CHAPTER 4

Basis and Dimensions

1. Introduction

In plane analytic geometry we have learnt that any point in the plane is associated with a pair of ordered numbers (a, b) called its coordinates. Conversely to every pair of ordered pair (a, b), there corresponds a point in the plane. We describe this by saying that there is **one-to-one correspondence** between the points in the plane and ordered pairs of real numbers. Although we generally use rectangular coordinate system, we may as well use oblique coordinate system *i.e.* system given by two non-parallel lines. Coordinates of a point in oblique coordinate system are obtained by drawing lines through P parallel to the axes (and not perpendicular to the axes). Similarly, in R^3 also we may have rectangular system or oblique coordinate system defined by three non-coplanar coordinate axes.

We shall soon extend the concept of coordinate system from R^2 and R^3 to R^n. For this we shall talk of vectors rather than coordinates. Instead of referring to the point P as $P(a, b)$, we shall denote it as a vector in terms of vectors of unit length in the directions of the axes. We have,

$$OP = av_1 + bv_2$$

where v_1 and v_2 are unit vectors in the positive direction of the axes. Thus, we have expressed a vector OP in R^2 as a **linear combination** of unit vectors v_1, v_2. Similarly, a vector OP in R^3 can be expressed as a linear combination of three vectors v_1, v_2, v_3 in the positive direction of the axes as

$$OP = av_1 + bv_2 + cv_3.$$

The vectors that specify the coordinate system are called "**basis vectors**". Any set of vectors can be basis vectors all that we need is that (i) they are linearly independent and (ii) they span the space V.

Although generally basis vectors of length 1 unit are used, it is not always very convenient. For instance, we may use 100 cms on one axes for length and 100 degrees for temperature on the other axis.

We also note that the x and y axes are linearly independent and they span the plane. Also the three coordinate axes x, y, z in R^3 are linearly independent and they span the spare of R^3.

We can now generalise the concept of coordinate system to general vector spaces.

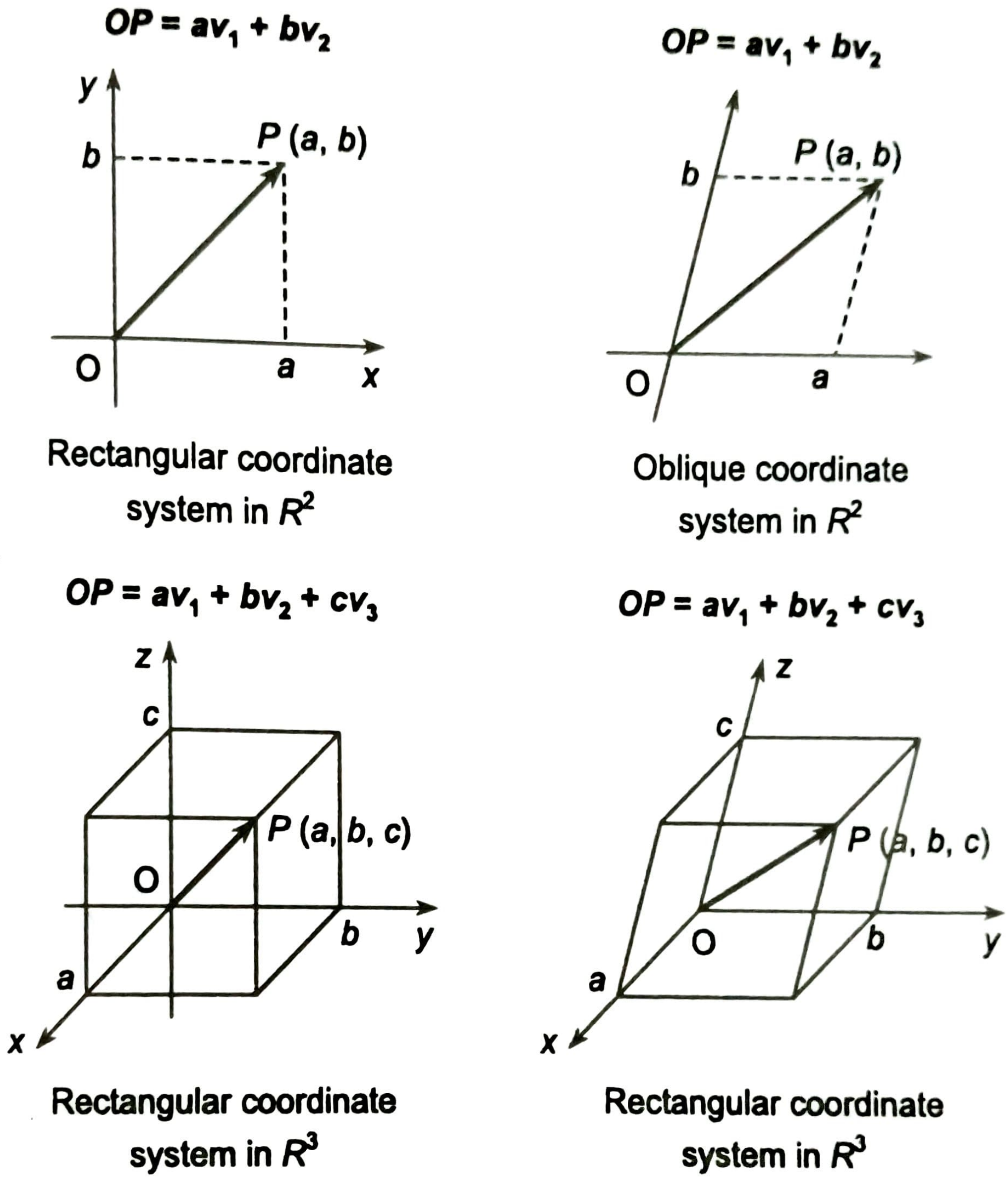

Fig. 4.1

2. Basis

Definition : If V is any vector space and $S = \{ v_1, v_2, \ldots, v_3 \}$ is a set of vectors in V then S is called a **basis** for V if the following two conditions hold good.

(1) ***S* is linearly independent,**

(2) ***S* spans *V*.**

The definition lays down two conditions for basis.

(i) The set $S = \{ v_1, v_2, \ldots, v_n \}$ must be linearly independent.

(ii) The set S spans the vector space V.

If the set S is not linearly independent, it cannot be the basis. Also if it has fewer than n vectors then it cannot span the space V and hence cannot be a basis for V. (See Ex. 1(ii), page 4-8)

We accept the following theorem about uniqueness of representation of a vector in V without proof.

Theorem : If $S = \{ v_1, v_2, ..., v_n\}$ is a basis for a vector space V, then every vector v in V can be expressed **uniquely** in the form

$$\boxed{v = c_1 v_1 + c_2 v_2 + ... + c_n v_n}$$

Definition : The coefficients $c_1, c_2, ..., c_n$ are called **coordinates** of the vector v with respect to the basis.

(a) Basis for R^2, R^3, ..., R^n

Example 1 : Standard Basis for R^2

We can easily show that if $i = (1, 0)$, $j = (0, 1)$ then the set $S = \{ i, j \}$ is linearly independent in R^2. Also any vector $v = (a, b)$ in R^2 can be expressed as $ai + bj$ i.e. as a linear combination of i and j.

$$v = ai + bj$$

Thus, $S = \{ i, j \}$ is a basis for R^2. It is called the **standard basis** for R^2.

A basis is a generalisation of coordinate system. In a plane any two linearly independent vectors can give us a coordinate system. Any vector in that plane can be expressed in terms of these vector. But the above set $S = \{ i, j \}$ is called the standard basis. Any vector v in the plane can be expressed as a linear combination of i and j as $ai + bj$. The coefficients (a, b) are called the coordinates of the point P.

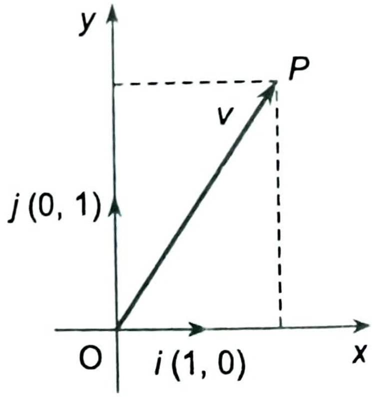

Fig. 4.2

Example 2 : Standard Basis for R^3

We can also show that if

$i = (1, 0, 0)$, $j = (0, 1, 0)$, $k = (0, 0, 1)$

then the set $S = \{ i, j, k \}$ is linearly independent in R^3. Also any vector $v = (a, b, c)$ in R^3 can be expressed as $ai + bj + ck$ i.e. as a linear combination of i, j and k.

$$v = ai + bj + ck$$

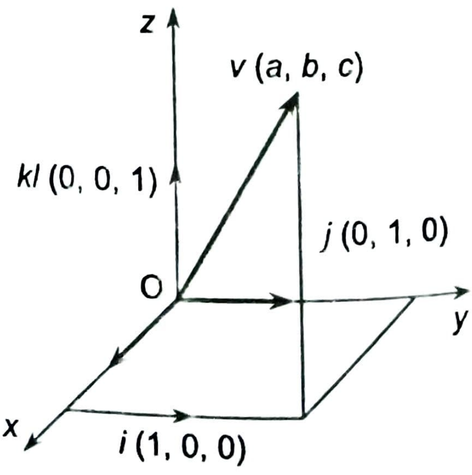

Fig. 4.3

Thus, $S = \{ i, j, k \}$ is a basis for R^3. It is called the **standard basis** for R^3. Any vector v can be expressed as $ai + bj + ck$.

Example 3 : Standard Basis for R^n

We can show that if $e_1 = (1, 0, ..., 0)$, $e_2 = (0, 1, ..., 0)$, ..., $e_n = (0, 0, ..., 1)$ then the set $S = \{ e_1, e_2, ..., e_n \}$ is linearly independent. Also any vector $v = (v_1, v_2, ..., v_n)$ can be expressed as a linear combination of $e_1, e_2, ..., e_n$ *i.e.* as

$$v = v_1 e_1 + v_2 e_2 + ... + v_n e_n$$

Thus, $S = \{ e_1, e_2, ..., e_n \}$ is a basis for R^n. It is called the **standard basis for R^n**.

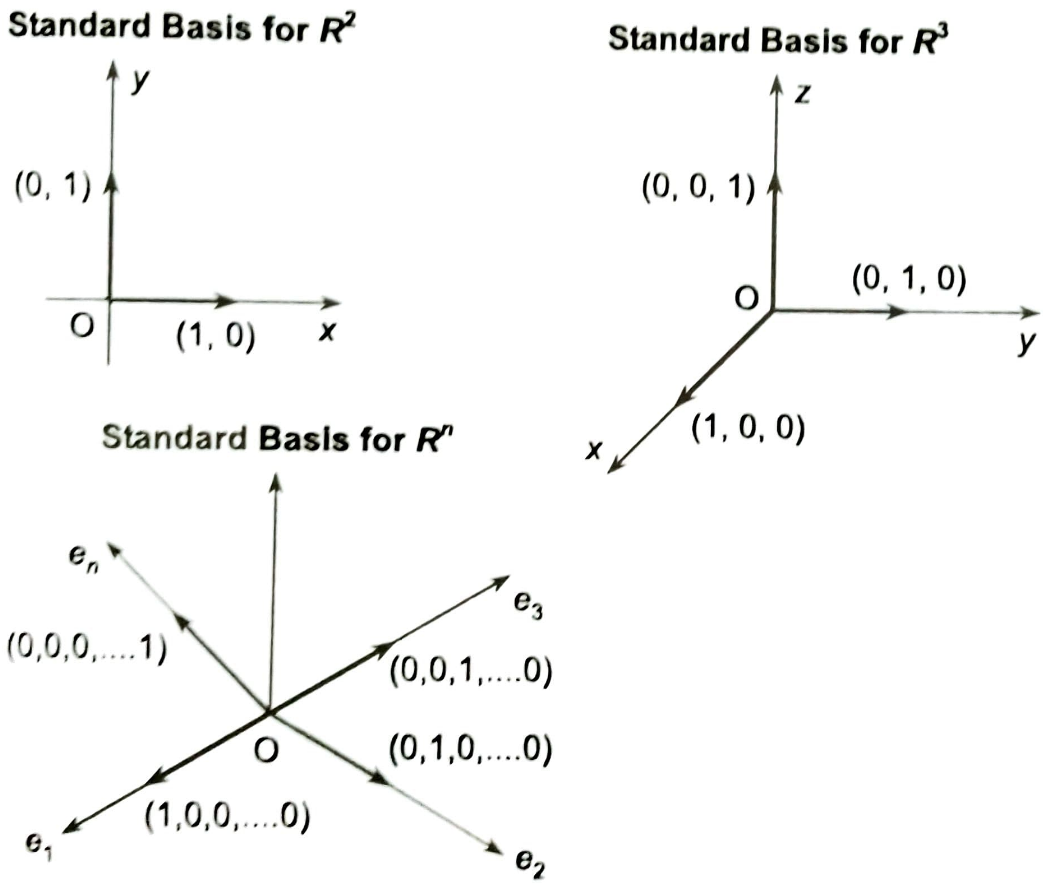

Fig. 4.4

Example 4 : If $v_1 = (1, 3)$, $v_2 = (2, 5)$ then check whether set $B = \{v_1, v_2\}$ is a basis of R^2.

Sol. : To prove a set B of vectors is a basis of V we have to show that (1) the vectors in the set are linearly independent and (2) the set spans V. *i.e.* any element of V can be expressed as a linear combination of members of B.

(i) To prove that B is linearly independent, consider

$$c_1 v_1 + c_2 v_2 = 0$$

$$\therefore \quad c_1 (1, 3) + c_2 (2, 5) = (0, 0)$$

$\therefore \quad c_1 + 2c_2 = 0, \quad 3c_1 + 5c_2 = 0$

$$\therefore \quad \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad R_2 - 3R_1 \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore \quad c_1 + 2c_2 = 0, \quad -c_2 = 0 \quad \therefore \quad c_1 = 0$

Since $c_1 = 0$, $c_2 = 0$, the vectors v_1, v_2 are linearly independent.

(ii) To prove that any vector $v = (a_1, a_2)$ in R^2 can be expressed as a linear combination of v_1, v_2, consider

$c_1 v_1 + c_2 v_2 = v$

$\therefore \quad c_1 (1, 3) + c_2 (2, 5) = (a, b)$

$\therefore \quad c_1 + 2c_2 = a_1, \quad 3c_1 + 5c_2 = a_2$

$$\therefore \quad \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\text{By} \quad R_2 - 3R_1 \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$\therefore \quad c_1 + 2c_2 = a_1, \quad -c_2 = a_2$

$\therefore \quad c_2 = -a_2 \qquad \therefore \quad c_1 = a_1 - 2c_2 = a_1 + 2a_2$

Thus, every vector v in R^2 can be expressed as a linear combination of v_1 and v_2.

Hence, $B = \{v_1, v_2\}$ is a basis of R^2.

Remark ....

As we shall see in the next example, the condition for both (a) and (b) is the same *viz.* the determinant of the coefficients of k_1, k_2 is non-zero *i.e.* $\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} \neq 0$ which is clearly true.

Example 5 : If $v_1 = (1, 0, 0)$, $v_2 = (2, 2, 0)$, $v_3 = (3, 3, 3)$, show that $S = \{v_1, v_2, v_3\}$ is a basis for R^3.

Sol. : To show that S is a basis for R^3, we have to show that (a) v_1, v_2, v_3 are linearly independent, (b) any vector b in R^3 can be expressed as a linear combination of v_1, v_2, v_3.

As we shall see that both the conditions need only one condition that the determinant of the coefficients is not singular *i.e.* $|A| \neq 0$.

(i) Now consider $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

$\therefore \quad c_1 (1, 0, 0) + c_2 (2, 2, 0) + c_3 (3, 3, 3) = (0, 0, 0)$

$$\therefore \quad (c_1 + 2c_2 + 3c_3) + (0c_1 + 2c_2 + 3c_3) + (0c_1 + 0c_2 + 3c_3) = (0, 0, 0)$$

$$\therefore \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The vectors v_1, v_2, v_3 will be linearly independent if $c_1 = 0, c_2 = 0, c_3 = 0$ *i.e.* the above set of equations has only trivial solution. It will have only trivial solution if the determinant of the coefficient is non-zero.

$$i.e. \quad \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{vmatrix} \neq 0$$

(ii) Now consider $c_1 v_1 + c_2 v_2 + c_3 v_3 = b$

$$\therefore \quad c_1 (1, 0, 0) + c_2 (2, 2, 0) + c_3 (3, 3, 3) = (b_1, b_2, b_3)$$

$$\therefore \quad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

If this is to be true for any b, the determinant of the coefficients must be non-zero *i.e.*

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{vmatrix} \neq 0$$

Thus the condition for (a) and (b) is the same *viz.*

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{vmatrix} \neq 0.$$

But expanding the above determinant by the first column we see that

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{vmatrix} = 1(2 \times 3 - 0 \times 3) = 6 \neq 0$$

Hence, S is a basis for R^3.

Example 6 : Determine whether the vectors $v_1 = (1, -1, 1)$, $v_2 = (0, 1, 2)$, $v_3 = (3, 0, -1)$ form the basis for R^3.

Sol. : We have to check whether the determinant of the coefficients of the following set of equations is non-singular.

$$c_1 (1, -1, 1) + c_2 (0, 1, 2) + c_3 (3, 0, -1) = (0, 0, 0)$$

$$\therefore \quad (c_1 + 0c_2 + 3c_3, \; -c_1 + c_2 + 0c_3, \; c_1 + 2c_2 - c_3) = (0, 0, 0)$$

$$\therefore \quad c_1 + 0\,c_2 + 3\,c_3 = 0$$
$$-\,c_1 + c_2 + 0\,c_3 = 0$$
$$c_1 + 2\,c_2 - c_3 = 0$$

$$\therefore \quad \begin{vmatrix} 1 & 0 & 3 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix} \quad \text{By } \begin{matrix} R_2 + R_1 \\ R_3 + R_2 \end{matrix} \begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{vmatrix}$$

$$\text{By} \quad R_3 - 3R_2 \begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & -8 \end{vmatrix} = 1(-8) \neq 0$$

Hence, the given set is the basis for R^3.

Example 7 : Let $v_1 = (1, 4, 8)$, $v_2 = (2, 5, 6)$, $v_3 = (3, 1, -4)$. Show that $S = (v_1, v_2, v_3)$ is a basis for R^3.

Sol. : As discussed above to show that S is a basis for R^3 it is enough to see that the determinant of the coefficients of the following set of equations is non-singular.

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$
$$c_1 (1, 4, 8) + c_2 (2, 5, 6) + c_3 (3, 1, -4) = (0, 0, 0)$$
$$\therefore \quad (c_1 + 2c_2 + 3c_3,\; 4c_1 + 5c_2 + c_3,\; 8c_1 + 6c_2 - 4c_3) = (0, 0, 0)$$

i.e.
$$c_1 + 2\,c_2 + 3\,c_3 = 0$$
$$4\,c_1 + 5\,c_2 + 1\,c_3 = 0$$
$$8\,c_1 + 6\,c_2 - 4\,c_3 = 0$$

$$\therefore \quad \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 8 & 6 & -4 \end{vmatrix}$$

$$\text{By} \quad \begin{matrix} R_2 - 4R_1 \\ R_3 - 8R_1 \end{matrix} \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -11 \\ 0 & -10 & -28 \end{vmatrix} = (-3)(-28) - (-10)(-11)$$

$$= 84 - 110 = -26 \neq 0$$

Hence, S is a basis of R^3.

3. Number of Vectors in a Basis; and Coordinate Vector

We now state an important theorem about the number of vectors in a basis.

Theorem : Let V be a finite-dimensional vector space and let $S = \{ v_1, v_2, \ldots, v_n \}$ be any basis.

(a) If a set has **more than n** vectors then the set is linearly dependent.

(b) If a set has **fewer than n** vectors then it does not span V.

In other words, the theorem states that if a basis of a vector space V has n vectors then any other set which **has more than n** vectors cannot be a basis for V and also any other vector which **has less n** vectors cannot be a basis for V.

This is shown in the following example; part (ii).

Example 1 : Check whether the following sets of vectors are basis for R^3.

(i) (2, –3, 1), (4, 1, 1), (0, –7, 1)

(ii) (1, 1, –1), (2, 3, 1)

Sol. : (i) A given set $S = \{ v_1, v_2, v_3 \}$ is a basis for R^3 if (i) they are linearly independent, (ii) they span V.

The given set is linearly independent if the determinant obtained from its components is not equal to zero.

Now the determinant

$$\Delta = \begin{vmatrix} 2 & 4 & 0 \\ -3 & 1 & -7 \\ 1 & 1 & 1 \end{vmatrix} \quad \text{By } R_{1,3} = \begin{vmatrix} 1 & 1 & 1 \\ -3 & 1 & -7 \\ 2 & 4 & 0 \end{vmatrix}$$

$$\text{By } \begin{matrix} R_2 + 3R_1 \\ R_3 - 2R_1 \end{matrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 4 & -4 \\ 0 & 2 & -2 \end{vmatrix} \quad \text{By } \begin{matrix} R_3 - \frac{1}{2}R_2 \\ \frac{1}{4}R_2 \end{matrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

Since the determinant is zero, the set S cannot be basis for R^3.

(ii) Again since the number of vectors is **less than three**, the set cannot span R^3.

Hence, it cannot be a basis for R^3.

Example 2 : Find by inspection the coordinate vector of $v = (4, 6)$ in R^2 relative to the basis $S = \{ u_1, u_2 \}$ in R^2 where $u_1 = (1, 0)$, $u_2 = (0, 1)$.

Sol. : Since u_1, u_2 are standard basis, it is clear that the coordinate vector of v will be v itself *i.e.* $v_s = (4, 6)$. [Fig. 4.5]

(v_s stands for coordinate vector v with respect to s).

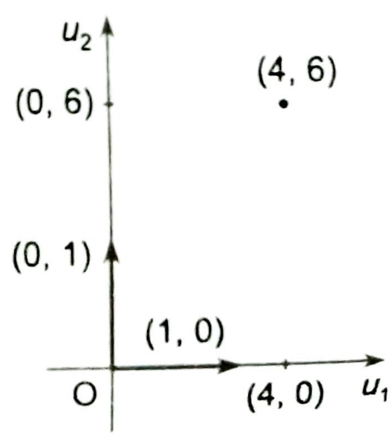

Fig. 4.5

Example 3 : Find the coordinate vector of (3, 5) in R^2 relative to (1, 1), (0, 2).

Sol. : Let the coordinate vector be $v = c_1 v_1 + c_2 v_2$

We find c_1, c_2 such that $(3, 5) = c_1 (1, 1) + c_2 (0, 2)$

Equating the components on the two sides, we get

$$c_1 + 0\, c_2 = 3$$
$$c_1 + 2\, c_2 = 5$$

$$\therefore \quad \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\text{By} \quad R_2 - R_1 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\therefore \quad c_1 = 3, \; 2c_2 = 2 \qquad \therefore \quad c_2 = 1$$
$$\therefore \quad c_1 = 3, \; c_2 = 1$$
$$\therefore \quad v = (3, 1)$$

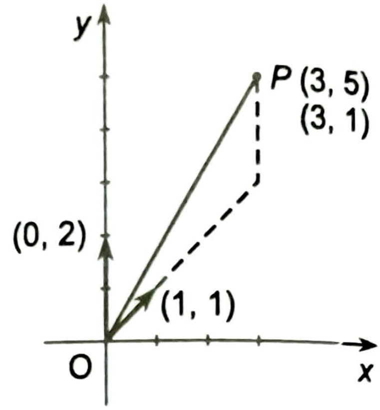

Fig. 4.6

[Verify that (3, 5) = 3 (1, 1) + 1 · (0, 2)]

Interpretation : The coordinates of P with respect to the standard axes Ox and Oy are (3, 5). If we take the vectors (1, 1) and (0, 2) as axes, the coordinates of the same point with respect to these vectors as axes are (3, 1).

Example 4 : Find the vector v in R^2 whose coordinate vector with respect to the basis $S = \{(1, 1), (0, 2)\}$ is $v_s = (4, 6)$.

Sol. : Using the definition of coordinate vector $(v)_S$, we have

$$\begin{aligned} v &= (4)\, v_1 + 6\, (v_2) \\ &= 4\, (1, 1) + 6\, (0, 2) \\ &= (4 \times 1 + 6 \times 0, \; 4 \times 1 + 6 \times 2) \\ &= (4, 16). \end{aligned}$$

Example 5 : Find the coordinate vector of $v = (2, 4, 7)$ relative to the basis $S = \{ u_1, u_2, u_3 \}$ in R^3 where $u_1 = (1, 0, 0)$, $u_2 = (0, 1, 0)$, $u_3 = (0, 0, 1)$.

Sol. : Since the u_1, u_2, u_3 are the standard basis in R^3, the coordinate vector of v is itself *i.e.* (2, 4, 7). [Fig. 4.7]

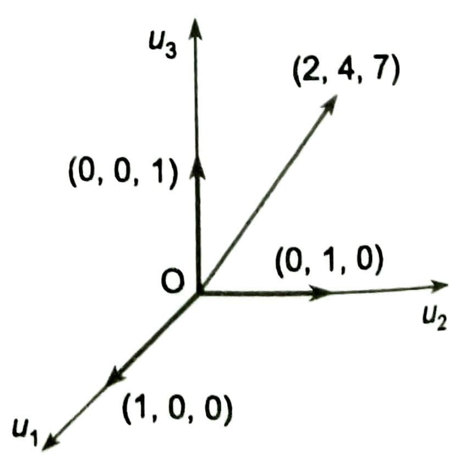

Fig. 7.7

Example 6 : Find the coordinate vector of $v = (1, 2, 5)$ relative to $v_1 = (1, 0, 0)$, $v_2 = (3, 3, 0)$, $v_3 = (5, 5, 5)$.

Sol. : Let the coordinate vector be

$$v = c_1\, v_1 + c_2\, v_2 + c_3\, v_3$$

We find c_1, c_2, c_3 such that

$$(1, 2, 5) = c_1\,(1, 0, 0) + c_2\,(3, 3, 0) + c_3 = (5, 5, 5)$$

Equating the components on the two sides, we get,

$$c_1 + 3\,c_2 + 5\,c_3 = 1$$
$$3\,c_2 + 5\,c_3 = 2$$
$$5\,c_3 = 5$$

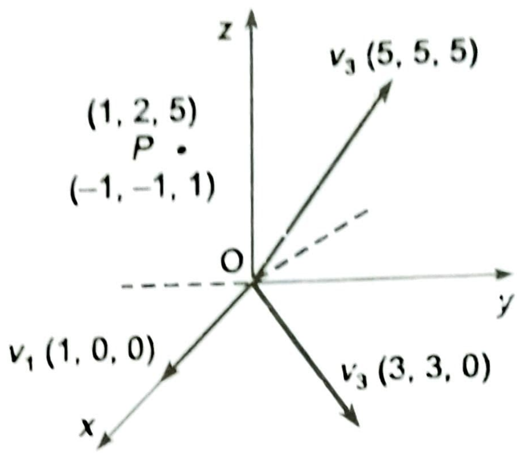

Fig. 4.8

[$\therefore \begin{bmatrix} 1 & 3 & 5 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$

$$\therefore \quad c_1 + 3\,c_2 + 5\,c_3 = 1$$
$$3\,c_2 + 5\,c_3 = 2$$
$$5\,c_3 = 5$$

Actually we don't need this.]

$$\therefore \quad c_3 = 1$$
$$\therefore \quad 3\,c_2 = 2 - 5\,c_3 = 2 - 5 = -3$$
$$\therefore \quad c_2 = -1$$
$$\therefore \quad c_1 = 1 - 3\,c_2 - 5\,c_3$$
$$\therefore \quad c_1 = 1 + 3 - 5 = -1$$

Hence, $v = (-1, -1, 1)$.

This means that the point P, whose coordinates with respect the usual rectangular axes (x, y, z) are $(1, 2, 5)$ has the coordinates $(-1, -1, 1)$ with respect to the vectors v_1, v_2, v_3.

[You can verify the result that

$$(1, 2, 5) = -1\,(v_1) - 1\,(v_2) + 1\,(v_3)$$]

Example 7 : Let $S = \{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ be a basis for R^3.

(i) Find the coordinates vector for $v = (1, 1, -1)$ with respect to s.

(ii) Find the vector v in R^3 whose coordinates vector with respect to the basis S is $v_s = (1, -2, 3)$.

Sol. : (I) Let the coordinate vector of $v = (1, 1, -1)$ be with respect to S.

$$\therefore \quad v = c_1\, v_1 + c_2\, v_2 + c_3\, v_3$$

We find c_1, c_2, c_3 such that

$$(1, 1, -1) = c_1\,(1, 1, 0) + c_2\,(1, 0, 1) + c_3\,(0, 1, 1)$$

Equating the components on the two sides,

$$c_1 + c_2 \qquad = 1$$
$$c_1 + \qquad c_3 = 1$$
$$c_2 + c_3 = -1$$

$$\therefore \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\text{By} \quad R_2 - R_1 \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\text{By} \quad R_3 + R_2 \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\therefore \quad c_1 + c_2 = 1, \qquad -c_2 + c_3 = 0$$
$$2c_3 = -1 \quad \therefore \quad c_3 = -1/2, \;\; c_2 = c_3 = -1/2.$$
$$\therefore \quad c_1 = 1 - c_2 = 3/2$$

Hence, the required coordinate vector v_s of v with respect to s is

$$v_s = (3/2, -1/2, -1/2)$$

(ii) Since the basis is S and $v_s = (1, -2, 3)$, by definition of the coordinate vector, we get

$$v = (1)\, v_1 + (-2)\, v_2 + 3\, (v_3)$$

Putting the values of v_1, v_2, v_3 from the data.

$$v = (1)\,(1, 1, 0) - 2\,(1, 0, 1) + 3\,(0, 1, 1)$$
$$= (1 - 2 + 0,\;\; 1 - 0 + 3,\;\; 0 - 2 + 3)$$
$$= (-1,\;\; 4,\;\; 1).$$

Example 8 : Let $S = \{(1, 2, 1), (2, 9, 0), (3, 3, 4)\}$ be the basis for R^3.

(i) Find the coordinate vector for $v = (5, -1, 9)$ with respect to S.

(ii) Find the vector v in R^3 whose coordinate vector with respect to the bais S is $v_s = (-1, 3, 2)$.

Sol. : (i) Let the coordinate vector of $v = (5, -1, 9)$ with respect to S be

$$v = c_1\, v_1 + c_2\, v_2 + c_3\, v_3$$

where $v_1 = (1, 2, 1)$, $v_2 = (2, 9, 0)$, $v_3 = (3, 3, 4)$.

We find c_1, c_2, c_3 such that

$$(5, -1, 9) = c_1\,(1, 2, 1) + c_2\,(2, 9, 0) + c_3\,(3, 3, 4)$$

Equating the components on the two sides, we get,

$$c_1 + 2c_2 + 3c_3 = 5$$
$$2c_1 + 9c_2 + 3c_3 = -1$$
$$c_1 + 0c_2 + 4c_3 = 9$$

$$\therefore \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 3 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 9 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & -3 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -11 \\ 4 \end{bmatrix}$$

$$\text{By} \quad R_2 + 3R_3 \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_3 - 2R_2 \\ -R_2 \end{matrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$$

$\therefore \quad c_1 + 2c_2 + 3c_3 = 5 \qquad \therefore \quad c_2 = -1, \; c_3 = 2.$

$\therefore \quad c_1 = 5 + 2 - 6 = 1$

Hence, the required coordinate vector v_s of v with respect to s is

$$v_s = (1, -1, 2).$$

(ii) Since the basis is S and $v_s = (-1, 3, 2)$, by **definition** of the coordinate vector, we get

$$v = (-1)\, v_1 + 3\,(v_2) + 2\, v_3$$

Putting the values of v_1, v_2, v_3 from data,

$$v = (-1)(1, 2, 1) + 3\,(2, 9, 10) + 2\,(3, 3, 4)$$
$$= (-1 \times 1 + 3 \times 2 + 2 \times 3, \; -1 \times 2 + 3 \times 9 + 2 \times 3, \; -1 \times 1 + 3 \times 10 + 2 \times 4)$$
$$= (11, 31, 37)$$

Explanation : In (i), $(5, -1, 9)$ are the coordinates of the point with respect to the usual rectangular axes in R^3. *i.e.* $S' = \{i, j, k\}$. The coordinates of this point P with respect to the new system with basis

$$S = \{(1, 2, 1), (2, 9, 0), (3, 3, 4)\} \text{ are } (1, -1, 2).$$

In (ii), $Q(-1, 3, 2)$ are the coordinates of a point Q with respect to the above basis S. The coordinates of this point are $(11, 31, 37)$ with respect to $S' = \{i, j, k\}$.

Example 9 : Prove that the set $S = \{(1, 1, 1), (1, -1, 1), (0, 1, 1)\}$ is a basis for the vector space R^3.

(i) Find the coordinate vector of $v = (1, 2, 3)$ with respect to this basis.

(ii) Find the vector v in R^3 whose coordinate vector with respect to the basis S is $(2, 4, -6)$.

Sol. : As discussed earlier, S is a basis for vector space R^3 if the determinant

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} \neq 0$$

By $R_2 - R_1,\ R_3 - R_1$

$$\Delta = \begin{vmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1(-2-0) = -2 \neq 0$$

Since $\Delta \neq 0$ the set S is a basis for R^3.

(i) To find the coordinate vector of $v = (1, 2, 3)$ with respect to the basis S, we have to find the constants, c_1, c_2, c_3 such that

$$v = c_1 v_1 + c_2 v_2 + c_3 v_3$$

$$\therefore \quad (1, 2, 3) = c_1 (1, 1, 1) + c_2 (1, -1, 1) + c_3 (0, 1, 1)$$

$\therefore$ Equating the components on the two sides

$$c_1 + c_2 + 0\, c_3 = 1$$

$$c_1 - c_2 + c_3 = 2$$

$$c_1 + c_2 + c_3 = 3$$

$$\therefore \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_3 - R_2 \\ R_2 - R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore \quad c_1 + c_2 = 1, \qquad -2c_2 + c_3 = 1$$

$$2c_2 = 1 \qquad \therefore \quad c_2 = 1/2$$

$$\therefore \quad c_3 = 1 + 2c_2 = 1 + 1 = 2$$

$$\therefore \quad c_1 = 1 - c_2 = 1 - (1/2) = 1/2$$

Hence, the coordinate vectors v_s of v is $v_s = (1/2, 1/2, 2)$.

(ii) Since the basis is S and $v_s = (2, 4, -6)$ by definition of the coordinate vector v_s, we get

$$v = 2(1, 1, 1) + 4(1, -1, 1) + (-6)(0, 1, 1)$$
$$= (2 \times 1 + 4 \times 1 - 6 \times 0,\ 2 \times 1 + 4(-1) + (-6) \times 1$$
$$+ 2 \times 1 + 4 \times 1 + (-6) \times 1)$$
$$= (6, -8, 0)$$

4. Standard Basis for P_n

It can be show that $S = \{1, x, x^2, \ldots, x^n\}$ is a basis for the vector space P_n of polynomials of the form $a_0 + a_1 x + \ldots + a_n x^n$. It is called standard bases for P_n and the set of coefficients $a_0, a_1, a_2, \ldots, a_n$ is called the coordinate vector relative to the standard basis.

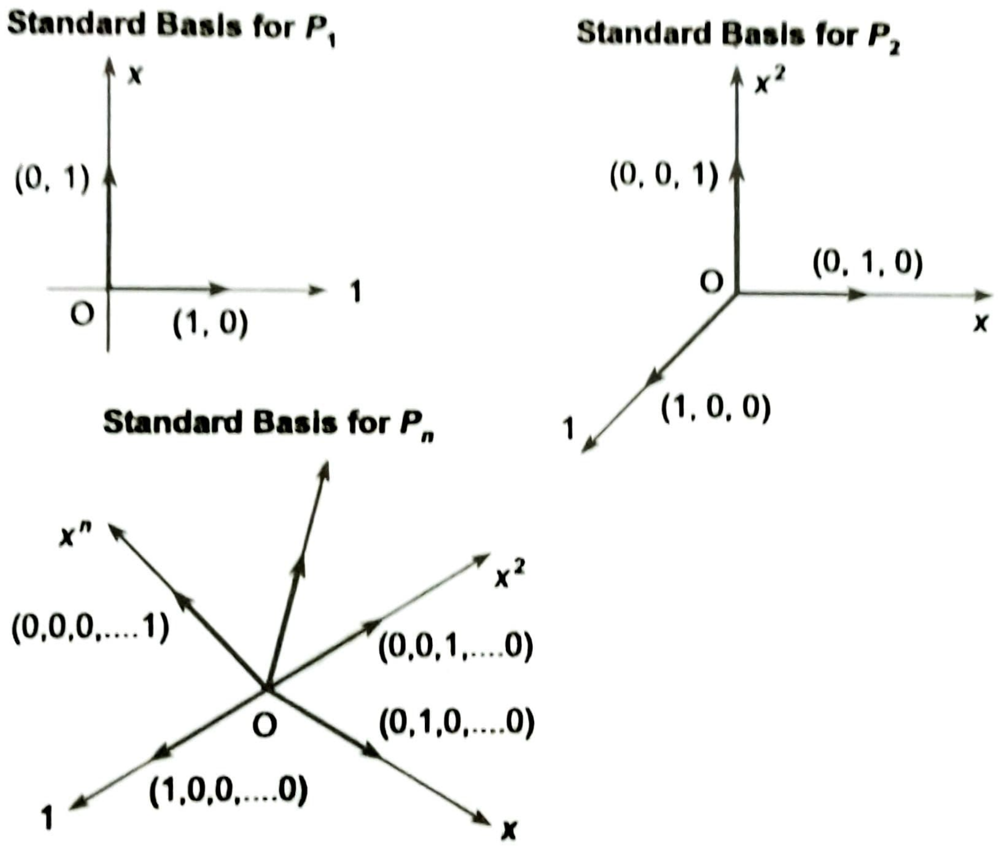

Fig. 4.9

For example, the coordinate vector of the polynomial $p = 1 - x + 2x^3$ relative to the basis $S = \{1, x, x^2, x^3\}$ is $p = (1, -1, 0, 2)$.

(The axis on which the constant term is plotted is denoted by 1, the axis on which the coefficient of x is plotted is denoted by x, ... the axis on which the coefficient of x^n is plotted is denoted by x^n.)

Example 1 : Find the coordinate vector of p in P_2 relative to the basis $S = \{p_1, p_2, p_3\}$ where $p_1 = 1$, $p_2 = x$, $p_3 = x^2$ and $p = 4 - 3x + x^2$.

Sol. : Let the coordinate vector of P be $c_1 p_1 + c_2 p_2 + c_3 p_3$.

We find c_1, c_2, c_3 such that

$$(4, -3, 1) = c_1 (1, 0, 0) + c_2 (0, 1, 0) + c_3 (0, 0, 1)$$

Equating the components on both sides, we have

$$4 = c_1, \quad -3 = c_2, \quad 1 = c_3$$

$$\therefore \quad p_s = (4, -3, 1) \qquad (i.e.\ p = 4p_1 - 3p_2 + p_3)$$

Example 2 : Find the coordinate vector of p in P_2 relative to the basis $S = \{ p_1, p_2, p_3 \}$ where $p_1 = 1 + x$, $p_2 = 1 + x^2$, $p_3 = x + x^2$ and $p = 2 - x - x^2$.

Sol. : Let the coordinate vector of p in P_2 be $c_1 p_1 + c_2 p_2 + c_3 p_3$.

We find c_1, c_2, c_3 such that

$$(2, -1, -1) = c_1 (1, 1, 0) + c_2 (1, 0, 1) + c_3 (0, 1, 1)$$

Equating the components on the two sides, we get,

$$\begin{aligned} c_1 + c_2 \quad\quad &= 2 \\ c_1 + \quad\quad c_3 &= -1 \\ c_2 + c_3 &= -1 \end{aligned}$$

$$\therefore \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

$$\text{By} \quad R_2 - R_1 \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$$

$$\text{By} \quad R_3 + R_2 \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix}$$

$$\therefore \quad c_1 + c_2 = 2, \qquad -c_2 + c_3 = -3$$

$$2c_3 = -4 \qquad \therefore \quad c_3 = -2$$

$$\therefore \quad c_2 = 1, \quad c_1 = 1$$

$$\therefore \quad p_s = (1, 1, -2) \qquad (i.e.\ p = p_1 + p_2 - 2p_3)$$

Example 3 : If the coordinate vector of P with respect to

$$S = \{ 1 + x, 1 + x^2, x + x^2 \} \text{ is } v_s = (2, 4, -6),$$

find the coordinate vector of P with respect to the standard basis for P_2.

Sol. : The standard basis for P_2 is $\{ 1, x, x^2 \}$.

By the definition of the coordinate vector v_s, we get

$$v = 2 (1, 1, 0) + 4 (1, 0, 1) - 6 (0, 1, 1)$$

$$\therefore \quad v = (2 + 4 - 0,\ 2 + 0 - 6,\ 0 + 4 - 6)$$
$$= (6, -4, -2)$$

Example 4 : Check whether the following set of vectors is a basis for P_2.

$$S = \{1 - 3x + 2x^2,\quad 1 + x + 4x^2,\quad 1 - 8x + x^2\}$$

Find the coordinate vector of $p = 1 - 2x + x^2$ with respect to the above basis.

Sol. : Let $v_1 = (1, -3, 2)$, $v_2 = (1, 1, 4)$, $v_3 = (1, -8, 1)$.

The set $S = \{v_1, v_2, v_3\}$ will be a basis for P_2 if (i) S is linearly independent and (ii) S spans P_2.

(i) To check whether S is linearly independent, consider

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$
$$\therefore \quad c_1 (1, -3, 2) + c_2 (1, 1, 4) + c_3 (1, -8, 1) = (0, 0, 0)$$
$$\therefore \quad c_1 + c_2 + c_3 = 0$$
$$-3c_1 + c_2 - 8c_3 = 0$$
$$2c_1 + 4c_2 + c_3 = 0$$
$$\therefore \quad \begin{bmatrix} 1 & 1 & 1 \\ -3 & 1 & -8 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The vectors v_1, v_2, v_3 will be linearly independent if the determinant is not equal to zero.

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ -3 & 1 & -8 \\ 2 & 4 & 1 \end{vmatrix}$$

By $R_2 + 3R_1$, $R_3 - 2R_1$

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 4 & -5 \\ 0 & 2 & -1 \end{vmatrix} = 1(-4 + 10) = 6 \neq 0$$

Hence, S is a basis for P_2.

(ii) To find the coordinate vector of p with respect to the above basis, let

$$p = c_1 p_1 + c_2 p_2 + c_3 p_3$$

We find c_1, c_2, c_3 such that

$$(1, -2, 1) = c_1 (1, -3, 2) + c_2 (1, 1, 4) + c_3 (1, -8, 1)$$

Equating the two sides,

$$c_1 + c_2 + c_3 = 1$$
$$-3c_1 + c_2 - 8c_3 = -2$$
$$2c_1 + 4c_2 + c_3 = 1$$

$$\therefore \quad \begin{bmatrix} 1 & 1 & 1 \\ -3 & 1 & -8 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_2 + 3R_1 \\ R_3 - 2R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & -5 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\text{By} \quad R_3 - \frac{1}{2}R_2 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & -5 \\ 0 & 0 & 3/2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -3/2 \end{bmatrix}$$

$$\therefore \quad c_1 + c_2 + c_3 = 1, \quad 4c_2 - 5c_3 = 1, \quad \frac{3}{2}c_3 = -\frac{3}{2}$$

$$\therefore \quad c_3 = -1 \qquad \therefore \quad 4c_2 = 1 - 5 = -4$$

$$\therefore \quad c_2 = -1, \qquad \therefore \quad c_1 = 1 - c_2 - c_3 = 3$$

$\therefore$ The coordinates of p are $(3, -1, -1)$ *i.e.* $p = 3p_1 - p_2 - p_3$.

Example 5 : Find the vector v in P_3 whose coordinate vector with respect to the basis $S = \{v_1, v_2, v_3\}$ and $v_1 = 1 - 3x + 2x^2$, $v_2 = 1 + x + 4x^2$, $v_3 = 1 - 8x + x^2$ is $v_s = (3, -1, 2)$.

Sol. : By definition of the coordinate vector v_s, we obtain

$$\begin{aligned} v &= 3v_1 - 1v_2 + 2v_3 \\ &= 3(1, -3, 2) - 1(1, 1, 4) + 2(1, -8, 1) \\ &= (4, -26, 4) \end{aligned}$$

Example 6 : Check whether the following set of vectors is a basis for P_2.

$$p_1 = 1 - 2x + 3x^2, \quad p_2 = 1 + 2x - x^2, \quad p_3 = 1 + 3x - 4x^2.$$

Find $p = 1 - 4x - 3x^2$ in terms of p_1, p_2, p_3.

Sol. : Let $v_1 = (1, -2, 3)$, $v_2 = (1, 2, -1)$, $v_3 = (1, 3, -4)$.

We have to check whether the determinant of the coefficients of c_1, c_2, c_3 in the following equation is zero.

$$c_1(1, -2, 3) + c_2(1, 2, -1) + c_3(1, 3, -4) = (0, 0, 0)$$

$$\therefore \quad \begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & 3 \\ 3 & -1 & -4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The vectors p_1, p_2, p_3 will be linearly independent if the determinant is not equal to zero.

$$\Delta = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & 3 \\ 3 & -1 & -4 \end{bmatrix}$$

By $R_2 + 2R_1,\ R_3 - 3R_1,$

$$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 4 & 5 \\ 0 & -4 & -7 \end{vmatrix} = 1(-28+20) = -8 \neq 0$$

Hence, S is a basis for P_2.

(ii) For the second part, let

$$p = c_1\, p_1 + c_2\, p_2 + c_3\, p_3$$

$$(1, -4, -3) = c_1\,(1, -2, 3) + c_2\,(1, 2, -1) + c_3\,(1, 3, -4)$$

$$\therefore \quad \begin{bmatrix} 1 & 1 & 1 \\ -2 & 2 & 3 \\ 3 & -1 & -4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_2 + 2R_1 \\ R_3 - 3R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 5 \\ 0 & -4 & -7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -6 \end{bmatrix}$$

$$\text{By} \quad R_3 + R_2 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 5 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -8 \end{bmatrix}$$

$$\therefore \quad c_1 + c_2 + c_3 = 1$$

$$4\,c_2 + 5\,c_3 = -2$$

$$-2\,c_3 = -8$$

$$\therefore \quad c_3 = 4, \quad 4\,c_2 = -2 - 20$$

$$\therefore \quad c_2 = -\frac{11}{2}$$

$$\therefore \quad c_1 = 1 - c_2 - c_3 = 1 + \frac{11}{2} - 4 = \frac{5}{3}$$

$$\therefore \quad p = c_1\, p_1 + c_2\, p_2 + c_3\, p_3$$

$$= \frac{5}{3} p_1 - \frac{11}{2} p_2 + 4\, p_3$$

(You can verify the above equality by putting the expressions for p_1, p_2, p_3.)

5. Standard Basis for M_{22}

The standard basis for the matrices M_{22} is $S = \{ M_1, M_2, M_3, M_4 \}$ where

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Generalising this we say that standard basis for the matrices M_{mn} are mn matrices of order mn whose only one element is one and the remaining elements are zero in order.

$$\begin{bmatrix} 1 & 0 & 0 & .... & 0 \\ 0 & 0 & 0 & .... & 0 \\ .... & .... & .... & .... & .... \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & .... & 0 \\ 0 & 0 & 0 & .... & 0 \\ .... & .... & .... & .... & .... \\ 0 & 0 & 0 & .... & 0 \end{bmatrix}, \; ....$$

$$........, \begin{bmatrix} 0 & 0 & 0 & .... & 0 \\ 0 & 0 & 0 & .... & 0 \\ .... & .... & .... & .... & .... \\ 0 & 0 & 0 & .... & 1 \end{bmatrix}$$

Example 1 : Show that

$$S = \left\{ \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix} \right\}$$

is a basis for M_{22}.

Sol. : As discussed above to show that S is a basis for M_{22}, it is enough to show that the only solution of

$$c_1 M_1 + c_2 M_2 + c_3 M_3 + c_3 M_4 = O$$

i.e., of $c_1 \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix} + c_2 \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

is a trivial solution.

Now from the above equality, we get

$$\begin{bmatrix} c_1 + c_2(0) + c_3(0) + c_4(0) & 2c_1 - c_2 + 2c_3 + c_4(0) \\ c_1 - c_2 + 3c_3 - c_4 & -2c_1 + c_2(0) + c_3 + 2c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \therefore \quad c_1 \qquad\qquad &= 0 \\ 2c_1 - c_2 + 2c_3 \qquad &= 0 \\ c_1 - c_2 + 3c_3 - c_4 &= 0 \\ -2c_1 + c_3 \qquad + 2c_4 &= 0 \end{aligned} \qquad \text{.......... (A)}$$

The matrix of the coefficients is

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & -1 & 3 & -1 \\ -2 & 0 & 1 & 2 \end{bmatrix}$$

and $$|A| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 2 & 0 \\ 1 & -3 & 3 & -1 \\ -2 & 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} -1 & 2 & 0 \\ -3 & 3 & -1 \\ 0 & 1 & 2 \end{vmatrix}$$

By $R_2 - 3\,R_1$,

$$|A| = \begin{vmatrix} -1 & 2 & 0 \\ 0 & -3 & -1 \\ 0 & 1 & 2 \end{vmatrix} = -1(-6+1) = 5 \neq 0$$

Since the determinant is not equal to zero, the only solution of (A) is a trivial solution.

Hence, S is a basis of M_{22}.

Example 2 : Show that the following set of vectors is a basis for M_{22}.

$$S = \left\{ \begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \right\}$$

Also find the coordinate vector of $\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$ relative to S.

Sol. : Consider the equation

$$c_1\, M_1 + c_2\, M_2 + c_3\, M_3 + c_4\, M_4 = O$$

$$\therefore \quad c_1 \begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix} + c_2 \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating the corresponding terms on both sides, we get,

$$\therefore \quad \begin{bmatrix} 3c_1 + c_4 & 6c_1 - c_2 - 8c_3 \\ 3c_1 - c_2 - 12c_3 - c_4 & -6c_1 - 4c_3 + 2c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \quad \left.\begin{aligned} 3c_1 + c_4 &= 0 \\ 6c_1 - c_2 - 8c_3 &= 0 \\ 3c_1 - c_2 - 12c_3 - c_4 &= 0 \\ -6c_1 - 4c_3 + 2c_4 &= 0 \end{aligned}\right\} \qquad \text{.......... (A)}$$

The matrix of the coefficients is

$$A=\begin{bmatrix} 3 & 0 & 0 & 1 \\ 6 & -1 & -8 & 0 \\ 3 & -1 & -12 & -1 \\ -6 & 0 & -4 & 2 \end{bmatrix}$$

Its determinant,

$$|A|=\begin{vmatrix} 3 & 0 & 0 & 1 \\ 6 & -1 & -8 & 0 \\ 3 & -1 & -12 & -1 \\ -6 & 0 & -4 & 2 \end{vmatrix}$$

By $\begin{matrix} R_2-2R_1 \\ R_3-R_1 \\ R_4+2R_1 \end{matrix}$ $$|A|=\begin{vmatrix} 3 & 0 & 0 & 1 \\ 0 & -1 & -8 & -2 \\ 0 & -1 & -12 & -2 \\ 0 & 0 & -4 & 4 \end{vmatrix}$$

By $\begin{matrix} R_3-R_2 \\ -R_2 \\ -R_4 \end{matrix}$ $$|A|=\begin{vmatrix} 3 & 0 & 0 & 1 \\ 0 & 1 & 8 & 2 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & -4 \end{vmatrix}$$

By $\begin{matrix} R_4-R_3 \\ -R_4 \end{matrix}$ $$|A|=\begin{vmatrix} 3 & 0 & 0 & 1 \\ 0 & 1 & 8 & 2 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix}$$

Expanding the determinant by the first column,

$$|A|=3\begin{vmatrix} 1 & 8 & 2 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{vmatrix}$$

Expanding again by the first column,

$$|A|=3\cdot 1\cdot\begin{vmatrix} 4 & 0 \\ 0 & 4 \end{vmatrix}=3\cdot 1\cdot 16=48\neq 0$$

Hence, S is a basis for M_{22}.

(ii) To find the coordinate vector let

$$c_1\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}+c_2\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}+c_3\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}+c_4\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}=\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 3c_1 + c_4 & 6c_1-c_2-8c_3 \\ 3c_1-c_2-12c_3-c_4 & -6c_1-4c_3+2c_4 \end{bmatrix}=\begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

Equating the corresponding terms on both sides, we get,

$$\therefore \quad \begin{aligned} 3c_1 + 0c_2 + 0c_3 + c_4 &= 2 \\ 6c_1 - c_2 - 8c_3 + 0c_4 &= 1 \\ 3c_1 - c_2 - 12c_3 - c_4 &= -1 \\ -6c_1 + 0c_2 - 4c_3 + 2c_4 &= 4 \end{aligned}$$

$$\therefore \quad \begin{bmatrix} 3 & 0 & 0 & 1 \\ 6 & -1 & -8 & 0 \\ 3 & -1 & -12 & -1 \\ -6 & 0 & -4 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 4 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 + 2R_1 \end{matrix} \quad \begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & -1 & -8 & -2 \\ 0 & -1 & -12 & -2 \\ 0 & 0 & -4 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -3 \\ 8 \end{bmatrix}$$

$$\text{By} \quad R_3 - R_2 \quad \begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & -1 & -8 & -2 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & -4 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 8 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} -R_2 \\ -R_3 \\ -R_4 \end{matrix} \quad \begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 1 & 8 & 2 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & -4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ -8 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_4 - R_3 \\ -R_4 \end{matrix} \quad \begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 1 & 8 & 2 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 8 \end{bmatrix}$$

$\therefore \quad 3c_1 + c_4 = 2, \quad c_2 + 8c_3 + 2c_4 = 3$

$4c_3 = 0, \quad 4c_4 = 8$

$\therefore \quad c_4 = 2, \; c_3 = 0, \; c_2 = -1 \quad \therefore \; c_1 = 0$

$\therefore \quad v_s = (0, -1, 0, 2).$

(Verify that $A = 0\,M_1 - M_2 + 0\,M_3 + 2\,M_4$)

Shortcut Method

In the above example, for verification of basis and for finding the coordinate vector we use the same matrix of c_1, c_2, c_3 and c_4. We can avoid the repetition by considering the coefficient matrix only once. This is illustrated below.

Consider the equations

$$c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = 0$$

and
$$c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = B$$

where B is the matrix $\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$.

We have $c_1 \begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix} + c_2 \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

and $c_1 \begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix} + c_2 \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix} + c_4 \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$

$$\therefore \begin{bmatrix} 3c_1 + c_4 & 6c_1 - c_2 - 8c_3 \\ 3c_1 - c_2 - 12c_3 - c_4 & -6c_1 - 4c_3 + 2c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and
$$\begin{bmatrix} 3c_1 + c_4 & 6c_1 - c_2 - 8c_3 \\ 3c_1 - c_2 - 12c_3 - c_4 & -6c_1 - 4c_3 + 2c_4 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix}$$

Equating the elements of the two matrices,

$$\begin{aligned} 3c_1 + 0c_2 + 0c_3 + c_4 &= 0,\ 2 \\ 6c_1 - c_2 - 8c_3 + 0c_4 &= 0,\ 1 \\ 3c_1 - c_2 - 12c_3 - c_4 &= 0,\ -1 \\ -6c_1 - 0c_2 - 4c_3 + 2c_4 &= 0,\ 4 \end{aligned}$$

The equations in the matrix form can be written as

$$\therefore \begin{bmatrix} 3 & 0 & 0 & 1 \\ 6 & -1 & -8 & 0 \\ 3 & -1 & -12 & -1 \\ -6 & 0 & -4 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \left[\begin{array}{c|c} 0 & 2 \\ 0 & 1 \\ 0 & -1 \\ 0 & 4 \end{array}\right]$$

By $\begin{matrix} R_2 - 2R_1 \\ R_3 - R_1 \\ R_4 + 2R_1 \end{matrix}$
$$\begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & -1 & -8 & -2 \\ 0 & -1 & -12 & -2 \\ 0 & 0 & -4 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \left[\begin{array}{c|c} 0 & 2 \\ 0 & -3 \\ 0 & -3 \\ 0 & 8 \end{array}\right]$$

By $\begin{matrix} R_3 - R_2 \\ -R_2 \\ -R_4 \end{matrix}$
$$\begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 1 & 8 & 2 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & -4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \left[\begin{array}{c|c} 0 & 2 \\ 0 & 3 \\ 0 & 0 \\ 0 & -8 \end{array}\right]$$

By $\begin{matrix} R_4 - R_3 \\ -R_4 \end{matrix}$
$$\begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 1 & 8 & 2 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \left[\begin{array}{c|c} 0 & 2 \\ 0 & 3 \\ 0 & 0 \\ 0 & 8 \end{array}\right]$$

Since the system is consistent, the given set S of matrices is a basis for M_{22}.

Further, we get

$$3c_1 + c_4 = 2, \quad c_2 + 8c_2 + 2c_4 = 3,$$
$$4c_3 = 0, \quad 4c_4 = 8.$$

$\therefore \quad c_3 = 0, \ c_4 = 2 \qquad \therefore \quad c_2 = -1$ and $c_1 = 0$

$\therefore \quad v_s = (0, -1, 0, 2)$

(You can verify the result by actual substitution.)

Example 3 : Show that the following set of vectors is a basis for M_{22}.

$$S = \left\{ \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Also find the coordinate vector of $\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$, relative to S.

Sol. : We shall use the above shortcut method.

Consider the equations

$$c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = 0$$

and $\quad c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = B$

where $B = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$ in the same matrix.

We have

$$c_1 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and $$c_1 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} -c_1 + c_2 & c_1 + c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and $$\begin{bmatrix} -c_1 + c_2 & c_1 + c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$$

$$\therefore \quad -c_1 + c_2 + 0c_3 + 0c_4 = 0, \ 2$$
$$c_1 + c_2 + 0c_3 + 0c_4 = 0, \ 0$$
$$0c_1 + 0c_2 + c_3 + 0c_4 = 0, \ -1$$
$$0c_1 + 0c_2 + 0c_3 + c_4 = 0, \ 3$$

$\therefore$ The equations in the matrix form can be written as

$$\therefore \quad \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \left[\begin{array}{c|c} 0 & 2 \\ 0 & 0 \\ 0 & -1 \\ 0 & 3 \end{array}\right]$$

$$\text{By} \quad \begin{matrix} R_2 + R_1 \\ -R_1 \end{matrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \left[\begin{array}{c|c} 0 & -2 \\ 0 & 2 \\ 0 & -1 \\ 0 & 3 \end{array}\right]$$

$$\therefore \quad |A| = |1 \cdot 2 \cdot 1| = 2 \neq 0$$

Hence, the system is consistent.

The given set S of matrices is a basis for M_{22}.

Further, we get

$$c_1 - c_2 = -2, \quad 2c_2 = 2 \quad \therefore \quad c_2 = 1, \quad c_3 = -1, \quad c_4 = 3$$

$$\therefore \quad c_1 = -2 + c_2 = -1$$

Hence, $v_s = (-1, 1, -1, 3)$.

(You can verify the above result by actual substitution.)

Example 4 : Verify whether the set

$$S = \left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$$

basis for M_{22}. Find the coordinates of $\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$ with respect to S.

Sol. : As discussed above in detail we have to solve the equations

$$c_1 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\text{and} \quad c_1 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\therefore \quad c_1 + c_2 + c_3 + 0\,c_4 = 0, \; 2$$

$$-c_1 + 0\,c_2 + 0\,c_3 - c_4 = 0, \; 1$$

$$0\,c_1 + 0\,c_2 + 0\,c_3 + c_4 = 0, \; 1$$

$$0\,c_1 - 1\,c_2 + 0\,c_3 + 0\,c_4 = 0, \; -1$$

$$\therefore \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \left[\begin{array}{c|c} 0 & 2 \\ 0 & 1 \\ 0 & 1 \\ 0 & -1 \end{array}\right]$$

By $R_2 + R_1$ $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \left[\begin{array}{c|c} 0 & 2 \\ 0 & 3 \\ 0 & 1 \\ 0 & -1 \end{array}\right]$

By $\begin{matrix} R_{3,4} \\ -R_3 \end{matrix}$ $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \left[\begin{array}{c|c} 0 & 2 \\ 0 & 3 \\ 0 & 1 \\ 0 & 1 \end{array}\right]$

By $R_3 - R_2$ $\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \left[\begin{array}{c|c} 0 & 2 \\ 0 & 3 \\ 0 & -2 \\ 0 & 1 \end{array}\right]$

Equations are clearly consistent ($|A| = 1 \times 1 \times (-2) \times 1 = -2 \neq 0$)

Further, $c_1 + c_2 + c_3 = 2$, $\quad c_2 + c_3 - c_4 = 3$

$-2c_3 + c_4 = -2$, $\quad c_4 = 1$

$\therefore \; 2c_3 = 3 \quad \therefore \; c_3 = 3/2$

$$c_2 = 3 - c_3 + c_4 = 3 - \frac{3}{2} + 1 = \frac{5}{2}$$

$$\therefore \; c_1 = 2 - c_2 - c_3 = 2 - \frac{5}{2} - \frac{3}{2} = -2$$

$$\therefore \; v_s = \left(-2, \frac{5}{2}, \frac{3}{2}, 1\right)$$

(You can verify the result by actual substitution.)

6. Checking for Basis

The following theorem helps us to check whether the given set of vectors forms a basis for the space or not.

Theorem : If V is an n-dimensional vector space and if S is a set in V with exactly n vectors, then S is a basis for V if either (i) S spans V or (ii) S is linearly independent.

For example, in the Fig. 4.10 (a) given on the next page, the **two** vectors in R^2 lie in the same line and as such are not independent. They do **not** form a basis for R^2.

In the Fig. 4.10 (b), the **two** vectors in R^2 are linearly independent and as such they form a basis for R^2.

In the Fig. 4.10 (c), the vector v_2 and v_3 are in the same line and hence, they are not independent. Therefore, $S = \{v_1, v_2, v_3\}$ does not form a basis for R^3.

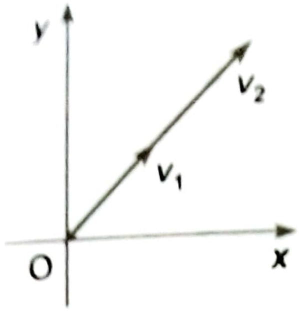

Fig. 4.10 (a)

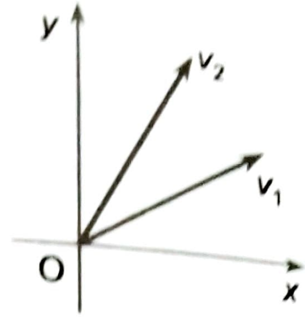

Fig. 4.10 (b)

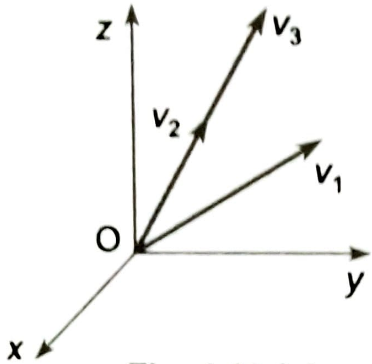

Fig. 4.10 (c)

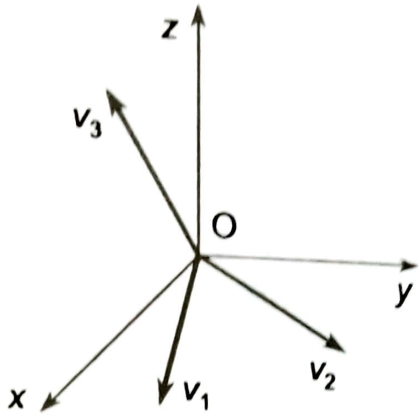

Fig. 4.10 (d)

In the Fig. 4.10 (d), the vectors v_1, v_2 are in the xy-plane and the vector v_3 is in the xz-plane. They are linearly independent and hence they form a basis for R^3.

Example 1 : Show by inspection whether the vectors

(a) $v_1 = (2, 5)$, $v_2 = (4, 10)$

(b) $v_1 = (3, 5)$, $v_2 = (6, 10)$

form a basis for R^2.

Sol. : (a) By inspection we see that v_2 is a scalar multiple of v_1 as $v_2 = 2v_1$, the two vectors lie in the same line. They are not linearly independent. Hence, they do not form a basis for R^2.

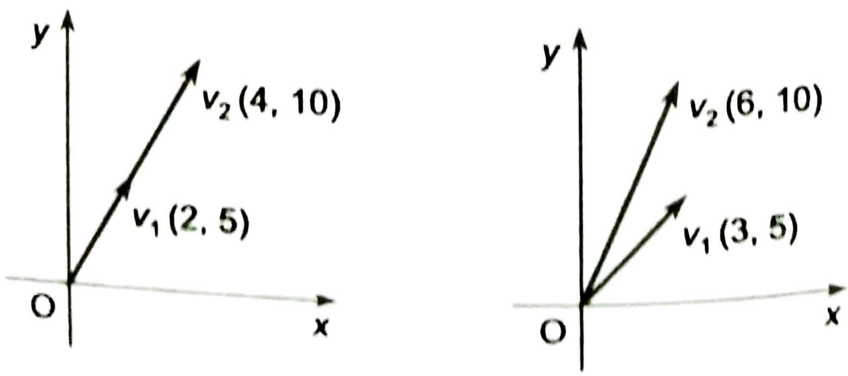

Fig. 4.11

(b) Since neither vector is a scalar multiple of the other, they are linearly independent.

Further any vector (a, b) in R^2 can be expressed as a linear combination of v_1, v_2 as shown below.

$$(a, b) = c_1 v_1 + c_2 v_2$$

$$= c_1 (3, 5) + c_2 (6, 11) \qquad \text{.......... (1)}$$

$$\therefore \quad \begin{bmatrix} 3 & 6 \\ 5 & 11 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\text{By} \quad R_2 - \frac{5}{3} R_1 \begin{bmatrix} 3 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a \\ b - (5/3)\,a \end{bmatrix}$$

$$\therefore \quad 3\,c_1 + 6\,c_2 = a$$

$$c_2 = b - \frac{5}{3} a = -\frac{5}{3} a + b$$

$$\therefore \quad 3c_1 = a + 10a - 6b$$

$$\therefore \quad c_1 = \frac{11a}{3} - 2b$$

$$\therefore \quad (a, b) = \left(\frac{11a}{3} - 2b\right) v_1 + \left(-\frac{5}{3} a + b\right) v_2$$

$$= \left(\frac{11a}{3} - 2\,b\right)(3, 5) + \left(-\frac{5}{3} a + b\right)(6, 11)$$

(You can verify the equality.)

Since the two vectors are linearly independent and they span R^2, they form a basis for R^2.

Example 2 : Show by inspection whether the vectors form a basis for R^3.

$$v_1 = (1, 3, 6), \ v_2 = (3, 0, 0)$$

and $v_3 = (6, 0, 5)$.

Sol. : It is easy to see that since y-coordinates of v_2 and v_3 are zero, they lie in the xz-plane. [Fig. 6.12]

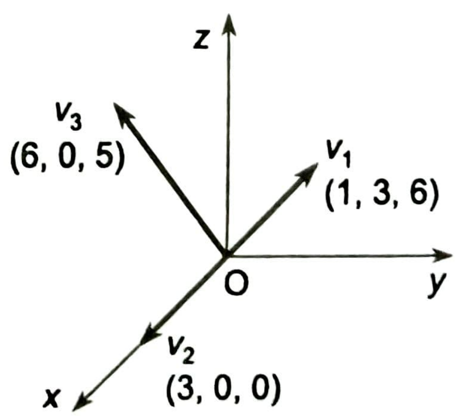

Fig. 4.12

Since y-coordinates of $v_1 \neq 0$, it does not lie in the xz-plane. Hence, the set $\{ v_1, v_2, v_3 \}$ is linearly independent.

$\therefore$ $S = \{v_1, v_2, v_3\}$ forms a basis for R^3.

[Further, if (l, m, n) is any vector in R^3 then let

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = (l, m, n) \qquad \text{.......... (1)}$$

$$\begin{bmatrix} 1 & 3 & 6 \\ 3 & 0 & 0 \\ 6 & 0 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_2 - 3R_1 \\ R_3 - 2R_2 \end{matrix} \begin{bmatrix} 1 & 3 & 6 \\ 0 & -9 & -18 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} l \\ m - 3l \\ n - 2m \end{bmatrix}$$

$$\therefore \quad c_1 + 3c_2 + 6c_3 = l$$

$$-9c_2 - 18c_3 = m - 3l \qquad \therefore \quad 9c_2 + 18c_3 = -m + 3l$$

$$5c_3 = n - 2m$$

$$\therefore \quad c_3 = \frac{n}{5} - \frac{2}{5}m$$

$$\therefore \quad 9c_2 = -m + 3l - 18\,c_3$$

$$= -m + 3l - \frac{18\,n}{5} + \frac{36}{5}m$$

$$= 3l + \frac{31}{5}m - \frac{18\,n}{5}$$

$$\therefore \quad c_2 = \frac{l}{3} + \frac{31}{45}m - \frac{18}{45}m$$

$$c_1 = l - 3\,c_2 - 6\,c_3$$

$$= l - l - \frac{31}{15}m + \frac{18}{15}n - \frac{6}{5}n + \frac{12}{5}m$$

$$= \frac{5}{15}m = \frac{1}{3}m.$$

Hence, putting these values in (1), we get

$$c_1\,v_1 + c_2\,v_2 + c_3\,v_3$$

$$= \frac{m}{3}(1,\ 3,\ 6) + \left(\frac{l}{3} + \frac{31}{45}m - \frac{18}{45}n\right)(3,\ 0,\ 0) + \left(\frac{n}{5} - \frac{2m}{5}\right)(6,\ 0,\ 5)$$

$$= \left(\frac{m}{3} + l + \frac{31m}{15} - \frac{18\,n}{15} + \frac{6\,n}{5} - \frac{12\,m}{5}\right) + (m + 0 + 0) + (2m + n - 2m)$$

$$= l + m + n\ ]$$

EXERCISE - I

1. State by inspection which of the following sets are not bases. Justify your answer.

(a) (i) $v_1 = (2,\ 3),\ \ v_2 = (3,\ -1),\ \ v_3 = (5,\ 2)$ for R^2.

(ii) $v_1 = (1,\ -3),\ v_2 = (3,\ 4),\ v_3 = (5,\ -2)$ for R^2.

(iii) $v_1 = (2,\ 4),\ \ v_2 = (1,\ -2),\ \ v_3 = (1,\ 0)$ for R^2.

[**Ans. : (i)** No. $v_1 + v_2 = v_3$. Not independent.

(ii) No. $2v_1 + v_2 = v_3$. Not independent.

(iii) Yes. They are independent. A basis for R^2 should have two linearly independent vectors.]

(b) (i) $v_1 = (2, 1, -4)$, $v_2 = (3, -2, 8)$ for R^3.

(ii) $v_1 = (2, 3, -2)$, $v_2 = (4, 6, -4)$, $v_3 = (5, -3, -9)$ for R^3.

(iii) $v_1 = (2, 4, -6)$, $v_2 = (1, 2, 2)$, $v_3 = (3, 6, -4)$

(iv) $v_1 = (2, 3, 1)$, $v_2 = (-1, 2, 3)$, $v_3 = (1, 2, 1)$

[**Ans. : (i)** No. **(ii)** No. $v_2 = 2v_1$ **(iii)** $v_1 + v_2 = v_3$

(iv) Yes. They are independent. A basis for R^3 should have three linearly independent vectors.]

2. State by inspection which of the following are not bases. Justify your answer.

(i) $p_1 = 1 + x - x^2$, $p_2 = 1 + 2x$

(ii) $p_1 = 1 - x + x^2$, $p_2 = 2 + x - 2x^2$, $p_3 = 3 - x^2$

(ii) $p_1 = 2$, $p_2 = 1 + x$, $p_3 = 1 - x^2$, $p_4 = 4 + x - x^2$

(iv) $p_1 = 1 + x + x^2$, $p_2 = 1 + x$, $p_3 = 1$.

[**Ans. : (i)** No. **(ii)** No. $p_1 + p_2 = p_3$

(iii) No. $p_1 + p_2 + p_3 = p_4$

(iv) Yes. A basis for P_2 must have three linearly independent vectors.]

3. Can the following sets of vectors be bases

(i) $(1, 2, -5)$, $(2, 4, 3)$ for R^3.

(ii) $(2, 4, -1)$, $(1, 3, -5)$, $(2, -4, -7)$ for R^4.

[**Ans. :** No. They cannot span R^3 and R^4 respectively as at least 3 (independent) vectors are required for R^3 and at least 4 (independent) vectors are required for R^4.]

4. Which of the following sets of vectors are bases for R^2 ?

(a) $(2, 1)$, $(4, 0)$ (b) $(4, 1)$, $(3, -2)$

(c) $(3, 9)$, $(-4, -12)$ (d) $(4, 6)$, $(6, 9)$ [**Ans. :** (a), (b)]

5. Which of the following sets of vectors are bases for R^3 ?

(a) $(1, 2, 1)$, $(2, 9, 0)$, $(3, 3, 4)$

(b) $(2, -3, 1)$, $(4, 1, 1)$, $(0, -7, 1)$

(c) $(1, 6, 4)$, $(2, 4, -1)$, $(-1, 2, 5)$ [**Ans. :** (a)]

6. Which of the following sets of vectors are bases for R^3 ?

(a) $(1, 0, 0)$, $(3, 3, 0)$, $(5, 5, 5)$

(b) $(1, 0, 0)$, $(a, b, 0)$, (l, m, n)

(c) $(2, 4, -1)$, $(-1, 2, 5)$, $(1, 6, 4)$

(d) $(1, 4, 2)$, $(2, 6, 3)$, $(-2, 3, 4)$ [**Ans.** : (a), (b), (d)]

7. Which of the following sets of vectors are basis for P_2

(i) $-1 + 4x + 2x^2$, $4 + 6x + x^2$, $5 + 2x - x^2$

(ii) $1 + x + x^2$, $2x + 3x^2$, $4x^2$

(iii) $6 + 5x + 2x^2$, $-4 + x + 3x^2$, $6 + 4x + x^2$

[**Ans.** : **(i)** Yes, **(ii)** No, **(iii)** No.]

8. Find the coordinate vector of w relative to the basis $S = \{ u_1, u_2 \}$ in R^2 where

(i) $u_1 = (1, 0)$, $u_2 = (0, 1)$, $w = (4, -6)$ [**Ans.** : $(4, -6)$]

(ii) $u_1 = (1, 1)$, $u_2 = (0, 2)$, $w_2 = (3, 4)$ [**Ans.** : $(3, 1/2)$]

9. Find the coordinate vector of v relative to the basis $S = \{ v_1, v_2, v_3 \}$ where

(i) $v_1 = (1, 2, 3)$, $v_2 = (-4, 5, 6)$, $v_3 = (7, -8, 9)$, $v = (5, -12, 3)$

(ii) $v_1 = (1, 0, 0)$, $v_2 = (2, 2, 0)$, $v_3 = (3, 3, 3)$, $v = (2, -1, 3)$

[**Ans.** : **(i)** $v_5 = (-2, 0, 1)$, **(ii)** $v_5 = (3, -2, 1)$]

10. Find the coordinate vector of $w =$ relative to $S = \{ v_1, v_2 \}$ for R^2.

(a) $v_1 = (1, 0)$, $v_2 = (0, 1)$, $w = (6, 9)$

(b) $v_1 = (1, 1)$, $v_2 = (0, 2)$, $w = (5, 7)$

(c) $v_1 = (2, -6)$, $v_2 = (4, 3)$, $w = (2, 9)$

(d) $v_1 = (1, 1)$, $v_2 = (-1, 2)$, $w = (a, b)$

[**Ans.** : **(a)** $(6, 9)$, **(b)** $(5, 2)$, **(c)** $(-1, 1)$, **(d)** $\left(\frac{b-a}{3} + a, \frac{b-a}{3} \right)$]

11. Find the coordinate vector v relative to the basis $S = \{ v_1, v_2, v_3 \}$

(i) $v = (2, -1, 3)$; $v_1 = (1, 0, 0)$, $v_2 = (2, 2, 0)$, $v_3 = (3, 3, 3)$

(ii) $v = (5, -12, 3)$; $v_1 = (1, 2, 3)$, $v_2 = (-4, 5, 6)$, $v_3 = (7, -8, 9)$

(iii) $v = (1, 2, 3)$; $v_1 = (1, 1, 1)$, $v_2 = (1, -1, 1)$, $v_3 = (0, 1, 1)$

(iv) $v = (1, 2, 3)$; $v_1 = (1, 1, 1)$, $v_2 = (1, -2, -1)$, $v_3 = (2, 1, 1)$

[**Ans.** : **(i)** $(3, -2, 1)$, **(ii)** $(-2, 0, 1)$, **(iii)** $(1/2, 1/2, 1)$, **(iv)** $(8, 1, -4)$]

12. Find the vector v in R^3 whose coordinate vector with respect to the basis S is

(i) $(3, -1, 4)$ and $S = \{ v_1, v_2, v_3 \}$ as in Ex. 11 (i) above.

(ii) $(4, 2, -7)$ and $S = \{ v_1, v_2, v_3 \}$ as in Ex. 11 (ii) above.

(iii) $(-4, 3, -1)$ and $S = \{ v_1, v_2, v_3 \}$ as in Ex. 11 (iii) above.

(iv) $(3, -1, -1)$ and $S = \{ v_1, v_2, v_3 \}$ as in Ex. 11 (iii) above.

[**Ans. :** **(i)** $(13, 10, 12)$, **(ii)** $(-53, 74, -39)$,
(iii) $(-1, -8, -2)$, **(iv)** $(0, 4, 3)$.]

13. Which of the following sets of vectors are bases for P_2 ?

(a) $1 + 3x + 2x^2,\ 1 - x + 4x^2,\ 1 + 7x$.

(b) $1 + 3x - 6x^2,\ -1 - 2x + 4x^2,\ 5$

(c) $4 + 6x + x^2,\ -1 + 4x + 2x^2,\ 5 + 2x - x^2$

(d) $1 + 2x + 3x^2,\ 3x + 4x^2,\ 5x^2$

(e) $1 + 3x + 4x^2,\ 2 + 6x + 9x^2,\ 1 + 2x + 3x^2$

(f) $x + x^2,\ 2x + x^2,\ 2 + 5x + 3x^2$

[**Ans.** ; (d), (e), (f)]

14. Find the coordinates of the vector p, with respect to the vectors p_1, p_2, p_3.

(i) $p_1 = x - x^2,\ p_2 = 1 + x,\ p_3 = 1 + 2x^2,\ p = 1 + x + x^2$.

(ii) $p_1 = 1 + x + x^2,\ p_2 = -1 + x,\ p_3 = 1 - x^2,\ p = 1$

(iii) $p_1 = 1 + x + x^2,\ p_2 = -1 + x,\ p_3 = 1 - x^2,\ p = x^2$.

[**Ans. : (i)** $(1, 0, 1)$, **(ii)** $\left(\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}\right)$, **(iii)** $\left(\frac{1}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$]

15. Show that S given below is a basis for M_{22} and find the coordinate vector of A relative to $S = \{A_1, A_2, A_3, A_4\}$ where

$$A_1 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix},\ A_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

(i) $A = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix}$

[**Ans. : (i)** $v_s = (3, -1, 4, 1)$; **(ii)** $v_s = (-2, 1, -1, 2)$]

7. Dimension

You might have observed that all basis for R^2 have two vectors, all bases for R^3 have three vectors, all bases for R^n have n vectors. This can be stated as a theorem.

Theorem : All bases for a finite dimensional vector space has the same number of vectors.

The number of vectors in a basis is called the dimension of the vector space.

We define the dimension of zero vector to be zero, the dimension of a line to be one, the dimension of a plane to be two and the dimension a cube to be three.

Definition : The number of a vectors in a basis for V is called the **dimension** of the vector V and is denoted by dim (V).

We have already seen that the standard basis for vector space R^n is

$$e_1 = (1, 0, ..., 0),\ e_2 = (0, 1, ..., 0), ..., e_n = (0, 0, ..., 1)$$

Hence, $$\boxed{\dim (R^n) = n}$$

We have noted above that the standard basis for vector space of P_n of polynomials of the form $a_0 + a_1 x + ... + a_n x^n$ is $S = \{1, x, x^2, ..., x^n\}$.

Hence, $$\boxed{\dim (P_n) = n + 1}$$

We have also noted above that for matrices of order mn,

$$\boxed{\dim (mn) = mn}$$

8. Dimension of A Solution Space

We have seen how to solve a system of linear equations. We have also noted that in certain systems of linear equations, the solution set contains independent variables called parameters. Given a system of linear equations by solving the system we can find the basis and the dimensions of the solution space as illustrated below.

To find the dimension and basis for the solution space of the system of equations

(i) Write the equations in matrix form and find the solution set.

(ii) The number of independent variables called parameters in the solution set is the dimension.

(iii) Vectors obtained by putting the parameters equal to one form the basis of the solution space.

Example 1 : Determine the dimension and basis for the solution space of the system

$$3x_1 + x_2 + x_3 + x_4 = 0$$
$$5x_1 - x_2 + x_3 - x_4 = 0$$

Sol. : We shall first solve the given system of equations. The equations in the matrix form can be written as

$$\begin{bmatrix} 3 & 1 & 1 & 1 \\ 5 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

By $R_2 + R_1$ $\begin{bmatrix} 3 & 1 & 1 & 1 \\ 8 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By $\frac{1}{8} R_2$ $\begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 0 & 1/4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By R_{21} $\begin{bmatrix} 1 & 0 & 1/4 & 0 \\ 3 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

By $R_2 - 3R_1$ $\begin{bmatrix} 1 & 0 & 1/4 & 0 \\ 0 & 1 & 1/4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Hence, we get

$$x_1 + \frac{1}{4} x_3 = 0$$

$$x_2 + \frac{1}{4} x_3 + x_4 = 0$$

Putting $x_4 = t$, $x_3 = s$, we get

$$x_2 = -\frac{1}{4} s - t \quad \text{and} \quad x_1 = -\frac{1}{4} s.$$

∴ The general solution is

$$x_1 = -\frac{1}{4} s, \quad x_2 = -\frac{1}{4} s - t, \quad x_3 = s, \quad x_4 = t.$$

We can write the solution vector as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -(1/4)s \\ -(1/4)s - t \\ s \\ t \end{bmatrix} = \begin{bmatrix} -(1/4)s + 0t \\ -(1/4)s - (1)t \\ 1(s) + 0(t) \\ 0(s) + 1(t) \end{bmatrix}$$

$$= \begin{bmatrix} (-1/4)s \\ (-1/4)s \\ s \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -t \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} -1/4 \\ -1/4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

This shows that vectors

$$v_1 = \begin{bmatrix} -1/4 \\ -1/4 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{span the solution space.}$$

Now, $k_1 v_1 + k_2 v_2 = 0$

$$\begin{bmatrix} -1/4 & 0 \\ -1/4 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{gives}$$

$$-\frac{1}{4} k_1 + 0\, k_2 = 0 \qquad \therefore \quad k_1 = 0$$

$$-\frac{1}{4} k_1 - k_2 = 0 \qquad \therefore \quad k_2 = 0$$

$\therefore$ v_1 and v_2 are linearly independent.

Hence, $\{ v_1, v_2 \}$ is a basis of the solution space and since the number of vectors is two, the dimension of the solution space is two.

Example 2 : Determine the dimension and basis for the solution space of the system

$$\begin{aligned} 2x_1 + 2x_2 - x_3 \qquad\quad + x_5 &= 0 \\ -x_1 - x_2 + 2x_3 - 3x_4 + x_5 &= 0 \\ x_1 + x_2 - 2x_3 \qquad\quad - x_5 &= 0 \\ x_3 + x_4 + x_5 &= 0 \end{aligned}$$

Sol. : We shall first solve the given system of equations. The equations in the matrix form can be written as

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By } \begin{matrix} R_1 + R_2 \\ R_2 + R_3 \end{matrix} \begin{bmatrix} 1 & 1 & 1 & -3 & 2 \\ 0 & 0 & 0 & -3 & 0 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By } R_3 - R_1 \begin{bmatrix} 1 & 1 & 1 & -3 & 2 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & -3 & 3 & -3 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

By $\begin{matrix} R_4 + (1/3)\,R_3 \\ -(1/3)\,R_3 \\ -(1/3)\,R_2 \end{matrix}$ $\begin{bmatrix} 1 & 1 & 1 & -3 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

By $\begin{matrix} R_3 + R_2 \\ R_4 - 2R_2 \end{matrix}$ $\begin{bmatrix} 1 & 1 & 1 & -3 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

By $R_1 + 3R_2$ $\begin{bmatrix} 1 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

By $\begin{matrix} R_1 - R_3 \\ R_{23} \end{matrix}$ $\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Hence, we get

$$\begin{aligned} x_1 + x_2 \quad\quad + x_5 &= 0 \\ x_3 \quad + x_5 &= 0 \\ x_4 \quad &= 0 \end{aligned}$$

Putting $x_5 = t$, we get

$$x_4 = 0,\ \ x_3 = -t,\ \ x_1 + x_2 + t = 0$$

Putting $x_2 = s$, we get $x_1 = -s - t$.

Thus, the general solution is

$$x_1 = -s - t,\ \ x_2 = s,\ \ x_3 = -t,\ \ x_4 = 0,\ \ x_5 = t.$$

We can write the solution vector as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} -1s-1t \\ 1s+0t \\ 0s-1t \\ 0s+0t \\ 0s+1t \end{bmatrix} = \begin{bmatrix} -s \\ s \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -t \\ 0 \\ -t \\ 0 \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

The vectors $v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ span the solution space.

Now, $k_1 v_1 + k_2 v_2 = 0$ gives $\begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore\ -k_1 - k_2 = 0$

$\therefore\ k_1 = 0 \qquad \therefore\ k_2 = 0$

$\therefore\ v_1, v_2$ are linearly independent.

Hence, $\{ v_1, v_2 \}$ is a basis of the solution space and since the number of vectors is two, the dimension of the solution space is two.

EXERCISE - II

Determine the dimension and basis for the solution space of the system.

(i)
$$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ 2x_1 + x_2 - 2x_3 &= 0 \\ x_1 - x_3 &= 0 \end{aligned}$$

(ii)
$$\begin{aligned} x_1 - 4x_2 + 2x_3 - x_4 &= 0 \\ 2x_1 - 8x_2 + 4x_3 - 2x_4 &= 0 \end{aligned}$$

(iii)
$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_1 - 4x_2 + 2x_3 &= 0 \\ 3x_1 - 6x_2 + 3x_3 &= 0 \end{aligned}$$

(iv)
$$\begin{aligned} x_1 + 3x_2 + 3x_3 + 2x_4 &= 0 \\ 2x_1 + 6x_2 + 9x_3 + 5x_4 &= 0 \\ x_1 + 3x_2 - 3x_3 &= 0 \end{aligned}$$

(v)
$$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 0 \\ 2x_1 + 4x_2 + 8x_3 &= 0 \\ 3x_1 + 6x_2 + 12x_3 &= 0 \end{aligned}$$

[**Ans. :** **(i)** Dim 1, $v_1 = \{ 1, 0, 1 \}$

(ii) Dim 3; $v_1 = \{ 4, 1, 0, 0 \}$, $v_2 = \{ -2, 0, 1, 0 \}$, $v_3 = \{ 1, 0, 0, 1 \}$

(iii) Dim 2; $v_1 = \{ 2, 1, 0 \}$, $v_2 = \{ -1, 0, 1 \}$

(iv) Dim 1; $v_1 = \{ 4, -5, 1 \}$

(v) Dim 2; $v_1 = \{ -3, 1, 0, 0 \}$, $v_2 = \{ -1, 0, -1/3, 1 \}$

(vi) Dim 2; $v_1 = \{ -2, 1, 0 \}$, $v_2 = \{ -4, 0, 1 \}$]

9. Row Space, Column Space, Null Space

Definition : Consider the matrix,

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

the vectors formed by the rows *viz.*

$$r_1 = [\ a_{11},\ a_{12},\ \ldots,\ a_{1n}\ ]$$
$$r_2 = [\ a_{21},\ a_{22},\ \ldots,\ a_{2n}\ ]$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$r_m = [\ a_{m1},\ a_{m2},\ \ldots,\ a_{mn}\ ]$$

in R^n are called **row vectors** of A and the vectors formed by the columns *viz.*

$$c_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \ldots \\ a_{m1} \end{bmatrix},\ c_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \ldots \\ a_{m2} \end{bmatrix},\ \ldots,\ c_m = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \ldots \\ a_{mn} \end{bmatrix}$$

in R^n are called **column vector** of A.

For example, if $A = \begin{bmatrix} 2 & 1 & 0 & 3 \\ 1 & 2 & 1 & -2 \\ 3 & -1 & 2 & 1 \end{bmatrix}$ then

$$r_1 = [\ 2, 1, 0, 3\ ],$$
$$r_2 = [\ 1, 2, 1, -2\ ],$$
$$r_3 = [\ 3, -1, 2, 1\ ]$$

are the row vectors of A in R^4 and

$$c_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix},\ c_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix},\ c_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},\ c_4 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

are the column vectors of A in R^3.

Definition : If A is an $m \times n$ matrix then the subspace of R^n spanned by the row vectors of A is called the **row space** of A.

The subspace of R^m spanned by the column vectors of A is called the **column space** of A.

The solution space of the homogeneous equation $AX = 0$ which is a subspace of R^n is called the **null-space** of A.

10. To Find Bais for Row Space, Column Space and Null Space

If a matrix A is in **row-echelon form**, then the **row vectors with leading ones** form a basis for row space and **column vectors (containing leading ones)** form a basis for column vectors.

The **solution vectors of A** form the basis for the null space of A.

Example 1 : Find a basis for row and column spaces of the matrix

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Sol. : The rows of the matrix A with leading ones form a basis for row space. There are three such rows, *viz.* r_1, r_2, r_3.

$$\therefore \quad r_1 = [\, 1, 3, -2, 0, 3 \,]$$

$$r_2 = [\, 0, 1, 3, 0, 2 \,]$$

$$r_3 = [\, 0, 0, 0, 1, 0 \,]$$

The columns which contain the leading ones form a basis for column space. There are three such columns *viz.* c_1, c_2 and c_3.

$$c_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Example 2 : Find a basis for row and column spaces of A where

$$A = \begin{bmatrix} 1 & 4 & 5 & 4 \\ 2 & 9 & 8 & 2 \\ 2 & 9 & 9 & 7 \\ -1 & -4 & -5 & -4 \end{bmatrix}$$

Sol. : We first reduce the above matrix to row-echelon form by elementary row operations.

$$\text{By } \begin{matrix} R_2 - 2R_1 \\ R_3 - 2R_1 \\ R_4 + R_1 \end{matrix} \begin{bmatrix} 1 & 4 & 5 & 4 \\ 0 & 1 & -2 & -6 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{By } R_3 - R_2 \begin{bmatrix} 1 & 4 & 5 & 4 \\ 0 & 1 & -2 & -6 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rows of the matrix with leading ones form a basis for row space. There are three such rows *viz.* r_1, r_2, r_3.

$\therefore$ The basis for row space is

$$r_1 = [\; 1, 4, 5, 4 \;]$$

$$r_2 = [\; 0, 1, -2, -6 \;]$$

$$r_3 = [\; 0, 0, 1, 5 \;]$$

The columns which contain the leading ones form a basis for column space. There are three such columns *viz.* c_1, c_2 and c_3.

$\therefore$ The basis for column space is

$$c_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

Example 3 : Find a basis for the row space and column space for

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

Sol. : We first reduce the above matrix to row-echelon form.

$$\text{By} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 - R_2 \\ R_4 + R_1 \end{matrix} \quad \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rows of the matrix with leading ones form a basi ; for row space. There are three such rows *viz.* r_1, r_3, r_5.

$\therefore$ A basis for row space is

$$r_1 = [\,1, -3, 4, -2, 5, 4\,]$$

$$r_2 = [\,0, 0, 1, 3, -2, -6\,]$$

$$r_3 = [\,0, 0, 0, 0, 1, 5\,]$$

The columns of the matrix which contain leading ones form a basis for column space. There are three such columns *viz.* c_1, c_3, c_5.

$\therefore$ A basis for column space is

$$c_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad c_3 = \begin{bmatrix} -5 \\ -2 \\ 1 \\ 0 \end{bmatrix}.$$

Example 4 : Find a basis for the null-space of

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & 4 \\ 7 & -6 & 2 \end{bmatrix}$$

Sol. : The null-space of A is the solution space of the system of homogeneous equations

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 0 \\ 5x_1 - 4x_2 - 4x_3 &= 0 \\ 7x_1 - 6x_2 + 2x_3 &= 0 \end{aligned}$$

We now solve this system of equations.

$$\begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $\begin{matrix} R_2 - 5R_1 \\ R_3 - 7R_1 \end{matrix}$ $\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 1 & -19 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By $R_3 - R_2$ $\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore \quad x_1 - x_2 + 3x_3 = 0, \qquad x_2 - 19x_3 = 0$

Putting $x_3 = t$, $x_2 = 19t$, then $x_1 = 16t$.

$$\therefore \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 16t \\ 19t \\ t \end{bmatrix} = t\begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$$

$\therefore$ The basis for the null-space is the vector $v = \begin{bmatrix} 16 \\ 19 \\ 1 \end{bmatrix}$.

Example 5 : Find a basis for the null-space of

$$A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix}$$

Sol. : The null-space of A is the solution space of the system of homogeneous equations

$$\begin{aligned} x_1 + 4x_2 + 5x_3 + 2x_4 &= 0 \\ 2x_1 + x_2 + 3x_3 &= 0 \\ -x_1 + 3x_2 + 2x_3 + 2x_4 &= 0 \end{aligned}$$

We now solve the system of equations

$$\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 + R_1 \end{matrix} \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{By} \quad \begin{matrix} R_3 + R_2 \\ -R_2 \end{matrix} \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 7 & 7 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \quad x_1 + 4x_2 + 5x_3 + 2x_4 = 0$$

$$7x_2 + 7x_3 + 4x_4 = 0$$

Putting $x_3 = s,\ x_4 = t$, we get

$$7x_2 = -7s - 4t \quad \therefore \quad x_2 = -s - \frac{4}{7}t$$

and $\quad x_1 = -4x_2 - 5x_3 - 2x_4$

$$= 4s + \frac{16}{7}t - 5s - 2t$$

$$= -s + \frac{2}{7}t$$

$$\therefore \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s + (2/7)t \\ -s - (4/7)t \\ s + 0 \\ 0 + t \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2/7 \\ -4/7 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ The basis for the null-space is $\{ v_1, v_2 \}$ where

$$v_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2/7 \\ -4/7 \\ 0 \\ 1 \end{bmatrix}$$

Example 6 : Find the basis for null-space, column space and row space for A.

$$A = \begin{bmatrix} 2 & -4 & 1 & 2 & -2 & -3 \\ -1 & 2 & 0 & 0 & 1 & -1 \\ 10 & -4 & -2 & 4 & -2 & 4 \end{bmatrix}$$

Sol. : We first solve the set of homogeneous equations

$$2x_1 - 4x_2 + x_3 + 2x_4 - 2x_5 - 3x_6 = 0$$

$$-x_1 + 2x_1 + 0x_3 + 0x_4 + x_5 - x_6 = 0$$

$$10x_1 - 4x_2 - 2x_3 + 4x_4 - 2x_5 + 4x_6 = 0$$

$$\begin{bmatrix} 2 & -4 & 1 & 2 & -2 & -3 \\ -1 & 2 & 0 & 0 & 1 & -1 \\ 10 & -4 & -2 & 4 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $\quad R_{2,1} \; -R_1$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & -1 & 1 \\ 2 & -4 & 1 & 2 & -2 & -3 \\ 10 & -4 & -2 & 4 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $\quad R_2 - 2R_1, \; R_3 - 10R_1$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 & -5 \\ 0 & 16 & -2 & 4 & 8 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $\quad \frac{1}{16} R_3$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 & -5 \\ 0 & 1 & -1/8 & 1/4 & 1/2 & -3/8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

By $\quad R_{23}$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1/8 & 1/4 & 1/2 & -3/8 \\ 0 & 0 & 1 & 2 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore$ The rows with leading ones form a basis for row space. There are three such rows viz. r_1, r_2, r_3.

$\therefore$ The basis for rows space is

$$r_1 = [\; 1, -2, 0, 0, -1, 1 \;]$$

$$r_2 = [\; 0, 1, -1/8, 1/4, 1/2, -3/8 \;]$$

$$r_3 = [\; 0, 0, 1, 2, 0, -5 \;]$$

The columns which contain the leading ones form a basis for column space. There are three such columns viz. c_1, c_2, c_3.

$\therefore$ The basis for column space is

$$c_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad c_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 0 \\ -1/8 \\ 1 \end{bmatrix}$$

Now, the solution is given by

$$x_1 - 2x_2 + 0x_3 + 0x_4 - x_5 + x_6 = 0$$

$$0x_1 + x_2 - \frac{1}{8}x_3 + \frac{1}{4}x_4 + \frac{1}{2}x_5 - \frac{3}{8}x_6 = 0$$

$$0x_1 + 0x_2 + x_3 + 2x_4 + 0x_5 - 5x_6 = 0$$

Since there are six unknowns and three equations, there are $6 - 3 = 3$ parameters.

Putting $x_4 = t_1$, $x_5 = t_2$, $x_6 = t_3$, we get

$$x_1 = 2x_2 + x_5 - x_6$$

$$x_2 = \frac{1}{8}x_3 - \frac{1}{4}x_4 - \frac{1}{2}x_5 + \frac{3}{8}x_6, \quad x_3 = -2x_4 + 5x_6$$

$$\therefore \quad x_3 = -2\,t_1 + 5\,t_3$$

$$\therefore \quad x_2 = \frac{1}{8}(-2t_1 + 5t_3) - \frac{1}{4}t_1 - \frac{1}{2}t_2 + \frac{3}{8}t_3$$

$$= -\frac{1}{2}t_1 - \frac{1}{2}t_2 + t_3$$

$$\therefore \quad x_1 = -t_1 - t_2 + 2\,t_3 + t_2 - t_3$$

$$= -t_1 + t_3$$

$$\therefore \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -t_1 + 0t_2 + t_3 \\ -(1/2)t_1 - (1/2)t_2 + t_3 \\ -2t_1 + 0t_2 + 5t_3 \\ t_1 + 0t_2 + 0t_3 \\ 0 + t_2 + 0t_3 \\ 0 + 0 + t_3 \end{bmatrix}$$

$$= t_1 \begin{bmatrix} -1 \\ -1/2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ -1/2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} 1 \\ 1 \\ 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ The basis for the null-space is $S = \{ v_1, v_2, v_3 \}$ where

$$v_1 = \begin{bmatrix} -1 \\ -1/2 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ -1/2 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 1 \\ 5 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

EXERCISE - III

1. List the row vectors and column vectors of the matrix

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & -1 & 1 & 0 \\ 3 & 4 & -1 & 2 \end{bmatrix}$$

[**Ans. :** $r_1 = [\,1, 2, 0, -1\,]$, $r_2 = [\,2, -1, 1, 0\,]$, $r_3 = [\,3, 4, -1, 2\,]$

$$c_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \quad c_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad c_4 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

2. Find a basis for the null-space of A where

(i) $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 5 & 6 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 2 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$

(iii) $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & -1 \\ 1 & 3 & 2 & 4 \end{bmatrix}$ (iv) $A = \begin{bmatrix} 4 & -1 & 2 & 1 \\ 2 & 3 & -1 & -2 \\ 0 & 7 & -4 & -5 \\ 2 & 11 & 7 & 8 \end{bmatrix}$

[**Ans. :** (i) $v = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ (ii) $v_1 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1/2 \\ 3/2 \\ 1 \\ 0 \end{bmatrix}$

(iii) $v = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ (iv) $v_1 = \begin{bmatrix} -5/14 \\ 4/7 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1/14 \\ 5/7 \\ 0 \\ 1 \end{bmatrix}$

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