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**References:**


1. Engineering Mathematics – D.C. Gokhroo, V.C. Jain & Kartar Singh, Unique Publication
2. Advanced Engineering Mathematics – Erwin Kreyszig, John Wiley & Sons
3. Higher Engineering Mathematics – B.V. Ramana, McGraw Hill Education
4. Advanced Engineering Mathematics – H.K. Das, S. Chand Pub.
5. Higher Engineering Mathematics – B.S. Grewal, Khanna Publishers
6. Engineering Mathematics – Y.N. Gaur & C.L. Koul, Jaipur Publishing House


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# UNIT-2 (DIFFERENTIAL CALCULUS-II)

# LECTURE NO.:- 9

## Introduction

Multivariable functions are used in many practical problems. The problems in statistics, probability, economics, operations research, mechanics, various fields of engineering etc. deal with such functions.

Some examples of multivariable functions are: area of an ellipse is $A(a, b) = \pi ab$; volume of a rectangular parallelepiped is $V(x, y, z) = xyz$ ; volume of right circular cylinder is $V(r, h) = \pi r^2 h$ ; amount by simple interest is $A(P, r, t) = P(1+rt)$; amount by compound interest is $A(P, r, \mathrm{t}, n) = P\left(1+\frac{r}{n}\right)^{nt}$ ; intelligence quotient is $Q(M, C) = \frac{M}{C} \times 100$; resistance for blood flow in a vessel (Poiseuille's law) is $R(l, r) = kl/r^4$; Cobb-Douglas function of productivity is $f(x, y) = kx^m y^n$; future value is $F(P, i, n) = P\frac{(1+i)^n - 1}{i}$ etc.

If $u = f(x, y, \mathrm{z}, \mathrm{t})$, then $x, y, \mathrm{z}, \mathrm{t}$ are known as independent variables or arguments and $u$ is known as dependent variable or value of the function.

If for every $x$ and $y$, a unique value of $f(x, y)$ is associated, then $f$ is said to be a function of the two independent variables $x$ and $y$, and may be denoted by $z = f(x, y)$.

The set of values of $x$ and $y$ for which the function $z = f(x, y)$ is defined is known as the domain of the function. For example, (i) $z = \sqrt{a^2 - x^2 - y^2}$ , domain is $x^2 + y^2 \le a^2$ ; (ii) $z = x^y + y^x$, domain is $x > 0$ and $y > 0$.

Multivariable calculus is the study of limit, continuity, differentiability of multivariable functions and their applications in various fields.

## Partial Differentiation

A partial derivative of a function of several variables is the ordinary derivative with respect to one of the variables when all the remaining variables are held constants. Partial differentiation is the process of finding partial derivatives. All the rules of differentiation applicable to function of a single independent variable are also applicable in partial differentiation with the only difference that while differentiating (partially) with respect to one variable, all the other variables are treated (temporarily) as constants.

Let a function of three variables $x, y, z$ be denoted as $u = f(x, y, z)$.

Keeping $y$ and $z$ constants and varying only $x$, the partial derivative of $u$ with respect to $x$ is denoted by $\frac{\partial u}{\partial x}$ and defined as

$$\frac{\partial u}{\partial x} = \frac{\partial f(x,y,z)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x,y,z)-f(x,y,z)}{\Delta x}.$$

Similarly the partial derivatives of $u$ w.r.t. $y$ and $z$ can be defined and are denoted by $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ respectively. These are called first order partial derivatives of $u = f(x, y, z)$.

**Higher Order Partial Derivatives**

Partial derivatives of higher order of a function $f(x,y,z)$ are calculated by successive differentiation. Thus if $u = f(x,y,z)$, then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} = f_{11}\ ,\ \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx} = f_{21}\ ,$$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy} = f_{12}\ ,\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy} = f_{22}\ ,$$

$$\frac{\partial^3 u}{\partial z^2 \partial y} = \frac{\partial^3 f}{\partial z^2 \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial^2 f}{\partial z \partial y}\right) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right)\right) = f_{yzz} = f_{233}\ ,\ \cdots\cdots$$

The partial derivative $\frac{\partial f}{\partial x}$ obtained by differentiating once is known as first order partial derivative, while $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial y^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$ which are obtained by differentiating twice are known as second order partial derivatives. Similarly 3rd order, 4th order derivatives involve 3, 4 times differentiation respectively.

**Notations:** If $z = f(x, y)$, then $\frac{\partial z}{\partial x} = p, \frac{\partial z}{\partial y} = q, \frac{\partial^2 z}{\partial x^2} = r, \frac{\partial^2 z}{\partial x \partial y} = s, \frac{\partial^2 z}{\partial y^2} = t$.

**Note 1.** The crossed or mixed partial derivatives $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are in general equal, i.e. $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$, i.e. the order of differentiation is immaterial, if the derivatives involved are continuous.

**Note 2.** A function of 2 variables has 2 derivatives of order 1, 4 derivatives of order 2 and $2^n$ derivatives of order $n$. A function of $n$ variables will have $n^m$ derivatives of order $m$.

**Examples on Partial Differentiation**

**Example 1.** If $z(x+y)=x^2+y^2$, show that $\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)$.

**Solution.** Here $z(x+y)=x^2+y^2 \Rightarrow z=\frac{x^2+y^2}{x+y}$. Differentiating it partially w.r.t. $x$ & $y$, we get

$$\frac{\partial z}{\partial x}=\frac{(x+y).2x-(x^2+y^2).1}{(x+y)^2}=\frac{x^2+2xy-y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial y}=\frac{(x+y).2y-(x^2+y^2).1}{(x+y)^2}=\frac{-x^2+2xy+y^2}{(x+y)^2}$$

$$\Rightarrow \left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=\left(\frac{x^2+2xy-y^2+x^2-2xy-y^2}{(x+y)^2}\right)^2=\left(\frac{2(x-y)(x+y)}{(x+y)^2}\right)^2=\frac{4(x-y)^2}{(x+y)^2} \quad \cdots\cdots\cdots (1)$$

$$\& \left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)=\left(1-\frac{(x^2+2xy-y^2)}{(x+y)^2}-\frac{(-x^2+2xy+y^2)}{(x+y)^2}\right)=\left(\frac{x^2+2xy+y^2-x^2-2xy+y^2+x^2-2xy-y^2}{(x+y)^2}\right)$$

$$\Rightarrow 4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)=4\frac{(x^2-2xy-y^2)}{(x+y)^2}=\frac{4(x-y)^2}{(x+y)^2} \quad \cdots\cdots\cdots (2)$$

From (1) and (2), we have

$$\boxed{\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)}.$$ Proved.

**Example 2.** If $u=x^2\tan^{-1}\frac{y}{x}-y^2\tan^{-1}\frac{x}{y}$, find $\frac{\partial^2 u}{\partial x\partial y}$.

**Solution.** Differentiating partially w.r.t. $y$, we have

$$\frac{\partial u}{\partial y}=x^2\frac{1}{1+\left(\frac{y}{x}\right)^2}.\frac{1}{x}-2y\tan^{-1}\frac{x}{y}-y^2\frac{1}{1+\left(\frac{x}{y}\right)^2}\left(-\frac{x}{y^2}\right)=\frac{x^3}{x^2+y^2}-2y\tan^{-1}\frac{x}{y}+\frac{xy^2}{x^2+y^2}$$

$$=\frac{x(x^2+y^2)}{x^2+y^2}-2y\tan^{-1}\frac{x}{y}=x-2y\tan^{-1}\frac{x}{y}$$

Now $$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}\left\{\frac{\partial u}{\partial y}\right\}=1-2y.\frac{1}{1+\left(\frac{x}{y}\right)^2}.\frac{1}{y}=1-\frac{2y^2}{x^2+y^2}=\frac{(x^2+y^2-2y^2)}{x^2+y^2}.$$

$$\therefore \boxed{\frac{\partial^2 u}{\partial x\partial y}=\frac{x^2-y^2}{x^2+y^2}}.$$ Ans.

**Example 3.** If $z = f\left(\frac{y}{x}\right)$, show that $x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) = 0$.

**Solution.** If $z = f\left(\frac{y}{x}\right)$, differentiating partially w.r.t. $x$, we have

$$\frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right).\left(-\frac{y}{x^2}\right) \Rightarrow x\frac{\partial z}{\partial x} = -\frac{y}{x}f'\left(\frac{y}{x}\right) \quad \cdots\cdots\cdots (1)$$

Similarly $\frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right).\frac{1}{x} \Rightarrow y\frac{\partial z}{\partial y} = \frac{y}{x}f'\left(\frac{y}{x}\right)$. $\cdots\cdots\cdots$ (2)

Adding (1) and (2), we obtain

$$x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) = -\frac{y}{x}f'\left(\frac{y}{x}\right) + \frac{y}{x}f'\left(\frac{y}{x}\right) = 0.$$ Proved.

**Example 4.** If $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$, show that $x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = 0$.

**Solution.** If $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$, differentiating partially w.r.t. $x$, we have

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}}.\frac{1}{y} + \frac{1}{1 + \frac{y^2}{x^2}}.\left(-\frac{y}{x^2}\right) = \frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{x^2 + y^2}$$

$$\therefore\ x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2} \quad \cdots\cdots\cdots (1)$$

Again $\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}}.\left(-\frac{x}{y^2}\right) + \frac{1}{1 + \frac{y^2}{x^2}}.\left(\frac{1}{x}\right) = -\frac{x}{y\sqrt{y^2 - x^2}} + \frac{x}{x^2 + y^2}$

$$\therefore\ y\frac{\partial u}{\partial y} = -\frac{x}{\sqrt{y^2 - x^2}} + \frac{xy}{x^2 + y^2} \quad \cdots\cdots\cdots (2)$$

Adding (1) & (2), we get,

$$\boxed{x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = 0}$$ Proved.

**Example 5.** If $u = \sin^{-1}\left(\frac{x^2 + y^2}{x + y}\right)$, show that $x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = \tan u$.

**Solution.** We have

$$u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right) \Rightarrow \sin u = \frac{x^2+y^2}{x+y} \qquad \cdots\cdots\cdots (1)$$

Differentiating (1) partially w.r.t. *x*, we get

$$\cos u \frac{\partial u}{\partial x} = \frac{(2x+0)(x+y)-(x^2+y^2)(1+0)}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2} \qquad \cdots\cdots\cdots (2)$$

Similarly $cos\, u \dfrac{\partial u}{\partial y} = \dfrac{y^2+2yx-x^2}{(y+x)^2} = \dfrac{y^2+2xy-x^2}{(x+y)^2}$ $\cdots\cdots\cdots$ (3)

Multiplying (2) by *x*, (3) by *y* and adding, we get

$$\cos u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = \frac{x(x^2+2xy-y^2)+y(y^2+2xy-x^2)}{(x+y)^2} = \frac{x^3+2x^2y-xy^2+y^3+2xy^2-x^2y}{(x+y)^2}$$

$$= \frac{x^3+x^2y+xy^2+y^3}{(x+y)^2} = \frac{(x+y)(x^2+y^2)}{(x+y)^2} = \frac{x^2+y^2}{x+y}$$

$$= \sin u \text{ , by (1)}$$

Dividing by cos *u*, we get

$$\boxed{x\left(\frac{\partial u}{\partial x}\right) + y\left(\frac{\partial u}{\partial y}\right) = \tan u}$$ Proved.

**Example 6.** If $u = \log(x^3+y^3+z^3-3xyz)$, show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z} \text{ and } \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}.$$

**Solution.** We have $u = \log(x^3+y^3+z^3-3xyz)$

$$\therefore \frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}, \frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}, \frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-xy-yz-zx)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{3}{(x+y+z)}.$$ Proved.

$$\& \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{(x+y+z)}\right)$$

$$= \left(-\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}\right) = -\frac{9}{(x+y+z)^2}.$$ Proved.

# LECTURE NO.:- 10

## Homogeneous Functions

Each term of the expression $f(x, y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + a_3 x^{n-3} y^3 + \cdots + a_n x^n$ is of degree $n$. Such an expression is called a homogeneous function of degree $n$.

The above expression can be written as

$$f(x, y) = x^n \left\{ a_0 + a_1 \left(\frac{y}{x}\right) + a_2 \left(\frac{y}{x}\right)^2 + a_3 \left(\frac{y}{x}\right)^3 + \ldots\ldots\ldots\ldots + a_n \left(\frac{y}{x}\right)^n \right\}$$

$$= x^n \phi\left(\frac{y}{x}\right).$$

Similarly it can be written as $f(x, y) = y^n \psi\left(\frac{x}{y}\right)$.

The expressions $ax^2 + 2hxy + by^2$ and $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ are second degree homogeneous functions of 2 and 3 independent variables respectively.

If a function $f(x, y)$ is written as $x^n F\left(\frac{y}{x}\right)$, then it is called a homogeneous function of $x$ and $y$ of degree $n$. For example, $x^2 \sin\left(\frac{y}{x}\right)$, $x^3 \exp\left(\frac{x}{y}\right)$ and $x^2 \tan^{-1}\left(\frac{y}{x}\right)$ are homogeneous function of $x$ and $y$ of degree 2, 3 and 2 respectively.

## Euler's Theorem on Homogeneous Functions

If $f(x, y)$ be a homogeneous function of degree $n$ in $x$ and $y$, then $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$.

**Proof.** If $f(x, y)$ be a homogeneous function of degree $n$ in $x$ and $y$, then it may be written as

$$f(x, y) = x^n F\left(\frac{y}{x}\right). \qquad \cdots\cdots\cdots (1)$$

Differentiating (1) partially w.r.t. $x$ and $y$, we have

$$\frac{\partial f}{\partial x} = nx^{n-1} F\left(\frac{y}{x}\right) + x^n F'\left(\frac{y}{x}\right) \times \left(-\frac{y}{x^2}\right) \;\; \& \;\; \frac{\partial f}{\partial y} = x^n F'\left(\frac{y}{x}\right) \times \left(\frac{1}{x}\right).$$

$$\Rightarrow x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nx^n F\left(\frac{y}{x}\right) - \frac{y}{x} x^n F'\left(\frac{y}{x}\right) + \frac{y}{x} x^n F'\left(\frac{y}{x}\right) = nx^n F\left(\frac{y}{x}\right) = nf$$, by (1). Hence Proved.

**Corollary.** If $f$ be a homogeneous function of degree $n$ in $x$ and $y$, then

(i) $x\frac{\partial^2 f}{\partial x^2} + y\frac{\partial^2 f}{\partial x \partial y} = (n-1)\frac{\partial f}{\partial x}$ [differentiating the equation of Euler's theorem w.r.t. $x$]

(ii) $x\frac{\partial^2 f}{\partial x \partial y} + y\frac{\partial^2 f}{\partial y^2} = (n-1)\frac{\partial f}{\partial y}$ [differentiating the equation of Euler's theorem w.r.t. $y$]

(iii) $x^2\dfrac{\partial^2 f}{\partial x^2}+2xy\dfrac{\partial^2 f}{\partial x\partial y}+y^2\dfrac{\partial^2 f}{\partial y^2}=n(n-1)f$ [multiplying above (i) by $x$, (ii) by $y$ and adding]

**A Deduction from Euler's Theorem**

If $u=\phi(H_n)$ so that $H_n=f(u)$, where $H_n$ is a homogeneous function of $x$ and $y$ of degree $n$, then using Euler's theorem, it is deduced that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=n\dfrac{f(u)}{f'(u)}$.

**Examples of Euler's Theorem on Homogeneous Functions**

**Example 1.** If $u=\tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\sin 2u$.

**Solution.** We have given $u=\tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$. $\Rightarrow$ $\tan u=\left(\dfrac{x^3+y^3}{x-y}\right)=f$ (say), ......... (1)

where $f$ is a homogeneous function of $x$ and $y$ of degree 3 – 1 = 2.

Hence by Euler's theorem, we have

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=2f$$

$$\Rightarrow x\frac{\partial(\tan u)}{\partial x}+y\frac{\partial(\tan u)}{\partial y}=2\tan u,\ \text{by (1)}$$

$$\Rightarrow x\left(\sec^2 u\frac{\partial u}{\partial x}\right)+y\left(\sec^2 u\frac{\partial u}{\partial y}\right)=2\tan u \Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2\tan u}{\sec^2 u}=2\sin u\cos u$$

$$\Rightarrow \boxed{x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u}.$$ Proved.

**Example 2.** If $u=\log\left(\dfrac{x^3+y^3}{x+y}\right)$, then show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=2$.

**Hint.** We have given $u=\log\left(\dfrac{x^3+y^3}{x+y}\right)\Rightarrow e^u=\dfrac{x^3+y^3}{x+y}=f$ (say). Solve as above example 1.

**Example 3.** If $u=x\sin^{-1}\left(\dfrac{y}{x}\right)$, prove that $x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}=0$.

**Solution.** Here given that $u=x\sin^{-1}\left(\dfrac{y}{x}\right)=xf\left(\dfrac{y}{x}\right)$.

So $u$ is a homogeneous function of $x$ and $y$ of degree 1. Hence by Euler's theorem, we have

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1.u=u. \qquad \text{......... (1)}$$

Differentiating (1) partially with respect to $x$, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial u}{\partial x}\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+xy\frac{\partial^2 u}{\partial x\partial y}=0. \qquad \cdots\cdots\cdots (2)$$

Again differentiating (1) with respect to $y$, we get

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=\frac{\partial u}{\partial y}\Rightarrow xy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial y^2}=0. \qquad \cdots\cdots\cdots (3)$$

Adding (2) and (3), we obtain,

$$\boxed{x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=0}$$ Proved.

**Example 4.** If $u=\sin^{-1}\left[(x^2+y^2)^{1/5}\right]$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{2}{25}\tan u(2\tan^2 u-3).$$

**Solution.** Here given that $u=\sin^{-1}\left[(x^2+y^2)^{1/5}\right]\Rightarrow \sin u=\left[(x^2+y^2)^{1/5}\right]=f\,(\text{say}) \qquad \cdots\cdots\cdots (1)$

Obviously $v$ is a homogeneous function in $x$ and $y$ of degree $\frac{2}{5}$. So by Euler's theorem,

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=\frac{2}{5}f \qquad \cdots\cdots\cdots (2)$$

We have $f=\sin u\Rightarrow \frac{\partial f}{\partial x}=\cos u\frac{\partial u}{\partial x}$ and $\frac{\partial f}{\partial y}=\cos u\frac{\partial u}{\partial y}$ .

Putting these values in (2), we get

$$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=\frac{2}{5}\sin u\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2}{5}\tan u \qquad \cdots\cdots\cdots (3)$$

Now differentiating (3) partially w.r.to $x$ and $y$ respectively, we get

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=\frac{2}{5}\sec^2 u\frac{\partial u}{\partial x} \qquad \cdots\cdots\cdots (4)$$

$$x\frac{\partial^2 u}{\partial y\partial x}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=\frac{2}{5}\sec^2 u\frac{\partial u}{\partial y} \qquad \cdots\cdots\cdots (5)$$

Multiplying (4) by $x$, (5) by $y$ and adding, we have

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2}{5}\sec^2 u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\frac{2}{5}\tan u=\frac{2}{5}\sec^2 u.\frac{2}{5}\tan u\text{, by (3)}$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{2}{5}\tan u\left(\frac{2}{5}\sec^2 u-1\right)=\frac{2}{25}\tan u\left(2\sec^2 u-5\right)=\frac{2}{25}\tan u\left\{2\left(1+\tan^2 u\right)-5\right\}$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{2}{25}\tan u\left(2\tan^2 u-3\right).$$ Proved.

**Example 5.** If $V=\log_e \sin\left\{\frac{\pi}{2}.\frac{(2x^2+y^2+zx)^{1/2}}{(x^2+xy+2yz+z^2)^{1/3}}\right\}$, find the value of

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z} \text{ when } x=0, y=1, z=2.$$

**Solution.** Let $u=\frac{\pi}{2}.\frac{(2x^2+y^2+zx)^{1/2}}{(x^2+xy+2yz+z^2)^{1/3}}$, then $V=\log_e \sin u$.

Therefore $\frac{\partial V}{\partial x}=\cot u\frac{\partial u}{\partial x}$, $\frac{\partial V}{\partial y}=\cot u\frac{\partial u}{\partial y}$ and $\frac{\partial V}{\partial z}=\cot u\frac{\partial u}{\partial z}$.

Thus $x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}=\cot u\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}\right]$ ……… (1)

Obviously $u$ is a homogeneous function in $x$, $y$ and $z$ of degree $1-2/3=1/3$.

Thus by Euler's theorem, $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=\frac{1}{3}u$ ……… (2)

Putting this value in (1), we obtain,

$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}=\frac{1}{3}u\cot u$ ……… (3)

Now when $x=0, y=1, z=2$, we get $u=\frac{\pi}{4}$. Substituting these values in (3), we obtain,

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z}=\frac{1}{3}.\frac{\pi}{4}\cot\frac{\pi}{4}=\frac{\pi}{12}.$$ Ans.

**Example 6.** If $u=\sin^{-1}\left(\frac{5x+3y+3z}{x^6+y^6+z^6}\right)$, find the value of $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}$.

**Solution.** Try yourself.

**Example 7.** Find $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$, when (i) $\sin z=\frac{x^2-y^2}{x^2+y^2}$, (ii) $z=\cos^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$.

**Solution.** Try yourself.

**Example 8.** If $u=\sec^{-1}\frac{x^2+y^2}{x+y}$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\cot u$.

**Solution.** Try yourself.

# LECTURE NO.:- 11

## Maxima & Minima of Two and More Independent Variables

A function $f(x, y)$ has a **maximum value** at a point $(x_0, y_0)$, if $f(x_0, y_0) \geq f(x, y)$ for all points $(x, y)$ in the neighborhood of $(x_0, y_0)$. Such a function has a **minimum value** at a point $(x_0, y_0)$, if $f(x_0, y_0) \leq f(x, y)$ for all points $(x, y)$ in the neighborhood of $(x_0, y_0)$.

A function $f(x_1, x_2, \ldots, x_n)$ of $n$ independent variables has a **maximum value** at a point $(x_1', x_2', \ldots, x_n')$, if $f(x_1', x_2', \ldots, x_n') \geq f(x_1, x_2, \ldots, x_n)$ at all points in the neighborhood of $(x_1', x_2', \ldots, x_n')$. Such a function has a **minimum value** at a point $(x_1', x_2', \ldots, x_n')$, if $f(x_1', x_2', \ldots, x_n') \leq f(x_1, x_2, \ldots, x_n)$ at all points in the neighborhood of $(x_1', x_2', \ldots, x_n')$.

For a function $f(x_1, x_2, \ldots, x_n)$ of $n$ variables, a **stationary (or critical) point** is a point at which $\frac{\partial f}{\partial x_1} = 0, \frac{\partial f}{\partial x_2} = 0, \cdots, \frac{\partial f}{\partial x_n} = 0$. These are the **necessary conditions** for stationary points.

A stationary point can be a maximum point or a minimum point or a saddle point (analogous to a point of inflection, which is neither a maximum point nor a minimum point).

To find the maximum or minimum points of a function, we first locate the stationary points. After locating the stationary points, we then examine each stationary point to determine that it is a maximum or a minimum or a saddle point. To determine if a point is a maximum or minimum, we may consider values of the function in the neighborhood of the point as well as the values of its first and second order partial derivatives.

### Sufficient Condition for a Maximum or Minimum

Let $z = f(x, y)$ have continuous first and second order partial derivatives in the neighborhood of point $(x_0, y_0)$.

If at the point $(x_0, y_0)$, $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$ and $\Delta = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$,

then the function $z = f(x, y)$ is maximum at $(x_0, y_0)$, if $\frac{\partial^2 f}{\partial x^2} < 0$

and minimum at $(x_0, y_0)$, if $\frac{\partial^2 f}{\partial x^2} > 0$.

If $\Delta < 0$, then the point $(x_0, y_0)$ is a saddle point (neither maximum nor minimum).

If $\Delta = 0$, then the nature of the point $(x_0, y_0)$ is undecided and more investigation is required.

**Note.** Every extreme value is a stationary value but the converse may not be true.

**Working Rule to Find the Maximum and Minimum Values of $z = f(x, y)$:**

1. Solve $p = \dfrac{\partial f}{\partial x} = 0$ and $q = \dfrac{\partial f}{\partial y} = 0$ simultaneously $\Rightarrow (x, y) = (a, b), (c, d), \cdots$ (say).
2. Calculate the value of $r = \dfrac{\partial^2 f}{\partial x^2}, s = \dfrac{\partial^2 f}{\partial x \partial y}, t = \dfrac{\partial^2 f}{\partial x^2}$ for each pair of values.
3. (i) if $rt - s^2 > 0$ and $r < 0$ at $(a, b)$, then $f(a,b)$ is a maximum value.

   (ii) if $rt - s^2 > 0$ and $r > 0$ at $(a, b)$, then $f(a,b)$ is a minimum value.

   (iii) if $rt - s^2 < 0$ at $(a, b)$, then $f(a,b)$ is not an extreme value i.e. $(a, b)$ is a saddle point.
4. If $rt - s^2 = 0$ at $(a, b)$, then the case is doubtful and needs further investigation.
5. Similarly examine at other points $(c, d), \cdots$ one by one.

**General Method**

If $u = f(x_1, x_2, \ldots, x_n)$, then all the stationary points $(x_1, x_2, \ldots, x_n)$ are obtained by the necessary conditions $\frac{\partial f}{\partial x_1} = 0, \frac{\partial f}{\partial x_2} = 0, \ldots, \frac{\partial f}{\partial x_n} = 0.$

For the nature of stationary points (points of maxima/ points of minima/ saddle points), following Hessian matrix is formed at all the stationary points:

$$\mathrm{H} = \begin{bmatrix} \frac{\partial^2 f}{\partial {x_1}^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial {x_2}^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial {x_n}^2} \end{bmatrix} = \begin{bmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1} & f_{n2} & \cdots & f_{nn} \end{bmatrix}.$$

Here first leading or principal minors of H are given below:

$$\mathrm{H}_1 = [f_{11}], \mathrm{H}_2 = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}, \ldots, \mathrm{H}_n = |\mathrm{H}|.$$

If at a stationary point,

(i) $H_1 > 0, H_2 > 0, \ldots, H_n > 0$ (all positive), then the stationary point will be a point of minima;

(ii) $H_1 < 0, H_2 > 0, H_3 < 0, \ldots$ (alternately negative and positive), then the stationary point will be a point of maxima;

(iii) Otherwise the stationary point will be a saddle point (neither maxima nor minima).

# LECTURE NO.:- 12

**Examples of Maxima and Minima**

**Example 1.** Discuss the maxima and minima of $f(x,y)=x^3y^2(1-x-y)$.

**Solution.** Differentiating $f(x,y)=x^3y^2(1-x-y)$ partially, we get

$$p=\frac{\partial f}{\partial x}=3x^2y^2-4x^3y^2-3x^2y^3,\ q=\frac{\partial f}{\partial y}=2x^3y-2x^4y-3x^3y^2,\ r=\frac{\partial^2 f}{\partial x^2}=6xy^2-12x^2y^2-6xy^3,$$

$$s=\frac{\partial^2 f}{\partial x\partial y}=6x^2y-8x^3y-9x^2y^2 \text{ and } t=\frac{\partial^2 f}{\partial y^2}=2x^3-2x^4-6x^3y.$$

Now $p=0$ and $q=0$ give $x^2y^2(3-4x-3y)=0$ and $x^3y(2-2x-3y)=0$.

Solving these, the stationary points are $O(0,0)$ and $A\left(\frac{1}{2},\frac{1}{3}\right)$ (say).

Now $rt-s^2=x^4y^2\left\{12(1-2x-y)(1-x-3y)-(6-8x-9y)^2\right\}$.

At $A\left(\frac{1}{2},\frac{1}{3}\right)$, $rt-s^2=\frac{1}{16}\cdot\frac{1}{9}\left\{12\left(1-1-\frac{1}{3}\right)\left(1-\frac{1}{2}-1\right)-(6-4-3)^2\right\}=\frac{1}{14}>0$

and $r=6\left(\frac{1}{2}\cdot\frac{1}{4}-\frac{2}{4}\cdot\frac{1}{9}-\frac{1}{2}\cdot\frac{1}{27}\right)=-\frac{1}{9}<0$.

Hence $f(x,y)$ has a maximum value at $A\left(\frac{1}{2},\frac{1}{3}\right)$ and $\max f=\frac{1}{8}\cdot\frac{1}{9}\left(1-\frac{1}{2}-\frac{1}{3}\right)=\frac{1}{432}$.

At $O(0,0)$, $rt-s^2=0$, and therefore, further investigation is needed. Now $f(0,0)=0$ and for the points along the line $y=x$, $f(x,y)=x^5(1-2x)$ which is positive for $x=0\cdot1$ and negative for $x=-0\cdot1$ i.e. in the neighbourhood of (0, 0), there are points where $f(x,y)>f(0,0)$ and there are points where $f(x,y)<f(0,0)$. So $f(0,0)$ is not an extreme value.

Hence the given function has the maximum value at $x=1/2$ and $y=1/3$. Ans.

**Example 2.** Examine the following function for extreme values:

$$f(x,y)=x^4+y^4-2x^2+4xy-2y^2.$$

**Solution.** Differentiating $f(x,y)=x^4+y^4-2x^2+4xy-2y^2$ partially, we get

$$p=\frac{\partial f}{\partial x}=4x^3-4x+4y,\ q=\frac{\partial f}{\partial y}=4y^3+4x-4y,$$

$$r=\frac{\partial^2 f}{\partial x^2}=12x^2-4,\ s=\frac{\partial^2 f}{\partial x\partial y}=4,\ t=\frac{\partial^2 f}{\partial y^2}=12y^2-4.$$

Now $p=0$ and $q=0$ give $x^3-x+y=0$ and $y^3+x-y=0$.

Adding these, we get $x^3 + y^3 = 0$ or $y = -x$.

Putting this value $y = -x$ in $p = 0$, we get

$x^3 - x - x = 0 \Rightarrow x^3 - 2x = 0$

$\Rightarrow x(x^2 - 2) = 0 \Rightarrow x = 0, x = \pm\sqrt{2} \Rightarrow y = -x = 0, \mp\sqrt{2}.$

$\Rightarrow$ O(0, 0), A$(\sqrt{2}, -\sqrt{2})$, B$(-\sqrt{2}, \sqrt{2})$ (say).

Now $rt - s^2 = 4(3x^2 - 1)4(3y^2 - 1) - 4^2 = 16[(3x^2 - 1)(3y^2 - 1) - 1]$.

At A $(\sqrt{2}, -\sqrt{2})$, $rt - s^2 = 20 \times 20 - 4^2 = +ve$ and $r$ is also positive. Hence $f(\sqrt{2}, -\sqrt{2})$ is a minimum value.

At B $(-\sqrt{2}, \sqrt{2})$, $rt - s^2$ and $r$ are positive. Hence $f(-\sqrt{2}, \sqrt{2})$ is also a minimum value.

At O (0, 0), $rt - s^2 = 0$, so further investigation is needed. Now $f(0,0) = 0$ and for the points along $x$-axis, where $y = 0$, $f(x, y) = x^4 - 2x^2 = x^2(x^2 - 2)$, which is negative for points in the neighbourhood of the origin. Again for points along the line $y = x$, $f(x, y) = 2x^4$, which is positive for points in the neighbourhood of the origin. Hence in the neighbourhood of (0, 0), there are points where $f(x, y) < f(0,0)$ and there are points where $f(x, y) > f(0,0)$. Thus $f(0,0)$ is not an extreme value i.e. it is a saddle point.

Therefore the given function has the minimum value at $x = -y = \pm\sqrt{2}$. Ans.

**Example 3.** Find the point $(x_1, x_2, x_3)$ at which the function

$f(x_1, x_2, x_3) = -x_1^2 - x_2^2 - x_3^2 + x_2 x_3 + x_1 + 2x_3$ has optimum value.

**Hint.** Use Hessian matrix for nature of stationary points.

**Example 4.** Show that the minimum value of $u = xy + \frac{a^3}{x} + \frac{a^3}{y}$ is $3a^2$.

**Solution.** Differentiating $u = xy + \frac{a^3}{x} + \frac{a^3}{y}$ partially, we get

$$p = \frac{\partial u}{\partial x} = y - \frac{a^3}{x^2},\ q = \frac{\partial u}{\partial y} = x - \frac{a^3}{y^2},\ r = \frac{\partial^2 u}{\partial x^2} = \frac{2a^3}{x^3},\ s = \frac{\partial^2 u}{\partial x \partial y} = 1 \text{ and } t = \frac{\partial^2 u}{\partial y^2} = \frac{2a^3}{y^3}.$$

For maxima and minima, we have

$$p = \frac{\partial u}{\partial x} = 0 \text{ and } q = \frac{\partial u}{\partial y} = 0 \Rightarrow y - \frac{a^3}{x^2} = 0 \text{ and } x - \frac{a^3}{y^2} = 0$$

Solving these equations, we get $x = a, y = a$.

At $x = a, y = a$; $r = 2, s = 1, t = 2 \Rightarrow rt - s^2 = 3 > 0$ and $r > 0$. So $u$ is minimum at $x = y = a$.

Hence the minimum value of $u = a.a + \frac{a^3}{a} + \frac{a^3}{a} = 3a^2$. Proved.

**Optimization with Equality Constraints:** By elimination of variables, it may be converted as an unconstrained optimization. It is illustrated in the following example:

**Example 5.** In a plane triangle, find the maximum value of $\cos A \cos B \cos C$.

**Solution.** We have $A + B + C = \pi$ so that $C = \pi - (A + B)$.

$$\therefore \cos A \cos B \cos C = \cos A \cos B \cos\{\pi - (A + B)\}$$

$$= -\cos A \cos B \cos(A + B) = f(A, B) \text{ (say)}$$

Differentiating it partially, we get

$$p = \frac{\partial f}{\partial A} = \cos B\{\sin A \cos(A+B) + \cos A \sin(A+B)\} = \cos B \sin(2A+B),$$

$$q = \frac{\partial f}{\partial B} = \cos A \sin(A+2B), \text{ by symmetry;}$$

$$r = \frac{\partial^2 f}{\partial A^2} = 2\cos B \cos(2A+B), \ s = \frac{\partial^2 f}{\partial A \partial B} = -\sin B \sin(2A+B) + \cos B \cos(2A+B) = \cos(2\text{A}+2\text{B}),$$

$t = 2\cos A \cos(A + 2B)$, by symmetry.

Now $p = 0$ and $q = 0$ give $2A + B = \pi$ and $A + 2B = \pi \Rightarrow A = B = \frac{\pi}{3}$.

At $A = B = \frac{\pi}{3}$, $r = -1, s = -\frac{1}{2}, t = -1$ so that $rt - s^2 = \frac{3}{4} > 0$.

These show that $f(A, B)$ is maximum for $A = B = \frac{\pi}{3}$.

Then $C = \pi - (A + B) = \frac{\pi}{3}$.

Hence $\cos A \cos B \cos C$ is maximum when $A = B = C = \frac{\pi}{3}$ i.e. the triangle is equilateral and the maximum value of $\cos A \cos B \cos C = \frac{1}{8}$. Ans.

**Example 6.** Find the maximum or minimum value of $xy(a - x - y)$.

**Solution.** Do yourself as example 1.

**Example 7.** Find the maximum value of $u$, where $u = \sin x \sin y \sin(x + y)$.

**Solution.** Do yourself as example 4.

# LECTURE NO.:- 13

## Introduction

Sometimes it is required to find the stationary values of a function of several variables which are not all independent but are connected by some given relation (s). If possible, given relations may be used to eliminate some variables to convert the given function to the function having least number of independent variables. Then solve it by the earlier method. When such a procedure becomes inconvenient, Lagrange's method may be used.

## Lagrange's Method of Multipliers

Let us have to maximize/minimize $u = f(x, y, z)$ ......... (1)

subject to the relation $\phi(x, y, z) = 0$ ......... (2)

In this method, a Lagrange's function is taken as $F = f(x, y, z) + \lambda\phi(x, y, z)$.

For stationary points, we have $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0, \frac{\partial F}{\partial \lambda} = \phi(x, y, z) = 0$.

Now solving the above equations simultaneously, we may obtain the values of $x, y, z$ and $\lambda$ for which $u$ is stationary.

The nature of the stationary points may be observed by the physical consideration of the problem.

**Example 1.** Given $x + y + z = a$, find the maximum value of $x^m y^n z^p$.

**Solution.** Let $f(x, y, z) = x^m y^n z^p$ and $\phi(x, y, z) = x + y + z - a$. ......... (1)

So we have to maximize $f(x, y, z) = x^m y^n z^p$ subject to $\phi(x, y, z) = x + y + z - a = 0$.

Let $F = f + \lambda\phi$ so that $F(x, y, z, \lambda) = x^m y^n z^p + \lambda(x + y + z - a)$. ......... (2)

For stationary points, we have $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0, \frac{\partial F}{\partial \lambda} = \phi(x, y, z) = 0$.

$\therefore mx^{m-1} y^n z^p + \lambda = 0, nx^m y^{n-1} z^p + \lambda = 0, px^m y^n z^{p-1} + \lambda = 0$ and $x + y + z = a$, by (2)

$\Rightarrow -\lambda = mx^{m-1} y^n z^p = nx^m y^{n-1} z^p = px^m y^n z^{p-1}$ and $x + y + z = a$

$$\Rightarrow \frac{m}{x} = \frac{n}{y} = \frac{p}{z} = \frac{m+n+p}{x+y+z} = \frac{m+n+p}{a}$$

So the maximum value of $f(x, y, z)$ occurs, when $x = \frac{ma}{m+n+p}, y = \frac{na}{m+n+p}, z = \frac{pa}{m+n+p}$.

Hence the maximum value of $x^m y^n z^p$ is $\frac{m^m n^n p^p a^{m+n+p}}{(m+n+p)^{m+n+p}}$. Ans.

**Example 2.** A rectangular box open at the top is to have volume of 32 cubic foot. Find the dimensions of the box requiring least material for its construction.

**Solution.** Let $x, y$ and $z$ be the edges of the box in foot, V be its volume and $S$ be its surface.

Then $V = xyz = 32$ (given) and $S = xy + 2yz + 2zx$ ......... (1)

So we have to minimize $S = xy + 2yz + 2zx$ subject to $\phi(x, y, z) = xyz - 32 = 0$.

Let $F = S + \lambda\phi$ so that $F = xy + 2yz + 2zx + \lambda(xyz - 32)$. ......... (2)

For stationary points, $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$, $\frac{\partial F}{\partial \lambda} = \phi(x, y, z) = 0$. Using (2), we have

$\Rightarrow$ $y + 2z + \lambda yz = 0,$ ......... (3)

$x + 2z + \lambda xz = 0,$ ......... (4)

$2y + 2x + \lambda xy = 0$ ......... (5)

and $xyz - 32 = 0.$ ......... (6)

Multiplying (3) by $x$ and (4) $y$ by and subtracting, we get

$2zx - 2zy = 0 \Rightarrow x = y$, since $z = 0$ is neglected as it does not satisfy (2).

Again multiplying (4) by $y$ and (5) by $z$ and subtracting, we get

$x(y - 2z) = 0 \Rightarrow y = 2z$, since $x = 0$ is neglected as it does not satisfy (2).

Putting these values in (6), $yyz - 32 = 0 \Rightarrow (2z)(2z)z = 32 \Rightarrow z^3 = 8 \Rightarrow z = 2.$

So we have $x = y = 2z = 4$, $z = 2$.

Hence $S$ is minimum when $x = y = 4$ feet and $z = 2$ feet, i.e. the dimensions of the box requiring least material for its construction are 4 feet, 4 feet and 2 feet. Ans.

**Example 3.** Show that the rectangular solid of maximum volume that can be inscribed in a sphere is a cube.

**Solution.** Let $2x, 2y, 2z$ be length, breadth and height of the rectangular solid so that its volume is $V = 8xyz$.

Let this rectangular solid of is inscribed in a sphere whose radius is $R$ and centre is origin. So equation of sphere will be $x^2 + y^2 + z^2 = R^2$.

So we have to maximize $V = 8xyz$ subject to $\phi(x, y, z) = x^2 + y^2 + z^2 - R^2 = 0.$ ......... (1)

Let $F = V + \lambda\phi$ so that $F(x, y, z, \lambda) = 8xyz + \lambda\left(x^2 + y^2 + z^2 - R^2\right)$. ......... (2)

For stationary points, we have $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$, $\frac{\partial F}{\partial \lambda} = \phi(x, y, z) = 0$.

$\Rightarrow$ $8yz + 2x\lambda = 0, 8zx + 2y\lambda = 0, 8xy + 2z\lambda = 0$ and $x^2 + y^2 + z^2 - R^2 = 0$, by (2)

$\Rightarrow 2x^2\lambda = -8xyz = 2y^2\lambda = 2z^2\lambda \Rightarrow x^2 = y^2 = z^2$

$\Rightarrow x = y = z$.

So for maximum volume $2x = 2y = 2z$, i.e. the sides of the rectangular solid are equal.

Hence the rectangular solid is a cube. <u>Proved.</u>

**Example 4.** Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

**Solution.** Let $V$ be the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, then $V = 2x2y2z = 8xyz$.

So we have to maximize $V = 8xyz$ subject to $\phi(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$. ……… (1)

Let $F = V + \lambda\phi$ so that $F(x,y,z,\lambda)=8xyz+\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)$. ……… (2)

For stationary points, we have $\frac{\partial F}{\partial x}=0, \frac{\partial F}{\partial y}=0, \frac{\partial F}{\partial z}=0$, $\frac{\partial F}{\partial \lambda}=\phi(x,y,z)=0$.

$\Rightarrow 8yz+\frac{2x\lambda}{a^2}=0, 8xz+\frac{2y\lambda}{b^2}=0, 8xy+\frac{2z\lambda}{c^2}=0$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$, by (2)

$\Rightarrow \frac{x^2}{a^2}=-\frac{4xyz}{\lambda}=\frac{y^2}{b^2}=\frac{z^2}{c^2}$ ……… (3)

and $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. ……… (4)

Putting the values from (3) in (4), we get

$3\left(-\frac{4xyz}{\lambda}\right)=1 \Rightarrow -\frac{4xyz}{\lambda}=\frac{1}{3}$.

Putting this value in (3), we get

$\Rightarrow \frac{x^2}{a^2}=\frac{1}{3}=\frac{y^2}{b^2}=\frac{z^2}{c^2}$

$\Rightarrow x=\frac{a}{\sqrt{3}}, y=\frac{b}{\sqrt{3}}, z=\frac{c}{\sqrt{3}}$, leaving negative sign for maximum $V$.

So maximum $V = 8\frac{a}{\sqrt{3}}\frac{b}{\sqrt{3}}\frac{c}{\sqrt{3}}$.

Hence the volume of the largest rectangular parallelopiped that can be inscribed in the given ellipsoid is $\frac{8abc}{3\sqrt{3}}$. <u>Ans.</u>

**Example 5.** Find the maxima and minima of $x^2+y^2+z^2$ subject to the conditions $ax^2+by^2+cz^2=1$ and $lx+my+nz=0$.

**Solution.** Let $u=x^2+y^2+z^2$, which to be maximize and minimize subject to the conditions $ax^2+by^2+cz^2=1$ and $lx+my+nz=0$.

Let $\phi_1(x,y,z)=ax^2+by^2+cz^2-1=0,\ \phi_2(x,y,z)=lx+my+nz=0$ and $F = u + \lambda_1\phi_1 + \lambda_2\phi_2$ so that $F(x,y,z,\lambda_1,\lambda_2)=x^2+y^2+z^2+\lambda_1\left(ax^2+by^2+cz^2-1\right)+\lambda_2(lx+my+nz)$.

For stationary points, we have

$$\frac{\partial F}{\partial x}=0,\frac{\partial F}{\partial y}=0,\frac{\partial F}{\partial z}=0,\ \frac{\partial F}{\partial \lambda_1}=\phi_1(x,y,z)=0\ ,\ \frac{\partial F}{\partial \lambda_2}=\phi_2(x,y,z)=0.$$

$$\Rightarrow 2x+2ax\lambda_1+l\lambda_2=0, \qquad \cdots\cdots\cdots (1)$$

$$2y+2by\lambda_1+m\lambda_2=0, \qquad \cdots\cdots\cdots (2)$$

$$2z+2cz\lambda_1+n\lambda_2=0, \qquad \cdots\cdots\cdots (3)$$

$$ax^2+by^2+cz^2=1 \qquad \cdots\cdots\cdots (4)$$

and $lx+my+nz=0$ $\cdots\cdots\cdots$ (5)

Multiplying (1) by $x$, (2) by $y$ and (3) by $z$, then adding, we get

$$2(x^2+y^2+z^2)+2\lambda_1(ax^2+by^2+cz^2)+\lambda_2(lx+my+nz)=0$$

$\Rightarrow 2(x^2+y^2+z^2)+2\lambda_1(1)+\lambda_2(0)=0$, by (4) and (5)

$\Rightarrow \lambda_1=-u$, since $u=x^2+y^2+z^2$.

Putting the value $\lambda_1$ in equations (1), (2) and (3), we get

$$2x-2axu+l\lambda_2=0,\ \ 2y-2byu+m\lambda_2=0,\ \ 2z-2czu+n\lambda_2=0$$

$\Rightarrow 2x(au-1)=l\lambda_2$, $2y(bu-1)=m\lambda_2$ and $2z(cu-1)=n\lambda_2$

$\Rightarrow x=\dfrac{l\lambda_2}{2(au-1)}$, $y=\dfrac{m\lambda_2}{2(bu-1)}$ and $z=\dfrac{n\lambda_2}{2(cu-1)}$

Putting these values of $x$, $y$ and $z$, in (3), we get

$$\frac{\lambda_2}{2}\left[\frac{l^2}{au-1}+\frac{m^2}{bu-1}+\frac{n^2}{cu-1}\right]=0$$

$$\Rightarrow \frac{l^2}{au-1}+\frac{m^2}{bu-1}+\frac{n^2}{cu-1}=0,$$

which gives the maximum and minimum values of $u=x^2+y^2+z^2$. Ans.

# LECTURE NO.:- 14

## Problems of Maxima, Minima and Lagrange's Method of Multipliers

**Example 1.** Find the extreme value of $x^2+y^2+z^2$ subject to the condition $ax+by+cz=p$.

**Solution.** Let $f(x,y,z)=x^2+y^2+z^2$ and $\phi(x,y,z)=ax+by+cz-p$. ......... (1)

So we have to find the extreme value of $f(x,y,z)=x^2+y^2+z^2$

subject to the condition $\phi(x,y,z)=ax+by+cz-p=0$.

Let $F=f+\lambda\phi$ so that $F(x,y,z,\lambda)=x^2+y^2+z^2+\lambda\left(ax+by+cz-p\right)$. ......... (2)

For stationary points, the necessary conditions are

$$\frac{\partial F}{\partial x}=0,\frac{\partial F}{\partial y}=0,\frac{\partial F}{\partial z}=0,\ \frac{\partial F}{\partial \lambda}=\phi(x,y,z)=0.$$

So using (2), we have

$$2x+\lambda a=0,\ 2y+\lambda b=0,\ 2z+\lambda c=0,\ ax+by+cz-p=0.$$

$$\Rightarrow x=-\lambda a/2, y=-\lambda b/2, z=-\lambda c/2 \qquad \cdots\cdots\cdots (3)$$

and $ax+by+cz=p.$ ......... (4)

Putting the values of $x$, $y$ and $z$ from (3) in (4), we get

$$\lambda=-\frac{2p}{a^2+b^2+c^2}$$

Putting this value of $\lambda$ in (3), we get

$$x=\frac{ap}{a^2+b^2+c^2}, y=\frac{bp}{a^2+b^2+c^2}, z=\frac{cp}{a^2+b^2+c^2}$$

So the maximum/minimum value of $f=x^2+y^2+z^2=\dfrac{a^2p^2+b^2p^2+c^2p^2}{\left(a^2+b^2+c^2\right)^2}=\dfrac{p^2}{\left(a^2+b^2+c^2\right)}$.

Hence the extreme value of $x^2+y^2+z^2$ is $\dfrac{p^2}{\left(a^2+b^2+c^2\right)}$. Ans.

**Example 2.** Determine any local maxima or local minima of the function $w=x^2+y^2+z^2$ subject to the constraint that $x+2y+3z=1$.

**Solution.** We have to maximize/minimize $w=x^2+y^2+z^2$ subject to $x+2y+3z=1$. ......... (1)

Let $\phi(x, y, z)=x+2y+3z-1$ and $F=w+\lambda\phi$ so that

$F(x, y, z, \lambda)=x^2+y^2+z^2+\lambda(x+2y+3z-1)$. ......... (2)

For stationary points, the necessary conditions are

$$\frac{\partial F}{\partial x}=0, \frac{\partial F}{\partial y}=0, \frac{\partial F}{\partial z}=0, \frac{\partial F}{\partial \lambda}=\phi(x,y,z)=0.$$

So using (2), we have

$2x+\lambda=0 \cdots$ (i), $y+\lambda=0 \cdots$ (ii), $2z+3\lambda=0 \cdots$ (iii) and $x+2y+3z=1 \cdots$ (iv).

From (i), (ii) and (iii), $x=-\lambda/2, y=-\lambda, z=-3\lambda/2 \cdots$ (v).

Putting these values in (iv), we get

$\lambda(-½-2-9/2)=1 \Rightarrow -\lambda(1+4+9)/2=1 \Rightarrow \lambda=-2/14=-1/7$.

Putting this value in (v), we get

$x=1/14, y=1/7, z=3/14 \cdots$ (vi).

Putting these values in (iv), we get

The corresponding value of $w=\left(\frac{1}{14}\right)^2+\left(\frac{1}{7}\right)^2+\left(\frac{3}{14}\right)^2=\frac{1+4+9}{14^2}=\frac{1}{14}$.

We observe that $x=1$, $y=0$, $z=0$ satisfy the given constraint and give $w=1>1/14$. So $1/14$ is the minimum value of $w$.

Hence the required local minima of the function $w$ is $\frac{1}{14}$ at $x=\frac{1}{14}, y=\frac{1}{7}, z=\frac{3}{14}$. Ans.

**Example 3.** Find the maximum of $w=xyz$ given that $xy+zx+yz=a$, for a given positive number $a$ and $x>0, y>0, z>0$.

**Solution.** We have to maximize $w=xyz$

such that $xy+zx+yz=a, a>0, x>0, y>0, z>0$. ……… (1)

Let $\phi(x,y,z)=xy+zx+yz-a$ and $F=w+\lambda\phi$ so that

$F(x,y,z,\lambda)=xyz+\lambda(xy+zx+yz-a)$. ……… (2)

For stationary points, the necessary conditions are

$$\frac{\partial F}{\partial x}=0, \frac{\partial F}{\partial y}=0, \frac{\partial F}{\partial z}=0, \frac{\partial F}{\partial \lambda}=\phi(x,y,z)=0.$$

So using (2), we have

$yz+\lambda(y+z)=0 \cdots$ (i), $xz+\lambda(x+z)=0 \cdots$ (ii), $xy+\lambda(x+y)=0 \cdots$ (iii)

and $xy+zx+yz-a=0$ i.e. $xy+zx+yz=a \cdots$ (iv).

Multiplying (i) by $x$, (ii) by $y$ and then subtracting, we get

$\lambda z(x-y)=0 \cdots$ (v).

If $\lambda=0$, then from (iii), either $x=0$ or $y=0$. But given $x>0, y>0$. So $\lambda\neq 0$. Also $z>0$.

So from (v), $x-y=0 \Rightarrow x=y$.

Similarly from (ii) and (iii), we get $y=z$.

Hence $x=y=z \cdots$ (vi).

Using it in (iv), we get

$3x^2 = a \Rightarrow x = \sqrt{a/3}$, since $x > 0$.

Putting it in (vi), we get

$x = y = z = \sqrt{a/3}$.

The corresponding value of $w = (a/3)^{3/2}$. It cannot be minimum, since minimum $w \to 0$, by $x \to 0, y = 1, z = a$ or $y \to 0, z = 1, x = a$ or $z \to 0, x = 1, y = a$.

Hence the maximum value of $w$ is $(a/3)^{3/2}$. Ans.

**Example 4.** Find points on the surface $z^2 = xy + 1$ whose distances from the origin are minimum.

**Solution.** Let P$(x, y, z)$ be the point on the given surface $z^2 = xy + 1$.

The distance of P from origin O(0, 0, 0) is $\text{OP} = \sqrt{x^2 + y^2 + z^2} = f$(say).

We have to minimize $f = \sqrt{x^2 + y^2 + z^2}$ subject to $z^2 = xy + 1$. ……… (1)

Let $\phi(x, y, z) = z^2 - xy - 1$ and $F = \text{OP} + \lambda\phi$ so that

$F(x, y, z, \lambda) = \sqrt{(x^2 + y^2 + z^2)} + \lambda(z^2 - xy - 1)$. ……… (2)

For stationary points, the necessary conditions are

$$\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0, \frac{\partial F}{\partial \lambda} = \phi(x, y, z) = 0.$$

So using (2), we have

$$\frac{x}{D} + \lambda(-y) = 0 \ \cdots \text{(i)}, \ \frac{y}{D} + \lambda(-x) = 0 \ \cdots \text{(ii)}, \ \frac{z}{D} + \lambda(2z) = 0 \ \cdots \text{(iii) and } z^2 = xy + 1 \ \cdots \text{(iv)},$$

where $D = \sqrt{x^2 + y^2 + z^2} \cdots$ (v).

From (iii), $\lambda = -\dfrac{1}{2D} \cdots$ (vi).

Putting this value in (i) and (ii), we get

$$\frac{x}{D} + \frac{1}{2D}(y) = 0 \Rightarrow 2x + y = 0 \ \cdots \text{(vii)}$$

and $\dfrac{y}{D} + \dfrac{1}{2D}(x) = 0 \Rightarrow x + 2y = 0 \ \cdots$ (viii).

Adding (vii) and (viii), we get

$3(x + y) = 0 \Rightarrow y = -x$.

Using it in (vii), we have

$x = 0, y = 0$.

Putting these values in (iv), we $z^2 = 1$ i.e. $z = \pm 1$. So we have obtained the stationary points (0, 0, −1) and (0, 0, 1) with OP = 1.

We observe that $x = 1$, $y = -1$, $z = 0$ satisfy the constraint and give OP = √2 > 1. So 1 is the minimum value of OP.

Hence the desired points are (0, 0, −1) and (0, 0, 1). Ans.

**Example 5.** Maximize the volume of a box of length $x$, width $y$ and height $z$ subject to the condition that $x + 2y + 2z = 108$.

**Solution.** Let the volume of the box of length $x$, width $y$ and height $z$ is $V$ so that $V = xyz$.

So we have to maximize $V = xyz$ subject to the condition $x + 2y + 2z = 108$.

This problem may be solved by Lagrange's multipliers method as above. This problem may also be solved by elimination method, by eliminating one variable from objective function with the help of constrained equation.

Let us eliminate $x$. So from condtion, $x = 108 - 2y - 2z$.

Putting this value in the objective function, it becomes

$V = (108 - 2y - 2z)yz = f(y, z)$ (say).

Hence the problem becomes: Maximize $f(y, z) = 108yz - 2y^2z - 2yz^2$.

Differentiating $f(y, z)$ partially w.r.t. $y$ and $z$, we get

$f_y = 108z - 4yz - 2z^2, f_z = 108y - 2y^2 - 4yz, f_{yy} = -4z, f_{yz} = 108 - 4y - 4z, f_{zz} = -4y.$

For stationary points, we have

$f_y = (108 - 4y - 2z)z = 0, f_z = (108 - 2y - 4z)y = 0.$

$\Rightarrow (z = 0$ or $108 - 4y - 2z = 0)$ and $(y = 0$ or $108 - 2y - 4z = 0)$.

Here we have following four possibilities:

(i) $z = 0, y = 0$

(ii) $z = 0, 108 - 2y - 4z = 0 \Rightarrow z = 0, y = 54$

(iii) $108 - 4y - 2z = 0, y = 0 \Rightarrow y = 0, z = 54$

(iv) $108 - 4y - 2z = 0, 108 - 2y - 4z = 0 \Rightarrow y = z = 18$

So the critical points are $(y, z) = (0, 0), (0, 54), (54, 0), (18, 18)$.

At the first three points, $f(y, z)$ is 0, which is clearly not the maximum value of $f(y, z)$. The only possibility is at (18, 18). To see that this is a point where $f(y, z)$ is maximum, we have

$f_{yy} = -4z, f_{yz} = 108 - 4y - 4z, f_{zz} = -4y.$

At (18, 18), $f_{yy} = -4\times18 < 0$ and $f_{yy}f_{zz} - (f_{yz})^2 = +4\times18\times4\times18 - (108 - 4\times18 - 4\times18)^2$

$= 4^2[18^2 - (27 - 18 - 18)^2] = 16[18^2 - (-9)^2] > 0.$

Hence $f$ is maximum, when $y = 18$, $z = 18$ and then $x = 108 - 2\times18 - 2\times18 = 36$ and maximum volume of the box is $V = 36\times18\times18 = 11664$ cubic units, i.e. the volume of the box is maximum, when its length $x = 36$ units, width $y = 18$ units and height $z = 18$ units. Ans.