Particle Physics

Preliminaries for Particle Physics

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Chapter 1 Notations and Conventions

1.1 General Notations

1.1.1 Vectors and Functions

A vector (in a non-relativistic context) is denoted by an arrow:

$$\vec{x}, \vec{r}, \vec{p}, \vec{F}, \vec{v}, \vec{a}, \ldots \tag{1.1}$$

If a function depends on multiple variables (e.g., $\{q_i\} = \{q_1, q_2, q_3, \ldots\}$), we may denote it in a short-handed way:

$$f(q_1, q_2, q_3, \ldots) \iff f(q_i) \tag{1.2}$$

A partial derivative by the variable x is denoted by:

$$\frac{\partial f}{\partial x} = \partial_x f, \tag{1.3}$$

For a general coordinate system $(x_1, x_2, x_3, \ldots)$, we denote the partial derivative by:

$$\frac{\partial f}{\partial x_i} = \partial_i f \tag{1.4}$$

1.2 Relativity Notations

1.2.1 Vectors(in Relativity)

For 4-vectors and tensors in relativity, sans-serif font is used:

$$\mathsf{x}, \mathsf{p}^\mu, \mathsf{g}_{\mu\nu} \tag{1.5}$$

Each component of a 4-vector is denoted by a superscript or subscript:

$$\mathsf{x}^\mu = x^0, x^1, x^2 \text{ or } x^3 \tag{1.6}$$

and without an index, it represents the entire vector:

$$\mathsf{x} = (x^0, x^1, x^2, x^3) = (ct, \vec{x}) \tag{1.7}$$

1.2.2 Einstein's Summation Convention

If an index such as i, j, k, μ, ν, etc., appears twice in a single term, it implies summation over that index. For example:

Example 1.2.1 (Examples)

$$x_i y_i = \vec{x} \cdot \vec{y} = \sum_{i=1}^{3} x_i y_i \tag{1.8}$$

$$g_{\mu\nu} u^{\mu} v^{\nu} = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} u^{\mu} v^{\nu} \tag{1.9}$$

Note that the greek index such as μ, ν usually runs from 0 to 3, while the latin index such as i, j, k usually runs from 1 to 3.

1.2.3 Metric Tensor

As per the convention in particle physics, we take the $(+, -, -, -)$ metric signature, and the Minkowski metric $\eta_{\mu\nu}$ is denoted by:

$$\eta_{\mu\nu} = \begin{pmatrix} \eta_{00} & \eta_{01} & \eta_{02} & \eta_{03} \\ \eta_{10} & \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{20} & \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{30} & \eta_{31} & \eta_{32} & \eta_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.10}$$

1.2.4 Units

In principle, we use SI units:

$$\text{Length} = [L] = \text{m} \tag{1.11}$$

$$\text{Time} = [T] = \text{s} \tag{1.12}$$

$$\text{Mass} = [M] = \text{kg} \tag{1.13}$$

$$\text{Charge} = [C] = \text{C} \tag{1.14}$$

However, for particle physics, we often use natural units, where:

$$\hbar = c = 1 \tag{1.15}$$

and the unit of energy in electronvolts (eV) is used.

Chapter 2

Analytical Mechanics

2.1 Lagrange Formalism

2.1.1 Newtonian Mechanics

In Newtonian mechanics, the motion of a particle is described through a few important quantities: for a particle of (inertial) mass m, position $\vec{r}$, we have

$$\text{velocity} : \vec{v} = \frac{d\vec{r}}{dt} \tag{2.1}$$

$$\text{acceleration} : \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} \tag{2.2}$$

$$\text{momentum} : \vec{p} = m\vec{v} = m\frac{d\vec{r}}{dt} \tag{2.3}$$

and the relations between these quantities, in the presence of external forces $\vec{F}^{(i)}_{\text{ext}}$ acting on the particle, are given by *Newton's second law*:

$$\frac{d\vec{p}}{dt} = m\frac{d^2\vec{r}}{dt^2} = \sum_i \vec{F}^{(i)}_{\text{ext}} \tag{2.4}$$

The work done by such forces is given by

$$W_{\text{total}} = \sum_i \int_l d\vec{x} \cdot \vec{F}^{(i)}_{\text{ext}}, \quad \text{where } l \text{ is the path of the particle.} \tag{2.5}$$

This is the energy change of the particle through the motion:

$$\begin{aligned} W_{\text{total}} &= \sum_i \int_l d\vec{x} \cdot \vec{F}^{(i)}_{\text{ext}} && (2.6)\\ &= \int_{t_i}^{t_f} dt\, \vec{v} \cdot m\frac{d\vec{v}}{dt} && (2.7)\\ &= \int_{t_i}^{t_f} dt\, \frac{m}{2}\frac{d}{dt}\vec{v}^{\,2} && (2.8)\\ &= \frac{m}{2}\vec{v}_f^{\,2} - \frac{m}{2}\vec{v}_i^{\,2} && (2.9) \end{aligned}$$

meaning that $m\vec{v}^2/2$ is the energy due to the motion of the particle: the **kinetic energy** T:

$$T = \frac{m}{2}\vec{v}^2 \tag{2.10}$$

Now, often, the external force acting on the particle is due to a potential V:

$$\vec{F}_{\text{ext}} = -\nabla V \tag{2.11}$$

For example, for a 1D spring, the potential is given by

Example 2.1.1 (1D spring/ Harmonic potential)

$$V = \frac{1}{2}kx^2 \implies F_{\text{ext}} = -kx \tag{2.12}$$

or the electrostatic potential:

Example 2.1.2 (Electrostatic potential)

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r} \implies F_{\text{ext}} = -\nabla V = -\frac{q_1 q_2}{4\pi\varepsilon_0 r^2}\hat{r} \tag{2.13}$$

Now, for a particle whose the external forces are given by a potential:

$$m\frac{d^2\vec{r}}{dt^2} = -\nabla V \iff -m\frac{d^2\vec{r}}{dt^2} - \nabla V = 0 \tag{2.14}$$

This looks as if the forces $-\nabla V$ and $m\ddot{\vec{r}}$ are in equilibrium. So, if we move a particle by an infinitisimal distance $\delta\vec{r}$, the total work done by these forces must be zero:

$$\left(-m\frac{d^2\vec{r}}{dt^2} - \nabla V\right)\cdot\delta\vec{r} = 0 \tag{2.15}$$

at any time t. We want to apply this for entire path of the motion of the particle, from t_i to t_f. Then, the integral of this equation over the time interval $[t_i, t_f]$ gives

$$\delta I = \int_{t_i}^{t_f} dt \left(-m\frac{d^2\vec{r}}{dt^2} - \nabla V\right)\cdot\delta\vec{r} = 0 \tag{2.16}$$

now, we can apply integration by parts:

$$\frac{d}{dt}\left[\dot{\vec{r}}\cdot\delta\vec{r}\right] = \ddot{\vec{r}}\cdot\delta\vec{r} + \dot{\vec{r}}\cdot\frac{d\,\delta\vec{r}}{dt} \tag{2.17}$$

$$= \ddot{\vec{r}}\cdot\delta\vec{r} + \dot{\vec{r}}\cdot\delta\dot{\vec{r}} \tag{2.18}$$

$$\Longleftrightarrow \quad -m\ddot{\vec{r}} = -\frac{d}{dt}\left(\dot{\vec{r}}\cdot\delta\vec{r}\right) + \dot{\vec{r}}\cdot\delta\dot{\vec{r}} \tag{2.19}$$

$$= -\frac{d}{dt}\left(m\dot{\vec{r}}\cdot\delta\vec{r}\right) + \delta\left(\frac{m}{2}\left(\frac{d\vec{r}}{dt}\right)^2\right), \tag{2.20}$$

where we used the commutitivity of $\frac{d(\cdot)}{dt}$ and $\delta(\cdot)$. Then the integral becomes

$$\delta I = \int_{t_i}^{t_f} dt\left(\delta\left[\frac{m}{2}\left(\frac{d\vec{r}}{dt}\right)^2\right] - \delta V\right) \tag{2.21}$$

$$= \int_{t_i}^{t_f} dt\,(\delta T - \delta V) = 0 \tag{2.22}$$

$$\Longleftrightarrow \quad \delta I = \delta\int_{t_i}^{t_f} dt\,(T - V) = 0 \tag{2.23}$$

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2.1.2 Lagrangian and Variational Principle

In Lagrangian mechanics, we will use a different approach to describe the motion of a particle than the Newtonian mechanics. Instead of using the usual Euclidean space, we will use a *configuration space* $\mathcal{C}$, which is the space of all possible positions of the particle. This space is spanned by the so-called **generalized coordinates** q_i, and their time derivatives (or generalized velocity) $\dot{q}_i := \frac{dq_i}{dt}$. Now, let us define quantities called the **Lagrangian** L and **action** S:

Definition 2.1: Lagrangian L and Action S

The Lagrangian L is defined as the difference between the kinetic energy T and potential energy V of a particle:

$$L := T - V, \qquad S[L] := \int_{t_i}^{t_f} dt\, L \tag{2.24}$$

Now, **variation** of the action δS is given by

$$\delta S = \delta\int_{t_i}^{t_f} dt\,(T - V) \tag{2.25}$$

which is an identical expression to (2.23). So, we postulate that the motion of the particle is such that the action is stationary:

Principle 2.1: Variational Principle

The motion of a particle is such that the action S is stationary, i.e., $\delta S = 0$.

$$\delta S = \delta \int_{t_i}^{t_f} dt\, L(q_i(t), \dot{q}_i(t), t) = 0 \tag{2.26}$$

As to show why this maybe useful, let us compare between Eq. (2.16):

$$\delta I = \delta S[L] = \int_{t_i}^{t_f} dt \left(-m \frac{d^2 \vec{r}}{dt^2} - \nabla V \right) \cdot \delta \vec{r} = 0 \tag{2.27}$$

and Eq. (**??**):

$$\delta I = \delta S[f] = \int_{B}^{A} dx \left(\frac{\partial F}{\partial f} - \frac{d}{dx} \frac{\partial F}{\partial \frac{df}{dx}} \right) \delta f = 0 \tag{2.28}$$

which immidiately gives that the EoM is the Euler-Lagrange equation, if we set $F = L(q_i, \dot{q}_i, t)$:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0 \iff m \frac{d^2 \vec{r}}{dt^2} = -\nabla V \tag{2.29}$$

The generalization to multiple particles is straightforward:

$$L = \sum_n \frac{m_n}{2} \dot{\vec{r}}_n^2 - V(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N) \tag{2.30}$$

The Euler-Lagrange equation is given by

$$\frac{\partial L}{\partial \vec{r}_i^{(N)}} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\vec{r}}_i^{(N)}} \right) = 0 \tag{2.31}$$

2.1.3 Harmonic Oscillator

Let us consider a particle of mass m in a 1D harmonic potential:

$$V = \frac{1}{2} k x^2 \implies F_{\text{ext}} = -\nabla V = -kx \tag{2.32}$$

then the Lagrangian is given by

$$L = T - V = \frac{m}{2} \dot{x}^2 - \frac{1}{2} k x^2 \tag{2.33}$$

The Euler-Lagrange equation yields

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0 \implies \frac{\partial L}{\partial x} = -kx, \quad \frac{\partial L}{\partial \dot{x}} = m\dot{x} \tag{2.34}$$

$$\implies \quad -kx - \frac{d}{dt}(m\dot{x}) = 0 \tag{2.35}$$

$$\implies \quad m\frac{d^2x}{dt^2} = -kx \tag{2.36}$$

which is the equation of motion of a harmonic oscillator in Newtonian mechanics. Here, notice that

$$\frac{\partial L}{\partial x} = -kx = -\nabla V = F_{\text{ext}} \tag{2.37}$$

and

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} = p \tag{2.38}$$

these derivatives of the Lagrangian are the **generalized force** and **conjugate momentum**, respectively.

$$\frac{\partial L}{\partial q} = F_{\text{g}}, \quad \frac{\partial L}{\partial \dot{q}} = p_{\text{g}} \tag{2.39}$$

2.2 Hamiltonian Formalism

2.2.1 Legendre Transform

Consider a function $f(x, y)$. The total differential of f is given by

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy \tag{2.40}$$

Often we want to find another function g such that

$$g = p \cdot y - f(x, y) \tag{2.41}$$

and the total differential of g is given by

$$\begin{aligned} dg &= \frac{\partial g}{\partial p}\,dp + \frac{\partial g}{\partial y}\,dy - df && (2.42)\\ &= y\,dp + p\,dy - \frac{\partial f}{\partial x}\,dx - \frac{\partial f}{\partial y}\,dy && (2.43)\\ &= y\,dp - \frac{\partial f}{\partial x}\,dx + \left(p - \frac{\partial f}{\partial y}\right)dy && (2.44) \end{aligned}$$

for this function g to be a function of x and p, we need to have

$$p = \frac{\partial f}{\partial y}, \quad \frac{\partial g}{\partial x} = -\frac{\partial f}{\partial x}. \tag{2.45}$$

This is called the **Legendre transform** of f.

Definition 2.2: Legendre transform(Analytical Mechanics)

The Legendre transform of a function $f(x, y)$ is defined as

$$g(p, x) = p \cdot y - f(x, y) \tag{2.46}$$

where

$$p = \frac{\partial f}{\partial y}, \quad \frac{\partial g}{\partial x} = -\frac{\partial f}{\partial x} \tag{2.47}$$

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2.2.2 Hamiltonian and Canonical Equations

Now, we define Hamiltonian $H = H(q, p)$ as the Legendre transform of Lagrangian $L = L(q, \dot{q})$:

Definition 2.3: Hamiltonian H

The Hamiltonian $H(q, p)$ is defined as the Legendre transform of the Lagrangian $L(q, \dot{q})$ $(\dot{q} \to p)$:

$$H(q, p) = p \cdot \dot{q} - L(q, \dot{q}) \tag{2.48}$$

where

$$p = \frac{\partial L}{\partial \dot{q}}, \quad \frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} \tag{2.49}$$

Note:

In Lagrange formalism, generalized coordinate span the configuration space, while in Hamilton formalism, generalized coordinates and **generalized momenta** or **conjugate momenta** span the **phase space**.

This is actually a physically intuitive quantity. Since p is defined as the derivative of Lagrangian L:

$$p = \frac{\partial L}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}}\left(\frac{1}{2}m\dot{q}^2 - V(q)\right) = m\dot{q} \tag{2.50}$$

and we can rewrite the Hamiltonian as

$$
\begin{aligned}
H(q,p) &= p \cdot \dot{q} - L(q,\dot{q}) && (2.51)\\
&= m \cdot \dot{q}^{\,2} - \left(\frac{1}{2}m\dot{q}^2 - V(q)\right) && (2.52)\\
&= \frac{1}{2}m\dot{q}^{\,2} + V(q) = T + V && (2.53)
\end{aligned}
$$

This is the total energy of the system, for the case of non-velocity dependent potential $V(q)$.

This relation is useful because we can obtain Lagrangian L from the Hamiltonian H as well:

$$
L(q,\dot{q}) = \dot{q} \cdot p - H(q,p) \implies \dot{q} = \frac{\partial H}{\partial p} \tag{2.54}
$$

and the Euler-Lagrange equation can be rewritten in terms of Hamiltonian:

$$
\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = 0 \implies -\frac{\partial H}{\partial q} - \frac{d}{dt}(p) = 0 \iff \quad \dot{p} = -\frac{\partial H}{\partial q} \tag{2.55}
$$

These two equations are called the **canonical equations** or **Hamilton's equations**:

Theorem 2.2.1 Canonical Equations

The relationship between a mechanical variable q and its canonical conjugate variable p is given by the canonical equations:

$$
\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q} \tag{2.56}
$$

For multiple degrees of freedom, we can write the Hamiltonian as

$$
H(\{q_i\},\{p_i\},t) = \sum_i p_i\dot{q}_i - L(\{q_i\},\{\dot{q}_i\},t) \tag{2.57}
$$

and the canonical equations as

$$
\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{2.58}
$$

where

$$
p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad \frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i} \tag{2.59}
$$

2.2.3 Poisson Bracket

Now, consider how a physical quantity $X(q_i, p_i, t)$ changes with time:

$$\frac{dX(q_i, p_i, t)}{dt} = \frac{\partial X}{\partial t} + \frac{\partial X}{\partial q_i}\dot{q}_i + \frac{\partial X}{\partial p_i}\dot{p}_i \tag{2.60}$$

from the canonical equations, the second part becomes:

$$\frac{\partial X}{\partial q_i}\dot{q}_i + \frac{\partial X}{\partial p_i}\dot{p}_i = \frac{\partial X}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial X}{\partial p_i}\frac{\partial H}{\partial q_i} \tag{2.61}$$

Since this term only depend on X and H, given q_i and p_i, we can define a new quantity called the **Poisson bracket**:

Definition 2.4: Poisson Bracket

The Poisson bracket of two physical quantities A and B is defined as

$$\{A,\, B\} := \sum_i \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \right) \tag{2.62}$$

If a physical quantity X does not explicitly depend on time, we can rewrite the time derivative as:

$$\frac{dX(q_i, p_i)}{dt} = \{X,\, H\} \tag{2.63}$$

Due to this property, we often call the Hamiltonian H the **generator** of time evolution.

Now, what happens if X happens to be q_i or p_i? A physical quantity q_i or p_i can be written as a function of q_i and p_i - simply as itself (noting that no explicit time dependence is present):

$$q_i(q_i, p_i) = q_i, \quad p_i(q_i, p_i) = p_i \tag{2.64}$$

Then, the Poisson bracket of q_i and H is given by

$$\{q_i,\, H\} = \sum_j \left(\frac{\partial q_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial q_i}{\partial p_j}\frac{\partial H}{\partial q_j} \right) \tag{2.65}$$

$$= \sum_j \left(\delta_{ij}\frac{\partial H}{\partial p_j} - 0 \right) = \frac{\partial H}{\partial p_i} \tag{2.66}$$

where Eq. (**??**) is used. Similarly, the Poisson bracket of p_i and H is given by

$$\{p_i,\, H\} = \sum_j \left(\frac{\partial p_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial p_i}{\partial p_j}\frac{\partial H}{\partial q_j} \right) \tag{2.67}$$

$$= \sum_j \left(0 - \delta_{ij}\frac{\partial H}{\partial q_j} \right) = -\frac{\partial H}{\partial q_i} \tag{2.68}$$

Thus, we can rewrite the canonical equations in terms of Poisson bracket:

Theorem 2.2.2 Canonical Equations with Poisson Bracket

The canonical equations can be rewritten in terms of Poisson bracket as follows:

$$\dot{q}_i = \{q_i, H\}, \quad \dot{p}_i = \{p_i, H\} \tag{2.69}$$

Finally, note that the Poisson bracket of q_i and p_j gives:

$$\{q_i, p_j\} = \sum_k \frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial p_i}{\partial q_k} \tag{2.70}$$

$$= \sum_k \delta_{ik}\delta_{jk} - 0 = \delta_{ij} \tag{2.71}$$

Theorem 2.2.3 Canonical Conjugate Relation in Analytical Mechanics

The Poisson bracket of the generalized coordinate q_i and its conjugate momentum p_j is given by

$$\{q_i, p_j\} = \delta_{ij} \tag{2.72}$$

2.3 Symmetry of a System

2.3.1 Symmetry

A fundamental concept in physics is **symmetry**. Let us see an example of symmetry in geometry:

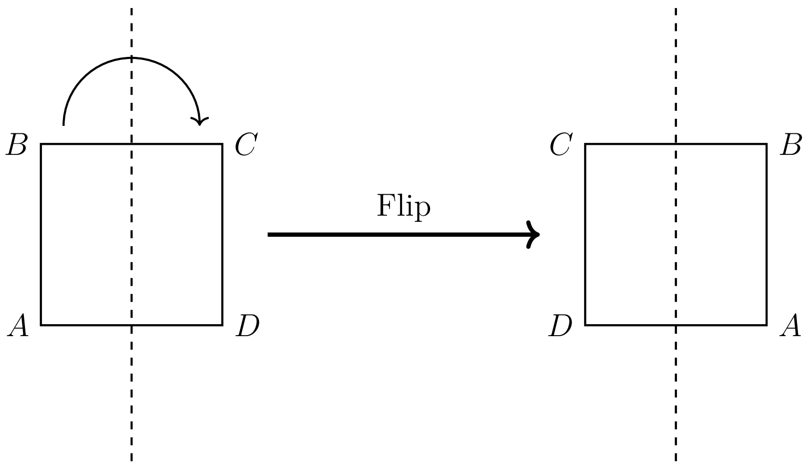

Figure 2.1: A square and its reflection.

Notice how we flipped the square, and it looks the same without the labels. In a more formal sense, we can say that the system(square) is **invariant** under the transformation(flip).

2.3.2 Transformations

For example, Newtonian mechanics(or Newton's equation of motion) has some symmetries.

Example 2.3.1 (Equation of Motion)

For example, the equation of motion is unchanged under Galilean transformation.

$$\sum \vec{F} = m\vec{a} \tag{2.73}$$

Definition 2.5: Galilean Transformation

The **Galilean transformation** is a transformation of coordinates from a stationary observer to a moving observer with constant velocity v.

$$\vec{\xi}(\vec{x}, t) = \vec{x} - \vec{v}t \tag{2.74}$$

where $\vec{\xi}$ is the new coordinate, $\vec{x}$ is the old coordinate, $\vec{v}$ is the velocity of the observer, and t is time.

and space/ time reversal:

Definition 2.6: Space/Time Reversal

The **space reversal** is a transformation of coordinates that flips the sign of the position vector:

$$\vec{x}' = -\vec{x} \tag{2.75}$$

The **time reversal** is a transformation of time that flips the sign of time:

$$t' = -t \tag{2.76}$$

In the context of analytical mechanics, the system is described by the action S. If the variation of the action δS is invariant under a transformation, we say that the system has a symmetry under that transformation.

Notice that the Lagrangian has a degree of freedom(that is, the Lagrangian can be modified by adding a total derivative and variation of action remains the same):

Theorem 2.3.1 Degree of Freedom in Lagrangian

Adding a total derivative of time to the Lagrangian does not change the action:

$$L'(q_i, \dot{q}_i, t) = L(q_i, \dot{q}_i, t) + \frac{d}{dt} f(q(t), t) \implies \delta S'[L'] = \delta S[L] \tag{2.77}$$

Proof: The action is given by

$$S'[L'] = \int_{t_i}^{t_f} dt\, L'(q_i(t), \dot{q}_i(t), t) \tag{2.78}$$

$$= \int_{t_i}^{t_f} dt\, L(q_i(t), \dot{q}_i(t), t) + \int_{t_i}^{t_f} dt\, \frac{d}{dt} f(q(t), t) \tag{2.79}$$

$$= S[L] + \Big[f(q_i, t)\Big]_{t_i}^{t_f} = S[L] + f(q_i(t_f), t_f) - f(q(t_i), t_i) \tag{2.80}$$

Since by taking the variation, $\delta q(t_f) = \delta q(t_i) = 0$, the last term is zero. Thus

$$\delta S'[L'] = \delta S[L] \tag{2.81}$$

□

2.3.3 Point Transformation

In Analytical Mechanics, there is a general family of transformations called **point transformation**.

Definition 2.7: Point Transformation

A **point transformation** is a transformation of the generalized coordinates q_i and time t to new coordinates Q_i

$$Q_j = Q_j(q_i, t) \tag{2.82}$$

Note: if there are N generalized coordinates q_i, then there must be N new coordinates Q_j.

Example 2.3.2 (Point Transformation)

For example, changing from Cartesian coordinates to polar coordinates is a point transformation:

$$(x, y) \to (r, \theta) \quad \Longleftrightarrow \quad \begin{cases} r &= \sqrt{x^2 + y^2} \\ \theta &= \arctan(\frac{y}{x}) \end{cases} \tag{2.83}$$

$$(r, \theta) \to (x, y) \quad \Longleftrightarrow \quad \begin{cases} x &= r\cos\theta \\ y &= r\sin\theta \end{cases} \tag{2.84}$$

notice how each new coordinate only depends on the old coordinates, not the velocity.

Point transformation does not change the action, i.e., the variation of the action is invariant under point transformation:

Theorem 2.3.2 Invariance of Euler-Lagrange Equation

The Euler-Lagrange equation is invariant under point transformation:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0 \implies \frac{\partial L}{\partial Q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_j}\right) = 0 \tag{2.85}$$

proof is given in Sec. **??**.

Now, using the point transformation, we can give the definition of symmetry, in the context of analytical mechanics:

Definition 2.8: Symmetry of the System

A system is said to have a **symmetry** under a point transformation $q_i \to Q_i$ if the change in Lagrangian is up to the degree of freedom:

$$L(Q_i, \dot{Q}_i, t) = L(q_i, \dot{q}_i, t) + \underbrace{\frac{d}{dt} f(q_i, t)}_{\text{change in } L} \tag{2.86}$$

where L is the Lagrangian of the system before the point transformation. Note that the Lagrangian of LHS has the same functional form as RHS(i.e. Q nad $\dot{Q}$ are substituted for q and $\dot{q}$). Or equivalently, the action is invariant under the point transformation:

$$S[Q_i, \dot{Q}_i, t] = S[q_i, \dot{q}_i, t] + \int_{t_i}^{t_f} dt \, \frac{d}{dt} f(q_i, t) = S[q_i, \dot{q}_i, t] \tag{2.87}$$

Importantly, the point transformation can be either discrete or continous, and collection of point transformations that preserve the action is called a **group**. In short, it means three things:

1. There is an identity transformation (i.e. no change)
2. If there are multiple transformations, there is a single transformation that is the result of applying those transformations in sequence.
3. If there is a transformation, there is an **inverse transformation** that undoes the transformation.

This concept is very important in particle physics.

2.3.4 Noether's Theorem

Theorem 2.3.3 Noether's Theorem

Every real paramter of a continuous symmetry of the action corresponds to a conserved quantity, called **Noether's charge** $\mathcal{Q}$. If the continous symmetry is given by an infinitesimal transformation: $Q_i(t) = q_i(t) + \epsilon F_i(q(t), t) + \mathcal{O}(\epsilon^2)$,

$$\mathcal{Q} = \sum_i F_i(q(t), t) \frac{\partial L(q(t), \dot{q}(t), t)}{\partial \dot{q}_i} - \Lambda(q(t), t) \tag{2.88}$$

where $\Lambda(q(t), t)$ is the change in the Lagrangian under the infinitesimal transformation.

Now, assume that a system has a continous symmetry. Thus if we consider a very ”small” transformation that is almost the identity transformation

[3]

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