Basis of a Vector

Definition

Let ***V*** be a vector space. A minimal set of vectors in ***V*** that spans ***V*** is called a **basis** for ***V***.

Equivalently, a **basis** for V is a set of vectors that

- is linearly independent;
- spans ***V***.

As a result, to check if a set of vectors form a basis for a vector space, one needs to check that it is linearly independent and that it spans the vector space. If at least one of these conditions fail to hold, then it is not a basis.

Procedure

Approach 1.

Get a spanning set for the vector space, then reduce this set to a basis.

Proposition Let $v_0, v_1, \ldots, v_k$ be a spanning set for a vector space V.

If v_0 is a linear combination of vectors $v_1, \ldots, v_k$ then $v_1, \ldots, v_k$ is also a spanning set for **V**. Indeed, if $v_0 = r_1v_1 + \cdots + r_kv_k$, then $t_0 v_0 + t_1v_1 + \cdots + t_kv_k = = (t_0r_1 + t_1) v_1 + \cdots + (t_0r_k + t_k)v_k$

Approach 2.

Build a maximal linearly independent set adding one vector at a time.

If the vector space **V** is trivial, it has the empty basis. If **V** $\neq \{0\}$, pick any vector $v_1 \neq 0$. If v_1 spans **V**, it is a basis.

Otherwise pick any vector $v_2 \in V$ that is not in the span of v_1. If v_1 and v_2 span **V**, they constitute a basis.

Otherwise pick any vector $v_3 \in V$ that is not in the span of v_1 and v_2. And so on.

History

The idea of a vector space developed from the notion of ordinary two- and three-dimensional spaces as collections of vectors ***{u, v, w, …}*** with an associated field of real numbers ***{a, b, c, …}.*** Vector spaces as abstract algebraic entities were first defined by the Italian mathematician ***Giuseppe Peano in 1888.*** Peano called his vector spaces "linear systems" because he correctly saw that one can obtain any vector in the space from a linear combination of finitely many vectors and scalars ***-av + bw + … + cz***. A set of vectors that can generate every vector in the space through such linear combinations is known as a spanning set. The dimension of a vector space is the number of vectors in the smallest spanning set. (For example, the unit vector in the x-direction together with the unit vector in the y-direction suffice to generate any vector in the two-dimensional Euclidean plane when combined with the real numbers.)

Significance

Every vector has a magnitude and a direction. Magnitudes and directions are easy to visualize as actual directions in ordinary space. But vector spaces are not confined to only 3 dimensions. We can generalize them to any dimension. For simplicity let us start from a 3 dimensional case . To describe a 3 d vector we can say "go x units in the +x direction, y units in the +y direction and z units in the

+ z direction if you want to reach the destination point." The amazing thing is that we can reach every possible destination by specifying only these three numbers (x, y, z). Now someone might say why stop at 3? Why not use 4 directions? The thing is that 3 is the smallest number of directions that can describe all points in our 3d space. 2 is too few and the 4th one is redundant as we can describe the fourth direction as a combination of the other three directions. Thus we say that we have 3 basis vectors (directions) which span (i.e. can describe every point) a 3 dimensional vector space. (This is no coincidence as the word dimension itself is the total number of directions required or the cardinality of the set of basis vectors) Now we have made a crucial assumption here. The assumption is that the 3 directions that we used are themselves independent of each other. For example N and E are independent directions but N and N-E aren't. Saying, " go 1.41 units in the N-E direction and 1 unit in the N" can be broken down to, " go 2 units towards N and 1 unit towards E. From the above example we may conclude that in order to be mutually independent, the directions should be at right angles to each other. But this simple test cannot work for higher dimensions where the definition of an angle itself needs upgradation.

Applications

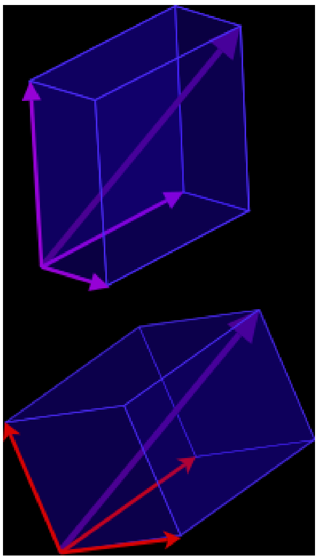
One of the most important applications of vector spaces in computer technology is image processing. The screen you are no doubt looking is nothing more than a dynamic 2D array or matrix of "pixels" each of which carries a "vector" of information representing brightness, colour, etc. This is generally encoded by "fast Fourier transforms" which is a "frequency-domain" basis, as opposed to the "spatial-domain" you are looking at. Transmission of these data from one computer to another represents a reorganization of this kind of multidimensional array into something more like a linear array (or parallel linear arrays) which again represent a sort of change of basis to a space of another dimension.

Geometric Representation and Interpretation

A set B is a basis if its elements are linearly independent and every element of V is a linear combination of elements of B. In other words, a basis is a linearly independent spanning set.

A vector space can have several bases; however all the bases have the same number of elements, called the dimension of the vector space.

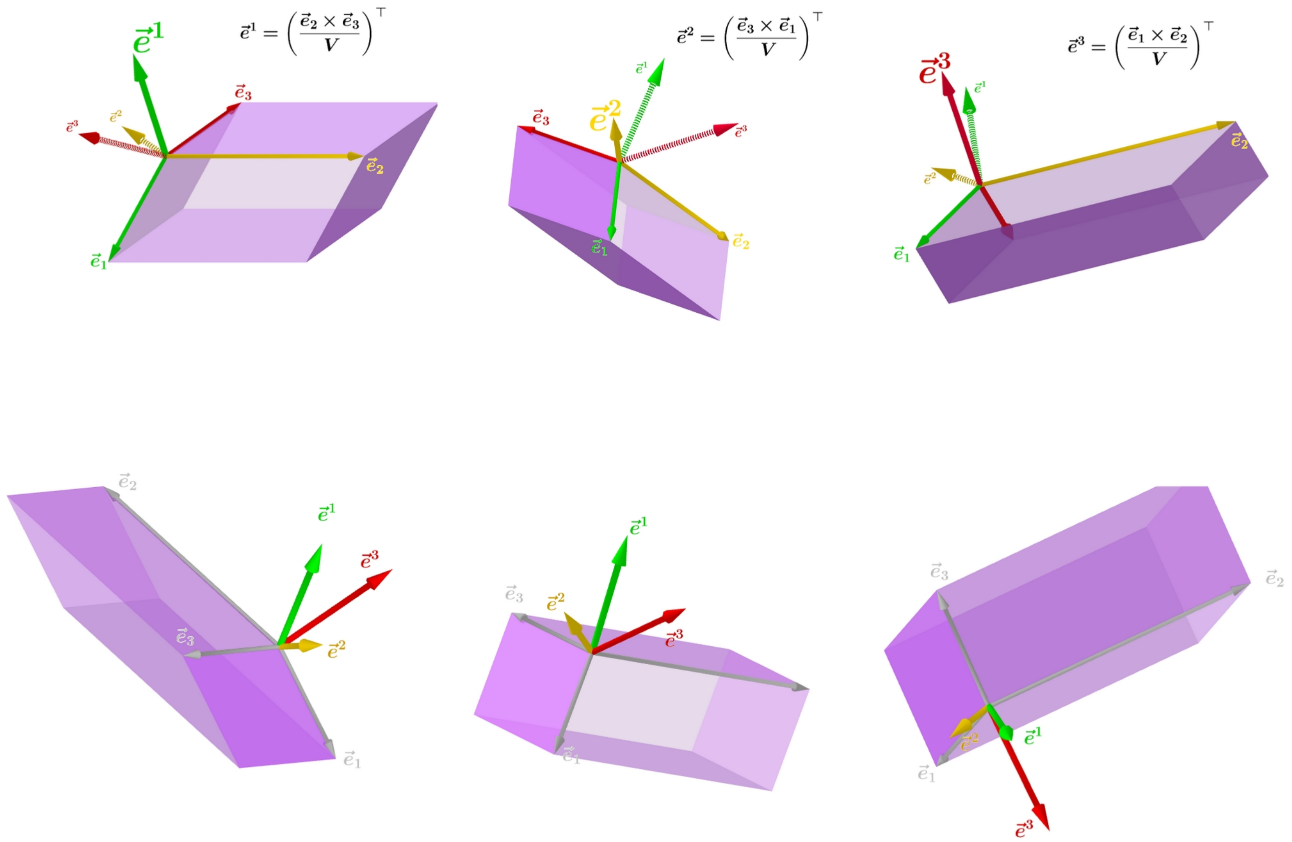
This article deals mainly with finite-dimensional vector spaces. However, many of the principles are also valid for infinite-dimensional vector spaces.



The same vector can be represented in two different bases (purple and red arrows).

V = (e$_1$;e$_2$;e$_3$) = e$_1$ · (e$_2$×e$_3$) = e$_2$ · (e$_3$×e$_1$) = e$_3$ · (e$_1$×e$_2$) is the volume of the parallelepiped formed by the basis vectors **e$_1$, e$_2$ and e$_3$.**

To avoid misunderstandings, says that the cross product of every combination of the basis vectors (for example **e$_2$ x e$_3$** , which would yield a vector with magnitude equal to the surface of the parallelogram defined by the vectors **e$_2$ and e$_3$**), scaled down by the volume of the parallelepiped **V** will result in a different basis vectors element for the dual space (after transposing).



* Basis

1) Is the set $S = \{(1,1), (1,-1)\}$ a basis of R^2?

We can write the set S as equations :

$$\therefore c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}x\\y\end{pmatrix}$$

$c_1 + c_2 = x$ ——— ①
$c_1 - c_2 = y$ ——— ②

$$\left[\begin{array}{cc|c}1 & 1 & x\\1 & -1 & y\end{array}\right]$$ $R_2 \to R_2 - R_1$, By Matrix Transformation

$$\left[\begin{array}{cc|c}1 & 1 & x\\0 & -2 & y-x\end{array}\right]$$, we obtain the ~~Basdoo~~ Row Echelon form of the Matrix

$\because$ This system is consistent for every x and y, therefore S spans in R^2.

Now, checking for linear dependency

$$c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$c_1 + c_2 = 0$ ——— ③
$c_1 - c_2 = 0$ ——— ④

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right) \quad R_2 \to R_2 - R_1$$

$$\left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & -2 & 0 \end{array}\right)$$ $\therefore$ The system has a unique solution $c_1 = c_2 = 0$ (Trivial).

Therefore S is linearly independent; Consequently, S is basis for R^2.

2) Find a basis for the vector space V spanned by vector $w_1 = (1,1,0)$, $w_2 = (0,1,1)$, $w_3 = (2,3,1)$ and $w_4 = (1,1,1)$

We need to find relation of the form $r_1 w_1 + r_2 w_2 + r_3 w_3 + r_4 w_4 = 0$, where $r_i \in R$ are not equal to zero.

$$\begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

By Row Reduction,

$$R_2 \to R_2 - R_1 \quad \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$R_3 \to R_3 - R_2$ $\begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$R_1 \to R_1 - R_3$ $\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ (Reduced Row echelon form)

$\therefore$ $r_1 + 2r_3 = 0$ $\quad r_1 = -2r_3$

$r_2 + r_3 = 0$ $\iff$ $r_2 = -r_3$

$r_4 = 0$ $\quad r_4 = 0$

$\therefore (r_1, r_2, r_3, r_4) = (-2t, -t, t, 0)$, $t \in R$

$\therefore$ Particular solution at $t = -1$,

$(r_1, r_2, r_3, r_4) = (2, 1, -1, 0)$

$\therefore$ We obtain $2w_1 + w_2 - w_3 = 0$ from Particular soln.

Hence, any of vectors w_1, w_2, w_3 can be dropped.

For instance $V = \text{Span}(w_1, w_2, w_4)$

Check whether w_1, w_2, w_4 are linearly independent

$\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} \neq 0$ $\therefore w_1, w_2, w_4$ are linearly Independent

$\Rightarrow V = R^3$ and (w_1, w_2, w_4) is a basis for V.

3) Is $S = \{1, t, t^2, t^3\}$ a basis for P_3?

We can write the set S as equations

$$c_1(1) + c_2 t + c_3 t^2 + c_4 t^3 = a + bt + ct^2 + dt^3$$

has a solution for every a, b, c and d.

$c_1 = a$, $c_2 = b$, $c_3 = c$ and $c_4 = d$

Therefore S spans P_3.

Now, checking the linear dependency

$c_1(1) + c_2(t) + c_3 t^2 + c_4 t^3 = 0$ can only be solved by $c_1 = c_2 = c_3 = c_4 = 0$ (Trivial)

Therefore, S is linearly Independent.

Consequently, S is a basis for P_3.

4) Is set $S = \left\{ \begin{pmatrix} a & b \\ b & d \end{pmatrix} \mid a, b, d \in \mathbb{R} \right\}$ is a basis for S?

Consider, the vectors $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

$$\because \begin{pmatrix} a & b \\ b & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{--- (1)}$$

Therefore, the vectors can be spanned as in Eq. (1) and it is clearly seen that this set is linearly Independent.

Hence, $\left\{\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\right\}$ is a basis of S.

HW 5) Find whether the set $\{1, t+1, t-1\}$ is a basis of P_2

Soln. Not a basis of P_2.