Jaiden Atterbury Homework 8

- **Exercise 1. (CS 4.7):**

Consider the matrix $A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$.

(a.) What is the characteristic polynomial of A?

$$\det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & -3 \\ -1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 3 = 4 - 4\lambda + \lambda^2 = \boxed{\lambda^2 - 4\lambda + 1}$$

(b.) Is A diagonalizable? Provide a brief justification.

i.) Eigenvalues of A:

$$(2-\lambda)^2 - 3 = 0 \Rightarrow (2-\lambda)^2 = 3 \Rightarrow 2 - \lambda = \pm\sqrt{3} \Rightarrow \lambda = 2 \mp \sqrt{3} \Rightarrow \lambda_1 = 2+\sqrt{3},\ \lambda_2 = 2 - \sqrt{3}.$$

Since there are no repeated eigenvalues, it is guaranteed that the algebraic multiplicity of each eigenvalue is equal to the geometric multiplicity of each eigenvalue $(AM(\lambda) = GM(\lambda) = 1)$ and hence $\boxed{\text{A is diagonalizable.}}$

(c.) Let $P(\lambda)$ denote the characteristic polynomial of A. Show that $P(A) = 0_{2\times 2}$, the 2x2 zero matrix.

Hint: The constant term should be multiplied by the identity matrix.

As found in part a., $P(\lambda) = \lambda^2 - 4\lambda + 1$, hence $P(A)$ is:

$$\begin{aligned} P(A) &= A^2 - 4A + I \\ &= \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 7 & -12 \\ -4 & 7 \end{bmatrix} + \begin{bmatrix} -8 & 12 \\ 4 & -8 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \boxed{0_{2\times 2}} \end{aligned}$$

- **Exercise 2. (CS 4.7):**

(a.) Prove that similar (square) matrices have the same eigenvalues.

Hint: Show that if $A = S^{-1}BS$, for some invertible matrix S, then A and B have the same characteristic polynomial.

Let $A = S^{-1}BS$, for some invertible matrix S, then

$$\det(A-\lambda I) = \det(S^{-1}BS - \lambda I) = \det(S^{-1}BS - \lambda S^{-1}S) \quad (\text{since } S^{-1}S = I_n)$$
$$= \det(S^{-1}(BS - \lambda S))$$
$$= \det(S^{-1}(B - \lambda I)S)$$
$$= \det(S^{-1})\det(B-\lambda I)\det(S) \quad (\text{since } \det(AB) = \det(A)\det(B))$$
$$= \frac{1}{\det(S)}\det(B-\lambda I)\det(S) \quad \left(\text{since } \det(S^{-1}) = \frac{1}{\det(S)}\right)$$
$$= \det(B-\lambda I) \quad \left(\text{since } \frac{1}{\det(S)}\det(S) = 1\right)$$

Since $\det(A-\lambda I) = \det(B-\lambda I)$, A and B have the same eigenvalues.

(b) How are the eigenvectors of similar square matrices related?

Let $A = S^{-1}BS$, for some invertible matrix S, then

$$A\vec{v} = \lambda\vec{v} \Rightarrow S^{-1}BS\vec{v} = \lambda\vec{v} \Rightarrow BS\vec{v} = \lambda S\vec{v}$$

thus if $\vec{v}$ is an eigenvector of A with eigenvalue λ, $S\vec{v}$ is an eigenvector of B with eigenvalue λ.

- Exercise 3. (cs4.6, cs5.1):

(a) Prove that for any $A \in \mathbb{R}^{n\times n}$,

$$\det(A) = \prod_{i=1}^{n} \lambda_i = \lambda_1 \cdots \lambda_n$$

Hint: writing out the characteristic polynomial, in both factored and unfactored form, we have

$$\det(A-\lambda I) = (-1)^n\lambda^n + \cdots + a_1\lambda + a_0 = (-1)^n(\lambda-\lambda_1)\cdots(\lambda-\lambda_n)$$

For both cases, what is the constant term, a_0?

As shown above, $\det(A-\lambda I) = (-1)^n(\lambda-\lambda_1)\cdots(\lambda-\lambda_n)$ thus letting $\lambda = 0$ since it is a variable, we obtain

$$\det(A) = (-1)^n(-\lambda_1)(-\lambda_2)\cdots(-\lambda_n)$$
$$= (-1)^n(-1)^n(\lambda_1)(\lambda_2)\cdots(\lambda_n)$$
$$= (-1)^{2n}\lambda_1\lambda_2\cdots\lambda_n$$

$= \lambda_1 \lambda_2 \cdots \lambda_n$ (since $(-1)^{2n} = 1$ as $2n$ is even)

$= \boxed{\prod_{i=1}^{n} \lambda_i}$

(b.) Prove that A is non-invertible if and only if $\lambda = 0$ is an eigenvalue.

i.) If A is non-invertible, then $\lambda = 0$ is an eigenvalue.

If A is non-invertible, then A has linearly independent columns, thus $A\vec{v} = \vec{0}$ has non-trivial solutions, hence $A\vec{v} = 0\vec{v}$ for some $\vec{v}$, and therefore 0 is an eigenvalue of A

ii.) If $\lambda = 0$ is an eigenvalue, then A is non-invertible.

Let $\lambda = 0$ be an eigenvalue of A, then it must satisfy $\det(A - \lambda I) = 0$. Plugging in $\lambda = 0$ we obtain $\det(A) = 0$. However, since $\det(A) = 0$, by Theorem 2, A is non-invertible.

(c.) Prove that if A is invertible and λ is an eigenvalue of A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}. What is the corresponding eigenvector?

Let A be invertible with eigenvalue λ, then

$A\vec{v} = \lambda\vec{v} \Rightarrow \vec{v} = \lambda A^{-1}\vec{v} \Rightarrow \frac{1}{\lambda}\vec{v} = A^{-1}\vec{v}$.

Thus, as seen above A has eigenvector $\vec{v}$ with eigenvalue λ and A^{-1} has eigenvector $\vec{v}$ with eigenvalue $\frac{1}{\lambda}$.

(d.) Prove that if A is symmetric, positive-definite, then A is invertible and A^{-1} is symmetric, positive definite.

Hint: use: $(A^{-1})^T = (A^T)^{-1}$.

i.) A is invertible:

Since A is positive-definite, all of its eigenvalues are positive, and hence $\lambda = 0$ is not an eigenvalue of A, which implies A is invertible.

ii) A^{-1} is symmetric, positive-definite:

▫ symmetric:

$A = A^T \Rightarrow A^{-1} = (A^T)^{-1} \Rightarrow A^{-1} = (A^{-1})^T$ (since $(A^{-1})^T = (A^T)^{-1}$). Hence, since $A^{-1} = (A^{-1})^T$, A^{-1} is symmetric.

▫ Positive-definite:

Since A is positive definite, it has all positive eigenvalues. Furthermore, since the eigenvalues of A^{-1} are $\frac{1}{\lambda}$, where λ is an eigenvalue of A, it follows that all of the eigenvalues of A^{-1} are also positive, and hence A^{-1} is positive-definite.

• Exercise 4. (cs4.7):

Solve the 2x2 system of ordinary differential equations
$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$
with initial condition $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

As found in lecture 20, the solution to this IVP is:
$$\vec{x}(t) = \exp(At)x_0$$

As found in Homework 7. Exercise 6 part:
$$A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} = Q \Lambda Q^T = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

We must now find $\exp(At)$:
$$At = Q(\Lambda t)Q^T \quad \Rightarrow \quad \exp(At) = Q\begin{pmatrix} e^{5t} & 0 \\ 0 & e^{-t} \end{pmatrix}Q^T$$

thus we can see:
$$\exp(At) = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} e^{5t} & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}}e^{5t} & \frac{1}{\sqrt{2}}e^{5t} \\ \frac{-1}{\sqrt{2}}e^{-t} & \frac{1}{\sqrt{2}}e^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}e^{5t} + \frac{1}{2}e^{-t} & \frac{1}{2}e^{5t} - \frac{1}{2}e^{-t} \\ \frac{1}{2}e^{5t} - \frac{1}{2}e^{-t} & \frac{1}{2}e^{5t} + \frac{1}{2}e^{-t} \end{bmatrix}$$

Our solution is:

$$\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \exp(At)x_0 = \begin{bmatrix} 1/2e^{5t} + 1/2e^{-t} & 1/2e^{5t} - 1/2e^{-t} \\ 1/2e^{5t} - 1/2e^{-t} & 1/2e^{5t} + 1/2e^{-t} \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3e^{5t} - e^{-t} \\ 3e^{5t} + e^{-t} \end{bmatrix}$$

or

$x(t) = 3e^{5t} - e^{-t}$ and $y(t) = 3e^{5t} + e^{-t}$.

Exercise 5. (cs5.2):

Obtain the singular value decomposition of $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$.

1.) Find eigenvalues of A^TA:

$$A^TA = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 1 = 0 \Rightarrow (2-\lambda)^2 = 1 \Rightarrow 2-\lambda = \pm 1 \Rightarrow \lambda = 2 \mp 1 \Rightarrow$$

$\lambda_1 = 3$, $\lambda_2 = 1$.

2.) Find singular values and form Σ.

$\sigma_1 = \sqrt{\lambda_1} = \sqrt{3}$, $\sigma_2 = \sqrt{\lambda_2} = \sqrt{1} = 1$

$$\Sigma = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

3.) Find eigenvectors of A^TA, normalize them, and form V^T

- $\lambda_1 = 3$

$$\text{null}(A - 3I) = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \overset{+R_1+R_2}{\sim} \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow v_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

- $\lambda_2 = 1$

$$\text{null}(A - I) = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \overset{-R_1+R_2}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow v_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \Rightarrow V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

4.) Find U matrix:

i.) Find $\vec{u}_j = \frac{1}{\sigma_j} A v_j$ for $1 \le j \le 2$:

$$\vec{u}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 2/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

ii.) Eigenspace for $\lambda = 0$ of AA^T:

$$AA^T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\text{null}(A) = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 \end{array}\right] \underset{\sim}{-R_1+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \underset{\sim}{-R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x_3 = x_3 \\ x_2 = -x_3 \\ x_1 = -x_3 \end{array}$$

$$\vec{u}_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \Rightarrow \vec{u}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

iii) Form U

$$U = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{bmatrix}$$

Thus $A = U \Sigma V^T = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$.

Multi-Step Problem

Consider the matrix $A = \begin{pmatrix} 7 & -6 & 3 \\ 9 & -8 & 3 \\ 9 & -6 & 1 \end{pmatrix}$

(a.) compute the exact eigenvalues/eigenvectors of A.

Hint: $\lambda = 4$ is an eigenvalue.

1.) Eigenvalues:

$$\det(A-\lambda I) = \det\begin{pmatrix} 7-\lambda & -6 & 3 \\ 9 & -8-\lambda & 3 \\ 9 & -6 & 1-\lambda \end{pmatrix} = 3\det\begin{pmatrix} 9 & -8-\lambda \\ 9 & -6 \end{pmatrix} - 3\det\begin{pmatrix} 7-\lambda & -6 \\ 9 & -6 \end{pmatrix} + (1-\lambda)\det\begin{pmatrix} 7-\lambda & -6 \\ 9 & -8-\lambda \end{pmatrix} = 0$$

$$= 3(-54 - 9(-8-\lambda)) - 3(-6(7-\lambda)+54) + (1-\lambda)((7-\lambda)(-8-\lambda)+54) = 0$$

$$= 3(-54+72+9\lambda) - 3(-42+6\lambda+54) + (1-\lambda)(-56-7\lambda+8\lambda+\lambda^2+54) = 0$$

$$= 3(9\lambda+18) - 3(6\lambda+12) + (1-\lambda)(\lambda^2+\lambda-2) = 0$$

$$= 27\lambda + 54 - 18\lambda - 36 + \lambda^2 + \lambda - 2 - \lambda^3 - \lambda^2 + 2\lambda = 0$$

$$= -\lambda^3 + 12\lambda + 16 = 0$$

Since $\lambda = 4$ is an eigenvalue we can divide $\lambda - 4$ out of $-\lambda^3 + 12\lambda + 16$.

$$\begin{array}{r l}
 & -\lambda^2 - 4\lambda - 4 \\
\lambda - 4 & \overline{)\,-\lambda^3 + 12\lambda + 16} \\
 & -(-\lambda^3 + 4\lambda^2) \\
\hline
 & -4\lambda^2 + 12\lambda + 16 \\
 & -(-4\lambda^2 + 16\lambda) \\
\hline
 & -4\lambda + 16 \\
 & -(-4\lambda + 16) \\
\hline
 & 0
\end{array}$$

Thus $\lambda_1 = 4$ and $-\lambda^2 - 4\lambda - 4 = 0 \Rightarrow (\lambda+2)^2 = 0 \Rightarrow \lambda_{2,3} = -2$.

2.) Eigenvectors

- $\lambda_1 = 4$

$$\text{null}(A-4I) = \left(\begin{array}{ccc|c} 3 & -6 & 3 & 0 \\ 9 & -12 & 3 & 0 \\ 9 & -6 & -3 & 0 \end{array}\right) \underset{\sim}{\overset{-3R_1+R_2,\ -3R_1+R_3}{}} \left(\begin{array}{ccc|c} 3 & -6 & 3 & 0 \\ 0 & 6 & -6 & 0 \\ 0 & 12 & -12 & 0 \end{array}\right) \underset{\sim}{\overset{-2R_2+R_3}{}} \left(\begin{array}{ccc|c} 3 & -6 & 3 & 0 \\ 0 & 6 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

- $\lambda_{2,3} = -2$

$$\text{null}(A+2I) = \left(\begin{array}{ccc|c} 9 & -6 & 3 & 0 \\ 9 & -6 & 3 & 0 \\ 9 & -6 & 3 & 0 \end{array}\right) \overset{-R_1+R_2,\ -R_1+R_3}{} \left(\begin{array}{ccc|c} 9 & -6 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow x_1 = \tfrac{2}{3}x_2 - \tfrac{1}{3}x_3 \Rightarrow v_2 = \begin{bmatrix} 2/3 \\ 1 \\ 0 \end{bmatrix},\ v_3 = \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix}.$$

(b.) Using an initial $b_0 = (1\ 2\ 2)^T$, code up the power iteration method in MATLAB.

(i) To which eigenvalue do you expect the code to converge to? To which eigenvector will b_K converge to?

The power method converges to the largest in magnitude eigenvalue, which is $\lambda_1 = 4$. Since b_K is not orthogonal to v_1, it will converge to the normalized version of v_1, the eigenvector corresponding to $\lambda_1 = 4$, namely b_K will converge to $(1/\sqrt{3}\ \ 1/\sqrt{3}\ \ 1/\sqrt{3})^T$.

(ii) To ten decimal places, what does the algorithm predict is the eigenvalue and eigenvector of A after 5 iterations? What about 10 iterations?

After 5 iterations: $\mu_K = 4.0653802497$

$$b_K = \begin{bmatrix} 0.6029941687 \\ 0.5640913191 \\ 0.5640913191 \end{bmatrix}$$

After 10 iterations: $\mu_K = 3.9863541559$

$$b_K = \begin{bmatrix} 0.5765499768 \\ 0.5777250499 \\ 0.5777250499 \end{bmatrix}$$

(iii) How many iterations are needed to guarantee that the eigenvalue is within 10^{-6} error of the true eigenvalue? What about within 10^{-9} error? Measuring the error of the eigenvector as:

```
error_b = max(abs(bK-v));
```

where v is the eigenvector found in part (i), what is the error in each case:

within 10^{-6} error: iterations = 24

error_b $= 4.5883671418 \times 10^{-8}$

within 10^{-9} error: iterations = 34

error_b = $4.4808370229 \times 10^{-11}$

(c) Code up the QR eigenvalue algorithm in MATLAB.

(i) To ten decimal places, what does this algorithm predict are the eigenvalues of A after 5 iterations? What about after 10 iterations?

After 5 iterations: $\lambda_1 = 3.8570886904$

$\lambda_2 = -2.0150460923$

$\lambda_3 = -1.8420425982$

After 10 iterations: $\lambda_1 = 4.0045601894$

$\lambda_2 = -1.9995351824$

$\lambda_3 = -2.0050250069$

(ii) How many iterations are needed to guarantee that all three eigenvalues computed by the QR algorithm are within 10^{-6} error of the true eigenvalues? What about within 10^{-9} error?

Within 10^{-6} error: iterations = 19

Within 10^{-9} error: iterations = 29

Multi-Step Problem

Table of Contents

Setup:

```
% Format long to show adequate number of decimal points:
format long;

% Define the matrix we will be working with:
A = [7, -6, 3; 9, -8, 3; 9, -6, 1];
```

Part b:

Using an initial b_0 = (1 2 2)^T. Code up the power iteration method in MATLAB

```
% Subpart (ii): To ten decimal places, what does the algorithm predict is
% the eigenvalue and eigenvector of A after 5 iterations? What about 10
% iterations?

% After 5 iterations:

% Initialize b:
b = [1; 2; 2];
for i = 1:5
    b = A*b / norm(A*b);
end

% Output b and mu:
b
mu = transpose(b)*A*b

% After 10 iterations:

% Initialize b:
b = [1; 2; 2];
for i = 1:10
    b = A*b / norm(A*b);
end

% Output b and mu:
b
mu = transpose(b)*A*b

% Subpart (iii): How many iterations are needed to guarentee that the
```

```
% eigenvalue is within 10^-6 error of the true eigenvalue? What about
% within 10^-9 error? Measuring the error of the eigenvetor as error_b =
% max(abs(b_k-v)); where v is the eigenvector found in part (i), what is
% the error of the eigenvector in each case?

% Define the eigenvector for use in computing error_b:
v = [1/sqrt(3); 1/sqrt(3); 1/sqrt(3)];

% Iterations for eigenvaue within 10^-6 error:

% Initialize b, mu, and the iteration count:
b_0 = [1; 2; 2];
mu_0 = transpose(b_0/norm(b_0))*A*(b_0/norm(b_0));
iterations = 0;
while abs(4-mu_0) > 10^-6
    b_0 = (A*b_0) / norm(A*b_0);
    mu_0 = transpose(b_0)*A*b_0;
    iterations = iterations + 1;
end

% Display the iteration count and error_b:
iterations
error_b = max(abs(b_0-v))

% Iterations for eigenvaue within 10^-9 error:

% Initialize b, mu, and the iteration count:
b_0 = [1; 2; 2];
mu_0 = transpose(b_0/norm(b_0))*A*(b_0/norm(b_0));
iterations = 0;
while abs(4-mu_0) > 10^-9
    b_0 = (A*b_0) / norm(A*b_0);
    mu_0 = transpose(b_0)*A*b_0;
    iterations = iterations + 1;
end

% Display the iteration count and error_b:
iterations
error_b = max(abs(b_0-v))

% Output:


b =

   0.602994168679282
   0.564091319087070
   0.564091319087070


mu =

   4.465380249716234
```

```
b =

   0.576599976837906
   0.577725049963444
   0.577725049963443


mu =

   3.986354155887673


iterations =

    24


error_b =

     4.588367141789007e-08


iterations =

    34


error_b =

     4.480837922926639e-11
```

Part c:

```
% Code up the QR eigenvalue alogrithm in MATLAB.

% Subpart (i): To ten decimal places, what does this algorithm predict are
% the eigenvalues of A after 5 iterations. What about after 10 iterations?

% After 5 iterations:

% initialize A_0
A_0 = A;
for i = 1:5
    [Q,R] = qr(A_0);
    A_0 = inv(Q)*A_0*Q;
end

% Display the eigenvalues of A_0 (the diagonal entries of A_0):
A_0

% After 10 iterations:
```

```
% initialize A_0
A_0 = A;
for i = 1:10
    [Q,R] = qr(A_0);
    A_0 = inv(Q)*A_0*Q;
end

% Display the eigenvalues of A_0 (the diagonal entries of A_0):
A_0

% Subpart (ii): How many iterations are needed to guarentee that all three
% eigenvalues compute by the QR algorithm are within 10^-6 error of the
% true eigenvalues? What about within 10^-9 error?

% Within 10^-6 error:

% initialize A_0 and the iteration count:
A_0 = A;
iterations = 0;
while (abs(4-A_0(1,1))>10^-6) & (abs(-2-A_0(2,2))>10^-6) & (abs(-2-
A_0(3,3))>10^-6)
    [Q,R] = qr(A_0);
    A_0 = inv(Q)*A_0*Q;
    iterations = iterations + 1;
end

% Display the number of iterations:
iterations

% Within 10^-9 error:

% initialize A_0 and the iteration count:
A_0 = A;
iterations = 0;
while (abs(4-A_0(1,1))>10^-9) & (abs(-2-A_0(2,2))>10^-9) & (abs(-2-
A_0(3,3))>10^-9)
    [Q,R] = qr(A_0);
    A_0 = inv(Q)*A_0*Q;
    iterations = iterations + 1;
end

% Display the number of iterations:
iterations

% Output:


A_0 =

   3.857088690496707  -7.796413543530077  16.818909075458247
   0.011303440533959  -2.015046092269428   0.032458367736341
   0.055007759858410  -0.073221230994027  -1.842042598227277
```

```
A_0 =

   4.004560189352390  -8.024901017915026 -16.659717432391378
  -0.000347795627530  -1.999535182361842   0.000964962744339
   0.001811132574674  -0.002420520268554  -2.005025006990547


iterations =

    19


iterations =

    29
```

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