AMATH 351 Homework 7: Series solutions of ODEs

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Exercise 1: First-order ODEs. By using the series expansion around zero for the solution $y(x) = \sum_{n=0}^{\infty} a_n x^n$, solve the ODEs:

1. $y' - y = 0$. Hint: the solution is $y(x) = a_0 \sum\limits_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x$.

To start we will replace y' and y with their series expansions around zero and simplify if possible.

$$\begin{aligned}
0 &= y' - y \\
&= \sum_{n=0}^{\infty} (n+1)a_{n+1}x^n - \sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} \left[(n+1)a_{n+1} - a_n\right] x^n
\end{aligned}$$

Hence as shown above, $\sum\limits_{n=0}^{\infty} \left[(n+1)a_{n+1} - a_n\right] x^n = 0$, thus in order for this relationship to hold for all values of x, $(n+1)a_{n+1} - a_n = 0 \implies a_{n+1} = \frac{1}{n+1}a_n$. Writing out the first three values of this recursive definition we see that: $a_1 = \frac{1}{2}a_0 = \frac{1}{2!}a_0$, $a_2 = \frac{1}{3}a_1 = \frac{1}{3 \cdot 2}a_0 = \frac{1}{3!}a_0$, and $a_3 = \frac{1}{4}a_2 = \frac{1}{4 \cdot 3 \cdot 2}a_0 = \frac{1}{4!}a_0$. Thus in general, we can see that $a_n = \frac{1}{n!}a_0$.

Going back to our solution form of $y(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ and plugging in our a_n term we can see that

$$\begin{aligned}
y(x) &= \sum_{n=0}^{\infty} \frac{1}{n!} a_0 x^n \\
&= a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} \\
&= a_0 e^x
\end{aligned}$$

Hence the general solution to the ODE $y' - y = 0$ is $a_0 e^x$.

2. $y' = 2xy$. Hint: the solution is $y(x) = a_0 \sum\limits_{n=0}^{\infty} \frac{x^{2n}}{n!} = a_0 e^{x^2}$.

To start we will subtract the $2xy$ term to the left-hand side of the equation and replace y' and y with their series expansions around zero and simplify if possible.

$$\begin{aligned}
0 &= y' - 2xy \\
&= \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - 2x\sum_{n=0}^{\infty}a_nx^n \\
&= \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - 2\sum_{n=0}^{\infty}a_nx^{n+1}
\end{aligned}$$

However, in order to solve this problem more easily, we are going to want to have the same power of x for all series in the ODE. Hence, in the second series we must introduce the change of index $k = n+1$. Hence the series becomes $2\sum_{k=1}^{\infty} a_{k-1}x^k$. However since k is just a "dummy" variable we will chance it back to n for convenience. Hence the series becomes $2\sum_{n=1}^{\infty} a_{n-1}x^n$. Putting this back into the equation we obtain the following

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n - 2\sum_{n=1}^{\infty}a_{n-1}x^n \\
&= a_1 + \sum_{n=1}^{\infty}\left[(n+1)a_{n+1} - 2a_{n-1}\right]x^n
\end{aligned}$$

Hence as shown above, $a_1 + \sum_{n=1}^{\infty}\left[(n+1)a_{n+1} - 2a_{n-1}\right]x^n = 0$, thus in order for this relationship to hold for all values of x, $a_1 = 0$ and $(n+1)a_{n+1} - 2a_{n-1} = 0 \implies a_{n+1} = \frac{2}{n+1}a_{n-1}$. Since this recursive definition steps by 2, we must split finding a closed form solution into two parts, one where n is even, and one where n is odd.

n is even $\implies$ n = 2k:
Writing out the first three values of this recursive definition we see that: $a_2 = \frac{2}{2}a_0 = \frac{1}{1!}a_0$, $a_4 = \frac{1}{2}a_2 = \frac{1}{2\cdot 1}a_0 = \frac{1}{2!}a_0$, and $a_6 = \frac{1}{3}a_4 = \frac{1}{3\cdot 2\cdot 1}a_0 = \frac{1}{3!}a_0$. Thus in general, we can see that $a_{2k} = \frac{1}{n!}a_0$.

n is odd $\implies$ n = 2k:
Writing out the first three values of this recursive definition we see that: $a_3 = \frac{2}{3}a_1 = 0$, $a_5 = \frac{2}{5}a_3 = 0$, and $a_7 = \frac{2}{7}a_5 = 0$. Thus in general, we can see that $a_{2k+1} = 0$.

Going back to our solution form of $y(x) = \sum_{n=0}^{\infty} a_nx^n$ and plugging in our a_2k and a_{2k+1} terms we can see that

$$\begin{aligned}
y(x) &= \sum_{k=0}^{\infty}\frac{1}{k!}a_0x^{2k} \\
&= a_0\sum_{k=0}^{\infty}\frac{x^{2k}}{k!} \\
&= a_0e^{x^2}
\end{aligned}$$

Hence the general solution to the ODE $y' = 2xy$ is $a_0e^{x^2}$.

Exercise 2: Hyperbolic cosine and sine. Find the general solution of the ODE

$$y'' - y = 0, \tag{1}$$

by using the series expansion of $y(t)$ around $t_0 = 0$:

$$y(t) = \sum_{n=0}^{\infty} a_n t^n.$$

In your solution, use the following Maclaurin series for the *hyperbolic cosine*

$$\cosh(t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}$$

and the *hyperbolic sine*

$$\sinh(t) = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}$$

To start we will replace y'' and y with their series expansions around zero and simplify if possible.

$$\begin{aligned} 0 &= y'' - y \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n - \sum_{n=0}^{\infty} a_n t^n \\ &= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_n\right] t^n \end{aligned}$$

Hence as shown above, $\sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_n\right] t^n = 0$, thus in order for this relationship to hold for all values of t, $(n+2)(n+1)a_{n+2} - a_n = 0 \implies \frac{1}{(n+1)(n+1)}a_0$. Since this recursive definition steps by 2, we must split finding a closed form solution into two parts, one where n is even, and one where n is odd.

n is even $\implies$ n = 2k:
Writing out the first three values of this recursive definition we see that: $a_2 = \frac{1}{2\cdot 1}a_0 = \frac{1}{2!}a_0$, $a_4 = \frac{1}{4\cdot 3}a_2 = \frac{1}{4\cdot 2\cdot 3\cdot 1}a_0 = \frac{1}{4!}a_0$, and $a_6 = \frac{1}{6\cdot 5}a_4 = \frac{1}{6\cdot 5\cdot 4\cdot 2\cdot 3\cdot 1}a_0 = \frac{1}{6!}a_0$. Thus in general, we can see that $a_{2k} = \frac{1}{(2k)!}a_0$.

n is odd $\implies$ n = 2k:
Writing out the first three values of this recursive definition we see that: $a_3 = \frac{1}{3\cdot 2\cdot 1}a_1 = \frac{1}{3!}a_1$, $a_5 = \frac{1}{5\cdot 4}a_3 = \frac{1}{5\cdot 4\cdot 3\cdot 2\cdot 1!}a_1 = \frac{1}{5!}a_1$, and $a_7 = \frac{1}{7\cdot 6}a_5 = \frac{1}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1!}a_1 = \frac{1}{7!}a_1$. Thus in general, we can see that $a_{2k+1} = \frac{1}{(2k+1)!}a_1$.

Going back to our solution form of $y(t) = \sum_{n=0}^{\infty} a_n t^n$ and plugging in our a_2k and a_{2k+1} terms we can see that

$$\begin{aligned} y(t) &= \sum_{k=0}^{\infty} \frac{1}{(2k)!}a_0 t^{2k} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}a_1 t^{2k+1} \\ &= a_0 \sum_{k=0}^{\infty} \frac{1}{(2k)!}t^{2k} + a_1 \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}t^{2k+1} \\ &= a_0 \cosh(t) + a_1 \sinh(t) \end{aligned}$$

Hence the general solution to the ODE $y'' - y = 0$ is $a_0 \cosh(t) + a_1 \sinh(t)$.

Exercise 3: By using the series expansion around $t_0 = 0$, find the general solution of the ODE

$$y'' + ty' + y = 0 \tag{2}$$

<u>Hint:</u> You should arrive to the closed form formulas for the series coefficients

$$a_{2k} = \frac{(-1)^k}{2 \cdot 4 \cdot 6 \ldots 2k} a_0, \quad a_{2k+1} = \frac{(-1)^k}{3 \cdot 5 \cdot 7 \ldots (2k+1)} a_1.$$

To start we will replace y'', y', and y with their series expansions around zero and simplify if possible.

$$\begin{aligned}
0 &= y'' + ty' + y \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}t^n + t\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n + \sum_{n=0}^{\infty} a_n t^n \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}t^n + \sum_{n=0}^{\infty}(n+1)a_{n+1}t^{n+1} + \sum_{n=0}^{\infty} a_n t^n
\end{aligned}$$

However, in order to solve this problem more easily, we are going to want to have the same power of t for all series in the ODE. Hence, in the second series we must introduce the change of index $k = n+1$. Hence the series becomes $\sum_{k=1}^{\infty} k a_k t^k$. However since k is just a "dummy" variable we will chance it back to n for convenience. Hence the series becomes $\sum_{n=1}^{\infty} n a_n t^n$. Putting this back into the equation we obtain the following

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}t^n \sum_{n=1}^{\infty} n a_n t^n + \sum_{n=0}^{\infty} a_n t^n \\
&= a_0 + 2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} + (n+1)a_n\right] t^n
\end{aligned}$$

Hence as shown above, $a_0 + 2a_2 + \sum_{n=1}^{\infty} \left[(n+2)(n+1)a_{n+2} + (n+1)a_n\right] t^n = 0$, thus in order for this relationship to hold for all values of t, $a_0 + 2a_2 = 0$ and $(n+2)(n+1)a_{n+2} + (n+1)a_n = 0 \implies a_{n+2} = \frac{-1}{n+2} a_n,\ n = 1, 2, 3, \cdots$. However, notice that $a_0 + 2a_2 = 0$ is $(n+2)(n+1)a_{n+2} + (n+1)a_n = 0$ at $n = 0$. Thus we can see that $a_{n+2} = \frac{-1}{n+2} a_n,\ n = 0, 1, 2, 3, \cdots$. Since this recursive definition steps by 2, we must split finding a closed form solution into two parts, one where n is even, and one where n is odd.

n is even $\implies$ n = 2k:
Writing out the first three values of this recursive definition we see that: $a_2 = \frac{-1}{2} a_0$, $a_4 = \frac{-1}{4} a_2 = \frac{1}{2 \cdot 4} a_0$, and $a_6 = \frac{-1}{6} a_4 = \frac{-1}{2 \cdot 4 \cdot 6} a_0$. Thus in general, we can see that $a_{2k} = \frac{(-1)^k}{2 \cdot 4 \cdot 6 \ldots 2k} a_0$.

n is odd $\implies$ n = 2k:
Writing out the first three values of this recursive definition we see that: $a_3 = \frac{-1}{3} a_1$, $a_5 = \frac{-1}{5} a_3 = \frac{1}{3 \cdot 5} a_1$, and $a_7 = \frac{-1}{7} a_5 = \frac{-1}{3 \cdot 5 \cdot 7} a_1$. Thus in general, we can see that $a_{2k+1} = \frac{(-1)^k}{3 \cdot 5 \cdot 7 \ldots (2k+1)} a_1$.

Going back to our solution form of $y(t) = \sum_{n=0}^{\infty} a_n t^n$ and plugging in our $a_2 k$ and a_{2k+1} terms we can see that

$$\begin{aligned}
y(t) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2 \cdot 4 \cdot 6 \ldots 2k} a_0 t^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^k}{3 \cdot 5 \cdot 7 \ldots (2k+1)} a_1 t^{2k+1} \\
&= a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{2 \cdot 4 \cdot 6 \ldots 2k} t^{2k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{3 \cdot 5 \cdot 7 \ldots (2k+1)} t^{2k+1}
\end{aligned}$$

Hence the general solution to the ODE $y'' + ty' + y = 0$ is $a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{2 \cdot 4 \cdot 6 \ldots 2k} t^{2k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{3 \cdot 5 \cdot 7 \ldots (2k+1)} t^{2k+1}$.

Exercise 4: By using the series expansion around $t_0 = 0$, find the general solution of the ODE

$$y'' - xy' + y = 0 \tag{3}$$

Hint: the solution is $y(x) = a_1 x + a_0 \left[1 - \frac{1}{2}x^2 - \sum_{k=2}^{\infty} \frac{(2k-3)!!}{(2k)!} x^{2k}\right]$. Double factorial $n!!$ is the product of all integers from 1 to n with the same parity (odd or even) as n; e.g. $9!! = 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1$.

To start we will replace y'', y', and y with their series expansions around zero and simplify if possible.

$$\begin{aligned} 0 &= y' - xy' + y \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - x\sum_{n=0}^{\infty}(n+1)a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}x^{n+1} + \sum_{n=0}^{\infty} a_n x^n \end{aligned}$$

However, in order to solve this problem more easily, we are going to want to have the same power of x for all series in the ODE. Hence, in the second series we must introduce the change of index $k = n+1$. Hence the series becomes $\sum_{k=1}^{\infty} k a_k x^k$. However since k is just a "dummy" variable we will chance it back to n for convenience. Hence the series becomes $\sum_{n=1}^{\infty} n a_n x^n$. Putting this back into the equation we obtain the following

$$0 = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n \quad = a_0 + 2a_2 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + (1-n)a_n\right]x^n$$

Hence as shown above, $a_0 + 2a_2 + \sum_{n=1}^{\infty}\left[(n+2)(n+1)a_{n+2} + (1-n)a_n\right]x^n = 0$, thus in order for this relationship to hold for all values of x, $a_0 + 2a_2 = 0$ and $(n+2)(n+1)a_{n+2} + (1-n)a_n = 0 \implies a_{n+2} = \frac{-(1-n)}{(n+2)(n+1)}a_{n-1},\ n = 1, 2, 3, \ldots$. However, notice that $a_0 + 2a_2 = 0$ is $(n+2)(n+1)a_{n+2} + (1-n)a_n = 0$ at $n = 0$. Thus we can see that $a_{n+2} = \frac{n-1}{(n+2)(n+1)}a_n,\ n = 0, 1, 2, 3, \cdots$. Since this recursive definition steps by 2, we must split finding a closed form solution into two parts, one where n is even, and one where n is odd.

n is even $\implies$ n = 2k:
Writing out the first four values of this recursive definition we see that: $a_2 = \frac{-1}{2}a_0 = \frac{-1}{2!}a_0$, $a_4 = \frac{1}{4\cdot 3}a_2 = \frac{-1}{4\cdot3\cdot2\cdot1}a_0 = \frac{-1}{4!}a_0$, $a_6 = \frac{3}{6\cdot5}a_4 = \frac{-3}{6\cdot5\cdot4\cdot3\cdot2\cdot1}a_0 = \frac{-3}{6!}a_0$, and $a_8 = \frac{5}{8\cdot7} = \frac{-5\cdot3}{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1} = \frac{-5\cdot3}{8!}a_0$.

n is odd $\implies$ n = 2k:
Writing out the first three values of this recursive definition we see that: $a_3 = 0 \cdot a_1 = 0$, $a_5 = \frac{2}{5\cdot4}a_3 = 0$, and $a_7 = \frac{4}{7\cdot6}a_5 = 0$. Thus in general, we can see that $a_{2k+1} = 0$.

Thus, if we expand our solution from these above calculations we obtain $a_0 + a_1 x - \frac{1}{2!}a_0x^2 - \frac{1}{4!}a_0x^4 - \frac{3}{6!}a_0x^6 - \frac{5\cdot3}{8!}a_0x^8 - \frac{7\cdot5\cdot3}{10!}x^{10} - \cdots = a_1x + a_0\left(1 - \frac{1}{2}x^2 - \frac{1}{4!}x^4 - \frac{3}{6!}x^6 - \frac{5\cdot3}{8!}x^8 - \frac{7\cdot5\cdot3}{10!}x^{10} - \ldots\right)$. However, notice that this simplifies to $a_1x + a_0\left[1 - \frac{1}{2}x^2 - \sum_{k=2}^{\infty}\frac{(2k-3)!!}{(2k)!}x^{2k}\right]$.

Hence the general solution to the ODE $y'' - xy' + y = 0$ is $a_1x + a_0\left[1 - \frac{1}{2}x^2 - \sum_{k=2}^{\infty}\frac{(2k-3)!!}{(2k)!}x^{2k}\right]$.