4) Cauchy stress tensor T is a 3x3 matrix that can predict the force of a fluid pushing against a flat surface. In particular if $x = [x_1; x_2; x_3]$ represents the normal vector to a flat surface in 3D space such that:

$$\|x\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

is the area of the surface, then the matrix multiplication Tx represents the force vector on that surface, i.e., $f = Tx$.

Suppose for a given fluid configuration that the cauchy stress tensor is:

$$T = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

(a) compute the force vector on a surface with normal vector $x = [2; 1; 1]$

$$f = Tx \Rightarrow f = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 + 2 + 1 \\ 4 + 5 + 3 \\ 2 + 3 + 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 12 \\ 9 \end{bmatrix}$$

(b) Suppose the force vector on an unknown surface is $f = [0, 0, 1]$. Derive a 3x3 system of equations for the component x_j of the normal vector to the unknown surface. Express your equations

(i) in linear system notation,

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ 2x_1 + 5x_2 + 3x_3 = 0 \\ x_1 + 3x_2 + 4x_3 = 1 \end{cases}$$

(ii) in matrix-vector form,

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 1 & 3 & 4 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\vec{f}}$$

(iii) as an augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ 1 & 3 & 4 & 1 \end{array}\right]$$

(c) Reduce your linear system in (b) to triangular form by Gaussian elimination. (You can use the augmented matrix if you prefer.)

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ 1 & 3 & 4 & 1 \end{array}\right] \underset{\sim}{\overset{-2R_1+R_2 \to R_2}{-R_1+R_3 \to R_3}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & 1 \end{array}\right]$$

$$\underset{\sim}{-R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

(d) Solve the triangular system from (c) by back substitution.

$2x_3 = 1 \Rightarrow x_3 = \frac{1}{2}$

$x_2 + x_3 = 0 \Rightarrow x_2 = -\frac{1}{2}$

$x_1 + 2x_2 + x_3 = 0 \Rightarrow x_1 - 1 + \frac{1}{2} = 0 \Rightarrow x_1 = \frac{1}{2}$

thus the solution is $(x_1, x_2, x_3) = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$

(e) What is the normal vector of the unknown surface from (b)?

normal vector is $x = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$

What is the surface area?

surface area is $\|x\| = \sqrt{\frac{1}{2}^2 + (-\frac{1}{2})^2 + (\frac{1}{2})^2}$

$= \sqrt{3/4}$

$= \sqrt{3}/2$