Homework 6 SHi 2028502 Jaiden Atterbury

Component Skills Exercises

Exercise 1. (CS 4.1):

From Olver and Shakiban, complete Exercise 7.1.1 (b), (d), (e) and Exercise 7.1.3 (b), (c), (d). Explain your answers.

7.1.1. Which of the following functions $F: \mathbb{R}^3 \to \mathbb{R}$ are linear?

(b.) $F(x, y, z) = y - 2$.

- STEP 1: Compute $F(\alpha x + \beta y)$:

Let $x = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $y = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in \mathbb{R}^3$, and $\alpha, \beta \in \mathbb{R}$. Hence $\alpha x + \beta y = \begin{bmatrix} \alpha x_1 + \beta x_2 \\ \alpha y_1 + \beta y_2 \\ \alpha z_1 + \beta z_2 \end{bmatrix}$.

$F(\alpha x + \beta y) = F(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) = \alpha y_1 + \beta y_2 - 2$

- STEP 2: compute $\alpha F(x) + \beta F(y)$:

$\alpha F(x) + \beta F(y) = \alpha F(x_1, y_1, z_1) + \beta F(x_2, y_2, z_2) = \alpha(y_1 - 2) + \beta(y_2 - 2) = \alpha y_1 + \beta y_2 - 2(\alpha + \beta)$

Since $F(\alpha x + \beta y) \neq \alpha F(x) + \beta F(y)$ for all $x, y \in \mathbb{R}^3$, $F(x, y, z) = y - 2$ is not linear.

(d.) $F(x, y, z) = x - y - z$

Let $x = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $y = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in \mathbb{R}^3$, and $\alpha, \beta \in \mathbb{R}$. Hence $\alpha x + \beta y = \begin{bmatrix} \alpha x_1 + \beta x_2 \\ \alpha y_1 + \beta y_2 \\ \alpha z_1 + \beta z_2 \end{bmatrix}$.

- STEP 1: compute $F(\alpha x + \beta y)$:

$$F(\alpha x + \beta y) = F(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) = \alpha x_1 + \beta x_2 - (\alpha y_1 + \beta y_2) - (\alpha z_1 + \beta z_2)$$
$$= \alpha x_1 + \beta x_2 - \alpha y_1 - \beta y_2 - \alpha z_1 - \beta z_2 = \alpha(x_1 - y_1 - z_1) + \beta(x_2 - y_2 - z_2)$$
$$= \alpha F(x_1, y_1, z_1) + \beta F(x_2, y_2, z_2) = \alpha F(x) + \beta F(y)$$

Since $F(\alpha x + \beta y) = \alpha F(x) + \beta F(y)$ for all $x, y \in \mathbb{R}^3$ and $\alpha, \beta \in \mathbb{R}$, $F(x, y, z) = x - y - z$ is linear.

(e.) $F(x,y,z) = xyz$.

Let $x = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $y = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \in \mathbb{R}^3$, and $\alpha, \beta \in \mathbb{R}$. Hence $\alpha x + \beta y = \begin{bmatrix} \alpha x_1 + \beta x_2 \\ \alpha y_1 + \beta y_2 \\ \alpha z_1 + \beta z_2 \end{bmatrix}$.

STEP 1: Compute $F(\alpha x + \beta y)$:

$$F(\alpha x + \beta y) = F(\alpha x_1 + \beta x_2, \alpha y_1 + \beta y_2, \alpha z_1 + \beta z_2) = (\alpha x_1 + \beta x_2)(\alpha y_1 + \beta y_2)(\alpha z_1 + \beta z_2)$$
$$= (\alpha^2 x_1 y_1 + \alpha\beta x_1 y_2 + \beta\alpha x_2 y_1 + \beta^2 x_2 y_2)(\alpha z_1 + \beta z_2)$$
$$= \alpha^3 x_1 y_1 z_1 + \alpha^2\beta x_1 y_2 z_1 + \beta\alpha^2 x_2 y_1 z_1 + \beta^2 \alpha x_2 y_2 z_1 +$$
$$\alpha^2\beta x_1 y_1 z_2 + \alpha\beta^2 x_1 y_2 z_2 + \beta^2\alpha x_2 y_1 z_2 + \beta^3 x_2 y_2 z_2.$$

STEP 2: compute $\alpha F(x) + \alpha F(y)$:

$$\alpha F(x) + \beta F(y) = \alpha F(x_1, y_1, z_1) + \beta F(x_2, y_2, z_2) = \alpha(x_1 y_1 z_1) + \beta(x_2 y_2 z_2) = \alpha x_1 y_1 z_1 + \beta x_2 y_2 z_2.$$

Since $F(\alpha x + \beta y) \neq \alpha F(x) + \beta F(y)$ $\forall x, y \in \mathbb{R}^3$ and $\alpha, \beta \in \mathbb{R}^3$, $F(x,y,z) = xyz$ is not linear

7.1.3 Which of the following functions $F: \mathbb{R}^2 \to \mathbb{R}^2$ are linear?

(b.) $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y + 1 \\ x - y - 1 \end{pmatrix}$

Let $x = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and $y = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$, and $\alpha, \beta \in \mathbb{R}$. Hence $\alpha x + \beta y = \begin{bmatrix} \alpha x_1 + \beta x_2 \\ \alpha y_1 + \beta y_2 \end{bmatrix}$.

STEP 1: compute $F(\alpha x + \beta y)$:

$$F(\alpha x + \beta y) = F\begin{pmatrix} \alpha x_1 + \beta x_2 \\ \alpha y_1 + \beta y_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 + \beta x_2 + \alpha y_1 + \beta y_2 + 1 \\ \alpha x_1 + \beta x_2 - \alpha y_1 - \beta y_2 - 1 \end{pmatrix}$$
$$= \begin{pmatrix} \alpha(x_1 + y_1) + \beta(x_2 + y_2) + 1 \\ \alpha(x_1 - y_1) + \beta(x_2 - y_2) - 1 \end{pmatrix}$$

STEP 2: Compute $\alpha F(x) + \beta F(y)$:

$$\alpha F(x) + \beta F(y) = \alpha F\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \beta F\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \alpha \begin{pmatrix} x_1 + y_1 + 1 \\ x_1 - y_1 - 1 \end{pmatrix} + \beta \begin{pmatrix} x_2 + y_2 + 1 \\ x_2 - y_2 - 1 \end{pmatrix}$$
$$= \begin{pmatrix} \alpha(x_1 + y_1) + \beta(x_2 + y_2) + \alpha + \beta \\ \alpha(x_1 - y_1) + \beta(x_2 - y_2) - \alpha - \beta \end{pmatrix}$$

Since $F(\alpha x+\beta y) \neq \alpha F(x)+\beta F(y)$ for all $x, y \in \mathbb{R}^2$, $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y+1 \\ x-y-1 \end{pmatrix}$ is not linear.

c. $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy \\ x-y \end{pmatrix}$

Let $x = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, $y = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$, and $\alpha, \beta \in \mathbb{R}$. Then $\alpha x+\beta y = \begin{pmatrix} \alpha x_1+\beta x_2 \\ \alpha y_1+\beta y_2 \end{pmatrix}$.

STEP 1: compute $F(\alpha x+\beta y)$:

$$F(\alpha x+\beta y) = F\begin{pmatrix} \alpha x_1+\beta x_2 \\ \alpha y_1+\beta y_2 \end{pmatrix} = \begin{pmatrix} (\alpha x_1+\beta x_2)(\alpha y_1+\beta y_2) \\ \alpha x_1+\beta x_2-(\alpha y_1+\beta y_2) \end{pmatrix} = \begin{pmatrix} \alpha^2 x_1 y_1+\alpha\beta x_1 y_2+\beta\alpha x_2 y_1+\beta^2 x_2 y_2 \\ \alpha(x_1-y_1)+\beta(x_2-y_2) \end{pmatrix}$$

STEP 2: compute $\alpha F(x)+\beta F(y)$:

$$\alpha F(x)+\beta F(y) = \alpha F\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}+\beta F\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \alpha\begin{pmatrix} x_1 y_1 \\ x_1-y_1 \end{pmatrix}+\beta\begin{pmatrix} x_2 y_2 \\ x_2-y_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 y_1+\beta x_2 y_2 \\ \alpha(x_1-y_1)+\beta(x_2-y_2) \end{pmatrix}$$

Since $F(\alpha x+\beta y) \neq \alpha F(x)+\beta F(y)$ for all $x, y \in \mathbb{R}^2$, $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} xy \\ x-y \end{pmatrix}$ is not linear.

d. $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3y \\ 2x \end{pmatrix}$

Let $x = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, $y = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in \mathbb{R}^2$, and $\alpha, \beta \in \mathbb{R}$. Then $\alpha x+\beta y = \begin{pmatrix} \alpha x_1+\beta x_2 \\ \alpha y_1+\beta y_2 \end{pmatrix}$.

STEP 1: compute $F(\alpha x+\beta y)$:

$$F(\alpha x+\beta y) = F\begin{pmatrix} \alpha x_1+\beta x_2 \\ \alpha y_1+\beta y_2 \end{pmatrix} = \begin{pmatrix} 3(\alpha y_1+\beta y_2) \\ 2(\alpha x_1+\beta x_2) \end{pmatrix} = \begin{pmatrix} 3\alpha y_1+3\beta y_2 \\ 2\alpha x_1+2\beta x_2 \end{pmatrix} = \alpha\begin{pmatrix} 3y_1 \\ 2x_1 \end{pmatrix}+\beta\begin{pmatrix} 3y_2 \\ 2x_2 \end{pmatrix} = \alpha F\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}+\beta F\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$
$$= \alpha F(x)+\beta F(y)$$

Since $F(\alpha x+\beta y) = \alpha F(x)+\beta F(y)$ for all $x, y \in \mathbb{R}^2$, $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3y \\ 2x \end{pmatrix}$ is linear.

As mentioned in Piazza, we also have to come up with a counterexample for the functions that are not linear. This is done below.

○ 7.1.1

(b) Since $F(x, y, z)$ does not pass through the origin, it is thus not linear.

(c) Let $x = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $y = \vec{0}$, $\alpha = 2$, $\beta = 0$

$F(\alpha x + \beta y) = (2+0)(2+0)(2+0) = 8$

$\alpha F(x) + \beta F(y) = 2 + 0 = 2$

Since $8 \neq 2$, F is not linear.

○ 7.1.3:

(b.) Let $x = \begin{pmatrix}1\\1\end{pmatrix}$, $y = \begin{pmatrix}1\\1\end{pmatrix}$, $\alpha = 1$, $\beta = 1$

$F(\alpha x + \beta y) = \begin{pmatrix}2 + 2 + 1\\ 0 + 0 - 1\end{pmatrix} = \begin{pmatrix}5\\-1\end{pmatrix}$

$\alpha F(x) + \beta F(y) = \begin{pmatrix}2 + 2 + 1 + 1\\ 0 + 0 - 1 - 1\end{pmatrix} = \begin{pmatrix}6\\-2\end{pmatrix}$

Since $\begin{pmatrix}5\\-1\end{pmatrix} \neq \begin{pmatrix}6\\-2\end{pmatrix}$, F is not linear.

(c.) Let $x = \begin{pmatrix}1\\1\end{pmatrix}$, $y = \begin{pmatrix}1\\1\end{pmatrix}$, $\alpha = 1$, $\beta = 1$

$F(\alpha x + \beta y) = \begin{pmatrix}2 \cdot 2\\ 0 + 0\end{pmatrix} = \begin{pmatrix}4\\0\end{pmatrix}$

$\alpha F(x) + \beta F(y) = \begin{pmatrix}1 + 1\\ 0 + 0\end{pmatrix} = \begin{pmatrix}2\\0\end{pmatrix}$

Since $\begin{pmatrix}4\\0\end{pmatrix} \neq \begin{pmatrix}2\\0\end{pmatrix}$, F is not linear.

Exercise 2 (CS 4.1):

In the subexercises that follow, $f: \mathbb{R}^3 \to \mathbb{R}^2$ such that

$$f\left(\begin{pmatrix}1\\0\\-1\end{pmatrix}\right) = \begin{pmatrix}1\\3\end{pmatrix},\quad f\left(\begin{pmatrix}1\\-1\\2\end{pmatrix}\right) = \begin{pmatrix}2\\2\end{pmatrix},\quad f\left(\begin{pmatrix}2\\0\\0\end{pmatrix}\right) = \begin{pmatrix}2\\-2\end{pmatrix}.$$

Since $\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^3$, in order to find f's effect on a given vector, we must (1) use Gaussian elimination and back substitution to find the coordinates of the linear combination of the given vector and the aforementioned basis, and (2) using linearity property of f, find what $f(\vec{v})$ is.

(a.) What is $f\left(\begin{pmatrix}2\\2\\2\end{pmatrix}\right)$

- STEP 1: Gaussian Elimination and Back Substitution:

$$\left[\begin{array}{ccc|c}1 & 1 & 2 & 2\\0 & -1 & 0 & 2\\-1 & 2 & 0 & 2\end{array}\right] \overset{R_1+R_3}{\sim} \left[\begin{array}{ccc|c}1 & 1 & 2 & 2\\0 & -1 & 0 & 2\\0 & 3 & 2 & 4\end{array}\right] \overset{3R_2+R_3}{\sim} \left[\begin{array}{ccc|c}1 & 1 & 2 & 2\\0 & -1 & 0 & 2\\0 & 0 & 2 & 10\end{array}\right] \Rightarrow \begin{array}{l}x_3 = 5\\x_2 = -2\\x_1 = \frac{2-10+2}{1} = -6\end{array}$$

- STEP 2: Apply f to both sides and use linearity of f:

$$\begin{bmatrix}2\\2\\2\end{bmatrix} = -6\begin{bmatrix}1\\0\\-1\end{bmatrix} - 2\begin{bmatrix}1\\-1\\2\end{bmatrix} + 5\begin{bmatrix}2\\0\\0\end{bmatrix}$$

$$f\left(\begin{bmatrix}2\\2\\2\end{bmatrix}\right) = f\left(-6\begin{bmatrix}1\\0\\-1\end{bmatrix} - 2\begin{bmatrix}1\\-1\\2\end{bmatrix} + 5\begin{bmatrix}2\\0\\0\end{bmatrix}\right)$$

$$= -6f\left(\begin{bmatrix}1\\0\\-1\end{bmatrix}\right) - 2f\left(\begin{bmatrix}1\\-1\\2\end{bmatrix}\right) + 5f\left(\begin{bmatrix}2\\0\\0\end{bmatrix}\right)$$

$$= -6\begin{pmatrix}1\\3\end{pmatrix} - 2\begin{pmatrix}2\\2\end{pmatrix} + 5\begin{pmatrix}2\\-2\end{pmatrix}$$

$$= \begin{pmatrix}0\\-32\end{pmatrix}$$

thus $f\left(\begin{pmatrix}2\\2\\2\end{pmatrix}\right) = \begin{pmatrix}0\\-32\end{pmatrix}$

(b.) What is $f\left(\begin{pmatrix}2\\4\\4\end{pmatrix}\right)$?

- STEP 1: Gaussian Elimination and Back Substitution:

$$\left[\begin{array}{ccc|c}1 & 1 & 2 & 2\\0 & -1 & 0 & 4\\-1 & 2 & 0 & 4\end{array}\right] \overset{R_1+R_3}{\sim} \left[\begin{array}{ccc|c}1 & 1 & 2 & 2\\0 & -1 & 0 & 4\\0 & 3 & 2 & 6\end{array}\right] \overset{3R_2+R_3}{\sim} \left[\begin{array}{ccc|c}1 & 1 & 2 & 2\\0 & -1 & 0 & 4\\0 & 0 & 2 & 18\end{array}\right] \Rightarrow \begin{array}{l}x_3 = 9\\x_2 = -4\\x_1 = \frac{2-18+4}{1} = -12\end{array}$$

∘STEP 2: Apply f to both sides and use linearity of f:

$$\begin{bmatrix}2\\4\end{bmatrix} = -12\begin{bmatrix}1\\0\\-1\end{bmatrix} - 4\begin{bmatrix}-1\\2\end{bmatrix} + 9\begin{bmatrix}2\\0\\0\end{bmatrix}$$

$$f\left(\begin{bmatrix}2\\3\\4\end{bmatrix}\right) = f\left(-12\begin{bmatrix}1\\0\\-1\end{bmatrix} - 4\begin{bmatrix}-1\\2\end{bmatrix} + 9\begin{bmatrix}2\\0\\0\end{bmatrix}\right)$$

$$= -12\, f\left(\begin{pmatrix}1\\0\\-1\end{pmatrix}\right) - 4\, f\left(\begin{pmatrix}-1\\2\end{pmatrix}\right) + 9\, f\left(\begin{pmatrix}2\\0\end{pmatrix}\right)$$

$$= -12\begin{pmatrix}1\\3\end{pmatrix} - 4\begin{pmatrix}2\\2\end{pmatrix} + 9\begin{pmatrix}2\\-2\end{pmatrix}$$

$$= \begin{pmatrix}-2\\-62\end{pmatrix}$$

Hence $f\left(\begin{pmatrix}2\\4\end{pmatrix}\right)$

(c.) What is $f\left(\begin{pmatrix}0\\0\end{pmatrix}\right)$?

As stated in Lecture 14 "the origin always gets mapped to itself" when a linear transformation is applied to $\vec{0}$. Thus we can automatically see that $f\left(\begin{pmatrix}0\\0\end{pmatrix}\right) = \begin{pmatrix}0\\0\end{pmatrix}$.

- Exercise 3. (CS4.1):

In th subexercises that follow, $f: \mathbb{R}^3 \to \mathbb{R}^2$ such that

$$f\left(\begin{pmatrix}1\\0\\-1\end{pmatrix}\right) = \begin{pmatrix}1\\3\end{pmatrix},\quad f\left(\begin{pmatrix}1\\-1\\2\end{pmatrix}\right) = \begin{pmatrix}2\\2\end{pmatrix},\quad f\left(\begin{pmatrix}2\\0\\0\end{pmatrix}\right) = \begin{pmatrix}2\\-2\end{pmatrix}$$

What is the canonical matrix representation of f?

To find the canonical basis/matrix representation of f, we must find how f acts on $\{e_1, e_2, e_3\}$, then combine the resulting vectors into a matrix A. We will do this using the method from the last problem.

1. Find $f(e_1) = f\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right)$.

- STEP 1: Gaussian Ellimination and Back Substitution:

$$\left[\begin{array}{ccc|c}1&1&2&1\\0&-1&0&0\\-1&2&0&0\end{array}\right] \overset{R_1+R_3}{\sim} \left[\begin{array}{ccc|c}1&1&2&1\\0&-1&0&0\\0&3&2&1\end{array}\right] \overset{3R_2+R_3}{\sim} \left[\begin{array}{ccc|c}1&1&2&1\\0&-1&0&0\\0&0&2&1\end{array}\right] \Rightarrow \begin{array}{l}x_3=\frac{1}{2}\\x_2=0\\x_1=\frac{1-1+0}{1}=0\end{array}$$

- STEP 2: Apply f to both sides and apply linearity of f:

$$\begin{bmatrix}1\\0\\0\end{bmatrix} = 0\begin{bmatrix}1\\0\\-1\end{bmatrix} + 0\begin{bmatrix}1\\-1\\2\end{bmatrix} + \frac{1}{2}\begin{bmatrix}2\\0\\0\end{bmatrix}$$

$$\begin{aligned}f\left(\begin{pmatrix}1\\0\\0\end{pmatrix}\right) &= f\left(0\begin{pmatrix}1\\0\\-1\end{pmatrix} + 0\begin{pmatrix}1\\-1\\2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}2\\0\\0\end{pmatrix}\right)\\ &= 0f\left(\begin{pmatrix}1\\0\\-1\end{pmatrix}\right) + 0f\left(\begin{pmatrix}1\\-1\\2\end{pmatrix}\right) + \frac{1}{2}f\left(\begin{pmatrix}2\\0\\0\end{pmatrix}\right)\\ &= 0\begin{bmatrix}1\\3\end{bmatrix} + 0\begin{bmatrix}2\\2\end{bmatrix} + \frac{1}{2}\begin{bmatrix}2\\-2\end{bmatrix}\\ &= \begin{bmatrix}1\\-1\end{bmatrix}\end{aligned}$$

hence $f(e_1) = \begin{bmatrix}1\\-1\end{bmatrix}$.

2. Find $f(e_2) = f\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right)$

- STEP 1: Gaussian Ellimination and Back Substitution:

$$\left[\begin{array}{ccc|c}1&1&2&0\\0&-1&0&1\\-1&2&0&0\end{array}\right] \overset{R_1+R_3}{\sim} \left[\begin{array}{ccc|c}1&1&2&0\\0&-1&0&1\\0&3&2&0\end{array}\right] \overset{3R_2+R_3}{\sim} \left[\begin{array}{ccc|c}1&1&2&0\\0&-1&0&1\\0&0&2&3\end{array}\right] \Rightarrow \begin{array}{l}x_3=3/2\\x_2=-1\\x_1=\frac{0-3+1}{1}=-2\end{array}$$

- STEP 2: Apply f to both sides and apply linearity of f:

$$\begin{bmatrix}0\\1\\0\end{bmatrix} = -2\begin{pmatrix}1\\0\\-1\end{pmatrix} - 1\begin{pmatrix}1\\-1\\2\end{pmatrix} + \frac{3}{2}\begin{pmatrix}2\\0\\0\end{pmatrix}$$

$$f\left(\begin{pmatrix}0\\1\\0\end{pmatrix}\right) = f\left(-2\begin{pmatrix}1\\0\\-1\end{pmatrix} - 1\begin{pmatrix}1\\-1\\2\end{pmatrix} + \frac{3}{2}\begin{pmatrix}2\\0\\0\end{pmatrix}\right)$$

$$= -2 f\left(\begin{pmatrix}1\\0\\-1\end{pmatrix}\right) - 1 f\left(\begin{pmatrix}1\\-1\\2\end{pmatrix}\right) + \frac{3}{2} f\left(\begin{pmatrix}2\\0\\0\end{pmatrix}\right)$$

$$= -2\begin{pmatrix}1\\3\end{pmatrix} - 1\begin{pmatrix}2\\2\end{pmatrix} + \frac{3}{2}\begin{pmatrix}2\\-1\end{pmatrix}$$

$$= \begin{pmatrix}-1\\-11\end{pmatrix}$$

Hence $f(e_2) = \begin{pmatrix}-1\\-11\end{pmatrix}$.

(3.) Find $f(e_3) = f\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right)$.

∘ STEP 1: Gaussian Elimination and Back substitution:

$$\left[\begin{array}{ccc|c}1&1&2&0\\0&-1&0&0\\-1&2&0&1\end{array}\right] \overset{R_1+R_3}{\sim} \left[\begin{array}{ccc|c}1&1&2&0\\0&-1&0&0\\0&3&2&1\end{array}\right] \overset{3R_2+R_3}{\sim} \left[\begin{array}{ccc|c}1&1&2&0\\0&-1&0&0\\0&0&2&1\end{array}\right] \Rightarrow \begin{array}{l}x_3 = 1/2\\ x_2 = 0\\ x_1 = \frac{0-1+0}{1} = -1\end{array}$$

∘ STEP 2: Apply f to both sides and use linearity of f:

$$\begin{bmatrix}0\\0\\1\end{bmatrix} = -1\begin{pmatrix}1\\0\\-1\end{pmatrix} + 0\begin{pmatrix}1\\-1\\2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}2\\0\\0\end{pmatrix}$$

$$f\left(\begin{pmatrix}0\\0\\1\end{pmatrix}\right) = f\left(-1\begin{pmatrix}1\\0\\-1\end{pmatrix} + 0\begin{pmatrix}1\\-1\\2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}2\\0\\0\end{pmatrix}\right)$$

$$= -1 f\left(\begin{pmatrix}1\\0\\-1\end{pmatrix}\right) + 0 f\left(\begin{pmatrix}1\\-1\\2\end{pmatrix}\right) + \frac{1}{2} f\left(\begin{pmatrix}2\\0\\0\end{pmatrix}\right)$$

$$= -1\begin{pmatrix}1\\3\end{pmatrix} + 0\begin{pmatrix}2\\2\end{pmatrix} + \frac{1}{2}\begin{pmatrix}2\\-2\end{pmatrix}$$

$$= \begin{pmatrix}0\\-4\end{pmatrix}$$

Hence $f(e_3) = \begin{pmatrix}0\\-4\end{pmatrix}$.

Therefore the canonical representation of f is

$$A = (\, f(e_1) \mid f(e_2) \mid f(e_3) \,)$$

$$= \begin{bmatrix}1 & -1 & 0\\ -1 & -11 & -4\end{bmatrix}$$

- Exercise 4. (CS4.2):

For each of the following subexercises, construct a 2×2 canonical matrix representation of the given linear transformation.

(a.) Rotate a vector in $\mathbb{R}^2$ by $\pi/4$ radians clockwise.

Using the general form of a rotation matrix in $\mathbb{R}^2$, but using a negative angle θ to account for clockwise rotation we obtain:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos(\frac{-\pi}{4}) & -\sin(\frac{-\pi}{4}) \\ \sin(\frac{-\pi}{4}) & \cos(\frac{-\pi}{4}) \end{bmatrix} = \begin{bmatrix} \cos(\frac{\pi}{4}) & \sin(\frac{\pi}{4}) \\ -\sin(\frac{\pi}{4}) & \cos(\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

(b.) Reflect a vector in $\mathbb{R}^2$ across the line $y = -\sqrt{3}x$.

As found in Homework 5, $H = 2qq^T - I_n$ reflects a vector about a line in $\mathbb{R}^2$. The line with slope $-\sqrt{3}$ in $\mathbb{R}^2$ is the span of $x = (-1, \sqrt{3})^T$. Hence $q = \frac{x}{\|x\|_2}$ which is $q = \frac{1}{\sqrt{(-1)^2 + (\sqrt{3})^2}}\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}$. Using the above formula

$$\begin{aligned} H &= 2qq^T - I_2 \\ &= 2\begin{pmatrix} -1/2 \\ \sqrt{3}/2 \end{pmatrix}\begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}\begin{pmatrix} -1/2 & \sqrt{3}/2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 3/2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix} \end{aligned}$$

(c.) Stretch the x component of a vector in $\mathbb{R}^2$ by three and contract its y component by $\frac{1}{3}$.

To see what the canonical matrix of f is, we must see how f acts on $\{e_1, e_2\}$:

$$f(e_1) = f\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$f(e_2) = f\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1/3 \end{bmatrix}$$

hence we can see that

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 1/3 \end{bmatrix}$$

• Exercise 5. (CS 4.2-4.3):

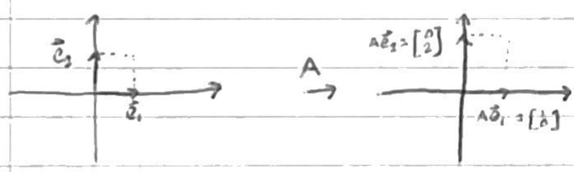
Consider the following 2x2 matrices:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},\quad B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},\quad C = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix},\quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

In the following subexercises, interpret these matrices as the canonical representation of a linear transformation in $\mathbb{R}^2$.

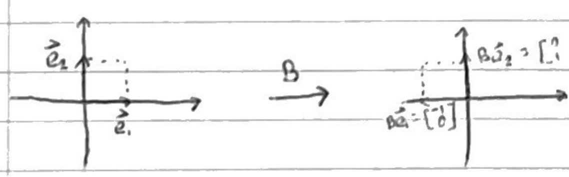
(a.) Describe in words how each of these matrices transforms the unit square in $\mathbb{R}^2$. A picture is helpful to include, but I am specifically looking for a few words that describe geometrically what has happened to the unit square, similar to the course notes.

(i) $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$



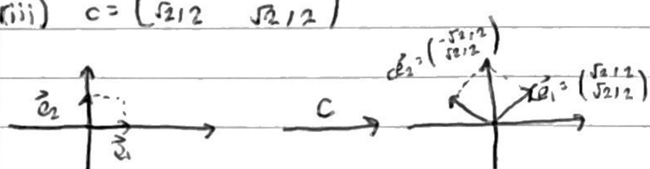
A stretches by a factor of 2 in the y-direction, while keeping the x-direction the same. In particular, A stretches $\vec{e}_2$ by 2 and keeps $\vec{e}_1$ the same.

(ii) $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$



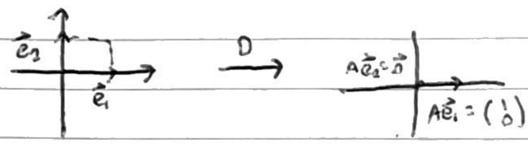
B reflects vectors across the y-axis. In particular, B changes the sign of $\vec{e}_1$ and keeps $\vec{e}_2$ the same

(iii) $C = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$



C rotates any vector by 45° counter-clockwise. In particular, it rotates the unit square 45° counter-clockwise.

(iv) $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$



D projects vectors onto the x-direction. In particular, D takes the y-coordinate/ $\vec{e}_2$ to zero, and keeps $\vec{e}_1$ the same.

(b.) suppose I want a linear transformation that first reflects a vector about the y axis and then stretches its y coordinate by two. using the matrices above, what is the canonical matrix representation of this linear transformation?

$$\begin{aligned} E &= (\text{stretch } y \text{ by } 2) * (\text{reflect about } y\text{-axis}) \\ &= A * B \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} * \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \end{aligned}$$

(c.) Suppose I want a linear transformation that first rotates a vector $\frac{\pi}{4}$ radians counter-clockwise and then reflects a vector about the y axis. using the matrices above, what is the canonical matrix representation of this linear transformation?

$$\begin{aligned} F &= (\text{reflect about } y\text{-axis}) * (\text{rotate by } \tfrac{\pi}{4} \text{ counter-clockwise}) \\ &= B * C \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} * \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \\ &= \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \end{aligned}$$

(d.) suppose I want a linear transformation that first projects a vector onto the x-axis and then rotates that vector $\pi/4$ radians counter-clockwise. Using the matrices above, what is the canonical matrix representation of this linear transformation.

$$\begin{aligned} G &= (\text{rotate by } \tfrac{\pi}{4} \text{ counter clockwise}) * (\text{project onto } x\text{-axis}) \\ &= C * D \end{aligned}$$

$$\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & 0 \\ \sqrt{2}/2 & 0 \end{bmatrix}$$

(e.) Show that B and C are orthogonal matrices. Remember two weeks ago we said that multiplying an orthogonal matrix to a vector does not change its length. Is this statement consistent with the geometric interpretation of B and C?

(i) Show B is orthogonal:

$B^T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow B^2 = BB = BB^T = B^TB = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$. Hence, B is orthogonal.

(ii) Show C is orthogonal:

$$C^T = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

$$C^TC = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$CC^T = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}\begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$C^TC = CC^T = I_2 \Rightarrow C$ is orthogonal.

(iii) Preservation of length and Geometry

Since B reflects vectors, and C rotates vectors, which simply move the unit square in space in a certain way, it is obvious to see that B and C don't change the length of vectors and hence multiplication by B and C is consistent with the properties of an orthogonal matrix.

(F.) One of these matrices above describe a linear transformation that is not invertible (meaning the linear transformation does not have an inverse to "undo" the transformation that has taken place). Which of these matrices is the canonical matrix representation of that linear transformation?

Remark: In general, any time the n dimensional unit square gets mapped to a polygonal shape that has a lower dimension, you know that the underlying linear transformation is not invertible.

Since D is singular, has 1 pivot, is not full rank, and $\det(D) = 1\cdot 0 - 0\cdot 0 = 0$, it follows that $D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible. In particular D turns the unit square into a line.

- Exercise 6 (CS4.4):

For the following matrices, find a basis for the range and the nullspace. Is the nullspace a point, a line, a plane, or all of $\mathbb{R}^3$? What are the dimensions of the range and nullspace? Show that these transformations satisfy the rank-nullity theorem.

(a.) $A = \begin{pmatrix} 4 & 1 & 5 \end{pmatrix}$

- STEP 1: range(A):

We start by row reducing A into echelon form which gives us

$$\begin{pmatrix} 4 & 1 & 5 \end{pmatrix}$$

the columns with pivot entries give our linearly independent vectors that span the range of A.

$$\text{range}(A) = \text{span}\{(4)\}$$

since there is one element in the basis for range(A), $\dim(\text{range}(A)) = 1$.

- STEP 2: Null(A):

To get a basis for null(A), we must find all x such that $Ax = 0$

$$[4 \quad 1 \quad 5 \mid 0]$$

x_2, x_3 are free variables

$$4x_1 + x_2 + 5x_3 = 0 \Rightarrow x_1 = -\tfrac{1}{4}x_2 - \tfrac{5}{4}x_3$$

thus

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4}x_2 - \frac{5}{4}x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1/4 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -5/4 \\ 0 \\ 1 \end{pmatrix}$$

Therefore,

$$\text{null}(A) = \text{span}\left\{ \begin{bmatrix} -1/4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5/4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

since there are two elements in the basis for null(A), $\dim(\text{null}(A)) = 2$.

- STEP 3: rank-nullity:

since, $\dim(\text{range}(A)) = 1$, $\dim(\text{null}(A)) = 2$ implies $\text{rank}(A) = 1$, $\text{nullity}(A) = 2$, and the number of columns $n = 3$, it follows that

$$\begin{aligned} \text{rank}(A) &+ \text{nullity}(A) = n \\ 1 &+ 2 = 3 \end{aligned}$$

hence the theorem holds / is satisfied.

(b.) $A = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & -2 \\ 3 & -1 & 1 \end{pmatrix}$

○ STEP 1: range(A):

We start by row-reducing A to echelon form which gives us

$$\begin{pmatrix} 2 & -1 & 1 \\ -1 & 1 & -2 \\ 3 & -1 & 1 \end{pmatrix} \overset{\frac{1}{2}R_1+R_2}{\sim} \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1/2 & -3/2 \\ 3 & -1 & 1 \end{pmatrix} \overset{-\frac{3}{2}R_2+R_3}{\sim} \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1/2 & -3/2 \\ 0 & 1/2 & -1/2 \end{pmatrix} \overset{-R_2+R_3}{\sim} \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1/2 & -3/2 \\ 0 & 0 & 1 \end{pmatrix}$$

The columns with pivot entries give our linearly independent vectors that span the range of A.

$$\text{range}(A) = \text{span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}$$

Since there are 3 elements in the basis for range(A), dim(range(A)) = 3.

○ STEP 2: Null (A):

Since the only x such that $Ax=0$ is $\vec{0}$, it turns out null(A) = $\{\vec{0}\}$. Since the only element in the nullspace is the zero vector, it turns out that dim(null(A)) = 0

○ STEP 3: rank-nullity:

Since dim(range(A)) = 3, dim(null(A)) = 0 implies rank(A) = 3, nullity(A) = 0, and the number of columns in A is $n=3$, it follows that

$$\begin{aligned} \text{rank}(A) + \text{nullity}(A) &= n \\ 3 \quad + \quad 0 \quad &= 3 \end{aligned}$$

thus the rank nullity theorem is satisfied.

(c.) $A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix}$

○ STEP 1: Range(A)

We start by row-reducing A to echelon form which gives us

$$\begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{bmatrix} \overset{2R_1+R_2}{\sim} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

The columns with pivot entries give our linearly independent vectors that span the range of A.

$$\text{range}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\}$$

since there is 1 element in the basis for range(A), dim(range(A)) = 1.

∘ STEP 2: Null(A):

To get a basis for null(A), we must find all x such that $Ax=0$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_2 and x_3 are free variables

$$x_1 - x_2 + 2x_3 = 0 \Rightarrow x_1 = x_2 - 2x_3$$

thus,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

therefore,

$$\text{null}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

since there are two elements in the basis for null(A), dim(null(A)) = 2.

∘ STEP 3: rank-nullity:

since dim(range(A)) = 1, dim(null(A)) = 2 implies rank(A) = 1, nullity(A) = 2, and the number of columns $n=3$, it follows that

$$\begin{array}{ccccc} \text{rank}(A) & + & \text{nullity}(A) & = & n \\ 1 & + & 2 & = & 3 \end{array}$$

hence the rank-nullity theorem is satisfied.

Multi-Step Problem

Typically we describe linear transformations $f: \mathbb{R}^n \to \mathbb{R}^n$ by their canonical matrix representations. In doing so we implicitly assume that vectors in $\mathbb{R}^n$ are expressed in terms of the canonical basis vectors, but you and I both know that there are many other bases we could choose for $\mathbb{R}^n$. If we choose to express our vectors with respect to a different basis, then the entries of the linear transformation change.

As you will see in this Multi-step problem, if a linear transformation is described by the canonical matrix representation A, then that same linear transformation can be equally well-described with respect to a new basis $\{b_1, \dots, b_n\}$ of $\mathbb{R}^n$ by the matrix

$$\tilde{A} = B^{-1}AB$$

where $B = (b_1 | \cdots | b_n)$. This formula is called the change-of-base formula.

To see why the change-of-base formula is true, you will wath the 3Blue1Brown YouTube vide "change of basis | chapter 13, Essence of linear algebra," which can be found at

https://youtube.com/watch?v=P2LTAUO1TdA. (click to access video)

As some of you may know, 3Blue1Brown is one of the most insightful and charismatic mathematicians on YouTube; you are in for a treat. As you watch his video, answer the following.

(a.) When associating the vector $(3\ \ 2)^T$ to an arrow in $\mathbb{R}^2$, what have we implicitly assumed?

We have implicitly assumed that $(3\ \ 2)^T$ is scaling the standard basis vectors $\hat{\imath}$ and $\hat{\jmath}$. In particular, $\hat{\imath}$ is of unit length pointing to the right, and $\hat{\jmath}$ is of unit length pointing upward.

(b.) What is a coordinate system?

A coordinate system is anyway to translate between vectors and sets of numbers.

(c.) What vector would Jennifer's coordinate system associate to the vector $(3 \ \ 2)^T$?

Jennifer uses the basis $\{\vec{b}_1, \vec{b}_2\}$, thus she would describe $(3 \ \ 2)^T$ with respect to the canonical basis as $(5/3 \ \ 1/3)^T$ with respect to her basis.

(d.) In Jennifer's coordinate system, what is the meaing of the first component of her vector, the second component?

Her first coordinate scales $\vec{b}_1$, her second coordinate scales $\vec{b}_2$, and the results are added up.

(e.) Expressed in our canonical coordinate system, what vectors does Jennifer use to define her basis? In Jennifers coordinate system, how are these vectors expressed?

Jennifers basis in canonical coordinates are

$$\vec{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

from her perspective, and in her system

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

they are what define the meaning of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in her coordinate system.

(f.) Space has no grid, true or false

True. A grid is a construct, a way to visualize our coordinate system. It depends on our choice of basis. Hence, space itself has no grid.

(g.) The vector $(-1 \ \ 2)^T$ expressed in Jennifer's coordinates is what vector expressed in our canonical coordinates?

$\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ in Jennifers coordinates is

$$-1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$$

expressed in our canonical coordinates.

(h.) What is the matrix that transforms a vector expressed in Jennifer's coordinates to that same vector expressed in our canonical coordinates?

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

(i.) What is the matrix that transforms a vector expressed in our canonical coordinates to that same vector expressed in Jennifer's coordinates?

$$A^{-1} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}$$

(j) The canonical representation of a 90° counter-clockwise rotation of $\mathbb{R}^2$ is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. How does Jennifer represent this linear transformation in her coordinate system?

$$\tilde{A} = B^{-1} A B$$

$$= \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & -2/3 \\ 5/3 & -1/3 \end{bmatrix}$$