# AMATH 353 Quiz 1

Jaiden Atterbury

2023-07-06

Note: In this quiz, I briefly collaborated with Kai Poffenbarger. In this collaboration we mainly compared answers to see if what we were getting for our solutions were similar.

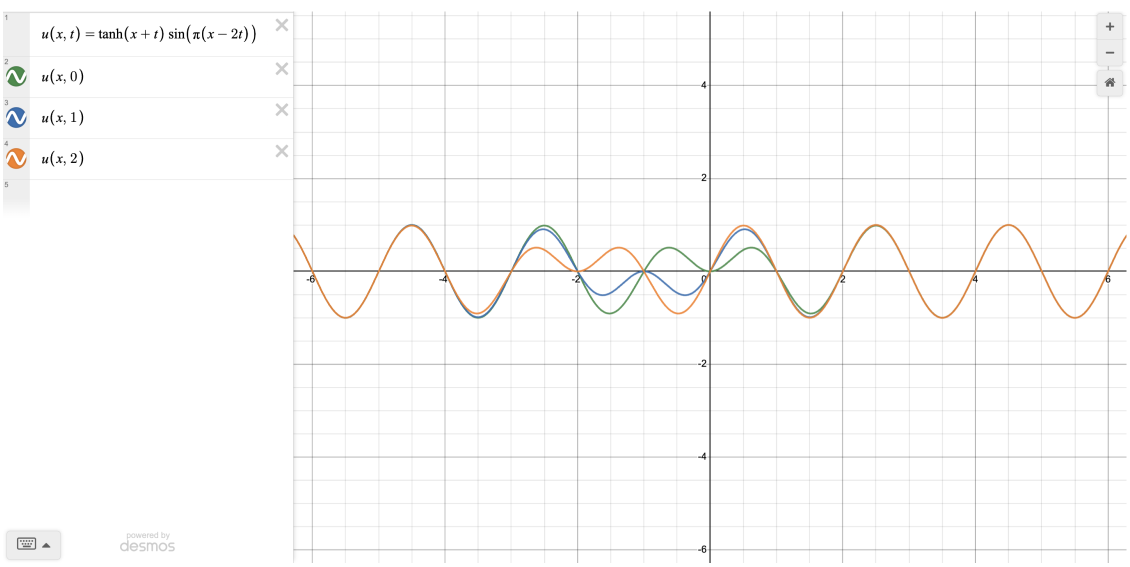
## Component Skills Practice

1. Consider a wave profile given by

$$u(x,t) = \tanh(x+t)\sin(\pi(x-2t))$$

a. Using software of your choice, make snapshot plots of this wave at $t = 0$, $t = 1$, and $t = 2$.

For this part of the problem, we will use Desmos to make snapshot plots of the wave $u(x,t) = \tanh(x+t)\sin(\pi(x-2t))$ at times $t = 0$, $t = 1$, and $t = 2$. The snapshot plots are shown below. In order to see the differences of the wave profiles at each individual time, we will set the axes of the snapshots to $-2\pi < x < 2\pi$ and $-2\pi < u < 2\pi$.




b. Compute $u_x$. What is $u_x(-2,1)$? What does this value mean physically?

To start off we will compute $u_x$, the partial derivative of $u$ with respect to $x$.

$$\begin{aligned}
u_x(x,t) &= \frac{\partial}{\partial x}\left(\tanh\left(x+t\right)\sin\left(\pi\left(x-2t\right)\right)\right) \\
&= \tanh\left(x+t\right)\frac{\partial}{\partial x}\sin\left(\pi\left(x-2t\right)\right)+\sin\left(\pi\left(x-2t\right)\right)\frac{\partial}{\partial x}\tanh\left(x+t\right) \\
&= \tanh\left(x+t\right)\cdot\pi\cos\left(\pi\left(x-2t\right)\right)+sech^2\left(x+t\right)\cdot\sin\left(\pi\left(x-2t\right)\right) \\
&= \pi\tanh\left(x+t\right)\cos\left(\pi\left(x-2t\right)\right)+sech^2\left(x+t\right)\sin\left(\pi\left(x-2t\right)\right)
\end{aligned}$$

Thus $u_x(x,t) = \pi\tanh\left(x+t\right)\cos\left(\pi\left(x-2t\right)\right) sech^2\left(x+t\right)\sin\left(\pi\left(x-2t\right)\right)$. Now we will find $u_x(-2,1)$.

$$\begin{aligned}
u_x(-2,1) &= \pi\tanh\left(-2+1\right)\cos\left(\pi\left(-2-2(1)\right)\right)+sech^2\left(-2+1\right)\sin\left(\pi\left(-2-2(1)\right)\right) \\
&= \pi\tanh\left(-1\right)\cos\left(-4\pi\right))+sech^2\left(-1\right)\sin\left(-4\pi\right)) \\
&= \pi\tanh\left(-1\right)\cdot 1+sech^2\left(-1\right)\cdot 0 \\
&= \pi\tanh(-1)
\end{aligned}$$

Thus $u_x(-2,1) = \pi\tanh(-1)$. We will now interpret this value in the context of waves. In general terms of partial derivatives, this value represents the instantaneous rate of change of the function $u$ treating $t$ as a constant and letting $x$ vary. Thus, the instantaneous rate of change of the function $u$ with respect to $x$ when $x=-2$ at $t=1$ is $\pi\tanh(-1)$. However, in the context of waves, $u_x(-2,1)$ tells us that the wave profile has steepness $\pi\tanh(-1)$ when $x=-2$ at $t=1$.

c. Compute $u_t$. What is $u_t(-2,1)$? What does this value mean physically?

To start off we will compute $u_t$, the partial derivative of $u$ with respect to $t$.

$$\begin{aligned}
u_t(x,t) &= \frac{\partial}{\partial t}\left(\tanh\left(x+t\right)\sin\left(\pi\left(x-2t\right)\right)\right) \\
&= \tanh\left(x+t\right)\frac{\partial}{\partial t}\sin\left(\pi\left(x-2t\right)\right)+\sin\left(\pi\left(x-2t\right)\right)\frac{\partial}{\partial t}\tanh\left(x+t\right) \\
&= \tanh\left(x+t\right)\cdot -2\pi\cos\left(\pi\left(x-2t\right)\right)+sech^2\left(x+t\right)\cdot\sin\left(\pi\left(x-2t\right)\right) \\
&= -2\pi\tanh\left(x+t\right)\cos\left(\pi\left(x-2t\right)\right)+sech^2\left(x+t\right)\sin\left(\pi\left(x-2t\right)\right)
\end{aligned}$$

Thus $u_t(x,t) = -2\pi\tanh\left(x+t\right)\cos\left(\pi\left(x-2t\right)\right) sech^2\left(x+t\right)\sin\left(\pi\left(x-2t\right)\right)$. Now we will find $u_t(-2,1)$.

$$\begin{aligned}
u_t(-2,1) &= -2\pi\tanh\left(-2+1\right)\cos\left(\pi\left(-2-2(1)\right)\right)+sech^2\left(-2+1\right)\sin\left(\pi\left(-2-2(1)\right)\right) \\
&= -2\pi\tanh\left(-1\right)\cos\left(-4\pi\right))+sech^2\left(-1\right)\sin\left(-4\pi\right)) \\
&= -2\pi\tanh\left(-1\right)\cdot 1+sech^2\left(-1\right)\cdot 0 \\
&= -2\pi\tanh(-1)
\end{aligned}$$

Thus $u_t(-2,1) = -2\pi\tanh(-1)$. We will now interpret this value in the context of waves. In general terms of partial derivatives, this value represents the instantaneous rate of change of the function $u$ treating $x$ as a constant and letting $t$ vary. Thus, the instantaneous rate of change of the function $u$ with respect to $t$ when $x=-2$ at $t=1$ is $-2\pi\tanh(-1)$. However, in the context of waves, $u_t(-2,1)$ tells us that the wave amplitude is changing at a rate of $-2\pi\tanh(-1)$ when $x=-2$ at $t=1$.

2. Provide an example of a PDE that fits each of the following descriptions below. Make sure that each of your example PDE's has partial derivatives with respect to $x$ and $t$. No ODE examples will receive credit. **Please keep your examples distinct from your collaborators**.

a. 2nd Order, Nonhomogeneous, Nonautonomous, Linear.

An example of a PDE that is 2nd Order, Nonhomogeneous, Nonautonomous, Linear, and has partial derivatives with respect to $x$ and $t$ is $tu_{xx} + u_t = 1$.

b. 3rd Order, Nonhomogeneous, Nonautonomous, Nonlinear.

An example of a PDE that is 3rd Order, Nonhomogeneous, Nonautonomous, Nonlinear, and has partial derivatives with respect to $x$ and $t$ is $tu_{xxx} + uu_t = 1$.

c. 1st Order, Homogeneous, Nonautonomous, Nonlinear.

An example of a PDE that is 1st Order, Homogeneous, Nonautonomous, Nonlinear, and has partial derivatives with respect to $x$ and $t$ is $tu_x + uu_t = 0$.

d. 2nd Order, Nonhomogenous, Autonomous, Nonlinear.

An example of a PDE that is 2nd Order, Nonhomogenous, Autonomous, Nonlinear, and has partial derivatives with respect to $x$ and $t$ is $u_{xx} + uu_t = 1$.

e. 1st Order, Homogeneous, Nonautonomous, Linear.

An example of a PDE that is 1st Order, Homogeneous, Nonautonomous, Linear, and has partial derivatives with respect to $x$ and $t$ is $tu_x + u_t = 0$.

3. Consider the following PDE:

$$u_{xt} = t^2 u_x + x^2 u_t.$$

This equation is homogeneous and linear, so it should satisfy the superposition principle. Show this directly using the definition of the superposition principle.

Suppose that $u^1$ and $u^2$ are solutions of the above PDE (using superscript notation for indexing like in the practice quiz). Then, we have that

$$u^1_{xt} = t^2 u^1_x + x^2 u^1_t \quad \text{and} \quad u^2_{xt} = t^2 u^2_x + x^2 u^2_t$$

Furthermore, if we let

$$u^3 = c_1 u^1 + c_2 u^2$$

where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants, then we can see by the linearity of the derivative that

$$\begin{aligned} u^3_t &= (c_1 u^1 + c_2 u^2)_t \\ &= c_1 u^1_t + c_2 u^2_t \end{aligned}$$

Again, using the linearity of the derivative, we see that

$$\begin{aligned} u^3_{xt} &= (c_1 u^1_t + c_2 u^2_t)_x \\ &= c_1 u^1_{xt} + c_2 u^2_{xt} \end{aligned}$$

Then, since $u^1$ and $u^2$ solve the above PDE, we have that

$$\begin{aligned} u^3_{xt} &= c_1 u^1_{xt} + c_2 u^2_{xt} \\ &= c_1 \left(t^2 u^1_x + x^2 u^1_t\right) + c_2 \left(t^2 u^2_x + x^2 u^2_t\right) \\ &= c_1 t^2 u^1_x + c_1 x^2 u^1_t + c_2 t^2 u^2_x + c_2 x^2 u^2_t \\ &= t^2 \left(c_1 u^1_x + c_2 u^2_x\right) + x^2 \left(c_1 u^1_t + c_2 u^2_t\right) \end{aligned}$$

Again, by the linearity of the derivative, we see that

$$\begin{aligned}u^3_{xt} &= t^2\left(c_1 u^1_x + c_2 u^2_x\right) + x^2\left(c_1 u^1_t + c_2 u^2_t\right)\\ &= t^2\left(c_1 u^1 + c_2 u^2\right)_x + x^2\left(c_1 u^1 + c_2 u^2\right)_t\\ &= t^2 u^3_x + x^2 u^3_t\end{aligned}$$

Thus $u^3$ solves the PDE too. Therefore, the PDE has a superposition principle.

# Multi-Step Problem

Consider the Black-Scholes equation:

$$u_t + \frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x - ru = 0. \tag{1}$$

This PDE models the price evolution of a European-style option $u$ as a function of the Stock price $x$ and time $t$. The constants $r \geq 0$ and $\sigma \geq 0$ represent the risk-free interest rate and volatility of the stock, respectively. This problem was taken from Ryan Creedon's version of this course offered in 2021.

1. Classify (1) by order, homogeneity, autonomy, and linearity. **Justify each of your choices in complete sentences.**

This PDE is a 2nd order, homogeneous, non-autonomous, and linear PDE. It is second order because our highest derivative is a second derivative in the form of the $u_{xx}$ term. It is homogeneous because the trivial solution $u = 0$ is in fact a solution for any value of $x$. It is non-autonomous because it has explicit $x$-dependence in the form of the $\frac{1}{2}\sigma^2 x^2 u_x x$ and $rxu_x$ terms. Lastly, it is linear because it does not have any products, powers, or functions of $u$ or its derivatives.

2. Suppose (1) has a solution of the form

$$u(x,t) = x^p$$

for some constant $p$. Determine the possible values of $p$. **Show all of your work, and simplify your final answer(s) as much as possible.**

To find the possible values of $p$ that make $u(x,t) = x^p$ a solution to the above PDE, we must take the corresponding derivatives of $u(x,t)$ and plug them into the above PDE and solve for $p$. Since $u(x,t)$ is not a complicated function it can easily be seen that $u_t(x,t) = 0$, $u_x(x,t) = px^{p-1}$, and $u_{xx} = p(p-1)x^{p-2}$. Plugging these values into the PDE we obtain

$$\begin{aligned}0 &= 0 + \frac{1}{2}\sigma^2 x^2 p(p-1)x^{p-2} + rxpx^{p-1} - rx^p\\ &= \frac{\sigma^2 x^2 p(p-1)}{2}x^p + rpx^p - rx^p\\ &= \left(\frac{\sigma^2 x^2 p(p-1)}{2} + rp - r\right)x^p\end{aligned}$$

however, the value of $p$ we chose must make $u(x,t)$ satisfy the PDE for all values of $x$ and $t$, hence we can drop the $x^p$ term and instead solve

$$\begin{aligned}0 &= \frac{\sigma^2 x^2 p(p-1)}{2} + rp - r\\ &= \frac{\sigma^2}{2}p^2 - \frac{\sigma^2}{2}p + rp - r\\ &= \frac{\sigma^2}{2}p^2 + (r - \frac{\sigma^2}{2})p - r\\ &= \sigma^2 p^2 + (2r - \sigma^2)p - 2r\end{aligned}$$

Using the quadratic formula we obtain the following solution solution form

$$\begin{aligned}
p &= \frac{-2r + \sigma^2 \pm \sqrt{(2r - \sigma^2)^2 - 4(\sigma^2)(-2r)}}{2\sigma^2} \\
&= \frac{-2r + \sigma^2 \pm \sqrt{4r^2 - 4r\sigma^2 + \sigma^4 + 8r\sigma^2}}{2\sigma^2} \\
&= \frac{-2r + \sigma^2 \pm \sqrt{4r^2 + 4r\sigma^2 + \sigma^4}}{2\sigma^2} \\
&= \frac{-2r + \sigma^2 \pm \sqrt{(2r + \sigma^2)^2}}{2\sigma^2} \\
&= \frac{-2r + \sigma^2 \pm 2r + \sigma^2}{2\sigma^2}
\end{aligned}$$

Now we must see what our value of $p$ is for each branch of the $\pm$ in the above expression. First, looking at the positive branch we see $p = \frac{-2r+\sigma^2+2r+\sigma^2}{2\sigma^2} = \frac{2\sigma^2}{2\sigma^2} = 1$. Lastly, looking at the negative branch we see $p = \frac{-2r+\sigma^2-2r+\sigma^2}{2\sigma^2} = \frac{-4r}{2\sigma^2} = \frac{-2r}{\sigma^2}$. Thus, as computed above, the two values of $p$ that make $u(x,t)$ a solution to the above PDE are $p = 1$ and $p = \frac{-2r}{\sigma^2}$.

3. The function

$$u(x,t) = e^{At} \sin\left(\ln(x) + \left(\frac{1}{2}\sigma^2 - r\right) t\right) \tag{2}$$

is a solution of (1) for some constant $A$. Determine $A$. **Show all of your work.**

To find the possible values of $A$ that make $u(x,t)$ a solution to the above PDE, we must take the corresponding derivatives of $u(x,t)$ and plug them into the above PDE and solve for $A$. These derivatives are more involved than the last part so we will show the derivations step-by-step. First of we will compute $u_t(x,t)$.

$$\begin{aligned}
u_t(x,t) &= Ae^{At} \sin\left(\ln(x) + \left(\frac{1}{2}\sigma^2 - r\right) t\right) + e^{At} \cos\left(\ln(x) + \left(\frac{1}{2}\sigma^2 - r\right) t\right) \cdot \left(\frac{1}{2}\sigma^2 - r\right) \\
&= Ae^{At} \sin\left(\ln(x) + \left(\frac{1}{2}\sigma^2 - r\right) t\right) + \left(\frac{1}{2}\sigma^2 - r\right) e^{At} \cos\left(\ln(x) + \left(\frac{1}{2}\sigma^2 - r\right) t\right)
\end{aligned}$$

As shown above, $u_t(x,t) = Ae^{At} \sin\left(\ln(x) + \left(\frac{1}{2}\sigma^2 - r\right) t\right) + \left(\frac{1}{2}\sigma^2 - r\right) e^{At} \cos\left(\ln(x) + \left(\frac{1}{2}\sigma^2 - r\right) t\right)$ Next we will compute $u_x(x,t)$.

$$\begin{aligned}
u_x(x,t) &= e^{At} \cos\left(\ln(x) + \left(\frac{1}{2}\sigma^2 - r\right) t\right) \cdot \frac{1}{x} \\
&= \frac{1}{x} e^{At} \cos\left(\ln(x) + \left(\frac{1}{2}\sigma^2 - r\right) t\right)
\end{aligned}$$

As shown above, $u_x(x,t) = \frac{1}{x} e^{At} \cos\left(\ln(x) + \left(\frac{1}{2}\sigma^2 - r\right) t\right)$. Lastly, we will compute $u_{xx}(x,t)$.

$$\begin{aligned}
u_{xx}(x,t) &= \frac{-e^{At}}{x^2} \cos\left(\ln(x) + \left(\frac{1}{2}\sigma^2 - r\right) t\right) + \frac{e^{At}}{x} \sin\left(\ln(x) + \left(\frac{1}{2}\sigma^2 - r\right) t\right) \cdot \frac{-1}{x} \\
&= \frac{-e^{At}}{x^2} \left(\cos\left(\ln(x) + \left(\frac{1}{2}\sigma^2 - r\right) t\right) + \sin\left(\ln(x) + \left(\frac{1}{2}\sigma^2 - r\right) t\right)\right)
\end{aligned}$$

As shown above, $u_{xx}(x,t) = \frac{-e^{At}}{x^2} \left(\cos\left(\ln(x) + \left(\frac{1}{2}\sigma^2 - r\right) t\right) + \sin\left(\ln(x) + \left(\frac{1}{2}\sigma^2 - r\right) t\right)\right)$. In order to make the below work less messy, and since it has no impact on the final answer, we will let $B = \ln(x) + \left(\frac{1}{2}\sigma^2 - r\right) t$.

Now we will plug in these derivatives into the PDE

$$
\begin{aligned}
0 &= Ae^{At}\sin(B) + \left(\frac{1}{2}\sigma^2 - r\right)e^{At}\cos(B) - \frac{1}{2}\sigma^2 x^2 \frac{e^{At}}{x^2}(\cos(B)+\sin(B)) + rx\frac{e^{At}}{x}\cos(B) - re^{At}\sin(B) \\
&= e^{At}\left(A\sin(B) + \frac{\sigma^2}{2}\cos(B) - r\cos(B) - \frac{\sigma^2}{2}\cos(B) - \frac{\sigma^2}{2}\sin(B) + r\cos(b) - r\sin(B)\right) \\
&= e^{At}\left(A\sin(B) - \frac{\sigma^2}{2}\sin(B) - r\sin(B)\right) \\
&= (A - \frac{\sigma^2}{2} - r)\sin(B) \quad (\text{Since } e^{At} \neq 0) \\
&= A - \frac{\sigma^2}{2} - r
\end{aligned}
$$

Thus as shown above, the value of $A$ that makes $u(x,t) = e^{At}\sin\left(\ln(x) + \left(\frac{1}{2}\sigma^2 - r\right)t\right)$ a solution to the above PDE is $A = r + \frac{\sigma^2}{2}$.

4. Is

$$u(x,t) = 1000000x\left(\pi + x^{-1-\frac{2r}{\sigma^2}}\right) - r\sigma e^{(r+\frac{1}{2}\sigma^2)t}\sin\left(\ln(x) + \left(\frac{1}{2}\sigma^2 - r\right)t\right)$$

   a solution of (1)? Provide a brief justification without directly substituting $u(x,t)$ into (1). (This justification can just be a sentence or two.)

First off, notice that the above equation can be rewritten as $u(x,t) = 1000000\pi x + 1000000x^{\left(-\frac{2r}{\sigma^2}\right)} - r\sigma e^{(r+\frac{1}{2}\sigma^2)t}\sin\left(\ln(x) + \left(\frac{1}{2}\sigma^2 - r\right)t\right)$. Again, notice that $x$, $x^{\left(-\frac{2r}{\sigma^2}\right)}$, and $e^{(r+\frac{1}{2}\sigma^2)t}\sin\left(\ln(x) + \left(\frac{1}{2}\sigma^2 - r\right)t\right)$ are all solutions that we found in parts b and c. Furthermore, in part a we found out that this PDE is linear and homogeneous, thus it must have a superposition principle. Since $u(x,t)$ above is just a linear combination of these solutions, $u(x,t) = 1000000x\left(\pi + x^{-1-\frac{2r}{\sigma^2}}\right) - r\sigma e^{(r+\frac{1}{2}\sigma^2)t}\sin\left(\ln(x) + \left(\frac{1}{2}\sigma^2 - r\right)t\right)$ is a solution of (1) by the superposition principle.