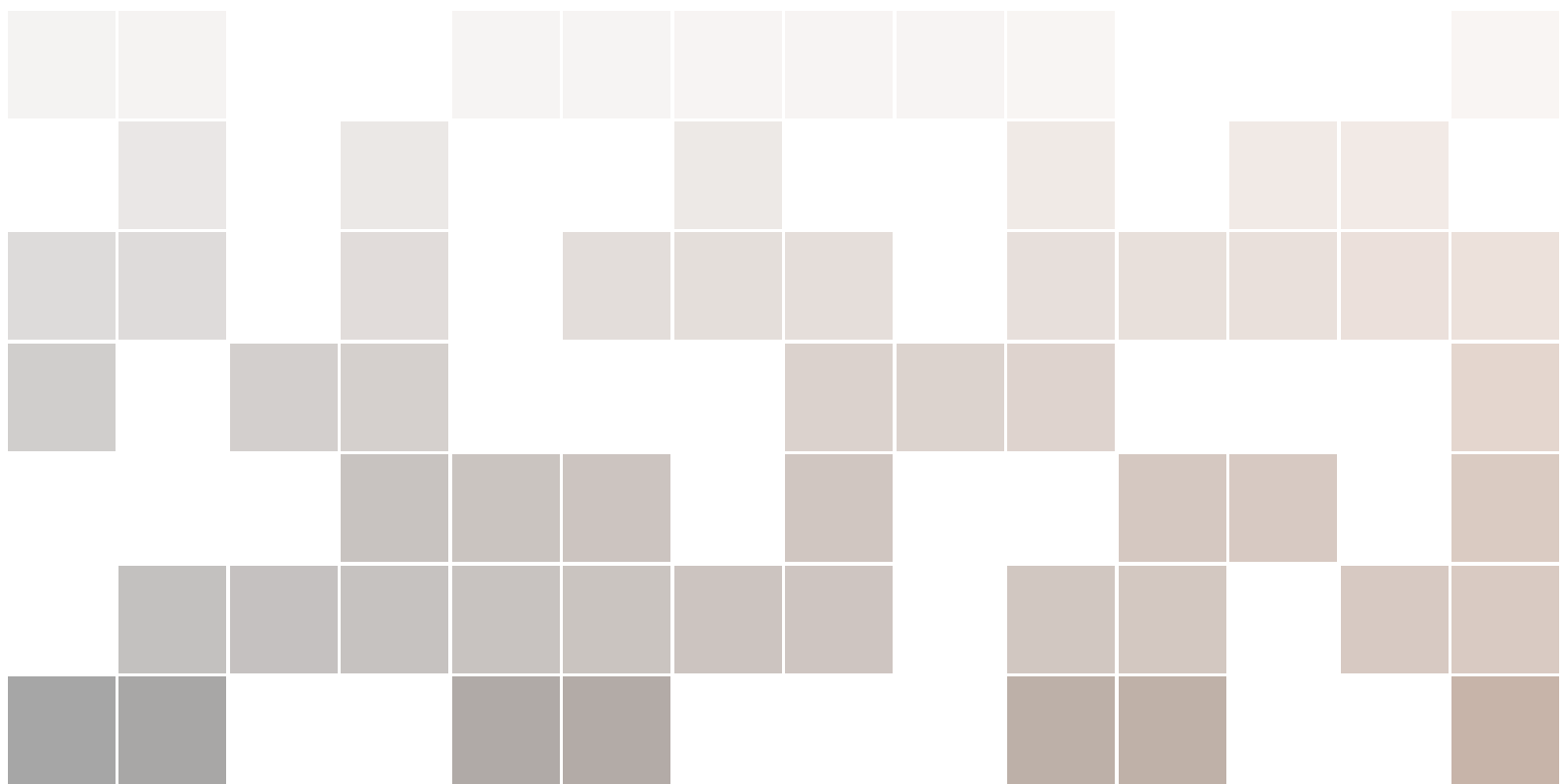


Particle Physics

# Preliminaries for Particle Physics

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# Preface

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## Preface

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This is a introductory text for preliminaries of particle physics:

- Analytical Mechanics
- Special Relativity
- Physics of Fields
- Quantum Mechanics
- Some Mathematics
- Extra topics

The intended audience is from a motivated high school student to a 1st/ 2nd year undergraduate student, who want a sneak peak/ a birds-eye view of what particle physics is about. I am working on this document as a personal study material and a reference for myself, so I can refer to it when I need to. I will likely be updating this document later, so if you find any mistakes, please let me know.

As a result, most of the content is very basic, so if you really want to dig into each subject, I strongly recommend that you read actual textbooks on each subjects. Some good/ known textbooks in English are listed at the end, so make sure to check them out. I would like to hear some suggestions too!

If you have any questions, please feel free to contact me via Discord, or through Github. I will try to answer/ fix any issues as it could be a good learning experience for myself too.

Lastly, Extra topics are kind of a miscellaneous section, where I will put some topics that I find interesting, but redundant/ not necessary for the main content. You can skip them if you want.

# CONTENTS

<b>Chapter 1</b>	<b>Notations and Conventions</b>	<b>Page 1</b>
1.1	General Notations	1
	Vectors and Functions — 1	
1.2	Relativity Notations	1
	Vectors(in Relativity) — 1 • Einstein's Summation Convention — 2 • Metric Tensor — 2	
	• Units — 2	
<b>Chapter 2</b>	<b>Analytical Mechanics</b>	<b>Page 3</b>
2.1	Lagrange Formalism	3
	Newtonian Mechanics — 3 • Lagrangian and Variational Principle — 5 • Harmonic Oscillator — 6	
2.2	Hamiltonian Formalism	7
	Legendre Transform — 7 • Hamiltonian and Canonical Equations — 8 • Poisson Bracket — 10	
2.3	Symmetry of a System	11
	Symmetry — 11 • Transformations — 12 • Point Transformation — 13 • Symmetry including Time — 14 • Noether's Theorem — 15	
<b>Chapter 3</b>	<b>Physics of Fields</b>	<b>Page 18</b>
3.1	Introduction: 1D String	18
	Newtonian Derivation — 18	
3.2	Lagrange Formalism	20
	Potential and Momentum Density — 20 • Generalization to General Field — 22	
3.3	Hamilton Formalism	24
	Legendre Transformation — 24 • Canonical Equations of Fields — 25 • Poisson Bracket — 26	
<b>Chapter 4</b>	<b>Quantum Mechanics</b>	<b>Page 29</b>
4.1	Hilbert Space in Quantum Mechanics	29
	Hilbert Space — 29 • Basis Vectors and Completeness Relation — 30	
4.2	Linear Algebra Approach	31
	Basic Concepts — 31 • Position and Momentum in Quantum Mechanics — 32 • Unitary Transformations: Shift Operator and Fourier Transform — 33	

4.3	Harmonic Oscillator	33
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## Chapter 5 Mathematical Remarks Page 34

5.1	Functional Derivative	34
	Definition — 34 • Two function case — 35 • Euler-Lagrange Equation — 36 • Important Property — 37 • In a n-dimensional space — 38	
5.2	Group Theory	39

## Chapter 6 Extra Page 40

6.1	Invariance of Euler-Lagrange Equation	40
6.2	Schrödinger Equation	43
	Historical Background — 43 • Schrödinger Equation — 43	
6.3	Derivative by Derivative	45
	An Interesting Problem — 45	
6.4	Variational Principle in Fluid Mechanics	49
	Overview: Cauchy's Equation of Motion — 49 • Lagrange Derivative — 50 • Derivation from Action Integral — 52 • Hamilton Formalism — 53	
6.5	Physical Constants	53
6.6	Textbooks	54
	General Textbooks — 54 • Analytical Mechanics — 54 • Quantum Mechanics — 54 • Particle Physics — 54	

# Notations and Conventions

## 1.1 General Notations

### 1.1.1 Vectors and Functions

A vector (in a non-relativistic context) is denoted by an arrow:

$$\vec{x}, \vec{r}, \vec{p}, \vec{F}, \vec{v}, \vec{a}, \dots \quad (1.1)$$

If a function depends on multiple variables (e.g.,  $\{q_i\} = \{q_1, q_2, q_3, \dots\}$ ), we may denote it in a short-handed way:

$$f(q_1, q_2, q_3, \dots) \iff f(q_i) \quad (1.2)$$

A partial derivative by the variable  $x$  is denoted by:

$$\frac{\partial f}{\partial x} = \partial_x f, \quad (1.3)$$

For a general coordinate system  $(x_1, x_2, x_3, \dots)$ , we denote the partial derivative by:

$$\frac{\partial f}{\partial x_i} = \partial_i f \quad (1.4)$$

## 1.2 Relativity Notations

### 1.2.1 Vectors(in Relativity)

For 4-vectors and tensors in relativity, sans-serif font is used:

$$x, p^\mu, g_{\mu\nu} \quad (1.5)$$

Each component of a 4-vector is denoted by a superscript or subscript:

$$x^\mu = x^0, x^1, x^2 \text{ or } x^3 \quad (1.6)$$

and without an index, it represents the entire vector:

$$x = (x^0, x^1, x^2, x^3) = (ct, \vec{x}) \quad (1.7)$$

### 1.2.2 Einstein's Summation Convention

If an index such as  $i, j, k, \mu, \nu$ , etc., appears twice in a single term, it implies summation over that index. For example:

#### Example 1.2.1 (Examples)

$$x_i y_i = \vec{x} \cdot \vec{y} = \sum_{i=1}^3 x_i y_i \quad (1.8)$$

$$g_{\mu\nu} u^\mu v^\nu = \sum_{\mu, \nu=0}^3 g_{\mu\nu} u^\mu v^\nu \quad (1.9)$$

Note that the greek index such as  $\mu, \nu$  usually runs from 0 to 3, while the latin index such as  $i, j, k$  usually runs from 1 to 3.

### 1.2.3 Metric Tensor

As per the convention in particle physics, we take the  $(+, -, -, -)$  metric signature, and the Minkowski metric  $\eta_{\mu\nu}$  is denoted by:

$$\eta_{\mu\nu} = \begin{pmatrix} \eta_{00} & \eta_{01} & \eta_{02} & \eta_{03} \\ \eta_{10} & \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{20} & \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{30} & \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad (1.10)$$

### 1.2.4 Units

In principle, we use SI units:

$$\text{Length} = [L] = \text{m} \quad (1.11)$$

$$\text{Time} = [T] = \text{s} \quad (1.12)$$

$$\text{Mass} = [M] = \text{kg} \quad (1.13)$$

$$\text{Charge} = [C] = \text{C} \quad (1.14)$$

However, for particle physics, we often use natural units, where:

$$\hbar = c = 1 \quad (1.15)$$

and the unit of energy in electronvolts (eV) is used.

# Analytical Mechanics

## 2.1 Lagrange Formalism

### 2.1.1 Newtonian Mechanics

In Newtonian mechanics, the motion of a particle is described through a few important quantities: for a particle of (inertial) mass  $m$ , position  $\vec{r}$ , we have

$$\text{velocity : } \vec{v} = \frac{d\vec{r}}{dt} \quad (2.1)$$

$$\text{acceleration : } \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \quad (2.2)$$

$$\text{momentum : } \vec{p} = m\vec{v} = m\frac{d\vec{r}}{dt} \quad (2.3)$$

and the relations between these quantities, in the presence of external forces  $\vec{F}_{\text{ext}}^{(i)}$  acting on the particle, are given by *Newton's second law*:

$$\frac{d\vec{p}}{dt} = m\frac{d^2\vec{r}}{dt^2} = \sum_i \vec{F}_{\text{ext}}^{(i)} \quad (2.4)$$

The work done by such forces is given by

$$W_{\text{total}} = \sum_i \int_l d\vec{x} \cdot \vec{F}_{\text{ext}}^{(i)}, \quad \text{where } l \text{ is the path of the particle.} \quad (2.5)$$

This is the energy change of the particle through the motion:

$$W_{\text{total}} = \sum_i \int_l d\vec{x} \cdot \vec{F}_{\text{ext}}^{(i)} \quad (2.6)$$

$$= \int_{t_i}^{t_f} dt \vec{v} \cdot m \frac{d\vec{v}}{dt} \quad (2.7)$$

$$= \int_{t_i}^{t_f} dt \frac{m}{2} \frac{d}{dt} \vec{v}^2 \quad (2.8)$$

$$= \frac{m}{2} \vec{v}_f^2 - \frac{m}{2} \vec{v}_i^2 \quad (2.9)$$

## 2 Analytical Mechanics

meaning that  $m\vec{v}^2/2$  is the energy due to the motion of the particle: the **kinetic energy**  $T$ :

$$T = \frac{m}{2} \vec{v}^2 \quad (2.10)$$

Now, often, the external force acting on the particle is due to a potential  $V$ :

$$\vec{F}_{\text{ext}} = -\nabla V \quad (2.11)$$

For example, for a 1D spring, the potential is given by

**Example 2.1.1** (1D spring/ Harmonic potential)

$$V = \frac{1}{2} kx^2 \implies F_{\text{ext}} = -kx \quad (2.12)$$

or the electrostatic potential:

**Example 2.1.2** (Electrostatic potential)

$$V = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r} \implies F_{\text{ext}} = -\nabla V = -\frac{q_1 q_2}{4\pi\epsilon_0 r^2} \hat{r} \quad (2.13)$$

Now, for a particle whose the external forces are given by a potential:

$$m \frac{d^2 \vec{r}}{dt^2} = -\nabla V \iff -m \frac{d^2 \vec{r}}{dt^2} - \nabla V = 0 \quad (2.14)$$

This looks as if the forces  $-\nabla V$  and  $m\ddot{\vec{r}}$  are in equilibrium. So, if we move a particle by an infinitesimal distance  $\delta\vec{r}$ , the total work done by these forces must be zero:

$$\left( -m \frac{d^2 \vec{r}}{dt^2} - \nabla V \right) \cdot \delta\vec{r} = 0 \quad (2.15)$$

at any time  $t$ . We want to apply this for entire path of the motion of the particle, from  $t_i$  to  $t_f$ . Then, the integral of this equation over the time interval  $[t_i, t_f]$  gives

$$\delta I = \int_{t_i}^{t_f} dt \left( -m \frac{d^2 \vec{r}}{dt^2} - \nabla V \right) \cdot \delta\vec{r} = 0 \quad (2.16)$$



now, we can apply integration by parts:

$$\frac{d}{dt} [\dot{\vec{r}} \cdot \delta \vec{r}] = \ddot{\vec{r}} \cdot \delta \vec{r} + \dot{\vec{r}} \cdot \frac{d \delta \vec{r}}{dt} \quad (2.17)$$

$$= \ddot{\vec{r}} \cdot \delta \vec{r} + \dot{\vec{r}} \cdot \dot{\delta \vec{r}} \quad (2.18)$$

$$\Leftrightarrow -m\ddot{\vec{r}} = -\frac{d}{dt} (\dot{\vec{r}} \cdot \delta \vec{r}) + \dot{\vec{r}} \cdot \dot{\delta \vec{r}} \quad (2.19)$$

$$= -\frac{d}{dt} (m\dot{\vec{r}} \cdot \delta \vec{r}) + \delta \left( \frac{m}{2} \left( \frac{d\vec{r}}{dt} \right)^2 \right), \quad (2.20)$$

where we used the commutativity of  $\frac{d(\cdot)}{dt}$  and  $\delta(\cdot)$ . Then the integral becomes

$$\delta I = \int_{t_i}^{t_f} dt \left( \delta \left[ \frac{m}{2} \left( \frac{d\vec{r}}{dt} \right)^2 \right] - \delta V \right) \quad (2.21)$$

$$= \int_{t_i}^{t_f} dt (\delta T - \delta V) = 0 \quad (2.22)$$

$$\Leftrightarrow \delta I = \delta \int_{t_i}^{t_f} dt (T - V) = 0 \quad (2.23)$$

[1]

## 2.1.2 Lagrangian and Variational Principle

In Lagrangian mechanics, we will use a different approach to describe the motion of a particle than the Newtonian mechanics. Instead of using the usual Euclidean space, we will use a *configuration space*  $\mathcal{C}$ , which is the space of all possible positions of the particle. This space is spanned by the so-called **generalized coordinates**  $q_i$ , and their time derivatives (or generalized velocity)  $\dot{q}_i := \frac{dq_i}{dt}$ . Now, let us define quantities called the **Lagrangian**  $L$  and **action**  $S$ :

### Definition 2.1: Lagrangian $L$ and Action $S$

The Lagrangian  $L$  is defined as the difference between the kinetic energy  $T$  and potential energy  $V$  of a particle:

$$L := T - V, \quad S[L] := \int_{t_i}^{t_f} dt L \quad (2.24)$$

Now, **variation** of the action  $\delta S$  is given by

$$\delta S = \delta \int_{t_i}^{t_f} dt (T - V) \quad (2.25)$$

which is an identical expression to (2.23). So, we postulate that the motion of the particle is such that the action is stationary:

**Principle 2.1: Variational Principle**

The motion of a particle is such that the action  $S$  is stationary, i.e.,  $\delta S = 0$ .

$$\delta S = \delta \int_{t_i}^{t_f} dt L(q_i(t), \dot{q}_i(t), t) = 0 \quad (2.26)$$

As to show why this maybe useful, let us compare between Eq. (2.16):

$$\delta I = \delta S[L] = \int_{t_i}^{t_f} dt \left( -m \frac{d^2 \vec{r}}{dt^2} - \nabla V \right) \cdot \delta \vec{r} = 0 \quad (2.27)$$

and Eq. (5.27):

$$\delta I = \delta S[f] = \int_B dx \left( \frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial \frac{df}{dx}} \right) \delta f = 0 \quad (2.28)$$

which immediately gives that the EoM is the Euler-Lagrange equation, if we set  $F = L(q_i, \dot{q}_i, t)$ :

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = 0 \iff m \frac{d^2 \vec{r}}{dt^2} = -\nabla V \quad (2.29)$$

The generalization to multiple particles is straightforward:

$$L = \sum_n \frac{m_n}{2} \dot{\vec{r}}_n^2 - V(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) \quad (2.30)$$

The Euler-Lagrange equation is given by

$$\frac{\partial L}{\partial \dot{\vec{r}}_i^{(N)}} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\vec{r}}_i^{(N)}} \right) = 0 \quad (2.31)$$

### 2.1.3 Harmonic Oscillator

Let us consider a particle of mass  $m$  in a 1D harmonic potential:

$$V = \frac{1}{2} kx^2 \implies F_{\text{ext}} = -\nabla V = -kx \quad (2.32)$$

then the Lagrangian is given by

$$L = T - V = \frac{m}{2} \dot{x}^2 - \frac{1}{2} kx^2 \quad (2.33)$$

The Euler-Lagrange equation yields

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) = 0 \implies \frac{\partial L}{\partial x} = -kx, \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad (2.34)$$

$$\implies -kx - \frac{d}{dt}(m\dot{x}) = 0 \quad (2.35)$$

$$\implies m \frac{d^2 x}{dt^2} = -kx \quad (2.36)$$

which is the equation of motion of a harmonic oscillator in Newtonian mechanics. Here, notice that

$$\frac{\partial L}{\partial x} = -kx = -\nabla V = F_{\text{ext}} \quad (2.37)$$

and

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} = p \quad (2.38)$$

these derivatives of the Lagrangian are the **generalized force** and **conjugate momentum**, respectively.

$$\frac{\partial L}{\partial q} = F_g, \quad \frac{\partial L}{\partial \dot{q}} = p_g \quad (2.39)$$

## 2.2 Hamiltonian Formalism

### 2.2.1 Legendre Transform

Consider a function  $f(x, y)$ . The total differential of  $f$  is given by

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \quad (2.40)$$

Often we want to find another function  $g$  such that

$$g = p \cdot y - f(x, y) \quad (2.41)$$

and the total differential of  $g$  is given by

$$dg = \frac{\partial g}{\partial p} dp + \frac{\partial g}{\partial y} dy - df \quad (2.42)$$

$$= y dp + p dy - \frac{\partial f}{\partial x} dx - \frac{\partial f}{\partial y} dy \quad (2.43)$$

$$= y dp - \frac{\partial f}{\partial x} dx + \left( p - \frac{\partial f}{\partial y} \right) dy \quad (2.44)$$

for this function  $g$  to be a function of  $x$  and  $p$ , we need to have

$$p = \frac{\partial f}{\partial y}, \quad \frac{\partial g}{\partial x} = -\frac{\partial f}{\partial x}. \quad (2.45)$$

This is called the **Legendre transform** of  $f$ .

**Definition 2.2: Legendre transform (Analytical Mechanics)**

The Legendre transform of a function  $f(x, y)$  is defined as

$$g(p, x) = p \cdot y - f(x, y) \quad (2.46)$$

where

$$p = \frac{\partial f}{\partial y}, \quad \frac{\partial g}{\partial x} = -\frac{\partial f}{\partial x} \quad (2.47)$$

[2]

### 2.2.2 Hamiltonian and Canonical Equations

Now, we define Hamiltonian  $H = H(q, p)$  as the Legendre transform of Lagrangian  $L = L(q, \dot{q})$ :

**Definition 2.3: Hamiltonian  $H$**

The Hamiltonian  $H(q, p)$  is defined as the Legendre transform of the Lagrangian  $L(q, \dot{q})$  ( $\dot{q} \rightarrow p$ ):

$$H(q, p) = p \cdot \dot{q} - L(q, \dot{q}) \quad (2.48)$$

where

$$p = \frac{\partial L}{\partial \dot{q}}, \quad \frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} \quad (2.49)$$

**Note:**

In Lagrange formalism, generalized coordinate span the configuration space, while in Hamilton formalism, generalized coordinates and **generalized momenta** or **conjugate momenta** span the **phase space**.

This is actually a physically intuitive quantity. Since  $p$  is defined as the derivative of Lagrangian  $L$ :

$$p = \frac{\partial L}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}} \left( \frac{1}{2} m \dot{q}^2 - V(q) \right) = m \dot{q} \quad (2.50)$$

and we can rewrite the Hamiltonian as

$$H(q, p) = p \cdot \dot{q} - L(q, \dot{q}) \quad (2.51)$$

$$= m \cdot \dot{q}^2 - \left( \frac{1}{2} m \dot{q}^2 - V(q) \right) \quad (2.52)$$

$$= \frac{1}{2} m \dot{q}^2 + V(q) = T + V \quad (2.53)$$

This is the total energy of the system, for the case of non-velocity dependent potential  $V(q)$ .

This relation is useful because we can obtain Lagrangian  $L$  from the Hamiltonian  $H$  as well:

$$L(q, \dot{q}) = \dot{q} \cdot p - H(q, p) \implies \dot{q} = \frac{\partial H}{\partial p} \quad (2.54)$$

and the Euler-Lagrange equation can be rewritten in terms of Hamiltonian:

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) = 0 \implies -\frac{\partial H}{\partial q} - \frac{d}{dt}(p) = 0 \iff \dot{p} = -\frac{\partial H}{\partial q} \quad (2.55)$$

These two equations are called the **canonical equations** or **Hamilton's equations**:

### Theorem 2.2.1 Canonical Equations

The relationship between a mechanical variable  $q$  and its canonical conjugate variable  $p$  is given by the canonical equations:

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \quad (2.56)$$

For multiple degrees of freedom, we can write the Hamiltonian as

$$H(\{q_i\}, \{p_i\}, t) = \sum_i p_i \dot{q}_i - L(\{q_i\}, \{\dot{q}_i\}, t) \quad (2.57)$$

and the canonical equations as

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad (2.58)$$

where

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad \frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} \quad (2.59)$$

### 2.2.3 Poisson Bracket

Now, consider how a physical quantity  $X(q_i, p_i, t)$  changes with time:

$$\frac{dX(q_i, p_i, t)}{dt} = \frac{\partial X}{\partial t} + \frac{\partial X}{\partial q_i} \dot{q}_i + \frac{\partial X}{\partial p_i} \dot{p}_i \quad (2.60)$$

from the canonical equations, the second part becomes:

$$\frac{\partial X}{\partial q_i} \dot{q}_i + \frac{\partial X}{\partial p_i} \dot{p}_i = \frac{\partial X}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial X}{\partial p_i} \frac{\partial H}{\partial q_i} \quad (2.61)$$

Since this term only depend on  $X$  and  $H$ , given  $q_i$  and  $p_i$ , we can define a new quantity called the **Poisson bracket**:

#### Definition 2.4: Poisson Bracket

The Poisson bracket of two physical quantities  $A$  and  $B$  is defined as

$$\{A, B\} := \sum_i \left( \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right) \quad (2.62)$$

If a physical quantity  $X$  does not explicitly depend on time, we can rewrite the time derivative as:

$$\frac{dX(q_i, p_i)}{dt} = \{X, H\} \quad (2.63)$$

Due to this property, we often call the Hamiltonian  $H$  the **generator** of time evolution.

Now, what happens if  $X$  happens to be  $q_i$  or  $p_i$ ? A physical quantity  $q_i$  or  $p_i$  can be written as a function of  $q_i$  and  $p_i$  - simply as itself (noting that no explicit time dependence is present):

$$q_i(q_i, p_i) = q_i, \quad p_i(q_i, p_i) = p_i \quad (2.64)$$

Then, the Poisson bracket of  $q_i$  and  $H$  is given by

$$\{q_i, H\} = \sum_j \left( \frac{\partial q_i}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial q_i}{\partial p_j} \frac{\partial H}{\partial q_j} \right) \quad (2.65)$$

$$= \sum_j \left( \delta_{ij} \frac{\partial H}{\partial p_j} - 0 \right) = \frac{\partial H}{\partial p_i} \quad (2.66)$$

where Eq. (5.31) is used. Similarly, the Poisson bracket of  $p_i$  and  $H$  is given by

$$\{p_i, H\} = \sum_j \left( \frac{\partial p_i}{\partial q_j} \frac{\partial H}{\partial p_j} - \frac{\partial p_i}{\partial p_j} \frac{\partial H}{\partial q_j} \right) \quad (2.67)$$

$$= \sum_j \left( 0 - \delta_{ij} \frac{\partial H}{\partial q_j} \right) = -\frac{\partial H}{\partial q_i} \quad (2.68)$$

Thus, we can rewrite the canonical equations in terms of Poisson bracket:

**Theorem 2.2.2 Canonical Equations with Poisson Bracket**

The canonical equations can be rewritten in terms of Poisson bracket as follows:

$$\dot{q}_i = \{q_i, H\}, \quad \dot{p}_i = \{p_i, H\} \quad (2.69)$$

Finally, note that the Poisson bracket of  $q_i$  and  $p_j$  gives:

$$\{q_i, p_j\} = \sum_k \frac{\partial q_i}{\partial q_k} \frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k} \frac{\partial p_j}{\partial q_k} \quad (2.70)$$

$$= \sum_k \delta_{ik} \delta_{jk} - 0 = \delta_{ij} \quad (2.71)$$

**Theorem 2.2.3 Canonical Conjugate Relation in Analytical Mechanics**

The Poisson bracket of the generalized coordinate  $q_i$  and its conjugate momentum  $p_j$  is given by

$$\{q_i, p_j\} = \delta_{ij} \quad (2.72)$$

## 2.3 Symmetry of a System

### 2.3.1 Symmetry

A fundamental concept in physics is **symmetry**. Let us see an example of symmetry in geometry:

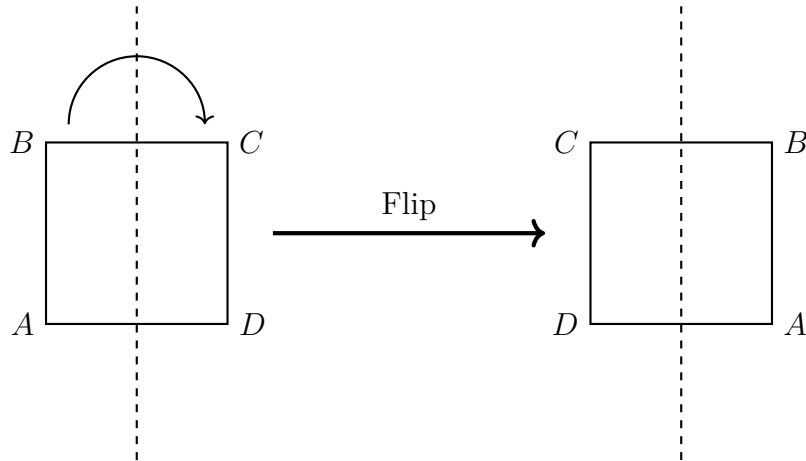


Figure 2.1: A square and its reflection.

Notice how we flipped the square, and it looks the same without the labels. In a more formal sense, we can say that the system(square) is **invariant** under the transformation(flip).

### 2.3.2 Transformations

For example, Newtonian mechanics(or Newton's equation of motion) has some symmetries.

#### Example 2.3.1 (Equation of Motion)

For example, the equation of motion is unchanged under Galilean transformation.

$$\sum \vec{F} = m\vec{a} \quad (2.73)$$

#### Definition 2.5: Galilean Transformation

The **Galilean transformation** is a transformation of coordinates from a stationary observer to a moving observer with constant velocity  $v$ .

$$\vec{\xi}(\vec{x}, t) = \vec{x} - \vec{v}t \quad (2.74)$$

where  $\vec{\xi}$  is the new coordinate,  $\vec{x}$  is the old coordinate,  $\vec{v}$  is the velocity of the observer, and  $t$  is time.

and space/ time reversal:

#### Definition 2.6: Space/Time Reversal

The **space reversal** is a transformation of coordinates that flips the sign of the position vector:

$$\vec{x}' = -\vec{x} \quad (2.75)$$

The **time reversal** is a transformation of time that flips the sign of time:

$$t' = -t \quad (2.76)$$

In the context of analytical mechanics, the system is described by the action  $S$ . If the variation of the action  $\delta S$  is invariant under a transformation, we say that the system has a symmetry under that transformation.

Notice that the Lagrangian has a degree of freedom(that is, the Lagrangian can be modified by adding a total derivative and variation of action remains the same):

#### Theorem 2.3.1 Degree of Freedom in Lagrangian

Adding a total derivative of time to the Lagrangian does not change the action:

$$L'(q_i, \dot{q}_i, t) = L(q_i, \dot{q}_i, t) + \frac{d}{dt}f(q(t), t) \implies \delta S'[L'] = \delta S[L] \quad (2.77)$$



**Proof:** The action is given by

$$S'[L'] = \int_{t_i}^{t_f} dt L'(q_i(t), \dot{q}_i(t), t) \quad (2.78)$$

$$= \int_{t_i}^{t_f} dt L(q_i(t), \dot{q}_i(t), t) + \int_{t_i}^{t_f} dt \frac{d}{dt} f(q(t), t) \quad (2.79)$$

$$= S[L] + \left[ f(q_i, t) \right]_{t_i}^{t_f} = S[L] + f(q_i(t_f), t_f) - f(q_i(t_i), t_i) \quad (2.80)$$

Since by taking the variation,  $\delta q(t_f) = \delta q(t_i) = 0$ , the last term is zero. Thus

$$\delta S'[L'] = \delta S[L] \quad (2.81)$$

□

### 2.3.3 Point Transformation

In Analytical Mechanics, there is a general family of transformations called **point transformation**.

#### Definition 2.7: Point Transformation

A **point transformation** is a transformation of the generalized coordinates  $q_i$  and time  $t$  to new coordinates  $Q_i$

$$Q_j = Q_j(q_i, t) \quad (2.82)$$

Note: if there are  $N$  generalized coordinates  $q_i$ , then there must be  $N$  new coordinates  $Q_j$ .

#### Example 2.3.2 (Point Transformation)

For example, changing from Cartesian coordinates to polar coordinates is a point transformation:

$$(x, y) \rightarrow (r, \theta) \quad \Longleftrightarrow \quad \begin{cases} r &= \sqrt{x^2 + y^2} \\ \theta &= \arctan\left(\frac{y}{x}\right) \end{cases} \quad (2.83)$$

$$(r, \theta) \rightarrow (x, y) \quad \Longleftrightarrow \quad \begin{cases} x &= r \cos \theta \\ y &= r \sin \theta \end{cases} \quad (2.84)$$

notice how each new coordinate only depends on the old coordinates, not the velocity.

Point transformation does not change the action, i.e., the variation of the action is invariant under point transformation:

### Theorem 2.3.2 Invariance of Euler-Lagrange Equation

The Euler-Lagrange equation is invariant under point transformation:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = 0 \implies \frac{\partial L}{\partial Q_j} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{Q}_j} \right) = 0 \quad (2.85)$$

proof is given in Sec. 6.1.

Now, using the point transformation, we can give the definition of symmetry, in the context of analytical mechanics:

#### Definition 2.8: Symmetry of the System

A system is said to have a **symmetry** under a point transformation  $q_i \rightarrow Q_i$  if the change in Lagrangian is up to the degree of freedom:

$$L(Q_i, \dot{Q}_i, t) = L(q_i, \dot{q}_i, t) + \underbrace{\frac{d}{dt} f(q_i, t)}_{\text{change in } L} \quad (2.86)$$

where  $L$  is the Lagrangian of the system before the point transformation. Note that the Lagrangian of LHS has the same functional form as RHS (i.e.  $Q$  and  $\dot{Q}$  are substituted for  $q$  and  $\dot{q}$ ). Or equivalently, the action is invariant under the point transformation:

$$S[Q_i, \dot{Q}_i, t] = S[q_i, \dot{q}_i, t] + \int_{t_i}^{t_f} dt \frac{d}{dt} f(q_i, t) = S[q_i, \dot{q}_i, t] \quad (2.87)$$

Importantly, the point transformation can be either discrete or continuous, and collection of point transformations that preserve the action is called a **group**. In short, it means three things:

1. There is an identity transformation (i.e. no change)
2. If there are multiple transformations, there is a single transformation that is the result of applying those transformations in sequence.
3. If there is a transformation, there is an **inverse transformation** that undoes the transformation.

This concept is very important in particle physics.

### 2.3.4 Symmetry including Time

So far we have seen symmetry under point transformation, but it is also useful to think about symmetry under transformations including time:

$$t' = t'(q_i, t) \quad (2.88)$$

$$Q_j(t') = Q_j(q_i, t) \quad (2.89)$$

and define

$$\dot{Q}_j := \frac{dQ_j(t')}{dt'} \quad (2.90)$$

For this transformation to be a symmetry of the system, the change in action must keep the variation of action invariant:

$$S[Q_i] = \int_{t'_i}^{t'_f} dt' L(Q_i(t'), \dot{Q}_i(t'), t') \quad (2.91)$$

$$= \int_{t_i}^{t_f} dt \frac{dt'}{dt} L(Q_i(t'), \dot{Q}_i(t'), t') \quad (2.92)$$

For the variation of action to be invariant, the change in Lagrangian must be up to the degree of freedom:

$$\frac{dt'}{dt} L(Q_i(t'), \dot{Q}_i(t'), t') = L(q_i, \dot{q}_i, t) + \frac{d}{dt} f(q_i, t) \quad (2.93)$$

Thus the symmetry under the transformation including time is defined as:

#### Definition 2.9: Symmetry of the System including Time

A system is said to have a **symmetry** under a transformation including time  $t' = t'(q_i, t)$  and  $Q_j(t') = Q_j(q_i, t)$  if the change in Lagrangian can be written as follows:

$$\frac{dt'}{dt} L(Q_i(t'), \dot{Q}_i(t'), t') = L(q_i, \dot{q}_i, t) + \frac{d}{dt} f(q_i, t) \quad (2.94)$$

### 2.3.5 Noether's Theorem

#### Theorem 2.3.3 Noether's Theorem (No time translation)

Every real parameter of a continuous symmetry of the action corresponds to a conserved quantity  $\mathcal{Q}$ . If the continuous symmetry is given by an infinitesimal transformation:  $Q_i(t) = q_i(t) + \epsilon F_i(q(t), t) + \mathcal{O}(\epsilon^2)$ ,

$$\mathcal{Q} = \sum_i F_i(q(t), t) \frac{\partial L(q(t), \dot{q}(t), t)}{\partial \dot{q}_i} - \Lambda(q(t), t) \quad (2.95)$$

where  $\Lambda(q(t), t)$  is the change in the Lagrangian under the infinitesimal transformation.

Now, assume that a system has a continuous symmetry. Thus if we consider a very "small" transformation that is almost the identity transformation:

$$Q_i(t) = q_i(t) + \epsilon F_i(q(t), t) + \mathcal{O}(\epsilon^2) \quad (2.96)$$

## 2 Analytical Mechanics

by differentiating w.r.t.  $t$ , we get

$$\dot{Q}_i(t) = \dot{q}_i(t) + \epsilon \frac{d}{dt} F_i(q(t), t) + \mathcal{O}(\epsilon^2) \quad (2.97)$$

Here, since the system has a symmetry under this transformation, the change in Lagrangian must be up to  $\frac{df}{dt}(q(t), t)$ :

$$L(Q_i, \dot{Q}_i, t) - L(q_i, \dot{q}_i, t) = \frac{d}{dt} f(q(t), t) \quad (2.98)$$

given that the transformation is infinitesimal, we can write the change in Lagrangian as:

$$\frac{d}{dt} f(q(t), t) = \epsilon \frac{d}{dt} \Lambda(q(t), t) + \mathcal{O}(\epsilon^2) \quad (2.99)$$

$$\Rightarrow L(Q_i, \dot{Q}_i, t) - L(q_i, \dot{q}_i, t) = \epsilon \frac{d}{dt} \Lambda(q(t), t) + \mathcal{O}(\epsilon^2) \quad (2.100)$$

Now, the new Lagrangian can be expanded as a Taylor series around the old Lagrangian with a small parameter  $\epsilon$ :

$$L(Q_i, \dot{Q}_i, t) = L(q_i, \dot{q}_i, t) + \sum_i \left( \epsilon F_i(q, t) \frac{\partial L(q_i, \dot{q}_i, t)}{\partial q_i} + \epsilon \frac{d}{dt} F_i(q, t) \frac{\partial L(q_i, \dot{q}_i, t)}{\partial \dot{q}_i} \right) + \mathcal{O}(\epsilon^2) \quad (2.101)$$

So the change in Lagrangian is:

$$L(Q_i, \dot{Q}_i, t) - L(q_i, \dot{q}_i, t) = \epsilon \sum_i \left( F_i(q, t) \frac{\partial L(q_i, \dot{q}_i, t)}{\partial q_i} + \frac{d}{dt} F_i(q, t) \frac{\partial L(q_i, \dot{q}_i, t)}{\partial \dot{q}_i} \right) + \mathcal{O}(\epsilon^2) \quad (2.102)$$

From the Euler-Lagrange equation, we know that

$$\frac{\partial L(q_i, \dot{q}_i, t)}{\partial q_i} = \frac{d}{dt} \left( \frac{\partial L(q_i, \dot{q}_i, t)}{\partial \dot{q}_i} \right) \quad (2.103)$$

the change in Lagrangian can be written as:

$$L(Q_i, \dot{Q}_i, t) - L(q_i, \dot{q}_i, t) = \epsilon \frac{d}{dt} \sum_i \left( F_i(q, t) \frac{\partial L(q_i, \dot{q}_i, t)}{\partial \dot{q}_i} \right) + \mathcal{O}(\epsilon^2) \quad (2.104)$$

By comparing Eq. (2.100) and Eq. (2.104), we can see that

$$\frac{d}{dt} \sum_i F_i(q(t), t) \frac{\partial L(q(t), \dot{q}(t), t)}{\partial \dot{q}_i} = \frac{d}{dt} \Lambda(q(t), t) \quad (2.105)$$

$$\Rightarrow \frac{d}{dt} \left[ \sum_i F_i(q(t), t) \frac{\partial L(q(t), \dot{q}(t), t)}{\partial \dot{q}_i} - \Lambda(q(t), t) \right] = 0 \quad (2.106)$$

Thus, the quantity:

$$\mathcal{Q} := \sum_i F_i(q(t), t) \frac{\partial L(q(t), \dot{q}(t), t)}{\partial \dot{q}_i} - \Lambda(q(t), t) \quad (2.107)$$

does not change with time, that is,  $\mathcal{Q}$  is conserved. This is **Noether's theorem**.

In the case where the transformation includes time translation:  $t' = t + \epsilon T(q(t), t) + \mathcal{O}(\epsilon^2)$ ,

$$L(Q_i(t'), \dot{Q}_i(t'), t') = L(Q_i(t), \dot{Q}_i(t), t) + \frac{d}{dt} f(Q_i(t), t) \quad (2.108)$$

[3]

# Physics of Fields

## 3.1 Introduction: 1D String

### 3.1.1 Newtonian Derivation

Now let us consider the motion of 1D wave on a very long string.

- Tension:  $T$
- Line density (mass per unit length):  $\mu$

Notice, that any point on the string is not moving in  $x$  direction, otherwise the string itself will be moving. Also, if we produce a "uniform" wave, then at all points on the string, the wave will have the same amplitude  $A$ , wavelength  $\lambda$ , and frequency  $f$  and velocity  $v$ . This means

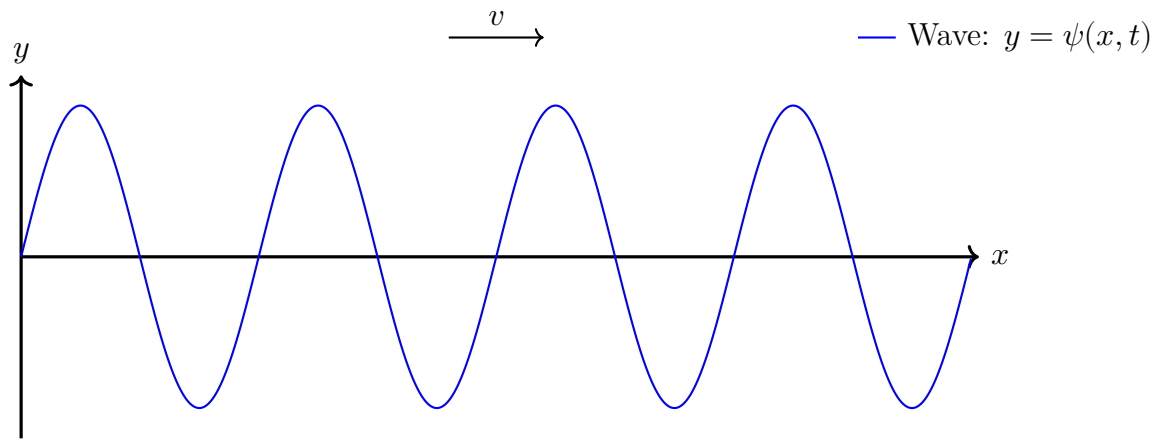


Figure 3.1: A plot of 1D wave on a string

that we can choose convenient point on the string, and calculate  $A$ ,  $\lambda$ ,  $f$ , and  $v$ , then the same values will be valid for all points on the string. For a moving wave, a stationary point of view is not very useful, so we will use a moving point of view: the Lagrange description in fluid mechanics.

Consider moving in the  $x$  direction, at the same constant velocity  $v$ , as the wave. Then, our new  $x$  coordinate is given by the Galilean transformation:

$$\xi(x, t) = x - vt \quad (3.1)$$

### 3 Physics of Fields

In this new perspective, we do not have to worry about the motion of the wave, since we are moving with the wave. In short, the wave is stationary in the new coordinate system. This means that vertical displacement of the wave at a point  $\xi$  only depends on the  $\xi$  coordinate, and not on time:

$$\psi(x, t) = \psi(\xi) \quad (3.2)$$

and any point on the string seems to move at velocity  $-v$ .

Then immediately,

$$\frac{\partial \psi(\xi)}{\partial x} = \frac{\partial \psi(\xi)}{\partial \xi} \frac{\partial \xi}{\partial x} = \frac{\partial \psi(\xi)}{\partial \xi}, \quad (3.3)$$

$$\frac{\partial \psi(\xi)}{\partial t} = \frac{\partial \psi(\xi)}{\partial \xi} \frac{\partial \xi}{\partial t} = -v \frac{\partial \psi(\xi)}{\partial \xi} \quad (3.4)$$

$$\Rightarrow \frac{\partial^2 \psi(\xi)}{\partial x^2} = \frac{\partial^2 \psi(\xi)}{\partial \xi^2}, \quad \frac{\partial^2 \psi(\xi)}{\partial t^2} = v^2 \frac{\partial^2 \psi(\xi)}{\partial \xi^2} \quad (3.5)$$

$$\Rightarrow \frac{\partial^2 \psi(x, t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi(x, t)}{\partial t^2} \quad (3.6)$$

we obtain the wave equation, but we should find its physical meaning.

#### Theorem 3.1.1 Wave Equation in 1D

The wave equation in 1D is given by:

$$\frac{\partial^2 \psi(x, t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi(x, t)}{\partial t^2} \quad (3.7)$$

where  $\psi(x, t)$  is the vertical displacement of the wave at point  $x$  and time  $t$ , and  $v$  is the velocity of the wave.

Now, let us focus on one of the peaks of the wave, at the point  $\xi_0$ . The tension on the

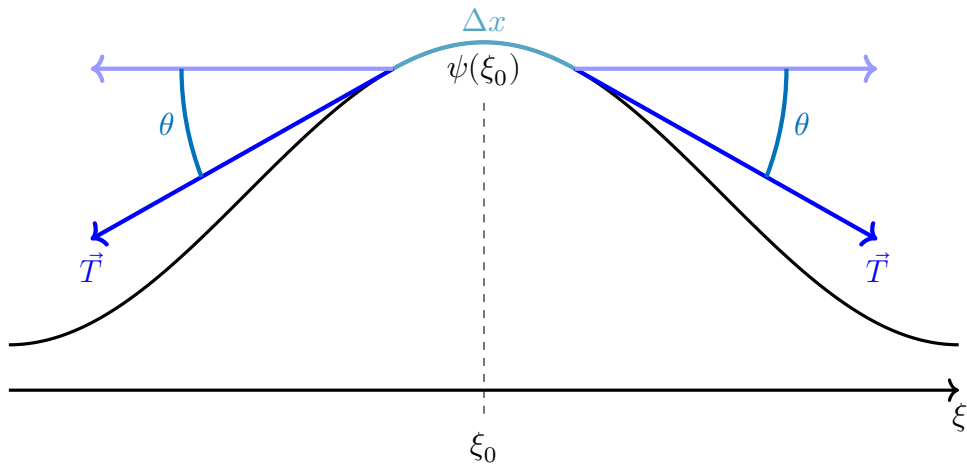


Figure 3.2: A peak of the wave

string at  $\xi_0$  cancels out horizontally, but adds vertically:

$$F_x = T \cos \theta - T \cos \theta = 0 \quad (3.8)$$

$$F_y = F_{-y} + F_{+y} = -T \partial_x \psi \left( \xi_0 + \frac{\Delta x}{2} \right) + T \partial_x \psi \left( \xi_0 - \frac{\Delta x}{2} \right) \quad (3.9)$$

Thus the equation of motion for the infinitesimal segment of the string at  $\xi_0$  is given by:

$$\vec{F} = m \ddot{\vec{r}} \iff T \left( \partial_x \psi \left( \xi_0 + \frac{\Delta x}{2} \right) - \partial_x \psi \left( \xi_0 - \frac{\Delta x}{2} \right) \right) = \mu \Delta x \partial_t^2 \psi(\xi_0) \quad (3.10)$$

by rearranging,

$$\frac{1}{\Delta x} \left( \partial_x \psi \left( \xi_0 + \frac{\Delta x}{2} \right) - \partial_x \psi \left( \xi_0 - \frac{\Delta x}{2} \right) \right) = \frac{\mu}{T} \partial_t^2 \psi(\xi_0) \quad (3.11)$$

as  $\Delta x \rightarrow 0$ , LHS becomes the derivative of the first derivative:

$$\partial_x^2 \psi(\xi_0) = \frac{\mu}{T} \partial_t^2 \psi(\xi_0) \quad (3.12)$$

Now, note that the infinitesimal section experiences force in  $y$  direction, while it moves in  $x$  direction, This is equivalent to a circular motion with radius  $r = \psi(\xi_0)$ :

$$F_c = \frac{mv^2}{r} = \frac{\mu \Delta x v^2}{\psi(\xi_0)} = F_y \quad (3.13)$$

In this limit of  $\Delta x \rightarrow 0$ ,  $\Delta x = 2\psi(\xi_0)\theta$ ,

$$\implies 2\theta \mu v^2 = 2T \sin \theta \approx 2T\theta \quad (3.14)$$

$$\implies v^2 = \frac{T}{\mu} \quad (3.15)$$

Thus, the wave equation can be written as:

$$\frac{\partial^2 \psi(x, t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \psi(x, t)}{\partial t^2}, \quad \text{where } v = \sqrt{\frac{T}{\mu}} \quad (3.16)$$

## 3.2 Lagrange Formalism

### 3.2.1 Potential and Momentum Density

Now, notice that the force on the infinitesimal section is given by

$$\dot{p} = \mu \partial_t^2 \psi \Delta x \quad (3.17)$$



since the extension of the string in  $\Delta x/2$  is given by

$$\Delta l - \frac{\Delta x}{2} = \frac{\Delta x}{2} \sqrt{1 + \left[ \partial_x \psi \left( \xi_0 - \frac{\Delta x}{2} \right) \right]^2} - \frac{\Delta x}{2} \quad (3.18)$$

$$\approx \frac{\Delta x}{2} \partial_x \psi \left( \xi_0 - \frac{\Delta x}{2} \right) \quad (3.19)$$

The force on the infinitesimal section due to the extension is then given by

$$F = -k \left( \Delta l - \frac{\Delta x}{2} \right) = -k \frac{\Delta x}{2} \partial_x \psi \left( \xi_0 - \frac{\Delta x}{2} \right) = -T \partial_x \psi \left( \xi_0 - \frac{\Delta x}{2} \right) \quad (3.20)$$

$$\Rightarrow V = \frac{k}{2} \left( \Delta l - \frac{\Delta x}{2} \right)^2 = \frac{k}{2} \left( \frac{\Delta x}{2} \partial_x \psi \left( \xi_0 - \frac{\Delta x}{2} \right) \right)^2 = \frac{T}{2} \frac{\Delta x}{2} \left( \partial_x \psi \left( \xi_0 - \frac{\Delta x}{2} \right) \right)^2 \quad (3.21)$$

We should define these quantity as the potential energy density and the momentum density:

**Definition 3.1: Momentum and Potential Energy Density**

The **momentum density** and **potential energy density** is defined as the momentum and potential energy per unit length of the string:

$$\pi = \mu \partial_t \psi, \quad \tilde{V} = \frac{T}{2} (\partial_x \psi)^2 \quad (3.22)$$

Now, since we have defined our potential energy (density) and momentum (density), we can write the Lagrangian:

$$L = T - V \quad (3.23)$$

$$= \int dx \tilde{T} - \int dx \tilde{V} \quad (3.24)$$

$$= \int dx \frac{\mu}{2} (\partial_t \psi)^2 - \frac{T}{2} (\partial_x \psi)^2 \quad (3.25)$$

thus we should define the Lagrangian density  $\mathcal{L}$  as:

**Definition 3.2: Lagrangian Density for 1D String**

The **Lagrangian density** is defined as the Lagrangian per unit length of the string:

$$\mathcal{L}[\partial_x \psi, \partial_t \psi] = \frac{\mu}{2} (\partial_t \psi)^2 - \frac{T}{2} (\partial_x \psi)^2 \quad (3.26)$$

### 3 Physics of Fields

Then the Euler-Lagrange equation for the Lagrangian density  $\mathcal{L}$  is given by:

$$\partial_t \left( \frac{\partial \mathcal{L}}{\partial(\partial_t \psi)} \right) + \partial_x \left( \frac{\partial \mathcal{L}}{\partial(\partial_x \psi)} \right) = \frac{\partial \mathcal{L}}{\partial \psi} \quad (3.27)$$

$$\Rightarrow \partial_t (\mu \partial_t \psi) + \partial_x (-T \partial_x \psi) = 0 \quad (3.28)$$

$$\Rightarrow \mu \partial_t^2 \psi - T \partial_x^2 \psi = 0 \quad (3.29)$$

$$\Rightarrow \partial_x^2 \psi = \frac{\mu}{T} \partial_t^2 \psi \quad (3.30)$$

which is the wave equation in 1D. Notice that our mechanical variable is now the field  $\psi(x, t)$  and its derivatives, rather than the position of a particle or the velocity of a particle.

#### 3.2.2 Generalization to General Field

Now, we would like to apply Lagrange formalism to a general field  $\psi(\vec{x}, t)$ . We can define the Lagrangian density for a general field  $\psi(\vec{x}, t)$  as:

##### Definition 3.3: Lagrangian Density of a Field

The **Lagrangian density** of a field  $\psi(\vec{x}, t)$  is defined such that the Lagrangian is given by the integral of the Lagrangian density over the whole space:

$$L[\psi, \partial_t \psi, \partial_i \psi] = \int d\vec{x} \mathcal{L}(\psi, \partial_\mu \psi) \quad (3.31)$$

where for  $\mu = 0, 1, 2, 3$ ,  $x^{0,1,2,3} = (t, \vec{x})$

$$\partial_\mu \psi := \frac{\partial \psi}{\partial x^\mu} = \left( \frac{\partial \psi}{\partial t}, \frac{\partial \psi}{\partial x^1}, \frac{\partial \psi}{\partial x^2}, \frac{\partial \psi}{\partial x^3} \right) \quad (3.32)$$

##### Definition 3.4: Action of a Field

The **action** of a field  $\psi(\vec{x}, t)$  is defined as the integral of the Lagrangian over time:

$$S[\psi] = \int dt L[\psi, \partial_t \psi, \partial_i \psi] = \int dt \int d^3 \vec{x} \mathcal{L}(\psi, \partial_\mu \psi) \quad (3.33)$$

where  $\mathcal{L}$  is the Lagrangian density of the field  $\psi(\vec{x}, t)$ .

and the stationary point of the action is given by:

$$\delta S[\mathcal{L}] = 0 \Rightarrow \int dt \int d^3 \vec{x} \frac{\partial \mathcal{L}}{\partial \psi} \delta \psi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \delta(\partial_\mu \psi) = 0 \quad (3.34)$$

where  $\delta \psi = 0$  at the boundary of the integration region. The second term can be integrated

by parts:

$$\int dt \int d^3\vec{x} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \delta(\partial_\mu \psi) = \int dt \int d^3\vec{x} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \partial_\mu (\delta\psi) \quad (3.35)$$

$$= \int dt \int d^3\vec{x} \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \delta\psi \right) - \int dt \int d^3\vec{x} \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) \delta\psi \quad (3.36)$$

Since  $\delta\psi = 0$  at the boundary of the integration region, the first term vanishes:

$$= - \int dt \int d^3\vec{x} \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) \delta\psi \quad (3.37)$$

Thus, the stationary point of the action is given by:

$$\int dt \int d^3\vec{x} \left( \frac{\partial \mathcal{L}}{\partial\psi} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) \right) \delta\psi = 0 \quad (3.38)$$

Since time and space are arbitrary, the integrand must be zero:

$$\frac{\partial \mathcal{L}}{\partial\psi} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) = 0 \iff \partial_t \left( \frac{\partial \mathcal{L}}{\partial(\partial_t \psi)} \right) + \sum_i \partial_i \left( \frac{\partial \mathcal{L}}{\partial(\partial_i \psi)} \right) = \frac{\partial \mathcal{L}}{\partial\psi} \quad (3.39)$$

This is the **Euler-Lagrange equation** for a field  $\psi(\vec{x}, t)$ :

**Theorem 3.2.1** Euler-Lagrange Equation for a Field

The **Euler-Lagrange equation** for a field  $\psi(\vec{x}, t)$  is given by:

$$\frac{\partial \mathcal{L}}{\partial\psi} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) = 0 \quad (3.40)$$

where  $\mathcal{L}$  is the Lagrangian density of the field  $\psi(\vec{x}, t)$ .

Additionally, we define the **canonical conjugate field**  $\pi(\vec{x}, t)$  as:

**Definition 3.5: Canonical Conjugate Field**

The **canonical conjugate field** or **canonical momentum density**  $\pi(\vec{x}, t)$  is defined as the derivative of the Lagrangian density with respect to the time derivative of the field  $\psi$ :

$$\pi(\vec{x}, t) := \frac{\partial \mathcal{L}}{\partial(\partial_t \psi(\vec{x}, t))} \quad (3.41)$$

### 3.3 Hamilton Formalism

#### 3.3.1 Legendre Transformation

Now, remember that we obtained the **Hamiltonian**  $H$  of the system through the **Legendre transformation** of the Lagrangian  $L$ :

$$H(q_i, p_i) = \sum_i p_i \dot{q}_i - L(q_i, \dot{q}_i) \quad (3.42)$$

where  $i$  represents each degree of freedom of the system.

In the field, the degree of freedom is infinite - each indexed by the spatial position  $\vec{x}$ , then,

##### Definition 3.6: Hamiltonian of a Field

The **Hamiltonian** of a field  $\psi(\vec{x}, t)$  (whose canonical conjugate field is  $\pi(\vec{x}, t)$ ) is defined as

$$H[\psi, \partial_i \psi, \pi] = \int d^3 \vec{x} \left[ \pi \cdot \partial_t \psi \right] - L = \int d^3 \vec{x} \left[ \pi \cdot \partial_t \psi - \mathcal{L} \right] \quad (3.43)$$

thus, we should define the **Hamiltonian density**  $\mathcal{H}$  as:

##### Definition 3.7: Hamiltonian Density

**Hamiltonian density**  $\mathcal{H}$  is defined as the Hamiltonian per unit volume of the field:

$$\mathcal{H}(\psi, \partial_i \psi, \pi, t) = \pi \cdot \partial_t \psi - \mathcal{L}(\psi, \partial_i \psi, \partial_t \psi, t) \quad (3.44)$$

which satisfies:

$$\int d^3 \vec{x} \mathcal{H}(\psi, \partial_i \psi, \pi, t) = H[\psi, \partial_i \psi, \pi, t] \quad (3.45)$$

which makes the Hamiltonian  $H$  a functional of the field  $\psi$ , its spatial derivatives  $\partial_i \psi$ , and the conjugate field  $\pi$ .

##### Note:

Similarly to the discrete case, we can write the Lagrangian density  $\mathcal{L}$  in terms of the Hamiltonian density  $\mathcal{H}$ :

$$\mathcal{L}[\psi(\vec{x}, t), \pi(\vec{x}, t)] = \pi \cdot \partial_t \psi - \mathcal{H}[\psi(\vec{x}, t), \pi(\vec{x}, t)] \quad (3.46)$$

where

$$\partial_t \psi = \frac{\partial \mathcal{H}[\psi, \pi]}{\partial \pi} \quad (3.47)$$

### 3.3.2 Canonical Equations of Fields

We are also interested in the **canonical equations** of fields, which are derived from the variational principle:

$$\delta S = 0 \iff \delta \int dt \int d^3\vec{x} \mathcal{L} = \delta \int dt \int d^3\vec{x} \pi \cdot \partial_t \psi - \mathcal{H} \quad (3.48)$$

The variation on this integral can be expanded as follows:

$$\delta S = \int dt \int d^3\vec{x} \delta \pi \partial_t \psi + \pi \delta(\partial_t \psi) - \delta \mathcal{H} \quad (3.49)$$

$$= \int dt \int d^3\vec{x} \delta \pi \partial_t \psi + \pi \partial_t(\delta \psi) - \frac{\partial \mathcal{H}}{\partial \psi} \delta \psi - \frac{\partial \mathcal{H}}{\partial \pi} \delta \pi - \frac{\partial \mathcal{H}}{\partial(\partial_i \psi)} \delta(\partial_i \psi) \quad (3.50)$$

$$= \int dt \int d^3\vec{x} \left( \partial_t \psi - \frac{\partial \mathcal{H}}{\partial \pi} \right) \delta \pi + \left( \pi \partial_t(\delta \psi) - \frac{\partial \mathcal{H}}{\partial \psi} \delta \psi - \frac{\partial \mathcal{H}}{\partial(\partial_i \psi)} \partial_i(\delta \psi) \right) \quad (3.51)$$

the  $\delta \psi$  term can be integrated by parts:

$$\int dt \int d^3\vec{x} \pi \partial_t(\delta \psi) - \frac{\partial \mathcal{H}}{\partial(\partial_i \psi)} \partial_i(\delta \psi) \quad (3.52)$$

$$= \int dt \int d^3\vec{x} \partial_t(\pi \delta \psi) - \int dt \int d^3\vec{x} \partial_t \pi \cdot \delta \psi \quad (3.53)$$

$$- \int dt \int d^3\vec{x} \partial_i \left( \frac{\partial \mathcal{H}}{\partial(\partial_i \psi)} \delta \psi \right) + \int dt \int d^3\vec{x} \partial_i \left( \frac{\partial \mathcal{H}}{\partial(\partial_i \psi)} \right) \delta \psi \quad (3.54)$$

$$= \int d^3\vec{x} [\pi \delta \psi]_{\text{boundary}} - \int dt \int d^3\vec{x} \partial_t \pi \delta \psi - \int dt \int_{\text{boundary}} dS \frac{\partial \mathcal{H}}{\partial(\partial_i \psi)} \delta \psi + \int dt \int d^3\vec{x} \partial_i \left( \frac{\partial \mathcal{H}}{\partial(\partial_i \psi)} \right) \delta \psi$$

the boundary terms vanish, so we end up with:

$$\int dt \int d^3\vec{x} \pi \partial_t(\delta \psi) - \frac{\partial \mathcal{H}}{\partial(\partial_i \psi)} \partial_i(\delta \psi) = - \int dt \int d^3\vec{x} \partial_t \pi \delta \psi + \int dt \int d^3\vec{x} \partial_i \left( \frac{\partial \mathcal{H}}{\partial(\partial_i \psi)} \right) \delta \psi \quad (3.55)$$

$$= - \int dt \int d^3\vec{x} \left[ \partial_t \pi - \nabla \cdot \left( \frac{\partial \mathcal{H}}{\partial(\nabla \psi)} \right) \right] \delta \psi \quad (3.56)$$

and thus the variation of the action becomes:

$$\delta S = \int dt \int d^3\vec{x} \left( \partial_t \psi - \frac{\partial \mathcal{H}}{\partial \pi} \right) \delta \pi - \left( \partial_t \pi - \nabla \cdot \left( \frac{\partial \mathcal{H}}{\partial(\nabla \psi)} \right) \right) \delta \psi \quad (3.57)$$

for the action to be stationary, the integrand must vanish:

$$\begin{cases} \partial_t \psi - \frac{\partial \mathcal{H}}{\partial \pi} = 0 \\ \partial_t \pi + \frac{\partial \mathcal{H}}{\partial \psi} - \partial_i \left( \frac{\partial \mathcal{H}}{\partial (\partial_i \psi)} \right) = 0 \end{cases} \iff \begin{cases} \frac{\partial \psi(\vec{x}, t)}{\partial t} = \frac{\partial \mathcal{H}[\psi, \pi]}{\partial \pi} \\ \frac{\partial \pi(\vec{x}, t)}{\partial t} = -\frac{\partial \mathcal{H}[\psi, \pi]}{\partial \psi} + \partial_i \left( \frac{\partial \mathcal{H}}{\partial (\partial_i \psi)} \right) \end{cases} \quad (3.58)$$

Thus we have the **canonical equations of fields**:

### Theorem 3.3.1 Canonical Equations of Fields

The variational principle in Hamilton formalism leads to the **canonical equations of fields**:

$$\frac{\partial \psi(\vec{x}, t)}{\partial t} = \frac{\partial \mathcal{H}(\psi, \partial_i \psi, \pi)}{\partial \pi}, \quad \frac{\partial \pi(\vec{x}, t)}{\partial t} = -\frac{\partial \mathcal{H}(\psi, \partial_i \psi, \pi)}{\partial \psi} + \partial_i \left( \frac{\partial \mathcal{H}(\psi, \partial_i \psi, \pi)}{\partial (\partial_i \psi)} \right) \quad (3.59)$$

### Corollary 3.3.1 Canonical Equations of Fields using $H$

Using Theorem 5.1.1, the **canonical equations of fields** can be re-written using the Hamiltonian  $H$  instead of the Hamiltonian density  $\mathcal{H}$ :

$$\frac{\partial \psi(\vec{x}, t)}{\partial t} = \frac{\delta H[\psi, \pi]}{\delta \pi}, \quad \frac{\partial \pi(\vec{x}, t)}{\partial t} = -\frac{\delta H[\psi, \pi]}{\delta \psi} \quad (3.60)$$

## 3.3.3 Poisson Bracket

In the discrete case, the time evolution of a physical quantity  $X(q_i, p_i, t)$ ,  $\dot{X}$  can be written as:

$$\dot{X} = \frac{dX}{dt} = \frac{\partial X}{\partial t} + \sum_i \left( \frac{\partial X}{\partial q_i} \dot{q}_i + \frac{\partial X}{\partial p_i} \dot{p}_i \right) = \frac{\partial X}{\partial t} + \{X, H\} \quad (3.61)$$

In the continuous case, a physical quantity  $X$  should be an integral of "density"  $\tilde{X}$  over some volume:

$$X = \int_V d^3 \vec{x}' \tilde{X}(\vec{x}', t) \quad (3.62)$$

and assume that  $\tilde{X}$  is a function of the field  $\psi(\vec{x}, t)$  and its conjugate field  $\pi(\vec{x}, t)$  (which makes  $X$  a functional of the fields):

$$\implies X[\psi, \pi, t] = \int_V d^3 \vec{x}' \tilde{X}(\psi(\vec{x}', t), \pi(\vec{x}', t), t) \quad (3.63)$$

Then the time evolution of  $X$  can be written as:

$$\frac{dX[\psi, \pi, t]}{dt} = \frac{d}{dt} \int_V d^3\vec{x}' \tilde{X}(\psi(\vec{x}', t), \pi(\vec{x}', t), t) = \int_V d^3\vec{x}' \frac{d\tilde{X}(\psi(\vec{x}', t), \pi(\vec{x}', t), t)}{dt} \quad (3.64)$$

$$= \int_V d^3\vec{x}' \frac{\partial \tilde{X}(\psi, \pi, t)}{\partial t} + \frac{\partial \tilde{X}(\psi, \pi, t)}{\partial \psi} \partial_t \psi + \frac{\partial \tilde{X}(\psi, \pi, t)}{\partial \pi} \partial_t \pi \quad (3.65)$$

$$= \int_V d^3\vec{x}' \frac{\partial \tilde{X}}{\partial t} + \int_V d^3\vec{x}' \left( \frac{\partial \tilde{X}}{\partial \psi} \frac{\partial \mathcal{H}}{\partial \pi} - \frac{\partial \tilde{X}}{\partial \pi} \frac{\partial \mathcal{H}}{\partial \psi} \right) \quad (3.66)$$

Using Theorem 5.1.1, the partial derivatives can be replaced with functional derivatives:

$$\frac{\partial \tilde{X}}{\partial t} = \frac{\delta X}{\delta t}, \quad \frac{\partial \tilde{X}}{\partial \psi} = \frac{\delta H}{\delta \psi}, \quad \frac{\partial \tilde{X}}{\partial \pi} = \frac{\delta H}{\delta \pi} \quad (3.67)$$

Thus, we can write the time evolution of  $X$  as:

$$\frac{dX[\psi, \pi, t]}{dt} = \frac{\partial X[\psi, \pi, t]}{\partial t} + \int d^3\vec{x}' \left( \frac{\delta X}{\delta \psi} \frac{\delta H}{\delta \pi} - \frac{\delta X}{\delta \pi} \frac{\delta H}{\delta \psi} \right) \quad (3.68)$$

If we were to write the coordinates explicitly,

$$\frac{dX[\psi, \pi, t]}{dt} = \frac{\partial X}{\partial t} + \int d^3\vec{x}' \left( \frac{\delta X[\psi, \pi, t]}{\delta \psi(\vec{x}', t)} \frac{\delta H[\psi, \pi, t]}{\delta \pi(\vec{x}', t)} - \frac{X[\psi, \pi, t]}{\delta \pi(\vec{x}', t)} \frac{\delta H[\psi, \pi, t]}{\delta \psi(\vec{x}', t)} \right) \quad (3.69)$$

Comparing with the discrete case, we can define the **Poisson bracket** of two physical quantities  $X$  and  $Y$  as:

#### Definition 3.8: Poisson Bracket of a Field

The **Poisson bracket** of two physical quantities  $X[\psi, \pi, t]$  and  $Y[\psi, \pi, t]$  is defined as:

$$\{X, Y\} = \int d^3\vec{x}' \left( \frac{\delta X}{\delta \psi(\vec{x}', t)} \frac{\delta Y}{\delta \pi(\vec{x}', t)} - \frac{\delta X}{\delta \pi(\vec{x}', t)} \frac{\delta Y}{\delta \psi(\vec{x}', t)} \right) \quad (3.70)$$

#### Theorem 3.3.2 Time Evolution of a Physical Quantity

The time evolution of a physical quantity  $X[\psi, \pi, t]$  can be written as:

$$\frac{dX}{dt} = \frac{\partial X}{\partial t} + \{X, H\} \quad (3.71)$$

#### Theorem 3.3.3 Poisson Brackets of Fields

For  $\psi$  and its conjugate field  $\pi$ , the Poisson bracket satisfies the following properties:

$$\left\{ \psi(\vec{x}, t), \pi(\vec{x}', t) \right\} = \delta^3(\vec{x} - \vec{x}') \quad (3.72)$$

$$\left\{ \psi(\vec{x}, t), \psi(\vec{x}', t) \right\} = 0 \quad (3.73)$$

$$\left\{ \pi(\vec{x}, t), \pi(\vec{x}', t) \right\} = 0 \quad (3.74)$$



# Quantum Mechanics

## 4.1 Hilbert Space in Quantum Mechanics

### 4.1.1 Hilbert Space

#### Definition 4.1: Hilbert Space

A **Hilbert space** is a complete inner product space, which is a vector space with an inner product that is complete with respect to the norm induced by the inner product. It is denoted as  $\mathbb{H}$ . We denote the basis vectors of the Hilbert space as  $|i\rangle$ , where  $i$  is an index.

#### Definition 4.2: (Hermitian) Inner Product and Norm

**Inner product** of a Hilbert space  $\mathbb{H}$  is a function  $\mathbb{H} \times \mathbb{H} \rightarrow \mathbb{C}$ , that satisfies the following properties:

1. Conjugate symmetry

$$\langle \psi | \phi \rangle = \langle \phi | \psi \rangle^* \quad (4.1)$$

2. Linearity

$$\langle \psi | (a|\varphi\rangle + b|\phi\rangle) = a\langle \psi | \varphi \rangle + b\langle \psi | \phi \rangle \quad (4.2)$$

3. Positivity

$$\langle \psi | \psi \rangle \geq 0, \langle \psi | \psi \rangle = 0 \iff |\psi\rangle = |0\rangle \quad (4.3)$$

where  $|\psi\rangle, |\phi\rangle, |\varphi\rangle \in \mathbb{H}$  and  $a, b \in \mathbb{C}$ .

The **norm** of a vector  $|\psi\rangle \in \mathbb{H}$  is defined as

$$\| |\psi\rangle \| := \sqrt{\langle \psi | \psi \rangle} \geq 0 \quad (4.4)$$

### Definition 4.3: Completeness

A vector space is said to be **complete** if every Cauchy sequence in the space converges to a limit in the space:

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : \forall m, n > N, \| |v_m\rangle - |v_n\rangle \| < \epsilon \quad (4.5)$$

where  $|v_m\rangle, |v_n\rangle \in \mathbb{H}$  are vectors in the space. Colloquially, this property ensures that any sum (even uncountably infinite) of vectors in  $\mathbb{H}$  will still be in  $\mathbb{H}$ .

## 4.1.2 Basis Vectors and Completeness Relation

### Definition 4.4: Basis Vectors

A set of vectors  $\{|i\rangle\}$  in a Hilbert space  $\mathbb{H}$  is said to be a **basis** if it is a set of vectors that spans the space, satisfying the following conditions:

1. Linear Independence

$$\sum_i c_i |i\rangle = |0\rangle \iff c_i = 0 \quad \forall i \quad (4.6)$$

2. Completeness

$$\forall |\psi\rangle \in \mathbb{H}, \exists \{c_i\} \in \mathbb{C} \text{ s.t. } |\psi\rangle = \sum_i c_i |i\rangle \quad (4.7)$$

3. Orthogonality

$$\langle i | j \rangle = 0 \quad \forall i \neq j \quad (4.8)$$

Especially, if the basis vectors satisfy:

$$\langle i | j \rangle = \delta_{ij} \quad \forall i, j \quad (4.9)$$

then the basis is said to be **orthonormal**.

### Theorem 4.1.1 Completeness Relation

The following relation holds for a complete set of orthonormal basis vectors  $\{|i\rangle\}$  in a Hilbert space  $\mathbb{H}$ :

$$|\psi\rangle = \sum_i c_i |i\rangle \iff \sum_i |i\rangle \langle i| = \hat{\mathbf{I}} \quad (4.10)$$

**Proof:** • (  $\Rightarrow$  )

$$\sum_i |i\rangle \langle i|\psi\rangle = \sum_i c_i |i\rangle = |\psi\rangle, \quad c_i := \langle i|\psi\rangle \quad (4.11)$$

• (  $\Leftarrow$  )

$$|\psi\rangle = \hat{\mathbf{I}}|\psi\rangle = \sum_i |i\rangle \langle i|\psi\rangle = \sum_i c_i |i\rangle \quad (4.12)$$

□

## 4.2 Linear Algebra Approach

### 4.2.1 Basic Concepts

A formal mathematical description of quantum mechanics is essentially linear algebra in a Hilbert space. The "state" of quantum particle is represented by a vector  $|\psi\rangle \in \mathbb{H}$ : a Hilbert space.

#### Definition 4.5: Observable, Hermitian Operator

A physical quantity (**observable**)  $A$  is represented by a Hermitian (self-adjoint) operator  $\hat{A}$  acting on the state vector  $|\psi\rangle$ :

$$\hat{A} = \hat{A}^\dagger, \quad \hat{A}|a\rangle = a|a\rangle, \quad a \in \mathbb{R} \quad (4.13)$$

where  $\hat{A}^\dagger$  is **Hermitian conjugate** of  $\hat{A}$ , and  $|a\rangle$  is an **eigenstate** of the operator  $\hat{A}$  with eigenvalue  $a$ .

and we postulate that the probability of finding the system in the state  $|\phi\rangle$  from another state  $|\psi\rangle$  is given by the inner product:

#### Principle 4.1: Born's Probability Interpretation

The probability of finding the system in the state  $|\phi\rangle$  from another state  $|\psi\rangle$  is given by:

$$P(\phi|\psi) = |\langle\phi|\psi\rangle|^2 \quad (4.14)$$

where  $\langle\phi|\psi\rangle$  is the inner product of the two state vectors.

Intuitively, the inner product  $\langle\phi|\psi\rangle$  indicates how "close" the two states  $|\phi\rangle$  and  $|\psi\rangle$  are: Note that the inner product is a complex number, and the probability is given by the square of the absolute value of the inner product.

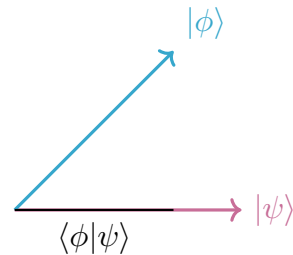


Figure 4.1: Inner product in a Hilbert space

### 4.2.2 Position and Momentum in Quantum Mechanics

The position and momentum of a particle are represented by operators  $\hat{x}$  and  $\hat{p}$ , and we impose the canonical commutation relation onto these operators:

#### Principle 4.2: Canonical commutation relation

The position and momentum operators satisfy the following commutation relation:

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij} \quad (4.15)$$

where  $\hbar$  is the reduced Planck's constant.

This implies that in the position basis, the momentum operator acts as a derivative operator (refer to Principle 6.3.1):

$$\hat{p}|x\rangle = -i\hbar\frac{\partial}{\partial x}|x\rangle \quad (4.16)$$

The eigenstates of  $\hat{x}$  and  $\hat{p}$  are denoted as  $|x\rangle$  and  $|p\rangle$ , respectively, and they satisfy the completeness relation:

#### Principle 4.3: Completeness relation of Continuous Eigenbasis

for  $|x\rangle$  and  $|p\rangle$ , the following completeness relation holds:

$$\langle x|x'\rangle = \delta(x - x'), \quad \langle p|p'\rangle = \delta(p - p') \quad (4.17)$$

$$\Leftrightarrow \int dx |x\rangle\langle x| = \hat{\mathbf{I}}, \quad \int dp |p\rangle\langle p| = \hat{\mathbf{I}} \quad (4.18)$$

Now, we can define the position and momentum wavefunctions  $\psi(x), \psi(p)$ , whose magnitude squared gives the probability density of finding the particle at a certain position  $x$  or momentum  $p$ :

#### Definition 4.6: Wavefunction

The **wavefunction**  $\psi(x)$  of a quantum particle is defined as the inner product of the state vector  $|\psi\rangle$  with the position eigenstate  $|x\rangle$ :

$$\psi(x) = \langle x|\psi\rangle \quad (4.19)$$

Colloquially, wavefunction represents the probability amplitude of finding the particle at position  $x$  in state  $|\psi\rangle$ .

#### Note:

We can equally define a wavefunction in the momentum basis:

$$\psi(p) = \langle p|\psi\rangle \quad (4.20)$$

in which the  $\hat{x}$  operator becomes a derivative operator w.r.t.  $p$ :

$$\hat{x}|p\rangle = i\hbar \frac{\partial}{\partial p}|p\rangle \quad (4.21)$$

### 4.2.3 Unitary Transformations: Shift Operator and Fourier Transform

Now, consider a shift operator  $\hat{S}(a)$  that shifts the position eigenstate  $|x\rangle$  by a constant  $a$ :

#### Definition 4.7: Shift Operator

The **shift operator**  $\hat{S}(a)$  is defined as:

$$\hat{S}(a)|x\rangle = |x + a\rangle \quad (4.22)$$

From the discussion in Sec. 6.3,

#### Theorem 4.2.1 Commutator of Shift Operator

For operators satisfying the commutation relation  $[\hat{D}, \hat{x}] = 1$ ,

$$[\hat{S}(a), \hat{x}] = a\hat{S}(a) \implies \hat{S}(a) = e^{a\hat{D}} \quad (4.23)$$

since  $\hat{p} = -i\hbar\hat{D} \iff \hat{D} = \frac{\hat{p}}{-i\hbar}$ , we can write the shift operator as:

$$\hat{S}(a) = e^{\frac{a\hat{p}}{-i\hbar}} = e^{i\frac{\hat{p}}{\hbar}a} \quad (4.24)$$

#### Definition 4.8: Fourier Transform in Quantum Mechanics

The position wavefunction  $\langle x|\psi\rangle$  can be expanded in the momentum basis  $|p\rangle$ , by the **Fourier transform**:

$$\langle x|\psi\rangle = \int dp \langle x|p\rangle \langle p|\psi\rangle = \int \frac{dp}{\sqrt{2\pi\hbar}} e^{i\frac{p}{\hbar}x} \langle p|\psi\rangle \quad (4.25)$$

## 4.3 Harmonic Oscillator

# Mathematical Remarks

## 5.1 Functional Derivative

### 5.1.1 Definition

Consider a quantity  $I$  defined as follows:

$$I := \int_A^B dx F(x) \quad (5.1)$$

Notice that  $I$  is not really a function of  $x$ , but if you had to say, it is more a "function" of  $F$  - may be  $F(x) = e^x$ , or  $F(x) = ax^2 + bx + c$ , or, etc. So, to denote the dependence of  $I$  on the function  $F$ , we write

$$I[f] := \int_A^B dx F(x) \quad (5.2)$$

This is called a **(linear) functional**. Now, imagine that  $F$  is a function of  $f$ , for example,  $F[f] = f(x)^2$ . By chain rule, a small change in  $F$ , denoted as  $\delta F$ , can be expressed as:

$$\delta F[f] = F[f + \delta f] - F[f] \quad (5.3)$$

$$= \frac{\partial F}{\partial f} \delta f \quad (5.4)$$

so, similarly,

$$\delta I[F] = I[F + \delta F] - I[F] \quad (5.5)$$

$$= \int_A^B dx \delta F[f] \quad (5.6)$$

$$= \int_A^B dx \frac{\partial F}{\partial f} \delta f \quad (5.7)$$

Then, the **functional derivative** of  $I$  with respect to  $f$ ,  $\frac{\delta I}{\delta f}$ , is defined as follows:

**Definition 5.1: Functional Derivative**

If a function  $\phi(x)$  exists, such that

$$\delta I = \int_A^B dx \phi(x) \delta f(x), \quad (5.8)$$

we say that  $\phi(x)$  is the **functional derivative** of  $I$  with respect to  $f$ , and denote it as

$$\frac{\delta I}{\delta f(x)} := \phi(x) \iff \delta I := \int_B^A dx \frac{\delta I}{\delta f(x)} \delta f(x). \quad (5.9)$$

Immediatly, by comparing Eq.(5.9) and Eq.(5.7), we see the following relation:

**Theorem 5.1.1 Functional-density relation**

For a quantity  $I[F]$  and its density  $F[f(x)]$ , the functional derivative satisfies the following relation:

$$I = \int_A^B dx F[f(x)] \implies \frac{\delta I}{\delta f(x)} = \frac{\partial F[f(x)]}{\partial f(x)} \quad (5.10)$$

**5.1.2 Two function case**

Consider a case where  $I$  is the functional of  $F$ , which is also a functional of  $f$  and  $g$ :

$$I[F[f, g]] = \int_B^A dx F[f(x), g(x)] \quad (5.11)$$

Or more generally, if a function  $D(x)$  satisfies the following Now, let us add some small change of  $f$ ,  $\delta f$ :

$$I[F[f + \delta f, g]] = \int_B^A dx F[f(x) + \delta f(x), g(x)] \quad (5.12)$$

$$= \int_B^A dx F[f, g] + \frac{\partial F}{\partial f} \delta f \quad (5.13)$$

and similarly, by adding  $\delta g$ ,

$$I[F[f, g + \delta g]] = \int_B^A dx F[f(x), g(x) + \delta g(x)] \quad (5.14)$$

$$= \int_B^A dx F[f, g] + \frac{\partial F}{\partial g} \delta g \quad (5.15)$$

Combining these two, we have

$$I[F[f + \delta f, g + \delta g]] = \int_B^A dx F[f(x) + \delta f(x), g(x) + \delta g(x)] \quad (5.16)$$

$$= \int_B^A dx F[f, g] + \frac{\partial F}{\partial f} \delta f + \frac{\partial F}{\partial g} \delta g \quad (5.17)$$

$$\Rightarrow I[F[f + \delta f, g + \delta g]] - I[F[f, g]] = \int_B^A dx \frac{\partial F}{\partial f} \delta f + \frac{\partial F}{\partial g} \delta g \quad (5.18)$$

In this case, we just like our normal derivative, we should denote the LHS as:

$$\delta I := \int_B^A dx \frac{\partial F}{\partial f} \delta f + \frac{\partial F}{\partial g} \delta g \quad (5.19)$$

or alternatively,

$$\delta I := \int_B^A dx \frac{\delta I}{\delta f} \delta f + \frac{\delta I}{\delta g} \delta g \quad (5.20)$$

### 5.1.3 Euler-Lagrange Equation

For the two function case, especially consider that  $g = \frac{df}{dx}$ , and let us see what happens. Specifically, let us set that  $\delta f(A) = \delta f(B) = 0$ . Then, we have:

$$\frac{\delta I}{\delta g} \delta g = \frac{\delta I}{\delta \frac{df}{dx}} \delta \frac{df}{dx} \quad (5.21)$$

we can change the order of the derivative:

$$= \frac{\delta I}{\delta \frac{df}{dx}} \frac{d \delta f}{dx} \quad (5.22)$$

from the differentiation of a product, we have

$$= \frac{d}{dx} \left( \frac{\delta I}{\delta \left( \frac{df}{dx} \right)} \delta f \right) - \frac{d}{dx} \frac{\delta I}{\delta \left( \frac{df}{dx} \right)} \delta f \quad (5.23)$$

Now, substituting this to the two function case, we get:

$$\delta I = \int_B^A dx \left[ \left( \frac{\delta I}{\delta f} - \frac{d}{dx} \frac{\delta I}{\delta \left( \frac{df}{dx} \right)} \right) \delta f + \frac{d}{dx} \left( \frac{\delta I}{\delta \left( \frac{df}{dx} \right)} \delta f \right) \right] \quad (5.24)$$



## 5 Mathematical Remarks

the total derivative term is zero, since  $\delta f(A) = \delta f(B) = 0$ . Thus, we have

$$\delta I = \int_B^A dx \left( \frac{\delta I}{\delta f} - \frac{d}{dx} \frac{\delta I}{\delta \left( \frac{df}{dx} \right)} \right) \delta f \quad (5.25)$$

and since  $I = \int_B^A dx F[f(x), g(x)]$ , we can say that

$$\frac{\partial F}{\partial f} = \frac{\delta I}{\delta f}, \quad \frac{\partial F}{\partial g} = \frac{\delta I}{\delta g} \quad (5.26)$$

Then

$$\delta I = \int_B^A dx \left( \frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial \left( \frac{df}{dx} \right)} \right) \delta f \quad (5.27)$$

And if we somehow want to find a minimum of  $I$ , we can set  $\delta I = 0$ :

$$\Rightarrow \quad \frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial \left( \frac{df}{dx} \right)} = 0 \quad (5.28)$$

This is called the **Euler-Lagrange equation**.

### **Theorem 5.1.2** Euler-Lagrange Equation

For a functional  $I \left[ F(f, \frac{df}{dx}) \right]$  to be stationary, ( $\delta I = 0$ ), the **Euler-Lagrange equation** must be satisfied:

$$\delta I \left[ F(f, \frac{df}{dx}) \right] \Leftrightarrow \frac{\partial F}{\partial f} - \frac{d}{dx} \left( \frac{\partial F}{\partial \left( \frac{df}{dx} \right)} \right) = 0 \quad (5.29)$$

### 5.1.4 Important Property

In general, consider that the functional  $F$  is a function of  $f_1(t), f_2(t), \dots, f_n(t)$ :

$$F[f_1, f_2, \dots, f_n] \Rightarrow \delta F = \sum_{j=1}^n \frac{\delta F}{\delta f_j(t)} \delta f_j(t) \quad (5.30)$$

If  $F = f_i$ , we expect that

$$\frac{\delta f_i}{\delta f_i} = 1 \Rightarrow \delta f_i = \sum_{j=1}^n \frac{\delta f_i}{\delta f_j} \delta f_j \Rightarrow \frac{\delta f_i}{\delta f_j} = \delta_{ij} \quad (5.31)$$

Similarly, consider a continuous case where  $F$  is a function of  $f(x)$ :

$$F[f(x, t)] \Rightarrow \delta F = \int_A^B dx' \frac{\delta F}{\delta f(x', t)} \delta f(x', t) \quad (5.32)$$

## 5 Mathematical Remarks

Note the distinction between the variable  $x$  and the integration variable  $x'$ . This is because  $x$  is an "index" of  $f(t)$ :  $f_i \rightarrow f(x)$ . Then, if we set  $F = f(x)$ , we expect that

$$\frac{\delta f(x, t)}{\delta f(x, t)} = 1 \implies \delta f(x, t) = \int_A^B dx' \frac{\delta f(x, t)}{\delta f(x', t)} \delta f(x', t) \quad (5.33)$$

Comparing with this with the definition of **Dirac delta function**:

### Definition 5.2: Dirac Delta Function

The **Dirac delta function**  $\delta(x)$  is defined as a function that satisfies the following property:

$$\int dx' \delta(x' - x) \varphi(x') = \varphi(x), \quad \forall \varphi(x') \in C^\infty \quad (5.34)$$

we have

### Theorem 5.1.3 Property of Functional Derivative

For a functional  $f_i(t)$  or  $f(x, t)$ , the functional derivative satisfies the following property:

$$\frac{\delta f_i(t)}{\delta f_j(t)} = \delta_{ij}, \quad \text{or} \quad \frac{\delta f(x, t)}{\delta f(x', t)} = \delta(x - x') \quad (5.35)$$

## 5.1.5 In a n-dimensional space

In the previous discussions, we have considered the functional  $I$  as an integral on 1D space represented by  $x$ . Here, we aim to generalize the discussion to  $n$ -dimensional space, e.g.  $\mathbb{R}^n$ . For  $\mathbb{R}^n$ , let us define a functional  $I$  and functional derivative as follows:

### Definition 5.3: Functional on $\mathbb{R}^n$

A quantity  $I \in \mathbb{R}$  defined on  $V \subset \mathbb{R}^n$  is called a **(linear) functional** if it can be expressed as follows:

$$I[f_i] := \int_{x \in V} d^n x F(f_i(x)) = \int_V d^n x F(f_i(x)), \quad i \in \mathbb{N} \quad (5.36)$$

### Definition 5.4: Functional Derivative on $\mathbb{R}^n$

The **functional derivative** of a functional  $I[f_i]$  with respect to  $f_i(x)$  is defined as follows:

$$\frac{\delta I[f_j]}{\delta f_i(x)} := \phi_i(x) \stackrel{\text{def}}{\iff} \delta I = \int_V d^n x' \frac{\delta I[f_j]}{\delta f_i(x')} \delta f_i(x') \quad (5.37)$$

where  $\phi_i(x)$  is a function of  $x$ .

## 5 Mathematical Remarks

Then, the variation of the functional  $\delta I$  can be expressed as:

$$\delta I = \delta \int_V d^n x' F(f_j(x')) = \int_V d^n x' \delta F(f_j(x')) = \int_V d^n x' \sum_j \frac{\partial F}{\partial f_j(x')} \delta f_j(x') \quad (5.38)$$

Now,

$$\delta f_j(x') = \int_V d^n x \delta^n(x - x') \delta f_j(x) \quad (5.39)$$

$$= \int_V d^n x \sum_i \delta_{ij} \delta^n(x - x') \delta f_i(x) \quad (5.40)$$

$$\Rightarrow \frac{\delta f_j(x')}{\delta f_i(x)} = \delta_{ij} \delta^n(x - x') \quad (5.41)$$

so,

$$\frac{\delta I}{\delta f_i(x)} = \int_V d^n x' \frac{\partial F}{\partial f_i(x')} \delta^n(x - x') = \frac{\partial F}{\partial f_i(x)} \quad (5.42)$$

thus we see that the functional-density relation still holds:

### **Theorem 5.1.4** Functional-density relation

For a quantity  $I[F]$  and its density  $F[f_i(x)]$ , the functional derivative satisfies the following relation:

$$I = \int_V d^n x F[f_j(x)] \quad \Rightarrow \quad \frac{\delta I[f_j]}{\delta f_i(x)} = \frac{\partial F[f_j]}{\partial f_i(x)} \quad (5.43)$$

[4]

## 5.2 Group Theory

## 6.1 Invariance of Euler-Lagrange Equation

Let us look at how the Euler-Lagrange Equation transforms under a point transformation of the generalized coordinates. The Euler-Lagrange Equation in the old coordinates  $q_i$  is given by:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = 0 \quad (6.1)$$

### Proposition 6.1.1 Euler-Lagrange Equation

In the new coordinates  $Q_j$ , the Euler-Lagrange Equation still holds:

$$\frac{\partial L}{\partial Q_j} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{Q}_j} \right) = 0 \quad (6.2)$$

where  $L(Q_j, \dot{Q}_j, t)$  is the Lagrangian written in the new coordinates.

Under the point transformation:

$$Q_j = Q_j(q_1, q_2, \dots, q_n, t) = Q_j(q_i, t) \quad (6.3)$$

the Lagrangian  $L$  transforms as follows:

$$\tilde{L}(Q, \dot{Q}, t) = L(q(Q, t), \dot{q}(Q, \dot{Q}, t), t) \quad (6.4)$$

Let us consider how the new coordinates  $\dot{Q}_j$  can be written in terms of the old coordinates  $\dot{q}_i$ :

$$\dot{Q}_j = \frac{dQ_j}{dt} = \frac{\partial Q_j}{\partial t} + \sum_k \frac{\partial Q_j}{\partial q_k} \dot{q}_k \implies \frac{\partial \dot{Q}_j}{\partial \dot{q}_i} = \frac{\partial Q_j}{\partial q_i} \quad (6.5)$$

using this,

$$\frac{d}{dt} \left( \frac{\partial \dot{Q}_j}{\partial \dot{q}_i} \right) = \frac{d}{dt} \left( \frac{\partial Q_j}{\partial q_i} \right) = \frac{\partial^2 Q_j}{\partial t \partial q_i} + \sum_k \frac{\partial^2 Q_j}{\partial q_k \partial q_i} \dot{q}_k = \frac{\partial^2 Q_j}{\partial q_i \partial t} + \sum_k \frac{\partial^2 Q_j}{\partial q_i \partial q_k} \dot{q}_k \quad (6.6)$$

and

$$\frac{\partial \dot{Q}_j}{\partial q_i} = \frac{\partial}{\partial q_i} \left( \frac{\partial Q_j}{\partial t} + \sum_k \frac{\partial Q_j}{\partial q_k} \right) = \frac{\partial^2 Q_j}{\partial q_i \partial t} + \sum_k \frac{\partial^2 Q_j}{\partial q_i \partial q_k} = \frac{d}{dt} \left( \frac{\partial \dot{Q}_j}{\partial \dot{q}_i} \right) \quad (6.7)$$

Now, we try to write the Euler-Lagrange Equation in the old coordinates in terms of the new coordinates:

$$\frac{\partial L(Q_j, \dot{Q}_j, t)}{\partial q_i} = \sum_j \frac{\partial L}{\partial Q_j} \frac{\partial Q_j}{\partial q_i} + \frac{\partial L}{\partial \dot{Q}_j} \frac{\partial \dot{Q}_j}{\partial q_i} + \frac{\partial L}{\partial t} \frac{\partial t}{\partial q_i} \quad (6.8)$$

Now, time  $t$  is an independent variable: a parameter that "we" set to describe the system, and crucially does not depend on the generalized coordinates  $q_i$  or the generalized velocities  $\dot{q}_i$ . Thus, we have  $\frac{\partial t}{\partial q_i} = 0$ .

$$\Rightarrow \frac{\partial L(Q_j, \dot{Q}_j, t)}{\partial q_i} = \sum_j \frac{\partial L}{\partial Q_j} \frac{\partial Q_j}{\partial q_i} + \frac{\partial L}{\partial \dot{Q}_j} \frac{\partial \dot{Q}_j}{\partial q_i} \quad (6.9)$$

and

$$\frac{\partial L(Q_j, \dot{Q}_j, t)}{\partial \dot{q}_i} = \sum_j \frac{\partial L}{\partial Q_j} \frac{\partial Q_j}{\partial \dot{q}_i} + \frac{\partial L}{\partial \dot{Q}_j} \frac{\partial \dot{Q}_j}{\partial \dot{q}_i} \quad (6.10)$$

since  $Q_j(q_i, t)$  is not a function of  $\dot{q}_i$ , we have  $\frac{\partial Q_j}{\partial \dot{q}_i} = 0$ .

$$\Rightarrow \frac{\partial L(Q_j, \dot{Q}_j, t)}{\partial \dot{q}_i} = \sum_j \frac{\partial L}{\partial \dot{Q}_j} \frac{\partial \dot{Q}_j}{\partial \dot{q}_i} \quad (6.11)$$

$$\Rightarrow \frac{d}{dt} \left( \frac{\partial \dot{Q}_j}{\partial \dot{q}_i} \right) = \sum_j \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{Q}_j} \right) \frac{\partial \dot{Q}_j}{\partial \dot{q}_i} + \frac{\partial L}{\partial \dot{Q}_j} \frac{d}{dt} \left( \frac{\partial \dot{Q}_j}{\partial \dot{q}_i} \right) \quad (6.12)$$

So, if we compare each term of the Euler-Lagrange Equation in the old coordinates,

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = \sum_j \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{Q}_j} \right) \frac{\partial \dot{Q}_j}{\partial \dot{q}_i} + \frac{\partial L}{\partial \dot{Q}_j} \frac{d}{dt} \left( \frac{\partial \dot{Q}_j}{\partial \dot{q}_i} \right) \quad (6.13)$$

$$\frac{\partial L}{\partial q_i} = \sum_j \frac{\partial L}{\partial Q_j} \frac{\partial Q_j}{\partial q_i} + \frac{\partial L}{\partial \dot{Q}_j} \frac{\partial \dot{Q}_j}{\partial q_i} \quad (6.14)$$

from Eq. (6.5) and Eq. (6.7), we can see that

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = \sum_j \frac{\partial Q_j}{\partial q_i} \left[ \frac{\partial L}{\partial Q_j} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{Q}_j} \right) \right] = 0 \quad (6.15)$$

Now, if we remember that the (i-th) component of the product of a matrix and a vector is

given by

$$(A\vec{x})_i = \sum_j A_{ij}x_j \quad (6.16)$$

So Eq. (6.15) can be rewritten as a product of a matrix and a vector:

$$\sum_j J_{ij}x_j = 0 \quad \text{where} \quad J_{ij} = \frac{\partial Q_j}{\partial q_i}, \quad x_j = \frac{\partial L}{\partial Q_j} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{Q}_j} \right) \quad (6.17)$$

As long as the point transformation is invertible (i.e. changing from 3D Cartesian coordinates to 3D spherical coordinates), the Jacobian matrix  $J_{ij}$  is invertible, and thus the only solution to the equation above is  $x_j = 0$  for all  $j$ . Thus, we have

$$\frac{\partial L}{\partial Q_j} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{Q}_j} \right) = 0 \quad (6.18)$$

[3]

## 6.2 Schrödinger Equation

### 6.2.1 Historical Background

In 1905, Albert Einstein found the photoelectric effect, suggesting that light has a particle like property, such as discrete energy [5]. Following this work, Arthur Compton in 1923 discovered the Compton effect, which is the scattering of X-rays by electrons, further supporting the particle nature of light and confirming the discrete momentum of photons [6]. Specifically, energy and momentum are each related to the frequency and wavelength of light, respectively, as follows:

#### Proposition 6.2.1 Planck-Einstein relation

For a photon with frequency  $\nu$  and wave length  $\lambda$ , the energy  $E$  and momentum  $p$  are given by

$$E = h\nu, \quad p = \frac{h}{\lambda} = \frac{h\nu}{c} = \frac{E}{c}, \quad (6.19)$$

In 1925, Louis de Broglie posulated that other particles (or any matter) also have a wave-like property, and the energy-frequency/ momentum-wavelength relation is given by the Planck-Einstein relation Eq. (6.19) [7].

Then in 1913, Niels Bohr proposed a model of the hydrogen atom, which describes the electron as a particle orbiting the nucleus in discrete energy levels [8]. The implication of this model is that electrons have a fixed, discrete(quantized) angular momentum  $\vec{L} := \vec{r} \times \vec{p}$ . For a particle orbiting in a circular orbit, the angular momentum is given by

$$\vec{L} = r\vec{p} = \frac{hr}{\lambda} \quad (6.20)$$

For the electron wave to be continous around the orbit of radius  $r$ , the wavelength must be an integer multiple of the circumference of the orbit:

$$\lambda = \frac{2\pi r}{n}, \quad n = 1, 2, 3, \dots \implies \vec{L} = n \frac{h}{2\pi} := n\hbar \quad (6.21)$$

### 6.2.2 Schrödinger Equation

Now, a classical wave obeys a wave equation:

$$\frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} - \nabla^2 \psi = 0 \quad (6.22)$$

but does the matter wave also obey wave equation?

Well, not a classical wave equation, but a quantum wave should follow a quantum wave equation, which must satisfy a few conditions that match with the physics as we know it:

- Planck-Einstein relation:  $E = \hbar\omega$ ,  $\vec{p} = \hbar\vec{k}$

## 6 Extra

- Energy equation:  $E = T + V$
- For  $V = V_0$ , the solution must be a plane wave:  $\psi(\vec{x}, t) = Ae^{i(\vec{k}\cdot\vec{x}-\omega t)}$  (i.e. constant momentum and constant energy).

The energy equation is given by

$$E = T + V = \frac{p^2}{2m} + V_0 \iff \hbar\omega = \frac{\hbar^2 \vec{k}^2}{2m} + V \quad (6.23)$$

If we multiply both sides by  $\psi(\vec{x}, t)$ , we get

$$\hbar\omega\psi(\vec{x}, t) = \frac{\hbar^2}{2m}\vec{k}^2\psi(\vec{x}, t) + V\psi(\vec{x}, t) \quad (6.24)$$

For a plane wave solution,

$$\frac{\partial}{\partial t}\psi(\vec{x}, t) = -i\omega\psi(\vec{x}, t) \implies i\hbar\frac{\partial}{\partial t}\psi(\vec{x}, t) = \hbar\omega\psi(\vec{x}, t) \quad (6.25)$$

$$\nabla^2\psi(\vec{x}, t) = -\vec{k}^2\psi(\vec{x}, t) \implies -\frac{\hbar^2}{2m}\nabla^2\psi(\vec{x}, t) = \frac{\hbar^2\vec{k}^2}{2m}\psi(\vec{x}, t) \quad (6.26)$$

Thus, we can write the wave equation as

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{x}, t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\vec{x}, t) + V_0\psi(\vec{x}, t) \quad (6.27)$$

And we postulate that for any potential  $V(\vec{x})$ , the wave equation is given by

### Principle 6.1: Schrödinger Equation

The Schrödinger equation is given by

$$i\hbar\frac{\partial}{\partial t}\psi(\vec{x}, t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\vec{x}, t) + V(\vec{x})\psi(\vec{x}, t) \quad (6.28)$$

where  $\psi(\vec{x}, t)$  is the wave function,  $V(\vec{x})$  is the potential, and  $m$  is the mass of the particle.



## 6.3 Derivative by Derivative

### 6.3.1 An Interesting Problem

#### Question 1: Derivative by derivative

What would the following operator mean?

$$\frac{d}{d\frac{d}{dx}} \quad (6.29)$$

Let us define a differential operator  $D$ :

#### Definition 6.1: Mysterious Operator

The operator  $D$  is defined as the derivative of the derivative with respect to  $x$ :

$$D := \frac{d}{dx} \quad (6.30)$$

Now let us actually apply  $D$  to some functions:

#### Example 6.3.1 (A power function)

Let us apply  $\frac{d}{dx}$  to a power function  $f(x) = x^n$ ,  $n \in \mathbb{Z}^+$ :

$$Df(x) = \frac{d}{dx}x^n = nx^{n-1} = \frac{n}{x}x^n \implies D = \frac{n}{x} \quad (6.31)$$

Then,

$$\frac{d}{dD}f(x) = \frac{d}{dD}\left(\frac{n}{D}\right)^n = n^n \cdot (-n) \cdot D^{-n-1} = -\left(\frac{n}{D}\right)^{n+1} = -x^{n+1} \quad (6.32)$$

#### Example 6.3.2 (Exponential Function)

Let us apply  $\frac{d}{dx}$  to the exponential function  $f(x) = e^{\alpha x}$ :

$$\frac{df}{dx} = \frac{d}{dx}e^{\alpha x} = \alpha f(x) \implies \frac{d}{dx} = \alpha \quad (6.33)$$

$$\implies Df(x) = \frac{de^{\alpha x}}{d\alpha} = xe^{\alpha x} \quad (6.34)$$

An important property of this operator is in the commutator with the  $x$  operator:

**Definition 6.2:  $x$  operator**

The  $x$  operator is defined as the operator that maps a function  $f$  to the function  $x \cdot f$ :

$$x : f \mapsto x \cdot f \quad (6.35)$$

**Definition 6.3: Commutator**

For two operators  $A : f \mapsto Af$  and  $B : f \mapsto Bf$ , the commutator is defined as:

$$[A, B] := AB - BA \quad (6.36)$$

Then the commutator of the  $x$  operator and the  $D$  operator is given by:

$$[x, D] = xD - Dx \quad (6.37)$$

applying this to a function  $f(x)$ , we have:

$$[x, D] f(x) = xDf(x) - D(xf(x)) \quad (6.38)$$

$$= xDf(x) - Dx f(x) - xDf(x) \quad (6.39)$$

$$= -Df(x) = -1 \cdot f(x) \quad (6.40)$$

$$\Rightarrow [x, D] = -1 \quad (6.41)$$

and notice since both  $x$  and  $D$  are just linear operators, if we define  $x = \frac{d}{dD}$ , then we have:

$$[x, D] f(D) = \frac{d}{dD} Df(D) - D \frac{d}{dD} f(D) \quad (6.42)$$

$$= f(D) + D \frac{df}{dD} - D \frac{df}{dD} \quad (6.43)$$

$$= f(D) \quad (6.44)$$

so in general,

$$\left[ \omega, \frac{d}{d\omega} \right] = -1 \quad (6.45)$$

Note the bi-linearity of the commutator:

**Theorem 6.3.1 Bi-linearity of Commutator**

For  $\alpha, \beta \in \mathbb{C}$  and operators  $A, B$ , the commutator satisfies:

$$[\alpha A, \beta B] = \alpha\beta [A, B] \quad (6.46)$$

This immediately leads to:

**Principle 6.2:** Canonical Commutation Relation

For the  $x$  operator and the  $D$  operator, the commutation relation is given by:

$$[x, -i\hbar D] = i\hbar = \left[ x, -i\hbar \frac{\partial}{\partial x} \right] = \left[ i\hbar \frac{\partial}{\partial p}, p \right] \quad (6.47)$$

Let us remark on other properties:

**Theorem 6.3.2** Commutator of General Function

For a general function  $f(x)$ , the commutator with the  $D$  operator is given by:

$$[D, f(x)] = \frac{df(x)}{dx} \quad (6.48)$$

**Proof:** Consider functions  $f(x)$  and  $g(x)$ , then:

$$[D, f(x)]g(x) = Df(x)g(x) - f(x)Dg(x) \quad (6.49)$$

$$= \frac{df(x)}{dx}g(x) + f(x)\frac{dg(x)}{dx} - f(x)\frac{dg(x)}{dx} \quad (6.50)$$

$$= \frac{df(x)}{dx}g(x) \quad (6.51)$$

$$\Rightarrow [D, f(x)] = \frac{df(x)}{dx} \quad (6.52)$$

□

**Theorem 6.3.3** Commutator of General Operator

For a general operator  $F$  that is a function of  $D$ , the commutator with the  $x$  operator is given by:

$$[F(D), x] = \frac{dF(D)}{dD} \quad (6.53)$$

**Proof:** Assume that  $F$  is a function of  $D$ , and we can write it as a Taylor series:

$$F(D) = \sum_{n=0}^{\infty} a_n D^n \quad (6.54)$$

Then,

$$[F, x] = Fx - xF \quad (6.55)$$

$$= \sum_{n=0}^{\infty} a_n D^n x - x \sum_{n=0}^{\infty} a_n D^n \quad (6.56)$$

$$= \sum_{n=0}^{\infty} a_n (D^n x - x D^n) \quad (6.57)$$

$$= \sum_{n=0}^{\infty} a_n n D^{n-1} = \frac{dF}{dD} \quad (6.58)$$

□

Finally, let us comment on the relationship with the **shift operator**:

**Definition 6.4: Shift Operator**

The **shift operator**  $S_a$  is defined as the operator that shifts a function by a constant  $a$ :

$$S_a[f](x) := f(x + a) \quad (6.59)$$

The commutator of the shift operator and the  $x$  operator is given by:

$$[S_a, x] f(x) = S_a[xf(x)] - xS_a[f(x)] \quad (6.60)$$

$$= (x + a)f(x + a) - xf(x + a) \quad (6.61)$$

$$= af(x + a) = aS_a[f(x)] \quad (6.62)$$

Thus,

**Theorem 6.3.4** Commutator of Shift Operator and  $x$

The commutator of the shift operator  $S_a$  and the  $x$  operator is given by:

$$[S_a, x] = aS_a \implies S_a = e^{aD} \quad (6.63)$$

## 6.4 Variational Principle in Fluid Mechanics

### 6.4.1 Overview: Cauchy's Equation of Motion

The equation of motion of a point mass is given by Newton's second law:

$$m \frac{d^2 \vec{r}}{dt^2} = \vec{F} \quad (6.64)$$

The equation of motion of a fluid (in general) is given by Cauchy's equation of motion:

$$\int_V \rho dV \frac{D\vec{u}}{Dt} = \oint_S dS \sigma \vec{n} + \int_V \rho dV \vec{F} \quad (6.65)$$

where

- $V$ : Volume of the fluid
- $S$ : Surface of  $V$
- $\rho(\vec{r}, t)$ : Density of the fluid at position  $\vec{r}$  and time  $t$
- $\vec{u}(\vec{r}, t)$ : Velocity of the fluid at position  $\vec{r}$  and time  $t$
- $\vec{n}$ : Normal vector of the surface  $S$  pointing outward
- $\sigma(\vec{r}, t)$ : Stress tensor of the fluid
- $\vec{F}$ : External force per unit volume acting on the infinitesimal volume  $dV$  of fluid

the stress tensor shows how the forces act on any surface within a fluid.

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \quad (6.66)$$

The Newtonian derivation of Cauchy's equation of motion is based on the conservation of the momentum of a fluid element:

$$\left( \frac{\partial p}{\partial t} + \Delta p \text{ flux per time} \right) = (\Delta p \text{ from the surface force}) + (\Delta p \text{ from the volume force}) \quad (6.67)$$

$$\Rightarrow \rho dV \frac{D\vec{u}}{Dt} = \nabla \cdot (\sigma \vec{n}) + \rho dV \vec{F} \quad (6.68)$$

[9]

### 6.4.2 Lagrange Derivative

Imagine that you want to measure the height of the water in a river at a certain point and time. Let us assume that you put a very light, small box with some measuring equipment on the water surface at some point  $\vec{r}_i$  at time  $t_i$ . Then we put another box at another point  $\vec{r}_f$  at the same time  $t_i$ , but we fix this box so that it does not move.

Since the water surface is not static, box 1 will be carried by the water flow. Let us denote the position of the box 1 at time  $t$  as  $\vec{\xi}(\vec{r}_i, t; t_i)$ , where the box 1 is placed at the point  $\vec{r}_i$  at time  $t_i$ .

Let us denote the height of the water surface at a point  $\vec{r}$  at time  $t$  by  $X(\vec{r}, t)$ . Then, we can write the height of the water surface at the positions of box 1 and 2 at time  $t$  as

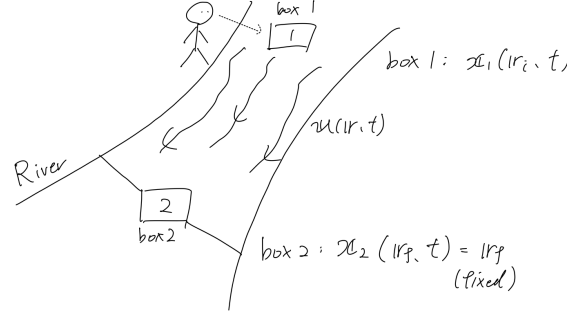


Figure 6.1: Boxes on a river

$$X(\vec{\xi}, t) = X_1(t) \quad (6.69)$$

$$X(\vec{r}_f, t) = X_2(t) \quad (6.70)$$

Finally, assume that the box 1 will reach  $\vec{r}_f$  at time  $t_f$ .

Now, at initial time  $t_i$ , the change in height of the water surface is given by:

$$X(\vec{r}_f, t_i) - X(\vec{r}_i, t_i) = \nabla X(\vec{r}_i, t_i) \cdot (\vec{r}_f - \vec{r}_i) \quad (6.71)$$

$$= \nabla X(\vec{r}_i, t_i) \cdot \Delta \vec{r} \quad (6.72)$$

assuming that  $\Delta \vec{r} := \vec{r}_f - \vec{r}_i$  is small enough. On the other hand, for box 2, the change in height of the water surface in a time interval  $\Delta t = t_f - t_i$  is given by:

$$X(\vec{r}_f, t_f) - X(\vec{r}_f, t_i) = \frac{\partial X(\vec{r}_f, t_i)}{\partial t} \Delta t \quad (6.73)$$

Thus, the total change of  $X$  measured by box 1, during  $\Delta t$ , moving at the velocity  $\vec{u}$  is given by

$$X(\vec{r}_f, t_f) - X(\vec{r}_i, t_i) = X(\vec{r}_f, t_f) - \underbrace{X(\vec{r}_f, t_i) + X(\vec{r}_f, t_i)}_{=0} - X(\vec{r}_i, t_i) \quad (6.74)$$

$$= \frac{\partial X}{\partial t} \Delta t + \nabla X(\vec{r}_i, t_i) \cdot \Delta \vec{r} \quad (6.75)$$

Now, since  $\vec{r}_i = \vec{\xi}(\vec{r}_i, t_i)$  and  $\vec{r}_f = \vec{\xi}(\vec{r}_i, t_f)$ , we can rewrite the equation above:

$$X(\vec{\xi}(\vec{r}_i, t_f), t_f) - X(\vec{\xi}(\vec{r}_i, t_i), t_i) = \frac{\partial X}{\partial t} \Delta t + \nabla X(\vec{\xi}(\vec{r}_i, t_i), t_i) \cdot \Delta \vec{\xi} \quad (6.76)$$

where  $\Delta \vec{\xi} = \vec{\xi}(\vec{r}_i, t_f) - \vec{\xi}(\vec{r}_i, t_i)$ .

## 6 Extra

Using  $\Delta t = t_f - t_i$ , we can rewrite the equation above as:

$$\frac{X(\vec{\xi} + \Delta\vec{\xi}, t_i + \Delta t) - X(\vec{\xi}, t_i)}{\Delta t} = \frac{\partial X}{\partial t} + \nabla X(\vec{\xi}, t_i) \cdot \frac{\Delta\vec{\xi}}{\Delta t} \quad (6.77)$$

$$(6.78)$$

Since the box 1 is a part of the fluid, the time derivative of  $\vec{\xi}$  is the velocity of the fluid at the position  $\vec{\xi}$ :

$$\frac{\partial \vec{\xi}(\vec{r}_i, t)}{\partial t} = \vec{u}(\vec{\xi}, t) \quad (6.79)$$

### Note:

In this derivative, we fix the starting position  $\vec{r}_i$ , we only care about the time evolution from the position  $\vec{r}_i$ . This is different from the total derivative, which takes into account a change in the starting position  $\vec{r}_i$ .

thus, by taking  $\Delta t \rightarrow 0$  ( $\Rightarrow \Delta x \rightarrow 0$ ), we can rewrite the equation above as:

$$\frac{DX(\vec{\xi}, t_i)}{Dt} := \frac{\partial X(\vec{\xi}, t_i)}{\partial t} + \vec{u}(\vec{\xi}, t_i) \cdot \nabla X(\vec{\xi}, t_i) \quad (6.80)$$

where  $\frac{D}{Dt}$  indicates that we are tracking the box 1 and its measurement of  $X$ . This is called the **Lagrange derivative**:

### Definition 6.5: Lagrange Derivative

The Lagrange description in Eulerian perspective is given as the **Lagrange derivative**:

$$\frac{DX}{Dt} := \frac{\partial X}{\partial t} + \vec{u} \cdot \nabla X \quad (6.81)$$

which tracks the change of a physical quantity  $X$  with the flow.

Now, define a new quantity called the "position function"  $\vec{x}(\vec{r}, t)$ :

$$\vec{x}(\vec{r}, t) := \vec{r} \quad (6.82)$$

If we measure this position function at the box 1 at  $t = t_i$ ,

$$\vec{x}(\vec{r}_i, t_i) = \vec{r}_i \quad (6.83)$$

and notice that tracking the position of box 1 is equivalent to observing the time-evolved position  $\vec{\xi}(\vec{r}_i, t_i)$ :

$$\frac{\partial \vec{\xi}}{\partial t} = \frac{D\vec{x}}{Dt} = \vec{u}(\vec{x}, t) \quad (6.84)$$

### 6.4.3 Derivation from Action Integral

In Newtonian mechanics, for a particle of mass  $m$ , the action integral  $S$  is given by:

$$S = \int dt L = \int dt m \mathcal{L}, \quad \mathcal{L} := \frac{L}{m} = \left[ \frac{1}{2} \dot{\vec{x}}^2 - \tilde{V}(\vec{x}) \right] \quad (6.85)$$

where  $\mathcal{L}$  is the Lagrangian density, and

$$\tilde{V}(\vec{x}) = \frac{V(\vec{x})}{m} \quad (6.86)$$

by assuming a similar form of the Lagrangian density for a fluid, we can write the action integral for a fluid as:

$$S = \int dt L = \int dt \int_V \rho(\vec{x}, t) dV(\vec{x}, t) \mathcal{L}\left(\vec{x}, \frac{D\vec{x}}{Dt}\right) \quad (6.87)$$

where the Lagrangian density is

$$\mathcal{L}\left(\vec{x}(\vec{r}, t), \frac{D\vec{x}(\vec{r}, t)}{Dt}\right) = \frac{1}{2} \left( \frac{D\vec{x}(\vec{r}, t)}{Dt} \right)^2 - \tilde{V}(\vec{x}(\vec{r}, t)) + \frac{1}{\rho(\vec{r}, t)} \nabla \cdot \sigma(\vec{r}, t) \vec{x}(\vec{r}, t) \quad (6.88)$$

Then the Euler-Lagrange equation for this Lagrangian density is given by:

$$\frac{\partial \mathcal{L}}{\partial \vec{x}} - \frac{D}{Dt} \left( \frac{\partial \mathcal{L}}{\partial \left( \frac{D\vec{x}}{Dt} \right)} \right) = 0 \quad (6.89)$$

calculating the partial derivative gives:

$$\frac{\partial \mathcal{L}}{\partial \vec{x}} = \frac{1}{\rho} \nabla \cdot \sigma - \nabla \tilde{V}(\vec{x}) = \frac{1}{\rho} \nabla \cdot \sigma + \vec{F} \quad (6.90)$$

$$\frac{\partial \mathcal{L}}{\partial \left( \frac{D\vec{x}}{Dt} \right)} = \frac{D\vec{x}}{Dt} = \vec{u}(\vec{x}, t) \quad (6.91)$$

hence the Euler-Lagrange equation becomes:

$$\frac{D\vec{u}}{Dt} = \frac{1}{\rho} \nabla \cdot \sigma + \vec{F} \quad (6.92)$$

by integrating over the volume  $V$ , we get

$$\int_V \rho dV \frac{D\vec{u}}{Dt} = \int_V dV \nabla \cdot \sigma + \int_V \rho dV \vec{F} \quad (6.93)$$

$$\Rightarrow \int_V \rho dV \frac{D\vec{u}}{Dt} = \int_V ds \vec{n} \cdot \sigma + \int_V \rho dV \vec{F} \quad (6.94)$$



### 6.4.4 Hamilton Formalism

Similarly to the point mass case, we should define the momentum density  $\vec{\pi}(\vec{r}, t)$  as the derivative of the Lagrangian density with respect to the velocity:

$$\vec{\pi}(\vec{r}, t) = \frac{\partial \mathcal{L}}{\partial(\frac{D\vec{x}}{Dt})} \quad (6.95)$$

then the Hamiltonian density should be defined by the Legendre transformation:

$$\mathcal{H}(\vec{\pi}, \vec{x}) = \vec{\pi} \cdot \frac{D\vec{x}}{Dt} - \mathcal{L}\left(\vec{x}, \frac{D\vec{x}}{Dt}\right) \quad (6.96)$$

$$= \frac{1}{2} \left( \frac{D\vec{x}}{Dt} \right)^2 + \tilde{V}(\vec{x}) - \frac{1}{\rho} \nabla \cdot \sigma \vec{x} \quad (6.97)$$

[10]

## 6.5 Physical Constants

All values are taken from [The NIST Reference on Constants, Units, and Uncertainty](#) [11].

Table 6.1: Physical Constants

Symbol	Constant	Value	Units
$c$	Speed of light in vacuum	$2.997\,924\,58 \times 10^8$	$\text{m s}^{-1}$
$e$	Elementary charge	$-1.602\,176\,634 \times 10^{-19}$	C
$G$	Gravitational constant	$6.674\,30 \times 10^{-11}$	$\text{N m}^2 \text{kg}^{-2}$
$h$	Planck constant	$6.626\,070\,15 \times 10^{-34}$	J s
$\hbar := \frac{h}{2\pi}$	Reduced Planck constant	$1.054\,571\,817 \times 10^{-34}$	J s

## 6.6 Textbooks

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### 6.6.1 General Textbooks

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- [University Physics with Modern Physics](#) by Hugh Young and Roger Freedman [12]
- [Feynman Lectures on Physics](#) by Richard P. Feynman [13]

### 6.6.2 Analytical Mechanics

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- [Classical Mechanics](#) by Goldstein, Poole, and Safko [14]
- [Mechanics](#) by Landau and Lifshitz [15]

### 6.6.3 Quantum Mechanics

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- [Quantum Mechanics](#) by Leonard Schiff [16]
- [Introduction to Quantum Mechanics](#) by David Griffiths [17]
- [Modern Quantum Mechanics](#) by J. J. Sakurai and Jim Napolitano [18]

### 6.6.4 Particle Physics

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- [Introduction to Elementary Particles](#) by David Griffiths [19]
- [An Introduction to Quantum Field Theory](#) by Michael Peskin and Daniel Schroeder [20]

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