Application of diagonalization to solve system of differential equations

Definition

Diagonalization Theorem: If the eigenvalues of an *nxn* matrix are real and distinct, then any set of corresponding eigenvectors $\{v_1, v_2, ..., v_n\}$ form a matrix $P = (v_1, v_2, ..., v_n)$ that is invertible and

$$\boldsymbol{P^{-1}AP = \Lambda}$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & \lambda_n \end{pmatrix}$$

and λ_i is the eigenvalue with eigenvector v_i.

Solution of Systems of Differential equations

Given the linear system

$$\dot{x} = Ax \qquad (\dot{x}=dx/dt)$$

Let $\{v_1, v_2, ..., v_n\}$ be a basis of eigenvectors with eigenvalues $\{\lambda_1, \lambda_2, ..., \lambda_n\}$
Let $P = (v_1, v_2, ..., v_n)$ and consider the change of coordinates

$$z = P^{-1}x$$

then

$$\dot{z} = P^{-1}\dot{x} = P^{-1}Ax = P^{-1}APz$$

then by the virtue of diagonalization theorem $\boldsymbol{P^{-1}AP = \Lambda}$, so that

$$\boldsymbol{\dot{z} = \Lambda z}$$

that is

$$\dot{z}_i = \Lambda z_i \qquad \text{for all i}$$

which have solutions

$$\dot{z}_i(t) = z_i(0)\, e^{\lambda_i t} \qquad \text{for all i}$$

so that

$$z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \\ \vdots \\ z_n(t) \end{pmatrix}$$

and

$$x(t) = Pz(t)$$

Applications

Markov processes, where the powers of some square matrix are used extensively. And Markov processes are really rich in applications, such ASN market and weather forecasting, genetics, diffusion of gasses (Ehrenfest Model) and the most famous one is probably Google's page ranking algorithm.

In quantum mechanical and quantum chemical computations matrix diagonalization is one of the most frequently applied numerical processes. The basic reason is that the time-independent Schrödinger equation is an eigenvalue equation, albeit in most of the physical situations on an infinite dimensional space (a Hilbert space).

A very common approximation is to truncate Hilbert space to finite dimension, after which the Schrödinger equation can be formulated as an eigenvalue problem of a real symmetric, or complex Hermitian matrix. Formally this approximation is founded on the variational principle, valid for Hamiltonians that are bounded from below.

First-order perturbation theory also leads to matrix eigenvalue problem for degenerate states.

Significance

The main purpose of diagonalization is determination of functions of a matrix.

If $P^{-1}AP = D$, where D is a diagonal matrix, then it is known that the entries of D are the eigen values of matrix A and P is the matrix of eigenvectors of A.

If the matrix A is invertible then $A = PDP^{-1}$

Then $f(A) = P*f(D)*P^{-1}$

This expression is very useful because f(D) is simply a diagonal matrix whose diagonal elements are $f(D_1)$, $f(D_2)$........$f(D_n)$ where D_1, D_2.......D_n are the eigenvalues of matrix A.

f can be any function such as sin, cosh, sqrt ...... etc. In essence diagonalization provides a simple pathway for evaluation of functions of a matrix.

If you need to transform a curve or a surface, x, using the transformation, y = Ax, and you want the new curve or surface to be extended or rotated by some amount, you can get the required transformation matrix A by the formula $A = PDP^{-1}$. The matrices P, $A = P^{-1}$ and D can be chosen with respect to the requirement.

* To solve differential equations using Diagonalization

1) Solve the system

$$y' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} y$$

We can write the system as

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$,

Now, Diagonilising A,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)^2 - 1 = 0$$

$$\therefore \lambda = 1, 3$$

$\therefore$ The eigenvectors are

$\lambda_2 = 3$ $\quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_1 = 1$ $\quad v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\therefore P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = [v_1 \ v_2]$$

$\therefore P^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$

$\therefore D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

$\because A = PDP^{-1}$

We can write, $y' = PDP^{-1}y$

Premultiply P^{-1} on both sides,

$(P^{-1}y)' = DP^{-1}y$

Let $z = P^{-1}y$,

$\therefore z' = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} z$

$$\begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$\therefore$ ~~zpz~~ $z_1' = z_1$ and

$z_2' = 3z_2$

On solving the ~~di~~ equations we get,

$z_1 = c_1 e^t$ and

$z_2 = c_2 e^{3t}$

$\because z = P\vec{y}$

Premultiplying P on both sides,

$\therefore Pz = y$

$$\therefore \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = P\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1e^{t} \\ c_2e^{3t} \end{bmatrix}$$

$$\therefore \vec{y} = \begin{bmatrix} c_2e^{3t} - c_1e^{t} \\ c_1e^{t} + c_2e^{3t} \end{bmatrix}$$

2) Solve the following system of equation with initial conditions $y_1(0) = 1$ and $y_2(0) = 6$.

$$y' = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} y$$

We can write the equation,

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$

Now,

$|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{vmatrix} = 0$$

$\lambda^2 + \lambda - 6 = 0$

$\therefore (\lambda - 2)(\lambda + 3) = 0$

$\therefore \lambda_1 = 2, \lambda_2 = -3$

∴ The eigenvectors are

$\lambda_1 = 2$
$$\begin{bmatrix} 1-\lambda & 1 \\ 4 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ 4 & -4 & 0 \end{array}\right] \xrightarrow{R_2 \to R_2 - 4R_1} \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\therefore V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = -3$
$$\left[\begin{array}{cc|c} 4 & 1 & 0 \\ 4 & 1 & 0 \end{array}\right] \quad \therefore V_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$$\therefore P = [V_1 \; V_2] = \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix}$$

$$\therefore D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$$

$$\therefore A = PDP^{-1}$$

We can write, $y' = PDP^{-1}y$
Premultiplying by P^{-1} on both sides.

$$(P^{-1}y)' = D(P^{-1}y)$$
Let $z = P^{-1}y$

$$\therefore z' = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} z$$

$\therefore \; z_1' = 2z_1$ and

$z_2' = -3z_2$

$\therefore \; z_1 = c_1 e^{2t}$

$z_2 = c_2 e^{-3t}$

$\because \; z = P^{-1} y$

$\therefore \; Pz = y$

$$\therefore \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = Pz = \begin{bmatrix} 1 & 1 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} c_1 e^{2t} \\ c_2 e^{-3t} \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} + c_2 e^{3t} \\ c_1 e^{2t} - 4c_2 e^{-3t} \end{bmatrix}$$

At $t=0$, $y_1(0) = 1$ and $y_2(0) = 6$

$\therefore \; 1 = c_1 + c_2$

$6 = c_1 - 4c_2$

$\therefore \; c_2 = -1, \; c_1 = 2$

$$\therefore \; \vec{y} = \begin{bmatrix} 2e^{2t} - e^{3t} \\ 2e^{2t} + 4e^{3t} \end{bmatrix}$$

3) Let $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$ and $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$. Solve

$$\frac{d\vec{x}}{dt} = A\vec{x} \quad \text{with } \vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We find the eigenvalues and eigenvectors

$\therefore$ $|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{vmatrix} = 0$$

$\lambda^2 - 8\lambda + 15 = 0$

$(\lambda - 3)(\lambda - 5) = 0$

$\therefore \lambda_1 = 3, \lambda_2 = 5$

$\therefore$ The corresponding eigenvectors are

$\lambda_1 = 3, \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 5 \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore P = [v_1 \ v_2] = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$\therefore D = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$

$\therefore A = PDP^{-1}$

We can write $\frac{dx}{dt} = PDP^{-1}x$

Premultiplying by P^{-1},

$$(P^{-1}x)' = D(P^{-1}x)$$

Let $z = P^{-1}x$.

$$\therefore \; z' = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} z$$

$$\therefore \; z_1' = 3z_1$$
$$z_2' = 5z_2$$

$$\therefore \; z_1 = c_1 e^{3t}$$
$$z_2 = c_2 e^{5t}$$

$\therefore \cancel{z = P^{-1}}$

$$\because \; z = P^{-1}x$$
$$Pz = x$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{3t} \\ c_2 e^{5t} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{3t} + c_2 e^{5t} \\ -c_1 e^{3t} + c_2 e^{5t} \end{bmatrix}$$

At $t = 0$,

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ -c_1 + c_2 \end{bmatrix}$$

$$\therefore \; c_1 = c_2 = \frac{1}{2}$$

$$\therefore \; \vec{x}(t) = \frac{1}{2} \begin{bmatrix} e^{3t} + e^{5t} \\ -e^{3t} + e^{5t} \end{bmatrix}$$

HW2

4) Let $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$, $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, Solve

$$\frac{d^2\vec{x}}{dt^2} = A\vec{x} \text{ with } \vec{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \text{ and } \vec{x}'(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Soln: $\vec{x}(t) = \begin{bmatrix} \cosh(\sqrt{3}t) + \cosh(\sqrt{5}t) \\ -\cosh(\sqrt{3}t) + \cosh(\sqrt{5}t) \end{bmatrix}$