AMATH 353 Final Quiz

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Note: In this quiz, I briefly collaborated with Kai Poffenbarger. In this collaboration we mainly compared answers to see if what we were getting for our solutions were similar.

Component Skills

1. Consider the PDE

$$u_t + x^2 u_x = -2ux$$

(i) Write the PDE in the form $u_t + \phi(u, x, t)_x = f(u, x, t)$ where $f = 0$.

Writing $u_t + x^2 u_x = -2ux$ in the form $u_t + \phi(u, x, t)_x = f(u, x, t)$ where $f = 0$, we obtain

$$\begin{aligned} u_t + x^2 u_x &= -2ux \\ u_t + x^2 u_x + 2ux &= 0 \end{aligned}$$

Since $(x^2 u)_x = x^2 \frac{d}{dx} u + \frac{d}{dx} x^2 u = x^2 u_x + 2xu$, we can see that

$$u_t + (x^2 u)_x = 0$$

Hence, the PDE in the required form is $u_t + (x^2 u)_x = 0$ with $\phi(u, x, t) = x^2 u$ and $f(u, x, t) = 0$.

(ii) Classify this conservation law: is it linear, semi-linear, or non-linear?

Since the PDE $u_t + x^2 u_x = -2ux$ is already in the form $u_t + c(u, x, t)u_x = g(x, t, u)$ with $c(x, t, u) = x^2$ and $g(x, t, u) = -2ux$, we can see that our conservation law is semi-linear. Our conservation law is semi-linear since c doesn't depend on u but g does depend on u.

(iii) Determine the form of the characteristics of this equation, and make a sketch of them for $x_0 = -3, -2, -1, 0, 1, 2, 3$.

We will start off by determining the form of the characteristics for this equation. To do so, we will solve the following IVP

$$\begin{cases} \frac{dx}{dt} = x^2 \\ x(0) = x_0 \end{cases}$$

We will first solve the ODE as show below

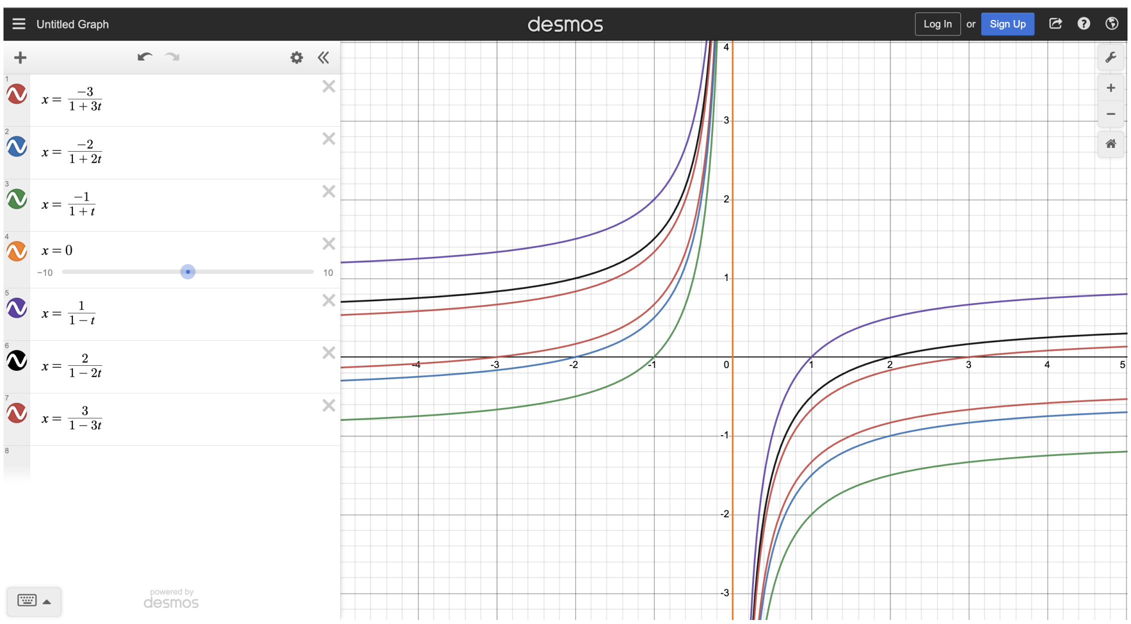
$$\frac{dx}{dt} = x^2$$
$$\int \frac{dx}{x^2} = \int dt + C$$
$$\frac{-1}{x} = t + C$$

We will now apply the initial condition

$$x(0) = x_0$$
$$\implies \frac{-1}{x_0} = 0 + C$$
$$\implies C = \frac{-1}{x_0}$$

Hence our characteristics take the form $x(t; x_0) = \frac{-1}{t-1/x_0} = \frac{x_0}{1-x+0t}$.

We will now sketch these characteristics for the following values of x_0: $x_0 = -3, -2, -1, 0, 1, 2, 3$. This is done below.



2. Consider the conservation law

$$u_t + (-sech^2(t)u)_x = -u$$

(i) Determine the form of the characteristics of this equation, and make a sketch of them for $x_0 = -3, -2, -1, 0, 1, 2, 3$.

Before we start with solving the associated IVP, we must write the conservation law in the form $u_t + c(u, x, t)u_x = g(u, x, t)$. This is done below

$$u_t + (-sech^2(t)u)_x = -u$$
$$u_t - sech^2(t)\frac{d}{dx}u - \frac{d}{dx}sech^2(t)u = -u$$
$$u_t - sech^2(t)u_x = -u$$

Hence we can see $c(x,t) = -sech^2(t)$ and $g(u,x,t) = -u$. We will now determine the form of the characteristics for this equation. To do so, we will solve the following IVP

$$\begin{cases} \frac{dx}{dt} = -sech^2(t) \\ x(0) = x_0 \end{cases}$$

We will first solve the ODE as show below

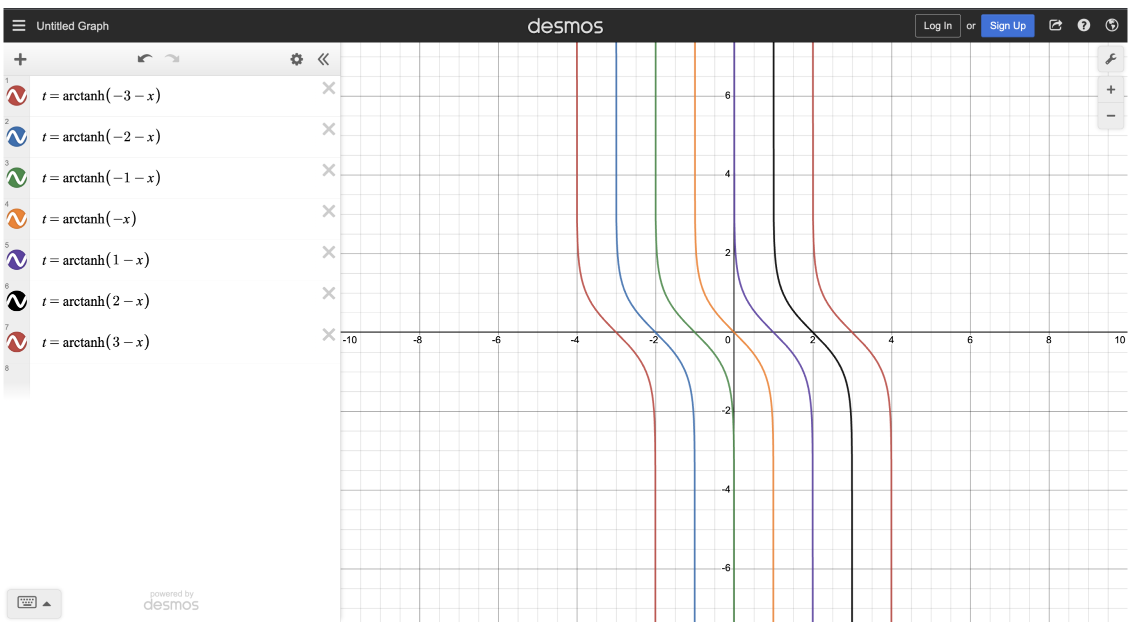
$$\begin{aligned} \frac{dx}{dt} &= -sech^2(t) \\ \int dx &= \int -sech^2(t)dt + C \\ x &= -\tanh(t) + C \end{aligned}$$

We will now apply the initial condition

$$\begin{aligned} x(0) &= x_0 \\ &\implies x_0 = -\tanh(0) + C \\ &\implies C = x_0 \end{aligned}$$

Hence our characteristics take the form $x(t; x_0) = -\tanh(t) + x_0$.

We will now sketch these characteristics for the following values of x_0: $x_0 = -3, -2, -1, 0, 1, 2, 3$. This is done below.



(ii) Determine the value of u along these characteristics to find a solution dependent on x_0.

To find the value of u along these characteristics to find a solution dependent on x_0, we must solve

$\frac{du(x(t:x_0),t)}{dt} = -u(x(t:x_0),t)$. This ODE is solved below

$$\begin{aligned}
\frac{du(x(t:x_0),t)}{dt} &= -u(x(t:x_0),t) \\
\int \frac{du(x(t:x_0),t)}{u(x(t:x_0)} &= -\int dt + I(x_0) \\
\ln|u(x(t:x_0),t)| &= -t + I(x_0) \\
|u(x(t:x_0),t)| &= e^{I(x_0)}e^{-t} \\
u(x(t:x_0),t) &= \pm e^{I(x_0)}e^{-t} \\
u(x(t:x_0),t) &= h(x_0)e^{-t} \quad \text{(Where } h(x_0) = \pm e^{I(x_0)}\text{)}
\end{aligned}$$

Thus as calculated above, the value of u along these characteristics is $u(x(t:x_0),t) = h(x_0)e^{-t}$.

(iii) Eliminate the dependence of x_0 in your solution in order to find an explicit solution $u(x,t)$ for a general initial condition $u(x,0) = f(x)$.

Applying the initial condition, we see that $f(x_0) = u(x_0,0) = u(x(0;x_0),0) = h(x_0)$, hence we can rewrite our expression for u as $u(x(t:x_0),t) = f(x_0)e^{-t}$. To eliminate the dependence of x_0, we must solve for x_0 in our expression for the characteristics found in part (i). $x(t;x_0) = -\tanh(t) + x_0 \implies x_0 = x(t;x_0) + \tanh(t)$. Plugging this in for x_0 in u and dropping the functional dependence, we obtain $u(x,t) = f(x+\tanh(t))e^{-t}$.

3. **Component Skills 6.1-6.2: Seasonal Migration of Orca Whales.** This problem taken from Ryan Creedon's 2021 class.

(i) Let $u(x,t)$ represent the population density of orca whales in the Pacific Ocean as a function of latitude x and time t. Suppose the whales migrate seasonally according to

$$u_t + c_0 \cos\left(\frac{2\pi}{T}t\right) u_x = 0, \tag{1}$$

where $c_0 > 0$ represents the maximum speed of migration and $T > 0$ controls the period of the migration. **Express (1) as a conservation law of the form $u_t + \phi_x = 0$. Is this conservation law linear, semi-linear, or nonlinear?**

Writing $u_t + x^2 u_x = -2ux$ in the form $u_t + \phi(u,x,t)_x = f(u,x,t)$ where $f = 0$, we obtain

$$\begin{aligned}
u_t + c_0 \cos\left(\frac{2\pi}{T}t\right) u_x &= 0 \\
u_t + \left(c_0 \cos\left(\frac{2\pi}{T}t\right) u\right)_x &= 0
\end{aligned}$$

Hence, the PDE in the required form is $u_t + \left(c_0 \cos\left(\frac{2\pi}{T}t\right) u\right)_x = 0$ with $\phi(u,x,t) = c_0 \cos\left(\frac{2\pi}{T}t\right) u$ and $f(u,x,t) = 0$.

Since the PDE $u_t + c_0 \cos\left(\frac{2\pi}{T}t\right) u_x = 0$ is already in the form $u_t + c(u,x,t)u_x = g(x,t,u)$ with $c(x,t,u) = c_0 \cos\left(\frac{2\pi}{T}t\right)$ and $g(x,t,u) = 0$, we can see that our conservation law is linear. Our conservation law is linear since c and g both don't depend on u.

(ii) Let $c_0 = \pi/2$ and $T = 4$. **Plot characteristics of (2) for** $x_0 = -2, -1, 0, 1,$ **and** 2 (You will want to use plotting software for this.)

We will start off by determining the form of the characteristics for this equation. To do so, we will solve the following IVP

$$\begin{cases} \frac{dx}{dt} = c_0 \cos\left(\frac{2\pi}{T}t\right) \\ x(0) = x_0 \end{cases}$$

We will first solve the ODE as show below

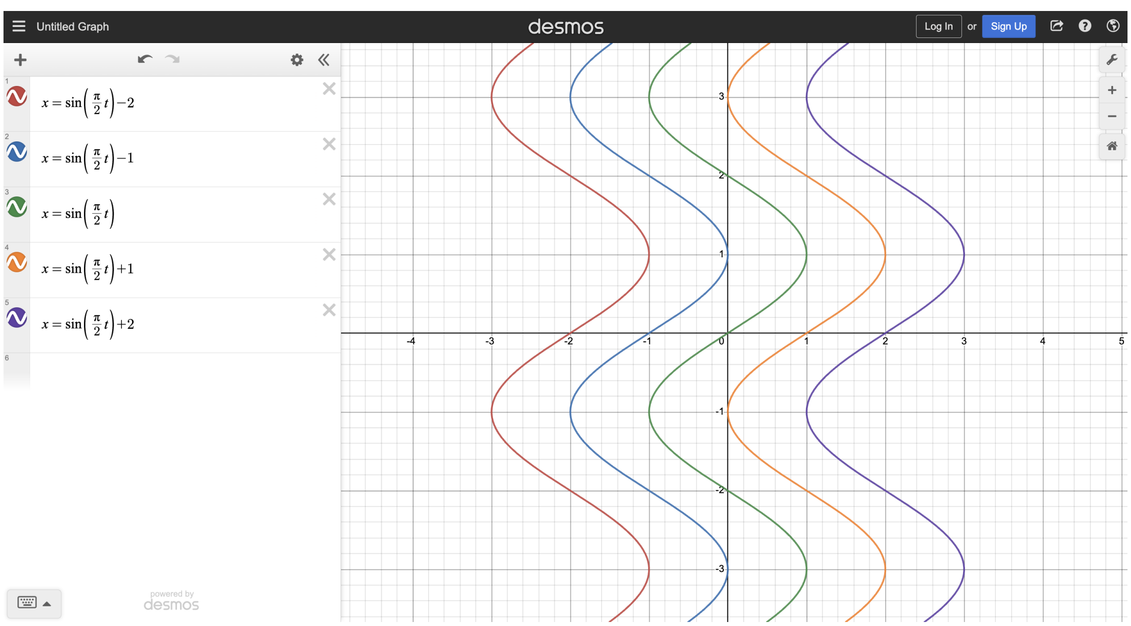
$$\begin{aligned} \frac{dx}{dt} &= c_0 \cos\left(\frac{2\pi}{T}t\right) \\ \int dx &= \int c_0 \cos\left(\frac{2\pi}{T}t\right) dt + C \\ x &= \frac{c_0 T}{2\pi} \sin\left(\frac{2\pi}{T}t\right) + C \end{aligned}$$

We will now apply the initial condition

$$\begin{aligned} & x(0) = x_0 \\ & \implies x_0 = \frac{c_0 T}{2\pi} \sin(0) + C \\ & \implies C = x_0 \end{aligned}$$

Hence our characteristics take the form $x(t; x_0) = \frac{c_0 T}{2\pi} \sin\left(\frac{2\pi}{T}t\right) + x_0$. In particular, when $c_0 = \pi/2$ and $T = 4$, the characteristics take the form $x(t; x_0) = \sin\left(\frac{\pi}{2}t\right) + x_0$

We will now sketch these characteristics for the following values of x_0: $x_0 = -2, -1, 0, 1, 2$. This is done below.



(iii) Suppose the initial population density of the orcas is

$$u(x, 0) = M sech(kx), \tag{2}$$

where $M > 0$ is the maximum initial population density and $k > 0$ controls the spread of the population density about this maximum value. **Solve the PDE (2) subject to the initial condition (3).**

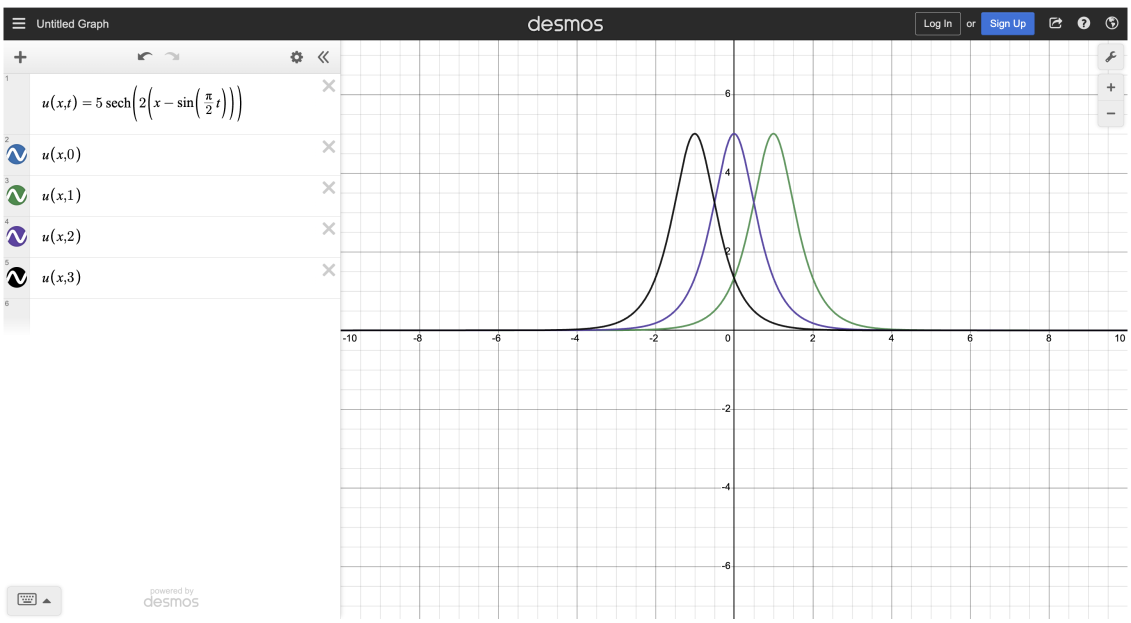
To solve this PDE subject to the initial conditions as described above, we will solve the following IVP

$$\begin{cases} \frac{du(x(t:x_0),t)}{dt} = 0 \\ u(x_0,0) = Msech(kx_0) \end{cases}$$

We can solve the ODE simply using direct integration, doing so we obtain $\frac{du(x(t:x_0),t)}{dt} = 0 \implies u(x(t;x_0),t) = I(x_0)$. Applying our initial condition we obtain $Msech(kx_0) = u(x_0,0) = u(x(0;x_0),0) = I(x_0)$. Rewriting $u(x(t;x_0),t)$ we obtain $u(x(t;x_0),t) = Msech(kx_0)$. To eliminate the dependence of x_0, we must solve for x_0 in our expression for the characteristics found in part (ii), which we found as $x(t;x_0) = \frac{c_0T}{2\pi}\sin\left(\frac{2\pi}{T}t\right) + x_0 \implies x_0 = x(t;x_0) - \frac{c_0T}{2\pi}\sin\left(\frac{2\pi}{T}t\right)$. Plugging this in for x_0 in u and dropping the functional dependence, we obtain our solution $u(x,t) = Msech\left(k\left(x - \frac{c_0T}{2\pi}\sin\left(\frac{2\pi}{T}t\right)\right)\right)$.

(iv) Let $c_0 = \pi/2$, $T = 4$, $M = 5$, and $k = 2$. **Create snapshot plots of your solution to part (c) at** $t = 0, 1, 2,$ **and** 3. (You will want to use plotting software for this.).

If we let $c_0 = \pi/2$, $T = 4$, $M = 5$, and $k = 2$, our solution from (iii) becomes $u(x,t) = 5sech\left(2\left(x - \sin\left(\frac{\pi}{2}t\right)\right)\right)$. We will now create snapshot plots of this solution for $t = 0, 1, 2,$ and 3.



Note: The snapshot plots at $t = 0$ and $t = 2$ overlap.

(v) In addition to seasonal migration, suppose the orca population is slowly declining due to effects of climate change. Equation (2) is modified to incorporate this effect:

$$u_t + c_0\cos\left(\frac{2\pi}{T}t\right)u_x = -\sigma u, \tag{3}$$

where $\sigma > 0$ controls the rate of decline of the population. **Solve the PDE (4) subject to the initial condition (3)**

To solve this PDE subject to the initial conditions as described in part (iii), we will solve the following IVP

$$\begin{cases} \frac{du(x(t:x_0),t)}{dt} = -\sigma u(x(t:x_0),t) \\ u(x_0,0) = Msech(kx_0) \end{cases}$$

We will start by solving this ODE as shown below

$$\begin{aligned}
\frac{du(x(t:x_0),t)}{dt} &= -\sigma u(x(t:x_0),t) \\
\int \frac{du(x(t:x_0),t)}{u(x(t:x_0)} &= -\sigma \int dt + I(x_0) \\
\ln|u(x(t:x_0),t)| &= -\sigma t + I(x_0) \\
|u(x(t:x_0),t)| &= e^{I(x_0)}e^{-\sigma t} \\
u(x(t:x_0),t) &= \pm e^{I(x_0)}e^{-\sigma t} \\
u(x(t:x_0),t) &= h(x_0)e^{-\sigma t} \quad (\text{Where } h(x_0) = \pm e^{I(x_0)})
\end{aligned}$$

Applying our initial condition we obtain $Msech(kx_0) = u(x_0,0) = u(x(0;x_0),0) = h(x_0)$. Rewriting $u(x(t;x_0),t)$ we obtain $u(x(t;x_0),t) = Msech(kx_0)e^{-\sigma t}$. To eliminate the dependence of x_0, we must solve for x_0 in our expression for the characteristics found in part (ii), which we found as $x(t;x_0) = \frac{c_0T}{2\pi}\sin\left(\frac{2\pi}{T}t\right) + x_0 \implies x_0 = x(t;x_0) - \frac{c_0T}{2\pi}\sin\left(\frac{2\pi}{T}t\right)$. Plugging this in for x_0 in u and dropping the functional dependence, we obtain our solution $u(x,t) = Msech\left(k\left(x - \frac{c_0T}{2\pi}\sin\left(\frac{2\pi}{T}t\right)\right)\right)e^{-\sigma t}$.

4. This problem taken from Ryan Creedon's 2021 class. Consider the following initial value problem of the modified inviscid Burger's (MIB) equation:

$$\begin{aligned}
u_t + u^p u_x &= 0, \\
u(x,0) &= \begin{cases} U & x \leq 0 \\ 0 & x > 0 \end{cases},
\end{aligned} \tag{4}$$

where $p \geq 1$ and $U > 0$. (Notice MIB reduces to the inviscid Burger's equation when $p = 1$.)

a. **Write the PDE as a conservation law of the form $u_t + \phi_x = 0$. Is this a linear, semi-linear, or nonlinear conservation law?**

Writing $u_t + u^p u_x = 0$ in the form $u_t + \phi(u,x,t)_x = f(u,x,t)$ where $f = 0$, we obtain

$$\begin{aligned}
u_t + u^p u_x &= 0 \\
u_t + \left(\frac{u^{p+1}}{p+1}\right)_x &= 0
\end{aligned}$$

Hence, the PDE in the required form is $u_t + \left(\frac{u^{p+1}}{p+1}\right)_x = 0$ with $\phi(u,x,t) = \frac{u^{p+1}}{p+1}$ and $f(u,x,t) = 0$.

Since the PDE $u_t + u^p u_x = 0$ is already in the form $u_t + c(u,x,t)u_x = g(x,t,u)$ with $c(x,t,u) = u^p$ and $g(x,t,u) = 0$, we can see that our conservation law is non-linear. Our conservation law is non-linear since c depends on u.

b. Compute the **breaking time** and **breaking location** of (4). You may compute the breaking time either via using a sequence of functions which approximates our initial condition or through looking at the sign of $c(f(u))$ to the left and the right of our initial discontinuity.

Since $f(x) = u(x,0) = \begin{cases} U & x \leq 0 \\ 0 & x > 0 \end{cases}$, it follows that the value of $cf(u)$ from the left of our initial discontinuity is $cf(0^-) = U$. Similarly, the value of $cf(u)$ from the right of our initial discontinuity is $cf(0^+) = 0$. Since $cf(0^-) > cf(0^+)$, it follows that we have a gradient catastrophe/shock immediately, hence $t_b = 0$ and $x_b = 0$.

c. According to the Rankine-Hugoniot condition, **what is the velocity of the shock wave modeled by (4)?** (Your answer will be a function of U and p.)

As shown in part (a), $\phi = \frac{u^{p+1}}{p+1}$. Furthermore, according to the Rankine-Hugoniot condition, the velocity of the shock wave modeled by (4) is given by $\frac{dx_s}{dt} = \frac{\phi(u^+)-\phi(u^-)}{u^+-u^-}$. In our case $u^- = U$ and $u^+ = 0$, thus the velocity is given by the following $\frac{dx_s}{dt} = \frac{\phi(U)-\phi(0)}{U-0} = \frac{\frac{U^{p+1}}{p+1}}{U} = \frac{U^p}{p+1}$.

d. Using your results from part (c), **what is the trajectory of the shock wave?**

Using the results from part (c), we can obtain the trajectory of the shock wave by solving the ODE: $\frac{dx_s}{dt} = \frac{U^p}{p+1}$. This ODE is solved below

$$\begin{aligned} \frac{dx_s}{dt} &= \frac{U^p}{p+1} \\ \int dx_s &= \int \frac{U^p}{p+1} dt + C \\ x_s(t) &= \frac{U^p}{p+1} t + C \end{aligned}$$

This ODE is paired with the appropriate initial condition $x_s(t_b) = x_b \implies x_s(0) = 0$ to from an IVP. Applying this initial condition we obtain $x_s(0) = 0 = C$. Hence the trajectory of the shock wave is given by $x_s(t) = \frac{U^p}{p+1} t$.

e. Using your results from part (c), **what is the velocity of the shock wave in the limit as** $p \to \infty$ when $0 < U \leq 1$?

To find the velocity of the shock wave in the limit as $p \to \infty$ when $0 < U \leq 1$, we will find $\lim_{p\to\infty} \frac{dx_s}{dt}$ for two cases, (i) $0 < U < 1$, and (ii) $U = 1$. For case (i) we can see that $\lim_{p\to\infty} \frac{dx_s}{dt} = \lim_{p\to\infty} \frac{U^P}{p+1}$, since $0 < U < 1$ it follows that $\lim_{p\to\infty} \frac{U^P}{p+1} = \frac{0^+}{\infty+1} = \frac{0}{\infty} = 0$. For case (ii) we can see that $\lim_{p\to\infty} \frac{dx_s}{dt} = \lim_{p\to\infty} \frac{1}{p+1} = \frac{1}{\infty+1} = \frac{1}{\infty} = 0$. Combining these two cases we can say that $\frac{dx_s}{dt} = 0$ as $p \to \infty$ for $0 < U \leq 1$.

f. If $U > \sqrt{e} \approx 1.64872$, it can be shown that the slowest traveling shock wave of (c) corresponds to $p = 1$. (You do not need to prove this fact.) If instead $1 < U < \sqrt{e}$, then there exists a $p > 1$ that minimizes the velocity of the shock wave computed in part (c). **What is this p value?** *Hint*: $z^r = e^{r\ln(z)}$.

To find the value of p that minimizes $\frac{dx_s}{dt}$, we will take the derivative of the velocity of the shock wave with respect to p. This calculation is done below

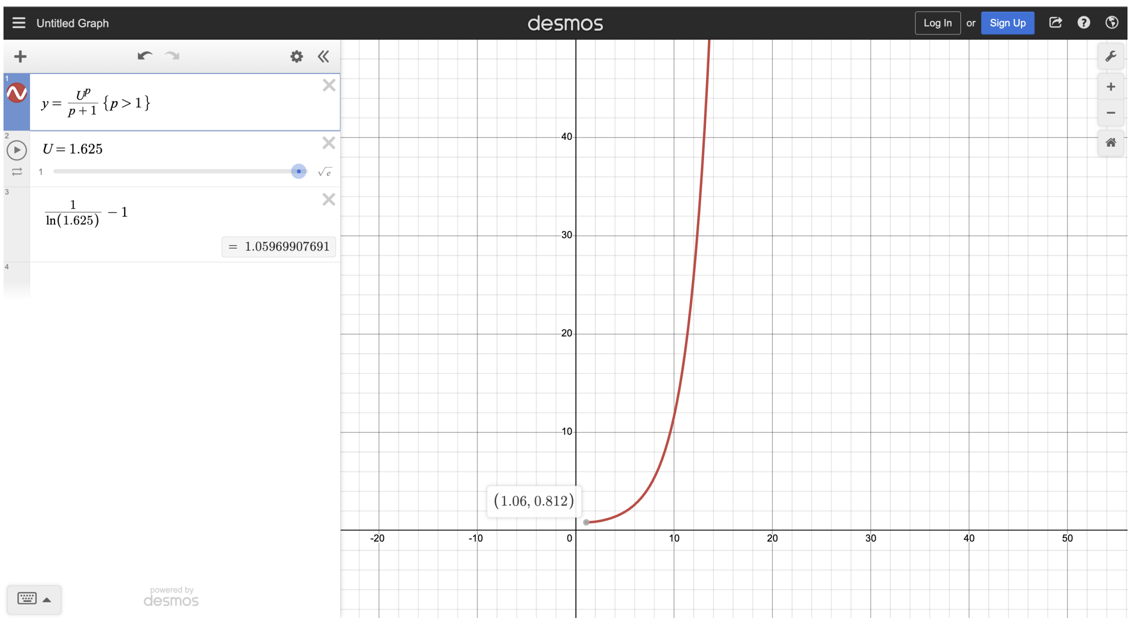
$$\begin{aligned} \frac{d}{dp}\left(\frac{dx_s}{dt}\right) &= \frac{(p+1)\ln(U)U^p - U^P}{(p+1)^2} \\ &= \frac{(p\ln(U) + \ln(U))U^p - U^P}{(p+1)^2} \\ &= \frac{(p\ln(U) + \ln(U) - 1)U^p}{(p+1)^2} \end{aligned}$$

Setting this expression equal to zero and solving for p we see that $\frac{(p\ln(U)+\ln(U)-1)U^p}{(p+1)^2} = 0 \implies (p\ln(U) +$

$\ln(U) - 1)U^p = 0$. Since when solving for p, U^p can't equal to zero, thus we see

$$\begin{aligned}
p\ln(U) + \ln(U) - 1 &= 0 \\
p\ln(U) + \ln(U) &= 1 \\
(p+1)\ln(U) &= 1 \\
p + 1 &= \frac{1}{\ln(U)} \\
p &= \frac{1}{\ln(U)} - 1
\end{aligned}$$

Thus as found above our one critical point is $p = \frac{1}{\ln(U)} - 1$. Instead of proceeding with the second derivative test, which would be tedious in this case, we will show the plot of $\frac{dx_s}{dt}$ versus p for $1 < U < \sqrt{e}$ and see that this value is in fact a minimum for all U in this range. (Plot inspired by Kai Poffenbarger.)



Hence, the value of p that minimizes the shock wave velocity is $p = \frac{1}{\ln(U)} - 1$.