MODERN INSTITUTE OF TECHNOLOGY AND RESEARCH CENTRE, ALWAR

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Subject & Subject Code: Engineering Mathematics-1, 1FY1-01

Semester: I **Session: 2022-23 (Odd Sem.)**

Branch: A & B

Unit No.: 4 **Date of Submission: 07/10/2022**

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UNIT-4 (DIFFERENTIAL EQUATIONS-II)

LECTURE NO.:- 25

Second Order Linear ODE with Variable Coefficients

A linear ordinary differential equation of second order with variable coefficients is of the form

$$\mathrm{P}_0 \frac{d^2y}{dx^2} + \mathrm{P}_1 \frac{dy}{dx} + \mathrm{P}_2\, y = \mathrm{Q}\,, \quad \cdots\cdots\cdots (1)$$

where P_0, P_1, P_2 and Q may be functions of x only. Here $\mathrm{P}_0 \neq 0$.

Equation (1) is general form. Its standard form is

$$\frac{d^2y}{dx^2} + \mathrm{P} \frac{dy}{dx} + \mathrm{Q}\, y = \mathrm{R}\,, \quad \cdots\cdots\cdots (2)$$

where P, Q and R may be functions of x only.

There is no any general method for solution of such equations but several methods are applicable under certain conditions.

Homogeneous Form of Linear ODE of Second Order

A linear differential equation of the form

$$a_0 x^2 \frac{d^2y}{dx^2} + a_1 x \frac{dy}{dx} + a_2\, y = \mathrm{Q}(x)\,, \quad \cdots\cdots\cdots (1)$$

where $a_0 (\neq 0)$, a_1, a_2 are constants and Q is a function of x only, is called a homogeneous differential equation of second order. This equation is also known as **Cauchy-Euler equation** of order two.

To solve such type equations, the independent variable x is changed into other variable, say z by substituting $x = e^z$, i.e. $z = \log x$ so that the equation (1) can be transformed into a linear ODE of second order with constant coefficients as

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{dy}{dz}\frac{1}{x} \Rightarrow x\frac{dy}{dx} = \frac{dy}{dz} = Dy \text{ , where } D = \frac{d}{dz}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz}\frac{1}{x}\right) = \left\{\frac{d}{dx}\left(\frac{dy}{dz}\right)\right\}\frac{1}{x} + \frac{dy}{dz}\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} + \frac{dy}{dz}\left(-\frac{1}{x^2}\right) = \frac{1}{x}\frac{d^2y}{dz^2}\frac{1}{x} - \frac{1}{x^2}\frac{dy}{dz} = \frac{1}{x^2}(D^2y - Dy)$$

$\Rightarrow \quad x^2 \frac{d^2y}{dx^2} = D(D-1)y.$

Putting these values in equation (1), we have

$a_0 D(D-1)y + a_1 Dy + a_2\, y = \mathrm{Q}(e^z)$

$\Rightarrow \quad (a_0 D^2 - a_0 D + a_1 D + a_2)\, y = \mathrm{Q}(e^z)$

$\Rightarrow \quad \{a_0 D^2 + (a_1 - a_0)D + a_2\}\, y = \mathrm{Q}(e^z).$ ………………………… (2)

Equation (2) is a linear ODE of second order with constant coefficients and can be solved easily. Then putting $z = \log x$, the complete solution of equation (1) is obtained.

Example 1. Solve $x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$.

Solution. Given differential equation is

$x^2 \frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x.$ ………………………………… (1)

It is a homogeneous differential equation of second order.

So putting $x = e^z$ and $\frac{d}{dz} = D$, equation (1) becomes

$D(D-1)y + 4Dy + 2y = e^{e^z}.$

$\Rightarrow \quad (D^2 + 3D + 2)y = e^{e^z}.$

Its A.E. is $m^2 + 3m + 2 = 0$.

$\Rightarrow \quad (m+1)(m+2) = 0 \Rightarrow m = -1, -2.$

$\therefore$ C.F. = $c_1 e^{-1z} + c_2 e^{-2z}$, where c_1 and c_2 are arbitrary constants.

Putting $e^z = x$, C.F. = $c_1 x^{-1} + c_2 x^{-2}$.

Now P.I. = $\frac{1}{D^2+3D+2} e^{e^z} = \frac{1}{(D+1)(D+2)} e^{e^z} = \left(\frac{1}{D+1} - \frac{1}{D+2}\right) e^{e^z}$.

$\Rightarrow$ P.I. = $\frac{1}{D+1} e^{e^z} - \frac{1}{D+2} e^{e^z} = \frac{1}{D-(-1)} e^{e^z} - \frac{1}{D-(-2)} e^{e^z}$

$= e^{-1z} \int e^{+1z} e^{e^z} dz - e^{-2z} \int e^{+2z} e^{e^z} dz$, by $\frac{1}{D-\alpha} Q = e^{\alpha x} \int e^{-\alpha x} Q\, dx$

$= x^{-1} \int e^x dx - x^{-2} \int x e^x dx$, by $e^z = x$

$= \frac{1}{x} e^x - \frac{1}{x^2}(x-1)e^x = \frac{1}{x} e^x - \frac{1}{x} e^x + \frac{1}{x^2} e^x$

$= \frac{1}{x^2} e^x.$

Hence the complete solution of the given differential equation is

$\boxed{y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{e^x}{x^2}.}$ Ans.

Example 2. Solve $x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}$.

Solution. Given differential equation is

$x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}.$ ………………………………… (1)

It is a homogeneous differential equation of second order.

So putting $x = e^z$ and $\frac{d}{dz} = D$, equation (1) becomes

$D(D-1)y + 3Dy + y = \frac{1}{(1-e^z)^2}.$

$\Rightarrow$ $(D^2 + 2D + 1)y = \frac{1}{(1-e^z)^2}.$

Its A.E. is $m^2 + 2m + 1 = 0.$

$\Rightarrow$ $(m+1)^2 = 0 \Rightarrow m = -1, -1.$

$\therefore$ C.F. = $(c_1 + c_2 z)e^{-1z}$, where c_1 and c_2 are arbitrary constants.

Putting $e^z = x$, i.e. $z = \log x$, C.F. = $(c_1 + c_2 \log x)x^{-1}$.

Now P.I. $= \frac{1}{D^2+2D+1}\frac{1}{(1-e^z)^2} = \frac{1}{(D+1)^2}\frac{1}{(1-e^z)^2} = \frac{1}{D+1}\left[\frac{1}{D+1}\frac{1}{(1-e^z)^2}\right]$

$= \frac{1}{D+1}\left[e^{-1z}\int e^{1z}\frac{1}{(1-e^z)^2}dz\right]$, by $\frac{1}{D-\alpha}Q = e^{\alpha x}\int e^{-\alpha x} Q dx$

$= \frac{1}{D+1}\left[e^{-z}\int \frac{-1}{t^2}dt\right]$, by $1 - e^z = t$

$= \frac{1}{D+1}\left[e^{-z}\frac{1}{t}\right] = \frac{1}{D+1}\left[e^{-z}\frac{1}{1-e^z}\right]$, by $t = 1 - e^z$

$= e^{-1z}\int e^{1z}\frac{e^{-z}}{1-e^z}dz$, again by above formula

$= e^{-z}\int \frac{1}{1-e^z}dz = e^{-z}\int \frac{e^{-z}}{e^{-z}-1}dz = e^{-z}\int \frac{-1}{u}du$, by $e^{-z} - 1 = u$

$$= -e^{-z}\log u = -e^{-z}\log(e^{-z}-1), \text{ by } u = e^{-z}-1$$
$$= -x^{-1}\log(x^{-1}-1), \text{ by } e^{z} = x$$
$$= -\frac{1}{x}\log\left(\frac{1}{x}-1\right) = -\frac{1}{x}\log\left(\frac{1-x}{x}\right) = \frac{1}{x}\log\left(\frac{1-x}{x}\right)^{-1}$$
$$= \frac{1}{x}\log\left(\frac{x}{1-x}\right).$$

Hence the complete solution of the given differential equation is

$$\boxed{y = \frac{1}{x}\left\{c_1 + c_2\log x + \log\left(\frac{x}{1-x}\right)\right\}}$$ Ans.

Example 3. Solve $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\ \log x}{x^2}$.

Solution. Given differential equation may be written as

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 12\log x. \quad \cdots\cdots\cdots\cdots\cdots\cdots (1)$$

It is a homogeneous differential equation of second order.

So putting $x = e^z$, i.e. $z = \log x$ and $\frac{d}{dz} = D$, equation (1) becomes

$$D(D-1)y + Dy = 12z.$$
$$\Rightarrow \quad (D^2 - D + D)y = 12z.$$
$$\Rightarrow \quad D^2y = 12z.$$

Integrating it two times w.r.t. z, we get

$y = 12 \cdot \frac{z^3}{2\times 3} + c_1z + c_2$, where c_1 and c_2 are arbitrary constants.

Putting $z = \log x$, the complete solution of the given differential equation is

$$\boxed{y = c_1\log x + c_2 + 2(\log x)^3}$$ Ans.

Example 4. Solve $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2\log x$.

Solution. Given differential equation is

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2\log x. \quad \cdots\cdots\cdots\cdots\cdots\cdots (1)$$

It is a homogeneous differential equation of second order.

So putting $x = e^z$, i.e. $\log x = z$ and $\frac{d}{dz} = D$, equation (1) becomes

$D(D-1)y - Dy - 3\,y = e^{2z}z.$

$\Rightarrow \quad (D^2 - 2D - 3)y = e^{2z}z.$

Its A.E. is $m^2 - 2m - 3 = 0.$

$\Rightarrow \quad (m+1)(m-3) = 0 \ \Rightarrow m = 3, -1.$

$\therefore$ C.F. = $c_1 e^{3z} + c_2 e^{-1z}$, where c_1 and c_2 are arbitrary constants.

Putting $e^z = x$, C.F. = $c_1 x^3 + c_2 x^{-1}$.

Now P.I. = $\frac{1}{D^2-2D-3} e^{2z} z.$

$\Rightarrow$ P.I. = $e^{2z} \frac{1}{(D+2)^2 - 2(D+2) - 3} z$, by $\frac{1}{f(D)} e^{ax} U = e^{ax} \frac{1}{f(D+a)} U$

$$= e^{2z} \frac{1}{D^2+2D-3} z = e^{2z} \frac{1}{-3\left(1-\frac{2D+D^2}{3}\right)} z = -\frac{1}{3} e^{2z} \left(1 - \frac{2D+D^2}{3}\right)^{-1} z$$

$$= -\frac{1}{3} e^{2z} \left(1 + \frac{2D+D^2}{3} + \cdots\right) z, \text{ by binomial theorem}$$

$$= -\frac{1}{3} e^{2z} \left(z + \frac{2D+D^2}{3} z + \cdots\right) = -\frac{1}{3} e^{2z} \left(z + \frac{2}{3} + \ 0\right)$$

$$= -\frac{1}{3} x^2 \left(\log x + \frac{2}{3}\right), \text{ by } e^z = x \text{ i.e. } z = \log x.$$

Hence the complete solution of the given differential equation is

$\boxed{y = c_1 x^3 + \frac{c_2}{x} - \frac{1}{3} x^2 \left(\log x + \frac{2}{3}\right)}$. Ans.

Example 5. Solve $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = x^m.$

Solution. Try yourself.

Example 6. Solve $x^2 \frac{d^2y}{dx^2} + 7x \frac{dy}{dx} + 13y = \log x.$

Solution. Try yourself.

Example 7. Solve $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + y = \frac{\log x \ \sin(\log x) + 1}{x}.$

Solution. Try yourself.

LECTURE NO.:- 26

Legendre Form of Linear ODE of Second Order
(Equations Reducible to Homogeneous Form)

Second form of homogeneous linear differential equation is of the form

$$a_0(ax+b)^2\frac{d^2y}{dx^2}+a_1(ax+b)\frac{dy}{dx}+a_2\,y=Q(x)\,, \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$$

where $a_0(\neq 0)$, a_1, a_2, $a(\neq 0)$, $b(\neq 0)$ are constants and Q is a function of *x* only. It is called a Legendre form of linear differential equation of second order.

Equation (1) can be reduced to homogeneous form, if we take $ax+b=t$ so that $adx=dt$ and $dx=dt/a$. Then equation (1) becomes

$$(a_0a^2)t^2\frac{d^2y}{dt^2}+(a_1a)t\frac{dy}{dt}+a_2\,y=Q\left(\frac{t-b}{a}\right), \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (2)$$

which is a homogeneous linear differential equation of second order and can be solved by substituting $t=e^z$ etc.

Then putting $z=\log(ax+b)$, the complete solution of equation (1) is obtained.

Note. For solution of equation (1), we can take $ax+b=e^z$ so that

$$(ax+b)\frac{dy}{dx}=aDy \text{ and } (ax+b)^2\frac{d^2y}{dx^2}=a^2D(D-1)y, \text{ where } D=\frac{d}{dz}.$$

Example 1. Solve $(1+2x)^2\frac{d^2y}{dx^2}-6(1+2x)\frac{dy}{dx}+16y=8(1+2x)^2$.

Solution. Given differential equation is

$$(1+2x)^2\frac{d^2y}{dx^2}-6(1+2x)\frac{dy}{dx}+16y=8(1+2x)^2. \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$$

It is a Legendre form of linear differential equation of second order.

So putting $1+2x=e^z$ and $\frac{d}{dz}=D$, equation (1) becomes

$2^2D(D-1)y-6\times 2Dy+16y=8e^{2z}$.

$\Rightarrow$ $4(D^2-D-3D+4)y=8e^{2z}$.

$\Rightarrow$ $(D^2-4D+4)y=2e^{2z}$. $\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$ (2)

Its A.E. is $m^2-4m+4=0$.

$\Rightarrow$ $(m-2)^2=0 \Rightarrow m=2,2$.

$\therefore$ C.F. = $(c_1+c_2z)e^{2z}$, where c_1 and c_2 are arbitrary constants.

Now P.I. $= \frac{1}{D^2-4D+4} 2e^{2z} = 2\frac{1}{(D-2)^2} e^{2z} \times 1.$

$\Rightarrow$ P.I. $= 2e^{2z} \frac{1}{(D+2-2)^2} 1$, by $\frac{1}{f(D)} e^{ax}\text{U} = e^{ax} \frac{1}{f(D+a)} \text{U}$

$= 2e^{2z} \frac{1}{D^2} 1 = 2e^{2z} \cdot \frac{z^2}{2} = z^2 e^{2z}.$

Hence the complete solution of the equation (2) is

$y = (c_1 + c_2 z)e^{2z} + z^2 e^{2z}$ i.e. $y = (c_1 + c_2 z + z^2)e^{2z}$.

Putting $e^z = 1 + 2x$, i.e. $z = \log(1 + 2x)$, we get

$\boxed{y = [c_1 + c_2 \log(1 + 2x) + \{\log(1 + 2x)\}^2](1 + 2x)^2}$. Ans.

This is the complete solution of the given differential equation.

Example 2. Solve $(1 + x)^2 \frac{d^2y}{dx^2} + (1 + x)\frac{dy}{dx} + y = 4 \cos \log(1 + x)$.

Solution. Given differential equation is

$(1 + x)^2 \frac{d^2y}{dx^2} + (1 + x)\frac{dy}{dx} + y = 4 \cos \log(1 + x)$. ………………… (1)

It is a Legendre form of linear differential equation of second order.

So putting $1 + x = e^z$, i.e. $\log(1 + x) = z$ and $\frac{d}{dz} = D$, equation (1) becomes

$D(D - 1)y + Dy + y = 4 \cos z$.

$\Rightarrow$ $(D^2 - D + D + 1)y = 4 \cos z$.

$\Rightarrow$ $(D^2 + 1)y = 4 \cos z$. ………………………………………… (2)

Its A.E. is $m^2 + 1 = 0$.

$\Rightarrow$ $m^2 = -1 \Rightarrow m = \pm i$.

$\therefore$ C.F. $= c_1 \cos 1z + c_2 \sin 1z$, where c_1 and c_2 are arbitrary constants.

Now P.I. $= \frac{1}{D^2+1} 4 \cos z = 4\frac{1}{D^2+1} \cos 1z$.

$\Rightarrow$ P.I. $= 4\frac{z}{2\times 1} \sin 1z$, by $\frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \sin ax$

$= 2z \sin z$.

Hence the complete solution of the equation (2) is

$y = c_1 \cos z + c_2 \sin z + 2z \sin z$ i.e. $y = c_1 \cos z + (c_2 + 2z) \sin z$.

Putting $e^z = 1 + x$, i.e. $z = \log(1 + x)$, we get

$\boxed{y = c_1 \cos\{\log(1 + x)\} + [c_2 + 2 \log(1 + x)] \sin\{\log(1 + x)\}}$. Ans.

This is the complete solution of the given differential equation.

Example 3. Solve $(3x+2)^2\frac{d^2y}{dx^2}+5(3x+2)\frac{dy}{dx}-3y=x^2+x+1.$

Solution. Given differential equation is

$$(3x+2)^2\frac{d^2y}{dx^2}+5(3x+2)\frac{dy}{dx}-3y=x^2+x+1. \quad \cdots\cdots\cdots\cdots (1)$$

It is a Legendre form of linear differential equation of second order.

So putting $3x+2=e^z$, i.e. $x=\frac{e^z-2}{3}$ and $\frac{d}{dz}=D$, equation (1) becomes

$$3^2D(D-1)y+5\times 3Dy-3y=\left(\frac{e^z-2}{3}\right)^2+\left(\frac{e^z-2}{3}\right)+1.$$

$\Rightarrow$ $(3D^2-3D+5D-1)y=\frac{1}{27}(e^{2z}-e^z+7).$

$\Rightarrow$ $(3D^2+2D-1)y=\frac{1}{27}(e^{2z}-e^z+7).$ $\cdots\cdots\cdots\cdots$ (2)

Its A.E. is $3m^2+2m-1=0.$

$\Rightarrow$ $(3m-1)(m+1)=0 \Rightarrow m=\frac{1}{3},-1.$

$\therefore$ C.F. $=c_1e^{(1/3)z}+c_2e^{-1z}$, where c_1 and c_2 are arbitrary constants.

Now P.I. $=\frac{1}{3D^2+2D-1}\frac{1}{27}(e^{2z}-e^z+7).$

$\Rightarrow$ P.I. $=\frac{1}{27}\left[\frac{1}{3D^2+2D-1}e^{2z}-\frac{1}{3D^2+2D-1}e^{1z}+7\frac{1}{3D^2+2D-1}1\right]$

$=\frac{1}{27}\left[\frac{1}{3\times2^2+2\times2-1}e^{2z}-\frac{1}{3\times1^2+2\times1-1}e^{1z}+7\frac{1}{3\times0^2+2\times0-1}e^{0z}\right],$

by $\frac{1}{f(D)}e^{ax}=\frac{1}{f(a)}e^{ax}$

$=\frac{1}{27}\left(\frac{1}{15}e^{2z}-\frac{1}{4}e^{1z}-7\right).$

Hence the complete solution of the equation (2) is

$$y=c_1e^{(1/3)z}+c_2e^{-1z}+\frac{1}{27}\left(\frac{1}{15}e^{2z}-\frac{1}{4}e^{1z}-7\right).$$

Putting $e^z=3x+2$, i.e. $z=\log(3x+2)$, we get

$$\boxed{y=c_1(3x+2)^{1/3}+\frac{c_2}{(3x+2)}+\frac{1}{27}\left\{\frac{1}{15}(3x+2)^2-\frac{1}{4}(3x+2)-7\right\}}$$ Ans.

This is the complete solution of the given differential equation.

Try yourself to solve the following examples:

Example 4. Solve $(x+1)^2\frac{d^2y}{dx^2}+(x+1)\frac{dy}{dx}=(2x+3)(2x+4).$

Example 5. Solve $(3x+2)^2\frac{d^2y}{dx^2}+3(3x+2)\frac{dy}{dx}-36y=3x^2+4x+1.$

Example 6. Solve $(x+a)^2\frac{d^2y}{dx^2}-4(x+a)\frac{dy}{dx}+6y=x.$

LECTURE NO.:- 27

Exact Form of Linear ODE of Second Order

A differential equation of second order is said to be exact, if it can be obtained by direct differentiation of a differential equation of first order, which is said to be its primitive.

A linear ODE of second order with variable coefficients is of the general form:

$$P_0 \frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + P_2\, y = Q\,, \quad \cdots\cdots\cdots (1)$$

where P_0, P_1, P_2 and Q may be functions of x only. Here $P_0 \neq 0$.

Equation (1) will be exact, if $P_2 - P_1{}' + P_0{}'' = 0$,

which is called the **condition of exactness** and then its **primitive** will be

$$P_0 \frac{dy}{dx} + (P_1 - P_0{}')\, y = \int Q dx + c\,, \quad \cdots\cdots\cdots (2)$$

where c is an arbitrary constant.

Equation (2) is a linear ODE of first order and can be solved with the help of an integrating factor and hence the complete solution of the equation (1) is obtained.

Example 1. Solve $\sin x \frac{d^2y}{dx^2} - \cos x \frac{dy}{dx} + 2y \sin x = \cos x$.

Solution. Given differential equation is

$$\sin x \frac{d^2y}{dx^2} - \cos x \frac{dy}{dx} + 2y \sin x = \cos x. \quad \cdots\cdots\cdots (1)$$

Here $P_0 = \sin x$, $P_1 = -\cos x$, $P_2 = 2 \sin x$ and $Q = \cos x$. So we have

$P_2 - P_1{}' + P_0{}'' = 2 \sin x - \sin x - \sin x = 0$.

So the condition of exactness is satisfied.

Hence the equation (1) is exact and its primitive is

$$P_0 \frac{dy}{dx} + (P_1 - P_0{}')\, y = \int Q dx + c$$

$$\Rightarrow \quad \sin x \frac{dy}{dt} + (-\cos x - \cos x)\, y = \int \cos x\, dx + c$$

$$\Rightarrow \quad \sin x \frac{dy}{dx} - 2 \cos x \cdot y = \sin x + c$$

$$\Rightarrow \quad \frac{dy}{dx} - 2y \cot x = 1 + c \operatorname{cosec} x. \quad \cdots\cdots\cdots (2)$$

It is a linear differential equation of first order.

$\therefore$ I.F. $= e^{\int \mathrm{P}dx} = e^{\int -2\cot x\, dx} = e^{-2\log\sin x} = e^{\log(\sin x)^{-2}} = (\sin x)^{-2} = \frac{1}{(\sin x)^2}$

$= \operatorname{cosec}^2 x.$

Solution of equation (2) is given by

$y \cdot \operatorname{cosec}^2 x = \int (1 + c\operatorname{cosec} x)\operatorname{cosec}^2 x\, dx + c'$

$\Rightarrow \quad y \cdot \operatorname{cosec}^2 x = -\cot x + c\int \operatorname{cosec}^3 x\, dx + c'. \quad \cdots\cdots\cdots\cdots (3)$

Now $\int \operatorname{cosec}^3 x\, dx = \int \operatorname{cosec} x \operatorname{cosec}^2 x\, dx$

$= -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x \cot^2 x\, dx$

$= -\operatorname{cosec} x \cot x - \int \operatorname{cosec} x\,(\operatorname{cosec}^2 x - 1)dx$

$= -\operatorname{cosec} x \cot x - \int \operatorname{cosec}^3 x\, dx + \int \operatorname{cosec} x\, dx$

$\Rightarrow \quad 2\int \operatorname{cosec}^3 x\, dx = -\operatorname{cosec} x \cot x + \log(\operatorname{cosec} x - \cot x)$

$\Rightarrow \quad \int \operatorname{cosec}^3 x\, dx = \frac{1}{2}\log(\operatorname{cosec} x - \cot x) - \frac{1}{2}\operatorname{cosec} x \cot x.$

Putting it in (3), we get

$y \cdot \operatorname{cosec}^2 x = -\cot x + \frac{c}{2}\{\log(\operatorname{cosec} x - \cot x) - \operatorname{cosec} x \cot x\} + c'.$

Hence the complete solution of the given differential equation is

$\boxed{y \cdot \operatorname{cosec}^2 x = -\cot x + c_1\{\log(\operatorname{cosec} x - \cot x) - \operatorname{cosec} x \cot x\} + c_2}$,

where c_1 and c_2 are arbitrary constants. Ans.

Example 2. Solve $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}$.

Solution. Given differential equation is

$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}. \quad \cdots\cdots\cdots\cdots (1)$

Here $\mathrm{P}_0 = x^2$, $\mathrm{P}_1 = 3x$, $\mathrm{P}_2 = 1$ and $\mathrm{Q} = \frac{1}{(1-x)^2}$. So we have

$\mathrm{P}_2 - \mathrm{P}_1' + \mathrm{P}_0'' = 1 - 3 + 2 = 0.$

So the condition of exactness is satisfied.

Hence the equation (1) is exact and its primitive is

$\mathrm{P}_0\frac{dy}{dx} + (\mathrm{P}_1 - \mathrm{P}_0')\,y = \int \mathrm{Q}dx + c_1$

$\Rightarrow \quad x^2\frac{dy}{dt} + (3x - 2x)\,y = \int \frac{1}{(1-x)^2}dx + c_1$

$\Rightarrow \quad x^2 \frac{dy}{dx} + xy = \frac{1}{1-x} + c_1$

$\Rightarrow \quad \frac{dy}{dx} + \frac{1}{x} y = \frac{1}{x^2(1-x)} + \frac{c_1}{x^2}.$ ………………………………………… (2)

It is a linear differential equation of first order.

$\therefore$ I.F. $= e^{\int \mathrm{P} dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x.$

Solution of equation (2) is given by

$$y \cdot x = \int x \left[\frac{1}{x^2(1-x)} + \frac{c_1}{x^2}\right] dx + c_2$$

$$= \int \frac{1}{x(1-x)} dx + c_1 \int \frac{1}{x} dx + c_2$$

$$= \int \left[\frac{1}{x} + \frac{1}{1-x}\right] dx + c_1 \log x + c_2$$

$$= \log x - \log(1-x) + c_1 \log x + c_2$$

Hence the complete solution of the given differential equation is

$\boxed{xy = \log\left(\frac{x}{1-x}\right) + c_1 \log x + c_2}$,

where c_1 and c_2 are arbitrary constants. Ans.

Example 3. Solve $(ax - bx^2)\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + 2by = x.$

Solution: Try yourself.

LECTURE NO.:- 28

Equations Reducible to Exact Form of Linear ODE of Second Order

Sometimes when an equation is not exact, it can be reduced to exact form by multiplying an integrating factor. If P_0, P_1, P_2 are polynomials in x, then **I.F.** will be taken as x^m and the value of m can be obtained by the condition of exactness; otherwise I.F. can be searched by **inspection**.

Example 1. Solve $\sqrt{x}\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + 3y = x.$

Solution. Given differential equation is

$$\sqrt{x}\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + 3y = x. \quad \cdots\cdots (1)$$

Here $P_0 = \sqrt{x}$, $P_1 = 2x$, $P_2 = 3$ and $Q = x$. So we have

$$P_2 - P_1' + P_0'' = 3 - 2 - \frac{1}{4}x^{-3/2} = 1 - \frac{1}{4}x^{-3/2} \neq 0.$$

So the condition of exactness is not satisfied.

Hence the equation (1) is not exact. To make it exact, multiplying equation (1) by I.F. $= x^m$, we get

$$x^{m+1/2}\frac{d^2y}{dx^2} + 2x^{m+1}\frac{dy}{dx} + 3x^m y = x^{m+1}. \quad \cdots\cdots (2)$$

Now $P_0 = x^{m+1/2}$, $P_1 = 2x^{m+1}$, $P_2 = 3x^m$ and $Q = x^{m+1}$.

If equation (2) is exact, then $P_2 - P_1' + P_0'' = 0$

$$\Rightarrow \quad 3x^m - 2(m+1)x^m + \left(m+\tfrac{1}{2}\right)\left(m-\tfrac{1}{2}\right)x^{m-3/2} = 0$$

$$\Rightarrow \quad -2\left(m+1-\tfrac{3}{2}\right)x^m + \left(m+\tfrac{1}{2}\right)\left(m-\tfrac{1}{2}\right)x^{m-3/2} = 0$$

$$\Rightarrow \quad \left(m-\tfrac{1}{2}\right)\left[-2x^m + \left(m+\tfrac{1}{2}\right)x^{m-3/2}\right] = 0$$

$$\Rightarrow \quad \left(m-\tfrac{1}{2}\right)\left[\left(m+\tfrac{1}{2}\right)x^{-3/2} - 2\right]x^m$$

$$\Rightarrow \quad \left(m-\tfrac{1}{2}\right) = 0 \quad \Rightarrow \quad m = \tfrac{1}{2}.$$

Putting it in equation (2), we get

$$x\frac{d^2y}{dx^2} + 2x^{3/2}\frac{dy}{dx} + 3x^{1/2}y = x^{3/2}. \quad \cdots\cdots (3)$$

Hence the equation (3) is exact and its primitive is

$$P_0\frac{dy}{dx} + (P_1 - P_0')\,y = \int Q\,dx + c_1$$

$\Rightarrow \quad x\frac{dy}{dt} + \left(2x^{3/2} - 1\right) y = \int x^{3/2} dx + c_1$

$\Rightarrow \quad \frac{dy}{dx} + \left(2x^{1/2} - \frac{1}{x}\right) y = \frac{2}{5} x^{3/2} + \frac{c_1}{x}$. ………………………………… (4)

It is a linear differential equation of first order.

$\therefore$ I.F. $= e^{\int \text{P}dx} = e^{\int\left(2x^{1/2} - \frac{1}{x}\right) dx} = e^{\frac{4}{3}x^{3/2}} e^{-\log x} = e^{\frac{4}{3}x^{3/2}} e^{\log x^{-1}}$

$= e^{\frac{4}{3}x^{3/2}} x^{-1}.$

Solution of equation (4) is given by

$y \cdot e^{\frac{4}{3}x^{3/2}} x^{-1} = \int e^{\frac{4}{3}x^{3/2}} x^{-1} \left(\frac{2}{5} x^{3/2} + \frac{c_1}{x}\right) dx + c_2$

$\Rightarrow \quad \frac{y}{x} e^{\frac{4}{3}x^{3/2}} = \frac{2}{5} \int e^{\frac{4}{3}x^{3/2}} x^{1/2} dx + c_1 \int e^{\frac{4}{3}x^{3/2}} x^{-1} \frac{1}{x} dx + c_2$

$= \frac{1}{5} \int e^t dt + c_1 \int e^{\frac{4}{3}x^{3/2}} x^{-2} dx + c_2$, by $\frac{4}{3} x^{3/2} = t$

$\Rightarrow \quad \frac{y}{x} e^{\frac{4}{3}x^{3/2}} = \frac{1}{5} e^{\frac{4}{3}x^{3/2}} + c_1 \int x^{-2} e^{\frac{4}{3}x^{3/2}} dx + c_2.$

Hence the complete solution of the given differential equation is

$\boxed{\frac{y}{x} e^{\frac{4}{3}x^{3/2}} = \frac{1}{5} e^{\frac{4}{3}x^{3/2}} + c_1 \int x^{-2} e^{\frac{4}{3}x^{3/2}} dx + c_2}$,

where c_1 and c_2 are arbitrary constants. <u>Ans.</u>

Example 2. Solve $x^4 y_2 + x^2(x-1)y_1 + xy = x^3 - 4$.

Solution: Try yourself.

Example 3. Solve $2x^2 \frac{d^2y}{dx^2} + 15x \frac{dy}{dx} - 7y = 3x^2$.

Solution: Try yourself.

Example 4. Solve $\frac{d^2y}{dx^2} = 2y \sec^2 x$.

Hint: I.F. = $\tan x$.

Example 5. Solve $\sin^2 x \frac{d^2y}{dx^2} - 2y = 0$.

Solution: Try yourself.

LECTURE NO.:- 29

Change of Dependent Variable-I

Complete Solution when one part of CF is known

Linear ODE of second order with variable coefficients in standard form is

$$\frac{d^2y}{dx^2} + \mathrm{P}\frac{dy}{dx} + \mathrm{Q}\,y = \mathrm{R}\,, \quad \cdots\cdots (1)$$

where P, Q and R may be functions of x only.

When a part of CF of equation (1) is u, i.e. $y = u$ is a solution or an integral of $\frac{d^2y}{dx^2} + \mathrm{P}\frac{dy}{dx} + \mathrm{Q}\,y = 0$, let $y = uv$ is the complete solution of equation (1).

Putting $y = uv$, $\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$ & $\frac{d^2y}{dx^2} = \frac{d^2u}{dx^2}v + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}$ in (1), we get

$$\left(\frac{d^2u}{dx^2}v + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}\right) + \mathrm{P}\left(\frac{du}{dx}v + u\frac{dv}{dx}\right) + \mathrm{Q}\,uv = \mathrm{R}$$

$$\Rightarrow \quad u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + \mathrm{P}u\right)\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + \mathrm{P}\frac{du}{dx} + \mathrm{Q}\,u\right)v = \mathrm{R}$$

$$\Rightarrow \quad u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + \mathrm{P}u\right)\frac{dv}{dx} + (0)\,v = \mathrm{R}, \text{ since } \frac{d^2u}{dx^2} + \mathrm{P}\frac{du}{dx} + \mathrm{Q}\,u = 0$$

$$\Rightarrow \quad u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + \mathrm{P}u\right)\frac{dv}{dx} = \mathrm{R}. \quad \cdots\cdots (2)$$

Putting $\frac{dv}{dx} = p$ so that $\frac{d^2v}{dx^2} = \frac{dp}{dx}$ in equation (2), we get

$$u\frac{dp}{dx} + \left(2\frac{du}{dx} + \mathrm{P}u\right)p = \mathrm{R}$$

$$\Rightarrow \quad \frac{dp}{dx} + \left(\frac{2}{u}\frac{du}{dx} + \mathrm{P}\right)p = \frac{\mathrm{R}}{\mathrm{u}}. \quad \cdots\cdots (3)$$

Equation (3) is a linear ODE of first order and can be solved for $p = \frac{dv}{dx}$, then integrating it w.r.t. x, the value of v is obtained and hence the complete solution of the equation (1) is obtained by $y = uv$.

Note. If R = 0, then the equation (3) is solved by separation of variables.

Search for one part of CF:

If $\mathrm{P} + \mathrm{Q}x = 0$, then $u = x$ is a part of CF.
If $1 + \mathrm{P} + \mathrm{Q} = 0$, then $u = e^{x}$ is a part of CF.
If $1 - \mathrm{P} + \mathrm{Q} = 0$, then $u = e^{-x}$ is a part of CF.
If $1 + \mathrm{P}/a + \mathrm{Q}/a^2 = 0$, then $u = e^{ax}$ is a part of CF.
If $m(m-1) + \mathrm{P}mx + \mathrm{Q}x^2 = 0$, then $u = x^m$ is a part of CF.

Example 1. Solve $x^2\frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3$.

Solution. Given differential equation may be written as

$$\frac{d^2y}{dx^2} - \frac{2}{x}(1+x)\frac{dy}{dx} + \frac{2}{x^2}(1+x)y = x. \quad \cdots\cdots\cdots (1)$$

Here $P = -\frac{2}{x}(1+x)$, $Q = \frac{2}{x^2}(1+x)$ and $R = x$. So we have $P + Qx = 0$.

Thus $u = x$ is a part of CF.

Let $y = uv = xv$ is the complete solution of equation (1).

$\Rightarrow$ $y = vx$, $\frac{dy}{dx} = v1 + x\frac{dv}{dx}$ and $\frac{d^2y}{dx^2} = 2\frac{dv}{dx} + x\frac{d^2v}{dx^2}$. Putting these in (1), we get

$$2\frac{dv}{dx} + x\frac{d^2v}{dx^2} - \frac{2}{x}(1+x)\left(v + x\frac{dv}{dx}\right) + \frac{2}{x^2}(1+x)vx = x.$$

$$\Rightarrow x\frac{d^2v}{dx^2} + [2 - 2(1+x)]\frac{dv}{dx} + \left[-\frac{2}{x}(1+x) + \frac{2}{x}(1+x)\right]v = x.$$

$$\Rightarrow x\frac{d^2v}{dx^2} - 2x\frac{dv}{dx} + [0]v = x$$

$$\Rightarrow \frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 1.$$

Let $\frac{dv}{dx} = p$. So above equation becomes

$$\frac{dp}{dx} - 2p = 1. \quad \cdots\cdots\cdots (2)$$

It is a linear differential equation of first order in p.

$\therefore$ I.F. $= e^{\int -2\,dx} = e^{-2x}$.

Solution of equation (2) is given by

$$p \cdot e^{-2x} = \int 1 e^{-2x}\,dx + c_1$$

$$= \frac{e^{-2x}}{-2} + c_1$$

$$\Rightarrow p = \frac{dv}{dx} = -\frac{1}{2} + c_1 e^{2x}.$$

Integrating it w.r.t. x, we get

$v = -\frac{1}{2}x + c_1\frac{e^{2x}}{2} + c_2$, where c_1 and c_2 are arbitrary constants.

Hence the complete solution of the given differential equation is

$$\boxed{y = -\frac{1}{2}x^2 + \frac{c_1}{2}xe^{2x} + c_2 x}$$ Ans.

Example 2. Solve $\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - (1 - \cot x)y = e^x \sin x$.

Solution. Given differential equation is

$$\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - (1 - \cot x)y = e^x \sin x. \quad \cdots\cdots\cdots (1)$$

Here P $= -\cot x$, Q $= -(1 - \cot x)$ and R $= e^x \sin x$.

So we have $1 + \mathrm{P} + \mathrm{Q} = 0$. Thus $u = e^x$ is a part of CF.

Let $y = uv = e^x v$ is the complete solution of equation (1).

$\Rightarrow \quad y = ve^x, \frac{dy}{dx} = ve^x + e^x \frac{dv}{dx}$ and $\frac{d^2y}{dx^2} = ve^x + 2e^x \frac{dv}{dx} + e^x \frac{d^2v}{dx^2}$.

Putting these values in equation (1), we get

$$ve^x + 2e^x \frac{dv}{dx} + e^x \frac{d^2v}{dx^2} - \cot x \left(ve^x + e^x \frac{dv}{dx}\right) - (1 - \cot x)ve^x = e^x \sin x.$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} + [2 - \cot x]\frac{dv}{dx} + [1 - \cot x - (1 - \cot x)]v = \sin x.$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} + (2 - \cot x)\frac{dv}{dx} + [0]v = \sin x \Rightarrow \frac{d^2v}{dx^2} + (2 - \cot x)\frac{dv}{dx} = \sin x.$$

Let $\frac{dv}{dx} = p$. So above equation becomes

$$\frac{dp}{dx} + (2 - \cot x)p = \sin x. \quad \cdots\cdots\cdots (2)$$

It is a linear differential equation of first order in p.

$$\therefore \text{ I.F.} = e^{\int (2 - \cot x)\, dx} = e^{2x} e^{-\log \sin x} = \frac{e^{2x}}{\sin x}.$$

Solution of equation (2) is given by

$p \cdot \frac{e^{2x}}{\sin x} = \int e^{2x} dx + c_1$, where c_1 is an arbitrary constant.

$$\Rightarrow \quad p \cdot \frac{e^{2x}}{\sin x} = \frac{e^{2x}}{2} + c_1$$

$$\Rightarrow \quad p = \frac{dv}{dx} = \frac{1}{2}\sin x + c_1 e^{-2x} \sin x.$$

Integrating it w.r.t. x, we get

$v = -\frac{1}{2}\cos x + c_1 \frac{e^{-2x}}{5}(-2\sin x - \cos x) + c_2$, c_2 is an arbitrary constant.

Hence the complete solution of the given differential equation is

$\boxed{y = -\frac{1}{2}e^x \cos x - \frac{c_1}{5}e^{-x}(2\sin x + \cos x) + c_2 e^x}$ <u>Ans.</u>

Example 3. Solve $\frac{d^2y}{dx^2} + (1-\cot x)\frac{dy}{dx} - y\cot x = \sin^2 x.$

Solution. Given differential equation is

$$\frac{d^2y}{dx^2} + (1-\cot x)\frac{dy}{dx} - y\cot x = \sin^2 x. \quad \cdots\cdots\cdots (1)$$

Here $\mathrm{P} = 1-\cot x$, $\mathrm{Q} = -\cot x$ and $\mathrm{R} = \sin^2 x$.

So we have $1 - \mathrm{P} + \mathrm{Q} = 0$. Thus $u = e^{-x}$ is a part of CF.

Let $y = uv = e^{-x}v$ is the complete solution of equation (1).

$\Rightarrow \quad y = ve^{-x},\ \frac{dy}{dx} = -ve^{-x} + e^{-x}\frac{dv}{dx}$ and $\frac{d^2y}{dx^2} = ve^{-x} - 2e^{-x}\frac{dv}{dx} + e^{-x}\frac{d^2v}{dx^2}$.

Putting these values in equation (1), we get

$$ve^{-x} - 2e^{-x}\frac{dv}{dx} + e^{-x}\frac{d^2v}{dx^2} + (1-\cot x)\left(-ve^{-x} + e^{-x}\frac{dv}{dx}\right) - ve^{-x}\cot x = \sin^2 x.$$

$\Rightarrow \quad \frac{d^2v}{dx^2} + [-2+1-\cot x]\frac{dv}{dx} + [1-(1-\cot x)-\cot x]v = e^x\sin^2 x.$

$\Rightarrow \quad \frac{d^2v}{dx^2} - (1+\cot x)\frac{dv}{dx} + [0]v = \sin x \ \Rightarrow \frac{d^2v}{dx^2} + (2-\cot x)\frac{dv}{dx} = e^x\sin^2 x.$

Let $\frac{dv}{dx} = p$. So above equation becomes

$$\frac{dp}{dx} - (1+\cot x)p = e^x\sin^2 x. \quad \cdots\cdots\cdots (2)$$

It is a linear differential equation of first order in p.

$\therefore$ I.F. $= e^{-\int(1+\cot x)\,dx} = e^{-x}e^{-\log\sin x} = \frac{e^{-x}}{\sin x}$.

Solution of equation (2) is given by

$p\cdot\frac{e^{-x}}{\sin x} = \int \sin x\,dx + c_1$, where c_1 is an arbitrary constant.

$\Rightarrow \quad p\cdot\frac{e^{-x}}{\sin x} = -\cos x + c_1$

$\Rightarrow \quad p = \frac{dv}{dx} = -e^x\sin x\cos x + c_1e^x\sin x = -\frac{1}{2}e^x\sin 2x + c_1e^x\sin x.$

Integrating it w.r.t. x, we get

$v = -\frac{1}{2}\frac{e^x}{5}(\sin 2x - 2\cos x) + c_1\frac{e^x}{5}(\sin x - \cos x) + c_2$, c_2 is an arb. const.

Hence the complete solution of the given differential equation is

$$\boxed{y = -\frac{1}{10}(\sin 2x - 2\cos 2x) - \frac{c_1}{5}(\sin x - \cos x) + c_2e^{-x}}$$ Ans.

Example 4. Solve $(x+2)\frac{d^2y}{dx^2}-(2x+5)\frac{dy}{dx}+2y=(x+1)e^x$.

Solution: Try yourself.

Example 5. Solve $x\frac{d^2y}{dx^2}-(2x-1)\frac{dy}{dx}+(x-1)y=0$.

Solution: Try yourself.

Example 6. Solve $x^2y_2+xy_1-y=0$, given that $x+\frac{1}{x}$ is one integral.

Solution: Try yourself.

Example 7. Solve $(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-a^2y=0$,

given that $y=e^{a\sin^{-1}x}$ is a solution.

Solution: Try yourself.

Example 8. Solve $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-9y=0$.

Solution: Try yourself.

LECTURE NO.:- 30

Change of Dependent Variable-II

Reduction to Normal Form by Removal of First Order Derivative

Linear ODE of second order with variable coefficients in standard form is

$$\frac{d^2y}{dx^2} + \mathrm{P}\frac{dy}{dx} + \mathrm{Q}\,y = \mathrm{R}\,, \quad \cdots\cdots (1)$$

where P, Q and R may be functions of x only.

Let $y = uv$ is the complete solution of equation (1).

Putting $y = uv$, $\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$ and $\frac{d^2y}{dx^2} = \frac{d^2u}{dx^2}v + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}$

in equation (1), we get

$$\left(\frac{d^2u}{dx^2}v + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}\right) + \mathrm{P}\left(\frac{du}{dx}v + u\frac{dv}{dx}\right) + \mathrm{Q}\,uv = \mathrm{R}$$

$$\Rightarrow \quad u\frac{d^2v}{dx^2} + \left(2\frac{du}{dx} + \mathrm{P}u\right)\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + \mathrm{P}\frac{du}{dx} + \mathrm{Q}\,u\right)\,v = \mathrm{R}. \quad \cdots\cdots (2)$$

In this method, u is taken such that the first order derivative $\frac{dv}{dx}$ is removed in equation (2), so we get

$$2\frac{du}{dx} + \mathrm{P}u = 0$$

$$\Rightarrow \quad u = e^{-\frac{1}{2}\int \mathrm{P}dx}. \quad \cdots\cdots (3)$$

By taking above value of u, equation (2) becomes

$$u\frac{d^2v}{dx^2} + \left(\mathrm{Q} - \frac{1}{2}\frac{d\mathrm{P}}{dx} - \frac{1}{4}\mathrm{P}^2\right)uv = \mathrm{R}$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} + \mathrm{I}v = \mathrm{S}, \quad \cdots\cdots (4)$$

where $\mathrm{I} = \mathrm{Q} - \frac{1}{2}\frac{d\mathrm{P}}{dx} - \frac{1}{4}\mathrm{P}^2$ and $\mathrm{S} = \frac{\mathrm{R}}{u}$. $\quad \cdots\cdots (5)$

Equation (4), which is called **normal form**, may be solved easily, if I is a constant or a constant divided by x square and hence the complete solution of the equation (1) is obtained by $y = uv$.

Note. If I = constant/x^2, then the equation (4) becomes homogeneous.

Working Rule: Write the values of P, Q and R.

Find $\mathrm{I} = \mathrm{Q} - \frac{1}{2}\frac{d\mathrm{P}}{dx} - \frac{1}{4}\mathrm{P}^2$.

If I = constant or constant/x^2, then this method is applicable otherwise not.

Find $u = e^{-\frac{1}{2}\int \mathrm{P}dx}$ and $\mathrm{S} = \frac{\mathrm{R}}{u}$.

Write normal form $\frac{d^2v}{dx^2} + \mathrm{I}v = \mathrm{S}$ and solve.

Finally $y = uv$ is the complete solution of the given differential equation.

Example 1. Solve $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2}\sin 2x$.

Solution. Given differential equation is

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2}\sin 2x. \quad \cdots\cdots (1)$$

Here $\mathrm{P} = -4x$, $\mathrm{Q} = 4x^2 - 1$ and $\mathrm{R} = -3e^{x^2}\sin 2x$.

So $\mathrm{I} = \mathrm{Q} - \frac{1}{2}\frac{d\mathrm{P}}{dx} - \frac{1}{4}\mathrm{P}^2 = 4x^2 - 1 - \frac{1}{2}(-4) - \frac{1}{4}16x^2 = 1 =$ constant.

Now $u = e^{-\frac{1}{2}\int \mathrm{P}dx} = e^{-\frac{1}{2}\int -4x dx} = u = e^{2\frac{x^2}{2}} = e^{x^2}$

and $\mathrm{S} = \frac{\mathrm{R}}{u} = \frac{-3e^{x^2}\sin 2x}{e^{x^2}} = -3\sin 2x$.

So equation (1) reduces to the normal form

$$\frac{d^2v}{dx^2} + \mathrm{I}v = \mathrm{S}$$

i.e. $\frac{d^2v}{dx^2} + 1v = -3\sin 2x$

$\Rightarrow$ $(D^2 + 1)v = -3\sin 2x$. $\cdots\cdots$ (2)

It is a linear ODE of second order with constant coefficient.

Its A.E. is $m^2 + 1 = 0$.

$\Rightarrow$ $m^2 = -1 \Rightarrow m^2 = i^2 \Rightarrow m = \pm i = 0 \pm i1$.

$\therefore$ C.F. $= c_1\cos 1x + c_2\sin 1x$, where c_1 and c_2 are arbitrary constants.

Now P.I. $= \frac{1}{D^2+1}(-3\sin 2x) = -3\frac{1}{D^2+1}\sin 2x$

$= -3\frac{1}{-2^2+1}\sin 2x$, by $\frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax$.

$\Rightarrow$ P.I. $= \sin 2x$.

So the complete solution of equation (2) is

$v = c_1\cos x + c_2\sin x + \sin 2x$.

Now the complete solution of the equation (1) is $y = uv = e^{x^2}v$.

Hence the complete solution of the given differential equation is

$\boxed{y = e^{x^2}(c_1 \cos x + c_2 \sin x + \sin 2x)}$ Ans.

Example 2. Solve $\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 5y = e^x \sec x$.

Solution: Try yourself.

Example 3. Solve $4x^2\frac{d^2y}{dx^2} + 4x^5\frac{dy}{dx} + (x^8 + 6x^4 + 4)y = 0$.

Solution. Given differential equation may be written as

$$\frac{d^2y}{dx^2} + x^3\frac{dy}{dx} + \frac{1}{4}\left(x^6 + 6x^2 + \frac{4}{x^2}\right)y = 0. \quad \cdots\cdots (1)$$

Here $\mathrm{P} = x^3$, $\mathrm{Q} = \frac{1}{4}x^6 + \frac{3}{2}x^2 + \frac{1}{x^2}$ and $\mathrm{R} = 0$.

So $\mathrm{I} = \mathrm{Q} - \frac{1}{2}\frac{d\mathrm{P}}{dx} - \frac{1}{4}\mathrm{P}^2 = \frac{1}{4}x^6 + \frac{3}{2}x^2 + \frac{1}{x^2} - \frac{1}{2}(3x^2) - \frac{1}{4}x^6$

$\Rightarrow \quad \mathrm{I} = \frac{1}{x^2} = \frac{\text{constant}}{x^2}$.

Now $u = e^{-\frac{1}{2}\int \mathrm{P}dx} = e^{-\frac{1}{2}\int x^3 dx} = u = e^{-\frac{x^4}{8}}$

and $\mathrm{S} = \frac{\mathrm{R}}{u} = \frac{0}{u} = 0$.

So equation (1) reduces to the normal form

$$\frac{d^2v}{dx^2} + \frac{1}{x^2}v = 0$$

i.e. $x^2\frac{d^2v}{dx^2} + 1v = 0. \quad \cdots\cdots (2)$

It is a homogeneous linear ODE of second order.

So putting $x = e^z$, i.e. $\log x = z$ and $\frac{d}{dz} = D$, equation (2) becomes

$D(D-1)v + v = 0$.

$\Rightarrow \quad (D^2 - D + 1)v = 0$.

Its A.E. is $m^2 - m + 1 = 0$.

$\Rightarrow \quad m = \frac{+1 \pm \sqrt{1-4}}{2} \Rightarrow m = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$.

$\therefore \quad$ C.F. $= c_1 e^{\frac{1}{2}z}\cos\left(\frac{\sqrt{3}}{2}z + c_2\right)$, where c_1 and c_2 are arbitrary constants.

Here P.I. = 0.

So the complete solution of equation (2) is

$$v = c_1 e^{\frac{1}{2}z}\cos\left(\frac{\sqrt{3}}{2}z + c_2\right) + 0.$$

Putting $e^z = x$ and $z = \log x$, we get

$$v = c_1 x^{\frac{1}{2}}\cos\left(\frac{\sqrt{3}}{2}\log x + c_2\right).$$

Now the complete solution of the equation (1) is $y = uv = e^{-\frac{x^4}{8}}v$.

Hence the complete solution of the given differential equation is

$$\boxed{y = c_1\sqrt{x}e^{-\frac{x^4}{8}}\cos\left(\frac{\sqrt{3}}{2}\log x + c_2\right)}$$ Ans.

Example 4. Solve $\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + (x^2 + 1)y = x^2 + 3x$.

Solution: Try yourself.

Example 5. Solve $\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 5y = 0$.

Solution: Try yourself.

Example 6. Solve $\frac{d^2y}{dx^2} - 2bx\frac{dy}{dx} + b^2x^2y = 0$.

Solution: Try yourself.

Example 7. Solve $\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 3y = 2\sec x$.

Solution: Try yourself.

Example 8. Solve $\frac{d^2y}{dx^2} + x^{-1/3}\frac{dy}{dx} + \left(\frac{1}{4}x^{-2/3} - \frac{1}{6}x^{-4/3} - 6x^{-2}\right)y = 0$.

Solution: Try yourself.

Example 9. Solve $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 3)y = e^{x^2}$.

Solution: Try yourself.

Example 10. Solve $\left(\frac{d^2y}{dx^2} + y\right)\cot x + 2\left(\frac{dy}{dx} + y\tan x\right) = \sec x$.

Solution. Try yourself.

LECTURE NO.:- 31

Method of Change of Independent Variable

A linear ODE of second order with variable coefficients in standard form is

$$\frac{d^2y}{dx^2} + \mathrm{P}\frac{dy}{dx} + \mathrm{Q}\, y = \mathrm{R}\,, \quad \cdots\cdots\cdots (1)$$

where P, Q and R may be functions of x only.

To solve the equation (1), let the independent variable x is changed into other variable, say z by such that $z = f(x)$ so that $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz}\frac{dz}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx} + \frac{dy}{dz}\cdot\frac{d}{dx}\left(\frac{dz}{dx}\right)$$

$$= \frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}\cdot\frac{dz}{dx} + \frac{dy}{dz}\cdot\frac{d^2z}{dx^2} = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2}.$$

Putting these values in equation (1), we have

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\left(\frac{d^2z}{dx^2} + \mathrm{P}\frac{dz}{dx}\right) + \mathrm{Q}\, y = \mathrm{R}.$$

It may be written as

$$\frac{d^2y}{dz^2} + \mathrm{P}_1\frac{dy}{dz} + \mathrm{Q}_1\, y = \mathrm{R}_1\,, \quad \cdots\cdots\cdots (2)$$

where $$\mathrm{P}_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2},\ \mathrm{Q}_1 = \frac{\mathrm{Q}}{\left(\frac{dz}{dx}\right)^2} \text{ and } \mathrm{R}_1 = \frac{\mathrm{R}}{\left(\frac{dz}{dx}\right)^2}. \quad \cdots\cdots\cdots (3)$$

The value $z = f(x)$ may be taken in following two ways:

(i) $z = f(x)$ may be taken in such a way that Q_1 = constant. Then equation (2) may be solved easily, when P_1 = constant.

(ii) $z = f(x)$ may be taken in such a way that $\mathrm{P}_1 = 0$. Then equation (2) becomes normal form and may be solved easily, when P_1 = constant or constant/z^2.

Finally putting $z = f(x)$, the complete solution of equation (1) is obtained.

Note. We can take Q_1 = constant (imperfect square constant of Q) so that

$$\left(\frac{dz}{dx}\right)^2 = \{f'(x)\}^2 \text{ (perfect square constant and whole function of } x \text{ in Q)}$$

$$\Rightarrow \quad \frac{dz}{dx} = f'(x) \Rightarrow z = f(x).$$

Example 1. Solve $\frac{d^2y}{dx^2} + \cot x \frac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0.$

Solution. Given differential equation is

$$\frac{d^2y}{dx^2} + \cot x \frac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0. \quad \cdots\cdots\cdots (1)$$

Here P = $\cot x$, Q = $4 \operatorname{cosec}^2 x$ and R = 0.

Let us take a variable z such that $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = 1$ i.e. $\left(\frac{dz}{dx}\right)^2 = Q$

$\Rightarrow \left(\frac{dz}{dx}\right)^2 = 4 \operatorname{cosec}^2 x \Rightarrow \frac{dz}{dx} = 2 \operatorname{cosec} x \Rightarrow z = 2 \log \tan(x/2).$

$\therefore \quad P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{-2 \operatorname{cosec} x \cot x + \cot x \cdot (2 \operatorname{cosec} x)}{4 \operatorname{cosec}^2 x} = 0$ and $R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{0}{\left(\frac{dz}{dx}\right)^2} = 0.$

Hence the equation (1) is transformed to $\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$

i.e. $\frac{d^2y}{dz^2} + 0\frac{dy}{dz} + 1y = 0$, which is a linear ODE with constant coefficients.

$\Rightarrow \quad (D^2 + 1)y = 0$, where $= \frac{d}{dz}$. $\quad \cdots\cdots\cdots (2)$

Its A.E. is $m^2 + 1 = 0$.

$\Rightarrow \quad m^2 = -1 \Rightarrow m = \pm i = 0 \pm i1.$

$\therefore$ C.F. $= c_1 \cos(1z + c_2)$, where c_1 and c_2 are arbitrary constants.

Here P.I. = 0.

So solution of equation (2) is $y = c_1 \cos(z + c_2) + 0$.

Putting $z = 2 \log \tan(x/2)$, we get

$$\boxed{y = c_1 \cos\{2 \log \tan(x/2) + c_2\}}$$

This is the complete solution of the given differential equation. Ans.

Example 2. Solve $x^6 \frac{d^2y}{dx^2} + 3x^5 \frac{dy}{dx} + a^2 y = \frac{1}{x^2}.$

Solution. Given differential equation may be written as

$$\frac{d^2y}{dx^2} + \frac{3}{x}\frac{dy}{dx} + \frac{a^2}{x^6} y = \frac{1}{x^8}. \quad \cdots\cdots\cdots (1)$$

Here P = $\frac{3}{x}$, Q = $\frac{a^2}{x^6}$ and R = $\frac{1}{x^8}$.

Let us take a variable z such that $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = 1$ i.e. $\left(\frac{dz}{dx}\right)^2 = Q$

$\Rightarrow \left(\frac{dz}{dx}\right)^2 = \frac{a^2}{x^6} \Rightarrow \frac{dz}{dx} = \frac{a}{x^3} \Rightarrow z = \frac{a}{-2x^2}$.

$\therefore$ $P_1 = \frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{-3\frac{a}{x^4}+\frac{3}{x}\cdot\frac{a}{x^3}}{\frac{a^2}{x^6}} = 0$ and $R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{\frac{1}{x^8}}{\frac{a^2}{x^6}} = \frac{1}{a^2x^2} = \frac{-2z}{a^2a}$.

Hence the equation (1) is transformed to $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1\, y = R_1$

i.e. $\frac{d^2y}{dz^2} + 0\frac{dy}{dz} + 1y = \frac{-2z}{a^3}$, which is a linear ODE with constant coefficients.

$\Rightarrow$ $(D^2 + 1)y = \frac{-2z}{a^3}$, where = $\frac{d}{dz}$. ………………………………… (2)

Its A.E. is $m^2 + 1 = 0$.

$\Rightarrow$ $m^2 = -1 \Rightarrow m = \pm i = 0 \pm i1$.

$\therefore$ C.F. = $c_1 \cos 1z + c_2 \sin 1z$, where c_1 and c_2 are arbitrary constants.

P.I. = $\frac{1}{D^2+1}\frac{-2z}{a^3} = -\frac{2}{a^3}\frac{1}{D^2+1}z = -\frac{2}{a^3}(1+D^2)^{-1}z = -\frac{2}{a^3}(1 - D^2 + \cdots)z$

$= -\frac{2}{a^3}(1z - D^2z + \cdots) = -\frac{2}{a^3}(z - 0 + 0) = -\frac{2z}{a^3}$.

So solution of equation (2) is $y = c_1 \cos z + c_2 \sin z - \frac{2z}{a^3}$.

Putting $z = -\frac{a}{2x^2}$, we get

$$\boxed{y = c_1 \cos\left(\frac{a}{2x^2}\right) - c_2 \sin\left(\frac{a}{2x^2}\right) + \frac{1}{a^2x^2}}$$

This is the complete solution of the given differential equation. <u>Ans.</u>

LECTURE NO.:- 32

Problems of Change of Independent Variable

Example 3. Solve $y_2 + (3\sin x - \cot x)y_1 + 2y\sin^2 x = \sin^2 x \cdot e^{-\cos x}$.

Solution. Given differential equation is

$$y_2 + (3\sin x - \cot x)y_1 + 2y\sin^2 x = \sin^2 x \cdot e^{-\cos x}. \quad \cdots\cdots\cdots (1)$$

Here P = $3\sin x - \cot x$, Q = $2\sin^2 x$ and R = $\sin^2 x \cdot e^{-\cos x}$.

Let us take a variable z such that $Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = 2$ i.e. $\left(\frac{dz}{dx}\right)^2 = \frac{Q}{2}$

$\Rightarrow$ $\left(\frac{dz}{dx}\right)^2 = \sin^2 x \Rightarrow \frac{dz}{dx} = \sin x \Rightarrow z = -\cos x$.

$\therefore$ $P_1 = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{\cos x + (3\sin x - \cot x)\sin x}{\sin^2 x} = \frac{3\sin^2 x}{\sin^2 x} = 3$

and $R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{\sin^2 x \cdot e^{-\cos x}}{\sin^2 x} = e^{-\cos x} = e^z$.

Hence the equation (1) is transformed to $\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1\, y = R_1$

i.e. $\frac{d^2y}{dz^2} + 3\frac{dy}{dz} + 2y = e^z$, which is a linear ODE with constant coefficients.

$\Rightarrow$ $(D^2 + 3D + 2)y = e^z$, where $= \frac{d}{dz}$. $\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (2)$

Its A.E. is $m^2 + 3m + 2 = 0$.

$\Rightarrow$ $(m+1)(m+2) = 0 \Rightarrow m = -1, -2$.

$\therefore$ C.F. = $c_1e^{-1z} + c_2e^{-2z}$, where c_1 and c_2 are arbitrary constants.

P.I. = $\frac{1}{D^2+3D+2}e^{1z} = \frac{1}{1^2+3\times1+2}e^{1z} = \frac{1}{6}e^z$.

So solution of equation (2) is $y = c_1e^{-1z} + c_2e^{-2z} + \frac{1}{6}e^z$.

Putting $z = -\cos x$, we get

$$\boxed{y = c_1e^{\cos x} + c_2e^{2\cos x} + \frac{1}{6}e^{-\cos x}.}$$

This is the complete solution of the given differential equation. Ans.

Example 4. Solve $x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 4x^3 \sin x^2$.

Hint. Take $Q_1 = -1$ so that $\left(\frac{dz}{dx}\right)^2 = 4x^2$, $z = x^2$ etc., then try yourself.

Example 5. Solve $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \frac{a^2}{x^4}y = 0$.

Solution. Try yourself.

Example 6. Solve $\cos x \frac{d^2y}{dx^2} + \sin x \frac{dy}{dx} - 2y\cos^3 x = 2\cos^5 x$.

Solution. Try yourself.

Example 7. Solve $(1+x^2)^2 \frac{d^2y}{dx^2} + 2x(1+x^2)\frac{dy}{dx} + 4y = 0$.

Solution. Try yourself.

Example 8. Solve $\frac{d^2y}{dx^2} + (\tan x - 3\cos x)\frac{dy}{dx} + 2y\cos^2 x = \cos^4 x$.

Solution. Try yourself.

Example 9. Solve $x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4x^3y = 8x^3 \sin x^2$.

Solution. Try yourself.

LECTURE NO.:- 33

Method of Variation of Parameters

A linear ODE of second order with variable coefficients in standard form is

$$\frac{d^2y}{dx^2} + \mathrm{P}\frac{dy}{dx} + \mathrm{Q}\,y = \mathrm{R}\,, \quad \cdots\cdots\cdots \quad (1)$$

where P, Q and R may be functions of x only.

For C.F. of the equation (1), we solve the following equation:

$$\frac{d^2y}{dx^2} + \mathrm{P}\frac{dy}{dx} + \mathrm{Q}\,y = 0. \quad \cdots\cdots\cdots \quad (2)$$

Solution of (2), i.e. C.F. of (1) will be obtained in the following form:

y = C.F. = $c_1u + c_2v$, where c_1 and c_2 are arbitrary constants.

Let the complete solution of the equation (1) is $y = \mathrm{A}u + \mathrm{B}v$, $\cdots\cdots$ (3)

where A and B are parameters.

For variation of parameters, we form following two equations:

$$\frac{d\mathrm{A}}{dx}u + \frac{d\mathrm{B}}{dx}v = 0 \quad \cdots\cdots\cdots \quad (4)$$

& $$\frac{d\mathrm{A}}{dx}\frac{du}{dx} + \frac{d\mathrm{B}}{dx}\frac{dv}{dx} = \mathrm{R}. \quad \cdots\cdots\cdots \quad (5)$$

Solving equations (4) and (5) simultaneously, the values of $\frac{dA}{dx}$ and $\frac{dB}{dx}$ are obtained. Integrating these w.r.t. x, the values of A and B are obtained. Putting these values in (3), the complete solution of the equation (1) is obtained.

Notes. (i) If equation (1) has constant coefficients, then find its C.F. by A.E.

(ii) If equation (1) is homogeneous, then find its C.F. by putting $x = e^z$ etc.

(iii) If one part u of C.F. of equation (1) is known, then find the other part of

C.F. by $v = u\int\left(\frac{1}{u^2}e^{-\int \mathrm{P}dx}\right)dx$.

(iv) If I = constant or constant/x^2, then reduce equation (1) to normal form.

Example 1. Solve $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$ by the method of variation of parameters.

Solution. Given differential equation is

$\frac{d^2y}{dx^2} + 4y = 4\tan 2x.$ ………………………………………… (1)

For C.F. of the equation (1), we solve the following equation:

$\frac{d^2y}{dx^2} + 4y = 0.$ ………………………………………… (2)

It is a linear ODE with constant coefficients.

$\Rightarrow$ $(D^2 + 4)y = 0$, where $= \frac{d}{dx}$.

Its A.E. is $m^2 + 4 = 0$.

$\Rightarrow$ $m^2 = -4 \Rightarrow m = \pm i2 = 0 \pm i2.$

$\therefore$ C.F. $= c_1 \cos 2x + c_2 \sin 2x$, where c_1 and c_2 are arbitrary constants.

Let the complete solution of the equation (1) is

$y = \mathrm{A}\cos 2x + \mathrm{B}\sin 2x,$ ………………………………………… (3)

where A and B are parameters.

For variation of parameters, we form following two equations:

$\frac{d\mathrm{A}}{dx}\cos 2x + \frac{d\mathrm{B}}{dx}\sin 2x = 0$ ………………………………………… (4)

& $\frac{d\mathrm{A}}{dx}(-2\sin 2x) + \frac{d\mathrm{B}}{dx}(2\cos 2x) = 4\tan 2x.$

i.e. $-\frac{d\mathrm{A}}{dx}\sin 2x + \frac{d\mathrm{B}}{dx}\cos 2x = 2\tan 2x.$ ………………………… (5)

Multiplying equation (4) by $\cos 2x$, equation (5) by $\sin 2x$ and subtracting, we get

$\frac{d\mathrm{A}}{dx} = -2\sin 2x \tan 2x.$ ………………………………………… (6)

Putting it in equation (4), we get

$\frac{d\mathrm{B}}{dx} = 2\cos 2x \tan 2x = 2\sin 2x.$ ………………………………… (7)

Integrating equations (6) and (7) w.r.t. x, we get

$\mathrm{A} = -2\int \frac{\sin^2 2x}{\cos 2x}dx + c_1 = -2\int \frac{1-\cos^2 2x}{\cos 2x}dx + c_1$

$= -2\int(\sec 2x - \cos 2x)dx + c_1$

$= -\frac{2}{2}\{\log(\sec 2x + \tan 2x) - \sin 2x\} + c_1$

$= \sin 2x - \log(\sec 2x + \tan 2x) + c_1$

and $B = -\cos 2x + c_2$, where c_1 and c_2 are arbitrary constants.

Putting the values of A and B in equation (3), we get

$y = \{\sin 2x - \log(\sec 2x + \tan 2x) + c_1\}\cos 2x + \{-\cos 2x + c_2\}\sin 2x$

$\Rightarrow \boxed{y = c_1 \cos 2x + c_2 \sin 2x - \cos 2x \cdot \log(\sec 2x + \tan 2x)}$

This is the complete solution of the given differential equation. Ans.

Example 2. Solve $(1-x)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = (1-x)^2$

by the method of variation of parameters.

Solution. Given differential equation may be written as

$$\frac{d^2y}{dx^2} + \frac{x}{1-x}\frac{dy}{dx} - \frac{1}{1-x}y = 1 - x. \quad \cdots\cdots (1)$$

For C.F. of the equation (1), we solve the following equation:

$$\frac{d^2y}{dx^2} + \frac{x}{1-x}\frac{dy}{dx} - \frac{1}{1-x}y = 0. \quad \cdots\cdots (2)$$

It has variable coefficients. Here $P = \frac{x}{1-x}$, $Q = -\frac{1}{1-x}$. We have

$P + Qx = 0$. So $u = x$ is a part of C.F.

Also $1 + P + Q = 0$. So $v = e^x$ is other part of C.F.

$\therefore$ C.F. $= c_1 u + c_2 v$

i.e. C.F. $= c_1 x + c_2 e^x$, where c_1 and c_2 are arbitrary constants.

Let the complete solution of the equation (1) is

$$y = A\,x + B\,e^x, \quad \cdots\cdots (3)$$

where A and B are parameters.

For variation of parameters, we form following two equations:

$$\frac{dA}{dx}\,x + \frac{dB}{dx}\,e^x = 0 \quad \cdots\cdots (4)$$

& $$\frac{dA}{dx}\,1 + \frac{dB}{dx}\,e^x = 1 - x. \quad \cdots\cdots (5)$$

Subtracting equation (4) from equation (5), we get

$\frac{dA}{dx}(1-x) = 1 - x - 0$

$\Rightarrow \frac{dA}{dx} = 1.$ ………………………………………………………… (6)

Putting it in equation (4), we get

$1x + \frac{dB}{dx}e^x = 0$

$\Rightarrow \frac{dB}{dx} = -xe^{-x}.$ ……………………………………………………… (7)

Integrating equations (6) and (7) w.r.t. x, we get

$A = x + c_1$

and $B = -\int xe^{-x} + c_2$

$= -\left[(x)\left(\frac{e^{-x}}{-1}\right) - (1)\left(\frac{e^{-x}}{+1}\right) + 0\right] + c_2$

$= (x+1)e^{-x} + c_2$, where c_1 and c_2 are arbitrary constants.

Putting the values of A and B in equation (3), we get

$y = (x + c_1)x + \{(x+1)e^{-x} + c_2\}e^x$

$\Rightarrow y = c_1x + c_2e^x + x^2 + x + 1$

or $\boxed{y = (c_1+1)x + c_2e^x + x^2 + 1}$

This is the complete solution of the given differential equation. Ans.

LECTURE NO.:- 34

Problems on Method of Variation of Parameters

Example 3. Solve $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=x^2e^x$

by the method of variation of parameters.

Solution. Given differential equation may be written as

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y=e^x. \quad \cdots\cdots (1)$$

For C.F. of the equation (1), we solve the following equation:

$$\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}-\frac{1}{x^2}y=0. \quad \cdots\cdots (2)$$

It has variable coefficients. Here $\mathrm{P}=\frac{1}{x}$, $\mathrm{Q}=-\frac{1}{x^2}$. So we have $\mathrm{P}+\mathrm{Q}x=0$.

Thus $u=x$ is a part of C.F. and the other part of C.F. is

$$v=u\int\left(\frac{1}{u^2}e^{-\int \mathrm{P}dx}\right)dx=x\int\left(\frac{1}{x^2}e^{-\int\frac{1}{x}dx}\right)dx=x\int\left(\frac{1}{x^2}e^{-\log x}\right)dx$$

$$=x\int\left(\frac{1}{x^2}e^{\log x^{-1}}\right)dx=x\int\left(\frac{1}{x^2}x^{-1}\right)dx=x\int\left(\frac{1}{x^3}\right)dx=x\left(\frac{1}{-2x^2}\right)$$

$$=-\frac{1}{2}\frac{1}{x}$$

$\therefore$ C.F. $=c_1x+c_2\frac{1}{x}$, where c_1 and c_2 are arbitrary constants.

Let the complete solution of the equation (1) is

$$y=\mathrm{A}x+\mathrm{B}\frac{1}{x}, \quad \cdots\cdots (3)$$

where A and B are parameters.

For variation of parameters, we form following two equations:

$$\frac{d\mathrm{A}}{dx}x+\frac{d\mathrm{B}}{dx}\frac{1}{x}=0 \quad \cdots\cdots (4)$$

& $$\frac{d\mathrm{A}}{dx}1+\frac{d\mathrm{B}}{dx}\left(-\frac{1}{x^2}\right)=\mathrm{e}^x. \quad \cdots\cdots (5)$$

Multiplying equation (5) by x and adding with equation (4), we get

$$\frac{d\mathrm{A}}{dx}=\frac{1}{2}\mathrm{e}^x. \quad \cdots\cdots (6)$$

Putting it in equation (4), we get

$\frac{1}{2}\mathrm{e}^{x}\, x + \frac{dB}{dx}\frac{1}{x} = 0$

$\Rightarrow \quad \frac{dB}{dx} = -\frac{1}{2}\mathrm{e}^{x}x^{2}.$ ………………………………………………… (7)

Integrating equations (6) and (7) w.r.t. x, we get

$A = \frac{1}{2}e^{x} + c_1$

and $B = -\frac{1}{2}\int x^2 e^x + c_2$

$= -\frac{1}{2}[(x^2)(e^x) - (2x)(e^x) + (2)(e^x) - 0] + c_2$

$= -\frac{1}{2}(x^2 - 2x + 2)e^x + c_2$, where c_1 and c_2 are arbitrary constants.

Putting the values of A and B in equation (3), we get

$y = \left(\frac{1}{2}e^x + c_1\right)x + \left\{\left(-\frac{1}{2}x^2 + x - 1\right)e^x + c_2\right\}\frac{1}{x}$

$\Rightarrow \quad \boxed{y = c_1 x + \frac{c_2}{x} + \left(1 - \frac{1}{x}\right)e^x}$

This is the complete solution of the given differential equation. Ans.

Example 4. Solve by the method of variation of parameters

$$(x+2)\frac{d^2y}{dx^2} - (2x+5)\frac{dy}{dx} + 2y = (x+1)e^x.$$

Solution. Given differential equation may be written as

$\frac{d^2y}{dx^2} - \frac{(2x+5)}{x+2}\frac{dy}{dx} + \frac{2}{x+2}y = \frac{(x+1)}{x+2}e^x.$ ………………………………… (1)

For C.F. of the equation (1), we take the equation

$\frac{d^2y}{dx^2} - \frac{(2x+5)}{x+2}\frac{dy}{dx} + \frac{2}{x+2}y = 0.$ ……………………………………… (2)

It has variable coefficients. Here $P = -\frac{(2x+5)}{x+2}$, $Q = \frac{2}{x+2}$. So we have

$1 + \frac{P}{a} + \frac{Q}{a^2} = 1 + \frac{P}{2} + \frac{Q}{2^2} = 1 + \frac{P}{2} + \frac{Q}{4}$

$= 1 + -\frac{(2x+5)}{2(x+2)} + \frac{1}{2(x+2)} = \frac{1}{2(x+2)}[2x + 4 - 2x - 5 + 1]$

$= 0.$

Thus $u = e^{ax} = e^{2x}$ is a part of C.F. and the other part of C.F. is

$v = u \int \left(\frac{1}{u^2} e^{-\int \mathrm{P}dx}\right) dx.$

Here $\int \mathrm{P}dx = \int \frac{(2x+5)}{x+2} dx = \int \frac{2(x+2)+1}{x+2} dx = \int \left(2 + \frac{1}{x+2}\right) dx = 2x + \log(x+2)$

$\Rightarrow \quad e^{-\int \mathrm{P}dx} = e^{-2x} e^{\log(x+2)} = e^{-2x}(x+2).$

$\Rightarrow \quad v = e^{2x} \int \left(\frac{1}{e^{4x}} e^{2x}(x+2)\right) dx = e^{2x} \int (x+2) e^{-2x} dx$

$= e^{2x}\left[(x+2)\left(\frac{e^{-2x}}{-2}\right) - (1)\left(\frac{e^{-2x}}{+2\times 2}\right) + 0\right] = -\frac{1}{4}[2(x+2)+1]$

$= -\frac{1}{4}(2x+5).$

$\therefore$ C.F. $= c_1 e^{2x} + c_2\,(2x+5)$, where c_1 and c_2 are arbitrary constants.

Let the complete solution of the equation (1) is

$y = \mathrm{A}e^{2x} + \mathrm{B}(2x+5)$, ………………………………………………… (3)

where A and B are parameters.

For variation of parameters, we form following two equations:

$\frac{d\mathrm{A}}{dx}\, e^{2x} + \frac{d\mathrm{B}}{dx}(2x+5) = 0$ ………………………………………………… (4)

& $\frac{d\mathrm{A}}{dx}\, 2e^{2x} + \frac{d\mathrm{B}}{dx}(2) = \frac{(x+1)}{x+2} \mathrm{e}^{x}.$ ……………………………………………… (5)

Multiplying equation (4) by 2 and subtracting from equation (5), we get

$\frac{d\mathrm{B}}{dx}(2 - 4x - 10) = \frac{(x+1)}{x+2} \mathrm{e}^{x} - 0 \Rightarrow -\frac{d\mathrm{B}}{dx}(4x+8) = \frac{(x+1)}{x+2} \mathrm{e}^{x} - 0$

$\Rightarrow \quad \frac{d\mathrm{B}}{dx} = -\frac{(x+1)}{4(x+2)^2} \mathrm{e}^{x}.$ ……………………………………………………… (6)

Putting it in equation (4), we get

$\frac{d\mathrm{A}}{dx}\, e^{2x} - \frac{(x+1)}{4(x+2)^2} e^{x}(2x+5) = 0$

$\Rightarrow \quad \frac{d\mathrm{A}}{dx} = \frac{(x+1)(2x+5)}{4(x+2)^2} \mathrm{e}^{-x}.$ ……………………………………………… (7)

Integrating equations (7) and (6) w.r.t. x, we get

$\mathrm{A} = \frac{1}{4} \int \frac{(x+1)(2x+5)}{(x+2)^2} \mathrm{e}^{-x} dx + c_1 = \frac{1}{4} \int \left[2 - \frac{1}{x+2} - \frac{1}{(x+2)^2}\right] e^{-x} dx + c_1$

$= \frac{1}{4}\left[2\frac{e^{-x}}{-1} - \int \frac{1}{x+2} e^{-x} dx - \int \frac{1}{(x+2)^2} e^{-x} dx\right] + c_1$

$$= \frac{1}{4}\left[-2e^{-x} - \frac{1}{x+2}\left(\frac{e^{-x}}{-1}\right) + \int -\frac{1}{(x+2)^2}\left(\frac{e^{-x}}{-1}\right)dx - \int \frac{1}{(x+2)^2}e^{-x}dx\right] + c_1$$

$$= \frac{1}{4}\left[-2e^{-x} + \frac{1}{x+2}e^{-x}\right] + c_1 = \frac{1}{4}\left(-2 + \frac{1}{x+2}\right)e^{-x} + c_1$$

and $B = -\frac{1}{4}\int \frac{(x+1)}{(x+2)^2}e^{x}dx + c_2 = -\frac{1}{4}\int\left[\frac{1}{x+2} - \frac{1}{(x+2)^2}\right]e^{x}dx + c_2$

$$= -\frac{1}{4}\left[\int \frac{1}{x+2}e^{x}dx - \int \frac{1}{(x+2)^2}e^{x}dx\right] + c_2$$

$$= -\frac{1}{4}\left[\frac{1}{x+2}(e^{x}) - \int -\frac{1}{(x+2)^2}(e^{x})dx - \int \frac{1}{(x+2)^2}e^{x}dx\right] + c_2$$

$= -\frac{1}{4}\left(\frac{1}{x+2}e^{x}\right) + c_2$, where c_1 and c_2 are arbitrary constants.

Putting the values of A and B in equation (3), we get

$$y = \left\{\frac{1}{4}\left(-2 + \frac{1}{x+2}\right)e^{-x} + c_1\right\}e^{2x} + \left\{-\frac{1}{4}\left(\frac{1}{x+2}e^{x}\right) + c_2\right\}(2x+5)$$

$$\Rightarrow \quad y = c_1e^{2x} + c_2(2x+5) + \frac{1}{4}\left(-2 + \frac{1}{x+2} - \frac{2x+5}{x+2}\right)e^{x}.$$

$$\Rightarrow \quad \boxed{y = c_1e^{2x} + c_2(2x+5) - e^{x}}$$

This is the complete solution of the given differential equation. Ans.

Extra Examples for Practice:

Example 5. Solve $\frac{d^2y}{dx^2} + a^2y = \sec ax$.

Example 6. Solve $x^2\frac{d^2y}{dx^2} - 2x(1+x)\frac{dy}{dx} + 2(1+x)y = x^3$.

Example 7. Solve $\frac{d^2y}{dx^2} - y = \frac{2}{1+e^{x}}$.

Example 8. Solve $\frac{d^2y}{dx^2} + (1-\cot x)\frac{dy}{dx} - y\cot x = \sin^2 x$.

Example 9. Solve $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$.

Example 10. Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^{x}\sin x$.

Solution. Try yourself.