AMATH 353 Homework 4

Jaiden Atterbury

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Component Skills Practice

Exercise 1. Skill 4.1.

Exercise 8.2 in Knobel.

Exercise 8.2 in Knobel states: Use direct substitution to verify that the function $u(x,t) = \cos(t)\sin(x)$ is a solution of $u_{tt} = u_{xx}$. According to the general form (8.1) for solutions of the wave equation, it must then be possible to write $u(x,t)$ as $u(x,t) = F(x-t) + G(x+t)$. What are the left and right moving waves in this case?

We will start by verifying the function $u(x,t) = \cos(t)\sin(x)$ is a solution of $u_{tt} = u_{xx}$ using direct substitution. First off, we need to calculate the partial derivatives, we can see that these are $u_t = -\sin(t)\sin(x)$, $u_{tt} = -\cos(t)\sin(x)$, $u_x = \cos(t)\cos(x)$, and lastly, $u_{xx} = -\cos(t)\sin(x)$. Thus substituting u_{tt} and u_{xx} into the wave equation $u_{tt} = u_{xx}$, we obtain $-\cos(t)\sin(x) = -\cos(t)\sin(x)$. Therefore we can see that, $u(x,t) = \cos(t)\sin(x)$ is a solution of $u_{tt} = u_{xx}$.

Next we will find the left and right moving waves of the solution $u(x,t) = \cos(t)\sin(x)$ by rewriting it in the form of $u(x,t) = F(x-t) + G(x+t)$. Using the product-to-sum identity $\cos(\alpha)\sin(\beta) = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$, and the fact that $\sin(x)$ is an odd function, which implies $\sin(-x) = -\sin(x)$, we can rewrite $u(x,t)$ as $u(x,t) = \frac{1}{2}[\sin(t+x) - \sin(t-x)] = \frac{1}{2}[\sin(x+t) + \sin(x-t)] = \frac{1}{2}\sin(x+t) + \frac{1}{2}\sin(x-t)$. Thus the right moving wave is $F(x-t) = \frac{1}{2}\sin(x-t)$, and the left moving wave is $G(x+t) = \frac{1}{2}\sin(x+t)$.

Exercise 2. Skill 4.2.

Consider the following initial value problem:

$$\begin{cases} u_{tt} = u_{xx} \\ u(x,0) = \frac{4}{1+x^2} \\ u_t(x,0) \equiv 0 \end{cases} .$$

a. Use d'Alembert's formula to find $u(x,t)$ for $t \geq 0$.

In order to find $u(x,t)$ for $t \geq 0$ of the above initial value problem, we must use d'Alembert's formula which states that $u(x,t) = \frac{1}{2}(\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} \psi(s)ds$, where $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$. In our case $c = 1$, and if we let $\phi(x) = \frac{4}{1+x^2}$, and $\psi(x) \equiv 0$, then d'Alembert's formula gives us the following:

$$\begin{aligned} u(x,t) &= \frac{1}{2}(\phi(x-ct)+\phi(x+ct)) + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} \psi(s)ds \\ &= \frac{1}{2}\left(\frac{4}{1+(x-t)^2} + \frac{4}{1+(x+t)^2}\right) + \frac{1}{2}\int\limits_{x-t}^{x+t} 0ds \\ &= \frac{2}{1+(x-t)^2} + \frac{2}{1+(x+t)^2} \end{aligned}$$

Thus, the solution to the initial value problem using d'Alembert's formula is $u(x,t) = \frac{2}{1+(x-t)^2} + \frac{2}{1+(x+t)^2}$.

b. Suppose instead $u_t(x,0) = 8sech(x)\tanh(x)$. If the initial conditions for $u(x,0)$ remains the same, what is $u(x,t)$ in this case?

If $u_t(x,0)$ is instead $u_t(x,0) = 8sech(x)\tanh(x)$, and $u(x,0)$ remains the same, then we can again use d'Alembert's formula to find $u(x,t)$ for $t \geq 0$. In our case $c = 1$, and if we let $\phi(x) = \frac{4}{1+x^2}$, and $\psi(x) = 8sech(x)\tanh(x)$, then d'Alembert's formula gives us the following:

$$\begin{aligned} u(x,t) &= \frac{1}{2}(\phi(x-ct)+\phi(x+ct)) + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} \psi(s)ds \\ &= \frac{1}{2}\left(\frac{4}{1+(x-t)^2} + \frac{4}{1+(x+t)^2}\right) + \frac{1}{2}\int\limits_{x-t}^{x+t} 8sech(s)\tanh(s)ds \\ &= \frac{2}{1+(x-t)^2} + \frac{2}{1+(x+t)^2} + 4\int\limits_{x-t}^{x+t} sech(s)\tanh(s)ds \\ &= \frac{2}{1+(x-t)^2} + \frac{2}{1+(x+t)^2} + 4\left[-sech(s)\right]_{x-t}^{x+t} \\ &= \frac{2}{1+(x-t)^2} + \frac{2}{1+(x+t)^2} - 4sech(x+t) + 4sech(x-t) \end{aligned}$$

Thus, the solution to the new initial value problem using d'Alembert's formula is $u(x,t) = \frac{2}{1+(x-t)^2} + \frac{2}{1+(x+t)^2} - 4sech(x+t) + 4sech(x-t)$.

Exercise 3. Skill 4.3.

Consider an infinitely long guitar string (purchased at your friendly neighborhood theoretical music store). Let $u = u(x,t)$ represent the displacement of the guitar string relative to its flat, unplucked equilibrium. Suppose the string is initially plucked so that

$$u(x,0) = \begin{cases} 1 & -1 < x < 0 \\ -1 & 0 < x < 1 \\ 0 & |x| > 1 \end{cases}$$

$$u_t(x,0) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases}.$$

Suppose the evolution of the string is modeled by the wave equation

$$u_{tt} = u_{xx}.$$

a. If we wish to know the value of u at $(x = 3, t = 5)$, what interval of initial conditions must we consider?

In order to know what interval of initial conditions we must consider if we wish to know the value of u at $(x = 3, t = 5)$, we must find the domain of dependence of the point $(\tilde{x}, \tilde{t}) = (3, 5)$. The domain of dependence of a point $(\tilde{x}, \tilde{t})$ is the interval $\left[\tilde{x} - c\tilde{t}, \tilde{x} + c\tilde{t}\right]$ such that the value of $u(\tilde{x}, \tilde{t})$ only depends on $\phi(x_0)$ and $\psi(x_0)$ for x_0 in the aforementioned interval.

Thus in our given scenario, $c = 1$ and $(\tilde{x}, \tilde{t}) = (3, 5)$. Hence the domain of dependence of $(\tilde{x}, \tilde{t}) = (3, 5)$ is $\left[\tilde{x} - c\tilde{t}, \tilde{x} + c\tilde{t}\right] = [3 - 5, 3 + 5] = [-2, 8]$. Therefore we must consider initial conditions $\phi(x_0)$ and $\psi(x_0)$ for $x_0 \in [-2, 8]$ if we wish to know the value of u at $(x = 3, t = 5)$.

b. Does the initial condition at $x_0 = 0$ influence the value of u at $(x = 3, t = 5)$?

Since $x_0 = 0$ is in the interval of dependence of $(\tilde{x}, \tilde{t}) = (3, 5)$, namely $x_0 \in [-2, 8]$, the initial conditions at $x_0 = 0$, $\phi(0)$ and $\psi(0)$, will influence the value of u at $(x = 3, t = 5)$.

c. Does the interval of initial conditions $-1 < x_0 < 1$ influence the value of u at $(x = 3, t = 5)$?

Since the interval of initial conditions we are looking at is $x_0 \in [-1, 1]$, which is a subset of the interval of dependence of $(\tilde{x}, \tilde{t}) = (3, 5)$, namely $x_0 \in [-1, 1] \subset [-2, 8]$, all of the initial conditions $\phi(x_0)$ and $\psi(x_0)$ for $x_0 \in [-1, 1]$ will influence the value of u at $(x = 3, t = 5)$.

d. When does the initial disturbance, i.e. the initial conditions along $-1 < x_0 < 1$, first arrive at $\tilde{x} = 3$?

Since characteristic lines of the domain of influence start at a t value of 0, we are only looking at rightward propagating lines, which are lines with positive slope. In particular, these lines take the form $t = \frac{1}{c}(x - \tilde{x}) + \tilde{t}$, which in our case reduces to $t = (x - \tilde{x}) + \tilde{t}$ since $c = 1$. The left-most characteristic line, which has an x-intercept of $x_0 = -1$, has an equation $t = x + 1$, hence it arrives at $\tilde{x} = 3$ at $t = 4$. Furthermore, the right-most characteristic line, which has an x-intercept of $x_0 = 1$, has an equation $t = x - 1$, hence it arrives at $\tilde{x} = 3$ at $t = 2$. Therefore the disturbance $-1 < x_0 < 1$ arrives at $\tilde{x} = 3$ at times $2 \leq t \leq 4$. Thus the initial disturbance first arrives at $\tilde{x} = 3$ at a time of $t = 2$.

Exercise 4. Skill 4.4.

Use the characteristic diagram to obtain a graphical solution to the initial value problem presented in **Exercise 3**:

$$\begin{cases} u_{tt} = u_{xx} \\ u(x,0) = \begin{cases} 1 & -1 < x < 0 \\ -1 & 0 < x < 1 \\ 0 & |x| > 1 \end{cases} \\ u_t(x,0) = \begin{cases} 1 & |x| < 1 \\ 0 & |x| > 1 \end{cases} \end{cases}.$$

Use Example 4.4.3 from the Lecture Notes as a template.

In order to create a characteristic diagram, we need to compute the domains of influence of the intervals $x_0 < -1$, $-1 < x_0 < 0$, $0 < x_0 < 1$, and $x_0 > 1$. These intervals correspond to the four cases over which $u(x_0, 0)$ is defined.

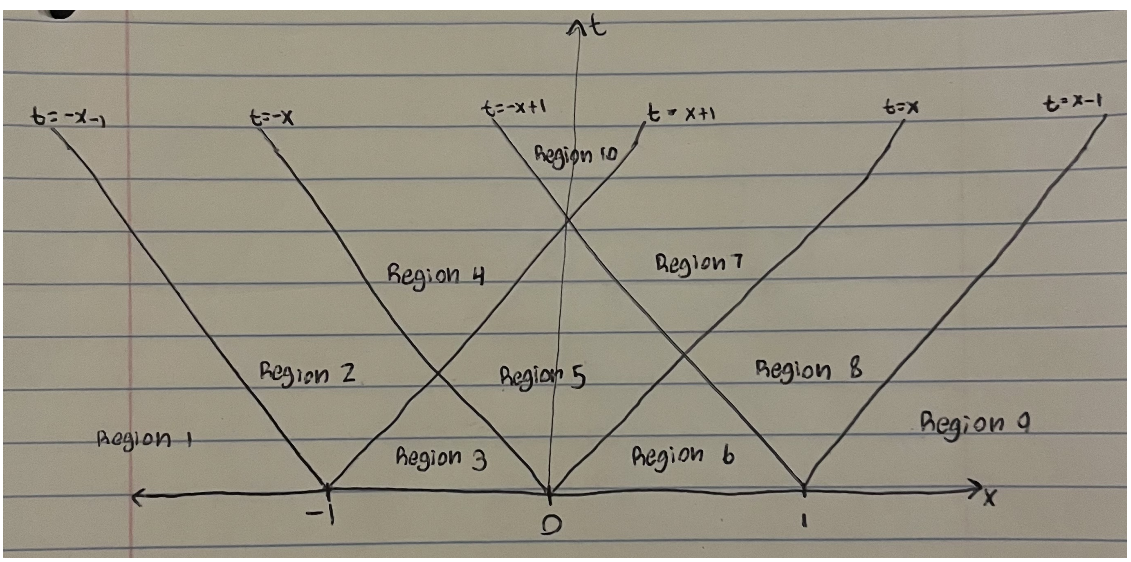
To find the domain of influence of the interval $x_0 < -1$, we must find the outward propagating characteristic at the point $x_0 = -1$. This characteristic must be $t = x + 1$, since this line passes through $x = -1$ and has slope $1/c = 1$ (since $c = 1$). The region bounded by the x-axis and to the left of $t = x + 1$ is the domain of influence of the interval $x_0 < -1$.

To find the domain of influence of the interval $-1 < x_0 < 0$, we must find the outward propagating characteristics at the points $x_0 = -1$ and $x_0 = 0$. The outward propagating characteristic at $x_0 = -1$ is $t = -x - 1$, since this line passes through $x = -1$ with slope $-1/c = -1$ (since $c = 1$). The outward propagating characteristic at the point $x_0 = 0$ is $t = x$, since this line passes through the origin and has slope $1/c = 1$ (since $c = 1$). The region bounded between the x-axis, the line $t = -x - 1$, and the line $t = x$ is the domain of influence of the interval $-1 < x_0 < 0$.

To find the domain of influence of the interval $0 < x_0 < 1$, we must find the outward propagating characteristics at the points $x_0 = 0$ and $x_0 = 1$. The outward propagating characteristic at $x_0 = 0$ is $t = -x$, since this line passes through the origin with slope $-1/c = -1$ (since $c = 1$). The outward propagating characteristic at the point $x_0 = 1$ is $t = x - 1$, since this line passes through $x = 1$ and has slope $1/c = 1$ (since $c = 1$). The region bounded between the x-axis, the line $t = x - 1$, and the line $t = -x$ is the domain of influence of the interval $0 < x_0 < 1$.

To find the domain of influence of the interval $x_0 > 1$, we must find the outward propagating characteristic at the point $x_0 = 1$. This characteristic must be $t = -x + 1$, since this line passes through $x = 1$ and has slope $-1/c = -1$ (since $c = 1$). The region bounded by the x-axis and to the right of $t = -x + 1$ is the domain of influence of the interval $x_0 > 1$.

Plotting each of the outward propagating characteristics in the xt-plane gives us the characteristic diagram below.



To obtain a graphical solution to the initial value problem presented in **Exercise 3**, we will now determine the value of u in each region.

- **Region 1.**

Let $(\tilde{x}, \tilde{t})$ be a point in this region. d'Alembert's formula tells us

$$u(\tilde{x}, \tilde{t}) = \frac{1}{2}\left(\phi(\tilde{x} - \tilde{t}) + \phi(\tilde{x} + \tilde{t})\right) + \frac{1}{2}\int_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} \psi(x)dx.$$

In this region, $\tilde{x} \pm \tilde{t} < -1$. We have $\phi(x_0) \equiv 0$ and $\psi(x_0) \equiv 0$ for $x_0 < -1$. Therefore,

$$\begin{aligned} u(\tilde{x}, \tilde{t}) &= \frac{1}{2}\left(\phi(\tilde{x} - \tilde{t}) + \phi(\tilde{x} + \tilde{t})\right) + \frac{1}{2}\int_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} \psi(x)dx \\ &= \frac{1}{2}[(0) + (0)] + \frac{1}{2}\int_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} (0)dx \\ &= 0. \end{aligned}$$

For any $(\tilde{x}, \tilde{t})$ in this region, we have $u(\tilde{x}, \tilde{t}) \equiv 0$.

- **Region 2.**

Let $(\tilde{x}, \tilde{t})$ be a point in this region. d'Alembert's formula tells us

$$u(\tilde{x}, \tilde{t}) = \frac{1}{2}\left(\phi(\tilde{x} - \tilde{t}) + \phi(\tilde{x} + \tilde{t})\right) + \frac{1}{2}\int_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} \psi(x)dx.$$

In this region, $\tilde{x} - \tilde{t} < -1$ and $-1 < \tilde{x} + \tilde{t} < 0$. We have $\phi(x_0) \equiv 0$ and $\psi(x_0) \equiv 0$ for $x_0 < -1$, and $\phi(x_0) \equiv 1$ and $\psi(x_0) \equiv 1$ for $-1 < x_0 < 0$. Therefore,

$$\begin{aligned} u(\tilde{x}, \tilde{t}) &= \frac{1}{2}\left(\phi(\tilde{x} - \tilde{t}) + \phi(\tilde{x} + \tilde{t})\right) + \frac{1}{2}\int_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} \psi(x)dx \\ &= \frac{1}{2}[(0) + (1)] + \frac{1}{2}\int_{\tilde{x}-\tilde{t}}^{-1} (0)dx + \frac{1}{2}\int_{-1}^{\tilde{x}+\tilde{t}} (1)dx \\ &= \frac{1}{2} + \frac{1}{2}(\tilde{x} + \tilde{t} + 1) \\ &= \frac{1}{2}(\tilde{x} + \tilde{t}) + 1. \end{aligned}$$

For any $(\tilde{x}, \tilde{t})$ in this region, we have $u(\tilde{x}, \tilde{t}) = \frac{1}{2}(\tilde{x} + \tilde{t}) + 1$.

- **Region 3.**

Let $(\tilde{x}, \tilde{t})$ be a point in this region. d'Alembert's formula tells us

$$u(\tilde{x}, \tilde{t}) = \frac{1}{2}\left(\phi(\tilde{x} - \tilde{t}) + \phi(\tilde{x} + \tilde{t})\right) + \frac{1}{2}\int_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} \psi(x)dx.$$

In this region, $-1 < \tilde{x} \pm \tilde{t} < 0$. We have $\phi(x_0) \equiv 1$ and $\psi(x_0) \equiv 1$ for $-1 < x_0 < 0$. Therefore,

$$\begin{aligned}
u(\tilde{x}, \tilde{t}) &= \frac{1}{2}\left(\phi(\tilde{x} - \tilde{t}) + \phi(\tilde{x} + \tilde{t})\right) + \frac{1}{2}\int\limits_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} \psi(x)dx \\
&= \frac{1}{2}[(1) + (1)] + \frac{1}{2}\int\limits_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} (1)dx \\
&= 1 + \frac{1}{2}(\tilde{x} + \tilde{t} - \tilde{x} + \tilde{t}) \\
&= \tilde{t} + 1
\end{aligned}$$

For any $(\tilde{x}, \tilde{t})$ in this region, we have $u(\tilde{x}, \tilde{t}) = \tilde{t} + 1$.

- **Region 4.**

Let $(\tilde{x}, \tilde{t})$ be a point in this region. d'Alembert's formula tells us

$$u(\tilde{x}, \tilde{t}) = \frac{1}{2}\left(\phi(\tilde{x} - \tilde{t}) + \phi(\tilde{x} + \tilde{t})\right) + \frac{1}{2}\int\limits_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} \psi(x)dx.$$

In this region, $\tilde{x} - \tilde{t} < -1$ and $0 < \tilde{x} + \tilde{t} < 1$. We have $\phi(x_0) \equiv 0$ and $\psi(x_0) \equiv 0$ for $x_0 < -1$, and $\phi(x_0) \equiv -1$ and $\psi(x_0) \equiv 1$ for $0 < x_0 < 1$. Therefore,

$$\begin{aligned}
u(\tilde{x}, \tilde{t}) &= \frac{1}{2}\left(\phi(\tilde{x} - \tilde{t}) + \phi(\tilde{x} + \tilde{t})\right) + \frac{1}{2}\int\limits_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} \psi(x)dx \\
&= \frac{1}{2}[(0) + (-1)] + \frac{1}{2}\int\limits_{\tilde{x}-\tilde{t}}^{-1} (0)dx + \frac{1}{2}\int\limits_{-1}^{\tilde{x}+\tilde{t}} (1)dx \\
&= \frac{-1}{2} + \frac{1}{2}(\tilde{x} + \tilde{t} + 1) \\
&= \frac{1}{2}(\tilde{x} + \tilde{t}).
\end{aligned}$$

For any $(\tilde{x}, \tilde{t})$ in this region, we have $u(\tilde{x}, \tilde{t}) = \frac{1}{2}(\tilde{x} + \tilde{t})$.

- **Region 5.**

Let $(\tilde{x}, \tilde{t})$ be a point in this region. d'Alembert's formula tells us

$$u(\tilde{x}, \tilde{t}) = \frac{1}{2}\left(\phi(\tilde{x} - \tilde{t}) + \phi(\tilde{x} + \tilde{t})\right) + \frac{1}{2}\int\limits_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} \psi(x)dx.$$

In this region, $-1 < \tilde{x} - \tilde{t} < 0$ and $0 < \tilde{x} + \tilde{t} < 1$. We have $\phi(x_0) \equiv 1$ and $\psi(x_0) \equiv 1$ for $-1 < x_0 < 0$, and

$\phi(x_0) \equiv -1$ and $\psi(x_0) \equiv 1$ for $0 < x_0 < 1$. Therefore,

$$\begin{aligned}
u(\tilde{x}, \tilde{t}) &= \frac{1}{2}\left(\phi(\tilde{x} - \tilde{t}) + \phi(\tilde{x} + \tilde{t})\right) + \frac{1}{2}\int_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} \psi(x)dx \\
&= \frac{1}{2}[(1) + (-1)] + \frac{1}{2}\int_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} (1)dx \\
&= \frac{1}{2}(\tilde{x} + \tilde{t} - \tilde{x} + \tilde{t}) \\
&= \tilde{t}.
\end{aligned}$$

For any $(\tilde{x}, \tilde{t})$ in this region, we have $u(\tilde{x}, \tilde{t}) = \tilde{t}$.

- **Region 6.**

Let $(\tilde{x}, \tilde{t})$ be a point in this region. d'Alembert's formula tells us

$$u(\tilde{x}, \tilde{t}) = \frac{1}{2}\left(\phi(\tilde{x} - \tilde{t}) + \phi(\tilde{x} + \tilde{t})\right) + \frac{1}{2}\int_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} \psi(x)dx.$$

In this region, $0 < \tilde{x} \pm \tilde{t} < 1$. We have $\phi(x_0) \equiv -1$ and $\psi(x_0) \equiv 1$ for $0 < x_0 < 1$. Therefore,

$$\begin{aligned}
u(\tilde{x}, \tilde{t}) &= \frac{1}{2}\left(\phi(\tilde{x} - \tilde{t}) + \phi(\tilde{x} + \tilde{t})\right) + \frac{1}{2}\int_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} \psi(x)dx \\
&= \frac{1}{2}[(-1) + (-1)] + \frac{1}{2}\int_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} (1)dx \\
&= -1 + \frac{1}{2}(\tilde{x} + \tilde{t} - \tilde{x} + \tilde{t}) \\
&= \tilde{t} - 1
\end{aligned}$$

For any $(\tilde{x}, \tilde{t})$ in this region, we have $u(\tilde{x}, \tilde{t}) = \tilde{t} - 1$.

- **Region 7.**

Let $(\tilde{x}, \tilde{t})$ be a point in this region. d'Alembert's formula tells us

$$u(\tilde{x}, \tilde{t}) = \frac{1}{2}\left(\phi(\tilde{x} - \tilde{t}) + \phi(\tilde{x} + \tilde{t})\right) + \frac{1}{2}\int_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} \psi(x)dx.$$

In this region, $-1 < \tilde{x} - \tilde{t} < 0$ and $\tilde{x} + \tilde{t} > 1$. We have $\phi(x_0) \equiv 1$ and $\psi(x_0) \equiv 1$ for $-1 < x_0 < 0$, and

$\phi(x_0) \equiv 0$ and $\psi(x_0) \equiv 0$ for $x_0 > 1$. Therefore,

$$\begin{aligned}
u(\tilde{x},\tilde{t}) &= \frac{1}{2}\left(\phi(\tilde{x}-\tilde{t}) + \phi(\tilde{x}+\tilde{t})\right) + \frac{1}{2}\int_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} \psi(x)dx \\
&= \frac{1}{2}[(1) + (0)] + \frac{1}{2}\int_{\tilde{x}-\tilde{t}}^{1} (1)dx + \frac{1}{2}\int_{1}^{\tilde{x}+\tilde{t}} (0)dx \\
&= \frac{1}{2} + \frac{1}{2}(1 - \tilde{x} + \tilde{t}) \\
&= \frac{1}{2}(\tilde{t} - \tilde{x}) + 1.
\end{aligned}$$

For any $(\tilde{x},\tilde{t})$ in this region, we have $u(\tilde{x},\tilde{t}) = \frac{1}{2}(\tilde{t} - \tilde{x}) + 1$.

- **Region 8.**

Let $(\tilde{x},\tilde{t})$ be a point in this region. d'Alembert's formula tells us

$$u(\tilde{x},\tilde{t}) = \frac{1}{2}\left(\phi(\tilde{x}-\tilde{t}) + \phi(\tilde{x}+\tilde{t})\right) + \frac{1}{2}\int_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} \psi(x)dx.$$

In this region, $0 < \tilde{x} - \tilde{t} < 1$ and $\tilde{x} + \tilde{t} > 1$. We have $\phi(x_0) \equiv -1$ and $\psi(x_0) \equiv 1$ for $0 < x_0 < 1$, and $\phi(x_0) \equiv 0$ and $\psi(x_0) \equiv 0$ for $x_0 > 0$. Therefore,

$$\begin{aligned}
u(\tilde{x},\tilde{t}) &= \frac{1}{2}\left(\phi(\tilde{x}-\tilde{t}) + \phi(\tilde{x}+\tilde{t})\right) + \frac{1}{2}\int_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} \psi(x)dx \\
&= \frac{1}{2}[(-1) + (0)] + \frac{1}{2}\int_{\tilde{x}-\tilde{t}}^{1} (1)dx + \frac{1}{2}\int_{1}^{\tilde{x}+\tilde{t}} (0)dx \\
&= \frac{-1}{2} + \frac{1}{2}(1 - \tilde{x} + \tilde{t}) \\
&= \frac{1}{2}(\tilde{t} - \tilde{x}).
\end{aligned}$$

For any $(\tilde{x},\tilde{t})$ in this region, we have $u(\tilde{x},\tilde{t}) = \frac{1}{2}(\tilde{t} - \tilde{x})$.

- **Region 9.**

Let $(\tilde{x},\tilde{t})$ be a point in this region. d'Alembert's formula tells us

$$u(\tilde{x},\tilde{t}) = \frac{1}{2}\left(\phi(\tilde{x}-\tilde{t}) + \phi(\tilde{x}+\tilde{t})\right) + \frac{1}{2}\int_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} \psi(x)dx.$$

In this region, $\tilde{x} \pm \tilde{t} > 1$. We have $\phi(x_0) \equiv 0$ and $\psi(x_0) \equiv 0$ for $x_0 > 1$. Therefore,

$$
\begin{aligned}
u(\tilde{x}, \tilde{t}) &= \frac{1}{2}\left(\phi(\tilde{x}-\tilde{t}) + \phi(\tilde{x}+\tilde{t})\right) + \frac{1}{2}\int\limits_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} \psi(x)dx \\
&= \frac{1}{2}[(0) + (0)] + \frac{1}{2}\int\limits_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} (0)dx \\
&= 0.
\end{aligned}
$$

For any $(\tilde{x}, \tilde{t})$ in this region, we have $u(\tilde{x}, \tilde{t}) \equiv 0$.

- **Region 10.**

Let $(\tilde{x}, \tilde{t})$ be a point in this region. d'Alembert's formula tells us

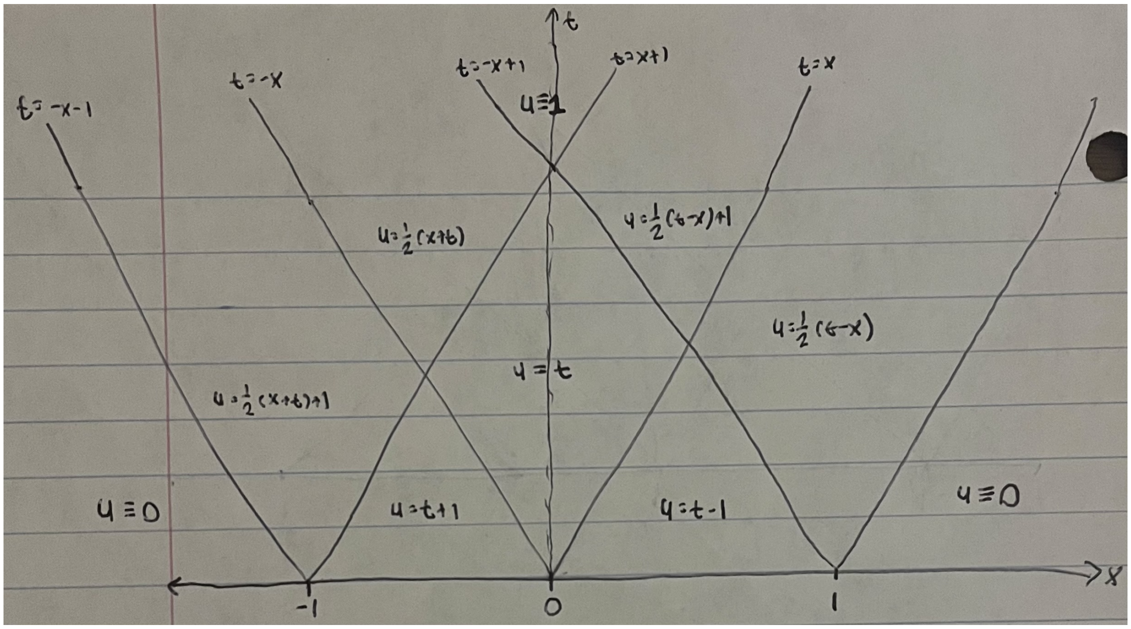
$$
u(\tilde{x}, \tilde{t}) = \frac{1}{2}\left(\phi(\tilde{x}-\tilde{t}) + \phi(\tilde{x}+\tilde{t})\right) + \frac{1}{2}\int\limits_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} \psi(x)dx.
$$

In this region, $\tilde{x} - \tilde{t} < -1$ and $\tilde{x} + \tilde{t} > 1$. We have $\phi(x_0) \equiv 0$ and $\psi(x_0) \equiv 0$ for $x_0 < -1$, and $\phi(x_0) \equiv 0$ and $\psi(x_0) \equiv 0$ for $x_0 > 0$. However, we also have $\phi(x_0) \equiv 1$ and $\psi(x_0) \equiv 1$ for $-1 < x_0 < 1$ Therefore,

$$
\begin{aligned}
u(\tilde{x}, \tilde{t}) &= \frac{1}{2}\left(\phi(\tilde{x}-\tilde{t}) + \phi(\tilde{x}+\tilde{t})\right) + \frac{1}{2}\int\limits_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} \psi(x)dx \\
&= \frac{1}{2}[(0) + (0)] + \frac{1}{2}\int\limits_{\tilde{x}-\tilde{t}}^{-1} (0)dx + \frac{1}{2}\int\limits_{-1}^{1} (1)dx + \frac{1}{2}\int\limits_{1}^{\tilde{x}+\tilde{t}} (0)dx \\
&= \frac{1}{2}(1+1) \\
&= 1.
\end{aligned}
$$

For any $(\tilde{x}, \tilde{t})$ in this region, we have $u(\tilde{x}, \tilde{t}) \equiv 1$.

Combining our analysis of each region above, the characteristic diagram becomes the graphical solution to the initial value problem as shown below.



Multi-Step Problem

This problem extends the methods we learned for the wave equation to other types of PDE's. For instance, consider the PDE given by

$$u_{tt} + 2u_{xt} - 3u_{xx} = 0. \tag{1}$$

(**S4.1**) Recall in lecture that we factored the wave equation as $(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0$. Perform a similar operation on (1).

In lecture we factored the wave equation as $(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0$. In our case notice that $u_{tt} + 2u_{xt} - 3u_{xx}$ reminds us of the polynomial $t^2 + 2xt - 3x^2$ which factors as $(t + 3x)(t - x)$. Using this same logic but with partial derivative operators we obtain $(\partial_t + 3\partial_x)(\partial_t - \partial_x)u = 0$.

(**S4.1**) From its factored form, we see that the wave equation is nothing but two coupled advection equations, $u_t - cu_x = w$ and $w_t + cw_x = 0$. This motivated us to use the change of variables $\hat{x} = x - ct$ and $\hat{t} = x + ct$ on the wave equation. By similar arguments, what would the appropriate change of variables be for (1)? (There are two possible answers; either is correct.)

Originally, I tried the change of variables $\hat{x} = x - t$ and $\hat{t} = x + 3t$, however this didn't work due to me not understanding the point of the change of variables we were doing in the first place, but after some help/inspiration on Piazza I realized that all I needed to do was flip my signs on the t variable in each expression to acquire the change of variable that solves the following problems: $\hat{x} = x + t$ and $\hat{t} = x - 3t$. However, this wasn't rigorous and more of a lucky guess, so Kai Poffenbarger and I came up with a way to come up with both change of variables only assuming the basic form of the change of variables and the end result we were looking for.

In the previous part we found the factored PDE in terms of partial derivative operators; $(\partial_t + 3\partial_x)(\partial_t - \partial_x)u = 0$. From here if we let $w = (\partial_t - \partial_x)u \implies w = u_t - u_x$. Thus plugging in w for $(\partial_t - \partial_x)u$ we obtain another equation; $(\partial_t + 3\partial_x)w = 0 \implies w_t + 3w_x = 0$. We know from the wave equation that we are looking for a change of variables of the form $\hat{x} = x + c_1 t$ and $\hat{t} = x + c_2 t$. From this change of variables we can see that

$$u_t = c_1 u_{\hat{x}} + c_2 u_{\hat{t}}$$
$$u_x = u_{\hat{x}} + u_{\hat{t}}$$

plugging into the first equation we see that

$$\begin{aligned} w &= c_1 u_{\hat{x}} + c_2 u_{\hat{t}} - u_{\hat{x}} - u_{\hat{t}} \\ &= (c_1 - 1)u_{\hat{x}} + (c_2 - 1)u_{\hat{t}} \end{aligned}$$

With these expression we can now calculate w_t and w_x. We will start by finding w_t.

$$\begin{aligned} w_t &= (c_1 - 1)\left[\frac{\partial u_{\hat{x}}}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial t} + \frac{\partial u_{\hat{x}}}{\partial \hat{t}}\frac{\partial \hat{t}}{\partial t}\right] + (c_2 - 1)\left[\frac{\partial u_{\hat{t}}}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial t} + \frac{\partial u_{\hat{t}}}{\partial \hat{t}}\frac{\partial \hat{t}}{\partial t}\right] \\ &= (c_1 - 1)(u_{\hat{x}\hat{x}}(c_1) + u_{\hat{x}\hat{t}}(c_2)) + (c_2 - 1)(u_{\hat{x}\hat{t}}(c_1) + u_{\hat{t}\hat{t}}(c_2)) \\ &= (c_1 - 1)(c_1 u_{\hat{x}\hat{x}} + c_2 u_{\hat{x}\hat{t}} + (c_2 - 1)(c_1 u_{\hat{x}\hat{t}} + c_2 u_{\hat{t}\hat{t}} \\ &= c_1^2 u_{\hat{x}\hat{x}} - c_1 u_{\hat{x}\hat{x}} + c_1 c_2 u_{\hat{x}\hat{t}} - c_2 u_{\hat{x}\hat{t}} + c_1 c_2 u_{\hat{x}\hat{t}} - c_1 u_{\hat{x}\hat{t}} + c_2^2 u_{\hat{t}\hat{t}} - c_2 u_{\hat{t}\hat{t}} \\ &= (c_1^2 - c_1)u_{\hat{x}\hat{x}} + (2c_1 c_2 - c_1 - c_2)u_{\hat{x}\hat{t}} + (c_2^2 - c_2)u_{\hat{t}\hat{t}} \end{aligned}$$

Similarly we can find an expression for w_x.

$$\begin{aligned} w_x &= (c_1 - 1)\left[\frac{\partial u_{\hat{x}}}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial x} + \frac{\partial u_{\hat{x}}}{\partial \hat{t}}\frac{\partial \hat{t}}{\partial x}\right] + (c_2 - 1)\left[\frac{\partial u_{\hat{t}}}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial x} + \frac{\partial u_{\hat{t}}}{\partial \hat{t}}\frac{\partial \hat{t}}{\partial x}\right] \\ &= (c_1 - 1)(u_{\hat{x}\hat{x}}(1) + u_{\hat{x}\hat{t}}(1)) + (c_2 - 1)(u_{\hat{x}\hat{t}}(1) + u_{\hat{t}\hat{t}}(1)) \\ &= (c_1 - 1)u_{\hat{x}\hat{x}} + (c_2 - 1)u_{\hat{t}\hat{t}} + (c_1 + c_2 - 2)u_{\hat{x}\hat{t}} \end{aligned}$$

Plugging these expressions into the second equation we see that

$$\begin{aligned} (c_1^2 - c_1)u_{\hat{x}\hat{x}} + (2c_1c_2 - c_1 - c_2)u_{\hat{x}\hat{t}} + (c_2^2 - c_2)u_{\hat{t}\hat{t}} + 3((c_1 - 1)u_{\hat{x}\hat{x}} + (c_2 - 1)u_{\hat{t}\hat{t}} + (c_1 + c_2 - 2)u_{\hat{x}\hat{t}}) = 0 \\ (c_1^2 - c_1)u_{\hat{x}\hat{x}} + (2c_1c_2 - c_1 - c_2)u_{\hat{x}\hat{t}} + (c_2^2 - c_2)u_{\hat{t}\hat{t}} + 3(c_1 - 1)u_{\hat{x}\hat{x}} + 3(c_2 - 1)u_{\hat{t}\hat{t}} + 3(c_1 + c_2 - 2)u_{\hat{x}\hat{t}} = 0 \\ (c_1^2 - c_1 + 3c_1 - 3)u_{\hat{x}\hat{x}} + (2c_1c_2 - c_1 - c_2 + 3c_1 + 3c_2 - 6)u_{\hat{x}\hat{t}} + (c_2^2 - c_2 + 3c_2 - 3)u_{\hat{t}\hat{t}} = 0 \end{aligned}$$

Based on the next problem we are looking to reduce the equation to $u_{\hat{x}\hat{t}} = 0$, thus we can set the coefficients of $u_{\hat{x}\hat{x}}$ and $u_{\hat{t}\hat{t}}$ to zero and solve for c_1 and c_2. This process is done below.

$$\begin{aligned} c_1^2 + 2c_1 - 3 = 0 &\implies c_1 = -3, 1 \\ c_2^2 + 2c_2 - 3 = 0 &\implies c_2 = -3, 1 \end{aligned}$$

Hence the two possible change of variables are $\hat{x} = x + t$, $\hat{t} = x - 3t$ and $\hat{x} = x - 3t$, $\hat{t} = x + t$. For the remaining problems we will use the change of variables $\hat{x} = x + t$ and $\hat{t} = x - 3t$ as stated in the beginning of the problem.

(**S2.1**) Carry out the appropriate change of variables on (1). You should find that $u_{\hat{x}\hat{t}} = 0$.

In this problem we will use the change of variables prescribed above, $\hat{x} = x + t$ and $\hat{t} = x - 3t$, to show that the PDE $u_{tt} + 2u_{xt} - 3u_{xx} = 0$ results in $u_{\hat{x}\hat{t}} = 0$ after the change of variables. We will start off by finding u_t using the multivariate chain rule.

$$\begin{aligned} u_t(x,t) &= \frac{\partial}{\partial t}u(x,t) \\ &= \frac{\partial u}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial t} + \frac{\partial u}{\partial \hat{t}}\frac{\partial \hat{t}}{\partial t} \\ &= u_{\hat{x}}(1) + u_{\hat{t}}(-3) \\ &= u_{\hat{x}} - 3u_{\hat{t}} \end{aligned}$$

As seen above, $u_t(x,t) = u_{\hat{x}}(x,t) - 3u_{\hat{t}}(x,t)$. We will now use the multivariate chain rule to find u_x.

$$\begin{aligned} u_x(x,t) &= \frac{\partial}{\partial x} u(x,t) \\ &= \frac{\partial u}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} + \frac{\partial u}{\partial \hat{t}} \frac{\partial \hat{t}}{\partial x} \\ &= u_{\hat{x}}(1) + u_{\hat{t}}(1) \\ &= u_{\hat{x}} + u_{\hat{t}} \end{aligned}$$

As seen above, $u_x(x,t) = u_{\hat{x}}(x,t) + u_{\hat{t}}(x,t)$. We will now use the multivariate chain rule to find u_{tt}.

$$\begin{aligned} u_{tt}(x,t) &= \frac{\partial}{\partial t} \left(u_{\hat{x}} - 3u_{\hat{t}} \right) \\ &= \left(\frac{\partial}{\partial \hat{x}} u_{\hat{x}} \frac{\partial \hat{x}}{\partial t} + \frac{\partial}{\partial \hat{t}} u_{\hat{x}} \frac{\partial \hat{t}}{\partial t} \right) - 3 \left(\frac{\partial}{\partial \hat{x}} u_{\hat{t}} \frac{\partial \hat{x}}{\partial t} + \frac{\partial}{\partial \hat{t}} u_{\hat{t}} \frac{\partial \hat{t}}{\partial t} \right) \\ &= \left(u_{\hat{x}\hat{x}}(1) + u_{\hat{x}\hat{t}}(-3) \right) - 3 \left(u_{\hat{x}\hat{t}}(1) + u_{\hat{t}\hat{t}}(-3) \right) \\ &= u_{\hat{x}\hat{x}} - 6u_{\hat{x}\hat{t}} + 9u_{\hat{t}\hat{t}} \end{aligned}$$

As seen above, $u_{tt} = u_{\hat{x}\hat{x}}(x,t) - 6u_{\hat{x}\hat{t}}(x,t) + 9u_{\hat{t}\hat{t}}(x,t)$. We will now use the multivariate chain rule to find u_{xx}.

$$\begin{aligned} u_{xx}(x,t) &= \frac{\partial}{\partial x} \left(u_{\hat{x}} + u_{\hat{t}} \right) \\ &= \left(\frac{\partial}{\partial \hat{x}} u_{\hat{x}} \frac{\partial \hat{x}}{\partial x} + \frac{\partial}{\partial \hat{t}} u_{\hat{x}} \frac{\partial \hat{t}}{\partial x} \right) + \left(\frac{\partial}{\partial \hat{x}} u_{\hat{t}} \frac{\partial \hat{x}}{\partial x} + \frac{\partial}{\partial \hat{t}} u_{\hat{t}} \frac{\partial \hat{t}}{\partial x} \right) \\ &= \left(u_{\hat{x}\hat{x}}(1) + u_{\hat{x}\hat{t}}(1) \right) + \left(u_{\hat{x}\hat{t}}(1) + u_{\hat{t}\hat{t}}(1) \right) \\ &= u_{\hat{x}\hat{x}} + 2u_{\hat{x}\hat{t}} + u_{\hat{t}\hat{t}} \end{aligned}$$

As seen above, $u_{xx} = u_{\hat{x}\hat{x}}(x,t) + 2u_{\hat{x}\hat{t}}(x,t) + u_{\hat{t}\hat{t}}(x,t)$. Our last derivative calculation will be using the multivariate chain rule to find u_{xt}.

$$\begin{aligned} u_{xt}(x,t) &= \frac{\partial}{\partial x} \left(u_{\hat{x}} - 3u_{\hat{t}} \right) \\ &= \left(\frac{\partial}{\partial \hat{x}} u_{\hat{x}} \frac{\partial \hat{x}}{\partial x} + \frac{\partial}{\partial \hat{t}} u_{\hat{x}} \frac{\partial \hat{t}}{\partial x} \right) + -3 \left(\frac{\partial}{\partial \hat{x}} u_{\hat{t}} \frac{\partial \hat{x}}{\partial x} + \frac{\partial}{\partial \hat{t}} u_{\hat{t}} \frac{\partial \hat{t}}{\partial x} \right) \\ &= \left(u_{\hat{x}\hat{x}}(1) + u_{\hat{x}\hat{t}}(1) \right) - 3 \left(u_{\hat{x}\hat{t}}(1) + u_{\hat{t}\hat{t}}(1) \right) \\ &= u_{\hat{x}\hat{x}} - 2u_{\hat{x}\hat{t}} - 3u_{\hat{t}\hat{t}} \end{aligned}$$

As seen above, $u_{xt} = u_{\hat{x}\hat{x}}(x,t) - 2u_{\hat{x}\hat{t}}(x,t) - 3u_{\hat{t}\hat{t}}(x,t)$. We will now plug these expressions into our PDE and simplify.

$$\begin{aligned} u_{\hat{x}\hat{x}} - 6u_{\hat{x}\hat{t}} + 9u_{\hat{t}\hat{t}} + 2(u_{\hat{x}\hat{x}} - 2u_{\hat{x}\hat{t}} - 3u_{\hat{t}\hat{t}}) - 3(u_{\hat{x}\hat{x}} + 2u_{\hat{x}\hat{t}} + u_{\hat{t}\hat{t}}) &= 0 \\ u_{\hat{x}\hat{x}} - 6u_{\hat{x}\hat{t}} + 9u_{\hat{t}\hat{t}} + 2u_{\hat{x}\hat{x}} - 4u_{\hat{x}\hat{t}} - 6u_{\hat{t}\hat{t}} - 3u_{\hat{x}\hat{x}} - 6u_{\hat{x}\hat{t}} - 3u_{\hat{t}\hat{t}} &= 0 \\ -16u_{\hat{x}\hat{t}} &= 0 \\ u_{\hat{x}\hat{t}} &= 0 \end{aligned}$$

Thus we have shown that the change of variables $\hat{x} = x + t$ and $\hat{t} = x - 3t$ results in $u_{\hat{x}\hat{t}} = 0$ once plugged into the PDE.

(**S4.1**) Solve $u_{\hat{x}\hat{t}} = 0$, and then transform back to unhatted variables. What is the general solution of (1)?

As shown above, using the change of variables $\hat{x} = x + t$ and $\hat{t} = x - 3t$ results in $u_{\hat{x}\hat{t}} = 0$ once plugged into the PDE. This is important, because we can solve this directly. Integrating with respect to $\hat{t}$, we see $u_{\hat{x}} = \int \partial_{\hat{t}}(u_{\hat{x}}) \partial_{\hat{t}} = \int 0 \partial_{\hat{t}} = \tilde{F}(\hat{x})$. Then, we integrate with respect to $\hat{x}$ and get that $u(\hat{x}, \hat{t}) = \int \partial_{\hat{x}}(u) \partial_{\hat{x}} = \int \tilde{F}(\hat{x}) \partial_{\hat{x}} = F(\hat{x}) + G(\hat{t})$. Changing back to our original coordinates, our general solution is given by $u(x,t) = F(x+t) + G(x-3t)$, for arbitrary functions F and G.