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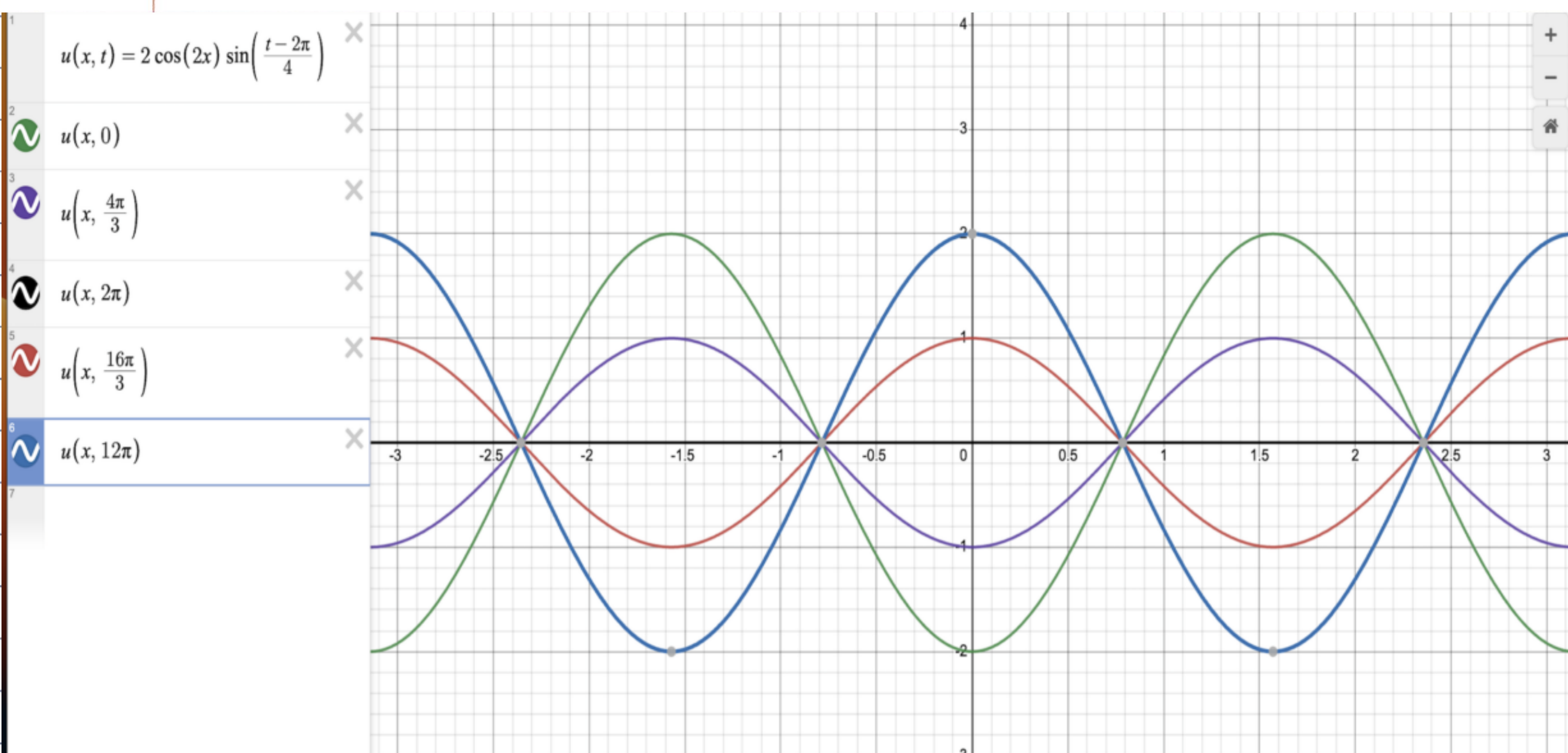
Exercise 1. Skill 1.1.

Let $u(x,t)$ represent the displacement of a guitar string from its equilibrium position, where x represents the horizontal coordinate along the string and t is time. Using software of your choice, make snapshot plots of

$$u(x,t) = 2\cos(2x)\sin\left(\frac{t-2\pi}{4}\right)$$

for $-\pi < x < \pi$ and $t = 0, \frac{4\pi}{3}, 2\pi, \frac{16\pi}{3}$, and 12π

For this problem, I will use Desmos to make snapshot plots of $u(x,t) = 2\cos(2x)\sin\left(\frac{t-2\pi}{4}\right)$ for $-\pi < x < \pi$ and $t = 0, \frac{4\pi}{3}, 2\pi, \frac{16\pi}{3}$, and 12π. These plots can be seen on the following page.



Exercise 2. Skill 1.2.

Suppose

$$u(x,t) = 4\,\text{sech}(x-t+4)e^{-\left(\frac{t}{2}\right)^2}$$

represents the intensity of a light pulse as it enters a one-dimensional optical fiber and slowly becomes absorbed. Here x represents the horizontal coordinate along the fiber and t represents time.

a.) When $t = t_0$ (for any $t_0 \geq 0$), at what value of x is the light pulse most intense? Your answer will be a function of t_0. What is $u_x(x, t_0)$ at this value of x?

We will start off by calculating $u_x(x,t)$.

$$u_x(x,t) = -4\,\text{sech}(x-t+4)\tanh(x-t+4)e^{-\left(\frac{t}{2}\right)^2}$$

thus at $t = t_0$ we have

$$u_x(x,t_0) = -4\,\text{sech}(x-t_0+4)\tanh(x-t_0+4)e^{-\left(\frac{t_0}{2}\right)^2}$$

finding the value of x where the light pulse is most intense is the same as finding the value of x such that

$$-4e^{-(t_0/2)^2}\text{sech}(x-t_0+4)\tanh(x-t_0+4) = 0$$

$$\Rightarrow \quad \text{sech}(x-t_0+4)\tanh(x-t_0+4) = 0$$

thus it follows that

$$\text{sech}(x-t_0+4) = 0 \quad \text{or} \quad \tanh(x-t_0+4) = 0$$

However $\text{sech}(x-t_0+4) \neq 0$, thus we must solve

$$\tanh(x-t_0+4) = 0$$

for x. However, since $\tanh(x-t_0+4)$ is 1-to-1, it is invertible so

$$x-t_0+4 = \tanh^{-1}(0)$$

$$\Rightarrow x-t_0+4 = 0$$

$$\Rightarrow x = t_0-4$$

thus at $x = t_0-4$ the light pulse is most intense.

$u_x(x,t_0)$ at this value of x is the same as computing $u_x(t_0-4, t_0)$

Hence we can see that:

$$u_x(t_0-4, t_0) = -4\operatorname{sech}(t_0-4+t_0+4)\tanh(t_0-4+t_0+4)\, e^{-(t_0/2)^2}$$
$$= -4\operatorname{sech}(0)\tanh(0)\, e^{-(t_0/2)^2}$$
$$= -4(1)(0)\, e^{-(t_0/2)^2}$$
$$= 0$$

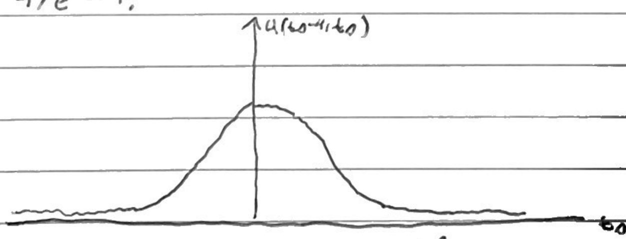
thus, as expected, $u_x(x, t_0) = 0$ at $x = t_0 - 4$.

b.) When $t = t_0$, what is the maximum intensity of the light pulse? Your answer will be a function of t_0. Is this intensity increasing or decreasing in time?

When $t = t_0$, the maximum intensity occurs at $x = t_0 - 4$, thus plugging $x = t_0 - 4$ and $t = t_0$ into $u(x,t)$ we will obtain the maximum intensity of the light pulse.

$$u(t_0-4, t_0) = 4\operatorname{sech}(t_0-4-t_0+4)\, e^{-(t_0/2)^2}$$
$$= 4\operatorname{sech}(0)\, e^{-(t_0/2)^2}$$
$$= 4e^{-(\frac{t_0}{2})^2}$$

thus when $t = t_0$ the maximum intensity of the light pulse is $4e^{-(\frac{t_0}{2})^2} = 4/e^{t_0^2/4}$.



looking at the plot of $4/e^{t_0^2/4}$, we can see that the maximum intensity of the light pulse is decreasing as t_0 increases, and thus decreases over time.

c.) compute u_t. Is the intensity of light increasing or decreasing at $x = -4$ when $t = 0$?

We will start off by computing $u_t(x,t)$.

$$u_t(x,t) = 4\operatorname{sech}(x-t+4)\cdot\left(-\tfrac{1}{2}te^{-(\frac{t}{2})^2}\right) + e^{-(\frac{t}{2})^2}\left(-4\operatorname{sech}(x-t+4)\tanh(x-t+4)\cdot -1\right)$$
$$= 4\operatorname{sech}(x-t+4)\cdot\left(-\tfrac{1}{2}te^{-(\frac{t}{2})^2}\right) + 4e^{-(\frac{t}{2})^2}\operatorname{sech}(x-t+4)\tanh(x-t+4)$$

hence at $x=-4$ when $t=0$

$$u_t(-4,0) = -2(0)\,\text{sech}(-4-0+4)\,e^{-(\frac{0}{2})^2} + e^{-(\frac{0}{2})^2}\,\text{sech}(-4-0+4)\tanh(-4-0+4)$$

$$= -2(0)\,\text{sech}(0)\,\cancel{\tanh}\,e + e^{0}\,\text{sech}(0)\tanh(0)$$

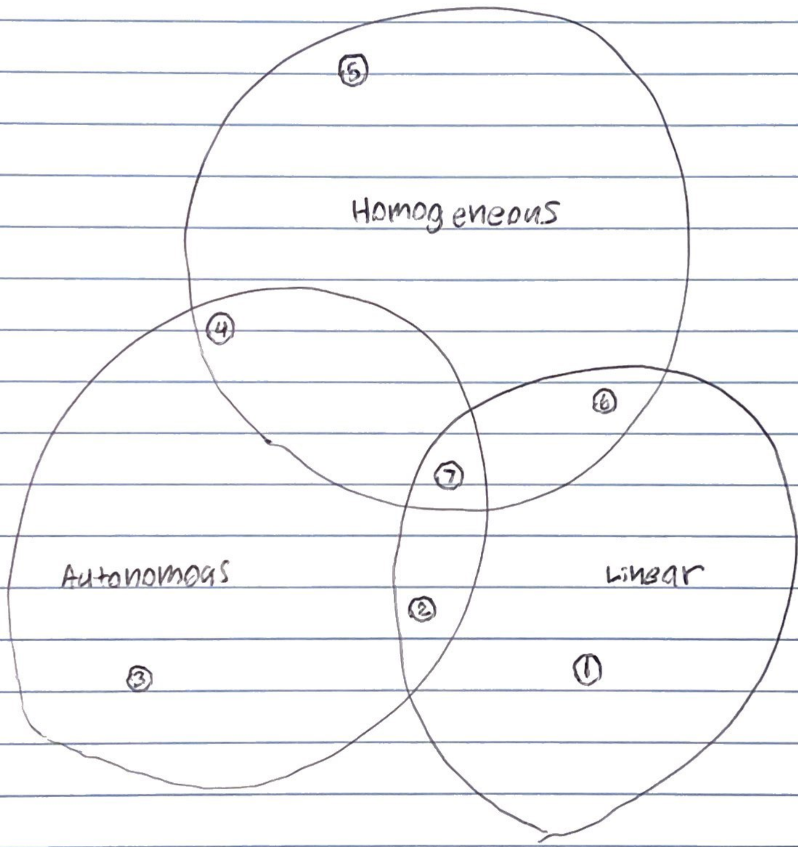
$$= -2(0)(1)(1) + (1)(1)(0)$$

$$= 0$$

since $u_t(-4,0) = 0$, the intensity of light is neither increasing or decreasing at $x=-4$ when $t=0$.

Exercise 3. Skill 1.3.

In each of the seven regions of the Venn diagram below, provide an example of a PDE that has all the properties which intersect in that region. Importantly, if a region excludes a property, you must ensure that your example does not have this property.



To make this more readable we will number each region and present the equation below:

1. $u_x + u_t = e^t$
2. $u_x + u_t = 1$
3. $u_x + 4u_t = 1$
4. $u_x + 4u_t = 0$
5. $tu^2 + u_{xt} = 0$
6. $tu + u_x + u_t = 0$
7. $u_x + u_t = 0$

Exercise 4. Skill 1.4.

Verify that

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}}$$

is a solution of the heat equation $u_t = u_{xx}$.

We will start by calculating $u_x(x,t)$.

$$u_x(x,t) = \frac{-x}{2t\sqrt{4\pi t}} e^{\frac{-x^2}{4t}}$$

next, we will calculate $u_{xx}(x,t)$.

$$u_{xx}(x,t) = \frac{\partial}{\partial x} u_x(x,t)$$

$$= \frac{-1}{2t\sqrt{4\pi t}} e^{\frac{-x^2}{4t}} + \frac{-x}{2t\sqrt{4\pi t}} e^{\frac{-x^2}{4t}} \cdot \left(\frac{-x}{2t}\right)$$

$$= e^{-x^2/4t} \left(\frac{x^2}{4t^2\sqrt{4\pi t}} - \frac{1}{2t\sqrt{4\pi t}}\right)$$

lastly, we will calculate $u_t(x,t)$

$$u_t(x,t) = \frac{-2\pi}{(4\pi t)^{3/2}} e^{\frac{-x^2}{4t}} + \frac{x^2}{4t^2\sqrt{4\pi t}} e^{\frac{-x^2}{4\pi t}}$$

$$= e^{\frac{-x^2}{4t}} \left(\frac{x^2}{4t^2\sqrt{4\pi t}} - \frac{2\pi}{(4\pi t)^{3/2}}\right)$$

$$= e^{\frac{-x^2}{4t}} \left(\frac{x^2}{4t^2\sqrt{4\pi t}} - \frac{2\pi}{4\pi t\sqrt{4\pi t}}\right)$$

$$= e^{-x^2/4t} \left(\frac{x^2}{4t^2\sqrt{4\pi t}} - \frac{1}{2t\sqrt{4\pi t}}\right)$$

As can be seen $u_t = u_{xx}$, thus $u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{\frac{-x^2}{4t}}$ is a solution of the heat equation.

Exercise 5. Skill 1.5.

One of the PDE's below satisfies the superposition principle. Explain which PDE this is, and explain why the others do not satisfy the superposition principle. For the PDE that satisfies the superposition principle, show explicitly that, if u_1 and u_2 are solutions of the PDE, then so is $c_1u_1 + c_2u_2$ for any constants c_1 and c_2.

a.) $u_{tt} = 2u_x - 7$

Since the above PDE is not homogeneous, the PDE does not satisfy the superposition principle.

b.) $2u_{tx} = \cos(u) - 1$

Since the above PDE is not linear, the PDE does not satisfy the superposition principle.

c.) $u_t = u_{tx} - 3u_{xx}$

Since the above PDE is linear and homogeneous, the PDE does satisfy the superposition principle.

Let $u = c_1u_1 + c_2u_2$, where u_1 and u_2 are solutions to PDE c. and c_1 and c_2 are constants then

$$u_t = (c_1u_1 + c_2u_2)_t = c_1(u_1)_t + c_2(u_2)_t$$
$$u_{tx} = (c_1(u_1)_t + c_2(u_2)_t)_x = c_1(u_1)_{tx} + c_2(u_2)_{tx}$$
$$u_x = (c_1u_1 + c_2u_2)_x = c_1(u_1)_x + c_2(u_2)_x$$
$$u_{xx} = (c_1(u_1)_x + c_2(u_2)_x)_x = c_1(u_1)_{xx} + c_2(u_2)_{xx}$$

Substituting these expressions into $u_t - u_{tx} + 3u_{xx} = 0$ we obtain

$$c_1(u_1)_t + c_2(u_2)_t - c_1(u_1)_{tx} - c_2(u_2)_{tx} + 3c_1(u_1)_{xx} + 3c_2(u_2)_{xx} = 0$$
$$\Rightarrow c_1((u_1)_t - (u_1)_{tx} + 3(u_1)_{xx}) + c_2((u_2)_t - (u_2)_{tx} + 3(u_2)_{xx}) = 0$$

Since u_1 and u_2 are solutions to the PDE, it follows that

$(u_1)_t - (u_1)_{tx} + 3(u_1)_{xx} = 0$ and $(u_2)_t - (u_2)_{tx} + 3(u_2)_{xx} = 0$

thus we can see that

$$c_1(0) + c_2(0) = 0$$

$$\Rightarrow \quad 0 = 0 \checkmark$$

Hence $c_1 u_1 + c_2 u_2$ is a solution to the PDE, and thus the superposition principle holds.

Multi-Step Problem:

The surface elevation of a wave you've made in an infinitely long, one dimensional pool (purchased at the theoretical physics store) depends on a horizontal coordinate x and time t as follows:

$$T(x,t) = \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x+2-ct)\right)$$

(S1.1) Compute T_x. What is the steepness of the wave profile at $x=0$ when $t=\frac{2}{c}$?

We will start by calculating $T_x(x,t)$.

$$T_x(x,t) = +2\frac{c}{2}\operatorname{sech}\left(\frac{\sqrt{c}}{2}(x+2-ct)\right)\cdot\left(\frac{-\sqrt{c}}{2}\operatorname{sech}\left(\frac{\sqrt{c}}{2}(x+2-ct)\right)\tanh\left(\frac{\sqrt{c}}{2}(x+2-ct)\right)\right)$$

$$= \frac{-c\sqrt{c}}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x+2-ct)\right)\tanh\left(\frac{\sqrt{c}}{2}(x+2-ct)\right)$$

thus the steepness of the wave profile at $x=0$ when $t=\frac{2}{c}$ is the same as computing $T_x(0,\frac{2}{c})$.

$$T_x(0,\tfrac{2}{c}) = \frac{-c\sqrt{c}}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(0+2-c\tfrac{2}{c})\right)\tanh\left(\frac{\sqrt{c}}{2}(0+2-c\tfrac{2}{c})\right)$$

$$= \frac{-c\sqrt{c}}{2}\operatorname{sech}^2(0)\tanh(0)$$

$$= \frac{-c\sqrt{c}}{2}(1^2)(0)$$

$$= 0$$

thus the steepness of the wave profile at $x=0$ when $t=\frac{2}{c}$ is 0.

(S1.2) Compute T_t at $x=-1$. What is the meaning of T_t at this fixed value of x? At which times will T reach its minimum value at $x=-1$?

Hint: Be sure you find all such times, and be sure that the times you have found correspond to a minimum. (or, if we're being particular, an infimum.)

We will start off by calculating $T_t(x,t)$

$$T_t(x,t) = \frac{2c}{2}\operatorname{sech}\left(\frac{\sqrt{c}}{2}(x+2-ct)\right)\cdot\left(\frac{-c\sqrt{c}}{2}\operatorname{sech}\left(\frac{\sqrt{c}}{2}(x+2-ct)\right)\tanh\left(\frac{\sqrt{c}}{2}(x+2-ct)\right)\cdot -1\right)$$

$$= \frac{c^2\sqrt{c}}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x+2-ct)\right)\tanh\left(\frac{\sqrt{c}}{2}(x+2-ct)\right)$$

thus at $x=-1$

$$T_t(-1,t) = \frac{c^2\sqrt{c}}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(1-ct)\right)\tanh\left(\frac{\sqrt{c}}{2}(1-ct)\right)$$

The meaning of T_t at this fixed value of x is, at the specific location $x=-1$, $T_t(-1,t)$ tells us the rate at which the surface elevation of the wave changes at time t.

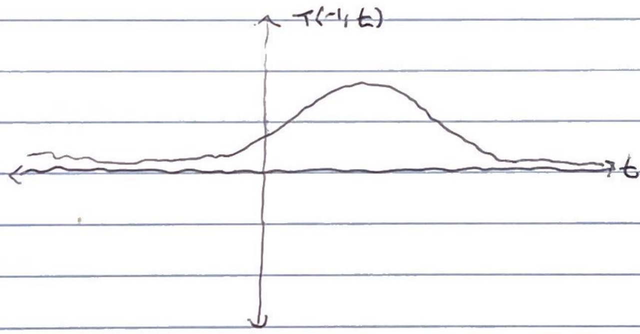
Since $T(-1,t) = \frac{c}{2}\operatorname{sech}^2(\frac{\sqrt{c}}{2}(x+2-ct)) = \frac{c}{2}\operatorname{sech}^2(\frac{\sqrt{c}}{2}(1-ct))$ has no minima, we will instead look at infima. Namely, we will look at

$\lim_{t\to-\infty} T(-1,t)$ and $\lim_{t\to\infty} T(-1,t)$

thus we will look at

$\lim_{t\to-\infty} \frac{c}{2}\operatorname{sech}^2(\frac{\sqrt{c}}{2}(1-ct))$ and $\lim_{t\to\infty} \frac{c}{2}\operatorname{sech}^2(\frac{\sqrt{c}}{2}(1-ct))$

to show what these limits equal, we will show the plot of $T(-1,t)$ for $c>0$, for any/arbitrary c, although it is important to note that this holds for $c\geq 0$.



As can be seen from the above plot, as t increases to ∞ $T(-1,t)$ gets closer and closer to zero, thus $\lim_{t\to\infty} T(-1,t)=0$. Similarly, as t decreases, $T(-1,t)$ gets closer and closer to zero, thus $\lim_{t\to-\infty} T(-1,t)=0$.

Thus at $x=-1$, $T(-1,t)=\frac{c}{2}\operatorname{sech}^2(\frac{\sqrt{c}}{2}(1-ct))$ reaches a minimum value of 0 as $t\to\pm\infty$. Since the function is strictly positive, we can be assured that these are truly the minimum/infimum.

(S1.3) Classify the famous Korteweg de Vries (KdV) equation

$$T_t + 6TT_x + T_{xxx} = 0$$

according to order, homogeneity, autonomy and linearity.

1.) order:

The KdV is third order due to the T_{xxx} term.

2.) Homogeneity:

The KdV equation is homogeneous since $T=0$ is a solution.

3.) Autonomy:

The KdV equation is autonomous since x and t don't explicitly appear in the equation.

4.) Linearity:

The KdV equation is non-linear due to the $6TT_x$ term.

(S1.4) Classify the much less famous and physically interesting equation

$$T_t + 6(TT_x)^4 + T_{xxx} = 0$$

According to order, homogeneity, autonomy and linearity

1.) order:

The above PDE is third order due to the T_{xxx} term.

2.) Homogeneity:

The above PDE is homogeneous since $T=0$ is a solution

3.) Autonomy:

The above PDE is autonomous since x and t don't explicitly appear in the equation.

4.) Linearity

The above PDE is non-linear due to the $6(TT_x)^4$ term.

(S1.5) show that $T(x,t)$ solves the KdV equation from S1.3.

Since we know T, T_t and T_x all that is left to do is to find T_{xxx}. First we must find T_{xx}:

$$T_{xx}(x,t) = \frac{\partial}{\partial x} T_x(x,t)$$
$$= \frac{c^2}{2} \operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x+2-ct)\right)\tanh^2\left(\frac{\sqrt{c}}{2}(x+2-ct)\right) - \frac{c^2}{4}\operatorname{sech}^4\left(\frac{\sqrt{c}}{2}(x+2-ct)\right)$$

now we can find T_{xxx}:

$$T_{xxx}(x,t) = \frac{\partial}{\partial x} T_{xx}(x,t)$$
$$= \frac{-c^2\sqrt{c}}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x+2-ct)\right)\tanh^3\left(\frac{\sqrt{c}}{2}(x+2-ct)\right) + \frac{c^2\sqrt{c}}{2}\operatorname{sech}^4\left(\frac{\sqrt{c}}{2}(x+2-ct)\right)\tanh\left(\frac{\sqrt{c}}{2}(x+2-ct)\right)\cdot 2$$
$$= \frac{c^2\sqrt{c}}{2}\left(\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x+2-ct)\right)\tanh\left(\frac{\sqrt{c}}{2}(x+2-ct)\right)\right)\left(2\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x+2-ct)\right) - \tanh^2\left(\frac{\sqrt{c}}{2}(x+2-ct)\right)\right)$$

Let $u = \frac{\sqrt{c}}{2}(x+2)-ct$ (so the work can fit on one line. Substituting T, T_t, T_x and T_{xxx} into the KdV equation we obtain

$$\frac{c^2\sqrt{c}}{2}\operatorname{sech}^2(u)\tanh(u) - \frac{3c^2\sqrt{c}}{2}\operatorname{sech}^4(u)\tanh(u) + \frac{c^2\sqrt{c}}{2}(\operatorname{sech}^2(u)\tanh(u))(2\operatorname{sech}^2(u) - \tanh^2(u)) = 0$$

factoring out a $\frac{c^2\sqrt{c}}{2}\operatorname{sech}^2(u)\tanh(u)$ term we obtain

$$\frac{c^2\sqrt{c}}{2}\operatorname{sech}^2(u)\tanh(u)\left(1 - 3\operatorname{sech}^2(u) + 2\operatorname{sech}^2(u) - \tanh^2(u)\right) = 0$$

$\Rightarrow$ $\frac{c^2\sqrt{c}}{2}\operatorname{sech}^2(u)\tanh(u)(1 - (\operatorname{sech}^2(u) + \tanh^2(u))) = 0$

$\Rightarrow$ $\frac{c^2\sqrt{c}}{2}\operatorname{sech}^2(u)\tanh(u)(1-1) = 0$ (since $\operatorname{sech}^2(u) + \tanh^2(u) \doteq 1$)

$\Rightarrow$ $0 = 0$ ✓

thus $T(x,t)$ solves the KdV equation.