AMATH 353 Quiz 3

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Note: In this quiz, I briefly collaborated with Kai Poffenbarger, as well as Robert Lu. In this collaboration we mainly compared answers to see if what we were getting for our solutions were similar.

Component Skills

1. Consider the following questions about the wave equation:

$$u_{tt} = c^2 u_{xx}, \quad c \neq 0. \tag{1}$$

(i) What change of variables simplifies the wave equation to $u_{\hat{x}\hat{t}} = 0$? (There is more than one answer.)

One of the change of variables that simplifies the wave equation to $u_{\hat{x}\hat{t}} = 0$ is $\hat{x} = x - ct$ and $\hat{t} = x + ct$.

We will use the change of variables $\hat{x} = x - ct$ and $\hat{t} = x + ct$, to show that the PDE $u_{tt} = c^2 u_{xx}$ results in $u_{\hat{x}\hat{t}} = 0$ after the change of variables. We will start off by finding u_t using the multivariate chain rule.

$$\begin{aligned} u_t(x,t) &= \frac{\partial}{\partial t} u(x,t) \\ &= \frac{\partial u}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial t} + \frac{\partial u}{\partial \hat{t}} \frac{\partial \hat{t}}{\partial t} \\ &= u_{\hat{x}}(-c) + u_{\hat{t}}(c) \\ &= -cu_{\hat{x}} + cu_{\hat{t}} \end{aligned}$$

As seen above, $u_t(x,t) = -cu_{\hat{x}}(x,t) + cu_{\hat{t}}(x,t)$. We will now use the multivariate chain rule to find u_x.

$$\begin{aligned} u_x(x,t) &= \frac{\partial}{\partial x} u(x,t) \\ &= \frac{\partial u}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} + \frac{\partial u}{\partial \hat{t}} \frac{\partial \hat{t}}{\partial x} \\ &= u_{\hat{x}}(1) + u_{\hat{t}}(1) \\ &= u_{\hat{x}} + u_{\hat{t}} \end{aligned}$$

As seen above, $u_x(x,t) = Au_{\hat{x}}(x,t) + Cu_{\hat{t}}(x,t)$. We will now use the multivariate chain rule to find u_{tt}.

$$\begin{aligned} u_{tt}(x,t) &= \frac{\partial}{\partial t} \left(-cu_{\hat{x}} + cu_{\hat{t}}\right) \\ &= -c\left(\frac{\partial}{\partial \hat{x}} u_{\hat{x}} \frac{\partial \hat{x}}{\partial t} + \frac{\partial}{\partial \hat{t}} u_{\hat{x}} \frac{\partial \hat{t}}{\partial t}\right) + c\left(\frac{\partial}{\partial \hat{x}} u_{\hat{t}} \frac{\partial \hat{x}}{\partial t} + \frac{\partial}{\partial \hat{t}} u_{\hat{t}} \frac{\partial \hat{t}}{\partial t}\right) \\ &= -c\left(u_{\hat{x}\hat{x}}(-c) + u_{\hat{x}\hat{t}}(c)\right) + c\left(u_{\hat{x}\hat{t}}(-c) + u_{\hat{t}\hat{t}}(c)\right) \\ &= c^2 u_{\hat{x}\hat{x}} - 2c^2 u_{\hat{x}\hat{t}} + c^2 u_{\hat{t}\hat{t}} \end{aligned}$$

As seen above, $u_{tt} = c^2 u_{\hat{x}\hat{x}}(x,t) - 2c^2 u_{\hat{x}\hat{t}}(x,t) + c^2 u_{\hat{t}\hat{t}}(x,t)$. We will now use the multivariate chain rule to find u_{xx}.

$$
\begin{aligned}
u_{xx}(x,t) &= \frac{\partial}{\partial x}\left(u_{\hat{x}} + u_{\hat{t}}\right) \\
&= \left(\frac{\partial}{\partial \hat{x}} u_{\hat{x}} \frac{\partial \hat{x}}{\partial x} + \frac{\partial}{\partial \hat{t}} u_{\hat{x}} \frac{\partial \hat{t}}{\partial x}\right) + c\left(\frac{\partial}{\partial \hat{x}} u_{\hat{t}} \frac{\partial \hat{x}}{\partial x} + \frac{\partial}{\partial \hat{t}} u_{\hat{t}} \frac{\partial \hat{t}}{\partial x}\right) \\
&= \left(u_{\hat{x}\hat{x}}(1) + u_{\hat{x}\hat{t}}(1)\right) + \left(u_{\hat{x}\hat{t}}(1) + u_{\hat{t}\hat{t}}(1)\right) \\
&= u_{\hat{x}\hat{x}} + 2u_{\hat{x}\hat{t}} + u_{\hat{t}\hat{t}}
\end{aligned}
$$

As seen above, $u_{xx} = u_{\hat{x}\hat{x}}(x,t) + 2u_{\hat{x}\hat{t}}(x,t) + u_{\hat{t}\hat{t}}(x,t)$. We will now plug these expressions into our PDE and simplify.

$$
\begin{aligned}
c^2 u_{\hat{x}\hat{x}} - 2c^2 u_{\hat{x}\hat{t}} + c^2 u_{\hat{t}\hat{t}} &= c^2 u_{\hat{x}\hat{x}} + 2c^2 u_{\hat{x}\hat{t}} + c^2 u_{\hat{t}\hat{t}} \\
-4c^2 u_{\hat{x}\hat{t}} &= 0 \\
u_{\hat{x}\hat{t}} &= 0
\end{aligned}
$$

Thus we have shown that the change of variables $\hat{x} = x - ct$ and $\hat{t} = x + ct$ results in $u_{\hat{x}\hat{t}} = 0$ once plugged into the PDE.

In general, the change of variables $\hat{x} = x \pm ct$ and $\hat{t} = x \mp ct$ simplifies the wave equation to $u_{\hat{x}\hat{t}} = 0$.

(ii) While solving a *gnarly* initial value problem for (1) over $\mathbb{R}$, student Smar T. Pants obtained the following solution:

$$
u(x,t) = \sqrt{\left(1 + e^{\cos\left(\frac{\tan^{-1}(x-2ct)}{1+sech(x-2ct)}\right)}\right)}.
$$

Is Smar correct? Justify your answer in a sentence or two.

The general solution of the wave equation, as computed in lecture, is $u(x,t) = F(x-ct) + G(x+ct)$. Where F and G are arbitrary functions to be determined by initial conditions. As can be seen from, $u(x,t) = \sqrt{\left(1 + e^{\cos\left(\frac{\tan^{-1}(x-2ct)}{1+sech(x-2ct)}\right)}\right)}$, $G \equiv 0$. However, we notice that the function F, can be written notationally as $F(x-2ct)$. No matter what we do, we cannot get F into the form $F(x-ct)$, hence Smar is incorrect and the above function is not a solution of an IVP of the wave equation.

(iii) One can readily show $u(x,t) = 2\cosh(x)\cosh(ct)$ is a solution of the wave equation (but you don't have to). Find F and G such that $u(x,t) = F(x-ct) + G(x+ct)$. *Hint*: Use the definition of cosh and properties of exponentials.

Since $u(x,t) = 2\cosh(x)\cosh(ct)$ is a solution of the wave equation, it can be rewritten in the form $u(x,t) = F(x-ct) + G(x+ct)$. We will find F and G for our particular $u(x,t)$ using the definition of cosh and properties of exponentials. Notice that by definition $\cosh(x) = \frac{e^x + e^{-x}}{2}$. We will use this fact to simplify the solution and derive F and G below.

$$\begin{aligned}
u(x,t) &= 2\cosh(x)\cosh(ct) \\
&= 2\left(\frac{e^x+e^{-x}}{2}\right)\left(\frac{e^{ct}+e^{-ct}}{2}\right) \\
&= \left(e^x+e^{-x}\right)\left(\frac{e^{ct}+e^{-ct}}{2}\right) \\
&= \frac{1}{2}\left(e^{x+ct}+e^{x-ct}+e^{-x+ct}+e^{-x-ct}\right) \\
&= \frac{1}{2}\left(e^{x+ct}+e^{x-ct}+e^{-(x-ct)}+e^{-(x+ct)}\right) \\
&= \frac{e^{x-ct}+e^{-(x-ct)}}{2}+\frac{e^{x+ct}+e^{-(x+ct)}}{2} \\
&= \cosh(x-ct)+\cosh(x+ct)
\end{aligned}$$

Hence, as computed above, $F(x-ct)=\cosh(x-ct)$, and $G(x-ct)=\cosh(x+ct)$.

(iv) True or false: there exists a real-valued constant A such that $u_{tt}=-c^2u_{xx}$ becomes $u_{\hat{t}\hat{t}}=c^2u_{\hat{x}\hat{x}}$ under the change of variables $\hat{x}=Ax$ and $\hat{t}=t$. Justify your answer with a brief calculation.

In this problem, we will use the change of variables $\hat{x}=Ax$ and $\hat{t}=t$ in order to find the real-valued choice for the constant A that turns $u_{tt}=-c^2u_{xx}$ into $u_{\hat{t}\hat{t}}=c^2u_{\hat{x}\hat{x}}$. We will first use the multivariate chain rule to find u_t.

$$\begin{aligned}
u_t(x,t) &= \frac{\partial}{\partial t}u(x,t) \\
&= \frac{\partial u}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial t}+\frac{\partial u}{\partial \hat{t}}\frac{\partial \hat{t}}{\partial t} \\
&= u_{\hat{x}}(0)+u_{\hat{t}}(1) \\
&= u_{\hat{t}}
\end{aligned}$$

As seen above, $u_t(x,t)=u_{\hat{t}}(x,t)$. We will now use the multivariate chain rule to find u_x.

$$\begin{aligned}
u_x(x,t) &= \frac{\partial}{\partial x}u(x,t) \\
&= \frac{\partial u}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial x}+\frac{\partial u}{\partial \hat{t}}\frac{\partial \hat{t}}{\partial x} \\
&= u_{\hat{x}}(A)+u_{\hat{t}}(0) \\
&= Au_{\hat{x}}
\end{aligned}$$

As seen above, $u_x(x,t)=Au_{\hat{x}}(x,t)$. We will now use the multivariate chain rule to find u_{tt}.

$$\begin{aligned}
u_{tt}(x,t) &= \frac{\partial}{\partial t}\left(u_{\hat{t}}\right) \\
&= \frac{\partial}{\partial \hat{x}}u_{\hat{t}}\frac{\partial \hat{x}}{\partial t}+\frac{\partial}{\partial \hat{t}}u_{\hat{t}}\frac{\partial \hat{t}}{\partial t} \\
&= u_{\hat{x}\hat{t}}(0)+u_{\hat{t}\hat{t}}(1) \\
&= u_{\hat{t}\hat{t}}
\end{aligned}$$

As seen above, $u_{tt} = u_{\hat{t}\hat{t}}(x,t)$. We will now use the multivariate chain rule to find u_{xx}.

$$\begin{aligned}
u_{xx}(x,t) &= \frac{\partial}{\partial x}\left(Au_{\hat{x}}\right) \\
&= A\left(\frac{\partial}{\partial \hat{x}}u_{\hat{x}}\frac{\partial \hat{x}}{\partial x} + \frac{\partial}{\partial \hat{t}}u_{\hat{x}}\frac{\partial \hat{t}}{\partial x}\right) \\
&= Au_{\hat{x}\hat{x}}(A) + Au_{\hat{t}\hat{x}}(0) \\
&= A^2 u_{\hat{x}\hat{x}}
\end{aligned}$$

As seen above, $u_{xx} = A^2 u_{\hat{x}\hat{x}}(x,t)$. With these expressions found, we can now see that we need $-c^2A^2 = c^2$. Below we will find the value(s) of A that make this happen.

$$\begin{aligned}
-c^2A^2 &= c^2 \\
-A^2 &= 1 \\
A^2 &= -1 \\
A &= \pm\sqrt{-1}
\end{aligned}$$

However, notice that $A = \pm\sqrt{-1}$ is a non real-valued constant, thus there does not exist a real-valued constant A such that the change of variables $\hat{x} = Ax$ and $\hat{t} = t$ turns $u_{tt} = -c^2u_{xx}$ into $u_{\hat{t}\hat{t}} = c^2u_{\hat{x}\hat{x}}$.

(v) True or false: if $u(x,t)$ solves (1) and is infinitely differentiable with respect to x and t (so that one can interchange the order of mixed partial derivatives), then u_t solves (1). *Hint*: What operations can you apply to both sides of (1)?

Since $u(x,t)$ solves (1), the wave equation, it follows that $u_{tt} = c^2u_{xx}$. Taking the partial derivative with respect to t on both sides of this equation leads to the relation $u_{ttt} = c^2u_{txx}$. Furthermore, if $u_t(x,t)$ is a solution of the wave equation (1), then it would satisfy $(u_t)_{tt} = c^2(u_t)_{xx} \implies u_{ttt} = c^2u_{xxt}$. Lastly, since $u(x,t)$ is infinitely differentiable with respect to x and t, so that we can interchange the order of mixed partial derivatives, it follows that assuming u_t is a solution leads to the following equation $u_{ttt} = c^2u_{txx}$. As can be seen, since we achieved the same form from taking the derivative with respect to t of $u(x,t)$ as we did from assuming u_t was a solution, it follows that if $u(x,t)$ solves (1) and is infinitely differentiable with respect to x and t, then u_t solves (1).

2. Consider the following initial value problem over $\mathbb{R}$:

$$\begin{cases} u_{tt} = 4u_{xx} \\ u(x,0) = 2sech^2(x) \\ u_t(x,0) = -\frac{4}{1+x^2} \end{cases} . \tag{2}$$

(i) Solve (2).

In order to find $u(x,t)$ for the above initial value problem, we must use d'Alembert's formula which states that $u(x,t) = \frac{1}{2}(\phi(x-ct)+\phi(x+ct)) + \frac{1}{2c}\int\limits_{x-ct}^{x+ct}\psi(s)ds$, where $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$. In our case

$c = 2$, and if we let $\phi(x) = 2sech^2(x)$, and $\psi(x) = \frac{-4}{1+x^2}$, then d'Alembert's formula gives us the following:

$$\begin{aligned}
u(x,t) &= \frac{1}{2}(\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} \psi(s)ds \\
&= \frac{1}{2}(\phi(x-2t) + \phi(x+2t)) + \frac{1}{4}\int\limits_{x-2t}^{x+2t} \psi(s)ds \\
&= \frac{1}{2}(2sech^2(x-2t) + 2sech^2(x+2t)) + \frac{1}{4}\int\limits_{x-2t}^{x+2t} \frac{-4}{1+s^2}ds \\
&= sech^2(x-2t) + sech^2(x+2t) - [\arctan(s)]_{x-2t}^{x+2t} \\
&= sech^2(x-2t) + sech^2(x+2t) - \arctan(x+2t) + \arctan(x-2t)
\end{aligned}$$

Thus, the solution to the initial value problem using d'Alembert's formula is $u(x,t) = sech^2(x-2t) + sech^2(x+2t) - \arctan(x+2t) + \arctan(x-2t)$.

(ii) Show that your solution from (i) satisfies $u_x(0,t) = 0$ for all t. Effectively, you have solved the following initial boundary value problem on $[0,\infty)$:

$$\begin{cases} u_{tt} = 4u_{xx} \\ u_x(0,t) = 0 \\ u(x,0) = 2sech^2(x) \\ u_t(x,0) = -\frac{4}{1+x^2} \end{cases}$$

Our first step in showing that our solution from (i) satisfies $u_x(0,t) = 0$ for all t is to take the partial derivative with respect to x of the solution of the IVP above. This turns out to be: $u_x(x,t) = -2sech^2(x-2t)\tanh(x-2t) - 2sech^2(x+2t)\tanh(x+2t) - \frac{1}{1+(x+2t)^2} + \frac{1}{1+(x-2t)^2}$. Hence evaluating this expression at $t = 0$ leads to the following result.

$$\begin{aligned}
u_x(x,0) &= \left[-2sech^2(x-2t)\tanh(x-2t) - 2sech^2(x+2t)\tanh(x+2t) - \frac{1}{1+(x+2t)^2} + \frac{1}{1+(x-2t)^2}\right]_{t=0} \\
&= -2sech^2(0-2t)\tanh(0-2t) - 2sech^2(0+2t)\tanh(0+2t) - \frac{1}{1+(0+2t)^2} + \frac{1}{1+(0-2t)^2} \\
&= -2sech^2(-2t)\tanh(-2t) - 2sech^2(2t)\tanh(2t) - \frac{1}{1+(2t)^2} + \frac{1}{1+(-2t)^2} \\
&= -2sech^2(-2t)\tanh(-2t) - 2sech^2(2t)\tanh(2t) - \frac{1}{1+4t^2} + \frac{1}{1+4t^2} \\
&= -2sech^2(-2t)\tanh(-2t) - 2sech^2(2t)\tanh(2t)
\end{aligned}$$

Since $sech(x)$ is even and $\tanh(x)$ is odd, it follows that

$$\begin{aligned}
u_x(x,0) &= -2sech^2(-2t)\tanh(-2t) - 2sech^2(2t)\tanh(2t) \\
&= 2sech^2(2t)\tanh(2t) - 2sech^2(2t)\tanh(2t) \\
&= 0
\end{aligned}$$

Hence we have shown that our solution from (i) satisfies $u_x(0,t) = 0$ for all t. Effectively, we have solved the above initial boundary value problem on $[0,\infty)$.

3. Consider the following heat equation with damping:

$$u_t + u = u_{xx}. \tag{3}$$

(i) Let $u(x,t) = X(x)T(t)$ be a (real-valued) standing wave solution of (3). Show that $X(x)$ and $T(t)$ satisfy the following ODE's:

$$X''(x) = \lambda X(x)$$
$$T'(t) = (\lambda - 1)T(t),$$

where λ is an arbitrary (real) constant.

Let $u(x,t) = X(x)T(t)$ be a standing wave solution of $u_t + u = u_{xx}$, we will use separation of variables to obtain ODE's for $X(x)$ and $T(t)$ letting λ denote our separation constant. This process is shown below.

$$\begin{aligned} u_t + u &= u_{xx} \\ \left(X(x)T(t)\right)_t + X(x)T(t) &= \left(X(x)T(t)\right)_{xx} \\ X(x)T'(t) + X(x)T(t) &= X''(x)T(t) \\ \frac{T'(t) + T(t)}{T(t)} &= \frac{X''(x)}{X(x)} = \lambda \end{aligned}$$

The last line from the above separation of variables process gives us ODE's for $X(x)$ and $T(t)$. These ODEs are $T'(t) + T(t) = \lambda T(t) \implies T'(t) = (\lambda - 1)T(t)$ and $X''(x) = \lambda X(x)$. Hence we have show that $X(x)$ and $T(t)$ satisfy the correct ODE's.

(ii) Find all the (real-valued) standing wave solutions of (3). (You will need to consider different cases for λ. The $X(x)$ equation will tell you what these cases are.)

To find all the (real-valued) standing wave solutions of (3), we will consider different cases for λ. The $X(x)$ equation tells us what these cases are. In particular, since the ODE for $X(x)$ is $X''(x) = \lambda X(x)$, we can see that the cases for lambda are (i) $\lambda < 0$, (ii) $\lambda = 0$, and (iii) $\lambda > 0$. In each of these cases we will solve the pair of ODEs for $X(x)$ and $T(t)$ and then combine the two resulting solutions to obtain the standing wave solution.

- **Case 1**, $\boldsymbol{\lambda < 0}$:

In this case our $X(x)$ ODE is $X''(x) - \lambda X(x) = 0$ with $\lambda < 0$. Since this is a linear second order homogeneous ODE with constant coefficients, it takes one of three forms. In particular, the discriminant of the characteristic polynomial is $\Delta = -4(1)(-\lambda) = 4\lambda$. However, $\lambda < 0 \implies 4\lambda < 0$. Hence the resulting polynomial has complex conjugate roots of the form $\sigma \pm i\omega$, where $\sigma = \frac{0}{2} = 0$ and $\omega = \frac{\sqrt{|4\lambda|}}{2} = \frac{\sqrt{-4\lambda}}{2} = \sqrt{-\lambda}$. The general solution of ODEs of this form is $f(x) = c_1 e^{\sigma x}\cos(\omega x) + c_2 e^{\sigma x}\sin(\omega x)$. Hence in our case, $X(x) = A\cos\left(x\sqrt{-\lambda}\right) + B\sin\left(x\sqrt{-\lambda}\right)$.

Furthermore, in this case our $T(t)$ ODE is $T'(t) = (\lambda - 1)T(t)$. We will solve this separable ODE below.

$$\begin{aligned} T'(t) &= (\lambda - 1)T(t) \\ \int \frac{T'(t)dt}{T(t)} &= \int (\lambda - 1)dt + C_1 \\ \int \frac{dT}{T} &= \int (\lambda - 1)dt + C_1 \\ \ln|T| &= (\lambda - 1)t + C_1 \\ |T| &= e^{C_1}e^{(\lambda-1)t} \\ T &= \pm e^{C_1}e^{(\lambda-1)t} \\ T &= Ce^{(\lambda-1)t} \quad (C = \pm e^{C_1}) \end{aligned}$$

Hence, $T(t) = Ce^{(\lambda-1)t}$.

Lastly, since $u(x,t) = X(x)T(t)$, we see that the real-valued standing wave solutions in this case are $u(x,t) = \left(A\cos\left(x\sqrt{-\lambda}\right) + B\sin\left(x\sqrt{-\lambda}\right)\right)\left(Ce^{(\lambda-1)t}\right)$

- **Case 2**, $\lambda = \mathbf{0}$:

In this case our $X(x)$ ODE is $X''(x) = 0$. Since this is a linear second order homogeneous ODE with constant coefficients, it takes one of three forms. In particular, the discriminant of the characteristic polynomial is $\Delta = 0$. Hence the resulting polynomial has one repeated root of the form $r = \frac{0}{2} = 0$. The general solution of ODEs of this form is $f(x) = c_1 e^{rx} + c_2 x e^{rx}$. Hence in our case, $X(x) = Dx + E$.

Furthermore, in this case our $T(t)$ ODE is $T'(t) = -T(t)$. We will solve this separable ODE below.

$$\begin{aligned}
T'(t) &= -T(t) \\
\int \frac{T'(t)dt}{T(t)} &= -\int dt + C_2 \\
\int \frac{dT}{T} &= -\int dt + C_2 \\
\ln|T| &= -t + C_2 \\
|T| &= e^{C_2} e^{-t} \\
T &= \pm e^{C_2} e^{-t} \\
T &= F e^{-t} \quad (F = \pm e^{C_2})
\end{aligned}$$

Hence, $T(t) = Fe^{-t}$.

Lastly, since $u(x,t) = X(x)T(t)$, we see that the real-valued standing wave solutions in this case are $u(x,t) = (Dx + E)\left(Fe^{-t}\right)$.

- **Case 3**, $\lambda > \mathbf{0}$:

In this case our $X(x)$ ODE is $X''(x) - \lambda X(x) = 0$ with $\lambda > 0$. Since this is a linear second order homogeneous ODE with constant coefficients, it takes one of three forms. In particular, the discriminant of the characteristic polynomial is $\Delta = -4(1)(-\lambda) = 4\lambda$. However, $\lambda > 0 \implies 4\lambda > 0$. Hence the resulting polynomial has two real roots of the form $r_{1,2} = \frac{\pm\sqrt{4\lambda}}{2} = \pm\sqrt{\lambda}$. The general solution of ODEs of this form is $f(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$. Hence in our case, $X(x) = Ge^{x\sqrt{\lambda}} + He^{-x\sqrt{\lambda}}$.

Furthermore, in this case our $T(t)$ ODE is $T'(t) = (\lambda - 1)T(t)$. We will solve this separable ODE below.

$$\begin{aligned}
T'(t) &= (\lambda - 1)T(t) \\
\int \frac{T'(t)dt}{T(t)} &= \int (\lambda - 1)dt + C_3 \\
\int \frac{dT}{T} &= \int (\lambda - 1)dt + C_3 \\
\ln|T| &= (\lambda - 1)t + C_3 \\
|T| &= e^{C_3} e^{(\lambda-1)t} \\
T &= \pm e^{C_3} e^{(\lambda-1)t} \\
T &= I e^{(\lambda-1)t} \quad (I = \pm e^{C_3})
\end{aligned}$$

Hence, $T(t) = Ce^{(\lambda-1)t}$.

Lastly, since $u(x,t) = X(x)T(t)$, we see that the real-valued standing wave solutions in this case are $u(x,t) = \left(Ge^{x\sqrt{\lambda}} + He^{-x\sqrt{\lambda}}\right)\left(Ie^{(\lambda-1)t}\right)$

Along with these three standing wave solution forms for the three different cases of λ, it follows that the trivial solutions $u(x,t) = 0$ is also a real-valued standing wave solution. Note that in the above cases A, B, C, D, E, F, G, H and I are arbitrary constants in $\mathbb{R}$.

Multi-Step Problem

A levy is a wall built to protect low-lying areas from flooding by a neighboring body of water. Suppose we have a body of water (say 10 feet deep, for concreteness) for $x < 0$ with a levy situated at $x = 0$ and dry land for $x > 0$. If $u(x, t)$ represents the depth of the water (in feet) at a displacement x from the levy (in hundreds of feet) at time t (in minutes), then we can approximate our set-up by

$$u(x, 0) = \begin{cases} 10 & x < 0 \\ 0 & x > 0 \end{cases}. \tag{4}$$

Suppose the levy fails at $t = 0$ so that the initial tendency of $u(x, t)$ is

$$u_t(x, 0) = \begin{cases} -4 & -1 < x < 0 \\ 4 & 0 < x < 1 \\ 0 & |x| > 1 \end{cases}. \tag{5}$$

Suppose $u(x, t)$ for $t \geq 0$ is governed by the wave equation

$$u_{tt} = 4u_{xx}. \tag{6}$$

(i) Smar T. Pants owns a home located at $x = 4$. When will their home begin to flood? (That is, when will the interval of nonzero initial conditions $-\infty < x_0 < 1$ first reach $x = 4$?)

Since characteristic lines of the domain of influence start at a t value of 0, we are only looking at rightward propagating lines, which are lines with positive slope. In particular, these lines take the form $t = \frac{1}{c}(x - \tilde{x}) + \tilde{t}$. Since $x = 4$ is to the right of the right-most characteristic line, we only need to find the equation of this characteristic. With that said, the right-most characteristic line, which has an x-intercept of $x_0 = 1$, has an equation $t = \frac{1}{2}(x - 1)$, hence it arrives at $x = 4$ at $t = \frac{1}{2}(4 - 1) = \frac{3}{2}$. Therefore Smar T. Pants' home will begin to flood at $t = \frac{3}{2}$ minutes.

(ii) Will the water reach Smar's friend at $x = 9$ by $t = 3$? (That is, does the interval of nonzero initial conditions $-\infty < x_0 < 1$ influence the depth of the water at $x = 9$ and $t = 3$?)

In order to know what interval of initial conditions we must consider if Smar's friend is located at $(x = 9, t = 3)$, we must find the domain of dependence of the point $(\tilde{x}, \tilde{t}) = (9, 3)$. The domain of dependence of a point $(\tilde{x}, \tilde{t})$ is the interval $\left[\tilde{x} - c\tilde{t}, \tilde{x} + c\tilde{t}\right]$ such that the value of $u(\tilde{x}, \tilde{t})$ only depends on $\phi(x_0)$ and $\psi(x_0)$ for x_0 in the aforementioned interval.

Thus in our given scenario, $c = 2$ and $(\tilde{x}, \tilde{t}) = (9, 3)$. Hence the domain of dependence of $(\tilde{x}, \tilde{t}) = (9, 3)$ is $\left[\tilde{x} - c\tilde{t}, \tilde{x} + c\tilde{t}\right] = [9 - 2(3), 9 + 2(3)] = [3, 15]$. Therefore we must consider initial conditions $\phi(x_0)$ and $\psi(x_0)$ for $x_0 \in [3, 15]$.

Since $-\infty < x_0 < 1$ is not a subset of the interval of dependence of $(\tilde{x}, \tilde{t}) = (9, 3)$, namely $-\infty < x_0 < 1 \not\subset [3, 15]$, the initial conditions at $-\infty < x_0 < 1$, $\phi(x_0)$ and $\psi(x_0)$, will not influence the value of u at $(x = 9, t = 3)$. Hence we know that the water won't reach Smar's friend at $x = 9$ by $t = 3$.

(iii) Construct the characteristic diagram of the initial value problem given by (4)-(6). Be sure to include expressions for u in each region.

In order to create a characteristic diagram, we need to compute the domains of influence of the intervals $x_0 < -1$, $-1 < x_0 < 0$, $0 < x_0 < 1$, and $x_0 > 1$. These intervals correspond to the four cases over which $u(x_0, 0)$ and $u_t(x_0, 0)$ is defined.

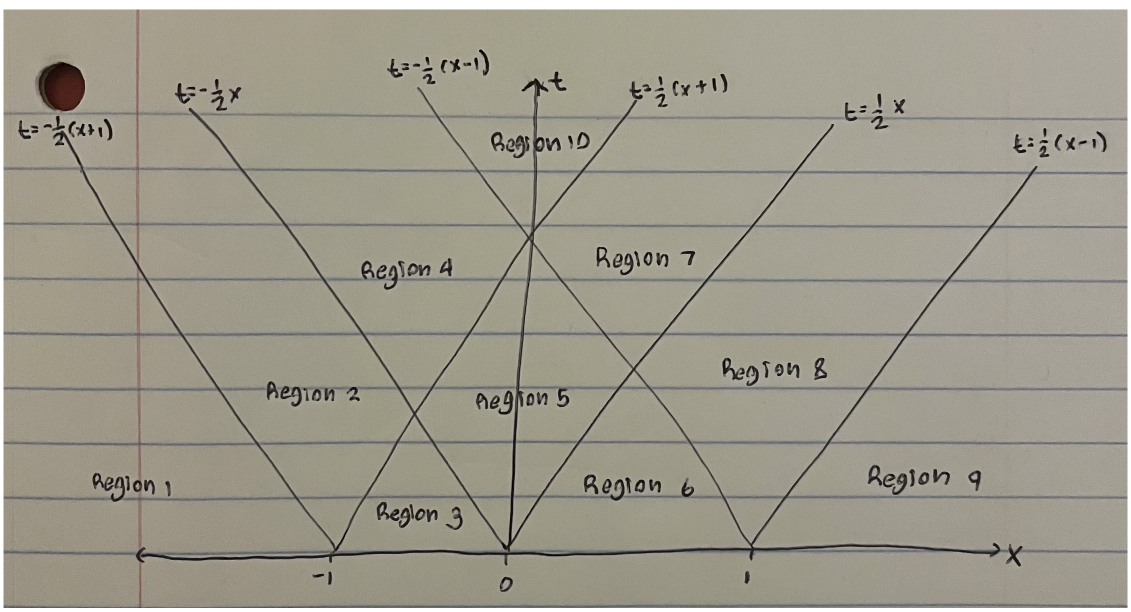
To find the domain of influence of the interval $x_0 < -1$, we must find the outward propagating characteristic at the point $x_0 = -1$. This characteristic must be $t = \frac{1}{2}(x+1)$, since this line passes through $x = -1$ and has slope $1/c = 1/2$ (since $c = 2$). The region bounded by the x-axis and to the left of $t = \frac{1}{2}(x+1)$ is the domain of influence of the interval $x_0 < -1$.

To find the domain of influence of the interval $-1 < x_0 < 0$, we must find the outward propagating characteristics at the points $x_0 = -1$ and $x_0 = 0$. The outward propagating characteristic at $x_0 = -1$ is $t = -\frac{1}{2}(x+1)$, since this line passes through $x = -1$ with slope $-1/c = -1/2$ (since $c = 2$). The outward propagating characteristic at the point $x_0 = 0$ is $t = \frac{1}{2}x$, since this line passes through the origin and has slope $1/c = 1/2$ (since $c = 2$). The region bounded between the x-axis, the line $t = -\frac{1}{2}(x+1)$, and the line $t = \frac{1}{2}x$ is the domain of influence of the interval $-1 < x_0 < 0$.

To find the domain of influence of the interval $0 < x_0 < 1$, we must find the outward propagating characteristics at the points $x_0 = 0$ and $x_0 = 1$. The outward propagating characteristic at $x_0 = 0$ is $t = -\frac{1}{2}x$, since this line passes through the origin with slope $-1/c = -1/2$ (since $c = 2$). The outward propagating characteristic at the point $x_0 = 1$ is $t = \frac{1}{2}(x-1)$, since this line passes through $x = 1$ and has slope $1/c = 1/2$ (since $c = 2$). The region bounded between the x-axis, the line $t = \frac{1}{2}(x-1)$, and the line $t = -\frac{1}{2}x$ is the domain of influence of the interval $0 < x_0 < 1$.

To find the domain of influence of the interval $x_0 > 1$, we must find the outward propagating characteristic at the point $x_0 = 1$. This characteristic must be $t = -\frac{1}{2}(x-1)$, since this line passes through $x = 1$ and has slope $-1/c = -1/2$ (since $c = 2$). The region bounded by the x-axis and to the right of $t = -\frac{1}{2}(x-1)$ is the domain of influence of the interval $x_0 > 1$.

Plotting each of the outward propagating characteristics in the xt-plane gives us the characteristic diagram below.



To obtain a graphical solution to the initial value problem presented above, we will now determine the value of u in each region.

- **Region 1.**

Let $(\tilde{x}, \tilde{t})$ be a point in this region. d'Alembert's formula tells us

$$u(\tilde{x}, \tilde{t}) = \frac{1}{2}\left(\phi(\tilde{x} - 2\tilde{t}) + \phi(\tilde{x} + 2\tilde{t})\right) + \frac{1}{4}\int_{\tilde{x}-2\tilde{t}}^{\tilde{x}+2\tilde{t}} \psi(x)dx.$$

In this region, $\tilde{x} \pm 2\tilde{t} < -1$. We have $\phi(x_0) \equiv 10$ and $\psi(x_0) \equiv 0$ for $x_0 < -1$. Therefore,

$$\begin{aligned}
u(\tilde{x}, \tilde{t}) &= \frac{1}{2}\left(\phi(\tilde{x} - 2\tilde{t}) + \phi(\tilde{x} + 2\tilde{t})\right) + \frac{1}{4}\int_{\tilde{x}-2\tilde{t}}^{\tilde{x}+2\tilde{t}} \psi(x)dx \\
&= \frac{1}{2}[(10) + (10)] + \frac{1}{4}\int_{\tilde{x}-2\tilde{t}}^{\tilde{x}+2\tilde{t}} (0)dx \\
&= 10.
\end{aligned}$$

For any $(\tilde{x}, \tilde{t})$ in this region, we have $u(\tilde{x}, \tilde{t}) \equiv 10$.

- **Region 2.**

Let $(\tilde{x}, \tilde{t})$ be a point in this region. d'Alembert's formula tells us

$$u(\tilde{x}, \tilde{t}) = \frac{1}{2}\left(\phi(\tilde{x} - 2\tilde{t}) + \phi(\tilde{x} + 2\tilde{t})\right) + \frac{1}{4}\int_{\tilde{x}-2\tilde{t}}^{\tilde{x}+2\tilde{t}} \psi(x)dx.$$

In this region, $\tilde{x} - 2\tilde{t} < -1$ and $-1 < \tilde{x} + 2\tilde{t} < 0$. We have $\phi(x_0) \equiv 10$ and $\psi(x_0) \equiv 0$ for $x_0 < -1$, and $\phi(x_0) \equiv 10$ and $\psi(x_0) \equiv -4$ for $-1 < x_0 < 0$. Therefore,

$$\begin{aligned}
u(\tilde{x}, \tilde{t}) &= \frac{1}{2}\left(\phi(\tilde{x} - 2\tilde{t}) + \phi(\tilde{x} + 2\tilde{t})\right) + \frac{1}{4}\int_{\tilde{x}-2\tilde{t}}^{\tilde{x}+2\tilde{t}} \psi(x)dx \\
&= \frac{1}{2}[(10) + (10)] + \frac{1}{4}\int_{\tilde{x}-2\tilde{t}}^{-1} (0)dx + \frac{1}{4}\int_{-1}^{\tilde{x}+2\tilde{t}} (-4)dx \\
&= 10 - \tilde{x} - 2\tilde{t} - 1 \\
&= 9 - \tilde{x} - 2\tilde{t}
\end{aligned}$$

For any $(\tilde{x}, \tilde{t})$ in this region, we have $u(\tilde{x}, \tilde{t}) = 9 - \tilde{x} - 2\tilde{t}$.

- **Region 3.**

Let $(\tilde{x}, \tilde{t})$ be a point in this region. d'Alembert's formula tells us

$$u(\tilde{x}, \tilde{t}) = \frac{1}{2}\left(\phi(\tilde{x} - 2\tilde{t}) + \phi(\tilde{x} + 2\tilde{t})\right) + \frac{1}{4}\int_{\tilde{x}-2\tilde{t}}^{\tilde{x}+2\tilde{t}} \psi(x)dx.$$

In this region, $-1 < \tilde{x} \pm 2\tilde{t} < 0$. We have $\phi(x_0) \equiv 10$ and $\psi(x_0) \equiv -4$ for $-1 < x_0 < 0$. Therefore,

$$\begin{aligned}
u(\tilde{x}, \tilde{t}) &= \frac{1}{2}\left(\phi(\tilde{x} - 2\tilde{t}) + \phi(\tilde{x} + 2\tilde{t})\right) + \frac{1}{4}\int_{\tilde{x}-2\tilde{t}}^{\tilde{x}+2\tilde{t}} \psi(x)dx \\
&= \frac{1}{2}[(10) + (10)] + \frac{1}{4}\int_{\tilde{x}-2\tilde{t}}^{\tilde{x}+2\tilde{t}} (-4)dx \\
&= 10 - (\tilde{x} + 2\tilde{t} - \tilde{x} + 2\tilde{t}) \\
&= 10 - 4\tilde{t}
\end{aligned}$$

For any $(\tilde{x}, \tilde{t})$ in this region, we have $u(\tilde{x}, \tilde{t}) = 10 - 4\tilde{t}$.

- **Region 4.**

Let $(\tilde{x}, \tilde{t})$ be a point in this region. d'Alembert's formula tells us

$$u(\tilde{x}, \tilde{t}) = \frac{1}{2}\left(\phi(\tilde{x}-2\tilde{t}) + \phi(\tilde{x}+2\tilde{t})\right) + \frac{1}{4}\int\limits_{\tilde{x}-2\tilde{t}}^{\tilde{x}+2\tilde{t}} \psi(x)dx.$$

In this region, $\tilde{x} - 2\tilde{t} < -1$ and $0 < \tilde{x} + 2\tilde{t} < 1$. We have $\phi(x_0) \equiv 10$ and $\psi(x_0) \equiv 0$ for $x_0 < -1$, and $\phi(x_0) \equiv 0$ and $\psi(x_0) \equiv 4$ for $0 < x_0 < 1$. Therefore,

$$\begin{aligned} u(\tilde{x}, \tilde{t}) &= \frac{1}{2}\left(\phi(\tilde{x}-2\tilde{t}) + \phi(\tilde{x}+2\tilde{t})\right) + \frac{1}{4}\int\limits_{\tilde{x}-2\tilde{t}}^{\tilde{x}+2\tilde{t}} \psi(x)dx \\ &= \frac{1}{2}[(10)+(0)] + \frac{1}{4}\int\limits_{\tilde{x}-2\tilde{t}}^{-1} (0)dx + \frac{1}{4}\int\limits_{-1}^{0}(-4)dx + \frac{1}{4}\int\limits_{0}^{\tilde{x}+2\tilde{t}}(4)dx \\ &= 5 + 0 - 1 + \tilde{x} + 2\tilde{t} \\ &= 4 + \tilde{x} + 2\tilde{t} \end{aligned}$$

For any $(\tilde{x}, \tilde{t})$ in this region, we have $u(\tilde{x}, \tilde{t}) = 4 + \tilde{x} + 2\tilde{t}$.

- **Region 5.**

Let $(\tilde{x}, \tilde{t})$ be a point in this region. d'Alembert's formula tells us

$$u(\tilde{x}, \tilde{t}) = \frac{1}{2}\left(\phi(\tilde{x}-2\tilde{t}) + \phi(\tilde{x}+2\tilde{t})\right) + \frac{1}{4}\int\limits_{\tilde{x}-2\tilde{t}}^{\tilde{x}+2\tilde{t}} \psi(x)dx.$$

In this region, $-1 < \tilde{x} - 2\tilde{t} < 0$ and $0 < \tilde{x} + 2\tilde{t} < 1$. We have $\phi(x_0) \equiv 10$ and $\psi(x_0) \equiv -4$ for $-1 < x_0 < 0$, and $\phi(x_0) \equiv 0$ and $\psi(x_0) \equiv 4$ for $0 < x_0 < 1$. Therefore,

$$\begin{aligned} u(\tilde{x}, \tilde{t}) &= \frac{1}{2}\left(\phi(\tilde{x}-2\tilde{t}) + \phi(\tilde{x}+2\tilde{t})\right) + \frac{1}{4}\int\limits_{\tilde{x}-2\tilde{t}}^{\tilde{x}+2\tilde{t}} \psi(x)dx \\ &= \frac{1}{2}[(10)+(0)] + \frac{1}{4}\int\limits_{\tilde{x}-2\tilde{t}}^{0} (-4)dx + \frac{1}{4}\int\limits_{0}^{\tilde{x}+2\tilde{t}}(4)dx \\ &= 5 + \tilde{x} - 2\tilde{t} + \tilde{x} + 2\tilde{t} \\ &= 5 + 2\tilde{x} \end{aligned}$$

For any $(\tilde{x}, \tilde{t})$ in this region, we have $u(\tilde{x}, \tilde{t}) = 5 + 2\tilde{x}$.

- **Region 6.**

Let $(\tilde{x}, \tilde{t})$ be a point in this region. d'Alembert's formula tells us

$$u(\tilde{x}, \tilde{t}) = \frac{1}{2}\left(\phi(\tilde{x}-2\tilde{t}) + \phi(\tilde{x}+2\tilde{t})\right) + \frac{1}{4}\int\limits_{\tilde{x}-2\tilde{t}}^{\tilde{x}+2\tilde{t}} \psi(x)dx.$$

In this region, $0 < \tilde{x} \pm 2\tilde{t} < 1$. We have $\phi(x_0) \equiv 0$ and $\psi(x_0) \equiv 4$ for $0 < x_0 < 1$. Therefore,

$$\begin{aligned} u(\tilde{x}, \tilde{t}) &= \frac{1}{2}\left(\phi(\tilde{x} - 2\tilde{t}) + \phi(\tilde{x} + 2\tilde{t})\right) + \frac{1}{4}\int\limits_{\tilde{x}-\tilde{t}}^{\tilde{x}+\tilde{t}} \psi(x)dx \\ &= \frac{1}{2}[(0) + (0)] + \frac{1}{4}\int\limits_{\tilde{x}-2\tilde{t}}^{\tilde{x}+2\tilde{t}} (4)dx \\ &= 0 + \tilde{x} + 2\tilde{t} - \tilde{x} + 2\tilde{t} \\ &= 4\tilde{t} \end{aligned}$$

For any $(\tilde{x}, \tilde{t})$ in this region, we have $u(\tilde{x}, \tilde{t}) = 4\tilde{t}$.

- **Region 7.**

Let $(\tilde{x}, \tilde{t})$ be a point in this region. d'Alembert's formula tells us

$$u(\tilde{x}, \tilde{t}) = \frac{1}{2}\left(\phi(\tilde{x} - 2\tilde{t}) + \phi(\tilde{x} + 2\tilde{t})\right) + \frac{1}{4}\int\limits_{\tilde{x}-2\tilde{t}}^{\tilde{x}+2\tilde{t}} \psi(x)dx.$$

In this region, $-1 < \tilde{x} - 2\tilde{t} < 0$ and $\tilde{x} + 2\tilde{t} > 1$. We have $\phi(x_0) \equiv 10$ and $\psi(x_0) \equiv -4$ for $-1 < x_0 < 0$, and $\phi(x_0) \equiv 0$ and $\psi(x_0) \equiv 4$ for $x_0 > 1$. Therefore,

$$\begin{aligned} u(\tilde{x}, \tilde{t}) &= \frac{1}{2}\left(\phi(\tilde{x} - 2\tilde{t}) + \phi(\tilde{x} + 2\tilde{t})\right) + \frac{1}{4}\int\limits_{\tilde{x}-2\tilde{t}}^{\tilde{x}+2\tilde{t}} \psi(x)dx \\ &= \frac{1}{2}[(10) + (0)] + \frac{1}{4}\int\limits_{\tilde{x}-2\tilde{t}}^{0} (-4)dx + \frac{1}{4}\int\limits_{0}^{1} (-4)dx + \frac{1}{4}\int\limits_{1}^{\tilde{x}+2\tilde{t}} (0)dx \\ &= 5 + \tilde{x} - 2\tilde{t} + 1 + 0 \\ &= 6 + \tilde{x} - 2\tilde{t} \end{aligned}$$

For any $(\tilde{x}, \tilde{t})$ in this region, we have $u(\tilde{x}, \tilde{t}) = 6 + \tilde{x} - 2\tilde{t}$.

- **Region 8.**

Let $(\tilde{x}, \tilde{t})$ be a point in this region. d'Alembert's formula tells us

$$u(\tilde{x}, \tilde{t}) = \frac{1}{2}\left(\phi(\tilde{x} - 2\tilde{t}) + \phi(\tilde{x} + 2\tilde{t})\right) + \frac{1}{4}\int\limits_{\tilde{x}-2\tilde{t}}^{\tilde{x}+2\tilde{t}} \psi(x)dx.$$

In this region, $0 < \tilde{x} - 2\tilde{t} < 1$ and $\tilde{x} + 2\tilde{t} > 1$. We have $\phi(x_0) \equiv 0$ and $\psi(x_0) \equiv 4$ for $0 < x_0 < 1$, and $\phi(x_0) \equiv 0$ and $\psi(x_0) \equiv 0$ for $x_0 > 0$. Therefore,

$$\begin{aligned} u(\tilde{x}, \tilde{t}) &= \frac{1}{2}\left(\phi(\tilde{x} - 2\tilde{t}) + \phi(\tilde{x} + 2\tilde{t})\right) + \frac{1}{4}\int\limits_{\tilde{x}-2\tilde{t}}^{\tilde{x}+2\tilde{t}} \psi(x)dx \\ &= \frac{1}{2}[(0) + (0)] + \frac{1}{4}\int\limits_{\tilde{x}-2\tilde{t}}^{1} (4)dx + \frac{1}{4}\int\limits_{1}^{\tilde{x}+2\tilde{t}} (0)dx \\ &= 0 + 1 - \tilde{x} + 2\tilde{t} + 0 \\ &= 1 - \tilde{x} + 2\tilde{t} \end{aligned}$$

For any $(\tilde{x}, \tilde{t})$ in this region, we have $u(\tilde{x}, \tilde{t}) = 1 - \tilde{x} + 2\tilde{t}$.

- **Region 9.**

Let $(\tilde{x}, \tilde{t})$ be a point in this region. d'Alembert's formula tells us

$$u(\tilde{x}, \tilde{t}) = \frac{1}{2}\left(\phi(\tilde{x} - 2\tilde{t}) + \phi(\tilde{x} + 2\tilde{t})\right) + \frac{1}{4}\int_{\tilde{x}-2\tilde{t}}^{\tilde{x}+2\tilde{t}} \psi(x)dx.$$

In this region, $\tilde{x} \pm 2\tilde{t} > 1$. We have $\phi(x_0) \equiv 0$ and $\psi(x_0) \equiv 0$ for $x_0 > 1$. Therefore,

$$\begin{aligned}
u(\tilde{x}, \tilde{t}) &= \frac{1}{2}\left(\phi(\tilde{x} - 2\tilde{t}) + \phi(\tilde{x} + 2\tilde{t})\right) + \frac{1}{4}\int_{\tilde{x}-2\tilde{t}}^{\tilde{x}+2\tilde{t}} \psi(x)dx \\
&= \frac{1}{2}[(0) + (0)] + \frac{1}{4}\int_{\tilde{x}-2\tilde{t}}^{\tilde{x}+2\tilde{t}} (0)dx \\
&= 0.
\end{aligned}$$

For any $(\tilde{x}, \tilde{t})$ in this region, we have $u(\tilde{x}, \tilde{t}) \equiv 0$.

- **Region 10.**

Let $(\tilde{x}, \tilde{t})$ be a point in this region. d'Alembert's formula tells us

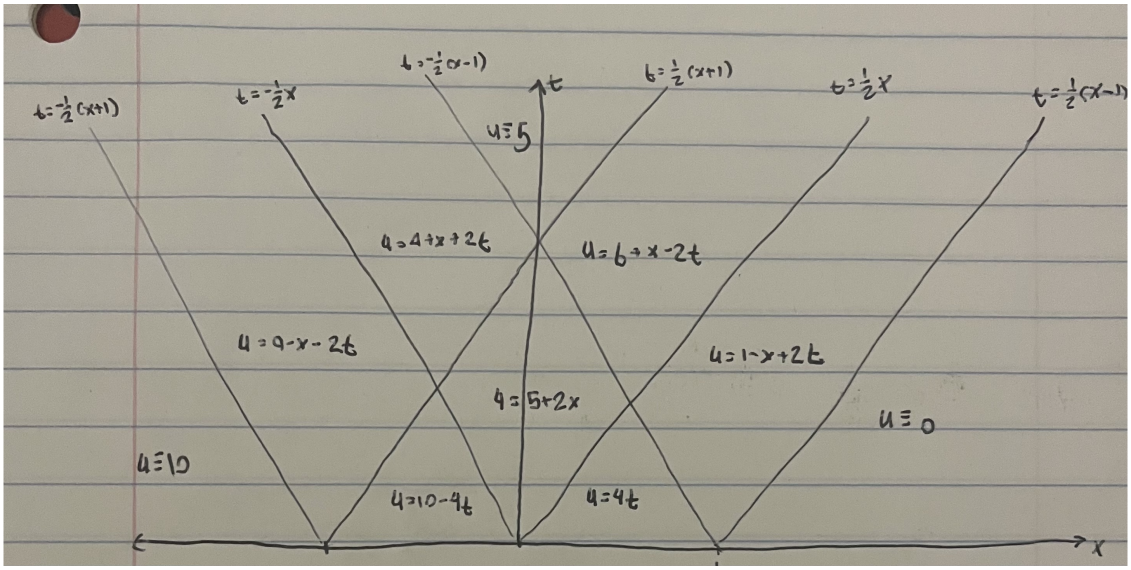
$$u(\tilde{x}, \tilde{t}) = \frac{1}{2}\left(\phi(\tilde{x} - 2\tilde{t}) + \phi(\tilde{x} + 2\tilde{t})\right) + \frac{1}{4}\int_{\tilde{x}-2\tilde{t}}^{\tilde{x}+2\tilde{t}} \psi(x)dx.$$

In this region, $\tilde{x} - 2\tilde{t} < -1$ and $\tilde{x} + 2\tilde{t} > 1$. We have $\phi(x_0) \equiv 10$ and $\psi(x_0) \equiv 0$ for $x_0 < -1$, and $\phi(x_0) \equiv 0$ and $\psi(x_0) \equiv 0$ for $x_0 > 1$. However, we also have $\phi(x_0) \equiv 10$ and $\psi(x_0) \equiv -4$ for $-1 < x_0 < 0$, as well as $\phi(x_0) \equiv 0$ and $\psi(x_0) \equiv 4$ for $0 < x_0 < 1$. Therefore,

$$\begin{aligned}
u(\tilde{x}, \tilde{t}) &= \frac{1}{2}\left(\phi(\tilde{x} - 2\tilde{t}) + \phi(\tilde{x} + 2\tilde{t})\right) + \frac{1}{4}\int_{\tilde{x}-2\tilde{t}}^{\tilde{x}+2\tilde{t}} \psi(x)dx \\
&= \frac{1}{2}[(0) + (0)] + \frac{1}{4}\int_{\tilde{x}-2\tilde{t}}^{-1} (0)dx + \frac{1}{4}\int_{-1}^{0} (-4)dx + \frac{1}{4}\int_{0}^{1} (4)dx + \frac{1}{4}\int_{1}^{\tilde{x}+2\tilde{t}} (0)dx \\
&= 5 + 0 - 1 + 1 + 0 \\
&= 5
\end{aligned}$$

For any $(\tilde{x}, \tilde{t})$ in this region, we have $u(\tilde{x}, \tilde{t}) \equiv 5$.

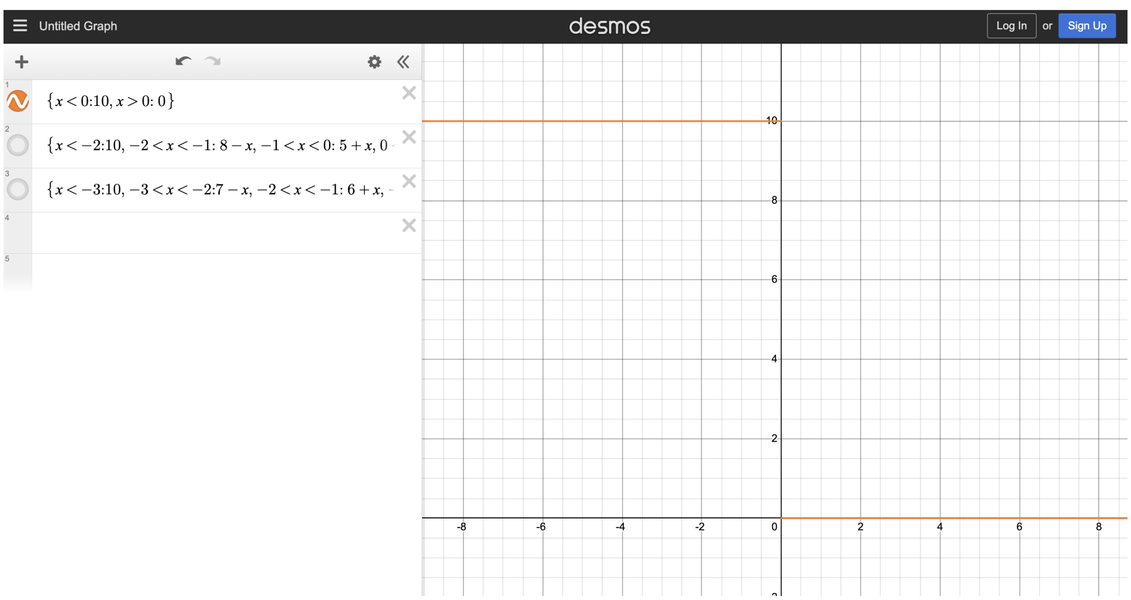
Combining our analysis of each region above, the characteristic diagram becomes the graphical solution to the initial value problem as shown below.



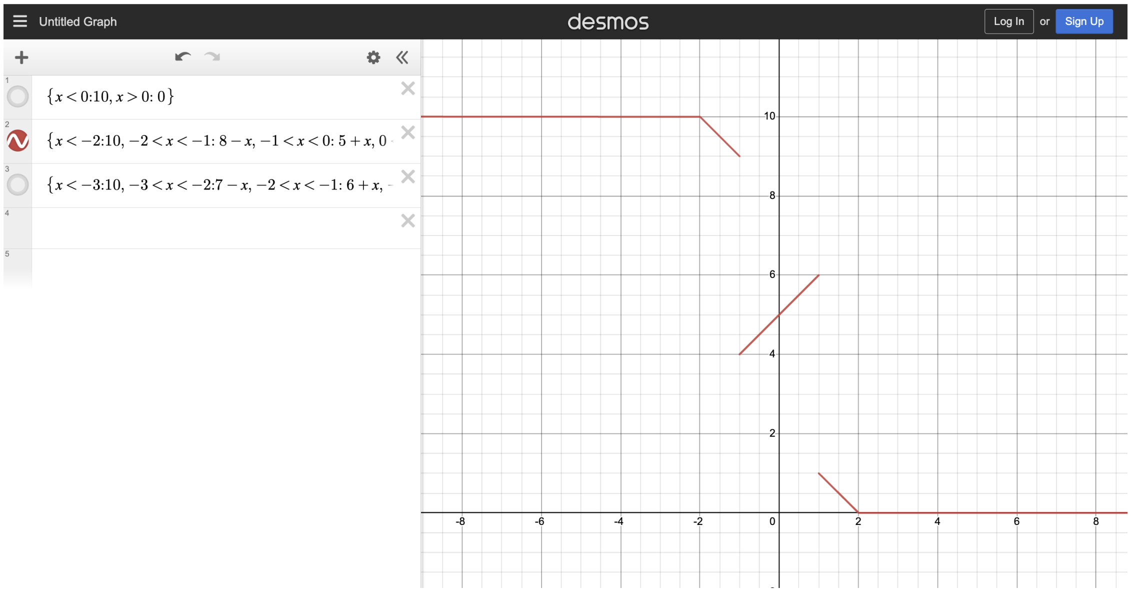
(iv) Obtain snapshot plots of your solution from (iii) for $t = 0, 1/2$, and 1.

We will now use the above characteristic plot/graphical solution to make snapshot plots for $t = 0, 1/2$, and 1. Give that these multi-part equations are hard to read we will break up these snapshot plots for the different values of t, as well as show all of the plots together at the end.

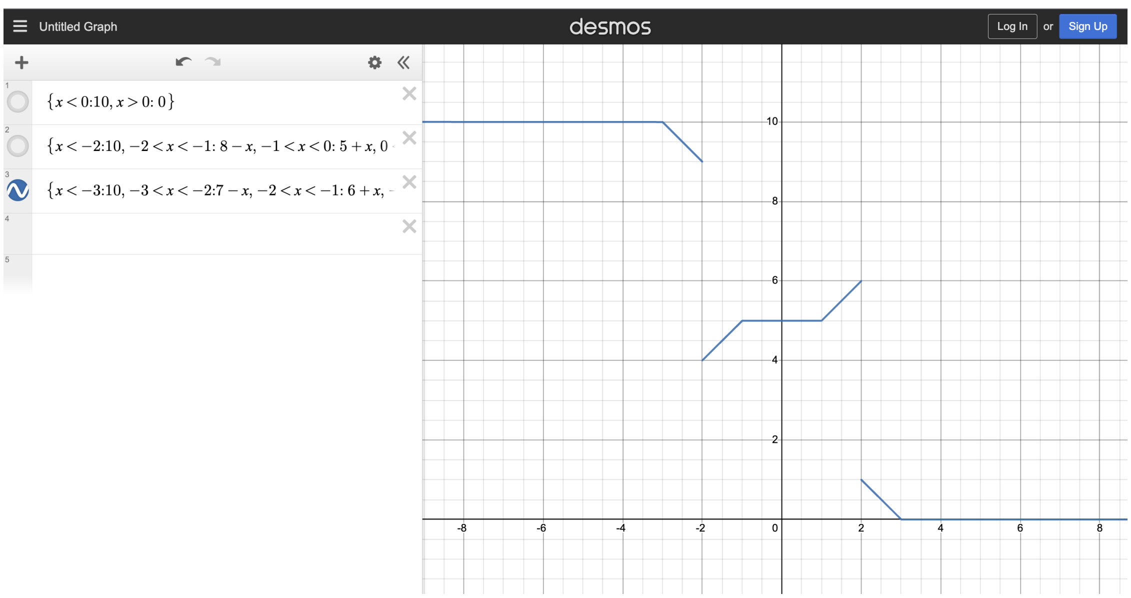
- Snapshot plot for $t = 0$.



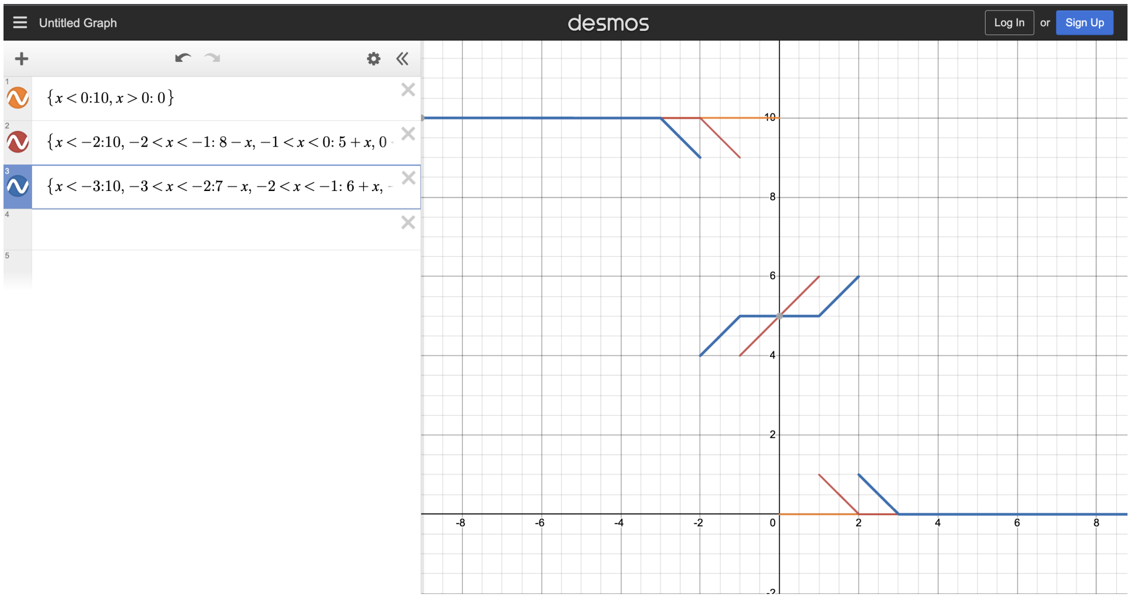
- Snapshot plot for $t = 1/2$.



- Snapshot plot for $t = 1$.



- Snapshot plots for all t.


Untitled Graph
desmos
Log In
or
Sign Up
{x < 0:10, x > 0: 0}
{x < −2:10, −2 < x < −1: 8 − x, −1 < x < 0: 5 + x, 0
{x < −3:10, −3 < x < −2:7 − x, −2 < x < −1: 6 + x,
10
8
6
4
2
-8
-6
-4
-2
0
2
4
6
8
-2