AMATH 351 Homework 6: Mechanical Vibrations

Jaiden Atterbury

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Underdamped oscillator under periodic external forcing. Let us consider the underdamped oscillator under a periodic force $F(t) = F_0 cos(\omega t)$ with amplitude F_0 and period ω:

$$my'' + cy' + ky = F_0 cos(\omega t), \quad (\text{with } c^2 < 4mk) \tag{1}$$

1. Find the general solution of the corresponding homogeneous ODE.

If $my'' + cy' + ky = F_0 cos(\omega t)$ is the ODE at hand, then the corresponding homogeneous ODE is thus $my'' + cy' + ky = 0$. To solve this we will first find its corresponding characteristic equation which is $mr^2 + cr + k = 0$. Since we are in the case that $c^2 < 4mk$, it turns out that the discriminant is less than zero, and hence we have two complex conjugate roots. These roots are of the form $r_{1,2} = \frac{-c \pm i\sqrt{4mk-c^2}}{2m} = \frac{-c}{2m} \pm i\frac{\sqrt{4mk-c^2}}{2m}$. Hence the two fundamental solutions are $y_1(t) = e^{\frac{-c}{2m}t}\cos\left(\frac{\sqrt{4mk-c^2}}{2m}t\right)$ and $y_2(t) = e^{\frac{-c}{2m}t}\sin\left(\frac{\sqrt{4mk-c^2}}{2m}t\right)$. Thus it follows that the general solution of the corresponding homogeneous solution is thus $c_1 \cdot e^{\frac{-c}{2m}t}\cos\left(\frac{\sqrt{4mk-c^2}}{2m}t\right) + c_2 \cdot e^{\frac{-c}{2m}t}\sin\left(\frac{\sqrt{4mk-c^2}}{2m}t\right)$.

2. By using the method of undetermined coefficients, see that one particular solution of ODE (1) is

$$\psi_p(t) = \frac{F_0}{(k-\omega^2 m)^2 + \omega^2 c^2}[\omega c \sin(\omega t) + (k - \omega^2 m)\cos(\omega t)]. \tag{2}$$

Since our ODE at hand is $my'' + cy' + ky = F_0 cos(\omega t$, it can be seen that the external forcing function is $F_0 cos(\omega t)$. Since our external forcing function is $F_0 cos(\omega t)$, it follows that the form of our particular solution is $\psi_p(t) = A\cos(\omega t) + B\sin(\omega t$. With the form of a particular solution identifies, we must take the first and second derivative of our particular solution then plug it back into our ODE and solve for A and B.

The first derivative of the particular solution form is $\psi'(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$, and the second derivative of the particular solution is $\psi''(t) = -A\omega^2\cos(\omega t) - B\omega^2\sin(\omega t)$. Plugging this back into the ODE we obtain: $-Am\omega^2\cos(\omega t) - Bm\omega^2\sin(\omega t) - Ac\omega\sin(\omega t) + Bc\omega\cos(\omega t) + Ak\cos(\omega t) + Bk\sin(\omega t) = F_0 cos(\omega t)$. This simplifies to, $(-Am\omega^2 + Bc\omega + Ak)\cos(\omega t) + (-Bm\omega^2 - Ac\omega + Bk)\sin(\omega t) = F_0 cos(\omega t)$. In order for this to hold for all t we must find the A and B that simultaneously solves the following equations: $-Am\omega^2 + Bc\omega + Ak = F_0$ and $-Bm\omega^2 - Ac\omega + Bk = 0$.

These two equations simplify to $(k - m\omega^2)A + c\omega B = F_0$ and $-c\omega A + (k - m\omega^2)B = 0$. This implies that $c\omega A = (k - m\omega^2)B$ and hence $B = \frac{c\omega}{k-m\omega^2}A$. Plugging this into the other equation we obtain $(k - m\omega^2)A + \frac{c^2\omega^2}{k-m\omega^2}A = F_0$ which simplifies to $\frac{(k-m\omega^2)^2+c^2\omega^2}{k-m\omega^2}A = F_0$. Hence solving for A we obtain $A = \frac{F_0(k-m\omega^2)}{(k-m\omega^2)^2+c^2\omega^2}$. Therefore this back into our expression for B we obtain: $B = \frac{c\omega}{k-m\omega^2} \cdot \frac{F_0(k-m\omega^2)}{(k-m\omega^2)^2+c^2\omega^2} = \frac{F_0 c\omega}{(k-m\omega^2)^2+c^2\omega^2}$. Hence the particular solution is:

$$\begin{aligned}\psi_p(t) &= \frac{F_0(k-m\omega^2)}{(k-m\omega^2)^2+c^2\omega^2}\cos(\omega t) + \frac{F_0 c\omega}{(k-m\omega^2)^2+c^2\omega^2}\sin(\omega t)\\ &= \frac{F_0}{(k-\omega^2 m)^2+\omega^2 c^2}[\omega c\sin(\omega t) + (k-\omega^2 m)\cos(\omega t)]\end{aligned}$$

3. By introducing

$$\sin(\phi) := \frac{\omega c}{\sqrt{(k-\omega^2 m)^2+\omega^2 c^2}},\ \cos(\phi) := \frac{k-\omega^2 m}{\sqrt{(k-\omega^2 m)^2+\omega^2 c^2}}$$

and employing the trigonometric identity

$$cos(a-b) = cos(a)cos(b) + sin(a) + sin(b)$$

see that solution (2) is expressed equivalently as

$$\psi_p(t) = \frac{F_0}{\sqrt{(k-\omega^2 m)^2+\omega^2 c^2}} \cos(\omega t - \phi) \tag{3}$$

with $\phi = \arctan(\omega c/(k-\omega^2 m))$.

To solve this problem, we will rely on turning one form of the particular solution, $y = A\cos(\omega t) + B\sin(\omega t)$, into another form, $y = R\cos(\omega t - \phi)$, using the following identity: $y = R\cos(\omega t-\phi) = R\cos(\phi)\cos(\omega t) + \sin(\phi) + \sin(\omega t)$. Setting the above identity equal to $y = A\cos(\omega t) + B\sin(\omega t)$, we conclude that $A = R\cos(\phi)$, and $B = R\sin(\phi)$. From these relationships we can conclude that $R^2 = A^2+B^2 \implies R = \sqrt{A^2+B^2}$.

Hence, if we remember that $A = \frac{F_0(k-m\omega^2)}{(k-m\omega^2)^2+c^2\omega^2}$ and $B = \frac{F_0 c\omega}{(k-m\omega^2)^2+c^2\omega^2}$, and if we introduce $\sin(\phi) := \frac{\omega c}{\sqrt{(k-\omega^2 m)^2+\omega^2 c^2}}$ and $\cos(\phi) := \frac{k-\omega^2 m}{\sqrt{(k-\omega^2 m)^2+\omega^2 c^2}}$, then we obtain the following system of equations: $\frac{F_0 c\omega}{(k-m\omega^2)^2+c^2\omega^2} = R\frac{\omega c}{\sqrt{(k-\omega^2 m)^2+\omega^2 c^2}}$ and $\frac{F_0(k-m\omega^2)}{(k-m\omega^2)^2+c^2\omega^2} = R\frac{k-\omega^2 m}{\sqrt{(k-\omega^2 m)^2+\omega^2 c^2}}$

Hence by isolation R in both equations we see that both equations lead to the same result. Namely, $R = \frac{F_0\sqrt{(k-\omega^2 m)^2+\omega^2 c^2}}{(k-m\omega^2)^2+c^2\omega^2}$. However, notice that $(k-m\omega^2)^2+c^2\omega^2$ is actually the same as $((k-m\omega^2)^2+c^2\omega^2)^1$ and $\sqrt{(k-\omega^2 m)^2+\omega^2 c^2} = ((k-\omega^2 m)^2+\omega^2 c^2)^{1/2}$. Hence we can see that $R = \frac{F_0}{\sqrt{(k-\omega^2 m)^2+\omega^2 c^2}}$. Lastly, if we see that $\frac{\sin(\phi)}{\cos(\phi)} = \frac{\frac{\omega c}{\sqrt{(k-\omega^2 m)^2+\omega^2 c^2}}}{\frac{k-\omega^2 m}{\sqrt{(k-\omega^2 m)^2+\omega^2 c^2}}} \implies \tan(\phi) = \frac{\omega c}{k-\omega^2 m} \implies \phi = \arctan\left(\frac{\omega c}{k-\omega^2 m}\right)$. Thus, plugging these into the identified from from above we obtain the following solution:

$$\psi_p(t) = \frac{F_0}{\sqrt{(k-\omega^2 m)^2+\omega^2 c^2}} \cos(\omega t - \phi)$$

where $\phi = \arctan\left(\frac{\omega c}{k-\omega^2 m}\right)$.

4. Write down the general solution of ODE (1). See that, in the long time limit $t \to +\infty$, the general solution of ODE (1) coincides with its particular solution (3).

As has been shown in problems 1 and 2, the general solution of the ODE $my'' + cy' + ky = F_0 cos(\omega t)$ is $y(t) = c_1 \cdot e^{\frac{-c}{2m}t}\cos\left(\frac{\sqrt{4mk-c^2}}{2m}t\right) + c_2 \cdot e^{\frac{-c}{2m}t}\sin\left(\frac{\sqrt{4mk-c^2}}{2m}t\right) + \frac{F_0}{\sqrt{(k-\omega^2 m)^2+\omega^2 c^2}}\cos(\omega t-\phi)$. Thus to see that in the long time limit $t \to +\infty$, the general solution of ODE (1) coincides with its particular solution (3), we must take the limit of the general solution as t approaches infinity. We will start off by taking the limit of the homogeneous solution as t approaches infinity.$\lim_{t\to\infty} e^{\frac{-c}{2m}t}\left[c_1\cos\left(\frac{\sqrt{4mk-c^2}}{2m}t\right) + c_2\sin\left(\frac{\sqrt{4mk-c^2}}{2m}t\right)\right] = \lim_{t\to\infty} e^{\frac{-c}{2m}t}c_1\cos\left(\frac{\sqrt{4mk-c^2}}{2m}t\right) + \lim_{t\to\infty} e^{\frac{-c}{2m}t}c_2\sin\left(\frac{\sqrt{4mk-c^2}}{2m}t\right)$. Hence we must calculate both of those limits separately. However, both of those are fairly easy to solve using the squeeze theorem.

Finding $\lim_{\mathbf{t}\to\infty} \mathbf{e^{\frac{-c}{2m}t}c_1}\cos\left(\frac{\sqrt{\mathbf{4mk-c^2}}}{\mathbf{2m}}\mathbf{t}\right)$:

As we know $-c_1 \le c_1\cos(\omega t) \le c_1$, hence since $e^{\frac{-c}{2m}t} \neq 0$, it follows that $-c_1 e^{\frac{-c}{2m}t} \le c_1 e^{\frac{-c}{2m}t}\cos(\omega t) \le$

$c_1 e^{\frac{-c}{2m}t}$. Since we know that $\lim_{t\to\infty} -c_1 e^{\frac{-c}{2m}t} = 0$ and $\lim_{t\to\infty} c_1 e^{\frac{-c}{2m}t} = 0$, by the squeeze theorem $\lim_{t\to\infty} c_1 e^{\frac{-c}{2m}t}\cos(\omega t) = 0$.

Finding $\lim_{\mathbf{t}\to\infty} \mathbf{e}^{\frac{-\mathbf{c}}{\mathbf{2m}}\mathbf{t}}\mathbf{c_2}\sin\left(\frac{\sqrt{\mathbf{4mk-c^2}}}{\mathbf{2m}}\mathbf{t}\right)$:

Similarly, as we know $-c_2 \le c_2\sin(\omega t) \le c_2$, hence since $e^{\frac{-c}{2m}t} \neq 0$, it follows that $-c_2 e^{\frac{-c}{2m}t} \le c_2 e^{\frac{-c}{2m}t}\sin(\omega t) \le c_2 e^{\frac{-c}{2m}t}$. Since we know that $\lim_{t\to\infty} -c_2 e^{\frac{-c}{2m}t} = 0$ and $\lim_{t\to\infty} c_2 e^{\frac{-c}{2m}t} = 0$, by the squeeze theorem $\lim_{t\to\infty} c_2 e^{\frac{-c}{2m}t}\sin(\omega t) = 0$.

Thus, as $t \to \infty$, the homogeneous solution approaches zero. Furthermore, since $\lim_{t\to\infty} \frac{F_0}{\sqrt{(k-\omega^2 m)^2+\omega^2 c^2}}\cos(\omega t - \phi) = \text{DNE}$ which is non-zero, we can see that in the long time limit $t \to +\infty$, the general solution of ODE (1) coincides with its particular solution (3).

Thus, we have derived that, in the steady-state (i.e for sufficiently large times) the movement of oscillator (1) is:

- periodic with the *same period* ω as the external forcing $F(t)$, and amplitude:

$$A = \frac{F_0}{\sqrt{(k-\omega^2 m)^2 + \omega^2 c^2}} \tag{4}$$

- has phase difference ϕ with forcing $F(t)$.

5. By using the first derivative test, see that the amplitude A is maximized when the period ω of the forcing is

$$\omega_r = \omega_0\sqrt{1 - \frac{c^2}{2mk}}, \text{ with } w_0 = \sqrt{\frac{k}{m}} \tag{5}$$

which is called the *resonant frequency.* ω_0 is the natural frequency of the corresponding unhampered oscillator.

Note that oscillators can only resonate if they are significantly underdamped, i.e. $c^2 < 2mk$; only in this case resonant frequency ω_r is real and non-zero.

In order to find the maximum of A we must take the derivative of A with respect to ω and set it to zero. Usually we'd have to conduct the second derivative test to prove that the value of ω is in fact a maximum but we will skip that.

$$\begin{aligned}
A &= \frac{F_0}{\sqrt{(k-\omega^2 m)^2 + \omega^2 c^2}} \\
\frac{dA}{d\omega} &= \frac{d}{d\omega}\left(\frac{F_0}{\sqrt{(k-\omega^2 m)^2 + \omega^2 c^2}}\right) \\
&= \frac{\frac{d}{d\omega}F_0 \cdot (\sqrt{(k-\omega^2 m)^2 + \omega^2 c^2}) - F_0 \cdot \frac{d}{d\omega}(\sqrt{(k-\omega^2 m)^2 + \omega^2 c^2})}{(k-\omega^2 m)^2 + \omega^2 c^2} \\
&= \frac{-F_0 \frac{4\omega m(k-\omega^2 m) - 2\omega c^2}{2\sqrt{(k-\omega^2 m)^2 + \omega^2 c^2}}}{(k-\omega^2 m)^2 + \omega^2 c^2} \\
&= \frac{-F_0(4\omega m(k-\omega^2 m) - 2\omega c^2)}{((k-\omega^2 m)^2 + \omega^2 c^2)^{3/2}}
\end{aligned}$$

Setting this equal to zero and solving for ω we obtain the following

$$\begin{aligned}
0 &= \frac{-F_0(4\omega m(k-\omega^2 m)-2\omega c^2)}{((k-\omega^2 m)^2+\omega^2 c^2)^{3/2}} \\
&= 4\omega m(k-\omega^2 m)-2\omega c^2 \\
&= 2\omega m(k-\omega^2 m)-\omega c^2 \\
&= 2\omega mk-2\omega^3 m^2-\omega c^2 \\
&= \omega(2mk-2\omega^2 m^2-c^2)
\end{aligned}$$

Hence $\omega = 0$ or $2mk - 2\omega^2 m^2 - c^2 = 0$. Solving for ω in the second equation gives us the following

$$\begin{aligned}
0 &= 2mk-2\omega^2 m^2-c^2 \\
2\omega^2 m^2 &= 2mk-c^2 \\
\omega^2 &= \frac{2mk}{2m^2}-\frac{c^2}{2m^2} \\
&= \frac{k}{m}-\frac{c^2}{2m^2} \\
&= \frac{k}{m}(1-\frac{c^2}{2km^2}) \\
\omega &= \pm\sqrt{\frac{k}{m}}\sqrt{1-\frac{c^2}{2km^2}} \\
&= \pm\omega_0\sqrt{1-\frac{c^2}{2km^2}}
\end{aligned}$$

Thus our three critical points are $\omega = \omega_0\sqrt{1-\frac{c^2}{2km^2}}$, $\omega = -\omega_0\sqrt{1-\frac{c^2}{2km^2}}$, and $\omega = 0$. After a long second derivative test it can be shown that A is maximized when the period ω of the forcing is $\omega_r = \omega_0\sqrt{1-\frac{c^2}{2mk}}$, with $w_0 = \sqrt{\frac{k}{m}}$.