Quiz 2 Jaiden Atterbury S#: 2028502

(4.) Consider the linear system $Ax=b$, where $A=\begin{pmatrix}1&0&1\\-1&1&0\\1&2&-1\end{pmatrix}$ and $b=\begin{pmatrix}1\\2\\1\end{pmatrix}$.
For the following, show and upload all your work. Show each elimination step for each column.

(a) compute the LU factorization of A. Is A regular is A invertible?

Step 1: Add row 1 to row 2:

$$A=\begin{pmatrix}1&0&1\\0&1&1\\1&2&-1\end{pmatrix} \quad L=\begin{pmatrix}1&0&0\\-1&1&0\\0&0&1\end{pmatrix}$$

Step 2: Add -1 times row 1 to row 3:

$$A=\begin{pmatrix}1&0&1\\0&1&1\\0&2&-2\end{pmatrix} \quad L=\begin{pmatrix}1&0&0\\-1&1&0\\1&0&1\end{pmatrix}$$

Step 3: Add -2 times row 2 to row 3

$$A=U=\begin{pmatrix}1&0&1\\0&1&1\\0&0&-4\end{pmatrix} \quad L=\begin{pmatrix}1&0&0\\-1&1&0\\1&2&1\end{pmatrix}$$

thus $A=LU \Rightarrow A=\begin{pmatrix}1&0&0\\-1&1&0\\1&2&1\end{pmatrix}\begin{pmatrix}1&0&1\\0&1&1\\0&0&-4\end{pmatrix}$.

Since A can be factored $A=LU$, A is thus regular. Furthermore, since A is square and nonsingular, it is thus invertible.

(b) Use the LU factorization from (a) to solve the linear system by forward/back substitution

Step 1: Let $c=Ux$ and solve $Lc=b$ using forward substitution

$$\begin{pmatrix}1&0&0\\-1&1&0\\1&2&1\end{pmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix}=\begin{pmatrix}1\\2\\1\end{pmatrix}$$

hence $c_1=1$, $c_2=3$, and $c_3=-6$.

Step 2: Solve $Ux = c$ by back substitution

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$$

Hence $x_3 = \frac{-6}{-4} = \frac{3}{2}$, $x_2 = 3 - \frac{3}{2} = \frac{3}{2}$, $x_1 = 1 - \frac{3}{2} = -\frac{1}{2}$. Hence the solution to the linear system is $(x_1, x_2, x_3)^T = (-\frac{1}{2}, \frac{3}{2}, \frac{3}{2})^T$.

(c) Compute A^{-1} by Gauss-Jordan elimination.

Gaussian

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \overset{R_1+R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \overset{-R_1+R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & -2 & -1 & 0 & 1 \end{array}\right] \overset{-2R_2+R_3}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -4 & -3 & -2 & 1 \end{array}\right]$$

Jordan

$$\overset{-\frac{1}{4}R_4}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & +\frac{3}{4} & +\frac{1}{2} & -\frac{1}{4} \end{array}\right] \overset{-R_3+R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 & 3/4 & 1/2 & -1/4 \end{array}\right] \overset{-R_3+R_1}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & -1/2 & 1/4 \\ 0 & 1 & 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 1 & 3/4 & 1/2 & -1/4 \end{array}\right]$$

thus $A^{-1} = \begin{bmatrix} 1/4 & -1/2 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 3/4 & 1/2 & -1/4 \end{bmatrix}$

(d) Use A^{-1} from (c) to solve the linear system

If $Ax = b$, and A^{-1} exist, then it follows that $x = A^{-1}b$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/4 & -1/2 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 3/4 & 1/2 & -1/4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 - 1 + 1/4 \\ 1/4 + 1 + 1/4 \\ 3/4 + 1 - 1/4 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ 3/2 \\ 3/2 \end{bmatrix}$$

Hence the solution to the linear system is $(x_1, x_2, x_3)^T = (\frac{-1}{2}, \frac{3}{2}, \frac{3}{2})^T$.

(e) Which method (LU decomposition or matrix inversion) has a lower flop count? Which method is preferred?

To create an LU decomposition takes $\sim\frac{2}{3}n^3$ to compute, and $\sim 2n^2$ per run to solve, while to create a matrix inverse takes $\sim\frac{4}{3}n^3$ to compute, hence, asymptotically, LU decomposition has a lower flop count. Hence, due to its lower asymptotic flop count and versatility in solving linear systems with changing $\vec{b}$ vectors, LU decomposition is preffered.