AMATH 351 Homework 8: Laplace Transform

Jaiden Atterbury

2023-05-21

Exercise 1: Calculation of Laplace transform. By using the definition of Laplace transform, prove that the Laplace transforms of hyperbolic cosine and sine are

$$\mathcal{L}\left[\cosh(at)\right] = \frac{s}{s^2 - a^2}, \quad \mathcal{L}\left[\sinh(at)\right] = \frac{a}{s^2 - a^2}$$

Hint: Perform integration by parts twice as we did for the Laplace transforms of $\cos(\omega t)$ and $\sin(\omega t)$. Note that $(\cosh(at))' = \sinh(at)$, $(\sinh(at))' = \cosh(at)$, and $\cosh(0) = 1$, $\sinh(0) = 0$.

Laplace transform of $\cosh(\mathbf{at})$:
To find the Laplace transform of $\cosh(at)$, we will make use of two facts; the first of those facts being that $\cosh(at) = \frac{1}{2}(e^{at} + e^{-at})$, and the second of those facts being that $\mathcal{L}\left[f(t)\right] = \int\limits_0^\infty f(t)e^{-st}dt$. Using both of these facts we will now solve the following integral

$$\begin{aligned}
\mathcal{L}\left[\cosh(at)\right] &= \int\limits_0^\infty \cosh(at)e^{-st}dt \\
&= \int\limits_0^\infty \frac{1}{2}(e^{at} + e^{-at})e^{-st}dt \\
&= \frac{1}{2}\int\limits_0^\infty e^{at-st}dt + \frac{1}{2}\int\limits_0^\infty e^{-at-st}dt \\
&= \frac{1}{2}\int\limits_0^\infty e^{(-s+a)t}dt + \frac{1}{2}\int\limits_0^\infty e^{(-s-a)t}dt \\
&= \frac{1}{2}\left[\frac{e^{(-s+a)t}}{-s+a}\right]_0^\infty + \frac{1}{2}\left[\frac{e^{(-s-a)t}}{-s-a}\right]_0^\infty \\
&= \frac{-1}{2}\cdot\frac{1}{-s+a} - \frac{1}{2}\cdot\frac{1}{-s-a} \\
&= \frac{-1}{2}\left(\frac{-s-a-s+a}{(-s+a)(-s-a)}\right) \\
&= \frac{-1}{2}\left(\frac{-2s}{s^2-a^2}\right) \\
&= \frac{s}{s^2-a^2}
\end{aligned}$$

Thus as shown above, the Laplace transform of $\cosh(at)$ is $\frac{s}{s^2-a^2}$.

Laplace transform of $\sinh(\mathbf{at})$:
To find the Laplace transform of $\cosh(at)$, we will make use of two facts; the first of those facts being that

$\sinh(at) = \frac{1}{2}(e^{at} - e^{-at})$, and the second of those facts being that $\mathcal{L}[f(t)] = \int\limits_0^\infty f(t)e^{-st}dt$. Using both of these facts we will now solve the following integral

$$\begin{aligned}
\mathcal{L}[\cosh(at)] &= \int\limits_0^\infty \sinh(at)e^{-st}dt \\
&= \int\limits_0^\infty \frac{1}{2}(e^{at} - e^{-at})e^{-st}dt \\
&= \frac{1}{2}\int\limits_0^\infty e^{at-st}dt - \frac{1}{2}\int\limits_0^\infty e^{-at-st}dt \\
&= \frac{1}{2}\int\limits_0^\infty e^{(-s+a)t}dt - \frac{1}{2}\int\limits_0^\infty e^{(-s-a)t}dt \\
&= \frac{1}{2}\left[\frac{e^{(-s+a)t}}{-s+a}\right]_0^\infty - \frac{1}{2}\left[\frac{e^{(-s-a)t}}{-s-a}\right]_0^\infty \\
&= \frac{-1}{2}\cdot\frac{1}{-s+a} + \frac{1}{2}\cdot\frac{1}{-s-a} \\
&= \frac{-1}{2}\left(\frac{-s-a+s-a}{(-s+a)(-s-a)}\right) \\
&= \frac{-1}{2}\left(\frac{-2a}{s^2-a^2}\right) \\
&= \frac{a}{s^2-a^2}
\end{aligned}$$

Thus as shown above, the Laplace transform of $\sinh(at)$ is $\frac{a}{s^2-a^2}$.

Exercise 2: Calculation of inverse Laplace transform. Find the inverse Laplace transform of the following functions

1. $\frac{4}{s(s+3)}$

Let $Y(s) = \frac{4}{s(s+3)}$. Using the partial fractions decomposition of the previous rational function we can see that $\frac{4}{s(s+3)} = \frac{A}{s} + \frac{B}{s+3}$. Clearing the denominators we get the result $4 = A(s+3) + Bs \implies 4 = As + 3A + Bs$. Matching the coefficients of the powers of s on both sides of the equation leads us to solving the following series of equations: $A + B = 0$ and $3A = 4$. We can easily solve this system by noticing that $A = \frac{4}{3} \implies B = \frac{-4}{3}$. Plugging in our values of A and B into the equation of $Y(s)$ we obtain $Y(s) = \frac{4}{3}\cdot\frac{1}{s} - \frac{4}{3}\cdot\frac{1}{s+3}$. Taking the inverse Laplace transform of both sides we obtain $\mathcal{L}^{-1}[Y(s)] = \frac{4}{3}\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{4}{3}\mathcal{L}^{-1}\left[\frac{1}{s+3}\right] = \frac{4}{3} - \frac{4}{3}e^{-3t}$ Hence we can see that the inverse Laplace transform of $Y(s) = \frac{4}{s(s+3)}$ is $y(t) = \frac{4}{3} - \frac{4}{3}e^{-3t}$.

2. $\frac{2}{3s+5}$

Let $Y(s) = \frac{2}{3s+5} = \frac{2}{3}\cdot\frac{1}{s+\frac{5}{3}}$. Taking the inverse Laplace transform of both sides we obtain $\mathcal{L}^{-1}[Y(s)] = \frac{2}{3}\mathcal{L}^{-1}\left[\frac{1}{s+\frac{5}{3}}\right] = \frac{2}{3}e^{-\frac{5}{3}t}$. Hence we can see that the inverse Laplace transform of $Y(s) = \frac{2}{3s+5}$ is $y(t) = \frac{2}{3}e^{-\frac{5}{3}t}$.

3. $\frac{s+3}{s^2+2s+2}$

Let $Y(s) = \frac{s+3}{s^2+2s+2} = \frac{s+1+2}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$. Taking the inverse Laplace transform of both sides we obtain $\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{s+1}{(s+1)^2+1}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{(s+1)^2+1}\right] = e^{-t}\cos(t) + 2e^{-t}\sin(t)$. Hence we can see that the inverse Laplace transform of $Y(s) = \frac{s+3}{s^2+2s+2}$ is $y(t) = e^{-t}\cos(t) + 2e^{-t}\sin(t)$.

4. $\frac{2s+1}{(s-1)(s-2)}$

Let $Y(s) = \frac{2s+1}{(s-1)(s-2)}$. Using the partial fractions decomposition of the previous rational function we can see that $\frac{2s+1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$. Clearing the denominators we get the result $2s+1 = A(s-2) + B(s-1) \implies 2s+1 = As-2A+Bs-B$. Matching the coefficients of the powers of s on both sides of the equation leads us to solving the following series of equations: $A+B = 2$ and $-2A-B = 1$. Isolating B in the top equation we obtain $B = 2-A$, plugging this into the bottom equation we see $-2A-2+A = 1 \implies A = -3$ and hence $B = 5$. Plugging in our values of A and B into the equation of $Y(s)$ we obtain $Y(s) = -3\cdot\frac{1}{s-1} + 5\cdot\frac{1}{s-2}$. Taking the inverse Laplace transform of both sides we obtain $\mathcal{L}^{-1}[Y(s)] = -3\mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + 5\mathcal{L}^{-1}\left[\frac{1}{s-2}\right] = -3e^t + 5e^{2t}$. Hence we can see that the inverse Laplace transform of $Y(s) = \frac{2s+1}{(s-1)(s-2)}$ is $y(t) = -3e^t + 5e^{2t}$.

Exercise 3: First-order ODEs. Using Laplace transform, find the solution to the IVPS:

1. $y' = -y + e^{-2t}, \quad y(0) = 1,$

Let $\mathcal{L}[y(t)] = Y(s)$. We will start off by taking the Laplace transform of both sides of the ODE

$$\begin{aligned}
\mathcal{L}[y'] &= \mathcal{L}[y] + \mathcal{L}\left[e^{-2t}\right] \\
sY(s) - y(0) &= -Y(s)\frac{1}{s+2} \\
sY(s) - 1 &= -Y(s)\frac{1}{s+2} \\
(s+1)Y(s) &= \frac{1}{s+2} + 1 \\
(s+1)Y(s) &= \frac{s+3}{s+2} \\
Y(s) &= \frac{s+3}{(s+2)(s+1)}
\end{aligned}$$

Using the partial fractions decomposition of the previous rational function we can see that $\frac{s+3}{(s+2)(s+1)} = \frac{A}{s+1} + \frac{B}{s+2}$. Clearing the denominators we get the result $s+3 = A(s+2)+B(s+1) \implies s+3 = As+2A+Bs+B$. Matching the coefficients of the powers of s on both sides of the equation leads us to solving the following series of equations: $A+B = 1$ and $2A+B = 3$. Isolating B in the top equation we obtain $B = 1-A$, plugging this into the bottom equation we see $2A+1-A = 3 \implies A = 2$ and hence $B = -1$. Plugging in our values of A and B into the equation of $Y(s)$ we obtain $Y(s) = 2\cdot\frac{1}{s+1} - \frac{1}{s+2}$. Taking the inverse Laplace transform of both sides we obtain $\mathcal{L}^{-1}[Y(s)] = 2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] = 2e^{-t} - e^{-2t}$. Hence we can see that the solution to our ODE is $y(t) = 2e^{-t} - e^{-2t}$.

2. $y' = -y + t^2, \quad y(0) = 0,$

Let $\mathcal{L}[y(t)] = Y(s)$. We will start off by taking the Laplace transform of both sides of the ODE

$$\begin{aligned}
\mathcal{L}[y'] &= -\mathcal{L}[y] + \mathcal{L}\left[t^2\right] \\
sY(s) - y(0) &= -Y(s)\frac{2}{s^3} \\
sY(s) &= -Y(s)\frac{2}{s^3} \\
(s+1)Y(s) &= \frac{2}{s^3} \\
Y(s) &= \frac{2}{s^3(s+1)}
\end{aligned}$$

Using the partial fractions decomposition of the previous rational function we can see that $\frac{2}{s^3(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^2} + \frac{D}{s+1}$. Clearing the denominators we get the result $2 = As^2(s+1) + Bs(s+1) + C(s+1) + Ds^3 \implies 2 = As^3 + As^2 + Bs^2 + Bs + Cs + C + Ds^3$. Matching the coefficients of the powers of s on both sides of the equation leads us to solving the following series of equations: $A+D=0$, $A+B=0$, $B+C=0$, and $C=2$. From those equations we can see that $B=-2$, $A=2$, $C=2$, and $D=-2$. Plugging in our values of A, B, C, and D into the equation of $Y(s)$ we obtain $Y(s) = 2\cdot\frac{1}{s} - 2\cdot\frac{1}{s^2} + \frac{2}{s^3} - 2\cdot\frac{1}{s+1}$. Taking the inverse Laplace transform of both sides we obtain $\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[2\cdot\frac{1}{s} - 2\cdot\frac{1}{s^2} + \frac{2}{s^3} - 2\cdot\frac{1}{s+1}\right] = 2 - 2t + t^2 - 2e^{-t}$. Hence we can see that the solution to our ODE is $y(t) = 2 - 2t + t^2 - 2e^{-t}$.

3. $y' + 4y = 2 + 3t, \quad y(0) = 0,$

Let $\mathcal{L}[y(t)] = Y(s)$. We will start off by taking the Laplace transform of both sides of the ODE

$$\begin{aligned}
\mathcal{L}[y'] + 4\mathcal{L}[y] &= \mathcal{L}[2] + 3\mathcal{L}[t] \\
sY(s) - y(0) + 4Y(s) &= \frac{2}{s} + \frac{3}{s^2} \\
(s+4)Y(s) &= \frac{2s+3}{s^2} \\
Y(s) &= \frac{2s+3}{s^2(s+2)}
\end{aligned}$$

Using the partial fractions decomposition of the previous rational function we can see that $\frac{2s+3}{s^2(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2}$. Clearing the denominators we get the result $2s+3 = As(s+4) + B(s+4) + cs^2 \implies 2s+3 = As^2 + 4As + Bs + 4B + cs^2$. Matching the coefficients of the powers of s on both sides of the equation leads us to solving the following series of equations: $A+C=0$, $4A+B=2$, and $4B=3$. Hence we can see that $B=\frac{3}{4}$, $A=\frac{5}{16}$, and $C=\frac{-5}{16}$. Plugging in our values of A, B, and C into the equation of $Y(s)$ we obtain $Y(s) = \frac{5}{16}\cdot\frac{1}{3} + \frac{3}{4}\cdot\frac{1}{s^2} - \frac{5}{16}\frac{1}{s+4}$. Taking the inverse Laplace transform of both sides we obtain $\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{5}{16}\cdot\frac{1}{3} + \frac{3}{4}\cdot\frac{1}{s^2} - \frac{5}{16}\frac{1}{s+4}\right] = \frac{5}{16} + \frac{3}{4}t - \frac{5}{16}e^{-4t}$. Hence we can see that the solution to our ODE is $y(t) = \frac{5}{16} + \frac{3}{4}t - \frac{5}{16}e^{-4t}$.

and the general solution of the ODE:

4. $\frac{dy}{dt} = 2y + 2e^{-3t}$.

Hint: For question 4, treat $y(0)$ that appears in the transformed equation as one undetermined coefficient; $y(0) = c$.

Let $\mathcal{L}[y(t)] = Y(s)$ and $y(0) = c$. We will start off by taking the Laplace transform of both sides of the ODE

$$\begin{aligned}
\mathcal{L}[y'] &= 2\mathcal{L}[y] + 2\mathcal{L}\left[e^{-3t}\right] \\
sY(s) - y(0) &= 2Y(s) + \frac{2}{s+3} \\
sY(s) - 2Y(s) &= c + \frac{2}{s+3} \\
(s-2)Y(s) &= \frac{c(s+3)+2}{s+3} \\
Y(s) &= \frac{cs+3c+2}{(s-2)(s+3)}
\end{aligned}$$

Using the partial fractions decomposition of the previous rational function we can see that $\frac{cs+3c+2}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}$. Clearing the denominators we get the result $cs+3c+2 = A(s+3)+B(s-2) \implies cs+3c+2 = As+3A+Bs-2B$. Matching the coefficients of the powers of s on both sides of the equation leads us to solving the following series of equations: $A+B=c$ and $3A-2B = 3c+2$. Isolating A in the top equation we obtain $A = 1-B$, plugging this into the bottom equation we see $3c-3B-2B = 3c+2 \implies B = \frac{-2}{5}$ and hence $A = c+\frac{2}{5}$. Plugging in our values of A and B into the equation of $Y(s)$ we obtain $Y(s) = (c+\frac{2}{5})\cdot\frac{1}{s-2} - \frac{2}{5}\cdot\frac{1}{s+3}$. Taking the inverse Laplace transform of both sides we obtain $\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[(c+\frac{2}{5})\cdot\frac{1}{s-2} - \frac{2}{5}\cdot\frac{1}{s+3}\right] = (c+\frac{2}{5})e^{2t} - \frac{2}{5}e^{-3t}$. Hence we can see that the solution to our ODE is $y(t) = (c+\frac{2}{5})e^{2t} - \frac{2}{5}e^{-3t}$.

Exercise 4: Second-order ODEs. Using Laplace transform, find the solution to the IVPs:

1. $y'' + 3y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 2,$

Let $\mathcal{L}[y(t)] = Y(s)$. We will start off by taking the Laplace transform of both sides of the ODE

$$\begin{aligned}
\mathcal{L}[y''] + 3\mathcal{L}[y'] + 3\mathcal{L}[y] &= \mathcal{L}[0] \\
s^2Y(s) - sy(0) - y'(0) + 3sY(s) - 3y(0) + 2Y(s) &= 0 \\
(s^2+3s+2)Y(s) - s - 2 - 3 &= 0 \\
(s^2+3s+2)Y(s) &= s+5 \\
Y(s) &= \frac{s+5}{s^2+3s+2} \\
Y(s) &= \frac{s+5}{(s+1)(s+2)}
\end{aligned}$$

Using the partial fractions decomposition of the previous rational function we can see that $\frac{s+5}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$. Clearing the denominators we get the result $s+5 = A(s+2)+B(s+1) \implies 2+5 = As+2A+Bs+B$. Matching the coefficients of the powers of s on both sides of the equation leads us to solving the following series of equations: $A+B = 1$, $2A+B = 5$. Isolating A in the top equation we obtain $A = 1-B$, plugging this into the bottom equation we see $2-2B+B = 5 \implies B = -3$ and hence $A = 4$. Plugging in our values of A, B, and C into the equation of $Y(s)$ we obtain $Y(s) = 4\cdot\frac{1}{s+1} - 3\cdot\frac{1}{s+2}$. Taking the inverse Laplace transform of both sides we obtain $\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[4\cdot\frac{1}{s+1} - 3\cdot\frac{1}{s+2}\right] = 4e^{-t} - 3e^{-2t}$. Hence we can see that the solution to our ODE is $y(t) = 4e^{-t} - 3e^{-2t}$.

2. $y'' + y = t, \quad y(0) = 0, \quad y'(0) = 1,$

Let $\mathcal{L}[y(t)] = Y(s)$. We will start off by taking the Laplace transform of both sides of the ODE

$$\begin{aligned}
\mathcal{L}[y''] + \mathcal{L}[y] &= \mathcal{L}[t] \\
s^2Y(s) - sy(0) - y'(0) + Y(s) &= \frac{1}{s^2} \\
(s^2+1)Y(s) - 1 &= \frac{1}{s^2} \\
(s^2+1)Y(s) &= \frac{1}{s^2} + 1 \\
(s^2+1)Y(s) &= \frac{s^2+1}{s^2} \\
Y(s) &= \frac{1}{s^2}
\end{aligned}$$

Taking the inverse Laplace transform of both sides we obtain $\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t$ Hence we can see that the solution to our ODE is $y(t) = t$.

3. $y'' - 3y' + 2y = 4t + e^{3t}, \quad y(0) = 0, \quad y'(0) = 0,$

Let $\mathcal{L}[y(t)] = Y(s)$. We will start off by taking the Laplace transform of both sides of the ODE

$$\begin{aligned}
\mathcal{L}[y''] - 3\mathcal{L}[y'] + 2\mathcal{L}[y] &= 4\mathcal{L}[t] + 2\mathcal{L}\left[e^{3t}\right] \\
s^2Y(s) - sy(0) - y'(0) - 3sY(s) + 3y(0) + 2Y(s) &= \frac{4}{s^2} + \frac{1}{s-3} \\
(s^2 - 3s + 2)Y(s) &= \frac{4(s-3)+s^2}{s^2(s-3)} \\
Y(s) &= \frac{4(s-3)+s^2}{s^2(s-3)(s^2-3s+2)} \\
Y(s) &= \frac{(s-2)(s+6)}{s^2(s-3)(s-1)(s-2)} \\
Y(s) &= \frac{s+6}{s^2(s-1)(s-3)}
\end{aligned}$$

Using the partial fractions decomposition of the previous rational function we can see that $\frac{s+6}{s^2(s-1)(s-3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s-3}$. Clearing the denominators we get the result $s + 6 = As(s-1)(s-3) + B(s-1)(s-3) + Cs^2(s-3) + Ds^2(s-1) \implies s + 6 = A(s^3 - 4s^2 + 3s) + B(s^2 - 4s + 3) + C(s^3 - 3s^2) + D(s^3 - s^2) \implies As^3 - 4As^2 + 3As + Bs^2 - 4Bs + 3B + Cs^3 - 3Cs^2 + Ds^3 - Ds^2$. Matching the coefficients of the powers of s on both sides of the equation leads us to solving the following series of equations: $A + C + D = 0$, $-4A + B - 3C - D = 0$, $3A - 4B = 1$, and $3B = 6$. Hence we can see that $B = 2$ and $A = 3$. We now have to solve the following system of equations for C and D, $C + D = -3$ and $-3c - D = 10$. Isolating D in the first equation we obtain $D = -C - 3$, plugging this into the second equation we see $-3C + C + 3 = 10 \implies C = \frac{-7}{2}$ and hence $D = \frac{1}{2}$. Plugging in our values of A, B, C, and D into the equation of $Y(s)$ we obtain $Y(s) = 3 \cdot \frac{1}{s} + 2 \cdot \frac{1}{s^2} - \frac{7}{2} \cdot \frac{1}{s-1} + \frac{1}{2} \cdot \frac{1}{s-3}$. Taking the inverse Laplace transform of both sides we obtain $\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[3 \cdot \frac{1}{s} + 2 \cdot \frac{1}{s^2} - \frac{7}{2} \cdot \frac{1}{s-1} + \frac{1}{2} \cdot \frac{1}{s-3}\right] = 3 + 2t - \frac{7}{2}e^t + \frac{1}{2}e^{3t}$. Hence we can see that the solution to our ODE is $y(t) = 3 + 2t - \frac{7}{2}e^t + \frac{1}{2}e^{3t}$.

4. $y'' + 4y' + 4y = t, \quad y(0) = 0, \quad y'(0) = 0.$

Let $\mathcal{L}[y(t)] = Y(s)$. We will start off by taking the Laplace transform of both sides of the ODE

$$\begin{aligned} \mathcal{L}[y''] + 4\mathcal{L}[y'] + 4\mathcal{L}[y] &= \mathcal{L}[t] \\ s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 4Y(s) &= \frac{1}{s^2} \\ (s^2+4s+4)Y(s) &= \frac{1}{s^2} \\ Y(s) &= \frac{1}{s^2(s^2+4s+4)} \\ Y(s) &= \frac{1}{s^2(s+2)^2} \end{aligned}$$

Using the partial fractions decomposition of the previous rational function we can see that $\frac{1}{s^2(s+2)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{(s+2)^2}$. Clearing the denominators we get the result $1 = As(s+2)^2 + B(s+2)^2 + Cs^2(s+2) + Ds^2 \implies 1 = A(s^3+4s^2+4) + B(s^2+4s+4) + Cs^3 + 2Cs^2 + Ds^2 \implies 1 = As^3 + 4As^2 + 4As + Bs^2 + 4Bs + 4B + Cs^3 + 2Cs^2 + Ds^2$. Matching the coefficients of the powers of s on both sides of the equation leads us to solving the following series of equations: $A + C = 0$, $4A + B + 2C + D = 0$, $4A + 4B = 0$, and $4B = 1$. Hence we can see that $B = \frac{1}{4}$ and $A = \frac{-1}{4}$, $C = \frac{1}{4}$, and lastly $D = \frac{1}{4}$. Plugging in our values of A, B, C, and D into the equation of $Y(s)$ we obtain $Y(s) = \frac{-1}{4} \cdot \frac{1}{s} + \frac{1}{4} \cdot \frac{1}{s^2} + \frac{1}{4} \cdot \frac{1}{s+2} + \frac{1}{4} \cdot \frac{1}{(s+2)^2}$. Taking the inverse Laplace transform of both sides we obtain $\mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{-1}{4} \cdot \frac{1}{s} + \frac{1}{4} \cdot \frac{1}{s^2} + \frac{1}{4} \cdot \frac{1}{s+2} + \frac{1}{4} \cdot \frac{1}{(s+2)^2}\right] = \frac{-1}{4} + \frac{1}{4}t + \frac{1}{4}e^{-2t} + \frac{1}{4}te^{-2t}$. Hence we can see that the solution to our ODE is $y(t) = \frac{-1}{4} + \frac{1}{4}t + \frac{1}{4}e^{-2t} + \frac{1}{4}te^{-2t}$.