Scientific Research

Report Assignment

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1.

In an non-interacting electron gas, the electrons fill up the momentum space from the lower energy states up to the Fermi energy E_F.

$$N(E_F) = N_{\text{tot}} \tag{1}$$

At the Fermi energy, the electrons have a corresponding wave number k_F:

$$E_F = \frac{\hbar^2 k_F^2}{2m} \iff k_F = \frac{\sqrt{2mE_F}}{\hbar} \tag{2}$$

and the total number of electrons in the gas is given by

$$N_{\text{tot}} = \int_{|k|<k_F} dN = \int_0^{k_F} dk\, \frac{V}{\pi^2} k^2 = \frac{V}{3\pi^2} k_F^3 \tag{3}$$

The energy of a relativistic particle is given by

$$E(p) = \sqrt{p^2c^2 + m^2c^4} \tag{4}$$

in terms of the wave number,

$$E(k) = \sqrt{(\hbar k)^2 c^2 + m^2 c^4} \tag{5}$$

So the total energy of the electron gas is given by

$$E_{\text{tot}} = \int_0^N dN\, E(k) \tag{6}$$

$$= \int_0^{k_F} dk\, \frac{V}{\pi^2} k^2 E(k) \tag{7}$$

$$\implies \varepsilon_e = \frac{1}{\pi^2} \int_0^{k_F} dk\, k^2 \sqrt{(\hbar k)^2 c^2 + m_e^2 c^4} \tag{8}$$

now set $k = \frac{m_e c}{\hbar} \sinh\theta$ (i.e. $x_F = \sinh\theta_F$), and so

$$dk = \frac{m_e c}{\hbar} \cosh\theta\, d\theta, \quad E(\theta) = m_e c^2 \cosh\theta \tag{9}$$

then the integral becomes

$$\varepsilon_e = \frac{1}{\pi^2} \int_0^{\theta_F} d\theta \left(\frac{m_e c}{\hbar} \cosh\theta\right) \left(\frac{m_e c}{\hbar} \sinh\theta\right)^2 \left(m_e c^2 \cosh\theta\right) \tag{10}$$

$$= \frac{m_e^4 c^5}{\pi^2 \hbar^2} \int_0^{\theta_F} d\theta\, \cosh^2\theta \sinh^2\theta \tag{11}$$

To calculate this integral, let us use a few formulas for hyperbolic functions:

$$\sinh x \cosh x = \frac{1}{2}\sinh(2x), \quad \sinh^2 x = \frac{1}{2}(\cosh(2x) - 1), \quad \cosh(2x) = 1 + 2\sinh^2 x \tag{12}$$

using these, we can rewrite the integral as

$$\int_0^{\theta_F} d\theta \; \cosh^2\theta \sinh^2\theta = \frac{1}{8}\int_0^{\theta_F} d\theta \; \cosh(4\theta) - 1 \tag{13}$$

$$= \frac{1}{8}\left[\frac{1}{4}\sinh(4\theta) - \theta\right]_0^{\theta_F} \tag{14}$$

$$= \frac{1}{8}\left[\frac{1}{4}\sinh(4\theta_F) - \theta_F\right] \tag{15}$$

and the expressino inside the brackets can be simplified further:

$$\frac{1}{4}\sinh(4\theta) = \frac{1}{2}\sinh(2\theta)\cosh(2\theta) \tag{16}$$

$$= \sinh\theta\cosh\theta\Big(1 + 2\sinh^2\theta\Big) \tag{17}$$

here, since $x_F = \sinh\theta_F$,

$$\cosh\theta_F = \sqrt{1 + x_F^2}, \quad \theta_F = \operatorname{arsinh} x_F = \ln\Big(x_F + \sqrt{1 + x_F^2}\Big) \tag{18}$$

thus,

$$\varepsilon_e = \frac{m_e^4 c^5}{8\pi^2\hbar^2}\left(\frac{1}{4}\sinh(4\theta_F) - \theta_F\right) \tag{19}$$

$$= \frac{m_e^4 c^5}{8\pi^2\hbar^2}\Big(\sinh\theta_F\cosh\theta_F(1 + 2x_F^2) - \ln(x_F + \sqrt{1 + x_F^2})\Big) \tag{20}$$

2.

Table 1: Properties of 12 sequential nuclides in the outer crust of a neutron star.

Nuclide	**Z**	**N**	**A**	n_B [fm^{-3}]	M_{cal} [MeV]	$n_N M_{\textbf{nuc}} c^2$	ϵ'_e [MeV fm^{-3}]	ϵ_L	Total $\epsilon(n_B)$
^{56}Fe	26	30	56	4.75×10^{-8}	-60.61	0.043916	1.83×10^{-5}	-2.71×10^{-7}	0.043940
^{62}Ni	28	34	62	1.61×10^{-6}	-66.75	0.047870	2.65×10^{-4}	-3.53×10^{-6}	0.048184
^{64}Ni	28	36	64	7.87×10^{-6}	-67.09	0.050479	8.84×10^{-4}	-1.04×10^{-5}	0.051448
^{86}Kr	36	50	86	1.83×10^{-5}	-82.90	0.053181	1.83×10^{-3}	-2.04×10^{-5}	0.055171
^{84}Se	34	50	84	6.29×10^{-5}	-82.35	0.060155	5.25×10^{-3}	-5.19×10^{-5}	0.065747
^{82}Ge	32	50	82	1.66×10^{-4}	-79.62	0.068225	1.18×10^{-2}	-1.10×10^{-4}	0.080517
^{80}Zn	30	50	80	3.60×10^{-4}	-74.96	0.077259	2.27×10^{-2}	-2.02×10^{-4}	0.100913
^{78}Ni	28	50	78	4.80×10^{-4}	-66.86	0.081182	2.80×10^{-2}	-2.43×10^{-4}	0.109893
^{124}Mo	42	82	124	7.30×10^{-4}	-85.73	0.084478	4.50×10^{-2}	-3.42×10^{-4}	0.130638
^{122}Zr	40	82	122	1.25×10^{-3}	-84.38	0.091391	7.04×10^{-2}	-5.16×10^{-4}	0.163351
^{120}Sr	38	82	120	2.10×10^{-3}	-81.98	0.100230	1.08×10^{-1}	-7.53×10^{-4}	0.210080
^{118}Kr	36	82	118	2.53×10^{-3}	-77.41	0.104646	1.25×10^{-1}	-8.54×10^{-4}	0.231142

The structure of a neutron star's outer crust is determined by several important physical processes. As the density increases, nuclei become progressively more neutron-rich through neutron capture proceses. As a result that can be seen in Table 1, the neutron/proton increases from 1.15 in ^{56}Fe to 2.16 in ^{118}Kr.

Regarding energy contributions, the nuclear mass energy is the largest component of the total energy density. However, the kinetic energy of the electrons grows much more rapidly with density. A much smaller component, the lattice energy, is a negative correction that accounts for the electrostatic arrangement of nuclei into a crystal lattice.

This sequence of increasingly dense and neutron-rich nuclei ends at the neutron drip point, where the neutron is no longer energetically bound to the nucleus.