Jaiden Atterbury Homework 7

Component Skills Exercises

Exercise 1. (CS4.5):

Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & 3 & 0 \\ 0 & 2 & 0 & -4 \\ 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

(a.) Compute det(A) by Laplace expansion about the third row. When computing the determinant of your 3x3 matrix, expand about the third row.

$$\det(A) = 3\det\begin{pmatrix} 1 & 3 & 0 \\ 2 & 0 & -4 \\ -1 & 0 & 0 \end{pmatrix} = -3\det\begin{pmatrix} 3 & 0 \\ 0 & -4 \end{pmatrix} = -3 \cdot -12 = \boxed{36}$$

(b.) Compute det(A) by Laplace expansion about the first column. When computing the determinant of your 3x3 matrix, expand about the second column.

$$\det(A) = 3\det\begin{pmatrix} 1 & 3 & 0 \\ 2 & 0 & -4 \\ -1 & 0 & 0 \end{pmatrix} = -9\det\begin{pmatrix} 2 & -4 \\ -1 & 0 \end{pmatrix} = -9 \cdot -4 = \boxed{36}$$

(c.) Compute det(A) by Lapace expansion about the third column. When computing the determinant of your 3x3 matrix, expand about the second row.

$$\det(A) = 3\det\begin{pmatrix} 0 & 2 & -4 \\ 3 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = -9\det\begin{pmatrix} 2 & -4 \\ -1 & 0 \end{pmatrix} = -9 \cdot -4 = \boxed{36}$$

(d.) Compute det(A) by reducing A to an upper-triangular matrix.

$$A = \underset{K=0}{\begin{pmatrix} 0 & 1 & 3 & 0 \\ 0 & 2 & 0 & -4 \\ 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}} \overset{R_3 \leftrightarrow R_1}{\sim} \underset{K=1}{\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & -4 \\ 0 & 1 & 3 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}} \overset{-\frac{1}{2}R_2 + R_3}{\sim} \underset{K=1}{\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & -4 \\ 0 & 0 & 3 & 2 \\ 0 & -1 & 0 & 0 \end{pmatrix}} \overset{\frac{1}{2}R_2 + R_4}{\sim} \underset{K=1}{\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & -4 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & -2 \end{pmatrix}} = U$$

$$\det(A) = (-1)^K u_{11} u_{22} u_{33} u_{44} = -1 \cdot 3 \cdot 2 \cdot 3 \cdot -2 = \boxed{36}$$

- Exercise 2. (CS4.5):

(a.) Prove that $\det(A^{-1}) = 1/\det(A)$ provided A^{-1} exists.

Hint: Use the definition of A^{-1}.

$$AA^{-1} = I_n$$

$$\det(AA^{-1}) = \det(I_n)$$

$$\det(A)\det(A^{-1}) = 1 \quad (\text{since } \det(AB) = \det(A)\det(B) \text{ and } \det(I_n) = 1)$$

$$\boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$$

(b.) Prove that $\det(A^T) = \det(A)$.

Hint: Consider the PLU decomposition of A. Note that the matrix P is orthogonal.

As shown in class, from the PLU decomposition of A, $\det(A) = (-1)^k u_{11} \cdots u_{nn}$. where k is the number of row swaps that took place, and $u_{11} \cdots u_{nn}$ are the diagonal entries of U. Now taking the transpose of the PLU decomposition and computing the determinant we see

$$(PA)^T = (LU)^T$$

$$A^T P^T = U^T L^T \quad (\text{since } (AB)^T = B^T A^T)$$

$$A^T = U^T L^T P \quad (\text{since } P \text{ is orthogonal})$$

$$\det(A^T) = \det(U^T L^T P)$$

$$\det(A^T) = \det(U^T)\det(L^T)\det(P) \quad (\text{since } \det(AB) = \det(A)\det(B))$$

From above we know $\det(P) = (-1)^k$. Since L is lower unitriangular (1's on the diagonal), L^T is upper unitriangular and hence $\det(L^T) = 1$. Also, since U is upper triangular, U^T is lower triangular with the same diagonal entries as U, hence $\det(U^T) = u_{11} \cdots u_{nn}$. Therefore, we have shown that

$$\det(A^T) = (-1)^k \cdot 1 \cdot u_{11} \cdots u_{nn}$$
$$= (-1)^k u_{11} \cdots u_{nn}$$

thus we can see that $\boxed{\det(A^T) = \det(A)}$.

(c.) Two $n \times n$ matrices A and B are called similar if there exists an $n \times n$ invertible matrix S such that $B = SAS^{-1}$. Prove that $\det(A) = \det(B)$ when A and B are similar.

If A and B are similar then

$$\det(B) = \det(SAS^{-1})$$
$$= \det(S)\det(A)\det(S^{-1}) \quad (\text{since } \det(AB) = \det(A)\det(B))$$
$$= \det(S)\det(A)\frac{1}{\det(S)} \quad (\text{since } \det(A^{-1}) = \frac{1}{\det(A)})$$
$$= \boxed{\det(A)}$$

(d.) Prove that $\det(cA) = c^n \det(A)$, where $c \in \mathbb{R}$ and A is $n \times n$.

Hint: write cA as CA, where C is an appropriately chosen diagonal matrix.

Let $c \in \mathbb{R}$ and $C \in \mathbb{R}^{n \times n}$, where $C = \operatorname{diag}(c)$ which is a matrix whose only non-zero entries are c on the diagonals. Hence we can see

$$\det(CA) = \det(C)\det(A) \quad (\text{since } \det(AB) = \det(A)\det(B)$$
$$= c \cdot c \cdots c \det(A) \quad (\text{since } C \text{ has } n\ c\text{'s on the diagonal})$$
$$= \boxed{c^n \det(A)}$$

- <u>Exercise 3. (CS 4.5):</u>

From Olver and Shakiban, complete Exercise 1.9.4. If a statement is true and the proof is already provided in the textbook or course notes, just cite the results. You do not need to rederive them. For (h), O represents the zero matrix.

1.9.4. True or false: If true, explain why. If false, give an explicit counterexample.

(a.) If $\det A \neq 0$ then A^{-1} exists.

By Theorem 2, an $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$. Thus this statement is $\boxed{\text{True}}$.

(b.) $\det(2A) = 2\det(A)$.

As shown in Exercise 2, $\det(cA) = c^n \det(A)$, thus we can see this statement is $\boxed{\text{False}}$.

Counter example: Let $A = I_2$, then $\det(A) = 1$, $2A = 2I_2 = \text{diag}(2) \Rightarrow \det(2A) = 4 \neq 1$.

(c) $\det(A+B) = \det A + \det B$

This statement is $\boxed{\text{False}}$ as shown by the counterexample

Counterexample: Let $A = B = I_2$, then $\det(I_2 + I_2) = \det\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4$, but $\det I_2 + \det I_2 = 1 + 1 = 2 \neq 4$.

(d.) $\det(A^{-T}) = \frac{1}{\det(A)}$

This statement is $\boxed{\text{True}}$ as shown below.

$$\begin{aligned} \det A^{-T} &= \det(A^{-1})^T \\ &= \det(A^{-1}) \quad (\text{since } \det(A^T) = \det(A)) \\ &= \boxed{\frac{1}{\det(A)}} \quad (\text{since } \det(A^{-1}) = \frac{1}{\det(A)}). \end{aligned}$$

(e.) $\det(AB^{-1}) = \frac{\det(A)}{\det(B)}$

This statement is $\boxed{\text{True}}$ as shown below.

$$\begin{aligned} \det(AB^{-1}) &= \det(A)\det(B^{-1}) \quad (\text{since } \det(AB) = \det(A)\det(B)) \\ &= \boxed{\frac{\det(A)}{\det(B)}} \quad (\text{since } \det(A^{-1}) = \frac{1}{\det(A)}) \end{aligned}$$

(f.) $\det[(A+B)(A-B)] = \det(A^2 - B^2)$

This statement is $\boxed{\text{False}}$ as shown below.

$\det[(A+B)(A-B)] = \det[(A+B)A - (A+B)B] = \det(A^2 + BA - AB + B^2)$, since $BA \neq AB$ in general, we can see this is False.

Counterexample: Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

$$\det[(A+B)(A-B)] = \det\left[\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)\right]$$
$$= \det\left[\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\right]$$
$$= \det\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$
$$= 2 - 2$$
$$= 0$$

$$\det(A^2 - B^2) = \det\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^2 - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^2\right)$$
$$= \det\left(\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$$
$$= \det\left(\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}\right)$$
$$= 2 - 4$$
$$= -2$$

g. If A is an $n \times n$ matrix with $\det A = 0$, then $\text{rank } A < n$.

If $\det A = 0$, then A is not invertible, which means A has linearly independent columns which implies $\text{rank}(A) < n$. Thus this statement is $\boxed{\text{true}}$

h. If $\det A = 1$ and $AB = 0$, then $B = 0$.

If $\det A = 1$, then A is invertible, thus

$$AB = 0 \implies B = A^{-1}0 \implies B = 0$$

hence this statement is $\boxed{\text{true}}$.

- Exercise 4. (CS 4.5):

Determine the volume of the parallelepiped (three-dimensional analogue of a parallelogram) that is formed by the vectors

$$a_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad a_2 = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$$

The volume of this parallelepiped is equal to the magnitude of the determinant of the matrix formed by these vectors. Namely,

$$\text{volume} = \left|\det\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & -2 \\ 1 & 0 & 3 \end{bmatrix}\right|$$

Expanding about the third row we obtain

$$\text{volume} = \left|\det\begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix} + 3\det\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\right|$$

$$= |2-6+3(3+2)|$$

$$= |2-6+15|$$

$$= \boxed{11}$$

- Exercise 5. (CS 4.6):

Find the Eigenvalues and eigenvectors of the following matrices:

(a.) $A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$

i.) Eigenvalues:

$$\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 4 = 0 \Rightarrow (1-\lambda)^2 = 4 \Rightarrow 1-\lambda = \pm 2 \Rightarrow \lambda = 1 \mp 2$$

$\therefore$ $\boxed{\lambda_1 = 3,\ \lambda_2 = -1.}$

ii.) Eigenvectors:

- $\lambda_1 = 3$:

$$\text{null}(A-3I) = \left[\begin{array}{cc|c} -2 & 4 & 0 \\ 1 & -2 & 0 \end{array}\right] \underset{\sim}{\overset{\frac{1}{2}R_1+R_2}{}} \left[\begin{array}{cc|c} -2 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \boxed{v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$$

- $\lambda_2 = -1$:

$$\text{null}(A+I) = \left[\begin{array}{cc|c} 2 & 4 & 0 \\ 1 & 2 & 0 \end{array}\right] \overset{-\frac{1}{2}R_1+R_2}{\sim} \left[\begin{array}{cc|c} 2 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \boxed{v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}}$$

$\therefore$ $\lambda_1 = 3$, $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\lambda_2 = -1$, $v_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

(b.) $B = \begin{pmatrix} 2 & 0 & -2 \\ -2 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

i.) Eigenvalues:

$$\det(B-\lambda I) = \det\begin{pmatrix} 2-\lambda & 0 & -2 \\ -2 & 1-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{pmatrix} = (1-\lambda)\det\begin{pmatrix} 2-\lambda & 0 \\ -2 & 1-\lambda \end{pmatrix} = (1-\lambda)^2(2-\lambda) \Rightarrow$$

$\boxed{\lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = 1;}$ $\lambda = 1$ has algebraic multiplicity of 2.

ii) Eigenvectors:

- $\lambda_1 = 2$:

$$\text{null}(A-2I) = \left[\begin{array}{ccc|c} 0 & 0 & -2 & 0 \\ -2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \overset{R_2 \leftrightarrow R_1}{\sim} \left[\begin{array}{ccc|c} -2 & -1 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \overset{-\frac{1}{2}R_2+R_3}{\sim} \left[\begin{array}{ccc|c} -2 & -1 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \boxed{v_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}}$$

- $\lambda_{2,3} = 1$:

$$\text{null}(A-I) = \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{2R_1+R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \boxed{v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

$\therefore$ $\lambda_1 = 2$, $v_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ and $\lambda_{2,3} = 1$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

- Exercise 6. (CS 4.6):

Consider the matrix $A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$.

(a) Find the eigenvalues of A.

$\det(A-\lambda I) = \det\begin{pmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 9 = 0 \Rightarrow (2-\lambda)^2 = 9 \Rightarrow 2-\lambda = \pm 3 \Rightarrow \lambda = 2 \mp 3 \Rightarrow$

$\boxed{\lambda_1 = 5, \quad \lambda_2 = -1.}$

(b) Find the corresponding eigenvectors of A.

- $\lambda_1 = 5$

$$\text{null}(A-5I) = \left(\begin{array}{cc|c} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right) \overset{R_1+R_2}{\sim} \left(\begin{array}{cc|c} -3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \boxed{v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

- $\lambda_2 = -1$

$$\text{null}(A+I) = \left(\begin{array}{cc|c} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right) \overset{-R_1+R_2}{\sim} \left(\begin{array}{cc|c} 3 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \boxed{v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}}$$

(c) Show that your eigenvectors are orthogonal. Why is this expected?

$v_1 \cdot v_2 = v_1^T v_2 = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -1 \cdot 1 + 1 \cdot 1 = -1 + 1 = 0 \Rightarrow v_1$ and v_2 are orthogonal.

This is expected because A is symmetric and thus has orthogonal eigenvectors.

(d.) Write out the spectral decomposition $A = Q\Lambda Q^T$, where Q is an orthogonal matrix.

1.) Normalize eigenvectors:

$q_1 = \frac{v_1}{\|v_1\|_2} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$

$q_2 = \frac{v_2}{\|v_2\|_2} = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$

2.) Form Q, Λ, Q^T:

$Q = (q_1 | q_2) = \begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix} \Rightarrow Q^T = \begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}, \Lambda = \begin{pmatrix}5 & 0\\ 0 & -1\end{pmatrix}$

$\boxed{\text{Thus } A = Q\Lambda Q^T = \begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}5 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}.}$

- Exercise 7. (CS4.6):

Consider the matrix $A = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$.

(a.) Find the eigenvalues of A.

$\det(A - \lambda I) = \det\begin{pmatrix}-\lambda & 1\\ -1 & -\lambda\end{pmatrix} = \lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1 \Rightarrow \lambda = \pm i \Rightarrow \boxed{\lambda_1 = i,\ \lambda_2 = -i}$

(b.) Find the corresponding eigenvectors of A.

- $\lambda_1 = i$

$\text{null}(A - iI) = \left(\begin{array}{cc|c}-i & 1 & 0\\ -1 & -i & 0\end{array}\right) \overset{iR_1 + R_2}{\sim} \left(\begin{array}{cc|c}-i & 1 & 0\\ 0 & 0 & 0\end{array}\right) \Rightarrow \boxed{v_1 = \begin{pmatrix}i\\-1\end{pmatrix}}$

- $\lambda_2 = -i$

$\text{null}(A + iI) = \left(\begin{array}{cc|c}i & 1 & 0\\ -1 & i & 0\end{array}\right) \overset{-iR_1 + R_2}{\sim} \left(\begin{array}{cc|c}i & 1 & 0\\ 0 & 0 & 0\end{array}\right) \Rightarrow \boxed{v_2 = \begin{pmatrix}i\\1\end{pmatrix}}$