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Exercise 1. (CS 1.1):

A chemist must determine the coefficients x_j (for $1 \le j \le 7$) that balance the reaction:

$$x_1 PCH_4 + x_2 KMnO_4 + x_3 H_2SO_4 \rightarrow x_4 PCH_2O_2 + x_5 K_2SO_4 + x_6 MnSO_4 + x_7 H_2O.$$

In order to balance this reaction, there must be an equal number of each element on both sides of the reaction. Use this information to derive a linear system satisfied by the unknown coefficients x_j. What are the dimensions of this system?

We will start out by equating both sides of the reaction for each element:

1.) Phosphorus (P):

$x_1 = x_4 \Rightarrow x_1 - x_4 = 0$

2.) Carbon (C):

$x_1 = x_4 \Rightarrow x_1 - x_4 = 0$

3.) Hydrogen (H):

$4x_1 + 2x_3 = 2x_4 + 2x_7 \Rightarrow 4x_1 + 2x_3 - 2x_4 - 2x_7 = 0$

4.) Potassium (K):

$x_2 = 2x_5 \Rightarrow x_2 - 2x_5 = 0$

5.) Manganese (Mn):

$x_2 = x_6 \Rightarrow x_2 - x_6 = 0$

6.) Oxygen (O):

$4x_2 + 4x_3 = 2x_4 + 4x_5 + 4x_6 + x_7 \Rightarrow 4x_2 + 4x_3 - 2x_4 - 4x_5 - 4x_6 - x_7 = 0$

7.) Sulfur (S)

$x_3 = x_5 + x_6 \Rightarrow x_3 - x_5 - x_6 = 0$

Thus, all of these equations form the following linear system:

$$\begin{cases} x_1 - x_4 = 0 \\ x_1 - x_4 = 0 \\ 4x_1 + 2x_3 - 2x_4 - 2x_7 = 0 \\ x_2 - 2x_5 = 0 \\ x_2 - x_6 = 0 \\ 4x_2 + 4x_3 - 2x_4 - 4x_5 - 4x_6 - x_7 = 0 \\ x_3 - x_5 - x_6 = 0 \end{cases}$$

Since the preceding linear system has 7 equations (7 rows), corresponding to the 7 included elements, and 7 unknowns corresponding to $x_1, x_2, \dots, x_7$. The dimensions of the system is 7 rows by 7 columns (7×7).

- Exercise 2 (CS 1.2):

From Olver and Shakiban, complete Exercises 1.1.1(b) and (g)

Solve the following systems of linear equations by reducing to triangular form and then using Back substitution.

1.1.1 (b):

$$\begin{cases} 6u + v = 5 \\ 3u - 2v = 5 \end{cases}$$

Multiplying equation 1 by $-\frac{1}{2}$ and adding it to equation 2, we obtain the equivalent system

$$\begin{cases} 6u + v = 5 \\ \quad -\frac{5}{2}v = \frac{5}{2} \end{cases}$$

Hence $-\frac{5}{2}v = \frac{5}{2} \Rightarrow v = -1$, and $6u + v = 5 \Rightarrow u = \frac{5+1}{6} = 6/6 = 1$.

Thus $v = -1$, $u = 1$.

1.1.1 (g):

$$\begin{cases} 3x_1 + x_2 = 1 \\ x_1 + 3x_2 + x_3 = 1 \\ x_2 + 3x_3 + x_4 = 1 \\ x_3 + 3x_4 = 1 \end{cases}$$

To make calculations easier we will find the corresponding augmented matrix of the linear system and use row operation notation to notate row operations.

Augmented matrix:

$$\left[\begin{array}{cccc|c} 3 & 1 & 0 & 0 & 1 \\ 1 & 3 & 1 & 0 & 1 \\ 0 & 1 & 3 & 1 & 1 \\ 0 & 0 & 1 & 3 & 1 \end{array}\right] \underset{\sim}{-\tfrac{1}{3}R_1+R_2 \to R_2} \left[\begin{array}{cccc|c} 3 & 1 & 0 & 0 & 1 \\ 0 & 8/3 & 1 & 0 & 2/3 \\ 0 & 1 & 3 & 1 & 1 \\ 0 & 0 & 1 & 3 & 1 \end{array}\right] \underset{\sim}{-\tfrac{3}{8}R_2+R_3 \to R_3} \left[\begin{array}{cccc|c} 3 & 1 & 0 & 0 & 1 \\ 0 & 8/3 & 1 & 0 & 2/3 \\ 0 & 0 & 21/8 & 1 & 3/4 \\ 0 & 0 & 1 & 3 & 1 \end{array}\right]$$

$$\underset{\sim}{-\tfrac{8}{21}R_3+R_4 \to R_4} \left[\begin{array}{cccc|c} 3 & 1 & 0 & 0 & 1 \\ 0 & 8/3 & 1 & 0 & 2/3 \\ 0 & 0 & 21/8 & 1 & 3/4 \\ 0 & 0 & 0 & 71/21 & 5/7 \end{array}\right]$$

Hence, using back substitution we see that $\frac{71}{21}x_4 = \frac{5}{7} \Rightarrow x_4 = \frac{3}{11}$, $\frac{21}{8}x_3 + x_4 = \frac{3}{4} \Rightarrow x_3 = \frac{3/4 - 3/11}{21/8} = 2/11$, $\frac{8}{3}x_2 + x_3 = \frac{2}{3} \Rightarrow x_2 = \frac{2/3 - 2/11}{8/3} = 2/11$, and $3x_1 + x_2 = 1 \Rightarrow x_1 = \frac{1-2/11}{3} = \frac{3}{11}$. Thus the solution is $x_1 = \frac{3}{11}, x_2 = \frac{2}{11}, x_3 = \frac{2}{11}, x_4 = \frac{3}{11}$.

Exercise 3. (CS 1.2):

Consider the following linear systems:

a. $\begin{cases} 2x+y-z=4 \\ 4x-y+z=8 \\ x-y+z=2 \end{cases}$

b. $\begin{cases} 2x+y-z=4 \\ 4x-y+z=8 \\ x-y+z=0 \end{cases}$

c. $\begin{cases} 2x+y-z=4 \\ 4x-y+z=8 \\ x-2y+z=0 \end{cases}$

Which of these systems has no solution? (There is only one.) What are the solutions of the two remaining systems.

For all three systems, we will find the corresponding augmented matrix to the linear system, use row operation notation, and back substitution to solve the system.

a.) $\left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 4 & -1 & 1 & 8 \\ 1 & -1 & 1 & 2 \end{array}\right] \underset{-2R_1+R_2 \to R_2}{\sim} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 0 & -3 & 3 & 0 \\ 1 & -1 & 1 & 2 \end{array}\right] \underset{-\frac{1}{2}R_1+R_3 \to R_3}{\sim} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 0 & -3 & 3 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & 0 \end{array}\right]$

$\underset{-\frac{1}{2}R_2+R_3 \to R_3}{\sim} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Let $z = s \in \mathbb{R}$, then $-3y+3z=0 \Rightarrow y=s$, and $2x+y-z=4 \Rightarrow 2x=4 \Rightarrow x=2$. Thus the solution is $(x,y,z)=(2,s,s)$ for any real number s.

b.) $\left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 4 & -1 & 1 & 8 \\ 1 & -1 & 1 & 0 \end{array}\right] \underset{-2R_1+R_2 \to R_2}{\sim} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 0 & -3 & 3 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right] \underset{-\frac{1}{2}R_1+R_3 \to R_3}{\sim} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 0 & -3 & 3 & 0 \\ 0 & -\frac{3}{2} & \frac{3}{2} & -2 \end{array}\right]$

$\underset{-\frac{1}{2}R_2+R_3 \to R_3}{\sim} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & -2 \end{array}\right]$

Since $0=-2$ is not possible, the system has no solution.

c.) $\left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 4 & -1 & 1 & 8 \\ 1 & -2 & 1 & 0 \end{array}\right] \underset{-2R_1+R_2 \to R_2}{\sim} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 0 & -3 & 3 & 0 \\ 1 & -2 & 1 & 0 \end{array}\right] \underset{-\frac{1}{2}R_1+R_3 \to R_3}{\sim} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 0 & -3 & 3 & 0 \\ 0 & -\frac{5}{2} & \frac{3}{2} & -2 \end{array}\right]$

$\underset{-\frac{5}{6}R_2+R_3 \to R_3}{\sim} \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & -1 & -2 \end{array}\right]$

Hence, $-z=-2 \Rightarrow z=2$, $-3y+3z=0 \Rightarrow y=2$, and $2x+y-z=4 \Rightarrow 2x=4 \Rightarrow x=2$. Thus the solution is $(x,y,z)=(2,2,2)$.

Exercise 4. (CS 1.2):

a. consider the following overdetermined system (more equations than unknowns):

$$\begin{cases} x+y=2 \\ 2x-3y=4 \\ 3x-2y=6 \\ 4x-2y=1 \end{cases}$$

Use Gaussian elimination to solve this system. Does this system have solutions? If so, what are they?

Multiplying equation 1 by -2 and adding it to equation 2, multiplying equation 1 by -3 and adding it to equation 3, and multiplying equation 1 by -4 and adding it to equation 4 we obtain the following equivalent linear system

$$\begin{cases} x+y=2 \\ -5y=0 \\ -5y=0 \\ -6y=-7 \end{cases}$$

since $-5y=0 \Rightarrow y=0$ and $-6y=-7 \Rightarrow y=\frac{7}{6}$, since $0 \neq \frac{7}{6}$ we can see that this system has no solutions.

b. consider the following underdetermined system (more unknowns than equations):

$$\begin{cases} x+y+z=2 \\ 2x-4y=4 \end{cases}$$

to obtain solutions to this system, we add one "trivial equation" to make the system square

$$\begin{cases} x+y+z=2 \\ 2x-4y=4 \\ 0x+0y+0z=0 \end{cases}$$

and then we perform Gaussian elimination on this new system. Does this new system have solutions. If so, what are they?

Again we will use the corresponding augmented matrices/matrix

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & -4 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{\sim}{-2R_1 + R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -6 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Let $z = s \in \mathbb{R}$, then $-6y - 2z = 0 \Rightarrow y = \frac{-1}{3}s$, and $x + y + z = 2$ $\Rightarrow x = 2 - \frac{2}{3}s$. Thus the solution to the system is $(x, y, z) = (2 - \frac{2}{3}s, \frac{-1}{3}s, s)$.

- **Exercise 5 (CS1.3)**

Express the linear system in Exercise 1 in matrix-vector format and as an augmented matrix:

- matrix-vector format:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 4 & 0 & 2 & -2 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 4 & 4 & -2 & -4 & -4 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{b}}$$

- augmented matrix:

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 4 & 0 & 2 & -2 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 4 & 4 & -2 & -4 & -4 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 \end{array}\right]$$

Exercise 6. (cs 1.4):

From Olver and Shakiban, complete Exercises 1.2.7(a), (b), (c).

Remark: Unless very special circumstances are at play, remember $AB \neq BA$.

Consider the matrices $A = \begin{bmatrix} 1 & -1 & 3 \\ -1 & 4 & -2 \\ 3 & 0 & 6 \end{bmatrix}$, $B = \begin{bmatrix} -6 & 0 & 3 \\ 4 & 2 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 3 \\ -3 & -4 \\ 1 & 2 \end{bmatrix}$

Compute the indicated combinations where possible.

1.2.7.(a): $3A - B$

Since A is 3×3 and B is 2×3, $3A-B$ is not possible. Namely since A and B aren't the same size $3A-B$ is not possible.

1.2.7.(b): AB

Since A is 3×3 and B is 2×3, AB is not possible. Namely, since the inner dimensions aren't the same, A is not compatible with B.

1.2.7.(c): BA:

$$BA = \begin{bmatrix} -6 & 0 & 3 \\ 4 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 3 \\ -1 & 4 & -2 \\ 3 & 0 & 6 \end{bmatrix}$$

let $D = BA$, then D is 2×3 and

$$d_1 = \begin{bmatrix} -6 & 0 & 3 \\ 4 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -6+0+9 \\ 4-2-3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$d_2 = \begin{bmatrix} -6 & 0 & 3 \\ 4 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 6+0+0 \\ -4+8+0 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$d_3 = \begin{bmatrix} -6 & 0 & 3 \\ 4 & 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix} = \begin{bmatrix} -18+0+18 \\ 12-4-6 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

thus $D = BA = \begin{bmatrix} 3 & 6 & 0 \\ -1 & 4 & 2 \end{bmatrix}$

Exercise 7. (CS 1.5):

Determine the output of each of the following MATLAB codes.

a.)
```
x = [0, -2, 2, -3, 3, -4, 4];
m = length(x);
SUM = 0
for j = m-1: -1 : 1   % j counts down from m-1 to 1 in intervals of 1
    if x(j) > 1
        SUM = SUM + 1;
    end
end
SUM % what is the value of SUM
```

This code chunk goes through the first 6 elements of x in descending order and counts the number of entries greater than 1, thus SUM = 2.

b.)
```
x = [1, 2, -3, 4, 5, -5, 3, -6, 7, 9, 9, 1, 2];
j = 1;
PROD = 1;
while abs(x(j)) < abs(x(j+1))   % abs(x(j)) is the absolute value of entry x(j)
    PROD = PROD * x(j);
    j = j+1;
end
PROD % what is the value of PROD
```

This code chunk goes through each element of x and computes a cumulative product until the j^{th} component of x (absolute value) is less than the absolute value of the $j+1^{st}$ component, thus PROD = -24.

c.)
```
A = [1, 2, -6, 4;  2, -3, 1, 0;  1, 5, -5, 3];
[m  n] = size(A);
SUM = 0;
for  i = 1:m
    for j = i:n
        SUM = SUM + A(i, j);
    end
end
SUM % what is the value of SUM?
```

This code goes through all 3 rows of the matrix A and for each of the i^{th} through 4^{th} column, adds the corresponding element, the following circled entries are added:

$$\begin{bmatrix} 1 & 2 & -6 & 4 \\ 2 & -3 & 1 & 0 \\ 1 & 5 & -5 & 3 \end{bmatrix}$$

thus, SUM = -3.

d.)
```
A = [1, 7, 0, 4;  2, -5, -8, 0;  3, 0, 9, 2];
B = [1, 2;  3, 0;  0, 8];
C = [B, B(:, 2)];
D = C * A;
E = D(2: end, 1:end);
F = E.^2 % What are the dimensions of F? What are its entries.
```

This code does a lot of matrix arithmetic, to show work I will show all the matrises:

$$A = \begin{bmatrix} 1 & 7 & 0 & 4 \\ 2 & -5 & -8 & 0 \\ 3 & 0 & 9 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 8 \end{bmatrix} \qquad E = \begin{bmatrix} 3 & 21 & 0 & 12 \\ 40 & -40 & 8 & 16 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 0 & 0 \\ 0 & 8 & 8 \end{bmatrix} \qquad D = \begin{bmatrix} 11 & -3 & 2 & 8 \\ 3 & 21 & 0 & 12 \\ 40 & -40 & 8 & 16 \end{bmatrix}$$

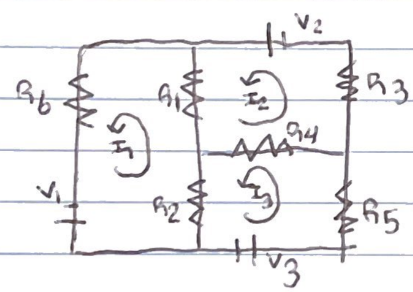
$F = E^2$ (element wise) thus

$$F = \begin{bmatrix} 9 & 441 & 0 & 144 \\ 1600 & 1600 & 64 & 256 \end{bmatrix}$$

The dimensions of F is 2x4.

- Multi-Step Problem:

Consider the circuit diagram below



Here, each V represents a change in voltage (measured in volts) at a battery, each R represents a resistance (measured in ohms) at a resistor, and each I represents a current (measured in amps) through a wire. These quantities obey two simple laws:

- Ohm's Law: The voltage drop across a resistor is $V = IR$.
- Kirchhoff's Second Law: The sum of all voltage drops in a closed loop is zero.

Using these two laws, we can construct the following system of equations

$$V_1 = I_1 R_6 + (I_1 - I_2) R_1 + (I_1 - I_3) R_2,$$
$$V_2 = I_2 R_3 + (I_2 - I_3) R_4 + (I_2 - I_1) R_1,$$
$$V_3 = I_3 R_5 + (I_3 - I_2) R_4 + (I_3 - I_1) R_2.$$

Given the values of the resistances R_j and voltage drops V_j, we want to calculate the currents I_j.

(a) Express the system above as a matrix equation $Ax = b$.
What are the dimensions of A, x, and b?

First off, we will rewrite the above system by removing parentheses:

$$V_1 = I_1 R_6 + I_1 R_1 - I_2 R_1 + I_1 R_2 - I_3 R_2$$
$$V_2 = I_2 R_3 + I_2 R_4 - I_3 R_4 + I_2 R_1 - I_1 R_1$$
$$V_3 = I_3 R_5 + I_3 R_4 - I_2 R_4 + I_3 R_2 - I_1 R_2$$

Furthermore, this system is equal to

$$V_1 = (R_1 + R_2 + R_6) I_1 + (-R_1) I_2 + (-R_2) I_3$$
$$V_2 = (-R_1) I_1 + (R_1 + R_3 + R_4) I_2 + (-R_4) I_3$$
$$V_3 = (-R_2) I_1 + (-R_4) I_2 + (R_2 + R_4 + R_5) I_3$$

thus in $Ax = b$ form the system is

$$\underbrace{\begin{bmatrix} R_1 + R_2 + R_6 & -R_1 & -R_2 \\ -R_1 & R_1 + R_3 + R_4 & -R_4 \\ -R_2 & -R_4 & R_2 + R_4 + R_5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}}_{b}$$

Where A is 3×3 and x, b are 3×1.

(b) Let

$R_1 = 30\,\Omega$, $R_2 = 5\,\Omega$, $R_3 = 10\,\Omega$, $R_4 = 20\,\Omega$, $R_5 = 15\,\Omega$, $R_6 = 15\,\Omega$.

Suppose we have no idea what the voltages of our batteries are what must these voltages be if a 1 amp current flows uniformly across all three loops of the circuit above?

This means $I_1 = I_2 = I_3 = 1$, thus using our values of $R_1, R_2, \rightarrow R_6$ into our $Ax = b$ equation we obtain

$$\begin{bmatrix} 50 & -30 & -5 \\ -30 & 60 & -20 \\ -5 & -20 & 40 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

Thus

$$\begin{bmatrix} 50 & -30 & -5 \\ -30 & +60 & -20 \\ -5 & -20 & +40 \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \\ 15 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

thus the voltages are $V_1 = 15$, $V_2 = 10$, and $V_3 = 15$.

(c) Let

$V_1 = 25V$, $V_2 = 30V$, $V_3 = 215V$,

and the resistances be as in (b). Using MATLAB'S backslash command, determine the currents I_1, I_2, and I_3 from your matrix equation $Ax = b$. Supposing any part of the circuit cannot handle currents beyond 10 amps, does this circuit break down?

We will use MATLAB to solve:

$$\begin{bmatrix} 50 & -30 & -5 \\ -30 & 60 & -20 \\ -5 & -20 & 40 \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 25 \\ 30 \\ 215 \end{bmatrix}$$

Code:

```
A = [50, -30, -5; -30, 60, -20; -5, -20, 40];
b = [25; 30; 215];
A\b;
```

Output: ans: 5 6 9

Thus as computed in MATLAB, $I_1 = 5$, $I_2 = 6$, $I_3 = 9$. Since I_1, I_2, I_3 are all less than 10, the circuit does not break down.