Component Skills Exercises

- Exercise 1. (CS3,4).

In lecture we defined the real inner-product on $\mathbb{R}^n$ as follows:

$$\varphi(u,v) = \|u\|_2 \|v\|_2 \cos(\theta),$$

where θ is the angle between u and v. This inner-product satisfies the following three properties:

- $\varphi(u,u) \geq 0$ and $\varphi(u,u)=0$ iff $u=0$ for all $u \in \mathbb{R}^n$,
- $\varphi(u,v) = \varphi(v,u)$ for all $u,v \in \mathbb{R}^n$, and
- $\varphi(u+\alpha w, v) = \varphi(u,v) + \alpha\varphi(w,v)$; where $\alpha \in \mathbb{R}$ and $u,v,w \in \mathbb{R}^n$.

Assuming these three properties are true, answer the following about φ.

(a) Let $\{e_1, \dots, e_n\}$ represent the canonical basis vectors in $\mathbb{R}^n$. Show that

$$\varphi(e_i, e_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{if } i \neq j, \end{cases}$$

where the symbol δ_{ij} is the Kronecker delta.

To show that $\varphi(e_i, e_j)$ equals the Kronecker delta δ_{ij}, we must calculate $\varphi(e_i, e_j)$ for two cases: (i) $i=j$, and (ii) $i \neq j$.

(i) Case 1: $i=j$:

If $i=j$, this implies we are looking at e_i and e_i. Hence it follows that the angle between these two identical vectors is $\theta=0$. Computing $\varphi(e_i, e_i)$ we see that:

$$\begin{aligned} \varphi(e_i, e_i) &= \|e_i\|_2 \|e_i\|_2 \cos(\theta) \\ &= 1 \cdot 1 \cdot \cos(\theta) && (\text{since } \|e_i\|_2 = 1) \\ &= \cos(0) && (\text{since } \theta = 0) \\ &= 1 \end{aligned}$$

(ii) Case 2: $i \neq j$:

If $i \neq j$, this implies we are looking at e_i and e_j. Hence it follows that the angle between these two perpendicular vectors is $\theta = \frac{\pi}{2}$. Computing $\varphi(e_i, e_j)$ we see that:

$$\begin{aligned} \varphi(e_i, e_j) &= \|e_i\|_2 \|e_j\|_2 \cos(\theta) \\ &= 1 \cdot 1 \cdot \cos(\theta) && (\text{since } \|e_i\|_2 = \|e_j\|_2 = 1) \\ &= \cos(\tfrac{\pi}{2}) && (\text{since } e_i \perp e_j \Rightarrow \theta = \pi/2) \\ &= 0 \end{aligned}$$

Putting these two cases together, we have confirmed that
$\varphi(e_i, e_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{if } i\neq j. \end{cases}$

(b.) Using the properties above, show that

$$\varphi(u, v+\alpha w) = \varphi(u,v) + \alpha\varphi(u,w)$$

for any $u, v, w \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

Below we will use property 2 and 3 from above to show that $\varphi(u, v+\alpha w) = \varphi(u,v) + \alpha\varphi(u,w)$. This calculation is shown below.

$$\begin{aligned}\varphi(u, v+\alpha w) &= \varphi(v+\alpha w, u) && \text{(property 2)} \\ &= \varphi(v,u) + \alpha\varphi(w,u) && \text{(property 3)} \\ &= \varphi(u,v) + \alpha\varphi(u,w) && \text{(property 2)}\end{aligned}$$

Hence we have shown that $\varphi(u, v+\alpha w) = \varphi(u,v) + \alpha\varphi(u,w)$ for any $u, v, w \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

(c.) Let $u = u_1 e_1 + \cdots + u_n e_n \in \mathbb{R}^n$ and $v = v_1 e_1 + \cdots + v_n e_n \in \mathbb{R}^n$. Using your answers from (a)-(b) as well as the properties of φ above, show that

$$\varphi(u,v) = u \cdot v = u_1 v_1 + \cdots + u_n v_n$$

Below we will use properties of φ and the results from (a)-(b) to show that $\varphi(u,v) = u \cdot v = u_1 v_1 + \cdots + u_n v_n$. This calculation is shown below.

$$\begin{aligned}\varphi(u,v) &= \varphi(u_1 e_1 + \cdots + u_n e_n, v_1 e_1 + \cdots + v_n e_n) \\ &= u_1 v_1 \varphi(e_1, e_1) + \cdots + u_1 v_n \varphi(e_1, e_n) + \cdots + u_n v_1 \varphi(e_n, e_1) + \cdots + \\ &\quad u_n v_n \varphi(e_n, e_n) \qquad \text{(property 3 and part b.)} \\ &= u_1 v_1 \delta_{11} + \cdots + u_1 v_n \delta_{1n} + \cdots + u_n v_1 \delta_{n1} + \cdots + u_n v_n \delta_{nn} \quad \text{(part a.)} \\ &= u_1 v_1 (1) + \cdots + u_1 v_n (0) + \cdots + u_n v_1 (0) + \cdots + u_n v_n (1) \quad \text{(part a.)} \\ &= u_1 v_1 + \cdots + 0 + \cdots + 0 + \cdots + u_n v_n \\ &= u_1 v_1 + \cdots + u_n v_n \\ &= u \cdot v\end{aligned}$$

Thus we have shown that $\varphi(u,v) = u \cdot v = u_1 v_1 + \cdots + u_n v_n$.

(d.) Let u and v be unit vectors (with respect to the 2-norm) in $\mathbb{R}^n$. Using your result from (c) together with the properties of φ above, show that u+v and u-v are always orthogonal to each other.

If u and v are unit vectors, then $\|u\|_2 = \|v\|_2 = 1$. If we let $u+v = (u_1+v_1, \dots, u_n+v_n)^T$ and $u-v = (u_1-v_1, \dots, u_n-v_n)^T$, then to show that u+v and u-v are always orthogonal to each other we must show that $\varphi(u+v, u-v) = 0$ for all $u, v \in \mathbb{R}^n$. This calculation is shown below.

$$
\begin{aligned}
\varphi(u+v, u-v) &= (u_1+v_1)(u_1-v_1) + \dots + (u_n+v_n)(u_n-v_n) \quad (\text{part c.}) \\
&= u_1^2 + u_1v_1 - u_1v_1 - v_1^2 + \dots + u_n^2 + u_nv_n - u_nv_n - v_n^2 \\
&= u_1^2 + \dots + u_n^2 - v_1^2 - \dots - v_n^2 \\
&= u_1^2 + \dots + u_n^2 - (v_1^2 + \dots + v_n^2) \\
&= \|u\|_2^2 - \|v\|_2^2 \quad (\text{since } \|u\|_2 = \sqrt{u_1^2+\dots+u_n^2} \text{ and } \|v\|_2 = \sqrt{v_1^2+\dots+v_n^2}) \\
&= 1 - 1 \quad (\text{since } \|u\|_2 = 1 \Rightarrow \|u\|_2^2 = 1 \text{ and } \|v\|_2 = 1 \Rightarrow \|v\|_2^2 = 1) \\
&= 0
\end{aligned}
$$

Hence, since $\varphi(u+v, u-v) = 0$ for all $u, v \in \mathbb{R}^n$, u+v and u-v are always orthogonal to each other.

Exercise 2. (CS 3.4):

For each of the following pairs of vectors, compute the 1-norm, 2-norm, and ∞-norm of each vector in the pair. Then, compute the angle between the vectors

Note: Please express your answers exactly (angles in radians) - no decimals.

In the following parts, we will use these key facts/definitions: $\|u\|_1 = |u_1| + \cdots + |u_n|$, $\|u\|_2 = \sqrt{u_1^2 + \cdots + u_n^2}$, $\|u\|_\infty = \max_{1 \le j \le n} |u_j|$, and $u \cdot v = u_1 v_1 + \cdots + u_n v_n = \|u_1\|_2 \|v\|_2 \cos(\theta)$ for any $u, v \in \mathbb{R}^n$.

(a.) $u = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$ and $v = \begin{pmatrix} 1+\sqrt{3} \\ 1-\sqrt{3} \end{pmatrix}$

(i) norms of u:

1.) 1-norm: $\|u\|_1 = |u_1| + |u_2|$

$= |\sqrt{3}| + |1|$

$= \sqrt{3} + 1$

2.) 2-norm: $\|u\|_2 = \sqrt{u_1^2 + u_2^2}$

$= \sqrt{(\sqrt{3})^2 + 1^2}$

$= \sqrt{4}$

$= 2$

3.) ∞-norm: $\|u\|_\infty = \max_{1 \le j \le 2} |u_j|$

$= \max\{|\sqrt{3}|, |1|\}$

$= \max\{\sqrt{3}, 1\}$

$= \sqrt{3}$

(ii) norms of v:

1.) 1-norm: $\|v\|_1 = |v_1| + |v_2|$

$= |1+\sqrt{3}| + |1-\sqrt{3}|$

$= 1 + \sqrt{3} - 1 + \sqrt{3}$

$= 2\sqrt{3}$

2.) 2-norm: $\|v\|_2 = \sqrt{v_1^2 + v_2^2}$

$= \sqrt{(1+\sqrt{3})^2 + (1-\sqrt{3})^2}$

$= \sqrt{1+2\sqrt{3}+3+1-2\sqrt{3}+3}$

$= \sqrt{8}$

3.) ∞-norm: $\|v\|_\infty = \max\limits_{1\le j\le 2} |v_j|$

$= \max\{|1+\sqrt{3}|, |1-\sqrt{3}|\}$

$= \max\{1+\sqrt{3}, \sqrt{3}-1\}$

$= 1+\sqrt{3}$

(iii) angle between u and v:

1.) start with $u \cdot v = u_1 v_1 + u_2 v_2$:

$u \cdot v = \sqrt{3}(1+\sqrt{3}) + 1(1-\sqrt{3})$

$= \sqrt{3} + 3 + 1 - \sqrt{3}$

$= 4$

2.) Move to $u \cdot v = \|u\|_2 \|v\|_2 \cos(\theta)$:

$u \cdot v = 2\sqrt{8}\cos(\theta)$

$4 = 2\sqrt{8}\cos(\theta)$

$\frac{4}{2\sqrt{8}} = \cos(\theta)$

$\frac{2\sqrt{3}}{8} = \cos(\theta)$

$\frac{\sqrt{3}}{2} = \cos(\theta)$

3.) Solve for θ:

$\frac{\sqrt{2}}{2} = \cos(\theta)$

$\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \theta$

$\theta = \frac{\pi}{4}$

(b) $u = \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}$ and $v = \begin{pmatrix} 4+\sqrt{6} \\ 2-4\sqrt{6} \\ 4+\sqrt{6} \end{pmatrix}$

(i) <u>norms of u:</u>

1.) 1-norm: $\|u\|_1 = |u_1| + |u_2| + |u_3|$

$= |1| + |-4| + |1|$

$= 1 + 4 + 1$

$= 6$

2.) 2-norm: $\|u\|_2 = \sqrt{u_1^2 + u_2^2 + u_3^2}$

$= \sqrt{1^2 + (-4)^2 + 1^2}$

$= \sqrt{1 + 16 + 1}$

$= \sqrt{18}$

3.) ∞-norm: $\|u\|_\infty = \max_{1 \le j \le 3} |u_j|$

$= \max \{|1|, |-4|, |1|\}$

$= \max \{1, 4, 1\}$

$= 4$

(ii) <u>norms of v:</u>

1.) 1-norm: $\|v\|_1 = |v_1| + |v_2| + |v_3|$

$= |4+\sqrt{6}| + |2-4\sqrt{6}| + |4+\sqrt{6}|$

$= 4+\sqrt{6} + 4\sqrt{6} - 2 + 4+\sqrt{6}$

$= 8 + 6\sqrt{6} - 2$

$= 6 + 6\sqrt{6}$

2.) 2-norm: $\|v\|_2 = \sqrt{v_1^2 + v_2^2 + v_3^2}$

$= \sqrt{(4+\sqrt{6})^2 + (2-4\sqrt{6})^2 + (4+\sqrt{6})^2}$

$= \sqrt{2(4+\sqrt{6})^2 + (2-4\sqrt{6})^2}$

$= \sqrt{2(16 + 8\sqrt{6} + 6) + 4 - 16\sqrt{6} + 96}$

$= \sqrt{44 + 16\sqrt{6} - 16\sqrt{6} + 100}$

$= \sqrt{144}$

$= 12$

3.) ∞-norm: $\|v\|_\infty = \max_{1 \le j \le 3} |v_j|$

$= \max\{|4+\sqrt{6}|, |2-4\sqrt{6}|, |4+\sqrt{6}|\}$

$= \max\{4+\sqrt{6}, 4\sqrt{6}-2\}$

$= 4\sqrt{6}-2$

iii) <u>angle between u and v:</u>

1.) start with $u \cdot v = u_1v_1 + u_2v_2 + u_3v_3$

$u \cdot v = 1(4+\sqrt{6}) - 4(2-4\sqrt{6}) + 1(4+\sqrt{6})$

$= 8 + 2\sqrt{6} - 8 + 16\sqrt{6}$

$= 18\sqrt{6}$

2.) Move to $u \cdot v = \|u\|_2 \|v\|_2 \cos(\theta)$

$u \cdot v = 12\sqrt{18} \cos(\theta)$

$18\sqrt{6} = 12\sqrt{18}\cos(\theta)$

$\frac{18\sqrt{6}}{12\sqrt{18}} = \cos(\theta)$

$\frac{3}{2\sqrt{3}} = \cos(\theta)$

$\frac{\sqrt{3}}{2} = \cos(\theta)$

3.) solve for θ:

$\frac{\sqrt{3}}{2} = \cos(\theta)$

$\cos^{-1}(\frac{\sqrt{3}}{2}) = \theta$

$\theta = \frac{\pi}{6}$

- Exercise 3. (CS 3.4):

(a.) Find a such that $u = \begin{pmatrix} 2 \\ 1 \\ a \end{pmatrix}$ is orthogonal to $v = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$.

To find this value of a, we must calculate $u \cdot v$, set it equal to zero, and solve for a. This calculation is done below.

$$\begin{aligned} u \cdot v = 0 &\Rightarrow 2(1) + 1(3) + a(-2) = 0 \\ &\Rightarrow 2 + 3 - 2a = 0 \\ &\Rightarrow 5 = 2a \\ &\Rightarrow a = \frac{5}{2} \end{aligned}$$

Thus $a = \frac{5}{2}$ makes $u = \begin{pmatrix} 2 \\ 1 \\ a \end{pmatrix}$ orthogonal to $v = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$.

(b.) For what values of a, b are the vectors $u = \begin{pmatrix} 2 \\ a \\ -1 \end{pmatrix}$ and $v = \begin{pmatrix} b \\ 3 \\ -1 \end{pmatrix}$ orthogonal?

To find these values of a and b, we must calculate $u \cdot v$, set it equal to zero, and solve for a or b. This calculation is done below.

$$\begin{aligned} u \cdot v = 0 &\Rightarrow 2(b) + a(3) - 1(-1) = 0 \\ &\Rightarrow 2b + 3a + 2 = 0 \\ &\Rightarrow 2b = -2 - 3a \\ &\Rightarrow b = -1 - \frac{3}{2}a \end{aligned}$$

Thus a and $b = -1 - \frac{3}{2}a$ make the vectors $u = \begin{pmatrix} 2 \\ a \\ -1 \end{pmatrix}$ and $v = \begin{pmatrix} b \\ 3 \\ -1 \end{pmatrix}$ orthogonal.

Exercise 4. (CS 3.4):

Recall the following operations on matrices:

- If A is a $m \times n$ matrix and B is a $m \times n$ matrix, then $A+B$ is a $m \times n$ matrix with entries

$$(A+B)_{ij} = A_{ij} + B_{ij}.$$

- If A is a $m \times n$ matrix and B is a $n \times p$ matrix, then AB is a $m \times p$ matrix with entries

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

Notice the variable k corresponding to the column of A and row of B is summed.

- If A is $m \times m$ and invertible, then there exists a $m \times m$ matrix A^{-1} such that

$$A^{-1}A = AA^{-1} = I_m$$

- If A is a $m \times n$ matrix, then its transpose denoted A^T, is a $n \times m$ matrix whose columns are the rows of A. In terms of the entries of A, we have

$$(A^T)_{ij} = A_{ji}.$$

Hint: To prove $A=B$, show that $A_{ij} = B_{ij}$ for all i, j.

(a.) Show that $(A^T)^T = A$.

If we let A be an $m \times n$ matrix with entries A_{ij}, then it follows that A^T is an $n \times m$ matrix with entries A_{ji}. Therefore $(A^T)^T$ is an $m \times n$ matrix with entries A_{ij}. Since $A_{ij} = (A^T)^T_{ij} = A_{ij}$ for all i, j, it follows that $(A^T)^T = A$.

(b.) Show that $(A+B)^T = A^T + B^T$ whenever the matrix addition is well-defined.

If we let A and B both be $m \times n$ matrices with entries A_{ij} and B_{ij} respectively, then $A+B$ is an $m \times n$ matrix with entries $A_{ij} + B_{ij}$. If we also let $C = A+B$, then its entries are C_{ij}. Notice that

$$(A+B)^T_{ij} = (A_{ij} + B_{ij})^T = (C_{ij})^T = C_{ji} = A_{ji} + B_{ji} = (A_{ij})^T + (B_{ij})^T = (A^T + B^T)_{ij}$$

Thus we have shown that $(A+B)^T$ has the same ij entries as $A^T + B^T$ and thus $(A+B)^T = A^T + B^T$.

(c.) Show that $(AB)^T = B^T A^T$, whenever the matrix multiplication is well-defined.

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix, then AB is a $p \times n$ matrix with entries $(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$. Thus we can see that

$$(AB)^T_{ij} = AB_{ji} = \sum_{k=1}^{n} A_{jk} B_{ki} = \sum_{k=1}^{n} A^T_{kj} B^T_{ik} = \sum_{k=1}^{n} B^T_{ik} A^T_{kj} = (B^T A^T)_{ij}.$$

Since $(AB)^T_{ij} = (B^T A^T)_{ij}$ for all i, j, then we can see that $(AB)^T = B^T A^T$.

(d) Show that $(A^{-1})^T = (A^T)^{-1}$, whenever A^{-1} exists.

Let A be an $n \times n$ invertible matrix, then it follows that

$$AA^{-1} = I_n$$
$$(AA^{-1})^T = I_n^T$$
$$(A^{-1})^T (A^T) = I_n \qquad (\text{since } (AB)^T = B^T A^T \text{ and } I_n^T = I_n)$$
$$(A^{-1})^T (A^T)(A^T)^{-1} = I_n (A^T)^{-1}$$
$$(A^{-1})^T I_n = (A^T)^{-1}$$
$$(A^{-1})^T = (A^T)^{-1}$$

hence we have shown that $(A^{-1})^T = (A^T)^{-1}$, whenever A^{-1} exists.

Exercise 5. (CS3.5):

(a.) Use Gram-Schmidt orthogonalization to determine an orthonormal basis for $\mathbb{R}^3$ starting with the following set of vectors:

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Let $v_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, We will now proceed with gram-schmidt.

1.) Find q_1 from v_1

$$q_1 = \frac{v_1}{\|v_1\|_2} = \frac{1}{\sqrt{1^2+(-2)^2+1^2}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} = \begin{bmatrix} \sqrt{6}/6 \\ -\sqrt{6}/3 \\ \sqrt{6}/6 \end{bmatrix}$$

2.) Find q_2 from q_1 and v_2:

i) First find $v_{2\perp}$:

$$v_{2\perp} = v_2 - (v_2 \cdot q_1) q_1$$

$$= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{6}/6 \\ -\sqrt{6}/3 \\ \sqrt{6}/6 \end{bmatrix} \right) \begin{bmatrix} \sqrt{6}/6 \\ -\sqrt{6}/3 \\ \sqrt{6}/6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \frac{\sqrt{6}}{3} \begin{bmatrix} \sqrt{6}/6 \\ -\sqrt{6}/3 \\ \sqrt{6}/6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 4/3 \\ 1/3 \\ -2/3 \end{bmatrix}$$

ii) Find q_2:

$$q_2 = \frac{v_{2\perp}}{\|v_{2\perp}\|_2} = \frac{1}{\sqrt{(\frac{4}{3})^2 + (\frac{1}{3})^2 + (-\frac{2}{3})^2}} \begin{bmatrix} 4/3 \\ 1/3 \\ -2/3 \end{bmatrix} = \frac{1}{\sqrt{\frac{7}{3}}} \begin{bmatrix} 4/3 \\ 1/3 \\ -2/3 \end{bmatrix} = \begin{bmatrix} 4\sqrt{21}/21 \\ \sqrt{21}/21 \\ -2\sqrt{21}/21 \end{bmatrix}$$

3.) Find q_3 from q_2, q_1, and v_3:

i.) First find $v_{3\perp}$:

$$v_{3\perp} = v_3 - (v_3 \cdot q_1) q_1 - (v_3 \cdot q_2) q_2$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{6}/6 \\ -\sqrt{6}/3 \\ \sqrt{6}/6 \end{bmatrix} \right) \begin{bmatrix} \sqrt{6}/6 \\ -\sqrt{6}/3 \\ \sqrt{6}/6 \end{bmatrix} - \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 4\sqrt{21}/21 \\ \sqrt{21}/21 \\ -2\sqrt{21}/21 \end{bmatrix} \right) \begin{bmatrix} 4\sqrt{21}/21 \\ \sqrt{21}/21 \\ -2\sqrt{21}/21 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{\sqrt{6}}{3} \begin{bmatrix} \sqrt{6}/6 \\ -\sqrt{6}/3 \\ \sqrt{6}/6 \end{bmatrix} - \frac{2\sqrt{21}}{21} \begin{bmatrix} 4\sqrt{21}/21 \\ \sqrt{21}/21 \\ -2\sqrt{21}/21 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/3 \\ -2/3 \\ 1/3 \end{bmatrix} - \begin{bmatrix} 8/21 \\ 2/21 \\ -4/21 \end{bmatrix}$$

$$= \begin{bmatrix} 2/7 \\ 4/7 \\ 6/7 \end{bmatrix}$$

ii.) Find q_3:

$$q_3 = \frac{v_{3\perp}}{\|v_{3\perp}\|_2} = \frac{1}{\sqrt{(\frac{2}{7})^2+(\frac{4}{7})^2+(\frac{6}{7})^2}} \begin{bmatrix} 2/7 \\ 4/7 \\ 6/7 \end{bmatrix} = \frac{1}{\sqrt{\frac{8}{7}}} \begin{bmatrix} 2/7 \\ 4/7 \\ 6/7 \end{bmatrix} = \begin{bmatrix} \sqrt{56}/28 \\ \sqrt{56}/14 \\ 3\sqrt{56}/28 \end{bmatrix} = \begin{bmatrix} \sqrt{14}/14 \\ \sqrt{14}/7 \\ 3\sqrt{14}/14 \end{bmatrix}$$

thus an orthonormal basis for $\mathbb{R}^3$ is:

$$\left\{ \begin{bmatrix} \sqrt{6}/6 \\ -\sqrt{6}/3 \\ \sqrt{6}/6 \end{bmatrix}, \begin{bmatrix} 4\sqrt{21}/21 \\ \sqrt{21}/21 \\ -2\sqrt{21}/21 \end{bmatrix}, \begin{bmatrix} \sqrt{14}/14 \\ \sqrt{14}/7 \\ 3\sqrt{14}/14 \end{bmatrix} \right\}$$

(b.) Use Gram-Schmidt orthogonalization to determine an orthonormal basis for the set of all vectors orthogonal to $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$.

Before we use Gram-Schmidt we must find a basis for all vectors orthogonal to $(3,-1,2)^T$. If we let $(x,y,z)^T \in \mathbb{R}^3$, then all vectors orthogonal to $(3,-1,2)^T$ satisfy

$$\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \Rightarrow 3x - y + 2z = 0 \Rightarrow z = \frac{y}{2} - \frac{3}{2}x$$

then the vectors in $\mathbb{R}^3$ in this plane take the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ -\frac{3}{2}x + \frac{y}{2} \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ -\frac{3}{2} \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

Here x and y are free so

$$\{(x,y,z) \mid z = \tfrac{y}{2} - \tfrac{3}{2}x\} = \text{span}\{[1,0,-3/2], [0,1,1/2]\}.$$

We have shown that the two vectors span the space, now we must form A and row reduce to show they are linearly independent

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{3}{2} & \frac{1}{2} \end{bmatrix} \overset{\frac{3}{2}R_1 + R_3}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \frac{1}{2} \end{bmatrix} \overset{-\frac{1}{2}R_2 + R_3}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

since the number of pivots is 2, the number of linearly independent vectors is 2, thus the vectors are linearly independent and we have a basis for the subspace. We may now use Gram-Schmidt to find an orthonormal basis for this subspace. Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ -3/2 \end{bmatrix}$, and $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix}$.

1.) Find q_1 from v_1:

$$q_1 = \frac{v_1}{\|v_1\|_2} = \frac{1}{\sqrt{1^2 + (-3/2)^2}} \begin{bmatrix} 1 \\ 0 \\ -3/2 \end{bmatrix} = \frac{1}{\sqrt{13/4}} \begin{bmatrix} 1 \\ 0 \\ -3/2 \end{bmatrix} = \begin{bmatrix} 2\sqrt{13}/13 \\ 0 \\ -3\sqrt{13}/13 \end{bmatrix}$$

2.) Find q_2 from q_1 and v_2:

i) First find $v_{2\perp}$

$$v_{2\perp} = v_2 - (v_2 \cdot q_1) q_1$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix} \cdot \begin{bmatrix} 2\sqrt{13}/13 \\ 0 \\ -3\sqrt{13}/13 \end{bmatrix} \right) \begin{bmatrix} 2\sqrt{13}/13 \\ 0 \\ -3\sqrt{13}/13 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix} + \frac{3\sqrt{13}}{26} \begin{bmatrix} 2\sqrt{13}/13 \\ 0 \\ -3\sqrt{13}/13 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1/2 \end{bmatrix} + \begin{bmatrix} 3/13 \\ 0 \\ -9/26 \end{bmatrix}$$

$$= \begin{bmatrix} 3/13 \\ 1 \\ 2/13 \end{bmatrix}$$

ii) Find q_2

$$q_2 = \frac{v_{2\perp}}{\|v_{2\perp}\|_2} = \frac{1}{\sqrt{(\frac{3}{13})^2 + 1^2 + (\frac{2}{13})^2}} \begin{bmatrix} 3/13 \\ 1 \\ 2/13 \end{bmatrix} = \frac{1}{\sqrt{14/13}} \begin{bmatrix} 3/13 \\ 1 \\ 2/13 \end{bmatrix} = \begin{bmatrix} 3\sqrt{182}/182 \\ \sqrt{182}/14 \\ \sqrt{182}/91 \end{bmatrix}$$

Thus an orthonormal basis for the set of all vectors orthogonal to $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ is

$$\left\{ \begin{bmatrix} \frac{2\sqrt{13}}{13} \\ 0 \\ -\frac{3\sqrt{13}}{13} \end{bmatrix}, \begin{bmatrix} 3\sqrt{182}/182 \\ \sqrt{182}/14 \\ \sqrt{182}/91 \end{bmatrix} \right\}$$

c.) Why does Gram-Schmidt orthogonalization not work on the set of vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}?$$

To see why Gram-Schmidt won't work on these vectors, we will form the matrix A from these vectors and row reduce.

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \overset{-R_1+R_3}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \overset{-2R_2+R_3}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

thus we can see that this matrix is not full rank, and hence the vectors are linearly dependent. since the vectors are linearly dependent, this is why Gram-Schmidt fails on these vectors.

- Exercise 6. (CS 3.6)!

Compute the reduced QR factorization of the following full rank matrices.

In the following two problems we will use the Gram-Schmidt process to compute the factorization: $A = \tilde{Q}\tilde{R}$, for full rank matrices.

(a.) $\begin{pmatrix} 1 & 0 \\ -1 & -2 \\ 1 & 1 \end{pmatrix}$

Let $v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$. We will now proceed with Gram-Schmidt.

1.) obtain q_1 from v_1:

$$q_1 = \frac{v_1}{\|v_1\|_2} = \frac{1}{\sqrt{1^2+(-1)^2+1^2}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = \begin{bmatrix} \sqrt{3}/3 \\ -\sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix}$$

2.) obtain q_2 from q_1 and v_2:

i) First find $v_{2\perp}$:

$$\begin{aligned} v_{2\perp} &= v_2 - (v_2 \cdot q_1) q_1 \\ &= \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \sqrt{3}/3 \\ -\sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix} \right) \begin{bmatrix} \sqrt{3}/3 \\ -\sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} - \sqrt{3} \begin{bmatrix} \sqrt{3}/3 \\ -\sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \end{aligned}$$

ii) find q_2:

$$q_2 = \frac{v_{2\perp}}{\|v_{2\perp}\|_2} = \frac{1}{\sqrt{(-1)^2+(-1)^2+0^2}} \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 \\ -\sqrt{2}/2 \\ 0 \end{bmatrix}$$

Thus $\hat{Q} = \begin{bmatrix} \sqrt{3}/3 & -\sqrt{2}/2 \\ -\sqrt{3}/3 & -\sqrt{2}/2 \\ \sqrt{3}/3 & 0 \end{bmatrix}$

Since $\|v_1\|_2 = \sqrt{3}$, $(v_2 \cdot q_1) = \sqrt{3}$, and $\|v_{2\perp}\|_2 = \sqrt{2}$ we can see $\tilde{R}$ is

$$\tilde{R} = \begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{2} \end{bmatrix}$$

Hence the reduced QR factorization is

$$A = \begin{bmatrix} \sqrt{3}/3 & -\sqrt{2}/2 \\ -\sqrt{3}/3 & -\sqrt{2}/2 \\ \sqrt{3}/3 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3} & \sqrt{3} \\ 0 & \sqrt{2} \end{bmatrix}$$

b. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & -6 & 1 \end{pmatrix}$

Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 3 \\ 2 \\ -6 \end{bmatrix}$, and $v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, We will now proceed with Gram-schmidt

1.) obtain q_1 from v_1:

$$q_1 = \frac{v_1}{\|v_1\|_2} = \frac{1}{\sqrt{1}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

2.) obtain q_2 from q_1 and v_2:

i.) First find $v_{2\perp}$:

$$\begin{aligned} v_{2\perp} &= v_2 - (v_2 \cdot q_1)q_1 \\ &= \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} - 0 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

ii.) Find q_2:

$$q_2 = \frac{v_{2\perp}}{\|v_{2\perp}\|_2} = \frac{1}{\sqrt{2^2+1^2}} \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 0 \\ 2\sqrt{5}/5 \\ 0 \\ \sqrt{5}/5 \end{bmatrix}$$

3.) obtain q_3 from q_2, q_1, and v_3:

i.) First find $v_{3\perp}$:

$$\begin{aligned} v_{3\perp} &= v_3 - (v_3 \cdot q_1)q_1 - (v_3 \cdot q_2)q_2 \\ &= \begin{bmatrix} 0 \\ 3 \\ 2 \\ -6 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 3 \\ 2 \\ -6 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 3 \\ 2 \\ -6 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2\sqrt{5}/5 \\ 0 \\ \sqrt{5}/5 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 2\sqrt{5}/5 \\ 0 \\ \sqrt{5}/5 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 3 \\ 2 \\ -6 \end{bmatrix} - 0 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - 0 \begin{bmatrix} 0 \\ 2\sqrt{5}/5 \\ 0 \\ \sqrt{5}/5 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 3 \\ 2 \\ -6 \end{bmatrix} \end{aligned}$$

ii.) Find q_3:

$$q_3 = \frac{v_{3\perp}}{\|v_{3\perp}\|_2} = \frac{1}{\sqrt{3^2+2^2+(-6)^2}} \begin{bmatrix} 0 \\ 3 \\ 2 \\ -6 \end{bmatrix} = \frac{1}{\sqrt{49}} \begin{bmatrix} 0 \\ 3 \\ 2 \\ -6 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 0 \\ 3 \\ 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 3/7 \\ 2/7 \\ -6/7 \end{bmatrix}$$

4.) obtain q_4 from q_3, q_2, q_1, and v_4:

i.) First find $v_{4\perp}$:

$$v_{4\perp} = v_4 - (v_4 \cdot q_1)q_1 - (v_4 \cdot q_2)q_2 - (v_4 \cdot q_3)q_3$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 2\sqrt{5}/5 \\ 0 \\ \sqrt{5}/5 \end{bmatrix}\right)\begin{bmatrix} 0 \\ 2\sqrt{5}/5 \\ 0 \\ \sqrt{5}/5 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 3/7 \\ 2/7 \\ -6/7 \end{bmatrix}\right)\begin{bmatrix} 0 \\ 3/7 \\ 2/7 \\ -6/7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - 0\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{\sqrt{5}}{5}\begin{bmatrix} 0 \\ 2\sqrt{5}/5 \\ 0 \\ \sqrt{5}/5 \end{bmatrix} + \frac{6}{7}\begin{bmatrix} 0 \\ 3/7 \\ 2/7 \\ -6/7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 2/5 \\ 0 \\ 1/5 \end{bmatrix} + \begin{bmatrix} 0 \\ 18/49 \\ 12/49 \\ -36/49 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -8/245 \\ 12/49 \\ 16/245 \end{bmatrix}$$

ii.) Find q_4

$$q_4 = \frac{v_{4\perp}}{\|v_{4\perp}\|_2} = \frac{1}{\sqrt{\left(\frac{-8}{245}\right)^2 + \left(\frac{12}{49}\right)^2 + \left(\frac{16}{245}\right)^2}}\begin{bmatrix} 0 \\ -8/245 \\ 12/49 \\ 16/245 \end{bmatrix} = \frac{1}{\frac{4\sqrt{5}}{35}}\begin{bmatrix} 0 \\ -8/245 \\ 12/49 \\ 16/245 \end{bmatrix} = \begin{bmatrix} 0 \\ -2\sqrt{5}/35 \\ 3\sqrt{5}/7 \\ 4\sqrt{5}/35 \end{bmatrix}$$

Thus we can see that

$$\tilde{Q} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{2\sqrt{5}}{5} & \frac{3}{7} & \frac{-2\sqrt{5}}{35} \\ 0 & 0 & \frac{2}{7} & \frac{3\sqrt{5}}{7} \\ 0 & \frac{\sqrt{5}}{5} & \frac{-6}{7} & \frac{4\sqrt{5}}{35} \end{bmatrix}$$

Since $\|v_1\|_2 = 1$, $(v_2 \cdot q_1) = 0$, $\|v_{2\perp}\|_2 = \sqrt{5}$, $(v_3 \cdot q_1) = 0$, $(v_3 \cdot q_2) = 0$, $\|v_{3\perp}\|_2 = 7$, $(v_4 \cdot q_1) = 0$, $(v_4 \cdot q_2) = \frac{\sqrt{5}}{5}$, $(v_4 \cdot q_3) = \frac{-6}{7}$, and $\|v_{4\perp}\|_2 = \frac{4\sqrt{5}}{35}$ we can see that

$$\tilde{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{5} & 0 & \frac{\sqrt{5}}{5} \\ 0 & 0 & 7 & -\frac{6}{7} \\ 0 & 0 & 0 & \frac{4\sqrt{5}}{35} \end{bmatrix}$$

Thus the reduced, and in this case full, QR decomposition is

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{2\sqrt{5}}{5} & \frac{3}{7} & \frac{-2\sqrt{5}}{35} \\ 0 & 0 & \frac{2}{7} & \frac{3\sqrt{5}}{7} \\ 0 & \frac{\sqrt{5}}{5} & \frac{-6}{7} & \frac{4\sqrt{5}}{35} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{5} & 0 & \frac{\sqrt{5}}{5} \\ 0 & 0 & 7 & \frac{-6}{7} \\ 0 & 0 & 0 & \frac{4\sqrt{5}}{35} \end{bmatrix}$$

Exercise 7. (cs3.6)

Recall that an orthogonal matrix Q is a $n \times n$ matrix such that $Q^T Q = I_n$.

(a) Show that Q preserves the dot product between vectors. That is,

$$(Qx) \cdot (Qy) = x \cdot y$$

for any $x, y \in \mathbb{R}^n$. Thus, Q preserves angles between vectors.

If Q is an orthogonal matrix such that $Q^T Q = I_n$, then we will use properties of the dot product, matrix transpose, and orthogonal matrices to show that $(Qx) \cdot (Qy) = x \cdot y$. This calculation is shown below.

$$\begin{aligned}
(Qx) \cdot (Qy) &= (Qx)^T (Qy) && \text{(matrix multiplication definition of dot product)} \\
&= x^T Q^T Q y && \text{(since } (AB)^T = B^T A^T\text{)} \\
&= x^T I_n y && \text{(since } Q \text{ is orthogonal} \Rightarrow Q^T = Q^{-1}\text{)} \\
&= x^T y \\
&= x \cdot y && \text{(matrix multiplication definition of dot product.}
\end{aligned}$$

thus Q preserves the dot product between vectors.

(b.) Show that Q preserves the 2-norm of a vector. That is,

$$\|Qx\|_2 = \|x\|_2$$

for any $x \in \mathbb{R}^n$.

If Q is an orthogonal matrix such that $Q^T Q = I_n$, then we will use properties of the dot product, matrix transpose, and orthogonal matrices to show that $\|Qx\|_2 = \|x\|_2$. This calculation is shown below.

$$\begin{aligned}
\|Qx\|_2 &= \sqrt{(Qx) \cdot (Qx)} && \text{(Definition of 2-norm)} \\
&= \sqrt{x \cdot x} && \text{(Part a} \Rightarrow Q \text{ preserves the dot product)} \\
&= \|x\|_2 && \text{(Definition of 2-norm)}
\end{aligned}$$

thus Q preserves the 2-norm of a vector.

Multi-Step Problem

Table of Contents

Setup

```
format long;
```

Part (a):

```
% In this part of the problem we will define the 5x5 Hilbert Matrix, we can
% do this by hand or by using hilb(n). In our case this is hilb(5).

A = hilb(5);

% We will also use the provided code to apply the classical Gram-Schmidt
% process to the above matrix. We will output the Q matrix to the Command
% Window.

% Run the classical Gram-Schmidt process:
[n,k] = size(A);
Q = zeros(n,k);

for j = 1:k
    v_j = A(:,j);
    for p = 1:j-1
        v_j = v_j - (transpose(Q(:,p))*A(:,j))*Q(:,p);
    end
    if norm(v_j,2)==0
        error('Matrix A is not full rank.');
    end
    Q(:,j) = v_j/norm(v_j,2);
end

% Output the Q matrix to the Command Window:
Q
```

```
Q =

  Columns 1 through 3
```

```
   0.826584298073692  -0.533354625741032   0.175305353859225
   0.413292149036846   0.374053540533383  -0.717262398141420
   0.275528099357897   0.462945978156888  -0.057664733696256
   0.206646074518423   0.443319111628251   0.352625606033429
   0.165316859614738   0.405913757574913   0.571955108009368

  Columns 4 through 5

  -0.039102073788536   0.005504737805264
   0.403345206656191  -0.110094734308526
  -0.678964622494018   0.495426233019006
  -0.206153706721816  -0.770662945733810
   0.576447189861588   0.385331440514156
```

Part (b):

```
% In this part, we define a notion of error for the classical Gram-Schmidt
% process as the maximum entry in absolute value of the matrix Q^T Q - I_n.
% We will compute and display this error in the command window. Based on
% this value we will find out roughly how many decimal places we can trust
% the Q obtained in a.

% Compute the matrix Q^T Q - I_5:
Z = (transpose(Q) * Q) - eye(5);

% Compute and display the maximum entry in absolute value of the matrix
% Q^T Q - I_n. To do this we will compute the maximum aboslute value entry
% in each row, then the maximum in the resulting row vector.
max(max(abs(Z)))

% Based on this value, we can trust the Q obtained in (a) by up to roughly 7
% decimal places.
```

```
ans =

     6.635477955985181e-08
```

Part (c):

```
% In this part, we will apply the modified Gram-Schmidt algorithm to the
% Hilbert matrix. We will then display the resulting Q matrix to the
% command window.

% Run the modified Gram-Schmidt process:
[n,k] = size(A);
for j = 1:k
    if norm(A(:,j),2)==0
        error('Matrix A is not full rank.');
    end
    A(:,j) = A(:,j)/norm(A(:,j),2);
```

```
    for p = j+1:k
        A(:,p) = A(:,p) - (transpose(A(:,j))*A(:,p))*A(:,j);
    end
end
Q = A;

% Output the Q matrix to the Command Window:
Q

% As can be seen from the following output, the Q matrix from the modified
% Gram-Schmidt is very similar to that of the classical Gram-Schmidt, with
% only minor differences farther down the numbers.
```

```
Q =

  Columns 1 through 3

   0.826584298073692  -0.533354625741032   0.175305353859193
   0.413292149036846   0.374053540533383  -0.717262398141397
   0.275528099357897   0.462945978156888  -0.057664733696227
   0.206646074518423   0.443319111628251   0.352625606033457
   0.165316859614738   0.405913757574913   0.571955108009393

  Columns 4 through 5

  -0.039102073792843   0.005504735413709
   0.403345206675991  -0.110094708423228
  -0.678964622492973   0.495426187907808
  -0.206153706732331  -0.770662958963769
   0.576447189844912   0.385331479484320
```

Part (d):

```
% In this part, we will apply the notion of error defined in part (b) for
% the above matrix. With this error value, we will again be able to say how
% may decimal places we can trust the Q obtained in part c.

% Compute the matrix Q^T Q - I_n:
Z = (transpose(Q) * Q) - eye(5);

% Compute and display the maximum entry in absolute value of the matrix
% Q^T Q - I_n.
max(max(abs(Z)))

% Based on this value, we can trust the Q obtained in (a) by roughly
% 11 decimal places.
```

```
ans =

     6.200845392712040e-12
```

Part (e):

```
% In this part, we will compute the condition number of the matrix A using
% cond(A) in order to see "how dependent" the columns of A are. We will
% then use this condition number to find approximately how many significant
% digits we will lose performing the modified Gram-Schmidt on A.

% Redefine the matrix A to ensure we are using the correct matrix in
% computations:
A = hilb(5);

% Compute and display the condition number of A:
cond_num = cond(A);
cond_num

% Compute and display the significant digits lost:
k = log10(cond_num);
k

% Based on computing the rule-of-thumb, we expect to lose approzimately 6
% significant digits (rounded) when perfroming Gram-Schmidt on the 5x5
% Hilbert matrix. Notice that the modified Gram-Schmidt algorithm lost
% approximately 5 digits, this is very close to the number we got using the
% rule of thumb calculation above which was approximately 5.68.
```

```
cond_num =

     4.766072502422425e+05


k =

   5.678160644632865
```

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