Jalden Atterbury
05-01-23
AMATH 351

Homework 5: Cauchy-Euler equation, nth order linear ODEs, method of undetermined coefficients

Exercise 1: Roots of the characteristic equation:

The characteristic equation of a homogeneous linear ODE with constant coefficients can be written as:

$$(r-2)^3(r^2-6r+25)^2(r-3)^2=0$$

1.) Determine the roots of the characteristic equation and their multiplicity:

i.) $(r-2)^3=0$ implies that $r_1=2$ with multiplicity 3.

ii.) $(r^2-6r+25)^2=0$ implies that $r_{2,3}=\sigma+i\omega=\frac{6\pm i\sqrt{64}}{2}=3\pm i8$ with multiplicity 2.

iii.) $(r-3)^2=0$ implies that $r_4=3$ with multiplicity 2.

2.) Find the general solution of the ODE.

i) Fundamental solutions:

a.) $r_1=2$ with multiplicity 3 implies $y_1(t)=e^{2t}$, $y_2(t)=te^{2t}$, $y_3(t)=t^2e^{2t}$.

b.) $r_{2,3}=3\pm 8i$ with multiplicity 2 implies $y_4(t)=e^{3t}\cos(8t)$, $y_5(t)=te^{3t}\cos(8t)$, $y_6(t)=e^{3t}\sin(8t)$, $y_7(t)=te^{3t}\sin(8t)$

c.) $r_4=3$ with multiplicity 2 implies $y_8(t)=e^{3t}$, $y_9(t)=te^{3t}$

ii.) General solution:

$$y(t)=c_1e^{2t}+c_2te^{2t}+c_3t^2e^{2t}+c_4e^{3t}\cos(8t)+c_5te^{3t}\cos(8t)+c_6e^{3t}\sin(8t)+c_7te^{3t}\sin(8t)+c_8e^{3t}+c_9te^{3t}$$

- Exercise 2: nth order homogeneous ODEs:

1.) Find the general solution of the ODE:

$$y''' - 3y'' + 4y' - 2y = 0$$

Since the sum of the coefficients is zero, one of the solutions of the characteristic equation $r^3 - 3r^2 + 4r - 2 = 0$ is $r_1 = 1$, Hence we can use polynomial long division to find the resulting equation.

$$\begin{array}{r|l} & r^2 - 2r + 2 \\ \hline r-1 & r^3 - 3r^2 + 4r - 2 \\ & -r^3 + r^2 \\ \hline & -2r^2 + 4r - 2 \\ & +2r - 2r \\ \hline & 2r - 2 \\ & -2r + 2 \\ \hline & 0 \end{array}$$

Hence, $r^3 - 3r^2 + 4r - 2 = 0 \Rightarrow (r-1)(r^2 - 2r + 2) = 0$

i.) $(r-1) = 0 \Rightarrow r_1 = 1 \Rightarrow y_1(t) = e^t$

ii.) $r^2 - 2r + 2 = 0 \Rightarrow r_{2,3} = \sigma + i\omega = \frac{2 \pm i\sqrt{4}}{2} = 1 \pm i1 \Rightarrow y_2(t) = e^t\cos(t),\ y_3(t) = e^t\sin(t)$

Thus the general solution is:

$$y(x) = c_1 e^t + c_2 e^t\cos(t) + c_3 e^t \sin(t)$$

2.) Find the solution of the IVP

$$y''' - 5y'' + 100y' - 500y = 0, \quad y(0) = 0,\ y'(0) = 10,\ y''(0) = 250$$

if we know that e^{5t} is one of the fundamental solutions of the ODE.

If we know that $y_1(t) = e^{5t}$ this implies that $r_1 = 5$ is a root of the corresponding characteristic equation $r^3 - 5r^2 + 100r - 500$. Hence we can use polynomial long division to find the resulting equation.

$$
\begin{array}{r|l}
 & r^2 + 100 \\
\hline
r-5 & r^3 - 5r^2 + 100r - 500 \\
 & -r^3 + 5r^2 \\
\hline
 & \qquad 100r - 500 \\
 & \qquad -100r + 500 \\
\hline
 & \qquad\quad 0
\end{array}
$$

Thus $r^3 - 5r^2 + 100r - 500 = 0 \Rightarrow (r-5)(r^2+100) = 0$

i.) $r - 5 = 0 \Rightarrow r_1 = 5 \Rightarrow y_1(t) = e^{5t}$

ii.) $r^2 + 100 = 0 \Rightarrow r_{2,3} = \sigma \pm i\omega = \frac{0 \pm i\sqrt{400}}{2} = i\frac{20}{2} = 10i \Rightarrow y_2(t) = \cos(10t)$,
$y_3(t) = \sin(10t)$

Hence the general solution is:
$y(t) = c_1 e^{5t} + c_2 \cos(10t) + c_3 \sin(10t)$

Now we will take derivatives of $y(t)$ in order to find c_1, c_2, and c_3.

$y'(t) = c_1 e^{5t} + c_2 \cos(10t) + c_3 \sin(10t)$

$y'(t) = 5c_1 e^{5t} - 10c_2 \sin(10t) + 10c_3 \cos(10t)$

$y''(t) = 25c_1 e^{5t} - 100c_2 \cos(10t) - 100c_3 \sin(10t)$

$y(0) = 0 \Rightarrow c_1 + c_2 = 0$

$y'(0) = 10 \Rightarrow 5c_1 + 10c_3 = 10 \Rightarrow c_1 + 2c_3 = 2$

$y''(0) = 250 \Rightarrow 25c_1 - 100c_2 = 250 \Rightarrow c_1 - 4c_2 = 10$

We can now use Gaussian elimination to find c_1, c_2, and c_3

$$
\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 2 & 2 \\ 1 & -4 & 0 & 10 \end{array}\right]
\underset{R_2 = R_2 - R_1}{\sim}
\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 1 & -4 & 0 & 10 \end{array}\right]
\underset{R_3 = R_3 - R_1}{\sim}
\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & -5 & 0 & 10 \end{array}\right]
\underset{R_3 = R_3 - 5R_2}{\sim}
$$

$$
\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & -10 & 0 \end{array}\right]
\Rightarrow c_3 = 0, \quad c_2 = -2, \; c_1 = 2
$$

Thus the solution to the IVP is:
$y(t) = 2e^{5t} - 2\cos(10t)$

- Exercise 3: Non-homogeneous Cauchy-Euler equation

Consider the equation:

$$t^2 \frac{d^2y(t)}{dt^2} + t\frac{dy(t)}{dt} + y(t) = t, \quad t > 0 \qquad (1)$$

1.) By performing the change of variable $x = \ln(t)$, see that ODE (1) is expressed as

$$\frac{d^2y(x)}{dx^2} + y(x) = e^x \qquad (2)$$

In general if $x = g(t)$ and we assume g^{-1} exists then $t = g^{-1}(x)$ and hence $y(t) = y(g^{-1}(x)) := \tilde{y}(x)$

Hence it follows that:

$$y'(t) = \frac{dy(t)}{dt} = \frac{dy(t)}{dt}\frac{dx}{dx} = \frac{d\tilde{y}(x)}{dx}\frac{dx}{dt} = \frac{d\tilde{y}}{dx}g'(t)$$

Similarly

$$y''(t) = \frac{d^2y(t)}{dt^2} = \frac{d}{dt}\left[\frac{dy(t)}{dt}\right] = \frac{d}{dt}\left[\frac{d\tilde{y}}{dx}g'(t)\right] = \frac{d^2\tilde{y}(x)}{dt\,dx}g'(t) + \frac{d\tilde{y}(x)}{dx}g''(t)$$

$$= \frac{d^2\tilde{y}(x)}{dt\,dx}\frac{dx}{dx}g'(t) + \frac{d\tilde{y}(x)}{dx}g''(x) = \frac{d^2\tilde{y}(x)}{dx^2}\frac{dx}{dt}g'(t) + \frac{d\tilde{y}(x)}{dx}g''(t)$$

$$= \frac{d^2\tilde{y}(x)}{dx^2}[g'(t)]^2 + \frac{d\tilde{y}(x)}{dx}g''(t)$$

In our case $x = \ln(t) \Rightarrow g(t) = \ln(t)$

$$g'(t) = \frac{1}{t}$$

$$g''(t) = \frac{-1}{t^2}$$

Plugging into our above formulas and into ODE (1) we obtain:

$$t^2\left[\frac{d^2\tilde{y}(x)}{dx^2}\left(\frac{1}{t^2}\right) + \frac{d\tilde{y}(x)}{dx}\left(\frac{-1}{t^2}\right)\right] + t\left[\frac{d\tilde{y}(x)}{dx}\cdot\frac{1}{t}\right] + \tilde{y}(x) = e^x$$

$$\frac{d^2\tilde{y}(x)}{dx^2} - \frac{d\tilde{y}(x)}{dx} + \frac{d\tilde{y}(x)}{dx} + \tilde{y}(x) = e^x$$

and hence $\frac{d^2\tilde{y}(x)}{dx^2} + \tilde{y}(x) = e^x$

2.) Determine the general solution of the homogeneous equation that corresponds to ODE (2).

$$\tilde{y}''(x) + \tilde{y}(x) = 0$$

corresponding characteristic equation:

$r^2 + 1 = 0$

discriminant calculation

$\Delta = -4(1)(1) = -4 < 0$

thus we have two complex conjugate roots

$r_{1,2} = \pm i\omega = \frac{\pm i\sqrt{4}}{2} = \pm 1i$

fundamental solutions:

$y_1(x) = \cos(x)$, $y_2(x) = \sin(x)$

general solution:

$y(x) = c_1 \cos(x) + c_2 \sin(x)$

3.) Find a particular solution of ODE (2) via the method of undetermined coefficients.

Since $g(x) = e^x$ our particular solution form is $\psi_p(x) = Ae^x$

We will now take derivatives of $\psi_p(x)$ to solve for A:

$\psi_p(x) = Ae^x$

$\psi_p'(x) = Ae^x$

$\psi_p''(x) = Ae^x$

Plugging these into the ODE we obtain:

$Ae^x + Ae^x = e^x$

$2Ae^x = e^x$

$2A = 1$

$A = \frac{1}{2}$

thus $\psi_p(x) = \frac{1}{2}e^x$

4.) Determine the general solution $y(x)$ of ODE (2). Then use $x = \ln(t)$ to return to the variable t, in order to determine the general solution $y(t)$ of the original ODE (1).

Based on parts 2.) and 3.) the general solution $y(x)$ of ODE (2) is:

$$y(x) = c_1\cos(x) + c_2\sin(x) + \tfrac{1}{2}e^{x}$$

Using the fact that $x = \ln(t)$ and $t = e^{x}$, the solution $y(t)$ of the original ODE (1) is:

$$y(t) = c_1\cos(\ln(t)) + c_2\sin(\ln(t)) + \tfrac{1}{2}t$$

- Exercise 4: undetermined coefficients and superposition principle.

Using the method of undetermined coefficients, find one particular solution for each of the following ODEs:

1.) $y'' - 3y' + 2y = 2e^{2t}$

$\boxed{\text{STEP 1:}}$ Find the corresponding homogeneous solution:

$y'' - 3y' + 2y = 0$

corresponding characteristic equation

$r^2 - 3r + 2 = 0 \Rightarrow (r-2)(r-1) = 0$

discriminant calculation:

$\Delta = 9 - 4(1)(2) = 1 > 0$

thus we have two distinct real roots

$r_1 = 1, \quad r_2 = 2$

fundamental solutions

$y_1(t) = e^t, \quad y_2(t) = e^{2t}$

general solution:

$y(t) = c_1 e^t + c_2 e^{2t}$

$\boxed{\text{STEP 2:}}$ Find a particular solution $\psi_p(t)$:

Since e^{2t} is a fundamental solution, and our $g(t) = 2e^{2t}$, it follows that our particular solution form is $\psi_p(t) = Ate^{2t}$

we will now take the derivatives of $\psi_p(t)$ to find A

$\psi_p(t) = Ate^{2t}$

$\psi_p'(t) = Ae^{2t} + 2Ate^{2t}$

$\psi_p''(t) = 2Ae^{2t} + 2Ae^{2t} + 4Ate^{2t}$

$= 4Ae^{2t} + 4Ate^{2t}$

Plugging these into the ode we obtain

$$4Ae^{2t} + 4Ate^{2t} - 3Ae^{2t} - 6Ate^{2t} + 2Ate^{2t} = 2e^{2t}$$

$$4A + 4At - 3A - 6At + 2At = 2$$

$$4A - 3A = 2$$

$$A = 2$$

Thus $\psi_p(t) = 2te^{2t}$

and the general solution is thus:

$$y(t) = c_1 e^{t} + c_2 e^{2t} + 2te^{2t}$$

2.) $y'' - 3y' + 2y = 2\sin(t)$

$\boxed{\text{STEP 1:}}$ Find the corresponding homogeneous solution

By part 1.) this homogeneous solution is $y(t) = c_1 e^{t} + c_2 e^{2t}$.

$\boxed{\text{STEP 2:}}$ Find a particular solution $\psi_p(t)$:

Since $g(t) = 2\sin(t)$ our particular solution form is $\psi_p(t) = A\sin(t) + B\cos(t)$.

We will now take derivatives of $\psi_p(t)$ to solve for A and B:

$$\psi_p(t) = A\sin(t) + B\cos(t)$$

$$\psi_p'(t) = A\cos(t) - B\sin(t)$$

$$\psi_p''(t) = -A\sin(t) - B\cos(t)$$

Plugging these into the ODE we obtain

$$-A\sin(t) - B\cos(t) - 3A\cos(t) + 3B\sin(t) + 2A\sin(t) + 2B\cos(t) = 2\sin(t)$$

$$A\sin(t) + B\cos(t) - 3A\cos(t) + 3B\sin(t) = 2\sin(t)$$

$$(A+3B)\sin(t) + (B-3A)\cos(t) = 2\sin(t)$$

$$\begin{aligned} A + 3B &= 2 \\ -3A + B &= 0 \end{aligned} \Rightarrow \left[\begin{array}{cc|c} 1 & 3 & 2 \\ -3 & 1 & 0 \end{array}\right] \overset{R_2 = R_2 + 3R_1}{\sim} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 10 & 6 \end{array}\right] \Rightarrow A = \frac{1}{5},\ B = \frac{3}{5}$$

thus $\psi_p(t) = \frac{1}{5}\sin(t) + \frac{3}{5}\cos(t)$

and the general solution is thus:

$$y(t) = c_1e^t + c_2e^{2t} + \frac{1}{5}\sin(t) + \frac{3}{5}\cos(t)$$

By using the particular solutions you found in the previous questions, and by invoking the superposition, find one particular solution of the ODE:

3.) $y'' - 3y' + 2y = 2e^{2t} + 2\sin(t)$

$\boxed{\text{STEP 1:}}$ Find the corresponding homogeneous solution:

By part 1.) this homogeneous solution is $y(t) = c_1e^t + c_2e^{2t}$

$\boxed{\text{STEP 2:}}$ Find a particular solution $\psi_p(t)$:

By parts 1.) and 2.) and by invoking the superposition principle it follows that:

$$\psi_p(t) = 2te^{2t} + \frac{1}{5}\sin(t) + \frac{3}{5}\cos(t)$$

and the general solution is thus:

$$y(t) = c_1e^t + c_2e^{2t} + 2te^{2t} + \frac{1}{3}\sin(t) + \frac{3}{5}\cos(t)$$