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AMATH 351

Homework 4: 2nd order linear ODEs with constant coefficients.

- Exercise 1: General solution of homogeneous ODEs!

1.) $y'' + 4y' + 4y = 0$

Corresponding characteristic equation:
$r^2 + 4r + 4 = 0 \Rightarrow (r+2)^2 = 0$

discriminant calculation:
$\Delta = 4^2 - 4(4)(1) = 16 - 16 = 0$
thus we have one double root: $r = -2$

fundamental solutions:
$y_1(t) = e^{-2t}$, $y_2(t) = te^{-2t}$

General solution:
$y(t) = c_1 e^{-2t} + c_2 te^{-2t}$

2.) $y'' + 6y' + 9y = 0$

Corresponding characteristic equation:
$r^2 + 6r + 9 = 0 \Rightarrow (r+3)^2 = 0$

discriminant calculation:
$\Delta = 6^2 - 4(9)(1) = 36 - 36 = 0$
thus we have one double root: $r = -3$

fundamental solutions:
$y_1(t) = e^{-3t}$, $y_2(t) = te^{-3t}$

General solution:
$y(t) = c_1 e^{-3t} + c_2 te^{-3t}$

3.) $y'' + 2y' + 5y = 0$

corresponding characteristic equation:
$r^2 + 2r + 5 = 0$

discriminant calculation:
$\Delta = 2^2 - 4(5)(1) = 4 - 20 = -16 < 0$

thus we have two complex conjugate roots:
$r_{1,2} = \sigma \pm i\omega = \frac{-2 \pm i\sqrt{16}}{2} = -1 \pm 2$

fundamental solutions:
$y_1(t) = e^{-t}\cos(2t)$, $\quad y_2(t) = e^{-t}\sin(2t)$

General solution:
$y(t) = c_1 e^{-t}\cos(2t) + c_2 e^{-t}\sin(2t)$

4.) $y'' + 9y = 0$

corresponding characteristic equation:
$r^2 + 9 = 0$

discriminant calculation:
$\Delta = -4(9)(1) = -36 < 0$

thus we have two complex conjugate roots:
$r_{1,2} = \sigma \pm i\omega = \frac{i\sqrt{36}}{2} = 3$

fundamental solutions:
$y_1(t) = \cos(3t)$, $\quad y_2(t) = \sin(3t)$

general solution:
$y(t) = c_1\cos(3t) + c_2\sin(3t)$

- Exercise 2: General solution of non-homogeneous ODES. The variation of parameters method.

1.) $y'' - 3y' + 2y = \sin(x)$

$\boxed{\text{STEP 1:}}$ Find the corresponding homogeneous solution:

$y'' - 3y' + 2y = 0$

corresponding characteristic equation:

$r^2 - 3r + 2 = 0 \Rightarrow (r+2)(r-1) = 0$

discriminant calculation:

$\Delta = (-3)^2 - 4(2)(1) = 9 - 8 = 1 > 0$

thus we have two distinct real roots:

$r_1 = 1$, $r_2 = 2$

fundamental solutions:

$y_1(x) = e^x$, $y_2(x) = e^{2x}$

general solution:

$$y(x) = c_1 e^x + c_2 e^{2x}$$

$\boxed{\text{STEP 2:}}$ Find the Wronskian $W(y_1, y_2)(x)$:

$y_1'(x) = e^x$

$y_2'(x) = 2e^{2x}$

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x}$$

$\boxed{\text{STEP 3:}}$ Find a particular solution of the form $y_p(x) = c_1(x) y_1(x) + c_2(x) y_2(x)$

i.) Finding $c_1(x)$:

$$c_1(x) = -\int \frac{y_2(x)\, b(x)}{W(y_1, y_2)(x)}\, dx$$

$$c_1(x) = -\int \frac{e^{2x}\sin(x)}{e^{3x}}\,dx$$

$$= \int -e^{-x}\sin(x)\,dx$$

use the DI method of integration by parts:

	D	I
+	$-e^{-x}$	$\sin(x)$
−	e^{-x}	$-\cos(x)$
+	$-e^{-x}$	$-\sin(x)$
−	e^{-x}	$\cos(x)$

$$\Rightarrow \int -e^{-x}\sin(x)\,dx = e^{-x}\cos(x) + e^{-x}\sin(x) - \int -e^{-x}\sin(x)\,dx$$

$$2\int -e^{-x}\sin(x)\,dx = e^{-x}\cos(x) + e^{-x}\sin(x)$$

$$\int -e^{-x}\sin(x)\,dx = \frac{e^{-x}}{2}(\cos(x) + \sin(x))$$

ii) Finding $C_2(x)$

$$C_2(x) = \int \frac{g_1(x)\,b(x)}{W(y_1,y_2)(x)}\,dx$$

$$= \int \frac{e^{x}\sin(x)}{e^{3x}}\,dx$$

$$= \int e^{-2x}\sin(x)\,dx$$

Use the DI method of integration by parts:

	D	I
+	e^{-2x}	$\sin(x)$
−	$-2e^{-2x}$	$-\cos(x)$
+	$4e^{-2x}$	$-\sin(x)$

$$\Rightarrow \int e^{-2x}\sin(x)\,dx = -e^{-2x}\cos(x) - 2e^{-2x}\sin(x) - 4\int e^{-2x}\sin(x)\,dx$$

$$5\int e^{-2x}\sin(x) = -e^{-2x}\cos(x) - 2e^{-2x}\sin(x)$$

$$\int e^{-2x}\sin(x) = \frac{-e^{-2x}}{5}(\cos(x) + 2\sin(x))$$

iii) Putting it all together:

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$$

$$= e^{x}\frac{e^{-x}}{2}(\cos(x) + \sin(x)) + e^{2x}\cdot\frac{-e^{-2x}}{5}(\cos(x) + 2\sin(x))$$

$$= +\frac{1}{2}\cos(x) + \frac{1}{2}\sin(x) - \frac{1}{5}\cos(x) - \frac{2}{5}\sin(x)$$

$$= \frac{3}{10}\cos(x) + \frac{1}{10}\sin(x)$$

$\boxed{\text{STEP 4:}}$ combine $y(x)$ and $y_p(x)$ to get the general solution:

$$y(x) = \frac{3}{10}\cos(x) + \frac{1}{10}\sin(x) + c_1e^{x} + c_2e^{2x}$$

2.) $y'' + 4y' + 4y = x - 2e^{2x}$

$\boxed{\text{STEP 2:}}$ Find the corresponding homogeneous solution:

$y'' + 4y' + 4y = 0$

general solution:

As calculated in Exercise 1 problem 1, the general solution is

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x}$$

$\boxed{\text{STEP 2:}}$ Find the Wronskian $W(y_1, y_2)(x)$:

$y_1'(x) = -2e^{-2x}$

$y_2'(x) = e^{-2x} - 2xe^{-2x}$

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x} - 2xe^{-2x} \end{vmatrix} = e^{-4x} - 2xe^{-4x} + 2xe^{-4x} = e^{-4x}$$

$\boxed{\text{STEP 3:}}$ Find a particular solution of the form $y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$

i) Finding $c_1(x)$:

$$c_1(x) = -\int \frac{y_2(x)\, b(x)}{W(y_1, y_2)(x)}\, dx$$

$$= -\int \frac{xe^{-2x}(x - 2e^{2x})}{e^{-4x}}$$

$$= -\int xe^{2x}(x - 2e^{2x})\, dx$$

$$= -\int x^2 e^{2x}\, dx + \int 2xe^{4x}\, dx$$

USE the DI method of integration by parts

	D	I
+	x^2	e^{2x}
−	$2x$	$\frac{1}{2}e^{2x}$
+	2	$\frac{1}{4}e^{2x}$
−	0	$\frac{1}{8}e^{2x}$

	D	I
+	$2x$	e^{4x}
−	2	$\frac{1}{4}e^{4x}$
+	0	$\frac{1}{16}e^{4x}$

$$c_1(x) = -\frac{1}{2}x^2 e^{2x} + \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + \frac{1}{2}xe^{4x} - \frac{1}{8}e^{4x}$$

ii) Finding $C_2(x)$:

$$c_2(x) = \int \frac{y_1(x)\, b(x)}{W(y_1, y_2)(x)}\, dx = \int \frac{e^{-2x}(x - 2e^{2x})}{e^{4x}}$$

$$= \int e^{2x}(x - 2e^{2x})\, dx$$

$$= \int x e^{2x}\, dx - \int 2e^{4x}\, dx$$

USE the DI method of integration by parts,

	D	I
+	x	e^{2x}
−	1	$\frac{1}{2} e^{2x}$
+	0	$\frac{1}{4} e^{2x}$

$$= \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} - \frac{1}{2} e^{4x}$$

iii) putting it all together:

$$\psi_p(x) = c_1(x) y_1(x) + c_2(x) y_2(x)$$

$$= e^{-2x}\left(-\frac{1}{2}x^2 e^{2x} + \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + \frac{1}{2} x e^{4x} - \frac{1}{3} e^{4x}\right) + x e^{-2x}\left(\frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} - \frac{1}{2} e^{4x}\right)$$

$$= -\frac{1}{2} x^2 + \frac{1}{2} x - \frac{1}{4} + \frac{1}{2} x e^{2x} - \frac{1}{8} e^{2x} + \frac{1}{2} x^2 - \frac{1}{4} x - \frac{1}{2} x e^{2x}$$

$$= \frac{1}{4} x - \frac{1}{4} - \frac{1}{8} e^{2x}$$

$\boxed{\text{STEP 4:}}$ combine $y(x)$ and $\psi_p(x)$ to get the general solution:

$$y(x) = \frac{1}{4} x - \frac{1}{4} - \frac{1}{8} e^{2x} + c_1 e^{-2x} + c_2 x e^{-2x}$$

3.) $8y'' + 4y' + y = \sin(x) - 2\cos(x)$

$y'' + \frac{1}{2}y' + \frac{1}{8}y = \frac{1}{8}\sin(x) - \frac{1}{4}\cos(x)$

$\boxed{\text{STEP 1:}}$ Find the corresponding homogeneous solution:

$y'' + \frac{1}{2}y' + \frac{1}{8}y = 0$

corresponding characteristic equation:

$r^2 + \frac{1}{2}r + \frac{1}{8} = 0$

discriminant calculation:

$\Delta = (\frac{1}{2})^2 - 4(\frac{1}{8})(1) = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4} < 0$

thus we have to complex conjugate roots

$r_{1,2} = \sigma \pm i\omega = \frac{-\frac{1}{2} \pm i\sqrt{\frac{1}{4}}}{2} = \frac{-\frac{1}{2} \pm \frac{1}{2}}{2} = -\frac{1}{4} \pm \frac{1}{4}$

fundamental solutions:

$y_1(x) = e^{-\frac{1}{4}x}\cos(\frac{1}{4}x)$, $y_2(x) = e^{-\frac{1}{4}x}\sin(\frac{1}{4}x)$

general solution:

$y(x) = c_1 e^{-\frac{1}{4}x}\cos(\frac{1}{4}x) + c_2 e^{-\frac{1}{4}x}\sin(\frac{1}{4}x)$

$\boxed{\text{STEP 2:}}$ Find the Wronskian $W(y_1, y_2)(x)$:

$y_1'(x) = -\frac{1}{4}e^{-\frac{1}{4}x}\cos(\frac{1}{4}x) - \frac{1}{4}e^{-\frac{1}{4}x}\sin(\frac{1}{4}x)$

$y_2'(x) = -\frac{1}{4}e^{-\frac{1}{4}x}\sin(\frac{1}{4}x) + \frac{1}{4}e^{-\frac{1}{4}x}\cos(\frac{1}{4}x)$

$$W(y_1,y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-\frac{1}{4}x}\cos(\frac{1}{4}x) & e^{-\frac{1}{4}x}\sin(\frac{1}{4}x) \\ -\frac{1}{4}e^{-\frac{1}{4}x}\cos(\frac{1}{4}x) - \frac{1}{4}e^{-\frac{1}{4}x}\sin(\frac{1}{4}x) & -\frac{1}{4}e^{-\frac{1}{4}x}\sin(\frac{1}{4}x) + \frac{1}{4}e^{-\frac{1}{4}x}\cos(\frac{1}{4}x) \end{vmatrix}$$

$= -\frac{1}{4}e^{-\frac{1}{2}x}\sin(\frac{1}{4}x)\cos(\frac{1}{4}x) + \frac{1}{4}e^{-\frac{1}{2}x}\cos^2(\frac{1}{4}x) - (-\frac{1}{4}e^{-\frac{1}{2}x}\cos(\frac{1}{4}x)\sin(\frac{1}{4}x) - \frac{1}{4}e^{-\frac{1}{2}x}\sin^2(\frac{1}{4}x))$

$= \frac{1}{4}e^{-\frac{1}{2}x}\cos^2(\frac{1}{4}x) + \frac{1}{4}e^{-\frac{1}{2}x}\sin^2(\frac{1}{4}x)$

$= \frac{1}{4}e^{-\frac{1}{2}x}(\cos^2(\frac{1}{4}x) + \sin^2(\frac{1}{4}x))$

$= \frac{1}{4}e^{-\frac{1}{2}x}$

STEP 3: Find a particular solution of the form $y_p(x) = c_1(x)\, y_1(x)\, c_2(x)\, y_2(x)$

i) Finding $c_1(x)$:

$$c_1(x) = -\int \frac{y_2(x)\,b(x)}{W(y_1, y_2)(x)}\,dx$$

$$= \int \frac{-\sin(\tfrac{1}{4}x)\,e^{-\frac{1}{4}x}\left(\tfrac{1}{8}\sin(x) - \tfrac{1}{4}\cos(x)\right)}{\tfrac{1}{4}e^{-\frac{1}{2}x}}\,dx$$

$$= \int -4e^{\frac{1}{2}x}\left(\sin(\tfrac{1}{4}x)\,e^{-\frac{1}{4}x}\right)\left(\tfrac{1}{8}\sin(x) - \tfrac{1}{4}\cos(x)\right)dx$$

$$= \int -4\sin(\tfrac{1}{4}x)\,e^{\frac{1}{4}x}\left(\tfrac{1}{8}\sin(x) - \tfrac{1}{4}\cos(x)\right)dx$$

$$= \int -\tfrac{1}{2}\sin(\tfrac{1}{4}x)\sin(x)\,e^{\frac{1}{4}x} + \sin(\tfrac{1}{4}x)\cos(x)\,e^{\frac{1}{4}x}\,dx$$

$$= \int -\tfrac{1}{4}e^{\frac{1}{4}x}\cos(-\tfrac{3}{4}x) + \tfrac{1}{4}e^{\frac{1}{4}x}\cos(\tfrac{5}{4}x) + \tfrac{1}{2}e^{\frac{1}{4}x}\sin(\tfrac{5}{4}x) + \tfrac{1}{2}e^{\frac{1}{4}x}\sin(-\tfrac{3}{4}x)\,dx$$

$$c_1(x) = \tfrac{1}{26}e^{\frac{1}{4}x}\left(7\sin(\tfrac{5}{4}x) - 4\cos(\tfrac{5}{4}x) - 13\sin(\tfrac{3}{4}x) + 13\cos(\tfrac{3}{4}x)\right)$$

ii) Finding $c_2(x)$:

$$c_2(x) = \int \frac{y_1(x)\,b(x)}{W(y_1, y_2)(x)}\,dx$$

$$= \int \frac{\left(e^{-\frac{1}{4}x}\cos(\tfrac{1}{4}x)\right)\left(\tfrac{1}{8}\sin(x) - \tfrac{1}{4}\cos(x)\right)}{\tfrac{1}{4}e^{-\frac{1}{2}x}}\,dx$$

$$= \int 4e^{\frac{1}{4}x}\cos(\tfrac{1}{4}x)\left(\tfrac{1}{8}\sin(x) - \tfrac{1}{4}\cos(x)\right)$$

$$= \int \tfrac{1}{2}e^{\frac{1}{4}x}\cos(\tfrac{1}{4}x)\sin(x) - e^{\frac{1}{4}x}\cos(\tfrac{1}{4}x)\cos(x)$$

$$= \int \tfrac{1}{4}e^{\frac{1}{4}x}\sin(\tfrac{5}{4}x) - \tfrac{1}{4}e^{\frac{1}{4}x}\sin(\tfrac{3}{4}x) - \tfrac{1}{2}e^{\frac{1}{4}x}\cos(\tfrac{5}{4}x) + \tfrac{-1}{2}e^{\frac{1}{4}x}\cos(-\tfrac{3}{4}x)\,dx$$

$$c_2(x) = -\tfrac{1}{26}e^{\frac{1}{4}x}\left(4\sin(\tfrac{5}{4}x) + 7\cos(\tfrac{5}{4}x) + 13\sin(\tfrac{3}{4}x) + 13\cos(\tfrac{3}{4}x)\right)$$

iii) putting it all together:

$$\psi_p(x) = c_1(x)y_1(x) + c_2 y_2(x)$$
$$= e^{-\frac{1}{4}x}\cos\left(\tfrac{1}{4}x\right)\left(\tfrac{1}{26}e^{\frac{1}{4}x}\left(7\sin\left(\tfrac{5}{4}x\right) - 9\cos\left(\tfrac{5}{4}x\right) - 13\sin\left(\tfrac{3}{4}x\right) + 13\cos\left(\tfrac{3}{4}x\right)\right)\right) +$$
$$e^{-\frac{1}{4}x}\sin\left(\tfrac{1}{4}x\right)\left(-\tfrac{1}{26}e^{\frac{1}{4}x}\left(9\sin\left(\tfrac{5}{4}x\right) + 7\cos\left(\tfrac{5}{4}x\right) + 13\sin\left(\tfrac{3}{4}x\right) + 13\sin\left(\tfrac{3}{4}x\right)\right)\right)$$

$\boxed{\text{STEP 4:}}$ combine $y(x)$ and $\psi_p(x)$:

$$y(x) = c_1 e^{-\frac{1}{4}x}\cos\left(\tfrac{1}{4}x\right) + c_2 e^{-\frac{1}{4}x}\sin\left(\tfrac{1}{4}x\right) + e^{-\frac{1}{4}x}\cos\left(\tfrac{1}{4}x\right)\left(\tfrac{1}{26}e^{\frac{1}{4}x}\left(7\sin\left(\tfrac{5}{4}x\right) - 9\cos\left(\tfrac{5}{4}x\right) - 13\sin\left(\tfrac{3}{4}x\right) + 13\cos\left(\tfrac{3}{4}x\right)\right)\right) +$$
$$e^{-\frac{1}{4}x}\sin\left(\tfrac{1}{4}x\right)\left(-\tfrac{1}{26}e^{\frac{1}{4}x}\left(9\sin\left(\tfrac{5}{4}x\right) + 7\cos\left(\tfrac{5}{4}x\right) + 13\sin\left(\tfrac{3}{4}x\right) + 13\sin\left(\tfrac{3}{4}x\right)\right)\right)$$

Which simplifies to:

$$y(x) = c_1 e^{-\frac{1}{4}x}\cos\left(\tfrac{1}{4}x\right) + c_2 e^{-\frac{1}{4}x}\sin\left(\tfrac{1}{4}x\right) + \tfrac{2}{13}\cos(x) - \tfrac{3}{13}\sin(x).$$

4.) $y'' - 2y' + y = 4e^x$

$\boxed{\text{STEP 1:}}$ Find the corresponding homogeneous solution:

$y'' - 2y' + y = 0$

corresponding characteristic equation:

$r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0$

discriminant calculation:

$\Delta = (-2)^2 - 4(1)(1) = 4 - 4 = 0$

thus we have one repeated root

$r = 1$

fundamental solutions:

$y_1(x) = e^x, \quad y_2(x) = xe^x$

general solution:

$y(x) = c_1 e^x + c_2 e^x x$

$\boxed{\text{STEP 2:}}$ Find the wronskian $W(y_1, y_2)(x)$

$y_1'(x) = e^x$

$y_2'(x) = e^x + xe^x$

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = xe^{2x} + e^{2x} - xe^{2x} = e^{2x}$$

$\boxed{\text{STEP 3:}}$ Find a particular solution of the form $y_p(x) = c_1(x) y_1(x) + c_2(x) y_2(x)$

1) Finding $c_1(x)$

$$c_1(x) = -\int \frac{y_2(x) b(x)}{W(y_1, y_2)(x)} dx = -\int \frac{xe^x \cdot 4e^x}{e^{2x}} = -\int 4x\,dx = -2x^2$$

ii.) Finding $c_2(x)$:

$$c_2(x) = \int \frac{y_1(x)\, b(x)}{W(y_1, y_2)(x)}\, dx = \int \frac{e^x \cdot 4e^x}{e^{2x}}\, dx = \int 4\, dx = 4x$$

iii.) Putting it all together:

$$\begin{aligned} y_p(t) &= c_1(x)\, y_1(x) + c_2(x)\, y_2(x) \\ &= -2x^2 e^x + 4x^2 e^x \\ &= 2x^2 e^x \end{aligned}$$

$\boxed{\text{STEP 4:}}$ Combine $y(x)$ and $y_p(x)$ to get the general solution

$$y(x) = 2x^2 e^x + c_2 x e^x + c_1 e^x$$

- Exercise 3: Wronskian determinant:

1.) $y_1(t) = e^{\sigma t}\cos(\omega t)$, $y_2(t) = e^{\sigma t}\sin(\omega t)$

To show that $y_1(t)$ and $y_2(t)$ are linearly independent, we must show that $W(y_1, y_2) \neq 0$.

$y_1'(t) = \sigma e^{\sigma t}\cos(\omega t) - \omega e^{\sigma t}\sin(\omega t)$
$y_2'(t) = \sigma e^{\sigma t}\sin(\omega t) + \omega e^{\sigma t}\cos(\omega t)$

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{\sigma t}\cos(\omega t) & e^{\sigma t}\sin(\omega t) \\ \sigma e^{\sigma t}\cos(\omega t) - \omega e^{\sigma t}\sin(\omega t) & \sigma e^{\sigma t}\sin(\omega t) + \omega e^{\sigma t}\cos(\omega t) \end{vmatrix}$$

$$= \sigma e^{2\sigma t}\cos(\omega t)\sin(\omega t) + \omega e^{2\sigma t}\cos^2(\omega t) - \left(\sigma e^{2\sigma t}\cos(\omega t)\sin(\omega t) - \omega e^{2\sigma t}\sin^2(\omega t)\right)$$
$$= \omega e^{2\sigma t}\cos^2(\omega t) + \omega e^{2\sigma t}\sin^2(\omega t)$$
$$= \omega e^{2\sigma t}\left(\cos^2(\omega t) + \sin^2(\omega t)\right)$$
$$= \omega e^{2\sigma t} \neq 0 \quad \forall t$$

$\therefore$ $y_1(t)$ and $y_2(t)$ are linearly independent.

2.) $y_1(t) = t^{\lambda}\cos(\omega\ln(t))$, $y_2(t) = t^{\lambda}\sin(\omega\ln(t))$, for $t > 0$

To show that $y_1(t)$ and $y_2(t)$ are linearly independent, we must show that $W(y_1, y_2) \neq 0$.

$$y_1'(t) = \lambda t^{\lambda-1}\cos(\omega\ln(t)) - t^{\lambda}\sin(\omega\ln(t))\cdot\frac{\omega}{t}$$
$$= -\omega t^{\lambda-1}\sin(\omega\ln(t)) + \lambda t^{\lambda-1}\cos(\omega\ln(t))$$

$$y_2'(t) = \lambda t^{\lambda-1}\sin(\omega\ln(t)) + t^{\lambda}\cos(\omega\ln(t))\cdot\frac{\omega}{t}$$
$$= \lambda t^{\lambda-1}\sin(\omega\ln(t)) + \omega t^{\lambda-1}\cos(\omega\ln(t))$$

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{\lambda}\cos(\omega\ln(t)) & t^{\lambda}\sin(\omega\ln(t)) \\ \lambda t^{\lambda-1}\cos(\omega\ln(t)) - \omega t^{\lambda-1}\sin(\omega\ln(t)) & \lambda t^{\lambda-1}\sin(\omega\ln(t)) + \omega t^{\lambda-1}\cos(\omega\ln(t)) \end{vmatrix}$$

$$= \lambda t^{\lambda}\cos(\omega\ln(t))\sin(\omega\ln(t)) + \omega t^{\lambda}\cos^2(\omega\ln(t)) - \left(\lambda t^{\lambda}\cos(\omega\ln(t))\sin(\omega\ln(t)) - \omega t^{\lambda}\sin^2(\omega\ln(t))\right)$$

$$= \omega t^{\lambda}\cos^2(\omega\ln(t)) + \omega t^{\lambda}\sin^2(\omega\ln(t))$$

$$= \omega t^{\lambda}\left(\cos^2(\omega\ln(t)) + \sin^2(\omega\ln(t))\right)$$

$$= \omega t^{\lambda} \neq 0 \quad \forall t > 0$$

$\therefore$ $y_1(t)$ and $y_2(t)$ are linearly independent.

Exercise 4: IVP solution:

$y'' + K^2 y = \cos(t)$, $y(0) = 0$, $y'(0) = 0$, $K \neq 0$

CASE 1: K=1

$y'' + y = \cos(t)$

STEP 1: Find the corresponding homogeneous solution:

$y'' + y = 0$

corresponding characteristic equation:

$r^2 + 1 = 0$

discriminant calculation:

$\Delta = -4(1)(1) = -4 < 0$

thus we have two complex conjugate roots:

$r_{1,2} = \sigma \pm i\omega = \frac{i\sqrt{4}}{2} = 1$

fundamental solutions:

$y_1(t) = \cos(t)$, $y_2(t) = \sin(t)$

general solution:

$y(t) = c_1 \cos(t) + c_2 \sin(t)$

STEP 2: Find a particular solution of the form $\psi_p(t) = c_1(t) y_1(t) + c_2(t) y_2(t)$

where $c_1(t) = -\int \frac{y_2(t) b(t)}{W(y_1, y_2)(t)} dt$ and $c_2(t) = \int \frac{y_1(t) b(t)}{W(y_1, y_2)(t)} dt$

i) Finding $c_1(t)$:

$c_1(t) = -\int \frac{y_1(t) b(t)}{W(y_1, y_2)(t)} dt$

$W(y_1, y_2)(t) = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} = \cos^2(t) + \sin^2(t) = 1$

$c_1(t) = -\int \sin(t)\cos(t)\,dt$

let $u = \sin(t) \Rightarrow du = \cos(t)\,dt$

$= -\int u\,du$

$= -\frac{1}{2}u^2$

$= -\frac{1}{2}\sin^2(t)$

ii.) Finding $c_2(t)$:

$c_2(t) = \int \frac{y_1(t)\,b(t)}{W(y_1,y_2)(t)}\,dt$

$= \int \cos^2(t)\,dt$

$= \int \frac{1}{2}(1+\cos(2t))\,dt$

$= \int \frac{1}{2}\,dt + \int \frac{1}{2}\cos(2t)\,dt$

$= \frac{1}{2}t + \frac{1}{2}\int \cos(2t)\,dt$

let $u = 2t \Rightarrow du = 2dt$

$= \frac{1}{2}t + \frac{1}{4}\int \cos(u)\,du$

$= \frac{1}{2}t + \frac{1}{4}\sin(u)$

$= \frac{1}{2}t + \frac{1}{4}\sin(2t)$

iii) Putting it all together:

$\psi_p(t) = c_1(t)y_1(t) + c_2(t)y_2(t)$

$= -\frac{1}{2}\sin^2(t)\cos(t) + \left(\frac{1}{2}t + \frac{1}{4}\sin(2t)\right)\sin(t)$

$= -\frac{1}{2}\sin^2(t)\cos(t) + \frac{1}{2}t\sin(t) + \frac{1}{4}\sin(2t)\sin(t)$

$= -\frac{1}{2}\sin^2(t)\cos(t) + \frac{1}{2}\sin^2(t)\cos(t) + \frac{1}{2}t\sin(t)$

$= \frac{1}{2}t\sin(t)$

$\boxed{\text{STEP 3:}}$ Combine $y(t)$ and $\psi_p(t)$ to get the general solution

$y(t) = c_1\cos(t) + c_2\sin(t) + \frac{1}{2}t\sin(t)$

$\boxed{\text{STEP 4:}}$ Find c_1 and c_2

$y(t) = c_1\cos(t) + c_2\sin(t) + \frac{1}{2}t\sin(t) \Rightarrow y(0) = c_1 = 0$

$y'(t) = -c_1\sin(t) + c_2\cos(t) + \frac{1}{2}\sin(t) + \frac{1}{2}t\cos(t) \Rightarrow y'(0) = c_2 = 0$

hence the solution to the IVP is $y(t) = \frac{1}{2}t\sin(t)$

CASE 2: $K \neq 1$

$y'' + K^2 y = \cos(t)$, $y(0) = 0$, $y'(0) = 0$, $K \neq 0$

$\boxed{\text{STEP 1:}}$ Find the corresponding homogeneous solution:

$y'' + K^2 y = 0$

corresponding characteristic equation:

$r^2 + K^2 = 0$

discriminant calculation:

$\Delta = -4(K^2)(1) = -4K^2 < 0$

Thus we have two complex conjugate roots:

$r_{1,2} = \sigma \pm i\omega = \frac{i\sqrt{4K^2}}{2} = \frac{2K}{2} = K$

fundamental solutions:

$y_1(t) = \cos(Kt)$, $y_2(t) = \sin(Kt)$

general solution:

$y(t) = c_1 \cos(Kt) + c_2 \sin(Kt)$

$\boxed{\text{STEP 2:}}$ Find the wronskian $W(y_1, y_2)(t)$:

$y_1'(t) = -K \sin(Kt)$

$y_2'(t) = K \cos(Kt)$

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(Kt) & \sin(Kt) \\ -K\sin(Kt) & K\cos(Kt) \end{vmatrix} = K\cos^2(Kt) + K\sin^2(Kt) = K(\cos^2(Kt) + \sin^2(Kt)) = K$$

$\boxed{\text{STEP 3:}}$ Find a particular solution of the form $\phi_p(t) = c_1(t) + c_2(t)$

1) Finding $c_1(t)$:

$$c_1(t) = -\int \frac{y_2(t)\, b(t)}{W(y_2, y_1)(t)}\, dt$$

$$c_1(t) = -\int \frac{\sin(kt)\cos(t)}{k}\,dt$$

$$= -\frac{1}{k}\int \sin(kt)\cos(t)\,dt$$

$$= -\frac{1}{k}\int \frac{1}{2}\left[\sin(kt+t) + \sin(kt-t)\right]dt$$

$$= -\frac{1}{2k}\int \sin((k+1)t)\,dt + -\frac{1}{2k}\int \sin((k-1)t)\,dt$$

let $u = (k+1)t$ let $u = (k-1)t$

$du = k+1\,dt$ $du = k-1\,dt$

$$= \frac{-1}{2k(k+1)}\int \sin(u)\,du + \frac{-1}{2k(k-1)}\int \sin(u)\,du$$

$$= +\frac{1}{2k(k+1)}\cos((k+1)t) + \frac{1}{2k(k-1)}\cos((k-1)t)$$

ii) Finding $c_2(t)$:

$$c_2(t) = \int \frac{y_1(t)\,b(t)}{W(y_1,y_2)(t)}\,dt$$

$$c_2(t) = \frac{1}{k}\int \cos(kt)\cos(t)\,dt$$

$$= \frac{1}{2k}\int\left[\cos((k-1)t) + \cos((k+1)t)\right]dt$$

$$= \frac{1}{2k}\int \cos((k-1)t) + \frac{1}{2k}\int \cos((k+1)t)\,dt$$

let $u = (k-1)t$ let $u = (k+1)t$

$du = k-1\,dt$ $du = k+1\,dt$

$$= \frac{1}{2k(k-1)}\int \cos(u)\,du + \frac{1}{2k(k+1)}\int \cos(u)\,du$$

$$= \frac{1}{2k(k-1)}\sin((k-1)t) + \frac{1}{2k(k+1)}\sin((k+1)t)$$

iii) Putting it all together:

$$\psi_p(t) = c_1(t)y_1(t) + c_2(t)y_2(t)$$

$$= \cos(kt)\left(\frac{1}{2k(k+1)}\cos((k+1)t) + \frac{1}{2k(k-1)}\cos((k-1)t)\right) + \sin(kt)\left(\frac{1}{2k(k-1)}\sin((k-1)t) + \frac{1}{2k(k+1)}\sin((k+1)t)\right)$$

$\boxed{\text{STEP 3:}}$ combine $y(t) + \psi_p(t)$ to get the general solution:

$$y(t) = c_1\cos(kt) + c_2\sin(kt) + \cos(kt)\left(\frac{1}{2k(k+1)}\cos((k+1)t) + \frac{1}{2k(k-1)}\cos((k-1)t)\right) + \sin(kt)$$
$$\left(\frac{1}{2k(k-1)}\sin((k-1)t) + \frac{1}{2k(k+1)}\sin((k+1)t)\right)$$

$\boxed{\text{STEP 4:}}$ Find c_1 and c_2:

$$y(0) = c_1 + \frac{1}{2K(K-1)} + \frac{1}{2K(K+1)} = 0$$

$$\Rightarrow c_1 = -\frac{1}{2K(K-1)} - \frac{1}{2K(K+1)}$$

$$\Rightarrow c_1 = \frac{-2K(K-1) - 2K(K+1)}{2K(K+1)\,2K(K-1)}$$

$$\Rightarrow c_1 = \frac{-4K^2}{4K^4 - 4K^2}$$

$$\Rightarrow c_1 = \frac{1}{1-K^2}$$

$$y'(t) = -c_1 K\sin(Kt) + c_2 K\cos(Kt) - K\sin(Kt)\left(\frac{1}{2K(K+1)}\cos((K+1)t) + \frac{1}{2K(K-1)}\cos((K-1)t)\right) + \cos(Kt)$$
$$\left(-\frac{1}{2K}\sin((K+1)t) - \frac{1}{2K}\sin((K-1)t)\right) + K\cos(Kt)\left(\frac{1}{2K(K-1)}\sin((K-1)t) + \frac{1}{2K(K+1)}\sin((K+1)t)\right) +$$
$$\sin(Kt)\left(\frac{1}{2K}\cos((K-1)t) + \frac{1}{2K}\cos((K+1)t)\right)$$

$$y'(0) = Kc_2 = 0$$

$$\Rightarrow c_2 = 0$$

Hence the solution of the IVP is:

$$y(t) = \frac{\cos(Kt)}{1-K^2} + \cos(Kt)\left(\frac{1}{2K(K+1)}\cos((K+1)t) + \frac{1}{2K(K-1)}\cos((K-1)t)\right) + \sin(Kt)$$
$$\left(\frac{1}{2K(K-1)}\sin((K-1)t) + \frac{1}{2K(K+1)}\sin((K+1)t)\right)$$

Which simplifies to:

$$y(t) = \frac{\cos(Kt)}{1-K^2} + \frac{\cos(t)}{K^2-1}$$