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Exercise 1. (CS 2.5):

Obtain the inverse of the matrix $A = \begin{bmatrix} 2 & 0 & 2 \\ 1 & 2 & 2 \\ 2 & 1 & 2 \end{bmatrix}$ by Gauss-Jordan elimination. Use this inverse to solve $Ax = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

First we will use Gauss-Jordan elimination to obtain the inverse of the matrix A:

Step 1: add $-\frac{1}{2}$ times row 1 to row 2:

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & -\frac{1}{2} & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Step 2: add -1 times row 1 to row 3:

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

Step 3: add $-\frac{1}{2}$ times row 2 to row 3:

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & -1/2 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} & -3/4 & -1/2 & 1 \end{array}\right]$$

Step 4: multiply row 3 by -2:

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & -2 \end{array}\right]$$

Step 5: mutiply row 2 by $\frac{1}{2}$:

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & -1/4 & 1/2 & 0 \\ 0 & 0 & 1 & 3/2 & 1 & -2 \end{array}\right]$$

Step 6: multiply row 1 by $\frac{1}{2}$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/2 & 0 & 0 \\ 0 & 1 & 1/2 & -1/4 & 1/2 & 0 \\ 0 & 0 & 1 & 3/2 & 1 & -2 \end{array}\right]$$

Step 7: add $-\frac{1}{2}$ times row 3 to row 2:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 3/2 & 1 & -2 \end{array}\right]$$

Step 8: add -1 times row 3 to row 1.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & 2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 3/2 & 1 & -2 \end{array}\right]$$

hence $A^{-1} = \begin{bmatrix} -1 & -1 & 2 \\ -1 & 0 & 1 \\ 3/2 & 1 & -2 \end{bmatrix}$

Now, we will use A^{-1} to solve $Ax = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. We will do this using the following formula: $Ax = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \Rightarrow x = A^{-1}\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

$$x = \begin{bmatrix} -1 & -1 & 2 \\ -1 & 0 & 1 \\ 3/2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 - 2 + 2 \\ -3 + 0 + 1 \\ 9/2 + 2 - 2 \end{bmatrix}$$

$$= \begin{bmatrix} -3 \\ -2 \\ 9/2 \end{bmatrix}$$

hence $x = \begin{bmatrix} -3 \\ -2 \\ 9/2 \end{bmatrix}$ solves $Ax = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$.

Exercise 2. (CS 3.1):

(a.) What linear combination of $v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ results in the vector $b = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$?

To find the linear combination such that $x_1 v_1 + x_2 v_2 = b \Rightarrow$ $x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ we will solve the following linear system for x_1 and x_2: (in matrix vector form):

$$\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

after adding $-\frac{1}{3}$ times row 1 to row 2 we obtain

$$\begin{bmatrix} 3 & 1 \\ 0 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{4}{3} \end{bmatrix}$$

after using back substitution we see that

$$x_2 = 4/3 \,/\, 2/3 = 2$$
$$x_1 = \frac{-1-2}{3} = -1$$

thus

$$-1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(b.) What linear combination of $v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $v_3 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ results in the vector $b = \begin{bmatrix} 0 \\ 3 \\ 8 \end{bmatrix}$?

To find the linear combination such that $x_1 v_1 + x_2 v_2 + x_3 v_3 = b \Rightarrow$ $x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 8 \end{bmatrix}$ we will solve the following linear system for x_1, x_2 and x_3 (in matrix vector form):

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 8 \end{bmatrix}$$

after adding -2 times row 1 to row 3 we obtain

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 8 \end{bmatrix}$$

after adding -1 times row 2 to row 3 we obtain

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -5 & 5 \end{array}\right]$$

after using back substitution we see that

$$x_3 = \frac{5}{-5} = -1$$

$$x_2 = 3$$

$$x_1 = 2$$

thus

$$2 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 8 \end{bmatrix}.$$

- Exercise 3: (CS 3.2):

The set $\mathbb{R}^n$ together with the usual vector addition and scalar multiplication operations is one of the most important examples of a vector space, but there are other examples. Complete Exercise 2.1.2 in Olver and Shakiban. Be sure to show that this set, together with its addition and scalar multiplication operations, satisfy all eight properties of a vector space.

Q2.1.2: Show that the positive quadrant $Q = \{(x,y) \mid x, y > 0\} \subset \mathbb{R}^2$ forms a vector space if we define addition by $(x_1, y_1) + (x_2, y_2) = (x_1 x_2, y_1 y_2)$ and scalar multiplication by $c(x,y) = (x^c, y^c)$.

Below we will use the addition and scalar multiplication operations defined above to show that Q satisfies all eight properties of a vector space.

- Properties of vector addition:

① Associativity of vector addition: $\forall u, v, w \in Q$, $(u+v)+w = u+(v+w)$

Let $u=(x_1, y_1)$, $v=(x_2, y_2)$, $w=(x_3, y_3)$ be elements of Q, then

$$(u+v)+w = ((x_1, y_1) + (x_2, y_2)) + (x_3, y_3) = (x_1 x_2, y_1 y_2) + (x_3, y_3) = ((x_1 x_2) x_3, (y_1 y_2) y_3)$$
$$= (x_1 x_2 x_3, y_1 y_2 y_3) = (x_1 (x_2 x_3), y_1 (y_2 y_3)) = (x_1, y_1) + (x_2 x_3, y_2 y_3) =$$
$$(x_1, y_1) + ((x_2, y_2) + (x_3, y_3)) = u + (v+w)).$$

② Commutativity of vector addition: $\forall u, v \in Q$, $u+v = v+u$

Let $u=(x_1, y_1)$, $v=(x_2, y_2)$ be elements of Q, then

$$u+v = (x_1, y_1) + (x_2, y_2) = (x_1 x_2, y_1 y_2) = (x_2 x_1, y_2 y_1) = (x_2, y_2) + (x_1, y_1)$$
$$= v+u.$$

③ Zero element for vector addition: $\exists \vec{0} \in Q$ s.t. $\forall v \in Q$, $v + \vec{0} = v$

Let $v = (x_1, y_1)$ and $\vec{0} = (1, 1)$ be elements of Q, then

$$v + \vec{0} = (x_1, y_1) + (1, 1) = (1x_1, 1y_1) = (x_1, y_1) = v.$$

④ Additive inverses: $\forall v \in Q$, $\exists (-v) \in Q$ s.t. $v + (-v) = \vec{0}$.

Let $v = (x_1, y_2)$, $-v = (\frac{1}{x_1}, \frac{1}{y_2})$, $\vec{0} = (1, 1)$ be elements of Q, then

$$v + (-v) = (x_1, y_1) + (\frac{1}{x_1}, \frac{1}{y_1}) = (x_1 \frac{1}{x_1}, y_1 \frac{1}{y_1}) = (1, 1) = \vec{0}.$$

- Properties of scalar multiplication:

⑤ Identity element for scalars: $\exists 1 \in \mathbb{R}$ s.t. $\forall v \in Q$, $1 \cdot v = v$.

Let $v = (x_1, y_1) \in Q$ and let $1 = 1 \in \mathbb{R}$, then

$$1 \cdot v = 1 \cdot (x_1, y_1) = (x_1^1, y_1^1) = (x_1, y_1) = v.$$

⑥ Associativity of scalar multiplication: $\forall a, b \in \mathbb{R}$ and $v \in Q$, $a \cdot (b \cdot v) = (ab) \cdot v$

Let $v = (x_1, y_1) \in Q$, and let $a, b \in \mathbb{R}$, then

$$a \cdot (b \cdot v) = a \cdot (b \cdot (x_1, y_1)) = a \cdot (x_1^b, y_1^b) = (x_1^{ab}, y_1^{ab}) = ab \cdot (x_1, y_1)$$
$$= (ab) \cdot v.$$

- Distribution properties:

⑦ Distribution of scalars: $\forall a \in \mathbb{R}$ and $u, v \in Q$, $a \cdot (u + v) = a \cdot u + a \cdot v$.

Let $a \in \mathbb{R}$ and $u = (x_1, y_1)$, $v = (x_2, y_2)$ be elements of Q, then

$$a \cdot (u + v) = a \cdot ((x_1, y_1) + (x_2, y_2)) = a \cdot (x_1 x_2, y_1 y_2) = ((x_1 x_2)^a, (y_1 y_2)^a)$$
$$= (x_1^a x_2^a, y_1^a y_2^a) = (x_1^a, y_1^a) + (x_2^a, y_2^a) = a \cdot (x_1, y_1) + a \cdot (x_2, y_2)$$
$$= a \cdot u + a \cdot v.$$

(8.) Distribution of vectors: $\forall a, b \in \mathbb{R}$ and $v \in Q$, $(a+b)v = av + bv$.

Let $v = (x_1, y_1) \in Q$, and let $a, b \in \mathbb{R}$, then

$$(a+b)v = (a+b)\cdot(x_1, y_1) = (x_1^{(a+b)}, y_1^{(a+b)}) = (x_1^a x_1^b, y_1^a y_1^b)$$

$$= (x_1^a, y_1^a) + (x_1^b, y_1^b) = a\cdot(x_1, y_1) + b(x_1, y_1) = a\cdot v + b\cdot v.$$

Thus the positive quadrant $Q = \{(x,y) \mid x, y > 0\}$ forms a vector space.

Exercise 4. (CS 3.2):

From Olver and Shakiban, complete Exercise 2.2.2 (a) - (e).

Which of the following are subspaces of $\mathbb{R}^3$? Justify your answers.

a. 2.2.2(a): The set of all vectors $(x, y, z)^T$ satisfying $x+y+z+1=0$.

Let V denote the set of all vectors $(x, y, z)^T$ satisfying $x+y+z+1=0$.

Since $(x, y, z)^T = (0, 0, 0)^T$ is the zero element of $\mathbb{R}^3$, but isn't an element of V since $0+0+0+1 = 1 \neq 0$, this set V is not a subspace of $\mathbb{R}^3$.

b. 2.2.2(b): The set of vectors of the form $(t, -t, 0)^T$ for $t \in \mathbb{R}$.

Let V denote the set of vectors of the form $(t, -t, 0)^T$ for $t \in \mathbb{R}$.

i) $\vec{0}$ vector:

Let $t=0$, then $(t, -t, 0)^T = (0, 0, 0)^T$ which is the zero element of $\mathbb{R}^3$. Since $(0, 0, 0) \in V$, V contains the zero element of $\mathbb{R}^3$.

ii) Vector addition:

Let $v_1 = (t_1, -t_1, 0)^T$, $v_2 = (t_2, -t_2, 0)^T$ be elements of V for $t_1, t_2 \in \mathbb{R}$. Then $v_1 + v_2 = (t_1 + t_2, -t_1 - t_2, 0)^T = (t_1 + t_2, -(t_1 + t_2), 0)^T$ where $t_1 + t_2 \in \mathbb{R}$. Let $\tilde{t} = t_1 + t_2$, then $v_1 + v_2 = (\tilde{t}, -\tilde{t}, 0)^T$ which is an element of V. Hence V is closed under vector addition.

iii) Scalar multiplication:

Let $v_1 = (t_1, -t_1, 0)^T \in V$ with $t_1 \in \mathbb{R}$, and let $c \in \mathbb{R}$, then $cv_1 = (ct_1, -ct_1, 0)^T$. Let $\tilde{t} = ct_1 \in \mathbb{R}$, then $cv_1 = (\tilde{t}, -\tilde{t}, 0)^T$ which is an element of V. Hence V is closed under scalar multiplication.

Therefore, since the 3 requirements hold true, V is a subspace of $\mathbb{R}^3$.

(c.) 2.2.2(c): The set of vectors of the form $(r-s, r+2s, -s)^T$ for $r, s \in \mathbb{R}$.

Let V denote the set of vectors of the form $(r-s, r+2s, -s)^T$ for $r, s \in \mathbb{R}$.

i) $\vec{0}$ vector:

Let $r, s = 0$, then $(r-s, r+2s, -s)^T = (0-0, 0+2(0), -0)^T = (0,0,0)^T$ which is the zero element of $\mathbb{R}^3$, is an element of V. Since $(0,0,0)^T \in V$, V contains the zero element of $\mathbb{R}^3$.

ii) vector addition:

Let $V_1 = (r_1 - s_1, r_1 + 2s_1, -s_1)^T$, $V_2 = (r_2 - s_2, r_2 + 2s_2, -s_2)^T$ be elements of V with $r_1, s_1, r_2, s_2 \in \mathbb{R}$, then $V_1 + V_2 = (r_1 - s_1 + r_2 - s_2, r_1 + 2s_1 + r_2 + 2s_2, -s_1 - s_2)^T = ((r_1 + r_2) - (s_1 + s_2), (r_1 + r_2) + 2(s_1 + s_2), -(s_1 + s_2))^T$. Let $\tilde{r} = r_1 + r_2$, $\tilde{s} = s_1 + s_2 \in \mathbb{R}$, then $V_1 + V_2 = (\tilde{r} - \tilde{s}, \tilde{r} + 2\tilde{s}, -\tilde{s})^T$ which is an element of V. Hence V is closed under vector addition.

iii) scalar multiplication

Let $V_1 = (r_1 - s_1, r_1 + 2s_1, -s_1)^T \in V$ with $r_1, s_1 \in \mathbb{R}$. Then $cV_1 = (c(r_1 - s_1), c(r_1 + 2s_1), c(-s_1))^T = (cr_1 - cs_1, cr_1 + 2cs_1, -cs_1)^T$. Let $\tilde{r} = cr_1$, $\tilde{s} = cs_1 \in \mathbb{R}$, then $cV_1 = (\tilde{r} - \tilde{s}, \tilde{r} + 2\tilde{s}, -\tilde{s}) \in V$. Hence V is closed under scalar multiplication.

Therefore, since the 3 requirements hold true, V is a subspace of $\mathbb{R}^3$.

(d.) 2.2.2(d): The set of vectors whose first component equals zero.

Let V denote the set of vectors whose first component equals zero i.e. of the form $(0, y, z)^T$, where $y, z \in \mathbb{R}$.

i) $\vec{0}$ vector:

Let $y, z = 0$, then $(0, y, z)^T = (0, 0, 0)^T$, which is the zero element of $\mathbb{R}^3$, is an element of V. Since $(0,0,0)^T \in V$, V contains the zero element of $\mathbb{R}^3$.

ii) vector addition:

Let $v_1 = (0, y_1, z_1)^T$, $v_2 = (0, y_2, z_2)^T$ be elements of V with $y_1, z_1, y_2, z_2 \in \mathbb{R}$, then $v_1 + v_2 = (0, y_1 + y_2, z_1 + z_2)^T$. Let $\tilde{y} = y_1 + y_2$, $\tilde{z} = z_1 + z_2 \in \mathbb{R}$, then $v_1 + v_2 = (0, \tilde{y}, \tilde{z})^T \in V$. Hence V is closed under vector addition.

iii) scalar multiplication

Let $v_1 = (0, y_1, z_1)^T \in V$ with $y_1 z_1 \in \mathbb{R}$, and let $c \in \mathbb{R}$, then $cv_1 = (0, cy_1, cz_1)^T$. Let $\tilde{y} = cy_1$, $\tilde{z} = cz_1 \in \mathbb{R}$, then $cv_1 = (0, \tilde{y}, \tilde{z})^T \in V$.

Hence V is closed under scalar multiplication.

Therefore, since the 3 requirements hold true, V is a subspace of $\mathbb{R}^3$.

(e.) 2.2.2(e): The set of vectors whose last component equals 1.

Let V denote the set of vectors whose last component equals 1. Since these vectors take the form $(x, y, 1)^T$ for $x, y \in \mathbb{R}$, $(0, 0, 0)^T \notin V$. Since the zero element of $\mathbb{R}^3$ is not in the set, V is not a subspace of $\mathbb{R}^3$.

Exercise 5. (CS 3.2):

From Olver and Shakiban, complete Exercise 2.3.3(a)-(b)

(a) 2.3.3(a): Determine whether $\begin{bmatrix}1\\-2\\-3\end{bmatrix}$ is in the span of $\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\1\end{bmatrix}$.

$b = \begin{bmatrix}1\\-2\\-3\end{bmatrix}$ is in the span of $v_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$ and $v_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}$ if $c_1 v_1 + c_2 v_2 = b$

$\Rightarrow c_1 \begin{bmatrix}1\\1\\0\end{bmatrix} + c_2 \begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\-2\\-3\end{bmatrix}$ has at least one solution. This is the same as solving the following linear system in matrix vector form:

$$\begin{bmatrix}1 & 0\\1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}1\\-2\\-3\end{bmatrix}$$

Adding -1 times row 1 to row 2 we obtain

$$\begin{bmatrix}1 & 0\\0 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}1\\-3\\-3\end{bmatrix}$$

using back substitution we see that

$c_2 = -3$

$c_1 = 1$

thus $\begin{bmatrix}1\\-2\\-3\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} - 3\begin{bmatrix}0\\1\\1\end{bmatrix}$, and hence $\begin{bmatrix}1\\-2\\-3\end{bmatrix} \in \text{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$

(b) Is $\begin{bmatrix}1\\-2\\-1\end{bmatrix}$ in the span of $\begin{bmatrix}1\\2\\2\end{bmatrix}$, $\begin{bmatrix}1\\-2\\0\end{bmatrix}$, $\begin{bmatrix}0\\3\\4\end{bmatrix}$?

$b = \begin{bmatrix}1\\-2\\-1\end{bmatrix}$ is in the span of $\begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}1\\-2\\0\end{bmatrix}, \begin{bmatrix}0\\3\\4\end{bmatrix}$ if $c_1 v_1 + c_2 v_2 + c_3 v_3 = b$

$\Rightarrow c_1 \begin{bmatrix}1\\2\\2\end{bmatrix} + c_2 \begin{bmatrix}1\\-2\\0\end{bmatrix} + c_3 \begin{bmatrix}0\\3\\4\end{bmatrix} = \begin{bmatrix}1\\-2\\-1\end{bmatrix}$ has at least one solution. This is the same as solving the following linear system in matrix vector form

$$\begin{bmatrix}1 & 1 & 0\\2 & -2 & 3\\2 & 0 & 4\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = \begin{bmatrix}1\\-2\\-1\end{bmatrix}$$

Adding -2 times row 1 to row 2 we obtain

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -4 & 3 \\ 2 & 0 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix}$$

adding -2 times row 1 to row 3 we obtain

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -4 & 3 \\ 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix}$$

adding $-\frac{1}{2}$ times row 2 to row 3 we obtain

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & -4 & 3 \\ 0 & 0 & \frac{5}{2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix}$$

using back substitution we see that

$$c_3 = \frac{-1}{5/2} = -\frac{2}{5}$$

$$c_2 = \frac{-4 + \frac{6}{5}}{-4} = \frac{7}{10}$$

$$c_1 = \frac{1 - \frac{7}{10}}{1} = \frac{3}{10}$$

thus $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} = \frac{3}{10}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + \frac{7}{10}\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} - \frac{2}{5}\begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}$, and hence $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \in \text{span}\left\{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}\right\}$.

Exercise 6. (cs 3.3)

Determine which of the following are bases of $\mathbb{R}^3$.

In order to determine if the following vectors are bases of $\mathbb{R}^3$, we must show that for any $\vec{b} \in \mathbb{R}^3$, $c_1 v_1 + c_2 v_2 + c_3 v_3 = \vec{b}$ has a unique solution (i.e. is linearly independent and spans $\mathbb{R}^3$. In other words, we are looking for a nonsingular 3×3 matrix after reducing the system $A\vec{c} = \vec{b}$, the matrix vector representation of the above linear combination, into row echelon form. Namely, if the number of pivots is 3, the vectors are linearly independent, and if the rank is 3 they span $\mathbb{R}^3$.

(a.) $\left\{ \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} \right\}$

The associated matrix A is written as

$$\begin{bmatrix} -1 & 0 & 5 \\ 5 & 1 & -1 \\ 0 & 3 & 0 \end{bmatrix}$$

adding 5 times row 1 to row 2 we obtain

$$\begin{bmatrix} -1 & 0 & 5 \\ 0 & 1 & 24 \\ 0 & 3 & 0 \end{bmatrix}$$

adding -3 times row 2 to row 3 we obtain

$$\begin{bmatrix} -1 & 0 & 5 \\ 0 & 1 & 24 \\ 0 & 0 & -72 \end{bmatrix}$$

Since there are 3 pivots and the rank is 3, there is a unique solution for all $\vec{b}$, and if $\vec{b} = \vec{0}$ then $c_1 = c_2 = c_3 = 0$.

Thus these vectors form a basis of $\mathbb{R}^3$.

(b.) $\left\{ \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix} \right\}$

The associated matrix A is written as

$$\begin{bmatrix} -1 & 0 & 3 \\ 2 & 4 & 8 \\ 1 & 2 & 4 \end{bmatrix}$$

adding 2 times row 1 to row 2 we obtain

$$\begin{bmatrix} -1 & 0 & 3 \\ 0 & 4 & 14 \\ 1 & 2 & 4 \end{bmatrix}$$

adding row 1 to row 3 we obtain

$$\begin{bmatrix} -1 & 0 & 3 \\ 0 & 4 & 14 \\ 0 & 2 & 7 \end{bmatrix}$$

adding $-\frac{1}{2}$ row 2 to row 3 we obtain

$$\begin{bmatrix} -1 & 0 & 3 \\ 0 & 4 & 14 \\ 0 & 0 & 0 \end{bmatrix}$$

Since there are free variable, the rank is $2 \neq 3$, the numer of pivots is $2 \neq 3$, etc. We can see that this set of vectors does not form a basis for $\mathbb{R}^3$.

(c.) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$

Based on proposition 2.2.5, a collection of k vectors in $\mathbb{R}^n$ spans $\mathbb{R}^n$ iff their $n \times k$ matrix has rank n/no all-zero rows. In particular, this requires $k \geq n$. However, this set of vectors, although linearly independent, does not span $\mathbb{R}^3$ since $k=2$ and $n=3 \Rightarrow 2 \not\geq 3$. Hence this set of vectors does not span $\mathbb{R}^3$ and is not a basis for $\mathbb{R}^3$.

• Exercise 7. (cs 3.3):

Find the dimension of and a basis for each of the following sets.

(a.) The plane given by the equation $x+3y-z=0$ in $\mathbb{R}^3$.

Since $x+3y-z=0 \Rightarrow z=x+3y$, vectors in $\mathbb{R}^3$ in this plane take the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ x+3y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}.$$

Here x and y are free variables, so

$$\{(x,y,z) \mid z=x+3y\} = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}\right\}$$

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$ span the space, to check linear independence we will form the matrix $A = [v_1 \mid v_2]$ and row reduce

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 3 \end{bmatrix} \overset{-R_1+R_3}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 3 \end{bmatrix} \overset{-3R_2+R_3}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Since the number of pivots is 2, the number of linearly independent vectors is 2, and the vectors are linearly independent. Thus $\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}\right\}$ forms a basis of the given plane and is two dimensional.

(b.) The hyperplane $2x+4y-z+w=0$ in $\mathbb{R}^4$.

Since $2x+4y-z+w=0 \Rightarrow w=-2x-4y+z$, vectors in $\mathbb{R}^4$ in this hyperplane take the form:

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ -2x-4y+z \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \\ -4 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Here x, y, and z are free variables, so

$$\{(x,y,z,w) \mid w = -2x-4y+z\} = \text{span}\left\{\begin{bmatrix}1\\0\\0\\-2\end{bmatrix}, \begin{bmatrix}0\\1\\0\\-4\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right\}$$

The vectors $v_1 = \begin{bmatrix}1\\0\\0\\-2\end{bmatrix}$, $v_2 = \begin{bmatrix}0\\1\\0\\-4\end{bmatrix}$, $v_3 = \begin{bmatrix}0\\0\\1\\1\end{bmatrix}$ span the space,

to check linear independence we will form the matrix

$A = [v_1 \mid v_2 \mid v_3]$ and row reduce:

$$A = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\\-2 & -4 & 1\end{bmatrix} \overset{2R_1+R_4}{\sim} \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\\0 & -4 & 1\end{bmatrix} \overset{4R_2+R_4}{\sim} \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 1\end{bmatrix} \overset{-R_3+R_4}{\sim} \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$$

Since the number of pivots is 3, the number of linearly independent vectors is 3, and the vectors are linearly independent. Thus $\left\{\begin{bmatrix}1\\0\\0\\-2\end{bmatrix}, \begin{bmatrix}0\\1\\0\\-4\end{bmatrix}, \begin{bmatrix}0\\0\\1\\1\end{bmatrix}\right\}$ forms a basis of the given hyperplane and is three-dimensional.

Exercise 8. (cs 3.3):

Every nth degree polynomial over the reals can be associated with a vector in $\mathbb{R}^{n+1}$ as follows:

$$P(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n \iff P = \begin{bmatrix} a_0 \\ \vdots \\ a_n \end{bmatrix}.$$

consequently, finding a set of n+1 linearly independent polynomials that span the set of all n^{th} degree polynomials is equivalent to finding a set of basis vectors for $\mathbb{R}^{n+1}$.

9. Show that the Legendre polynomials

$P_0(x) = 1$

$P_1(x) = x$

$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$

form a basis for the set of all real-valued quadratic polynomials

$$P^{(2)} = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$$

That is, show that any quadratic polynomial can be expressed as a linear combination of the three Legendre polynomials above, and show that the legendre polynomials are linearly independent.

Hint: convert the Legendre polynomials to vectors in $\mathbb{R}^3$. Do these vectors form a basis for $\mathbb{R}^3$.

To show that any quadratic polynomial can be expressed as a linear combination of the three Legendre polynomials, and that these polynomials are linearly independent we will convert these Legendre polynomials to vectors in $\mathbb{R}^3$ and show that they form a basis for $\mathbb{R}^3$.

Converting these polynomials into vectors in $\mathbb{R}^3$ we see

$$P_0(x) = 1 \Rightarrow v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$P_1(x) = x \Rightarrow v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2} \Rightarrow v_3 = \begin{bmatrix} -1/2 \\ 0 \\ 3/2 \end{bmatrix}$$

To show that v_1, v_2, and v_3 form a basis for $\mathbb{R}^3$, we will construct the matrix $A = [v_1 | v_2 | v_3]$ and row reduce:

$$A = \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 3/2 \end{bmatrix}$$

Notice, however, that this matrix is already row reduced and is nonsingular, thus for any vector $\vec{b} \in \mathbb{R}^3$, $\vec{b}$ can be uniquely expressed as a linear combination of $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 0 \\ 3/2 \end{bmatrix} \right\}$, thus this set spans $\mathbb{R}^3$. Furthermore, if $\vec{b} = \vec{0}$, the unique solution is $(0,0,0)^T$, thus v_1, v_2, v_3 are linearly independent and this set forms a basis for $\mathbb{R}^3$. Hence the Legendre polynomials form a basis for $P^{(2)}$.

(b.) Similarly, show that the Hermite polynomials

$$h_0(x) = 1$$

$$h_1(x) = x$$

$$h_2(x) = x^2 - 1$$

$$h_3(x) = x^2 - 3x$$

form a basis for the set of all real-valued cubic polynomials

$$P^{(3)} = \{ax^3 + bx^2 + cx + d : a, b, c, d \in \mathbb{R}\}.$$

To show this, we will use the exact same process as part a.

Converting these polynomials into vectors in $\mathbb{R}^4$ we see

$$h_0(x)=1 \Rightarrow v_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$h_1(x)=x \Rightarrow v_2 = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}$$

$$h_2(x)=x^2-1 \Rightarrow v_3 = \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}$$

$$h_3(x)=x^3-3x \Rightarrow v_4 = \begin{bmatrix}0\\-3\\0\\1\end{bmatrix}$$

To show that v_1, v_2, v_3, and v_4 form a basis for $\mathbb{R}^4$, we will construct the matrix $A = [v_1 \mid v_2 \mid v_3 \mid v_4]$ and row reduce

$$A = \begin{bmatrix}1 & 0 & -1 & 0\\0 & 1 & 0 & -3\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

Notice, however, that this matrix is already in row echelon form and is nonsingular, thus for any vector $\vec{b} \in \mathbb{R}^4$, $\vec{b}$ can be uniquely expressed as a linear combination of $\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-3\\0\\1\end{bmatrix}\right\}$, thus this set spans $\mathbb{R}^4$. Furthermore, if $\vec{b}=\vec{0}$, the unique solution is $(0,0,0,0)^T$, thus v_1, v_2, v_3, and v_4 are linearly independent and this set forms a basis for $\mathbb{R}^4$. Hence the Hermite polynomials form a basis for $\mathcal{P}^{(3)}$.

Note: Both parts a. and b. use theory that depends on the matrix be ~~ing~~ $n \times n$; in both cases this holds.