Homework 3: Qualitative theory of ODEs
04/16/23

Jaiden Atterbury

- Exercise 1:

(1.) $y' = 2.1y(1-y)$

The roots of $f(y)$ are equilibrium points. Thus $y=0$ and $y=1$ are equilibrium points.

To use the linearization criterion, we must find $f'(y)$. It follows that $f'(y) = 2.1 - 4.2y$.

Since $f'(0) = 2.1 > 0$, $y=0$ is unstable.
Since $f'(1) = -2.1 < 0$, $y=1$ is asymptotically stable.

(2.) $y' = 1 + y^2$

The roots of $f(y) = 1+y^2$ are equilibrium points. However, $f(y)$ has no real roots. Thus there are no equilibrium points.

(3) $y' = 1 - y^2$

The roots of $f(y) = 1-y^2$ are equilibrium points. Thus $y=-1$ and $y=1$ are equilibrium points.

To use the linearization criterion, we must find $f'(y)$. It follows that $f'(y) = -2y$.

Since $f'(-1) = 2 > 0$, $y=-1$ is unstable.
Since $f'(1) = -2 < 0$, $y=1$ is asymptoticaly stable.

(4) $y' = 2y^2 - y^3$

The roots of $f(y) = 2y^2 - y^3 = y^2(2-y)$ are equilibrium points. Thus $y=0$ and $y=2$ are equilibrium points.

To use the linearization criterion, we must find $f'(y)$. It follows that $f'(y) = 4y - 3y^2$.

Since $f'(0) = 0$, it follows from the linearization criterion that the stability of $y=0$ is inconclusive.

Since $f'(2) = -4 < 0$, $y=2$ is asymptotically stable.

Exercise 2:

$\frac{dy}{dt} = \sin(y)$

The roots of $f(y) = \sin(y)$ are equilibrium points. Thus $y = K\pi$ where $K \in \mathbb{Z}$ (the integers) are all equilibrium points.

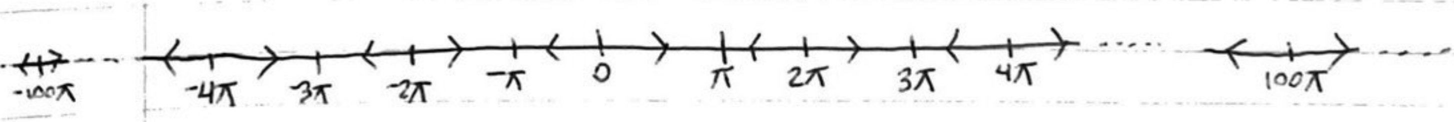
To use the linearization criterion, we must find $f'(y)$. It follows that $f'(y) = \cos(y)$.

If we let $\mathbb{O}$ represent the odd integers $\{\cdots, -3, -1, 1, 3, \cdots\}$, and let $\mathbb{E}$ represent the even integers $\{\cdots, -2, 0, 2, \cdots\}$. Then using the fact that $\cos(x) = \cos(-x)$ then we can see that:

If $K \in \mathbb{O}$, then $f'(K\pi) = -1 < 0$, thus all of these points are asymptotically stable.

If $K \in \mathbb{E}$, then $f'(K\pi) = 1 > 0$, thus all of these points are unstable.

The following phase-line plot shows this trend:



- Exercise 3:

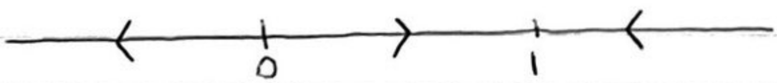
① $y' = 3y(1-y)$

The roots of $f(y) = 3y(1-y)$ are equilibrium points. Thus $y=0$ and $y=1$ are equilibrium points. To asses the stability of these points, we will use the linearization criterion.

To use the linearization criterion, we must find $f'(y)$. It follows that $f'(y) = 3-6y$.

Since $f'(0) = 3 > 0$, $y=0$ is unstable.
Since $f'(1) = -3 < 0$, $y=1$ is asymptotically stable

Phase-line plot:



② $y' = y^2 - 6y - 16$

The roots of $f(y) = y^2 - 6y - 16 = (y+2)(y-8)$ are equilibrium points. Thus $y=-2$ and $y=8$ are equilibrium points. To asses the stability of these points, we will first use the linearization criterion.

To use the linearization criterion, we must find $f'(y)$. It follows that $f'(y) = 2y-6$.

Since $f'(-2) = -10 < 0$, $y=-2$ is asymptotically stable.
Since $f'(8) = 10 > 0$, $y=8$ is unstable.

Phase-line plot:

• Exercise 4:

$y' = y(y^2 - \mu)$ where $\mu \in \mathbb{R}$

We will consider three cases: $\mu = 0$, $\mu > 0$, $\mu < 0$.

In all three cases, we will find all of the equilibrium points, assess their stability, and draw the phase-line plot.

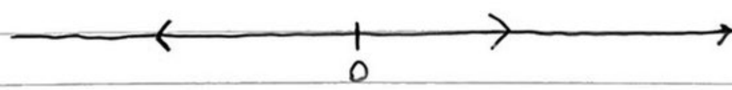
∘ Case 1: $\mu = 0$

If $\mu = 0$, then $y' = y(y^2 - \mu)$ becomes $y' = y^3$.

The roots of $f(y) = y^3$ are the equilibrium points. Thus the one equilibrium point is $y = 0$.

Since $f'(y) = 3y^2$, and $f'(0) = 0$, it follows that the linearization criterion is inconclusive. However, for $y > 0$, $f(y)$ is negative, thus $\frac{dy}{dt}$ will be negative and any solution in this region will be decreasing. For all values of $y > 0$, $f(y)$ is positive, thus $\frac{dy}{dt}$ will be positive and any solution in this region will be increasing.

Hence, $y = 0$ is unstable and the phase-line plot looks like



∘ Case 2: $\mu > 0$.

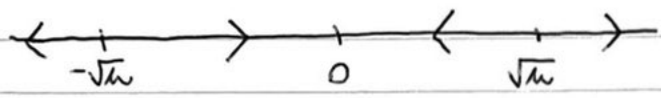
If $\mu > 0$, then $y' = y(y^2 - \mu)$ stays as it is.

The roots of $f(y) = y(y^2 - \mu)$ are equilibrium points. Thus $y = -\sqrt{\mu}$, $y = 0$, and $y = \sqrt{\mu}$ are equilibrium points.

To use linearization, we must find $f'(y)$. It follows that $f'(y) = 3y^2 - \mu$

Since $f'(-\sqrt{\mu}) = 3(-\sqrt{\mu})^2 - \mu = 3\mu - \mu = 2\mu > 0$, $y = -\sqrt{\mu}$ is unstable.
Since $f'(0) = -\mu < 0$, $y = 0$ is asymptotically stable.
Since $f'(\sqrt{\mu}) = 3(\sqrt{\mu})^2 - \mu = 3\mu - \mu = 2\mu > 0$, $y = \sqrt{\mu}$ is unstable.

Phase-line plot:



∘ <u>Case 3:</u> $\mu < 0$.

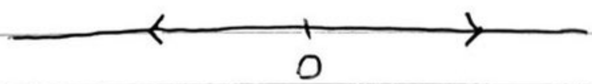
If $\mu < 0$, then $y' = y(y^2 + \mu)$ becomes $y(y^2 + \mu)$.

The roots of $f(y) = y(y^2 + \mu)$ are equilibrium points. Thus the one equilibrium point is $y = 0$.

To use linearization we must find $f'(y)$. It follows that $f'(y) = 3y^2 + \mu$.

Since $f'(0) = \mu > 0$, $y = 0$ is unstable.

Phase-line plot:



Exercise 5:

Consider the system of ODEs that models the spread of various bacterial infections in a population:

$$\frac{dS(t)}{dt} = \frac{-\lambda}{N} S(t) I(t) + \gamma I(t),$$
$$\frac{dI(t)}{dt} = \frac{\lambda}{N} S(t) I(t) - \gamma I(t).$$

Where:

N is the total population.

$S(t) > 0$ is the number of susceptible individuals at time t.

$I(t) > 0$ is the number of infected individuals at time t.

$\lambda > 0$ is the infectivity rate.

$\gamma > 0$ is the curing rate.

Under the assumption that the population stays constant, it follows that $S(t) + I(t) = N$ for all t.

The system of ODEs is expressed equivalently as one ODE

$$\frac{dx(t)}{dt} = \lambda(1 - x(t))\, x(t) - \gamma x(t) \qquad (1)$$

Where:

$x(t) = \frac{I(t)}{N}$ is the infected population fraction at time t.

Our goal is to find the equilibrium points of ODE (1) and assess/interpret in context their stability for different values of the basic reproduction number $R_0 = \frac{\lambda}{\gamma}$.

We will start by rewriting the equation for $\frac{dx}{dt}$ in terms of R_0:

$$\begin{aligned} x' &= \lambda(1-x)x - \gamma x \\ &= \lambda x - \lambda x^2 - \gamma x \\ &= \gamma x\left(\frac{\lambda}{\gamma} - \frac{\lambda}{\gamma}x - 1\right) \\ &= \gamma x\,(R_0 - R_0 x - 1) \end{aligned}$$

With this new equation for x', if we let $f(x) = \gamma x(R_0 - R_0 x - 1)$, then the roots of $f(x)$ are the equilibrium solutions/points.

Thus the equilibrium points are $x=0$ and $-R_0x+R_0-1=0 \Rightarrow$ $R_0x = R_0-1 \Rightarrow x = 1-\frac{1}{R_0}$. We can divide by R_0 since it will never equal zero, in fact, $R_0>0$.

Now we will use linearization to get a general expression to assess stability. Since $f(x)=\gamma x(R_0-R_0x-1)$ it follows that $f'(x)=\lambda-2\lambda x-\gamma$.

Thus $f'(0)=\lambda-\gamma=\lambda-\frac{\lambda}{R_0}=\lambda(1-\frac{1}{R_0})$ and $f'(1-\frac{1}{R_0})=\lambda-\gamma-2\lambda+\frac{2\lambda}{R_0}=$ $\lambda-2\lambda-\frac{\lambda}{R_0}+\frac{2\lambda}{R_0}=\frac{\lambda}{R_0}-\lambda=\lambda(\frac{1}{R_0}-1)$.

We will now analyze stability for three cases: $R_0>1$, $R_0=1$, and $0<R_0<1$.

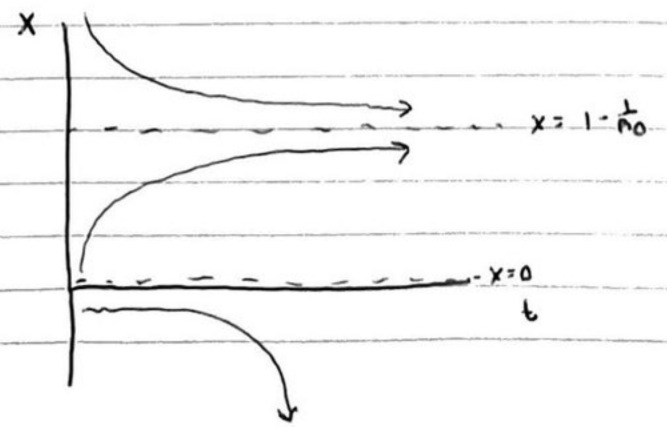
◦ Case 1: $R_0>1$

When $R_0>1$, it follows that $\lambda>\gamma$.

Going back to the linearization expressions we see:
$f'(0)=\lambda(1-\frac{1}{R_0})$. Since $\lambda>0$, and $R_0>1$, it follows that $1-\frac{1}{R_0}>0$.
Thus $f'(0)>0$, $x=0$ is unstable.

$f'(1-\frac{1}{R_0})=\lambda(\frac{1}{R_0}-1)$, since $\lambda>0$, and $R_0>1$, it follows that $\frac{1}{R_0}-1<0$
Thus $f'(1-\frac{1}{R_0})<0$, $x=1-\frac{1}{R_0}$ is stable. In this case $x>0$.

Here is a theoretical plot of solutions for x:



Thus in the case where $\lambda > \gamma$, the fraction of individuals infected will reach a steady-state proportion of $1-\frac{1}{R_0}$ as $t \to \infty$.

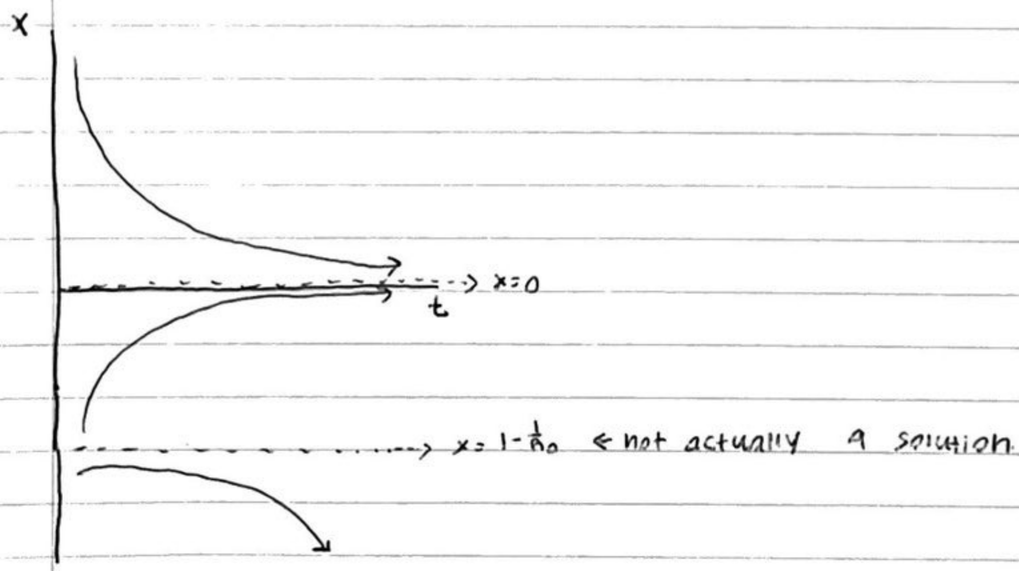
°Case 2: $0 < R_0 < 1$.

When $0 < R_0 < 1$, it follows that $\gamma > \lambda$.

Going back to the linearization expressions we see that:
$f'(0) = \lambda(1-\frac{1}{R_0})$. Since $\lambda > 0$, and $0 < R_0 < 1$, it follows that $1-\frac{1}{R_0} < 0$.
Thus $f'(0) < 0$; $x=0$ is asymptotically stable.

Also, $f'(1-\frac{1}{R_0}) = \lambda(\frac{1}{R_0}-1)$. Since $\lambda > 0$, and $0 < R_0 < 1$, it follows that $\frac{1}{R_0}-1 > 0$.
Thus $f'(1-\frac{1}{R_0}) > 0$, $x = 1-\frac{1}{R_0}$ is unstable. In this case $x < 0$. However $x \in [0,1]$, thus $1-\frac{1}{R_0}$ is not a solution in this case.

Here is a theoretical plot of solutions for x:



Thus in the case where $\gamma > \lambda$, the fraction of individuals infected will go to 0 as $t \to \infty$. Thus in the population the disease will go extinct.

Case 3: $R_0 = 1$

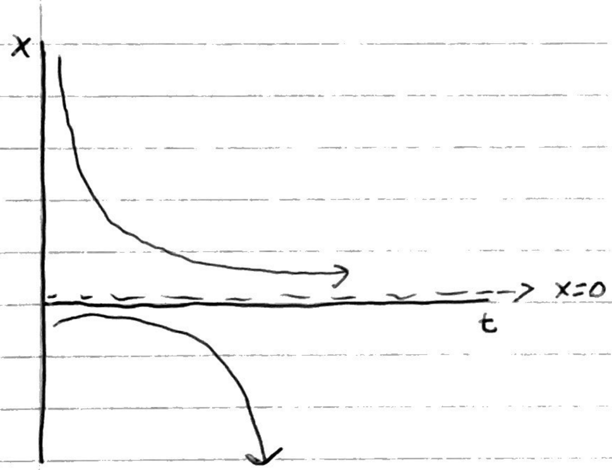
When $R_0 = 1$, it follows that $\lambda = \gamma$

Hence the equilibrium solution $x = 1 - \frac{1}{R_0}$ becomes $x = 0$. Thus we now only have one equilibrium solution.

Going back to the linearization we see:
$f'(0) = \lambda(1 - \frac{1}{R_0}) = 0$. Thus the criterion is inconclusive.

However, for $x > 0$, $f(x)$ is negative, thus $\frac{dx}{dt}$ will be negative so any solution in this region will be decreasing. Also, for $x < 0$, $f(x)$ is negative, thus the same logic holds. $x = 0$ is therefore semistable.

Here is a theoretical plot of solutions for x:



Thus in the case where $\lambda = \gamma$, the fraction of individuals infected will go to 0 as $t \to \infty$. Thus in the population the disease will go extinct.

NOTE: In the above plots for the solution of x, negative solutions can be disregarded since it is a population fraction and thus positive.