

Jaiden Atterbury Final Exam student number: 2028502 date: 08-18-23

- 1.) Answer the following true-false questions. Answer the questions carefully. You do not need to justify your answers.

a.) Given $A \in \mathbb{R}^{m \times n}$, the zero vector in \mathbb{R}^n is always in $\text{range}(A)$.

This statement is False.

b.) There exists $A \in \mathbb{R}^{3 \times 3}$ with eigenvalues $\lambda_1, \lambda_2 = -2$ and $\lambda_3 = 6$.

This statement is False.

c.) If $A \in \mathbb{R}^{n \times n}$, then $\text{rank}(A) = \text{rank}(A^2)$.

This statement is False.

d.) If the j^{th} column of $A \in \mathbb{R}^{n \times n}$ is the j^{th} canonical basis vector, then $\lambda=1$ is an eigenvalue of A .

This statement is True.

e.) If $A \in \mathbb{R}^{n \times n}$ has a repeated eigenvalue, then A is not diagonalizable.

This statement is False.

f.) If $A \in \mathbb{R}^{m \times n}$ is full rank, then the eigenvalues of $A^T A$ and $A A^T$ are positive real numbers.

This statement is True.

g.) If $A \in \mathbb{R}^{m \times n}$ has K non-zero singular values, then $\text{rank}(A)=K$

This statement is True.

h) If $A \in \mathbb{R}^{n \times n}$, the singular values of $\exp(A)$ are the exponentials of the singular values of A .

This statement is True.

i) The singular values of a symmetric matrix are its eigenvalues.

This statement is False.

j) Let $A \in \mathbb{R}^{m \times m}$ and $A = U\Sigma V^T$ be a SVD of A . If $\text{rank}(A) = k$, then the first k columns of V are an orthonormal basis for $\text{null}(A)$.

This statement is False.

k.) Let $u \neq 0 \in \mathbb{R}^m$ and $v \neq \mathbb{R}^n$. Then $\text{rank}(uv^T) = 1$.

This statement is True.

l) The singular values of an orthogonal matrix are always equal to 1.

This statement is True.

m) Let $A \in \mathbb{R}^{m \times n}$ and $A = U\Sigma V^T$ be a SVD of A . If $\text{rank}(A) = k$, then the last k columns of U are an orthonormal basis for the left nullspace of A .

This statement is False.

n.) Suppose σ is a singular value of $A \in \mathbb{R}^{m \times n}$. Then, $c\sigma$ is a singular value of $cA \in \mathbb{R}^{m \times n}$ for any non-zero $c \in \mathbb{R}$.

This statement is True.

0) If σ is a singular value of an invertible matrix, $A \in \mathbb{R}^{n \times n}$, then
 $1/\sigma$ is a singular value of A^{-1} .

This statement is True.

2. Consider the following data

X	Y
0	1
1	2
2	-1
3	2

We want to fit this data to the polynomial $y = a_1 x^4 + a_2 x^3 + a_3 x^2 + a_4 x + a_5$.

You may use MATLAB for the calculations. You do not need to submit your code.

- a.) Determine a linear system for the coefficients, a_j , write it in matrix-vector form.

Let $y = f(x)$, then

$$\begin{cases} f(0) = 1 \\ f(1) = 2 \\ f(2) = -1 \\ f(3) = 2 \end{cases}$$

which can be written as the linear system

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 81 & 27 & 9 & 3 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \end{pmatrix}$$

$A \quad \vec{x} \quad \vec{b}$

- b.) since the system is underdetermined, we choose to minimize $\|Ax-b\|_2^2 + \|x\|_2^2$. Write down the linear system that solves this least-squares problem. You may use MATLAB to compute the matrix multiplications, but write out the matrices explicitly.

If $K > n$, minimizing $\|Ax-b\|_2^2 + \|x\|_2^2$ leads to the least-squares problem

$$\begin{pmatrix} A_{n \times K} \\ I_{K \times K} \end{pmatrix} \vec{x} = \begin{pmatrix} \vec{b}_{n \times 1} \\ \vec{0}_{K \times 1} \end{pmatrix}$$

which in our case:

$$\begin{pmatrix} A_{4 \times 5} \\ I_{5 \times 5} \end{pmatrix} \vec{x} = \begin{pmatrix} \vec{b}_{4 \times 1} \\ \vec{0}_{5 \times 1} \end{pmatrix}$$

which equals

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 81 & 27 & 9 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We will compute the least squares solution using the normal equation

$$A^T A \vec{x} = A^T \vec{b}$$

using MATLAB for matrix multiplication the corresponding linear system is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 16 & 8 & 4 & 2 & 1 \\ 81 & 27 & 9 & 3 & 1 \\ 0 & 1 & 8 & 27 & 0 \\ 0 & 1 & 9 & 27 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \vec{x} = \begin{pmatrix} 0 & 16 & 81 & 100 & 0 \\ 0 & 1 & 8 & 27 & 0 \\ 0 & 1 & 9 & 27 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 6814 & 2316 & 794 & 276 & 48 \\ 2316 & 795 & 276 & 98 & 36 \\ 794 & 276 & 99 & 36 & 19 \\ 276 & 98 & 36 & 15 & 6 \\ 48 & 36 & 14 & 6 & 5 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 148 \\ 48 \\ 16 \\ 6 \\ 4 \end{pmatrix}$$

C.) Solve the linear system from part (b). You may use MATLAB, write your answers out to 6 decimal places.

Using the backslash command in MATLAB, the least-squares solution from part (b) is:

$$\vec{x} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} 23.0541 \\ -6.648091 \\ -0.153946 \\ 0.404280 \\ 0.893556 \end{pmatrix}$$

d.) Solve the least-squares problem using the pseudoinverse. The MATLAB code for the pseudoinverse of a matrix A is $\text{pinv}(A)$. Write your answers out to 6 decimal places.

In MATLAB, using $A^+ = \text{pinv}(A)$ and solving $x^* = A^+ b$, where A^+ is the pseudo-inverse of A , we minimize $\|Ax - b\|_2^2$ and obtain the least squares solution

$$\hat{x}^* = \begin{pmatrix} 0.644330 \\ -2.199313 \\ 0.087629 \\ 2.467354 \\ 1.000000 \end{pmatrix}$$

e.) In MATLAB, make a scatter plot of the original data (code: `scatter(x, y, 'ro')`). Overlay this (code: `hold on;`) with your two solutions from (c) and (d). Restrict to the range $-1 \leq x \leq 4$ and $-2 \leq y \leq 3$ (code: `axis([xmin xmax ymin ymax])`).

See plot attached to the end of the problem.

f.) It should be obvious from your plot that the residual $\|Ax - b\|_2$ is smaller for your answer in (d) than in (c). What are the two-norms $\|x\|_2$. Answer to 3 decimals. Which is smaller?

Computing these norms in MATLAB we see:

$$\|\hat{x}\|_2 = 1.208 \quad (\text{norm for answer from (c)})$$

whereas

$$\|\hat{x}^*\|_2 = 3.514 \quad (\text{norm for answer from (d)})$$

Thus, the normal least squares solution from (c) has the smaller 2-norm, but has a larger residual.

③ Given $A \in \mathbb{R}^{m \times n}$, the following code computes the product of all of the elements of A above the diagonal.

% A is a given $m \times n$ matrix.

[m, n] = size(A);

Prod = 1;

for i = 1:m

 for j = i:n

 Prod = Prod * A(i, j);

 end

end

Prod

④ What is the total flop count of the code

i) inner for loop:

flops: 1

runs: $n-i+1$

total: $(n-i+1)$

ii) outer for loop:

flops: $n-i+1$

runs: $m-1+1 = m$

$$\begin{aligned}\text{total: } \sum_{i=1}^m (n-i+1) &= \sum_{i=1}^m n+1 - \sum_{i=1}^m i = m(n+1) - \frac{m(m+1)}{2} = mn+m - \frac{m^2+m}{2} \\ &= mn+m - \frac{m^2}{2} - \frac{m}{2} = -\frac{m^2}{2} + (n+1-\frac{1}{2})m = -\frac{m^2}{2} + (n+\frac{1}{2})m\end{aligned}$$

hence the total flop count of the code is

$$\left\{ \frac{-m^2}{2} + \left(n + \frac{1}{2}\right)m \right\}$$

- (b) If $m=n$, what is the asymptotic flop count?

$$\text{If } m=n, \frac{-m^2}{2} + \left(n + \frac{1}{2}\right)m \Rightarrow \frac{-n^2}{2} + \left(n + \frac{1}{2}\right)n = \frac{-n^2}{2} + n^2 + \frac{n}{2} = \frac{n^2}{2} + \frac{n}{2}.$$

Since $\lim_{n \rightarrow \infty} \frac{\frac{1}{2}n^2 + \frac{1}{2}n}{\frac{1}{2}n^2} = 1$, it follows that the asymptotic flop count is $\left(\frac{1}{2}n^2\right)$.

i.) consider the following matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

Note that the rows of A are orthogonal.

a.) Compute a reduced SVD of A^T . Recall a reduced SVD $A^T = \tilde{U}\tilde{\Sigma}V^T$ has $\tilde{U} \in \mathbb{R}^{4 \times 3}$ has orthogonal columns, $\tilde{\Sigma} \in \mathbb{R}^{3 \times 3}$ is diagonal with non-negative decreasing entries and $V^T \in \mathbb{R}^{3 \times 3}$ is an orthogonal matrix.

Notice that

$$A^T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

We will start by finding the singular values, which are the square roots of the eigenvalues of AAT .

$$AAT = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

since AAT is diagonal, we can see that $\lambda_{1,2,3} = 3 \Rightarrow \sigma_{1,2,3} = \sqrt{3}$.

since the rows of A are orthogonal, it follows that the columns of A^T are orthogonal. we can normalize these columns to find

$$A^T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} = \tilde{U}\tilde{\Sigma}V^T$$

$$= \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & 1 & 1 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the reduced SVD of A^T .

b.) Guess $x_4 = e_4$. Use Gram-Schmidt to find a fourth orthonormal vector u_4 .

If we guess $x_4 = e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, running Gram-Schmidt with the columns of \tilde{U} being our u_1, u_2, u_3 , we see

i.) Find $x_{4\perp}$:

$$x_{4\perp} = x_4 - (x_4 \cdot u_3) u_3 - (x_4 \cdot u_2) u_2 - (x_4 \cdot u_1) u_1$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \right) \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} - \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 0 \\ 0 \end{pmatrix} \right) \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 0 \\ 0 \end{pmatrix} - \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix} \right) \begin{pmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{3}} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} - \frac{1}{\sqrt{3}} \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 0 \\ 0 \end{pmatrix} - 0 \begin{pmatrix} 0 \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \\ 0 \end{pmatrix} - \begin{pmatrix} 1/3 \\ -1/3 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/3 & -1/3 \\ 1/3 & 1/3 \\ 0 & -1/3 \\ 0 & 0 \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1/3 \\ 0 \\ -1/3 \\ -1/3 \end{pmatrix}$$

ii.) Find u_4

$$u_4 = \frac{x_{4\perp}}{\|x_{4\perp}\|_2} = \frac{1}{\sqrt{\frac{1}{3}^2 + 0^2 + (-\frac{1}{3})^2 + (-\frac{1}{3})^2}} \begin{pmatrix} 1/3 \\ 0 \\ -1/3 \\ -1/3 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1/3 \\ 0 \\ -1/3 \\ -1/3 \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{3} \\ 0 \\ -1/\sqrt{3} \\ -1/\sqrt{3} \end{pmatrix}$$

c.) Compute the full SVD of A.

To get the full SVD of A^T , we need to find

$$A^T = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{3} & 1/\sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Where u_4 is orthonormal to the first three columns of U. We found u_4 in part (b), thus the full SVD of A^T is

$$A^T = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 0 & 1/\sqrt{3} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{3} & 0 & -1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To find the full SVD of A, we simply transpose the full SVD of A^T

$$A = U \Sigma V^T = (A^T)^T = (U \Sigma V^T)^T = V \Sigma^T U^T$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 0 & 1/\sqrt{3} \\ 1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \\ 1/\sqrt{3} & 0 & -1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}$$

Which is the full SVD of A.

5. Consider the following matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 4 & 2 & 7 \\ 1 & 3 & 0 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix}$$

(a) Compute the PLU factorization of A. Use partial pivoting to determine whether to exchange rows.

Since 2 has the highest magnitude in column 1, interchange row 1 and 4:

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 4 & 2 & 7 \\ 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Add $-1/2$ times row 1 to row 2:

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 7/2 & 2 & 13/2 \\ 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Add $-1/2$ times row 1 to row 3:

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 7/2 & 2 & 13/2 \\ 0 & 5/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Add $-\frac{5}{7}$ times row 2 to row 3:

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 7/2 & 2 & 13/2 \\ 0 & 0 & -10/7 & -58/14 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Add $\frac{3}{10}$ times row 3 to row 4:

$$U = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 7/2 & 2 & 13/2 \\ 0 & 0 & -10/7 & -58/14 \\ 0 & 0 & 0 & -58/20 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Thus the PLU factorization of A is:

$$PA = LU \Rightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 4 & 2 & 7 \\ 1 & 3 & 0 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 7/2 & 2 & 13/2 \\ 0 & 0 & -10/7 & -58/14 \\ 0 & 0 & 0 & -58/20 \end{pmatrix}$$

(b) Based on your PLU factorization, is A invertible? This should be a one sentence answer.

Since the PLU decomposition has all non-zero elements on the diagonal of U, it is nonsingular. Lastly, since A is nonsingular that implies it is invertible as well.

(c) compute the determinant of A.

Using the PLU factorization of A, $\det(A) = (-1)^k u_{11} \cdots u_{nn}$, hence in our case:

$$\begin{aligned}\det(A) &= (-1)^1 \cdot 2 \cdot \frac{7}{2} \cdot -\frac{10}{7} \cdot -\frac{58}{20} \\ &= -29\end{aligned}$$

(d) compute the volume of the parallelepiped to which A maps the 4D unit hyper-cube.

The volume of this parallelepiped is equal to the magnitude of the determinant of A, which is

$$\det(A) = -29 \Rightarrow |\det(A)| = |-29| \Rightarrow \boxed{\text{volume} = 29}$$