AMATH 351 Homework 9: Laplace Transform II

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Exercise 1: The boxcar function. Using Laplace transform, solve the IVP

$$y'' + y = g(t),\ y(0) = 0,\ y'(0) = 1,$$

where

$$g(t) = \begin{cases} 0, & 0 \leq t < 1 \\ 1, & 1 \leq t < 2 \\ 0, & t \geq 2. \end{cases}$$

Hint: Function $g(t)$ is zero everywhere, except for the interval $[1, 2)$ where it is equal to 1. Due to its shape, $g(t)$ is called the rectangular or boxcar function. It is easy to see that the boxcar function can be expressed as the difference of two Heaviside step functions: $g(t) = u_1(t) - u_2(t)$.

Solution:
To solve the above IVP, we must first notice that the boxcar function can be rewritten as $g(t) = u_1(t) - u_2(t)$, hence our IVP becomes: $y'' + y = u_1(t) - u_2(t),\ y(0) = 0,\ y'(0) = 1$. Now we will take the Laplace transform of both sides. $\mathcal{L}[y'' + y] = \mathcal{L}[u_1(t) - u_2(t)]$. Using the linearity of the Laplace transform, this becomes $\mathcal{L}[y''] + \mathcal{L}[y] = \mathcal{L}[u_1(t)] - \mathcal{L}[u_2(t)]$. Solving these Laplace transforms using Laplace pairs and letting $\mathcal{L}[y(t)] = Y(s)$ we can see that the equation becomes $s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{e^{-s}}{s} - \frac{e^{-2s}}{s}$. Plugging in our initial conditions and isolating $Y(s)$ on the left hand side we see that the equation becomes $Y(s) = \frac{1}{s^2+1} + \frac{e^{-s}}{s(s^2+1)} - \frac{e^{-2s}}{s(s^2+1)}$.

In order to take the inverse Laplace transform of both sides we must first use partial fractions decomposition to put the functions on the right hand side into a nicer form. First off, see that we can rewrite the last two functions in the right hand side of the equation (ignoring the exponential terms) as $\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$. After clearing the denominators the equation becomes $1 = A(s^2+1)+(Bs+C)s \implies 1 = As^2+A+Bs^2+Cs$. Hence we must simultaneously solve the following system of equations: $A + B = 0$, $C = 0$, and $A = 1$. It can easily be seen that the three coefficients that simultaneously solve this system are $A = 1$, $B = -1$, and $C = 0$.

With this said we can now rewrite our equation for $Y(s)$ as $Y(s) = \frac{1}{s^2+1} + e^{-s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) - e^{-2s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$. Hence taking the inverse Laplace transform of of both sides and using the linearity of the Laplace transform, we obtain the following solution. $\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{1}{s^2+1} + e^{-s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right) - e^{-2s}\left(\frac{1}{s} - \frac{s}{s^2+1}\right)\right] = \sin(t) + u_1(t)(1 - \cos(t-1) - u_2(t)(1 - \cos(t-2))$

Hence as calculated above the solution to the IVP is $y(t) = \sin(t) + u_1(t)(1 - \cos(t-1) - u_2(t)(1 - \cos(t-2))$

Exercise 2: The RL circuit. We consider the Resistor-Inductor (RL) electrical circuit shown in figure 1. By *Kirchhoffs's voltage law*, and using the *constitutive relations* for the resistor: $V_R(t) = I(t)R$, and the inductor: $V_{L(t)} = LdI(t)/dt$, we obtain an ODE for the current $I(t)$ flowing in the circuit:

$$V_0(t) = V_R(t) + V_L(t) \implies L\frac{dI(t)}{dt} + RI(t) = V_0(t)$$

When the switch is open, there is no current in the circuit; that means that we have the initial condition

$$I(0) = 0.$$

We also consider that after the switch is closed, source voltage decreases exponentially as $V_0(t) = 2e^{-t}$ *volts*.

Thus, the IVP for $I(t)$, for the case where the switch closes at $t = 4s$, and for $R = 1$ *ohm*, $L = 1/3$ *henrys*, reads

$$\frac{1}{3}I'(t) + I(t) = 2u_4(t)e^{-(t-4)},\ I(0) = 0. \tag{1}$$

Solve IVP (1) by using Laplace transform. The SI unit of electric current is the *Ampere*.

Solution:
To solve this IVP, we will start by taking the Laplace transform of both sides, doing this we obtain $\mathcal{L}\left[\frac{1}{3}I'(t) + I(t)\right] = \mathcal{L}\left[2u_4(t)e^{-(t-4)}\right]$. Using the linearity of the Laplace transform we can simplify the above equation to $\frac{1}{3}\mathcal{L}[I'(t)] + \mathcal{L}[I(t)] = 2\mathcal{L}\left[u_4(t)e^{-(t-4)}\right]$. If we let $\mathcal{L}[I(t)] = Y(s)$ and use Laplace pairs to solve each individual Laplace transform we obtain, $\frac{1}{3}sY(s) - \frac{1}{3}I(0) + Y(s) = 2e^{-4s}\mathcal{L}[e^{-t}]$. If we plug in our initial conditions and isolate $Y(s)$ on th left hand side of the equation we obtain the following expression for $Y(s)$: $Y(s) = \frac{2e^{-4s}}{\left(\frac{1}{3}s+1\right)(s+1)}$. Multiplying the top and bottom by 3 we can simplify the equation to $Y(s) = \frac{6e^{-4s}}{(s+3)(s+1)}$.

In order to take the inverse Laplace transform of both sides we must first use partial fractions decomposition to put the function on the right hand side into a nicer form. First off, see that we can rewrite the function in the right hand side of the equation (ignoring the exponential terms) as $\frac{6}{(s+3)(s+1)} = \frac{A}{s+3} + \frac{B}{s+1}$. After clearing the denominators the equation becomes $6 = A(s+3) + B(s+1) \implies 1 = As + 3A + Bs + B$. Hence we must simultaneously solve the following system of equations: $A + B = 0$ and $3A + b = 6$ It can easily be seen that the three coefficients that simultaneously solve this system are $A = 3$, $B = -3$.

With this said we can now rewrite our equation for $Y(s)$ as $Y(s) = e^{-4s}\left(\frac{3}{s+1} - \frac{3}{s+3}\right)$. Hence taking the inverse Laplace transform of of both sides and using the linearity of the Laplace transform, we obtain the following solution. $\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[e^{-4s}\left(\frac{3}{s+1} - \frac{3}{s+3}\right)\right] = u_4(t)\left(3e^{-(t-4)} - 3e^{-3(t-4)}\right)$

Hence as calculated above the solution to IVP (1) is $y(t) = u_4(t)\left(3e^{-(t-4)} - 3e^{-3(t-4)}\right)$

Exercise 3: Calculating inverse Laplace transforms using convolution. By using convolution, prove that

1. $\mathcal{L}^{-1}\left[\frac{2\omega s}{(s^2+\omega^2)^2}\right] = t\sin(\omega t)$. Hint: $\frac{2\omega s}{(s^2+\omega^2)^2} = 2 \cdot \frac{\omega}{(s^2+\omega^2)^2} \cdot \frac{s}{(s^2+\omega^2)^2}$

To show that $\mathcal{L}^{-1}\left[\frac{2\omega s}{(s^2+\omega^2)^2}\right] = t\sin(\omega t)$, we will use convolutions, the following hint: $\frac{2\omega s}{(s^2+\omega^2)^2} = 2 \cdot \frac{\omega}{(s^2+\omega^2)^2} \cdot$

$\frac{s}{(s^2+\omega^2)^2}$, and the following trigonometric identity: $\sin(a)cos(b) = \frac{1}{2}\sin(a+b) + \frac{1}{2}\sin(a-b)$.

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{2\omega s}{(s^2+\omega^2)^2}\right] &= \mathcal{L}^{-1}\left[2\cdot\frac{\omega}{(s^2+\omega^2)^2}\cdot\frac{s}{(s^2+\omega^2)^2}\right] \\
&= 2\mathcal{L}^{-1}\left[\frac{\omega}{(s^2+\omega^2)^2}\cdot\frac{s}{(s^2+\omega^2)^2}\right] \\
&= 2\int_0^t \mathcal{L}^{-1}\left[\frac{\omega}{(s^2+\omega^2)^2}\right](t-\tau)\cdot\mathcal{L}^{-1}\left[\frac{s}{(s^2+\omega^2)^2}\right](\tau)d\tau \\
&= 2\int_0^t \sin(\omega(t-\tau))\cos(\omega\tau)d\tau \\
&= 2\int_0^t \frac{1}{2}\sin(\omega(t-\tau)+\omega\tau)d\tau + 2\int_0^t \frac{1}{2}\sin(\omega(t-\tau)-\omega\tau)d\tau \\
&= \int_0^t \sin(\omega t)d\tau + \int_0^t \sin(\omega(t-2\tau))d\tau
\end{aligned}$$

Solving the first integral we obtain $\int_0^t \sin(\omega t)d\tau = \sin(\omega t)\int_0^t d\tau = t\sin(\omega t)$. Using the substitution $u = t - 2\tau \implies du = -2d\tau$, we can simplify the second integral to $\frac{-1}{2}\int_t^{-t}\sin(\omega u)du = \frac{1}{2}\int_{-t}^{t}\sin(\omega u)du$. Using the fact that $\cos(t) = \cos(-t)$, we can solving this second integral and obtain $\frac{-1}{2\omega}[\cos(\omega u)]_0^t = 0$. Hence we have proven that $\mathcal{L}^{-1}\left[\frac{2\omega s}{(s^2+\omega^2)^2}\right] = t\sin(\omega t)$.

2. $\mathcal{L}^{-1}\left[\frac{s^2-\omega^2}{(s^2+\omega^2)^2}\right] = t\cos(\omega t)$. <u>Hint</u>: $\frac{s^2-\omega^2}{(s^2+\omega^2)^2} = \frac{s}{s^2+\omega^2}\cdot\frac{s}{s^2+\omega^2} - \frac{\omega}{s^2+\omega^2}\cdot\frac{\omega}{s^2+\omega^2}$

To show that $\mathcal{L}^{-1}\left[\frac{s^2-\omega^2}{(s^2+\omega^2)^2}\right] = t\cos(\omega t)$, we will use convolutions, the following hint: $\frac{s^2-\omega^2}{(s^2+\omega^2)^2} = \frac{s}{s^2+\omega^2}\cdot\frac{s}{s^2+\omega^2} - \frac{\omega}{s^2+\omega^2}\cdot\frac{\omega}{s^2+\omega^2}$, and the following trigonometric identities: $\cos(a)\cos(b) = \frac{1}{2}\cos(a+b) + \frac{1}{2}\cos(a-b)$ and $\sin(a)\sin(b) = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b)$.

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{s^2-\omega^2}{(s^2+\omega^2)^2}\right] &= \mathcal{L}^{-1}\left[\frac{s}{s^2+\omega^2}\cdot\frac{s}{s^2+\omega^2} - \frac{\omega}{s^2+\omega^2}\cdot\frac{\omega}{s^2+\omega^2}\right] \\
&= \mathcal{L}^{-1}\left[\frac{s}{s^2+\omega^2}\cdot\frac{s}{s^2+\omega^2}\right] - \mathcal{L}^{-1}\left[\frac{\omega}{s^2+\omega^2}\cdot\frac{\omega}{s^2+\omega^2}\right] \\
&= \int_0^t \mathcal{L}^{-1}\left[\frac{s}{s^2+\omega^2}\right](\tau)\cdot\mathcal{L}^{-1}\left[\frac{s}{s^2+\omega^2}\right](t-\tau)d\tau - \int_0^t \mathcal{L}^{-1}\left[\frac{\omega}{s^2+\omega^2}\right](\tau)\cdot\mathcal{L}^{-1}\left[\frac{\omega}{s^2+\omega^2}\right](t-\tau)d\tau \\
&= \int_0^t \cos(\omega\tau)\cos(\omega(t-\tau))d\tau - \int_0^t \sin(\omega\tau)\sin(\omega(t-\tau))d\tau
\end{aligned}$$

Using the above trigonometric identities these integral become $\frac{1}{2}\int_0^t \cos(\omega\tau+\omega(t-\tau))d\tau + \frac{1}{2}\int_0^t \cos(\omega\tau-\omega(t-\tau))d\tau - \frac{1}{2}\int_0^t \cos(\omega\tau+\omega(t-\tau))d\tau + \frac{1}{2}\int_0^t \cos(\omega\tau+\omega(t-\tau))d\tau$. Hence we can simplify this to

$$= \int_0^t \cos(\omega\tau + \omega(t-\tau))d\tau$$

$$= \int_0^t \cos(\omega\tau + \omega t - \omega\tau)d\tau$$

$$= \int_0^t \cos(\omega t)d\tau$$

$$= \cos(\omega t)\int_0^t d\tau$$

$$= t\cos(\omega t)$$

Hence we have proven that $\mathcal{L}^{-1}\left[\frac{s^2-\omega^2}{(s^2+\omega^2)^2}\right] = t\cos(\omega t)$.

Exercise 4: Duhamel's integral. Consider the IVP

$$y'' + 4y = f(t),\ y(0) = y'(0) = 0, \tag{2}$$

where $f(t)$ is an unspecified continuous function.

1. Determine the transfer function $W(s)$ of the IVP (2).

In order to find the transfer function we must first take the Laplace transform of both sides, this results in the following equation $\mathcal{L}[y'' + 4y] = \mathcal{L}[f(t)]$. Using the linearity of the Laplace transform this becomes $\mathcal{L}[y''] + 4\mathcal{L}[y] = \mathcal{L}[f(t)]$. If we let $F(s) = \mathcal{L}[f(t)]$ and $Y(s) = \mathcal{L}[y(t)]$, then the following equation becomes $s^2Y(s) - sy(0) - y'() + 4Y(s) = F(s)$. After plugging in the initial conditions and isolating $Y(s)$ on the left hand side of the equation we obtain $Y(s) = \frac{1}{s^2+4}F(S)$. Thus as can be seen from the previous expression of $Y(s)$ the transfer function $W(s)$ of the IVP (2) is $W(S) = \frac{1}{s^2+4}$.

2. Determine the unit impulse response function $w(t)$ of the IVP (2).

To determine the unit impulse response function $w(t)$ of the IVP (2), we must take the inverse Laplace transform of the transfer function. $w(t) = \mathcal{L}^{-1}[W(s)] = \mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right] = \mathcal{L}^{-1}\left[\frac{1}{2}\frac{2}{s^2+4}\right] = \frac{1}{2}\sin(2t)$. Hence the impulse response function of IVP (2) is $w(t) = \frac{1}{2}\sin(2t)$.

3. Express the solution $y(t)$ of the IVP (2) as a convolution, for a general forcing function $f(t)$.

To express the solution $y(t)$ of the IVP (2) as a convolution, for a general forcing function $f(t)$, we must use the general convolution formula $\int_0^t f(t-\tau)g(\tau)d\tau$ for our given functions. Hence we can see that the solution $y(t)$ of the IVP (2) as a convolution for a general forcing function $f(t)$ is $\frac{1}{2}\int_0^t \sin(2\tau)f(t-\tau)d\tau$.

4. By using the convolution form of $y(t)$, calculate the solution of the IVP (2) for $f(t) = t$. <u>Hint</u>: Integration by parts may be needed.

Using the above expression for the solution of $y(t)$ in convolution form for $f(t) = t$ we must solve $\frac{1}{2}\int\limits_0^t \sin(2\tau)(t - \tau)d\tau$ in order to find the solution of IVP (2). Expanding the integral out using the distributive property we obtain $\frac{1}{2}t\int\limits_0^t \sin(2\tau)d\tau - \frac{1}{2}\int\limits_0^t \tau\sin(2\tau)d\tau$. Solving the first of these integrals we can see that $\frac{1}{2}t\int\limits_0^t \sin(2\tau)d\tau = \frac{-1}{4}t\cos(2\tau)|_0^t = \frac{t}{4} - \frac{t}{4}\cos(2t)$. Solving the second of these integrals requires integration by parts twice, using the tabular method we can see that $-\frac{1}{2}\int\limits_0^t \tau\sin(2\tau)d\tau = \frac{-1}{2}\left(\frac{-\tau}{2}\cos(2\tau) + \frac{1}{4}\sin(2\tau)\right)_0^t = \frac{t}{4}\cos(2t) - \frac{1}{8}\sin(2t)$. Adding the two integral solutions together we obtain the solution to IVP (2) which is $y(t) = \frac{t}{4} - \frac{1}{8}\sin(2t)$.