AMATH 353 Quiz 4

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Note: In this quiz, I briefly collaborated with Kai Poffenbarger. In this collaboration we mainly compared answers to see if what we were getting for our solutions were similar.

Component Skills

1. Consider the function $f(x) = x^2$ on $-\pi < x < \pi$.

(i) Compute the Fourier series of $f(x)$. Simplify expressions involving $\cos(n\pi)$ and $\sin(n\pi)$.

In order to determine the Fourier series representation of $f(x) = x^2$ on $-\pi < x < \pi$, we must solve for the coefficients a_0, a_n, and b_n of the Fourier series representation of $f(x)$ on $[-L, L]$ defined as $f(x) = \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right)$. The formulas for these coefficients are defined as, $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$, $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$, and $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$.

Step 1: Solve for a_0:
Since in our case $L = \pi$, $f(x) = x^2$ and $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$, we have $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx$. Since x^2 is an even function we can instead solve $\frac{2}{\pi}\int_0^{\pi} x^2 dx$. This calculation is simple and hence $a_0 = \frac{2}{\pi}\left[\frac{1}{3}x^3\right]_0^{\pi} = \frac{2\pi^2}{3}$.

Step 2: Solve for a_n:
Since in our case $L = \pi$, $f(x) = x^2$ and $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$, we have $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)\,dx$. Hence we must solve, since x^2 and $\cos(nx)$ are even functions, their product is even and we can instead solve $\frac{2}{\pi}\int_0^{\pi} x^2\cos(nx)dx$. Using integration by parts, we can see that $\int x^2\cos(nx)dx = \frac{x^2\sin(nx)}{n} + \frac{2x\cos(nx)}{n^2} - \frac{2\sin(nx)}{n^3}$. Hence in our case $\frac{2}{\pi}\int_0^{\pi} x\cos(nx)dx = \frac{2}{\pi}\left[\frac{x^2\sin(nx)}{n} + \frac{2x\cos(nx)}{n^2} - \frac{2\sin(nx)}{n^3}\right]_0^{\pi} = \frac{4\cos(n\pi)}{n^2}$. Since $\cos(n\pi) = (-1)^n$, it follows that $a_n = \frac{4(-1)^n}{n^2}$.

Step 3: Solve for b_n:
Since in our case $L = \pi$, $f(x) = x^2$ and $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$, we have $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin(nx)\,dx$. Since x^2 is an even function, and $\sin(nx)$ is an odd function, it follows that their product is an odd function. With that said, since the integral over a symmetric interval of an odd function equals zero, we can see that $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin(nx)\,dx = 0 \implies b_n = 0$, $\forall n \in \mathbb{N}$.

Replacing the Fourier series form of $f(x)$ with our values of a_0, a_n and b_n, we can see that the Fourier series representation of $f(x) = x^2$ on $[-\pi, \pi]$ is $f(x) = \frac{\pi^2}{3} + \sum\limits_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos(nx)$.

(ii) Evaluate your series at $x = \pi$. Show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

This was a famous problem solved by Euler in the mid-18th century.

Since $f(x) = x^2$ and $f(x) = \frac{\pi^2}{3} + \sum\limits_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx)$, in order to show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, we will compute $f(\pi)$ for both expressions of f, set them equal to each other, then solve for $\sum_{n=1}^{\infty} \frac{1}{n^2}$. First off, $f(x) = x^2 \implies f(\pi) = \pi^2$. We will now move onto finding $f(\pi)$ based on the other expression.

$$\begin{aligned} f(\pi) &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(n\pi) \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} (-1)^n \\ &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \end{aligned}$$

Setting these two expressions for $f(\pi)$ equal to each other and solving for $\sum_{n=1}^{\infty} \frac{1}{n^2}$ we obtain the expected result.

$$\begin{aligned} \pi^2 &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \\ \pi^2 - \frac{\pi^2}{3} &= \sum_{n=1}^{\infty} \frac{4}{n^2} \\ \frac{\pi^2 - \frac{\pi^2}{3}}{4} &= \sum_{n=1}^{\infty} \frac{1}{n^2} \\ \frac{2\pi^2}{12} &= \sum_{n=1}^{\infty} \frac{1}{n^2} \\ \sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{\pi^2}{6} \end{aligned}$$

Hence we have shown that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

2. Consider the function $f(x) = x^2$ on $0 < x < \pi$.

(i) Compute the Fourier sine series of $f(x)$. Simplify expressions involving $\cos(n\pi)$ and $\sin(n\pi)$.

Since the function we are looking at is on a non-symmetric interval, we must extend $f(x) = x^2$, so that it is on a symmetric interval. In particular, since we are looking for the Fourier sine series of $f(x) = x^2$, it turns out that we are looking at the odd extension of f, which in our case is defined as

$$f_{odd}(x) = \begin{cases} x^2 & x \in (0, \pi) \\ -x^2 & x \in (-\pi, 0) \end{cases}$$

We can now proceed in finding the Fourier series of $f_{odd}(x)$. By definition, $f_{odd}(x)$ is an odd function, hence it follows that $a_n = 0$, $\forall n \in \mathbb{N}$. Thus all we have to find is b_n. Since $f_{odd}(x)$ is an odd function, and $\sin(nx)$

is an odd function, this integral simplifies to $\frac{2}{\pi}\int_0^\pi x^2\sin(nx)dx$. Using integration by parts, we can see that $\int x^2\sin(nx)dx = \frac{-x^2\cos(nx)}{n} + \frac{2x\sin(nx)}{n^2} + \frac{\cos(nx)}{n^3}$. This integral is solved below.

$$\begin{aligned} b_n &= \frac{2}{\pi}\left[\frac{-x^2\cos(nx)}{n} + \frac{2x\sin(nx)}{n^2} + \frac{\cos(nx)}{n^3}\right]_0^\pi \\ &= \frac{-2\pi\cos(n\pi)}{n} + \frac{4\cos(n\pi)}{n^3\pi} - \frac{4}{n^3\pi} \\ &= \frac{4\cos(n\pi) - 2\pi^2n^2\cos(n\pi) - 4}{n^3\pi} \\ &= \frac{4(-1)^n - 2\pi^2n^2(-1)^n - 4}{n^3\pi} \\ &= \frac{(4-2\pi^2n^2)(-1)^n - 4}{n^3\pi} \end{aligned}$$

Since $b_n = \frac{(4-2\pi^2n^2)(-1)^n-4}{n^3\pi}$ and $a_n = 0$, $\forall n \in \mathbb{N}$, the Fourier sine series of $f(x) = x^2$ on $0 < x < \pi$ is $f(x) = \sum\limits_{n=1}^{\infty} \frac{(4-2\pi^2n^2)(-1)^n-4}{n^3\pi}\sin(nx)$.

(ii) Compute the Fourier cosine series of $f(x)$ Simplify expressions involving $\cos(n\pi)$ and $\sin(n\pi)$. (Do yo need to compute anything? Look at the previous problem.)

Since the function we are looking at is on a non-symmetric interval, we must extend $f(x) = x^2$, so that it is on a symmetric interval. In particular, since we are looking for the Fourier cosine series of $f(x) = x^2$, it turns out that we are looking at the even extension of f, which in our case is defined as

$$f_{even}(x) = \begin{cases} x^2 & x \in (0,\pi) \\ x^2 & x \in (-\pi, 0) \end{cases}$$

We can now proceed in finding the Fourier series of $f_{even}(x)$. By definition, $f_{even}(x)$ is an even function, hence it follows that $b_n = 0\,\forall n \in \mathbb{N}$. Thus all we have to find is a_n and a_0.

Notice that $f_{even}(x)$ is simply just $f(x) = x^2$ on $-\pi < x < \pi$. Thus we are finding the normal Fourier series of $f(x) = x^2$ on $-\pi < x < \pi$. This is exactly what we did in problem 1 part (i), hence the Fourier cosine series of $f(x)$ is $f(x) = \frac{\pi^2}{3} + \sum\limits_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx)$. Below we will show that these a_0, a_n, and b_n are the same from problem 1 part (i).

Step 1: Solve for a_0:
Since in our case $L = \pi$, $f(x) = f_{even}(x)$ and $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$, we have $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f_{even}(x)dx$. Since $f_{even}(x)$ is an even function, we must solve $\frac{2}{\pi}\int_0^\pi x^2dx$.

Step 2: Solve for a_n:
Since in our case $L = \pi$, $f(x) = f_{even}(x)$ and $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$, we have $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f_{even}(x)\cos(nx)\,dx$. Since $f_{even}(x)$ is an even function, and $\cos(nx)$ is an even function, it follows that their product is an even function function, hence we can simplify the integral to $a_n = \frac{2}{\pi}\int_0^\pi x^2\cos(nx)dx$.

As can be seen since the coefficients a_0, a_n, and b_n are the same from problem 1 part (i), we have reassurance that we didn't need to calculate anything new from this problem and that the Fourier cosine series of $f(x)$ is $f(x) = \frac{\pi^2}{3} + \sum\limits_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx)$.

(iii) Which of these series converges to $f(x)$ faster? Give a brief justification of your choice.

As can be seen from the forms of $f_{odd}(x)$ and $f_{even}(x)$, we can see that there are no obvious removable, jump or infinite discontinuities in either of the two functions. However, as mentioned in Piazza, since $f_{odd}(-\pi) = -\pi^2 \neq f_{odd}(\pi) = \pi^2$, but $f_{even}(-\pi) = f_{even}(\pi) = \pi^2$, it has been shown that $f_{odd}(x)$ has a boundary discontinuity, while $f_{even}(x)$ has no discontinuities. Since f_{odd} has more discontinuities than f_{even}, the Fourier cosine series should converge to $f(x)$ faster.

3. Solve the following initial boundary value problem *without using Fourier series to satisfy the initial condition*:

$$\begin{cases} u_t = u_{xx}, \quad 0 < x < L, \\ u(0,t) = u(L,t) = 0, \\ u(x,0) = \sin(\frac{\pi}{L}x) - 2\sin(\frac{2\pi}{L}x) + 3\sin(\frac{3\pi}{L}x). \end{cases}$$

As found in the Homework 6 Multi-Step Problem, the BVP

$$\begin{aligned} u_t = u_{xx}, \quad 0 < x < L \\ u(0,t) = u(L,t) = 0, \end{aligned}$$

has a countably infinite number of standing wave solutions of the form.

$$u_n(x,t) = H_n \sin\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2\pi^2}{L^2}t}.$$

Since the PDE is linear and homogeneous it has a superposition principle thus any linear combination of these above solutions is also a solution. Hence we can write the above solutions into one as shown below.

$$u(x,t) = \sum_{n=1}^{\infty} H_n \sin\left(\frac{n\pi x}{L}\right) e^{\frac{-n^2\pi^2}{L^2}t}$$

As shown below, this new solution form also satisfies the boundary conditions.

$$u(0,t) = \sum_{n=1}^{\infty} H_n \sin(0)\, e^{\frac{-n^2\pi^2}{L^2}t} = 0$$

$$u(L,t) = \sum_{n=1}^{\infty} H_n \sin(n\pi)\, e^{\frac{-n^2\pi^2}{L^2}t} = 0$$

Applying the initial condition $u(x,0) = \sin(\frac{\pi}{L}x) - 2\sin(\frac{2\pi}{L}x) + 3\sin(\frac{3\pi}{L}x)$, we obtain the following.

$$\begin{aligned} \sin\left(\frac{\pi}{L}x\right) - 2\sin\left(\frac{2\pi}{L}x\right) + 3\sin\left(\frac{3\pi}{L}x\right) &= u(x,0) \\ &= \sum_{n=1}^{\infty} H_n \sin\left(\frac{n\pi x}{L}\right) \\ &= H_1 \sin\left(\frac{\pi x}{L}\right) + H_2 \sin\left(\frac{2\pi x}{L}\right) + H_3 \sin\left(\frac{3\pi x}{L}\right) + \dots \end{aligned}$$

Matching the forms of both sides of the initial condition, we can see that $H_1 = 1$, $H_2 = -2$, $H_3 = 3$, and $H_n = 0$, $\forall n \geq 4$. Hence, plugging in these values of n and H_n into our general solution from, we can see that the solution to the above IBVP is

$$u(x,t) = \sin\left(\frac{\pi x}{L}\right) e^{\frac{-\pi^2}{L^2}t} - 2\sin\left(\frac{2\pi x}{L}\right) e^{\frac{-4\pi^2}{L^2}t} + 3\sin\left(\frac{3\pi x}{L}\right) e^{\frac{-9\pi^2}{L^2}t}.$$

4. Find the solution of the following initial boundary value problem *using Fourier series*:

$$\begin{cases} u_{tt} = 9u_{xx}, \quad 0 < x < \pi, \\ u_x(0,t) = 0 = u_x(\pi,t), \\ u(x,0) = x^2(x-\pi)^2, \\ u_t(x,0) \equiv 0. \end{cases}$$

Feel free to use software of your choice to evaluate any integrals that appear in your solution. Be sure to simplify expressions involving $\cos(n\pi)$ and $\sin(n\pi)$.

As found in Lecture 11 Example 5.6.2 the solution to the IBVP

$$\begin{cases} u_{tt} = c^2u_{xx}, \quad 0 < x < \pi, \\ u_x(0,t) = 0 = u_x(\pi,t), \\ u(x,0) = f(x), \\ u_t(x,0) = g(x), \end{cases}$$

is $u(x,t) = \frac{1}{2}A_0 + \frac{1}{2}B_0t + \sum\limits_{n=1}^{\infty} \cos(nx)\left[A_n\cos(nct) + B_n\sin(nct)\right]$. Where $A_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)dx, n = 0,1,2,\ldots$, $B_0 = \frac{2}{\pi}\int_0^\pi g(x)dx$, and $B_n = \frac{2}{nc\pi}\int_0^\pi g(x)\cos(nx)dx$, $n = 1,2,3,\ldots$. Hence in our case $c = 3$, and our solution becomes $u(x,t) = \frac{1}{2}A_0 + \frac{1}{2}B_0t + \sum\limits_{n=1}^{\infty} \cos(nx)\left[A_n\cos(3nt) + B_n\sin(3nt)\right]$. Below we will find values for the coefficients A_0, A_n, B_0, and B_n for the solutions above.

Step 1: Solve for A_0:
As shown above, the form of our coefficient A_0 is, $A_0 = \frac{2}{\pi}\int_0^\pi f(x)dx$, hence in our case $A_0 = \frac{2}{\pi}\int_0^\pi x^4 - 2\pi x^3 + \pi^2x^2dx$. This integral is solved below.

$$\begin{aligned} A_0 &= \frac{2}{\pi}\int_0^\pi x^4 - 2\pi x^3 + \pi^2x^2dx \\ &= \frac{2}{\pi}\left[\frac{x^5}{5} - \frac{\pi x^4}{2} + \frac{\pi^2x^3}{3}\right]_0^\pi \\ &= \frac{2}{\pi}\left[\frac{\pi^5}{5} - \frac{\pi^5}{2} + \frac{\pi^5}{3}\right] \\ &= \frac{2}{\pi}\left[\frac{\pi^5}{30}\right] \\ &= \frac{\pi^4}{15} \end{aligned}$$

Step 2: Solve for A_n:
As shown above, the form of our coefficient A_n is, $A_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)dx$, hence in our case $A_n = \frac{2}{\pi}\int_0^\pi (x^4 - 2\pi x^3 + \pi^2x^2)\cos(nx)dx$. This integral is solved below.

$$\begin{aligned} A_n &= \frac{2}{\pi}\int_0^\pi (x^4 - 2\pi x^3 + \pi^2x^2)\cos(nx)dx \\ &= \frac{2}{\pi}\int_0^\pi x^4\cos(nx)dx - 4\int_0^\pi x^3\cos(nx)dx + 2\pi\int_0^\pi x^2\cos(nx)dx \end{aligned}$$

Using integration by parts on the 3 above integrals we can see that

$$\int x^4 \cos(nx)dx = \frac{x^4 \sin(nx)}{n} + \frac{4x^3 \cos(nx)}{n^2} - \frac{12x^2 \sin(nx)}{n^3} - \frac{24x \cos(nx)}{n^4} + \frac{24 \sin(nx)}{n^5}$$
$$\int x^3 \cos(nx)dx = \frac{x^3 \sin(nx)}{n} + \frac{3x^2 \cos(nx)}{n^2} - \frac{6x \sin(nx)}{n^3} - \frac{6 \cos(nx)}{n^4}$$
$$\int x^2 \cos(nx)dx = \frac{x^2 \sin(nx)}{n} + \frac{2x \cos(nx)}{n^2} - \frac{2 \sin(nx)}{n^3}$$

Resuming our calculation of A_n we see we must solve $\frac{2}{\pi}\left[\frac{x^4 \sin(nx)}{n} + \frac{4x^3 \cos(nx)}{n^2} - \frac{12x^2 \sin(nx)}{n^3} - \frac{24x \cos(nx)}{n^4} + \frac{24 \sin(nx)}{n^5}\right]_0^\pi - 4\left[\frac{x^3 \sin(nx)}{n} + \frac{3x^2 \cos(nx)}{n^2} - \frac{6x \sin(nx)}{n^3} - \frac{6 \cos(nx)}{n^4}\right]_0^\pi + 2\pi\left[\frac{x^2 \sin(nx)}{n} + \frac{2x \cos(nx)}{n^2} - \frac{2 \sin(nx)}{n^3}\right]_0^\pi$. This calculation is shown below.

$$\begin{aligned} A_n &= \frac{8\pi^2 \cos(n\pi)}{n^2} - \frac{48 \cos(n\pi)}{n^4} - \frac{12\pi^2 \cos(n\pi)}{n^2} + \frac{24 \cos(n\pi)}{n^4} - \frac{24}{n^4} + \frac{4\pi^2 \cos(n\pi)}{n^2} \\ &= \frac{-24 \cos(n\pi)}{n^4} - \frac{24}{n^4} \\ &= \frac{-24(-1)^n - 24}{n^4} \end{aligned}$$

Step 3: Solve for B_0:
As shown above, the form of our coefficient B_0 is, $B_0 = \frac{2}{\pi}\int_0^\pi g(x)dx$, hence in our case $B_n = \frac{2}{\pi}\int_0^\pi 0dx = 0$.

Step 4: Solve for B_n:
As shown above, the form of our coefficient B_n is, $B_n = \frac{2}{nc\pi}\int_0^\pi g(x)\cos(nx)dx$, hence in our case $B_n = \frac{2}{3n\pi}\int_0^\pi 0 \cdot \cos(nx)dx = \frac{2}{3n\pi}\int_0^\pi 0dx = 0$.

Plugging in our values of A_0, B_0, A_n, and B_n into our solution form above we obtain the solution to the above IBVP which is $u(x,t) = \frac{\pi^4}{30} + \sum_{n=1}^{\infty} \frac{-24(-1)^n - 24}{n^4}\cos(nx)\cos(3nt)$. Furthermore, since all of the odd numbers in this summation equal 0, we can “rename” and replace n as $2n$ in our summation to only account for the even numbers. Doing this leads to an equivalent result of $u(x,t) = \frac{\pi^4}{30} + \sum_{n=1}^{\infty} \frac{-3}{n^4}\cos(2nx)\cos(6nt)$.