Jaiden Atterbury Homework 7

Component Skills Practice

① Exercise 1:

(a) $u_t + u^2 u_x = u_{xxx}$

$u_t + u^2 u_x - u_{xxx} = 0$

$u_t + \left(\frac{u^3}{3}\right)_x - u_{xxx} = 0$ (since $f^2 f' = \left(\frac{f^3}{3}\right)'$

$$\boxed{\begin{array}{l} u_t + \left(\frac{u^3}{3} - u_{xx}\right)_x = 0 \\ \phi = \frac{u^3}{3} - u_{xx} \end{array}}$$

(b) $u_t + (\cos(u) - u^3)u_x = u_x$

$u_t + \cos(u) u_x - u^3 u_x - u_x = 0$

$u_t + (\sin(u))_x - \left(\frac{u^4}{4}\right)_x - (u)_x = 0$ (since $f^3 f' = \left(\frac{f^4}{4}\right)'$)

$$\boxed{\begin{array}{l} u_t + \left(\sin(u) - \frac{u^4}{4} - u\right)_x = 0 \\ \phi = \sin u - \frac{u^4}{4} - u \end{array}}$$

(c) $u_t + 2xu u_x = -u^2$

$u_t + 2xuu_x + u^2 = 0$

$$\boxed{\begin{array}{l} u_t + (xu^2)_x = 0 \\ \phi = xu^2 \end{array}}$$

(since $(xu^2)_x = x\frac{d}{dx}u^2 + \frac{d}{dx}x\, u^2 = 2xuu_x + u^2$)

② Exercise 2:

(a) $u_t + (uxt^2)_x = xu^3$

$u_t + u\frac{d}{dx}xt^2 + \frac{d}{dx}u\, xt^2 = xu^3$

$u_t + t^2 u + xt^2 u_x = xu^3$

$u_t + xt^2 u_x = xu^3 - t^2 u$

$c(x,t,u) = xt^2$, $g(x,t,u) = xu^3 - t^2 u$

since only $g(x,t,u)$ depends on u, this conservation law is semi-linear

(b) $u_t + (xu)_x = u_x + u$

$u_t + x\frac{d}{dx}u + \frac{d}{dx}x\, u = u_x + u$

$u_t + xu_x + u = u_x + u$

$u_t + xu_x - u_x = 0$

$u_t + (x-1)u_x = 0$

$c(x,t,u) = x-1, \quad g(x,t,u) \equiv 0$

Since $c(x,t,u)$ and $g(x,t,u)$ don't depend on u, this conservation law is linear.

(c) $u_t - [(x^2 - e^u)u]_x = e^u(1+u)u_x$

$u_t - (x^2u - e^u u)_x = e^u u_x + e^u u u_x$

$u_t - x^2\frac{d}{dx}u - \frac{d}{dx}x^2 u + \frac{d}{dx}u + \frac{d}{dx}e^u u = e^u u_x + e^u u u_x$

$u_t - x^2u_x - 2xu + e^u u_x + e^u u u_x = e^u u_x + e^u u u_x$

$u_t - x^2u_x - 2xu = 0$

$u_t + (-x^2)u_x = 2xu$

$c(x,t,u) = -x^2, \quad g(x,t,u) = 2xu$

since only $g(x,t,u)$ depends on u, this conservation law is semi-linear.

(3.) Exercise 3: *see end of problem for plots*

$$\begin{cases} u_t + xu_x = x \\ u(x,0) = \tanh(x) \end{cases}$$

(a) $\begin{cases} \frac{dx}{dt} = x \\ x(0) = x_0 \end{cases}$

ODE $\frac{dx}{dt} = x \Rightarrow \int \frac{1}{x}dx = \int dt + C \Rightarrow \ln|x| = t + C \Rightarrow |x| = ce^t \Rightarrow x = \pm ce^t \Rightarrow x = De^t$

IC $x(0) = x_0 \Rightarrow x_0 = De^0 \Rightarrow D = x_0 \Rightarrow \boxed{x(t; x_0) = x_0 e^t}$

$x = x_0 e^t \Rightarrow \frac{x}{x_0} = e^t \Rightarrow t = \ln\left(\frac{x}{x_0}\right)$

(b) $\begin{cases} \dfrac{d(u(x(t;x_0),t)}{dt} = x(t;x_0) \\ u(x_0,0) = \tanh(x_0) \end{cases} = \begin{cases} \dfrac{d[u(x(t;x_0),t))}{dt} = x_0 e^t \\ u(x_0,0) = \tanh(x_0) \end{cases}$

ODE $\dfrac{d(u(x(t;x_0),t)}{dt} = x_0 e^t \Rightarrow \int du(x(t;x_0),t) = x_0 \int e^t dt + I(x_0) \Rightarrow u(x(t;x_0),t) = x_0 e^t + I(x_0)$

IC $u(x_0,0) = \tanh(x_0) \Rightarrow \tanh(x_0) = u(x_0,0) = u(x(0;x_0),0) = x_0 + I(x_0) \Rightarrow I(x_0) = \tanh(x_0) - x_0$

$u(x(t;x_0),t) = x_0 e^t + \tanh(x_0) - x_0$

$x(t;x_0) = x_0 e^t \Rightarrow x_0 = \dfrac{x(t;x_0)}{e^t}$

$u(x(t;x_0),t) = x(t;x_0) + \tanh\left(\dfrac{x(t;x_0)}{e^t}\right) - \dfrac{x(t;x_0)}{e^t}$

$\boxed{u(x,t) = x + \tanh\left(\frac{x}{e^t}\right) - \frac{x}{e^t}}$

4. Exercise 4: * see end of problem for plots *

$\begin{cases} u_t + cu_x = -2tu \\ u(x,0) = e^{-x^2} \end{cases}$

(a) $\begin{cases} \dfrac{dx}{dt} = c \\ x(0) = x_0 \end{cases}$

ODE $\dfrac{dx}{dt} = c \Rightarrow \int dx = \int c\,dt + D \Rightarrow x = ct + D$

IC $x(0) = x_0 \Rightarrow x_0 = c(0) + D \Rightarrow D = x_0 \Rightarrow \boxed{x(t;x_0) = ct + x_0}$

$x = ct + x_0 \Rightarrow ct = x - x_0 \Rightarrow t = \frac{1}{c}x - \frac{x_0}{c}$

(b) $\begin{cases} \dfrac{du(x(t;x_0),t)}{dt} = -2tu(x(t;x_0),t) \\ u(x_0,0) = e^{-x_0^2} \end{cases}$

ODE $\dfrac{du(x(t;x_0),t)}{dt} = -2tu(x(t;x_0),t) \Rightarrow \int \dfrac{1}{u(x(t;x_0),t)} du(x(t;x_0),t) = \int -2t\,dt + I(x_0) \Rightarrow \ln|u(x(t;x_0),t)| = -t^2 + I(x_0)$

$|u(x(t;x_0),t)| = e^{I(x_0)} e^{-t^2} \Rightarrow u(x(t;x_0),t) = \pm e^{I(x_0)} e^{-t^2} \Rightarrow u(x(t;x_0),t) = g(x_0) e^{-t^2}$

IC $u(x_0,0) = e^{-x_0^2} \Rightarrow e^{-x_0^2} = u(x_0,0) = u(x(0;x_0),0) = g(x_0) \Rightarrow g(x_0) = e^{-x_0^2}$

$u(x(t;x_0),t) = e^{-x_0^2} e^{-t^2}$

$x(t;x_0) = ct + x_0 \Rightarrow x(t;x_0) - ct = x_0$

$u(x(t;x_0),t) = e^{-(x(t;x_0)-ct)^2} e^{-t^2}$

$\boxed{u(x,t) = e^{-(x-ct)^2} e^{-t^2}}$

$\boxed{\text{Same general solution as HW 2 MSP.}}$

(5.) Exercise 5:

$$\begin{cases} u_t + cu_x = -\sigma u \\ u(x,0) = f(x) \end{cases}$$

(a) $\begin{cases} \frac{dx}{dt} = c \\ x(0) = x_0 \end{cases}$

ODE $\frac{dx}{dt} = c \Rightarrow \int dx = \int c\,dt + D \Rightarrow x = ct + D$

IC $x(0) = x_0 \Rightarrow x_0 = c(0) + D \Rightarrow D = x_0 \Rightarrow \boxed{x = ct + x_0}$

$x = ct + x_0 \Rightarrow ct = x - x_0 \Rightarrow t = \frac{1}{c}x - \frac{x_0}{c} \Rightarrow$ $\boxed{\text{Slopes in the } xt \text{ plane are } \frac{1}{c}.}$

(b) $\begin{cases} \frac{du(x(t;x_0),t)}{dt} = -\sigma u(x(t;x_0),t) \\ u(x_0, 0) = f(x_0) \end{cases}$

ODE $\frac{du(x(t;x_0),t)}{dt} = -\sigma u(x(t;x_0),t) \Rightarrow \int \frac{1}{u(x(t;x_0),t)} du(x(t;x_0),t) = \int -\sigma\,dt + I(x_0) \Rightarrow$

$\ln|u(x(t;x_0);t)| = -\sigma t + I(x_0) \Rightarrow |u(x(t;x_0),t)| = e^{I(x_0)} e^{-\sigma t} \Rightarrow u(x(t;x_0),t) = \pm e^{I(x_0)} e^{-\sigma t}$

$\Rightarrow u(x(t;x_0),t) = g(x_0) e^{-\sigma t}$

IC $u(x_0,0) = f(x_0) \Rightarrow f(x_0) = u(x_0,0) = u(x(0;x_0),0) = g(x_0) \Rightarrow u(x(t;x_0),t) = f(x_0) e^{-\sigma t}$

x_0 $x(t;x_0) = ct + x_0 \Rightarrow x_0 = x(t;x_0) - ct \Rightarrow u(x(t;x_0),t) = f(x(t;x_0) - ct) e^{-\sigma t}$

$\Rightarrow \boxed{u(x,t) = f(x-ct) e^{-\sigma t}}$

AMATH 353 Homework 7

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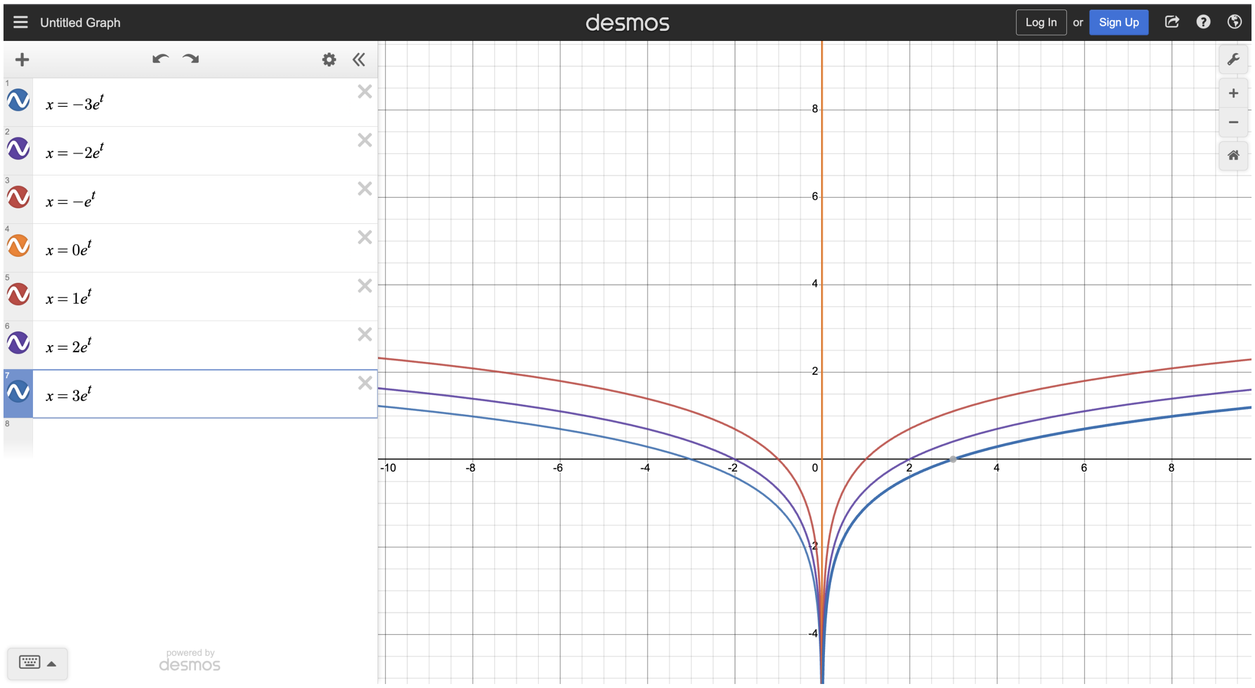
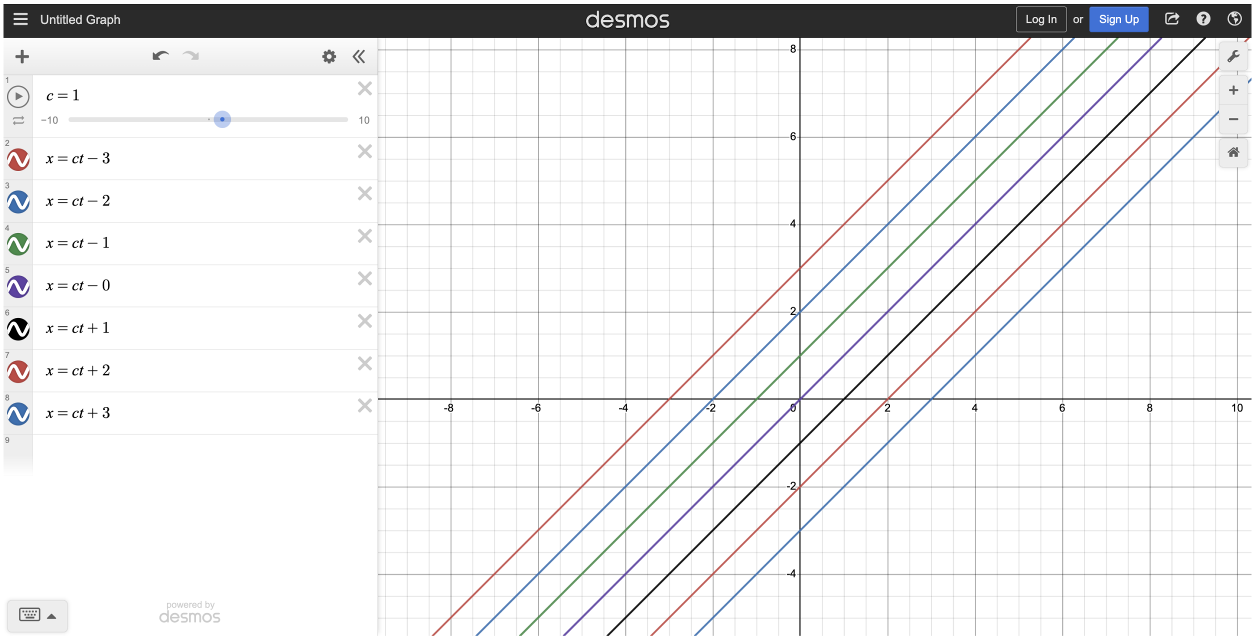
2023-08-15

Exercise 3 Part a:



Exercise 4 Part a:



Exercise 4 Part b: