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(1.) Answer the following true-false questions. Answer the questions carefully. You do not need to justify your answers.

a.) Given $A \in \mathbb{R}^{m \times n}$, the zero vector in $\mathbb{R}^n$ is always in range(A).

This statement is $\boxed{\text{False}}$.

b.) There exists $A \in \mathbb{R}^{3 \times 3}$ with eigenvalues λ_1, $\lambda_2 = -2$ and $\lambda_3 = i$.

This statement is $\boxed{\text{False}}$.

c.) If $A \in \mathbb{R}^{n \times n}$, then $\text{rank}(A) = \text{rank}(A^2)$.

This statement is $\boxed{\text{False}}$.

d.) If the j^{th} column of $A \in \mathbb{R}^{n \times n}$ is the j^{th} canonical basis vector, then $\lambda = 1$ is an eigenvalue of A.

This statement is $\boxed{\text{True}}$.

e.) If $A \in \mathbb{R}^{n \times n}$ has a repeated eigenvalue, then A is not diagonalizable.

This statement is $\boxed{\text{False}}$.

f.) If $A \in \mathbb{R}^{m \times n}$ is full rank, then the eigenvalues of A^TA and AA^T are positive real numbers.

This statement is $\boxed{\text{True}}$.

g.) If $A \in \mathbb{R}^{m \times n}$ has K non-zero singular values, then $\text{rank}(A) = k$

This statement is $\boxed{\text{True}}$.

h) If $A \in \mathbb{R}^{n \times n}$, the singular values of $\exp(A)$ are the exponentials of the singular values of A.

This statement is $\boxed{\text{True}}$.

i) The singular values of a symmetric matrix are its eigenvalues.

This statement is $\boxed{\text{False}}$.

j.) Let $A \in \mathbb{R}^{m \times n}$ and $A = U \Sigma V^T$ be a SVD of A. If $\text{rank}(A) = k$, then the first k columns of V are an orthonormal basis for $\text{null}(A)$.

This statement is $\boxed{\text{False}}$.

k.) Let $u \neq 0 \in \mathbb{R}^m$ and $v \neq \mathbb{R}^n$. Then $\text{rank}(uv^T) = 1$.

This statement is $\boxed{\text{True}}$.

l.) The singular values of an orthogonal matrix are always equal to 1.

This statement is $\boxed{\text{True}}$

m) Let $A \in \mathbb{R}^{m \times n}$ and $A = U \Sigma V^T$ be a SVD of A. If $\text{rank}(A) = k$, then the last k columns of U are an orthonormal basis for the left nullspace of A

This statement is $\boxed{\text{False}}$

n.) suppose σ is a singular value of $A \in \mathbb{R}^{m \times n}$. Then, $|c|\sigma$ is a singular value of $cA \in \mathbb{R}^{m \times n}$ for any non-zero $c \in \mathbb{R}$.

This statement is $\boxed{\text{True}}$.

0) If σ is a singular value of an invertible matrix, $A \in \mathbb{R}^{n \times n}$, then $1/\sigma$ is a singular value of A^{-1}.

This statement is $\boxed{\text{True}}$

(2.) Consider the following data

x	y
0	1
1	2
2	-1
3	2

we want to fit this data to the polynomial $y = a_1x^4 + a_2x^3 + a_3x^2 + a_4x + a_5$.

You may use MATLAB for the calculations. You do not need to submit your code.

a.) Determine a linear system for the coefficients, a_j, write it in matrix-vector form.

Let $y = f(x)$, then

$$\begin{cases} f(0) = 1 \\ f(1) = 2 \\ f(2) = -1 \\ f(3) = 2 \end{cases}$$

which can be written as the linear system

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 81 & 27 & 9 & 3 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \end{pmatrix}}_{\vec{b}}$$

b.) Since the system is underdetermined, we choose to minimize $\|Ax-b\|_2^2 + \|x\|_2^2$. Write down the linear system that solves this least-squares problem. You may use MATLAB to compute the matrix multiplications, but write out the matrices explicitly.

If $k > n$, minimizing $\|Ax-b\|_2^2 + \|x\|_2^2$ leads to the least-squares problem

$$\begin{pmatrix} A_{n \times k} \\ I_{k \times k} \end{pmatrix} \vec{x} = \begin{pmatrix} \vec{b}_{n \times 1} \\ \vec{0}_{k \times 1} \end{pmatrix}$$

which in our case:

$$\begin{pmatrix} A_{4\times 5} \\ I_{5\times 5} \end{pmatrix} \vec{x} = \begin{pmatrix} \vec{b}_{4\times 1} \\ \vec{0}_{5\times 1} \end{pmatrix}$$

which equals

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 81 & 27 & 9 & 3 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We will compute the least squares solution using the normal equation

$$A^TAx = A^Tb$$

Using MATLAB for matrix multiplication the corresponding linear system is

$$\begin{pmatrix} 0 & 1 & 16 & 81 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 8 & 27 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 9 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 81 & 27 & 9 & 3 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \vec{x} = \begin{pmatrix} 0 & 1 & 16 & 81 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 8 & 27 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 9 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 6819 & 2316 & 794 & 276 & 98 \\ 2316 & 795 & 276 & 98 & 36 \\ 794 & 276 & 99 & 36 & 14 \\ 276 & 98 & 36 & 15 & 6 \\ 98 & 36 & 14 & 6 & 5 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 148 \\ 48 \\ 16 \\ 6 \\ 4 \end{pmatrix}$$

C.) Solve the linear system from part (b). You may use MATLAB. Write your answers out to 6 decimal places.

Using the backslash command in MATLAB, the least-squares solution from part (b) is:

$$\vec{x} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 0.230541 \\ -0.648091 \\ -0.153946 \\ 0.404280 \\ 0.893556 \end{pmatrix}$$

d.) Solve the least-squares problem using the pseudoinverse. The MATLAB code for the pseudoinverse of a matrix A is pinv(A). Write your answers out to 6 decimal places.

In MATLAB, using A^+= pinv(A) and solving $x^* = A^+ b$, where A^+ is the pseudo-inverse of A, we minimize $\|Ax-b\|_2^2$ and obtain the least squares solution

$$\vec{x}^* = \begin{pmatrix} 0.644330 \\ -2.199313 \\ 0.087629 \\ 2.467354 \\ 1.000000 \end{pmatrix}$$

e.) In MATLAB, make a scatter plot of the original data (code: scatter(x, y, 'ro')). Overlay this (code: hold on;) with your two solutions from (c) and (d). Restrict to the range $-1 \leq x \leq 4$ and $-2 \leq y \leq 3$ (code: axis([xmin xmax ymin ymax])).

See plot attached to the end of the problem.

f.) It should be obvious from your plot that the residual $\|Ax-b\|_2$ is smaller for your answer in (d) than in (c). What are the two-norms $\|x\|_2$. Answer to 3 decimals. Which is smaller?

Computing these norms in MATLAB we see:

$\|\vec{x}\|_2 = 1.208$ (norm for answer from (c))

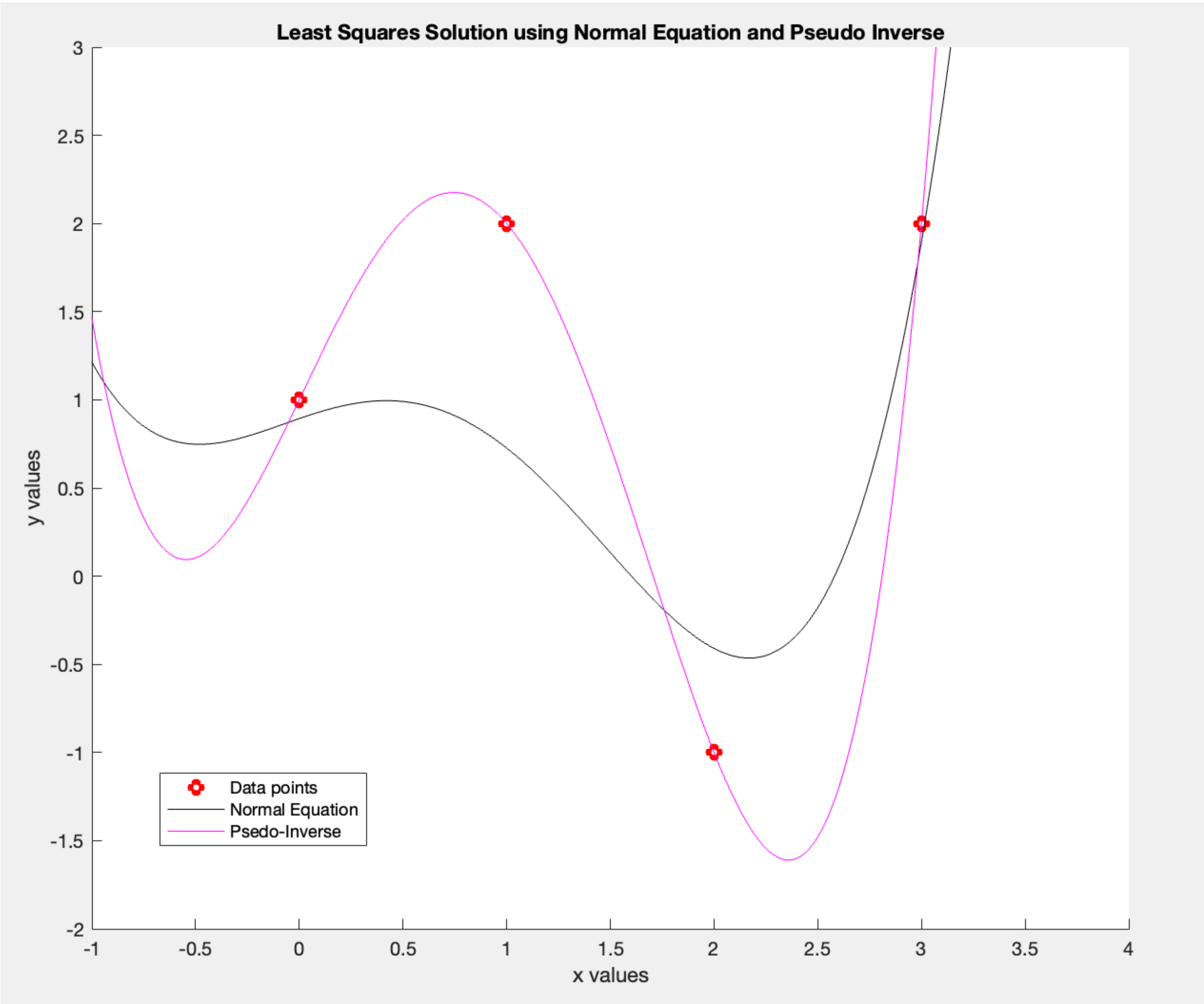
whereas

$\|\vec{x}^*\|_2 = 3.514$ (norm for answer from (d))

Thus, the normal least squares solution from (c) has the smaller 2-norm, but has a larger residual.

Final Exam Problem 2 Part e

Final Exam Problem 2 Part e:



(3.) Given $A \in \mathbb{R}^{m \times n}$, the following code computes the product of all of the elements of A above the diagonal.

```
% A is a given mxn matrix.
[m, n] = size(A);
Prod = 1;
for i = 1:m
   for j = i:n
      Prod = prod * A(i, j);
      end
end
Prod
```

(a.) What is the total flop count of the code

i) inner for loop:

flops: 1

runs: $n-i+1$

total: $n-i+1$

ii) outer for loop:

flops: $n-i+1$

runs: $m-1+1 = m$

total: $\sum_{i=1}^{m} n-i+1 = \sum_{i=1}^{m} n+1 - \sum_{i=1}^{m} i = m(n+1) - \frac{m(m+1)}{2} = mn+m - \frac{m^2+m}{2}$

$= mn+m - \frac{m^2}{2} - \frac{m}{2} = -\frac{m^2}{2} + (n+1-\frac{1}{2})m = -\frac{m^2}{2} + (n+1/2)m$

hence the total flop count of the code is

$$\boxed{-\frac{m^2}{2} + (n+\frac{1}{2})m}$$

(b.) If $m=n$, what is the asymptotic flop count?

If $m=n$, $-\frac{m^2}{2} + (n+\frac{1}{2})m \Rightarrow -\frac{n^2}{2} + (n+\frac{1}{2})n = -\frac{n^2}{2} + n^2 + \frac{n}{2} = \frac{n^2}{2} + \frac{n}{2}$.

Since $\lim_{n\to\infty} \frac{\frac{1}{2}n^2 + \frac{1}{2}n}{\frac{1}{2}n^2} = 1$, it follows that the asymptotic flop count is $\boxed{\frac{1}{2}n^2}$.

(ii.) consider the following matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

Note that the rows of A are orthogonal.

a.) Compute a reduced SVD of A^T. Recall a reduced SVD $A^T = \tilde{U}\tilde{\Sigma}V^T$ has $\tilde{U} \in \mathbb{R}^{4\times3}$ has orthgonal columns, $\tilde{\Sigma} \in \mathbb{R}^{3\times3}$ is diagonal with non-negative decreasing entries and $V^T \in \mathbb{R}^{3\times3}$ is an orthogonal matrix.

Notice that

$$A^T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

We will start by finding the singular values, which are the square roots of the eigenvalues of AA^T.

$$AA^T = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

since AA^T is diagonal, we can see that $\lambda_{1,2,3} = 3 \Rightarrow \sigma_{1,2,3} = \sqrt{3}$.

since the rows of A are orthogonal, it follows that the columns of A^T are orthogonal. We can normalize these columns to find

$$A^T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} = \tilde{U}\tilde{\Sigma}V^T$$

$$= \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{3} & 0 & -1/\sqrt{3} \end{pmatrix}\begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the reduced SVD of A^T.

b.) Guess $x_4 = e_4$. Use Gram-schmidt to find a fourth orthonormal vector u_4.

If we guess $x_4 = e_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$, running Gram-schmidt with the columns of $\tilde{U}$ being our u_1, u_2, u_3, we see

i.) Find $x_{4\perp}$:

$$x_{4\perp} = x_4 - (x_4 \cdot u_3)u_3 - (x_4 \cdot u_2)u_2 - (x_4 \cdot u_1)u_1$$

$$= \begin{pmatrix}1\\0\\0\\0\end{pmatrix} - \left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \cdot \begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\0\\1/\sqrt{3}\end{pmatrix}\right)\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\0\\1/\sqrt{3}\end{pmatrix} - \left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \cdot \begin{pmatrix}1/\sqrt{3}\\-1/\sqrt{3}\\1/\sqrt{3}\\0\end{pmatrix}\right)\begin{pmatrix}1/\sqrt{3}\\-1/\sqrt{3}\\1/\sqrt{3}\\0\end{pmatrix} - \left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \cdot \begin{pmatrix}0\\1/\sqrt{3}\\1/\sqrt{3}\\-1/\sqrt{3}\end{pmatrix}\right)\begin{pmatrix}0\\1/\sqrt{3}\\1/\sqrt{3}\\-1/\sqrt{3}\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\\0\\0\end{pmatrix} - \frac{1}{\sqrt{3}}\begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\0\\1/\sqrt{3}\end{pmatrix} - \frac{1}{\sqrt{3}}\begin{pmatrix}1/\sqrt{3}\\-1/\sqrt{3}\\1/\sqrt{3}\\0\end{pmatrix} - 0\begin{pmatrix}0\\1/\sqrt{3}\\1/\sqrt{3}\\-1/\sqrt{3}\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\\0\\0\end{pmatrix} - \begin{pmatrix}1/3\\1/3\\0\\1/3\end{pmatrix} - \begin{pmatrix}1/3\\-1/3\\1/3\\0\end{pmatrix}$$

$$= \begin{pmatrix}1 - 1/3 - 1/3\\0 - 1/3 + 1/3\\0 - 0 - 1/3\\0 - 1/3 - 0\end{pmatrix}$$

$$= \begin{pmatrix}1/3\\0\\-1/3\\-1/3\end{pmatrix}$$

ii.) Find u_4

$$u_4 = \frac{x_{4\perp}}{\|x_{4\perp}\|_2} = \frac{1}{\sqrt{\frac{1}{3}^2 + 0^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2}}\begin{pmatrix}1/3\\0\\-1/3\\-1/3\end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix}1/3\\0\\-1/3\\-1/3\end{pmatrix}$$

$$= \begin{pmatrix}1/\sqrt{3}\\0\\-1/\sqrt{3}\\-1/\sqrt{3}\end{pmatrix}$$

c.) compute the full SVD of A.

To get the full SVD of A^T, we need to find

$$A^T = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 0 & \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} & u_4 \\ 0 & 1/\sqrt{3} & 1/\sqrt{3} & \\ 1/\sqrt{3} & 0 & -1/\sqrt{3} & \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Where u_4 is orthonormal to the first three columns of U. We found u_4 in part (b), thus the full SVD of A^T is

$$A^T = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 0 & 1/\sqrt{3} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{3} & 0 & -1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To find the full SVD of A, we simply transpose the full SVD of A^T

$$A = U\Sigma V^{T*} = (A^T)^T = (U\Sigma V^T)^T = V\Sigma^T U^T$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 0 & 1/\sqrt{3} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{3} & 0 & -1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}$$

Which is the full SVD of A.

5. Consider the following matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 4 & 2 & 7 \\ 1 & 3 & 0 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix}$$

(a) compute the PLU factorization of A. Use partial pivoting to determine whether to exchange rows.

Since 2 has the highest magnitude in column 1, interchange row 1 and 4:

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 4 & 2 & 7 \\ 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Add $-1/2$ times row 1 to row 2:

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 7/2 & 2 & 13/2 \\ 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Add $-1/2$ times row 1 to row 3:

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 7/2 & 2 & 13/2 \\ 0 & 5/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Add $-\frac{5}{7}$ times row 2 to row 3:

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 7/2 & 2 & 13/2 \\ 0 & 0 & -10/7 & -58/14 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/2 & 5/7 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Add $7/10$ times row 3 to row 4:

$$U = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 7/2 & 2 & 13/2 \\ 0 & 0 & -10/7 & -58/14 \\ 0 & 0 & 0 & -58/20 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/2 & 5/7 & 1 & 0 \\ 0 & 0 & -7/10 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Thus the PLU factorization of A is:

$$PA = LU \Rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 4 & 2 & 7 \\ 1 & 3 & 0 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ 1/2 & 5/7 & 1 & 0 \\ 0 & 0 & -7/10 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 7/2 & 2 & 13/2 \\ 0 & 0 & -10/7 & -58/14 \\ 0 & 0 & 0 & -58/20 \end{pmatrix}$$

(b) Based on your PLU factorization, is A invertible? This should be a one sentence answer.

Since the PLU decomposition has all non-zero elements on the diagonal of U, it is nonsingular. Lastly, since A is nonsingular, that implies it is invertible as well.

(c) Compute the determinant of A.

Using the PLU factorization of A, $\det(A) = (-1)^k u_{11} \cdots u_{nn}$, hence in our case:

$$\det(A) = (-1)^1 \cdot 2 \cdot \frac{7}{2} \cdot \frac{-10}{7} \cdot \frac{-58}{20}$$
$$= \boxed{-29}$$

(d) Compute the volume of the parallelepiped to which A maps the 4D unit hyper-cube.

The volume of this parallelepiped is equal to the magnitude of the determinant of A, which is

$$\det(A) = -29 \Rightarrow |\det(A)| = |-29| \Rightarrow \boxed{\text{volume} = 29}$$