0 & 5 & 2\\ 0 & -10 & -4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

By $R_3+2R_2\begin{bmatrix}1 & 2 & 1\\ 0 & 5 & 2\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\therefore\ x_1+2x_2+x_3=0,\ 5x_2+2x_3=0$

Putting $x_3=-5t$, $5x_2=10t\ \ \therefore\ x_2=2t.$

$\therefore\ x_1=-2x_2-x_3=-4t+5t=t.$

$\therefore\ X_1=\begin{bmatrix}t\\2t\\-5t\end{bmatrix}=t\begin{bmatrix}1\\2\\-5\end{bmatrix}\qquad\therefore\ X_1=[1,2,-5]'.$

For $\lambda=10$, we have

$$\begin{bmatrix}0 & 0 & 2\\ 0 & 0 & 4\\ 2 & 4 & -8\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\therefore\ 2x_3 = 0,\ 4x_3 = 0,\ 2x_1 + 4x_2 - 8x_3 = 0$

$\therefore\ x_3 = 0$ and $x_1 + 2x_2 - 4x_3 = 0$

$\therefore\ x_3 = 0$ and putting $x_2 = -t$, we get $x_1 = 2t$.

$$\therefore\ X_2 = \begin{bmatrix} 2t \\ -t \\ 0 \end{bmatrix} = t\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

or $\quad X_2 = [2, -1, 0]'$.

For $\lambda = 12$, we have

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -2 & 4 \\ 2 & 4 & -10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore\ -2x_1 + 2x_3 = 0 \qquad \therefore\ x_1 = x_3$

$-2x_2 + 4x_3 = 0 \qquad \therefore\ x_2 = 2x_3$

Putting $x_3 = t,\ x_1 = t,\ x_2 = 2t$.

$$\therefore\ X_3 = \begin{bmatrix} t \\ 2t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

or $\quad X_3 = [1, 2, 1]'$.

Writing these vectors, in desceding order of the size of the eigen values, we have

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 2 \\ 1 & 0 & -5 \end{bmatrix}$$

We now convert this matrix into an orthonormal matrix by using Gram-Schmidt orthonormalisation process.

Here, we have

$$u_1 = (1, 2, 1),\ u_2 = (2, -1, 0),\ u_3 = (1, 2, -5)$$

$$v_1 = u_1 = (1, 2, 1)$$

$$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2}\cdot v_1$$

$\therefore\ \langle u_2, v_1\rangle = (2, -1, 0)\,(1, 2, 1) = 0$

$\therefore\ v_2 = (2, -1, 0) - 0 = (2, -1, 0)$

$$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2}\cdot v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2}\cdot v_2$$

Now, $\langle u_3, v_1 \rangle = (1, 2, -5) \cdot (1, 2, 1) = 0$

$\langle u_3, v_2 \rangle = (1, 2, -5) \cdot (2, -1, 0) = 0$

$\therefore \quad v_3 = (1, 2, -5) - 0 - 0 = (1, 2, -5)$

Now, norms of these vectors are

$$\| v_1 \| = \sqrt{1+4+1} = \sqrt{6}$$

$$\| v_2 \| = \sqrt{4+1+0} = \sqrt{5}$$

$$\| v_3 \| = \sqrt{1+4+25} = \sqrt{30}$$

$\therefore$ The normalised vectors are

$$v_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$v_2 = \frac{v_2}{\|v_2\|} = \left(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, 0\right)$$

$$v_3 = \frac{v_3}{\|v_3\|} = \left(\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, -\frac{5}{\sqrt{30}}\right)$$

$$\therefore \quad V = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ \frac{1}{\sqrt{6}} & 0 & -\frac{5}{\sqrt{30}} \end{bmatrix}$$

We take the transpose of this

$$\therefore \quad V' = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & -\frac{5}{\sqrt{30}} \end{bmatrix}$$

Step 5 : Now, we obtain the diagonal matrix D, by taking square roots of non-zero eigen values *i.e.*, by taking D_{11} as the square root of the largest eigen value *i.e.*, $\sqrt{12}$, D_{22} as the square root of the next eigen value *i.e.*, $\sqrt{10}$. Further we have to take a zero column vector, so as to make multiplication of matrices possible.

$$\therefore \quad D = \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix}$$

Thus, we have

$$A_{mn} = U_{mm} D_{mn} V_{nn}'$$

$$\therefore\ A_{mn} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{12} & 0 & 0 \\ 0 & \sqrt{10} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & -\frac{5}{\sqrt{30}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{12}}{\sqrt{2}} & \frac{\sqrt{10}}{\sqrt{2}} & 0 \\ \frac{\sqrt{12}}{\sqrt{2}} & -\frac{\sqrt{10}}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & -\frac{5}{\sqrt{30}} \end{bmatrix}$$

Thus, we have decomposed A.

$$\begin{bmatrix} \sqrt{6} & \sqrt{5} & 0 \\ \sqrt{6} & -\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{30}} & \frac{2}{\sqrt{30}} & -\frac{5}{\sqrt{30}} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

(You can verify the above equality.)

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