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UNIT-3

DIFFERENTIAL EQUATIONS-I

LECTURE NO.:- 15

Differential Equations of First Order and First Degree

Linear Form

A differential equation is called a linear differential equation, if the dependent variable and its derivative occur in the first degree only.

The differential equation of the form

$$\frac{dy}{dx} + \mathrm{P}\,y = \mathrm{Q} \quad \cdots\cdots\cdots\cdots (1)$$

is called linear differential equation of first order in y, provided that P and Q are functions of x only.

To solve equation (1), an integrating factor (I.F.) is taken as

I.F. $= e^{\int \mathrm{P}dx}$.

Solution of equation (1) is given by

$y\cdot(\text{I.F.}) = \int \mathrm{Q}\cdot(\text{I.F.})dx + \mathrm{c}$,

where c is an arbitrary constant.

Similarly the differential equation of the form

$$\frac{dx}{dy} + \mathrm{P}\,x = \mathrm{Q} \quad \cdots\cdots\cdots\cdots (2)$$

is called linear differential equation of first order in x, provided that P and Q are functions of y only.

To solve equation (2), an integrating factor (I.F.) is taken as

I.F. $= e^{\int \mathrm{P}dy}$.

Solution of equation (2) is given by

$x\cdot(\text{I.F.}) = \int \mathrm{Q}\cdot(\text{I.F.})dy + \mathrm{c}$,

where c is an arbitrary constant.

Example 1. Solve $\cos^2 x \frac{dy}{dx} + y = \tan x$.

Solution. Given differential equation is

$$\cos^2 x \frac{dy}{dx} + y = \tan x.$$

It can be written in the following standard form:

$$\frac{dy}{dx} + (\sec^2 x)y = \sec^2 x \tan x. \quad \cdots\cdots\cdots (1)$$

It is a linear differential equation in y. Here P = $\sec^2 x$ and Q = $\sec^2 x \tan x$.

$\therefore$ I.F. $= e^{\int \mathrm{P} dx} = e^{\int \sec^2 x \, dx} = e^{\tan x}$.

Solution of equation (1) is given by

$y \cdot$(I.F.) = $\int$ Q$\cdot$(I.F.)dx + c, where c is an arbitrary constant.

$$\Rightarrow y \cdot e^{\tan x} = \int \sec^2 x \tan x \cdot e^{\tan x} dx + \mathrm{c}. \quad \cdots\cdots\cdots (2)$$

Let $\tan x = t$ so that $\sec^2 x \, dx = dt$. Then we have

$$\int \sec^2 x \tan x \cdot e^{\tan x} dx = \int t \cdot e^t \, dt = t \cdot e^t - \int 1 \cdot e^t dt = t \cdot e^t - e^t = (t-1)e^t$$
$$= (\tan x - 1)\, e^{\tan x}.$$

Putting it in equation (2), we get

$\boxed{y \cdot e^{\tan x} = (\tan x - 1)\, e^{\tan x} + \mathrm{c}}$ Ans.

This is the solution of the given differential equation.

Example 2. Solve $x \log x \frac{dy}{dx} + y = 2 \log x$.

Solution. Given differential equation is

$$x \log x \frac{dy}{dx} + y = 2 \log x.$$

It can be written in the following standard form:

$$\frac{dy}{dx} + \frac{1}{x \log x} y = \frac{2}{x}. \quad \cdots\cdots\cdots (1)$$

It is a linear differential equation in y. Here P = $\frac{1}{x \log x}$ and Q = $\frac{2}{x}$.

$\therefore$ I.F. $= e^{\int P dx} = e^{\int \frac{1}{x \log x} dx} = e^{\int \frac{1}{t} dt}$, by putting $\log x = t$ so that $\frac{1}{x} dx = dt$

$= e^{\log t} = t = \log x.$

Solution of equation (1) is given by

$y \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) dx + c$, where c is an arbitrary constant.

$\Rightarrow y \cdot \log x = \int \frac{2}{x} \cdot \log x \, dx + c$

$= \int 2t \, dt + c$, by putting $\log x = t$ so that $\frac{1}{x} dx = dt$

$= t^2 + c$

$= (\log x)^2 + c.$

Hence the solution of the given differential equation is

$\boxed{y \cdot \log x = (\log x)^2 + c.}$ Ans.

Example 3. Solve $(1 + y^2) + \left(x - e^{-\tan^{-1} y}\right) \frac{dy}{dx} = 0.$

Solution. Given differential equation is

$(1 + y^2) + \left(x - e^{-\tan^{-1} y}\right) \frac{dy}{dx} = 0.$

It can be written as $(1 + y^2) \frac{dx}{dy} + \left(x - e^{-\tan^{-1} y}\right) = 0.$

$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{-\tan^{-1} y}}{1+y^2}.$ ………………………………… (1)

It is a linear differential equation in x. Here P $= \frac{1}{1+y^2}$ and Q $= \frac{e^{-\tan^{-1} y}}{1+y^2}$.

$\therefore$ I.F. $= e^{\int P dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}.$

Solution of equation (1) is given by

$x \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) dy + c$, where c is an arbitrary constant.

$\Rightarrow x \cdot e^{\tan^{-1} y} = \int \frac{e^{-\tan^{-1} y}}{1+y^2} \cdot e^{\tan^{-1} y} \, dy + c$

$$= \int \frac{1}{1+y^2}\, dy + c$$

$$= \tan^{-1} y + c.$$

Hence the solution of the given differential equation is

$\boxed{x \cdot e^{\tan^{-1} y} = \tan^{-1} y + c.}$ Ans.

Example 4. Solve $(x + 2y^3)\frac{dy}{dx} = y$.

Solution. Given differential equation is

$$(x + 2y^3)\frac{dy}{dx} = y.$$

It can be written as $(x + 2y^3) = y\frac{dx}{dy}$

$$\Rightarrow \frac{dx}{dy} - \frac{x}{y} = 2y^2. \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (1)$$

It is a linear differential equation in x. Here $P = -\frac{1}{y}$ and $Q = 2y^2$.

$$\therefore \text{I.F.} = e^{\int P dy} = e^{\int -\frac{1}{y} dy} = e^{-\log y} = e^{\log y^{-1}} = y^{-1}.$$

Solution of equation (1) is given by

$x \cdot (\text{I.F.}) = \int Q \cdot (\text{I.F.}) dy + c$, where c is an arbitrary constant.

$$\Rightarrow x \cdot y^{-1} = \int 2y^2 \cdot y^{-1}\, dy + c$$

$$= 2\int y\, dy + c$$

$$= y^2 + c.$$

Hence the solution of the given differential equation is

$\boxed{x = y(y^2 + c).}$ Ans.

LECTURE NO.:- 16

Differential Equations Reducible to Linear Form (Bernoulli's Equations)

The differential equation of the type

$$\frac{dy}{dx} + \mathrm{P}\,y = \mathrm{Q}\,y^n \ (n \neq 1) \quad \cdots\cdots\cdots \quad (1)$$

is called Bernoulli's equation in y, provided that P and Q are functions of x only.

It can be reduced into linear equation as given below:

Dividing equation (1) by y^n, we get

$$y^{-n}\frac{dy}{dx} + \mathrm{P}\,y^{1-n} = \mathrm{Q}. \quad \cdots\cdots\cdots \quad (2)$$

Now let $y^{1-n} = v$. $\quad \cdots\cdots\cdots \quad (3)$

Differentiating (3) w.r.t. x, we get

$(1-n)y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$, i.e. $y^{-n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{dv}{dx}$.

Putting these values in equation (2), we get

$$\frac{1}{(1-n)}\frac{dv}{dx} + \mathrm{P}\,v = \mathrm{Q}.$$

$$\Rightarrow \frac{dv}{dx} + (1-n)\mathrm{P}\,v = (1-n)\mathrm{Q}.$$

This is a linear equation in v and can be solved as usual method.

Remark. Generally a differential equation of the type

$$\frac{dy}{dx} + \mathrm{P}\,f(y) = \mathrm{Q}\,\mathrm{g}(y)$$

can be reduced into linear equation, when P and Q are functions of x only and dividing it by $g(y)$, it becomes

$\mathrm{k}\cdot\phi'(y)\frac{dy}{dx} + \mathrm{P}\,\phi(y) = \mathrm{Q}$, where k is some constant.

By putting $\phi(y) = v$ so that $\phi'(y)\frac{dy}{dx} = \frac{dv}{dx}$, above equation becomes

$$\frac{dv}{dx} + \frac{\mathrm{P}}{\mathrm{k}}\,v = \frac{\mathrm{Q}}{\mathrm{k}}.$$

This is a linear equation in v and can be solved as usual method.

Example 1. Solve $y(2xy + e^x)dx = e^x dy$.

Solution. Given differential equation is

$y(2xy + e^x)dx = e^x dy$.

It can be written as:

$e^x \frac{dy}{dx} = 2xy^2 + ye^x$.

$\Rightarrow \frac{dy}{dx} - y = 2xe^{-x}y^2$.

It is a Bernoulli's equation in *y*. So dividing it by y^2, we get

$y^{-2}\frac{dy}{dx} - y^{-1} = 2xe^{-x}$. .... (1)

Now let $y^{-1} = v$ so that $-1y^{-2}\frac{dy}{dx} = \frac{dv}{dx}$, i.e. $y^{-2}\frac{dy}{dx} = -\frac{dv}{dx}$.

Putting these values in equation (1), we get

$-\frac{dv}{dx} - v = 2xe^{-x}$.

$\Rightarrow \frac{dv}{dx} + v = -2xe^{-x}$. .... (2)

It is a linear differential equation in *v*.

Here P = 1 and Q = $-2xe^{-x}$.

$\therefore$ I.F. $= e^{\int \mathrm{P}dx} = e^{\int 1\, dx} = e^x$.

Solution of equation (2) is given by

$v\cdot(\text{I.F.}) = \int \mathrm{Q}\cdot(\text{I.F.})dx + c$,

i.e. $v\cdot e^x = -2\int xe^{-x}e^{x}\, dx + c$

$= -2\int x\, dx + c$

$= -x^2 + c$, where c is an arbitrary constant.

Putting $v = 1/y$, we get

$\boxed{e^x/y + x^2 = c}$ Ans.

This is the solution of the given differential equation.

Example 2. Solve $\frac{dy}{dx} = \frac{1}{xy(x^2y^2+1)}$.

Solution. Given differential equation is $\frac{dy}{dx} = \frac{1}{xy(x^2y^2+1)}$.

It can be written as: $\frac{dx}{dy} = x^3y^3 + xy$.

$\Rightarrow \frac{dx}{dy} - yx = y^3x^3$.

It is a Bernoulli's equation in x. So dividing it by x^3, we get

$x^{-3}\frac{dx}{dy} - yx^{-2} = y^3$. ………………………………… (1)

Now let $x^{-2} = v$ so that $-2x^{-3}\frac{dx}{dy} = \frac{dv}{dy}$, i.e. $x^{-3}\frac{dx}{dy} = -\frac{1}{2}\frac{dv}{dy}$.

Putting these values in equation (1), we get

$-\frac{1}{2}\frac{dv}{dy} - yv = y^3$.

$\Rightarrow \frac{dv}{dy} + 2yv = -2y^3$. …………………………………… (2)

It is a linear differential equation in v. Here P = $2y$ and Q = $-2y^3$.

$\therefore$ I.F. $= e^{\int P dy} = e^{\int 2y\,dy} = e^{y^2}$.

Solution of equation (2) is given by $v\cdot$(I.F.) $= \int$ Q$\cdot$(I.F.)dy + c,

i.e. $v\cdot e^{y^2} = \int(-2y^3)e^{y^2}dy + c$

$= -\int y^2e^{y^2}2y\,dy + c$

$= -\int t\cdot e^t\,dt + c$, by putting $y^2 = t$ so that $2ydy = dt$

$= -t\cdot e^t + \int 1\cdot e^t dt + c = -t\cdot e^t + e^t + c$

$= (1-t)e^t + c$, where c is an arbitrary constant.

Putting $v = 1/x^2$ and $t = y^2$, we get

$\frac{1}{x^2}e^{y^2} = (1-y^2)e^{y^2} + c$

$\boxed{\left(\frac{1}{x^2} + y^2 - 1\right)e^{y^2} = c}$ <u>Ans.</u>

This is the solution of the given differential equation.

LECTURE NO.:- 17

Miscellaneous Problems

Example 1. Solve the following differential equations:

(i) $(x+1)\frac{dy}{dx} - y = e^x(x+1)^2$

(ii) $x(x-1)\frac{dy}{dx} - (x-2)y = x^2(2x-1)$

(iii) $x^2\frac{dy}{dx} + (1-2x)y = x^2$

(iv) $(1+y^2)dx = (\tan^{-1}y - x)dy$

(v) $(y-x)\frac{dy}{dx} = a^2$

(vi) $y\log y\,\frac{dx}{dy} + x = \log y$

Solution. Try yourself.

Example 2. If $\frac{dy}{dx} + 2y\tan x = \sin x$ and $y = 0$ for $x = \frac{\pi}{3}$,

show that the maximum value of y is 1/8.

Solution. Given differential equation is

$$\frac{dy}{dx} + (2\tan x)y = \sin x \quad \cdots\cdots\cdots (1)$$

with $y = 0$ for $x = \frac{\pi}{3}$. $\cdots\cdots\cdots$ (2)

Equation (1) is a linear ODE in y. Here P = $2\tan x$ and Q = $\sin x$.

$\therefore$ I.F. $= e^{\int P dx} = e^{2\int \tan x\,dx} = e^{2\log\sec x} = e^{\log(\sec x)^2} = \sec^2 x$.

Solution of equation (1) is given by

$y\cdot$(I.F.) $= \int$ Q$\cdot$(I.F.)dx + c

i.e. $y\cdot(\sec^2 x) = \int \sin x\cdot(\sec^2 x)dx + c$

$$= \int \frac{\sin x}{\cos x}\cdot(\sec x)dx + c = \int \sec x\cdot\tan x\cdot dx + c$$

$\Rightarrow$ $y\cdot(\sec^2 x) = \sec x + c$, where c is an arbitrary constant.

$\Rightarrow$ $y = \cos x + c\cos^2 x$. $\cdots\cdots\cdots$ (3)

Using condition (2) in equation (3), we get

$$0 = \cos\frac{\pi}{3} + c\left(\cos\frac{\pi}{3}\right)^2 \Rightarrow 0 = \frac{1}{2} + c\left(\frac{1}{2}\right)^2 \Rightarrow 0 = 2 + c \Rightarrow c = -2.$$

Putting the value of c in (3), we get

$$y = \cos x - 2\cos^2 x. \quad \cdots\cdots\cdots (4)$$

For maximum value of y, differentiating (4) w.r.t. x, we get

$$\frac{dy}{dx} = -\sin x - 2\times 2\cos x\,(-\sin x) = -\sin x\,(1 - 4\cos x) = 0 \quad \cdots (5)$$

and $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(-\sin x + 2\sin 2x) = -\cos x + 2\times 2\cos 2x = -\cos x + 4\cos 2x$

i.e. $\dfrac{d^2y}{dx^2} = -\sqrt{1-\sin^2 x} + 4(1 - 2\sin^2 x) \quad \cdots\cdots\cdots (6)$

or $\dfrac{d^2y}{dx^2} = -\cos x + 4(2\cos^2 x - 1). \quad \cdots\cdots\cdots (7)$

From (5), $\sin x = 0$ and $\cos x = ¼$.

Putting $\sin x = 0$ in (6), we get

$\Rightarrow \quad \dfrac{d^2y}{dx^2} = -\sqrt{1-0} + 4(1-0) = 3 > 0.$

Putting $\cos x = ¼$ in (7), we get

or $\dfrac{d^2y}{dx^2} = -\dfrac{1}{4} + 4\left(\dfrac{2}{16} - 1\right) = -\dfrac{15}{4} < 0.$

So y is maximum, when $\cos x = ¼$ and then from (4), we have

the maximum value of $y = \dfrac{1}{4} - 2\times\dfrac{1}{16} = \dfrac{1}{8}.$

Hence the maximum value of y is 1/8. <u>Proved.</u>

Example 3. Solve $\sin y\dfrac{dy}{dx} = \cos y\ (1 - x\cos y).$

Solution. Given differential equation is $\sin y\dfrac{dy}{dx} = \cos y\ (1 - x\cos y).$

It can be written as: $\sin y\dfrac{dy}{dx} - \cos y = -x\cos^2 y.$

Dividing above equation by $\cos^2 y$, we get

$\sec y \tan y \frac{dy}{dx} - \sec y = -x.$ ………………… (1)

Now let $\sec y = v$ so that $\sec y \tan y \frac{dy}{dx} = \frac{dv}{dx}$.

Putting these in (1), we get

$\frac{dv}{dx} - v = -x.$ …………………………………… (2)

It is a linear differential equation in v. Here P = – 1 and Q = $-x$.

$\therefore$ I.F. = $e^{\int P dx} = e^{-\int dx} = e^{-x}$.

Solution of equation (2) is given by

$v \cdot$(I.F.) = $\int$ Q$\cdot$(I.F.)dx + c, where c is an arbitrary constant.

$\Rightarrow v \cdot e^{-x} = -\int x \cdot e^{-x}\, dx + c$

$= -x \cdot \left(\frac{e^{-x}}{-1}\right) + 1 \cdot \left(\frac{e^{-x}}{+1}\right) + c. = (x+1)e^{-x} + c.$

$\Rightarrow v = x + 1 + c\, e^{x}$

Putting $v = \sec y$, we get

$\boxed{\sec y = x + 1 + c\, e^{x}}$ Ans.

This is the solution of the given differential equation.

Example 4. Solve the following differential equations:

(i) $3x(1-x^2)y^2\frac{dy}{dx} + (2x^2-1)y^3 = ax^3$

(ii) $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$

(iii) $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$

(iv) $x\frac{dy}{dx} + y\log y = xy\, e^{x}$

(v) $(\sec x \tan x \tan y - e^{x})dx + \sec x \sec^2 y\, dy = 0$

(vi) $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^{x}\sec y$

(vii) $\tan y\frac{dy}{dx} + \tan x = \cos y \cos^2 x$

(viii) $\sec^2 y\frac{dy}{dx} + x\tan y = x^3$

(ix) $\frac{dy}{dx} = e^{x-y}(e^{x} - e^{y})$

Solution. Try yourself.

LECTURE NO.:- 18

Exact Form of Differential Equations of First Order

The differential equation Mdx + Ndy = 0 ································· (1)

is called an exact differential equation, if M = M(x, y) and N = N(x, y) satisfy the condition of exactness $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

In this case, the solution of the differential equation (1) is given by

$$\int_{y\ constant} Mdx + \int \left\{N - \frac{\partial}{\partial y}\int_{y\ constant} Mdx\right\} dy = \text{c}$$

or $$\int_{x\ constant} Ndy + \int \left\{M - \frac{\partial}{\partial x}\int_{x\ constant} Ndy\right\} dx = \text{c},$$

where c is an arbitrary constant.

Remark. Above condition of exactness is necessary and sufficient.

Proof. If the differential equation Mdx + Ndy = 0 is exact, then

Mdx + Ndy = exact differential = $df(x, y)$ (say)

$$= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

$\Rightarrow$ M = $\frac{\partial f}{\partial x}$, N = $\frac{\partial f}{\partial y}$ $\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y\partial x} = \frac{\partial^2 f}{\partial x\partial y} = \frac{\partial N}{\partial x}$. So the condition is necessary.

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then let $\int_{y\ constant} Mdx$ = U(x, y). So M = $\frac{\partial U}{\partial x}$.

Now $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{\partial^2 U}{\partial y\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial y}\right)$. So N = $\frac{\partial U}{\partial y} + \phi(y)$ (an arbitrary function).

$\Rightarrow$ Mdx + Ndy = $\frac{\partial U}{\partial x}dx + \left\{\frac{\partial U}{\partial y} + \phi(y)\right\} dy = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + d\int \phi(y)dy$

$= dU + d\int \phi(y)dy = d[U(x,y) + \int \phi(y)dy]$

= exact differential = $df(x, y)$ (say)

$\Rightarrow$ Mdx + Ndy = 0 is exact differential equation. So the condition is sufficient.

Hence the solution of the differential equation Mdx + Ndy = 0 is given by

$f(x, y)$ =c, i.e. $\int_{y\ constant} Mdx + \int \left\{N - \frac{\partial}{\partial y}\int_{y\ constant} Mdx\right\} dy$ = c etc.

Working Rule: Write given differential equation as Mdx + Ndy = 0. ⋯⋯⋯ (1)

Write M = M(x, y) and N = N(x, y).

Find $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}$. If $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0$, then equation (1) is exact differential equation.

Write the solution of the differential equation (1) as

$$\int_{y\ constant} Mdx + \int\{terms\ of\ N\ free\ from\ x\}dy = \text{c}$$

or $$\int_{x\ constant} Ndy + \int\{terms\ of\ M\ free\ from\ y\}dx = \text{c},$$

where c is an arbitrary constant.

Example 1. Solve $e^y dx + (xe^y + 2y)dy = 0$.

Solution. Given differential equation is

$$e^y dx + (xe^y + 2y)dy = 0. \quad \cdots\cdots\cdots \quad (1)$$

Here M = e^y and N = $xe^y + 2y$.

$\Rightarrow$ $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = e^y - e^y - 0 = 0$.

So the equation (1) is an exact differential equation.

The solution of the differential equation (1) is

$$\int_{y\ constant} Mdx + \int\{terms\ of\ N\ free\ from\ x\}dy = \text{c}$$

i.e. $\int_{y\ constant} e^y dx + \int\{2y\}dy = \text{c}$

i.e. $\boxed{xe^y + y^2 = \text{c, where c is an arbitrary constant}}$ Ans.

Example 2. Solve $\left(3x^2y + \frac{y}{x}\right)dx + (x^3 + \log x)dy = 0$.

Solution. Given differential equation is

$$\left(3x^2y + \frac{y}{x}\right)dx + (x^3 + \log x)dy = 0. \quad \cdots\cdots\cdots \quad (1)$$

Here M = $3x^2y + \frac{y}{x}$ and N = $x^3 + \log x$.

$\Rightarrow$ $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3x^2 + \frac{1}{x} - 3x^2 - \frac{1}{x} -= 0$.

So the equation (1) is an exact differential equation.

The solution of the differential equation (1) is

$\int_{y\ constant} Mdx + \int\{terms\ of\ N\ free\ from\ x\}dy = \mathrm{c}$

i.e. $\int_{y\ constant} \left\{3x^2y + \frac{y}{x}\right\} dx + \int\{0\}dy = \mathrm{c}$

i.e. $x^3y + y\log x + 0 = \mathrm{c}$

i.e. $(x^3 + \log x)y = \mathrm{c}$, where c is an arbitrary constant. Ans.

Example 3. Solve $\left(1 + e^{x/y}\right)dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0.$

Solution. Given differential equation is

$\left(1 + e^{x/y}\right)dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0.$ ······························· (1)

Here $\mathrm{M} = 1 + e^{x/y}$ and $\mathrm{N} = e^{x/y}\left(1 - \frac{x}{y}\right)$.

$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0 + e^{x/y}\left(-\frac{x}{y^2}\right) - e^{x/y}\left(\frac{1}{y}\right)\left(1 - \frac{x}{y}\right) - e^{x/y}\left(0 - \frac{1}{y}\right) = 0.$

So the equation (1) is exact differential equation.

The solution of the differential equation (1) is

$\int_{y\ constant} Mdx + \int\{terms\ of\ N\ free\ from\ x\}dy = \mathrm{c}$

i.e. $\int_{y\ constant}\left(1 + e^{x/y}\right)dx + \int 0dy = \mathrm{c}$

i.e. $x + \frac{e^{x/y}}{1/y} + 0 = \mathrm{c}$

i.e. $x + ye^{x/y} = \mathrm{c}$, where c is an arbitrary constant. Ans.

Example 4. Solve $(\cos x - x\cos y)dy - (\sin y + y\sin x)dx = 0.$

Solution. Given differential equation is

$(\cos x - x\cos y)dy - (\sin y + y\sin x)dx = 0.$ ························· (1)

Here $\mathrm{M} = -(\sin y + y\sin x)$ and $\mathrm{N} = \cos x - x\cos y$.

$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -\cos y - 1\sin x + \sin x + 1\cos y = 0.$

So the equation (1) is an exact differential equation.

The solution of the differential equation (1) is

$\int_{y\ constant} M dx + \int\{terms\ of\ N\ free\ from\ x\}dy = c$

i.e. $\int_{y\ constant}[-(\sin y + y\sin x)]dx + \int 0 dy = c$

i.e. $-x\sin y - y(-\cos x) + 0 = c$

i.e. $y\cos x = x\sin y + c$, where c is an arbitrary constant. Ans.

Example 5. Solve the following differential equations:

(i) $\left(y^2 e^{xy^2} + 4x^3\right)dx + \left(2xye^{xy^2} - 3y^2\right)dy = 0$

(ii) $(2ax + by)ydx + (ax + 2by)xdy = 0$

(iii) $(\sin x \cdot \sin y - xe^y)dy = (e^y + \cos x \cdot \cos y)dx$

(iv) $(x^2 - ay)dx = (ax - y^2)dy$

(v) $(1 + 4xy + 2y^2)dx + (1 + 4xy + 2x^2)dy = 0$

(vi) $x(x^2 + 3y^2)dx + y(y^2 + 3x^2)dy = 0$

(vii) $(e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0$

(viii) $\left\{y\left(1 + \frac{1}{x}\right) + \cos y\right\}dx + (x + \log x - x\sin y)dy = 0$

LECTURE NO.:- 19

Equations Reducible to Exact Form

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the differential equation Mdx + Ndy = 0 is not exact and it can be reduced to exact differential equation by multiplying some suitable function of x and y, which is called an integrating factor (I.F.).

Rules for finding I.F. of the differential equation Mdx + Ndy = 0

(i) If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$, then I.F. = $e^{\int f(x)dx}$.

(ii) If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = F(y)$, then I.F. = $e^{\int F(y)dy}$.

(iii) If Mdx + Ndy = 0 is homogeneous, then I.F. = $\frac{1}{Mx+Ny}$, provided Mx + N$y \neq 0$.

(iv) If the differential equation has the form $f_1(xy)ydx + f_2(xy)xdy = 0$, then I.F. = $\frac{1}{Mx-Ny}$, provided Mx – N$y \neq 0$.

(v) If the equation has the form $x^a y^b(mydx + nxdy) + x^c y^d(pydx + qxdy) = 0$, then I.F. = $x^h y^k$, where h and k are obtained by the condition of exactness.

(vi) Sometimes I.F. can be found by inspection.

Example 1. Solve $ydx - xdy + (1 + x^2)dy + x^2 \sin y\, dy = 0$.

Solution. Given differential equation is

$$ydx - xdy + (1 + x^2)dx + x^2 \sin y\, dy = 0.$$

It may be written as

$$(y + 1 + x^2)dx + (-x + x^2 \sin y)dy = 0. \quad \cdots\cdots (1)$$

Here M = $y + 1 + x^2$ and N = $-x + x^2 \sin y$.

$$\Rightarrow \quad \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 1 + 0 + 0 - (-1 + 2x \sin y) = 2(1 - x \sin y) \neq 0.$$

So the equation (1) is not an exact differential equation.

Now we have

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2(1-x\sin y)}{-x(1-x\sin y)} = -\frac{2}{x} = \text{a function of } x \text{ only} = f(x) \text{ (say)}.$$

$\therefore$ I.F. $= e^{\int f(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$.

Multiplying it in equation (1), we get

$\left(\frac{y+1}{x^2}+1\right)dx + \left(-\frac{1}{x}+\sin y\right)dy = 0.$ ······································ (2)

So the equation (2) is an exact differential equation.

Here M $= \frac{y+1}{x^2}+1$ and N $= -\frac{1}{x}+\sin y$.

The solution of the differential equation (2) is

$\int_{y\ constant} Mdx + \int\{terms\ of\ N\ free\ from\ x\}dy = c$

i.e. $\int_{y\ constant}\left(\frac{y+1}{x^2}+1\right)dx + \int \sin y\, dy = c$

i.e. $-\frac{y+1}{x} + x - \cos y = c$

i.e. $\boxed{x = \frac{y+1}{x} + \cos y + c, \text{ where c is an arbitrary constant.}}$

This is the solution of the given differential equation. Ans.

Remark. By inspection, I.F. $= \frac{1}{x^2}$ so that the given differential equation becomes

$\frac{ydx - xdy}{x^2} + \left(\frac{1}{x^2}+1\right)dx + \sin y\, dy = 0$, which is an exact differential equation

$d\left[-\frac{y}{x} - \frac{1}{x} + x - \cos y\right] = 0$. Its integration gives $-\frac{y+1}{x} + x - \cos y = c$.

Example 2. Solve $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$.

Solution. Given differential equation is

$(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0.$ ································· (1)

Here M $= xy^3 + y$ and N $= 2(x^2y^2 + x + y^4)$.

$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3xy^2 + 1 - 2(2xy^2 + 1 + 0) = -(1 + xy^2) \neq 0.$

So the equation (1) is not an exact differential equation.

Now we have

$\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{+(1+xy^2)}{y(xy^2+1)} = \frac{1}{y}$ = a function of y only = $F(y)$ (say).

$\therefore$ I.F. $= e^{\int F(y)dy} = e^{\int \frac{1}{y}dy} = e^{\log y} = y.$

Multiplying it in equation (1), we get

$(xy^4 + y^2)dx + 2(x^2y^3 + xy + y^5)dy = 0.$ ······························ (2)

So the equation (2) is an exact differential equation.

Here M = $xy^4 + y^2$ and N = $2(x^2y^3 + xy + y^5)$.

The solution of the differential equation (2) is

$\int_{y\ constant} Mdx + \int\{terms\ of\ N\ free\ from\ x\}dy$ = c

i.e. $\int_{y\ constant}(xy^4 + y^2)dx + \int 2y^5dy$ = c

i.e. $\frac{x^2}{2}y^4 + y^2x + 2\frac{y^6}{6}$ = c

i.e. $y^2(3x^2y^2 + 6x + 2y^4)$ = c, where c is an arbitrary constant.

This is the solution of the given differential equation. Ans.

LECTURE NO.:- 20

Problems of Equations Reducible to Exact Form

Example 3. Solve $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$.

Solution. Given differential equation is

$$(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0. \quad \cdots\cdots\cdots (1)$$

Here M = $x^2y - 2xy^2$ and N = $-(x^3 - 3x^2y)$.

$$\Rightarrow \quad \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = x^2 - 4xy + (3x^2 - 6xy) = 2x(2x - 5y) \neq 0.$$

So the equation (1) is not an exact differential equation but it is homogeneous.

$$\therefore \text{ I.F.} = \frac{1}{Mx+Ny} = \frac{1}{(x^2y-2xy^2)x-(x^3-3x^2y)y} = \frac{1}{x^2y^2}.$$

Multiplying it in equation (1), we get

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0. \quad \cdots\cdots\cdots (2)$$

So the equation (2) is an exact differential equation.

Here M = $\frac{1}{y} - \frac{2}{x}$ and N = $-\frac{x}{y^2} + \frac{3}{y}$.

The solution of the differential equation (2) is

$$\int_{y\ constant} Mdx + \int\{terms\ of\ N\ free\ from\ x\}dy = \text{c}$$

i.e. $\int_{y\ constant} \left(\frac{1}{y} - \frac{2}{x}\right)dx + \int \frac{3}{y}dy = \text{c}$

i.e. $\frac{x}{y} - 2\log x + 3\log y = \text{c}$, where c is an arbitrary constant.

This is the solution of the given differential equation. Ans.

Example 4. Solve $(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)dy = 0$.

Solution. Given differential equation is

$$(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)dy = 0. \quad \cdots\cdots\cdots (1)$$

Here M = $xy^2 + 2x^2y^3$ and N = $x^2y - x^3y^2$.

$$\Rightarrow \quad \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2xy + 6x^2y^2 - 2xy + 3x^2y^2 = 9x^2y^2 \neq 0.$$

So the equation (1) is not an exact differential equation. It can be written as

$(xy + 2x^2y^2)ydx + (xy - x^2y^2)xdy = 0$ i.e. $f_1(xy)ydx + f_2(xy)xdy = 0$.

$\therefore$ I.F. $= \frac{1}{Mx - Ny} = \frac{1}{(xy + 2x^2y^2 - xy + x^2y^2)xy} = \frac{1}{3x^3y^3}$.

Multiplying it in equation (1), we get

$$\frac{1}{3}\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \frac{1}{3}\left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0. \quad \cdots\cdots (2)$$

So the equation (2) is an exact differential equation.

Here M $= \frac{1}{3}\left(\frac{1}{x^2y} + \frac{2}{x}\right)$ and N $= \frac{1}{3}\left(\frac{1}{xy^2} - \frac{1}{y}\right)$.

The solution of the differential equation (2) is

$\int_{y\ constant} Mdx + \int \{terms\ of\ N\ free\ from\ x\}dy$ = arbitrary constant

i.e. $\int_{y\ constant} \frac{1}{3}\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \int -\frac{1}{3y}dy$ = c' i.e. $\frac{1}{3}\left(-\frac{1}{xy} + 2\log x - \log y\right)$ = c',

i.e. $-\frac{1}{xy} + \log x^2 - \log y$ = 3c' = log c i.e. $\log\frac{x^2}{cy} = \frac{1}{xy}$

i.e. $x^2 = cye^{\frac{1}{xy}}$, where c is an arbitrary constant.

This is the solution of the given differential equation. Ans.

Example 5. Solve $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$.

Solution. Given differential equation is

$$(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0. \quad \cdots\cdots (1)$$

Here M $= y^2 + 2x^2y$ and N $= 2x^3 - xy$.

$\Rightarrow \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y + 2x^2 - 6x^2 + y = 3y - 4x^2 \neq 0$.

So the equation (1) is not an exact differential equation. It can be written as

$y(ydx - xdy) + x^2(2ydx + 2xdy) = 0$,

which is the form $x^ay^b(mydx + nxdy) + x^cy^d(pydx + qxdy) = 0$.

$\therefore$ I.F. $= x^hy^k$, where h and k are constants to be determined.

Multiplying it in equation (1), we get

$(x^h y^{k+2} + 2x^{h+2}y^{k+1})dx + (2x^{h+3}y^k - x^{h+1}y^{k+1})dy = 0.$ ……… (2)

So the equation (2) will be an exact differential equation, if

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, where M = $x^h y^{k+2} + 2x^{h+2}y^{k+1}$ and N = $2x^{h+3}y^k - x^{h+1}y^{k+1}$.

$\Rightarrow$ $(k+2)x^h y^{k+1} + 2(k+1)x^{h+2}y^k = 2(h+3)x^{h+2}y^k - (h+1)x^h y^{k+1}$

Equating the coefficients of like powers of x and y, we get

$\Rightarrow$ $k + 2 = -(h + 1)$, $2(k + 1) = 2(h + 3) \Rightarrow h + k = -3$, $h - k = -2$

$\Rightarrow$ $h = -5/2$, $k = -½$.

Putting these values in equation (2), we get

$\left(x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2}\right)dx + \left(2x^{1/2}y^{-1/2} - x^{-3/2}y^{1/2}\right)dy = 0$

This is an exact differential equation. Its solution is

$\int_{y\ constant} Mdx + \int\{terms\ of\ N\ free\ from\ x\}dy$ = arbitrary constant

i.e. $\int_{y\ constant}\left(x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2}\right)dx + \int 0dy$ = c'

i.e. $\frac{x^{-3/2}}{-3/2}y^{3/2} + 2\frac{x^{1/2}}{1/2}y^{1/2} + 0$ = c', i.e. $-x^{-3/2}y^{3/2} + 6x^{1/2}y^{1/2}$ = 3c'/2 = c

i.e. $\sqrt{xy}\left(6 - \frac{y}{x^2}\right)$ = c, where c is an arbitrary constant.

This is the solution of the given differential equation. Ans.

Example 6. Solve $(x^2 + y^2 + x)dx - (2x^2 + 2y^2 - y)dy = 0.$

Solution. Given differential equation is

$(x^2 + y^2 + x)dx - (2x^2 + 2y^2 - y)dy = 0.$ ………………………… (1)

Here M = $x^2 + y^2 + x$ and N = $2x^2 + 2y^2 - y$.

$\Rightarrow$ $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0 + 2y + 0 - 4x - 0 + 0 = 2(y - 2x) \neq 0.$

So the equation (1) is not an exact differential equation. It can be written as

$(x^2 + y^2)dx + xdx - 2(x^2 + y^2)dy + ydy = 0$

By inspection, I.F. = $\frac{1}{x^2+y^2}$ so that the given differential equation becomes

$dx - 2dy + \frac{xdx+ydy}{x^2+y^2} = 0$, which is an exact differential equation.

Its integration gives $x - 2y + \frac{1}{2}\int \frac{2xdx+2ydy}{x^2+y^2} = c'$

i.e. $2(x-2y) + \log(x^2+y^2) = c$, where c is an arbitrary constant.

This is the solution of the given differential equation. Ans.

Example 7. Solve the following differential equations:

(i) $(xy\sin xy + \cos xy)ydx + (xy\sin xy - \cos xy)xdy = 0$

(ii) $\left(y + \frac{1}{3}y^3 + \frac{1}{2}x^2\right)dx + \frac{1}{4}(x + xy^2)dy = 0$

(iii) $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)dy = 0$

(iv) $(x^4y^4 + x^2y^2 + xy)ydx + (x^4y^4 - x^2y^2 + xy)xdy = 0$

(v) $(y^3 - 2yx^2)dx + (2xy^2 - x^3)dy = 0$

(vi) $\frac{dy}{dx} = \frac{x^3+y^3}{xy^2}$

(vii) $(2ydx + 3xdy) + 2xy(3ydx + 4xdy) = 0$

Solution. Try yourself.

LECTURE NO.:- 21

Linear ODE of Higher Order with Constant Coefficients

A linear ordinary differential equation of order n with constant coefficients is of the following form:

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = \mathrm{Q}\,, \cdots\cdots\cdots\cdots\cdots\cdots (1)$$

where a_0 ($\neq 0$), a_1, a_2, $\cdots$, a_n are constants and Q is the function of x only.

Equation (1) may be written as

$$a_0 D^n y + a_1 D^{n-1} y + \cdots + a_{n-1} D y + a_n y = \mathrm{Q}\,, \text{ where } D \equiv \frac{d}{dx}.$$

$\Rightarrow$ $(a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n) y = \mathrm{Q}.$

$\Rightarrow$ $f(D) y = \mathrm{Q}$, where $f(D) = a_0 D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n.$

Its complete or general solution is given by

y = C.F. + P.I.,

where C.F. is the **complementary function**, containing n arbitrary constants and P.I. is the **particular integral**, which does not contain any arbitrary constant.

For **C.F.**, an auxiliary equation (**A.E.**): $f(m) = 0$ is taken.

Let the roots of A.E. are $m = m_1, m_2, \cdots , m_n$.

Here following cases may arise:

(i) When all the roots of A.E. are real and distinct:

C.F. = $c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$,

where c_1, c_2, $\cdots$, c_n are arbitrary constants.

(ii) When some roots of A.E. are real and repeated as $m = m_1, m_1, \cdots$:

C.F. = $(c_1 + c_2 x) e^{m_1 x} + \cdots$,

where c_1, c_2, $\cdots$ are arbitrary constants.

(iii) When more roots of A.E. are real and repeated as $m = m_1, m_1, m_1, \cdots$:

C.F. = $(c_1 + c_2x + c_3x^2)e^{m_1x} + \cdots\cdots$,

where c_1, c_2, c_3, $\cdots\cdots$ are arbitrary constants and so on.

(iv) When some roots of A.E. are complex and distinct as $m = \alpha \pm i\beta$, $\cdots\cdots$:

C.F. = $e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x) + \cdots\cdots$

or C.F. = $c_1e^{\alpha x}\cos(\beta x + c_2) + \cdots\cdots$ or C.F. = $c_1e^{\alpha x}\sin(\beta x + c_2) + \cdots\cdots$,

where c_1, c_2, $\cdots\cdots$ are arbitrary constants.

(v) When some roots of A.E. are complex and repeated as $m = \alpha \pm i\beta$, $\alpha \pm i\beta$, $\cdots$:

C.F. = $e^{\alpha x}[(c_1 + c_2x)\cos\beta x + (c_3 + c_4x)\sin\beta x] + \cdots\cdots$,

where c_1, c_2, c_3, c_4, $\cdots\cdots$ are arbitrary constants.

(vi) When some roots of A.E. are real and in surds as $m = \alpha \pm \sqrt{\beta}$, $\cdots\cdots$:

C.F. = $e^{\alpha x}\left(c_1 \cosh\sqrt{\beta}x + c_2 \sinh\sqrt{\beta}x\right) + \cdots\cdots$

or C.F. = $c_1e^{\alpha x}\cosh\left(\sqrt{\beta}x + c_2\right) + \cdots\cdots$

or C.F. = $c_1e^{\alpha x}\sinh\left(\sqrt{\beta}x + c_2\right) + \cdots\cdots$,

where c_1, c_2, $\cdots\cdots$ are arbitrary constants.

P.I. of $f(D)y = Q$ is $y = \frac{1}{f(D)}Q$ i.e. P.I. = $\frac{1}{f(D)}Q$.

Following formulas of P.I. are used in various cases:

(i) General Formula: $\frac{1}{D-\alpha}Q = e^{\alpha x}\int Qe^{-\alpha x}dx$,

where α is a constant and $D - \alpha$ is a linear factor of $f(D)$.

(ii) $\frac{1}{f(D)}K = \frac{1}{f(0)}K$,

where K is a constant and $f(0) \neq 0$.

(iii) $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$,

where a is a constant and $f(a) \neq 0$.

(iv) $\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$,

$\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax$,

where a is a constant and $f(-a^2) \neq 0$.

(v) When Q = P(x) = a polynomial of x, then

$\frac{1}{f(D)} \mathrm{P}(x) = \frac{1}{A[1+\phi(D)]^n} \mathrm{P}(x) = \frac{1}{A}[1+\phi(D)]^{-n} \mathrm{P}(x)$,

where A is lowest degree term of D or constant and $f(D) = A[1+\phi(D)]^n$. Expand $[1+\phi(D)]^{-n}$ by binomial theorem.

(vi) $\frac{1}{f(D)} e^{ax} \mathrm{U} = e^{ax} \frac{1}{f(D+a)} \mathrm{U}$,

where a is a constant and U is a function of x.

(vii) $\frac{1}{f(D)} x\mathrm{U} = x \frac{1}{f(D)} \mathrm{U} - \frac{f'(D)}{[f(D)]^2} \mathrm{U} = x \frac{1}{f(D)} \mathrm{U} + \frac{d}{dD} \frac{1}{f(D)} \mathrm{U}$,

$\frac{1}{f(D)} x^2 \mathrm{U} = x^2 \frac{1}{f(D)} \mathrm{U} + 2x \frac{d}{dD} \frac{1}{f(D)} \mathrm{U} + \frac{d^2}{dD^2} \frac{1}{f(D)} \mathrm{U}$ and so on,

where U is a function of x.

Remark. Failure cases of the formulas (ii), (iii) and (iv) may be resolved by the formula (vi) as $e^{ax} = e^{ax} \times 1$ and $e^{iax} = \cos ax + i \sin ax$. After using (vi), following formulas are obtained, which may be used directly:

(viii) $\frac{1}{D-a} e^{ax} = x e^{ax}$, $\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$,

where a is a constant and $r = 1, 2, 3, \cdots$.

(ix) $\frac{1}{D^2+a^2} \sin ax = -\frac{x}{2a} \cos ax$, $\frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \sin ax$,

$$\frac{1}{(D^2+a^2)^r}\sin ax = \frac{(-1)^r x^r}{(2a)^r r!}\sin\left(ax + r\frac{\pi}{2}\right),$$

$$\frac{1}{(D^2+a^2)^r}\cos ax = \frac{(-1)^r x^r}{(2a)^r r!}\cos\left(ax + r\frac{\pi}{2}\right),$$

where a is a constant and $r = 1, 2, 3, \cdots$.

Note 1. Following form of the binomial theorem is used for formula (v):

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \cdots .$$

Note 2. If Q = 0, then P.I. = 0. So in this case, the complete solution is given by

y = C.F. + 0 i.e. y = C.F.

LECTURE NO.:- 22

Examples of Linear ODE of Higher Order with Constant Coefficients-Evaluation of Complementary Functions only

Example 1. Solve $\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 26\frac{dy}{dx} - 24y = 0.$

Solution. Given differential equation is

$$\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 26\frac{dy}{dx} - 24y = 0.$$

$\Rightarrow$ $(D^3 - 9D^2 + 26D - 24)y = 0$, where $D \equiv \frac{d}{dx}$.

Its A.E. is $m^3 - 9m^2 + 26m - 24 = 0.$

Factorizing, $(m-2)(m-3)(m-4) = 0.$

So the roots of A.E. are $m = 2, 3, 4$.

$\Rightarrow$ C.F. $= c_1e^{2x} + c_2e^{3x} + c_3e^{4x}$,

where c_1, c_2 and c_3 are arbitrary constants.

Here P.I. = 0.

Hence the complete solution of the given equation is

$\boxed{y = c_1e^{2x} + c_2e^{3x} + c_3e^{4x}}$ Ans.

Example 2. Solve $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = 0.$

Solution. Given differential equation is

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = 0.$$

$\Rightarrow$ $(D^3 - 4D^2 + 5D - 2)y = 0$, where $D \equiv \frac{d}{dx}$.

Its A.E. is $m^3 - 4m^2 + 5m - 2 = 0.$

Factorizing, $(m-1)(m-1)(m-2) = 0.$

So the roots of A.E. are $m = 1, 1, 2$.

$\Rightarrow$ C.F. $= (c_1 + c_2x)e^{1x} + c_3e^{2x}$, where c_1, c_2 and c_3 are arbitrary constants.

Here P.I. = 0.

Hence the complete solution of the given equation is

$y = (c_1 + c_2 x)e^x + c_3 e^{2x}$. Ans.

Example 3. Solve $(D^4 - D^3 - 9D^2 - 11D - 4)y = 0$.

Solution. Given differential equation is

$$(D^4 - D^3 - 9D^2 - 11D - 4)y = 0.$$

Its A.E. is $m^4 - m^3 - 9m^2 - 11m - 4 = 0$.

$\Rightarrow$ $(m+1)^3(m-4) = 0$.

So the roots of A.E. are $m = -1, -1, -1, 4$.

$\Rightarrow$ C.F. = $(c_1 + c_2 x + c_3 x^2)e^{-1x} + c_4 e^{4x}$,

where c_1, c_2, c_3 and c_4 are arbitrary constants.

Here P.I. = 0.

Hence the complete solution of the given equation is

$y = (c_1 + c_2 x + c_3 x^2)e^{-x} + c_4 e^{4x}$. Ans.

Example 4. Solve $\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 26y = 0$.

Solution. Given differential equation is

$$\frac{d^2y}{dx^2} + 10\frac{dy}{dx} + 26y = 0.$$

$\Rightarrow$ $(D^2 + 10D + 26)y = 0$, where $D \equiv \frac{d}{dx}$.

Its A.E. is $m^2 + 10m + 26 = 0$.

$\Rightarrow$ $m = \frac{-10 \pm \sqrt{100 - 4 \times 26}}{2} = -5 \pm i1$.

$\therefore$ C.F. = $e^{-5x}(c_1 \cos 1x + c_2 \sin 1x)$,

where c_1 and c_2 are arbitrary constants.

Here P.I. = 0.

Hence the complete solution of the given equation is

$y = e^{-5x}(c_1 \cos x + c_2 \sin x)$. Ans.

Example 5. Solve $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0$.

Solution. Given differential equation is

$$\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0.$$

$\Rightarrow$ $(D^4 + 8D^2 + 16)y = 0$, where $D \equiv \frac{d}{dx}$.

Its A.E. is $m^4 + 8m^2 + 16 = 0$.

$\Rightarrow$ $(m^2 + 4)^2 = 0 \Rightarrow m^2 = -4, -4 \Rightarrow m = \pm i2, \pm i2 \Rightarrow m = 0 \pm i2, 0 \pm i2$.

$\therefore$ C.F. = $e^{0x}[(c_1 + c_2x)\cos 2x + (c_3 + c_4x)\sin 2x]$,

where c_1, c_2, c_3 and c_4 are arbitrary constants.

Here P.I. = 0.

Hence the complete solution of the given equation is

$y = (c_1 + c_2x)\cos 2x + (c_3 + c_4x)\sin 2x$. Ans.

Example 6. Solve $(D^2 + 3D + 1)y = 0$.

Hint. $m = -\frac{3}{2} \pm \frac{\sqrt{5}}{2} \Rightarrow y = e^{-\frac{3}{2}x}\left(c_1 \cosh\frac{\sqrt{5}}{2}x + c_2 \sinh\frac{\sqrt{5}}{2}x\right)$. Ans.

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Examples of Linear ODE with Constant Coefficients- Non-zero P.I.

Example 7. Solve $\frac{d^2y}{dx^2} + a^2y = \sec ax$.

Solution. Given differential equation is

$$\frac{d^2y}{dx^2} + a^2y = \sec ax.$$

$\Rightarrow$ $(D^2 + a^2)y = \sec ax$, where $D \equiv \frac{d}{dx}$.

Its A.E. is $m^2 + a^2 = 0$.

$\Rightarrow$ $m^2 = -a^2 \Rightarrow m^2 = i^2 a^2 \Rightarrow m = \pm ia$.

$\therefore$ C.F. $= c_1 \cos ax + c_2 \sin ax$,

where c_1 and c_2 are arbitrary constants.

Now P.I. $= \frac{1}{D^2+a^2} \sec ax = \frac{1}{(D-ia)(D+ia)} \sec ax$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right] \sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia} \sec ax - \frac{1}{D+ia} \sec ax\right]. \quad \cdots\cdots\cdots (1)$$

Now we have

$\frac{1}{D-ia} \sec ax = e^{iax} \int e^{-iax} \sec ax \, dx$, by $\frac{1}{D-\alpha} Q = e^{\alpha x} \int Q e^{-\alpha x} dx$

$= e^{iax} \int (\cos ax - i \sin ax) \frac{1}{\cos ax} dx$, by Euler's theorem

$$= e^{iax} \int \left(1 - i \frac{\sin ax}{\cos ax}\right) dx$$

$$= e^{iax}\left(x + \frac{i}{a} \log \cos ax\right).$$

Similarly $\frac{1}{D+ia} \sec ax = e^{-iax}\left(x - \frac{i}{a} \log \cos ax\right)$.

Putting these values in (1), we get

P.I. $= \frac{1}{2ia}\left[e^{iax}\left(x + \frac{i}{a} \log \cos ax\right) - e^{-iax}\left(x - \frac{i}{a} \log \cos ax\right)\right]$

$$= \frac{x}{a}\left(\frac{e^{iax} - e^{-iax}}{2i}\right) + \frac{1}{a^2}\left(\frac{e^{iax} + e^{-iax}}{2}\right)\log\cos ax$$

$$= \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax \cdot \log\cos ax, \text{ by Euler's theorem.}$$

Hence the complete solution of the given differential equation is

$$\boxed{y = c_1\cos ax + c_2\sin ax + \frac{x}{a}\sin ax + \frac{1}{a^2}\cos ax \cdot \log\cos ax}$$ <u>Ans.</u>

Example 8. Solve $\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$.

Solution. Try yourself as above example 1.

Example 9. Solve $\frac{d^2y}{dx^2} + 4y = \tan 2x$.

Solution. Try yourself as above example 1.

Example 10. Solve $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2 = 0$.

Solution. Given differential equation is

$$\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2 = 0.$$

It may be written as

$$D^3y - 4D^2y + 5Dy = 2, \text{ where } D = \frac{d}{dx}.$$

$\Rightarrow \quad (D^3 - 4D^2 + 5D)y = 2.$

Its A.E. is $m^3 - 4m^2 + 5m = 0$.

$\Rightarrow \quad m(m^2 - 4m + 5) = 0 \Rightarrow m = 0, m^2 - 4m + 5 = 0.$

$\Rightarrow \quad m = 0, m = \frac{4 \pm \sqrt{16-20}}{2} \Rightarrow m = 0, 2 \pm i1.$

$\therefore \quad$ C.F. $= c_1e^{0x} + c_2e^{2x}\cos(1x + c_3)$,

where c_1 , c_2 and c_3 are arbitrary constants.

Now P.I. $= \frac{1}{D^3 - 4D^2 + 5D}2 = 2\frac{1}{D(D^2 - 4D + 5)}1 = 2\frac{1}{D}\frac{1}{(D^2 - 4D + 5)}1$

$= 2\frac{1}{D}\frac{1}{(0^2 - 4\times 0 + 5)}1$, by $\frac{1}{f(D)}\text{K} = \frac{1}{f(0)}\text{K}$

$$=\frac{2}{5}\frac{1}{D}1=\frac{2}{5}D^{-1}1=\frac{2}{5}\int 1\,dx$$
$$=\frac{2}{5}\,x.$$

Hence the complete solution of the given differential equation is

$y=c_1+c_2e^{2x}\cos(x+c_3)+\frac{2}{5}\,x.$ Ans.

Example 11. Solve $(D^2-3D+2)y=e^x$.

Solution. Given differential equation is

$(D^2-3D+2)y=e^x$.

Its A.E. is $m^2-3m+2=0.\ \Rightarrow (m-1)(m-2)=0 \Rightarrow m=1,2.$

$\therefore$ C.F. $=c_1e^{1x}+c_2e^{2x}$, where c_1 and c_2 are arbitrary constants.

Now P.I. $=\frac{1}{(D^2-3D+2)}e^x=\frac{1}{(D-1)(D-2)}e^{1x}=\frac{1}{(D-1)}\frac{1}{(D-2)}e^{1x}$

$=\frac{1}{(D-1)}\frac{1}{(1-2)}e^{1x}$, by $\frac{1}{f(D)}e^{ax}=\frac{1}{f(a)}e^{ax}$

$=-\frac{1}{D-1}e^x$

$=-xe^x$, by $\frac{1}{D-a}e^{ax}=xe^{ax}$.

Hence the complete solution of the given differential equation is

$y=c_1e^x+c_2e^{2x}-xe^x.$ Ans.

Example 12. Solve $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=\sin 2x$.

Solution. Given differential equation is

$\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=\sin 2x$

$\Rightarrow$ $(D^2+D+1)y=\sin 2x$, where $D=\frac{d}{dx}$.

It's A.E. is $m^2+m+1=0.\ \Rightarrow m=\frac{-1\pm\sqrt{1-4}}{2}\Rightarrow m=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$.

$\therefore$ C.F. $=e^{-\frac{1}{2}x}\left\{c_1\cos\left(\frac{\sqrt{3}}{2}x\right)+c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right\}$,
where c_1 and c_2 are arbitrary constants.

Now P.I. $= \frac{1}{(D^2+D+1)} \sin 2x = \frac{1}{-2^2+D+1} \sin 2x$, by $\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$

$= \frac{1}{D-3} \sin 2x = \frac{1}{D-3} \frac{D+3}{D+3} \sin 2x = (D+3) \frac{1}{D^2-9} \sin 2x$

$= (D+3) \frac{1}{-2^2-9} \sin 2x$, by $\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$

$= -\frac{1}{13}(D+3) \sin 2x = -\frac{1}{13}(D \sin 2x + 3 \sin 2x)$

$= -\frac{1}{13}(2 \cos 2x + 3 \sin 2x)$.

Hence the complete solution of the given differential equation is

$y = e^{-\frac{x}{2}} \left\{c_1 \cos\left(\frac{\sqrt{3}}{2} x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2} x\right)\right\} - \frac{1}{13}(2 \cos 2x + 3 \sin 2x)$. Ans.

Example 13. Solve $(D^2 - 4D + 4)y = e^{-4x} + 5 \cos 3x$.

Solution. Given differential equation is

$(D^2 - 4D + 4)y = e^{-4x} + 5 \cos 3x$.

Its A.E. is $m^2 - 4m + 2 = 0$. $\Rightarrow (m-2)^2 = 0 \Rightarrow m = 2, 2$.

$\therefore$ C.F. $= (c_1 + c_2 x)e^{2x}$, where c_1 and c_2 are arbitrary constants.

Now P.I. $= \frac{1}{(D^2-4D+4)}[e^{-4x} + 5 \cos 3x] = \frac{1}{D^2-4D+4} e^{-4x} + 5 \frac{1}{D^2-4D+4} \cos 3x$

$= \frac{1}{(-4)^2-4(-4)+4} e^{-4x} + 5 \frac{1}{-3^2-4D+4} \cos 3x$, by $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$

and $\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax$

$= T_1 - 5 \frac{1}{4D+5} \frac{4D-5}{4D-5} \cos 3x$, where $T_1 = \frac{1}{36} e^{-4x}$ (say)

$= T_1 - 5(4D-5) \frac{1}{16D^2-25} \cos 3x = T_1 - 5(4D-5) \frac{1}{16(-3^2)-25} \cos 3x$

$= T_1 + \frac{5}{169}(4D-5) \cos 3x = T_1 + \frac{5}{169}(-4 \times 3 \sin 3x - 5 \cos 3x)$

$= \frac{1}{36} e^{-4x} - \frac{5}{169}(12 \sin 3x + 5 \cos 3x)$.

Hence the complete solution of the given differential equation is

$y = (c_1 + c_2 x)e^{2x} + \frac{1}{36} e^{-4x} - \frac{5}{169}(12 \sin 3x + 5 \cos 3x)$. Ans.

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Examples of Linear ODE with Constant Coefficients- Continued

Example 14. Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2$.

Solution. Given differential equation is

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2$$

$\Rightarrow$ $(D^2 - 4D + 4)y = x^2$, where $D = \frac{d}{dx}$.

Its A.E. is $m^2 - 4m + 2 = 0$.

$\Rightarrow$ $(m-2)^2 = 0 \Rightarrow m = 2, 2$.

$\therefore$ C.F. = $(c_1 + c_2\,x)e^{2x}$, where c_1 and c_2 are arbitrary constants.

Now P.I. $= \frac{1}{(D^2-4D+4)}x^2 = \frac{1}{(D-2)^2}x^2 = \frac{1}{(-2)^2\left(1-\frac{D}{2}\right)^2}x^2 = \frac{1}{4}\left(1 - \frac{D}{2}\right)^{-2} x^2$.

Expanding by binomial theorem, we get

$$\text{P.I.} = \frac{1}{4}\left[1 + 2\left(\frac{D}{2}\right) + \frac{2\times3}{2}\left(\frac{D}{2}\right)^2 + \frac{2\times3\times4}{2\times3}\left(\frac{D}{2}\right)^3 + \cdots\right]x^2$$

$$= \frac{1}{4}\left[x^2 + Dx^2 + \frac{3}{4}D^2x^2 + \frac{1}{2}D^3x^2 + \cdots\right] = \frac{1}{4}\left[x^2 + 2x + \frac{3}{4}2 + \frac{1}{2}0 + \cdots\right]$$

$\therefore$ P.I. $= \frac{1}{4}(x^2 + 2x + 3/2)$.

Hence the complete solution of the given differential equation is

$y = (c_1 + c_2\,x)e^{2x} + \frac{1}{8}(2x^2 + 4x + 3)$. Ans.

Example 15. Solve $(D^2 - 4D + 4)y = 8x^2e^{2x}\sin 2x$.

Solution. Given differential equation is

$$(D^2 - 4D + 4)y = 8x^2e^{2x}\sin 2x.$$

Its A.E. is $m^2 - 4m + 2 = 0$.

$\Rightarrow$ $(m-2)^2 = 0 \Rightarrow m = 2, 2$.

$\therefore$ C.F. = $(c_1 + c_2\,x)e^{2x}$, where c_1 and c_2 are arbitrary constants.

Now P.I. $= \frac{1}{(D^2-4D+4)} 8x^2 e^{2x} \sin 2x = 8 \frac{1}{(D-2)^2} e^{2x} x^2 \sin 2x$

$= 8e^{2x} \frac{1}{(D+2-2)^2} x^2 \sin 2x$, by $\frac{1}{f(D)} e^{ax} \mathrm{U} = e^{ax} \frac{1}{f(D+a)} \mathrm{U}$

$= 8e^{2x} \frac{1}{D^2} x^2 \sin 2x = 8e^{2x} \frac{1}{D} \int x^2 \sin 2x \, dx$

$= 8e^{2x} \frac{1}{D} \left[x^2 \left(-\frac{\cos 2x}{2} \right) - 2x \left(-\frac{\sin 2x}{2^2} \right) + 2 \left(\frac{\cos 2x}{2^3} \right) \right]$

$= \frac{8}{4} e^{2x} \int [-2x^2 \cos 2x + 2x \sin 2x + \cos 2x] dx$

$= 2e^{2x} \left[-2x^2 \left(\frac{\sin 2x}{2} \right) + 4x \left(-\frac{\cos 2x}{2^2} \right) - 4 \left(-\frac{\sin 2x}{2^3} \right) \right.$

$\left. +2x \left(-\frac{\cos 2x}{2} \right) - 2 \left(-\frac{\sin 2x}{2^2} \right) + \frac{\sin 2x}{2} \right]$

$= \frac{2}{2} e^{2x} [(-2x^2 + 1 + 1 + 1) \sin 2x + (-2x - 2x) \cos 2x]$

$= - e^{2x} \{(2x^2 - 3) \sin 2x + 4x \cos 2x\}$

Hence the complete solution of the given differential equation is

$y = \{c_1 + c_2 x - (2x^2 - 3) \sin 2x - 4x \cos 2x\} e^{2x}$. Ans.

Example 16. Solve $(D^2 - 2D + 1)y = xe^x$.

Solution. Try yourself as above example 6.

Example 17. Solve $(D^2 + 2D + 1)y = x \cos x$.

Solution. Given differential equation is

$(D^2 + 2D + 1)y = x \cos x$.

Its A.E. is $m^2 + 2m + 1 = 0$.

$\Rightarrow$ $(m + 1)^2 = 0 \Rightarrow m = -1, -1$.

$\therefore$ C.F. $= (c_1 + c_2 x)e^{-1x}$, where c_1 and c_2 are arbitrary constants.

Now P.I. $= \frac{1}{(D^2+2D+1)} x \cos x = \frac{1}{(D+1)^2} x \cos x$

$= x \frac{1}{(D+1)^2} \cos x - \frac{2(D+1)}{(D+1)^4} \cos x$, by $\frac{1}{f(D)} x\mathrm{U} = x \frac{1}{f(D)} \mathrm{U} - \frac{f'(D)}{[f(D)]^2} \mathrm{U}$

$= x \frac{1}{D^2 + 2D + 1} \cos 1x - 2(D + 1) \frac{1}{(D^2+2D+1)^2} \cos 1x$

$$= x\frac{1}{-1^2+2D+1}\cos 1x - 2(D+1)\frac{1}{(-1^2+2D+1)^2}\cos 1x,$$

by $\frac{1}{f(D^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax$

$$= \frac{x}{2}\frac{1}{D}\cos x - \frac{2}{2^2}(D+1)\frac{1}{D^2}\cos x$$

$$= \frac{x}{2}\int \cos x\, dx - \frac{1}{2}\left(\frac{1}{D}\cos x + \frac{1}{D^2}\cos x\right) = \frac{x}{2}\sin x - \frac{1}{2}(\sin x - \cos x)$$

$$= \frac{x}{2}\sin x + \frac{1}{2}(-\sin x - \cos x) = \frac{1}{2}\{(x-1)\sin x + \cos x\}.$$

Hence the complete solution of the given differential equation is

$y = (c_1 + c_2 x)e^{-x} + \frac{1}{2}\{(x-1)\sin x + \cos x\}.$ Ans.

Example 18. Solve $(D^2 + 2D + 1)y = e^{-x}$.
Solution. Try yourself.

Example 19. Solve $(D^2 + 4)y = \sin 2x$.
Solution. Given differential equation is

$$(D^2 + 4)y = \sin 2x.$$

Its A.E. is $m^2 + 4 = 0$.

$\Rightarrow$ $m^2 = -2^2 \Rightarrow m^2 = i^2\, 2^2 \Rightarrow m = \pm i2.$

$\therefore$ C.F. $= c_1 \cos 2x + c_2 \sin 2x$, where c_1 and c_2 are arbitrary constants.

Now P.I. $= \frac{1}{D^2+4}\sin 2x = \frac{1}{D^2+2^2}\sin 2x$

$= -\frac{x}{2(2)}\cos 2x$, by $\frac{1}{D^2+a^2}\sin ax = -\frac{x}{2a}\cos ax.$

Hence the complete solution of the given differential equation is

$y = c_1\cos 2x + c_2 \sin 2x - \frac{x}{4}\cos 2x.$ Ans.

Example 20. Solve $(D-1)^2(D^2+1)^2 y = \sin^2\frac{x}{2} + e^x$.

Example 21. Solve $(D^3 + 2D^2 + D)y = e^{-x} + \cos x + x^2$.

Example 22. Solve $(D^2 - 2D + 1)y = xe^x \sin x$.

Example 23. Solve $(D^2 - 1)y = xe^x + \cos^2 x$.

Solution. Try yourself.