Homework 5

Graded

Student

Jaiden Rain Atterbury

Total Points

39 / 40 pts

Question 1

Question 1

10 / 10 pts

- ✔ **– 0 pts** Correct, line $y \approx 1.25424x + 0.67797$ found using projection technique, line and points plotted, and the value the line minimizes is ≈ 1.7966 (with work shown at each step!)

Question 2

Question 2

9 / 10 pts

- ✔ **– 1 pt** b.iii

1 Why?

2 This is not correct.

3 There's a very short proof using the results from both (iii) and (iv).

4 It's not necessary to include this.

5 What about the general case?

Question 3

Question 3

10 / 10 pts

- ✔ **– 0 pts** correct

6 The question is asking what the column space is specifically.

Question 4

Question 4

10 / 10 pts

- ✔ **– 0 pts** All parts sufficiently answered (a 2pt, b 5pt, c 3pt)

Question assigned to the following page: 1

Jaiden Atterbury 04/23/2022 Homework 5

(1) Linear Regression

(a) Goal: Find a line $C + Dx = y$ going through 4 points:

$C + 0D = 0$ Point 1

$C + 1D = 3$ Point 2

$C + 3D = 4$ Point 3

$C + 5D = 7$ Point 4

$$\underset{A}{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 5 \end{bmatrix}} \underset{x}{\begin{bmatrix} C \\ D \end{bmatrix}} = \underset{b}{\begin{bmatrix} 0 \\ 3 \\ 4 \\ 7 \end{bmatrix}}$$

has no solutions

$b \notin Col(A)$

therefore we must solve $A\hat{x} = b_V$ for $\hat{x}$, where b_V is the orthogonal projection of b on $V = Col(A)$.

Solution: $\hat{x} = \begin{bmatrix} C \\ D \end{bmatrix} = (A^TA)^{-1}A^Tb$

$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 5 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 5 \end{bmatrix}$ $A^TA = \begin{bmatrix} 4 & 9 \\ 9 & 35 \end{bmatrix}$ $(A^TA)^{-1} = \begin{bmatrix} 35/59 & -9/59 \\ -9/59 & 4/59 \end{bmatrix}$

$$\hat{x} = \left(\begin{bmatrix} 4 & 9 \\ 9 & 35 \end{bmatrix}\right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 4 \\ 7 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} 35/59 & -9/59 \\ -9/59 & 4/59 \end{bmatrix} \begin{bmatrix} 14 \\ 50 \end{bmatrix}$$

$$\hat{x} = \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 40/59 \\ 74/59 \end{bmatrix}$$

$\therefore$ Linear regression line is $\boxed{y = \frac{40}{59} + \frac{74}{59}x}$

(b) $E = \|e\|^2 = e_1^2 + e_2^2 + e_3^2 + e_4^2$ where $e = [e_1\ e_2\ e_3\ e_4]^T$

$e = b - p$

$b = \begin{bmatrix} 0 \\ 3 \\ 4 \\ 7 \end{bmatrix}$ $p = A\hat{x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 40/59 \\ 74/59 \end{bmatrix} = \begin{bmatrix} 40/59 \\ 114/59 \\ 262/59 \\ 410/59 \end{bmatrix}$

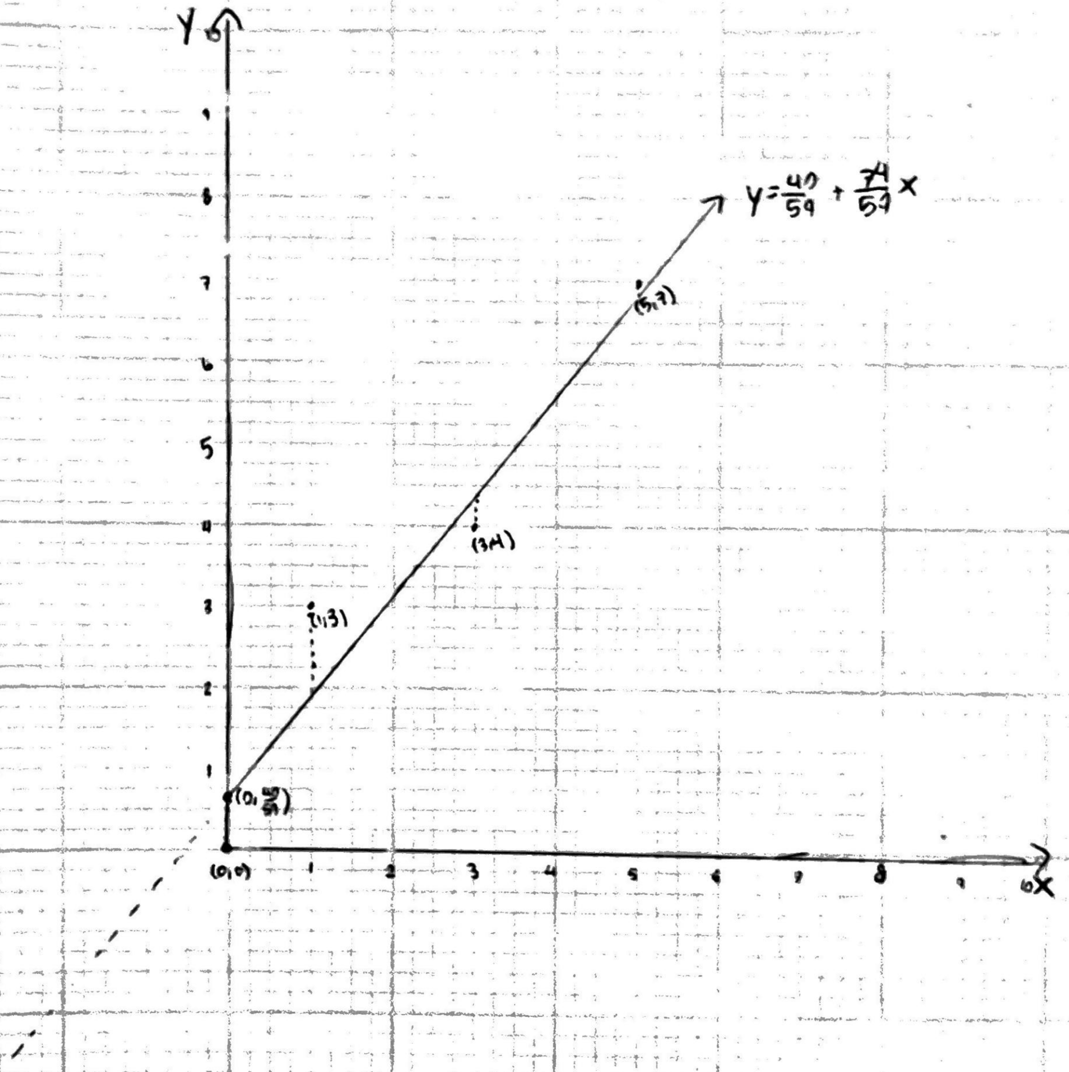
$$e = \begin{bmatrix} 0 \\ 3 \\ 4 \\ 7 \end{bmatrix} - \begin{bmatrix} 40/59 \\ 114/59 \\ 262/59 \\ 410/59 \end{bmatrix} = \begin{bmatrix} -40/59 \\ 63/59 \\ -26/59 \\ 3/59 \end{bmatrix}$$

sum of squared errors

$E = \|e\|^2 = \left(\frac{-40}{59}\right)^2 + \left(\frac{63}{59}\right)^2 + \left(\frac{-26}{59}\right)^2 + \left(\frac{3}{59}\right)^2 = \boxed{\frac{106}{59}}$

(1) Linear Regression

(a)



Question assigned to the following page: 2

(2) Oblique Projectors

(a) $P = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

(i) $P^2 = \underset{P}{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}} \cdot \underset{P}{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}} = \begin{matrix} 0\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ 1\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{matrix} \Bigg\rangle \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = P \checkmark$

$\therefore$ P is a projector

(ii) $P = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow Col(P) = Span\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

$P = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{matrix} x_1 = s \\ x_2 = 0 \end{matrix} \rightarrow null(P) = Span\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

(iii) $\vec{x} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$P\vec{x} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = 5\begin{bmatrix} 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\vec{x} - P\vec{x} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$

Picture on seperate page

(b)(i) If $\vec{v} \in col(P)$

$\vec{v} = P\vec{w}$ (linear combination of columns of P)

then $PP\vec{w} = P^2\vec{w} = P\vec{w} = \vec{v}$

$\therefore P\vec{v} = \vec{v}$

Since $P(P\vec{w}) = P\vec{w}$

$P^2\vec{w} = Pw$

$P\vec{w} = P\vec{w}$

$\vec{v} = \vec{v} \checkmark$

CHECK: $\vec{v} \in col(P)$ since $\vec{v} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $col(P) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$P\vec{v} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \vec{v} \checkmark$

$\therefore$ If $\vec{v} \in col(P)$ then $P\vec{v} = \vec{v}$

(ii) For all $\vec{v}$, then $\vec{v} - P\vec{v} \in Null(P)$

Proof: for $\vec{v} - P\vec{v}$ to be in Null(P), $P(\vec{v} - P\vec{v})$ must equal $\vec{0}$:

$P(\vec{v} - P\vec{v}) = P\vec{v} - P^2\vec{v} = P\vec{v} - P\vec{v} = \vec{0}$

$\therefore \vec{v} - P\vec{v} \in Null(P)$

Question assigned to the following page: 2

CHECK: $\vec{v} = \begin{bmatrix}2\\2\end{bmatrix}$, $P\vec{v} = \begin{bmatrix}2\\2\end{bmatrix}$, $\vec{v} - P\vec{v} = \begin{bmatrix}2\\2\end{bmatrix} - \begin{bmatrix}2\\2\end{bmatrix} = \vec{0} \in \text{Null}(P)$

Also, $\vec{v} = P\vec{v} + (\vec{v} - P\vec{v})$, if $\vec{v} = \begin{bmatrix}2\\2\end{bmatrix}$

$$\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix} + \left(\begin{bmatrix}2\\2\end{bmatrix} - \begin{bmatrix}2\\2\end{bmatrix}\right) = \begin{bmatrix}2\\2\end{bmatrix} \checkmark$$

$\therefore$ $P\vec{v}$ is the oblique projection/shadow of $\vec{v}$ into $\text{col}(P)$ along $\text{Null}(P)$

(iii) $I - P = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix} \rightarrow$ complementary projector to P

$$\left.\begin{array}{l} (I-P)^2 = \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix} \cdot \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix} = 1\begin{bmatrix}1\\0\end{bmatrix} + 0\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \\ \qquad\quad (I-P)\cdot(I-P) \qquad -1\begin{bmatrix}1\\0\end{bmatrix} + 0\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix} \end{array}\right\} \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix} = I - P$$

$\therefore$ $I - P$ is also a projector (since $(I-P)^2 = (I-P)$)

(iv) $\text{Col}(I-P) = \text{Null}(P)$

Proof: ① $\vec{v} \in \text{Null}(P)$ then $P\vec{v} = \vec{0}$

then $(I-P)\vec{v} = \vec{v} - P\vec{v} = \vec{v} - \vec{0} = \vec{v}$,

since $(I-P)\vec{v} = \vec{v}$, $\vec{v} \in \text{col}(I-P)$

$\therefore$ $\text{Null}(P) \subseteq \text{col}(I-P)$

② $\vec{w} \in \text{col}(I-P)$ then $(I-P)\vec{w} = \vec{w}$

then $(I-P)\vec{w} = \vec{w} - P\vec{w} = \vec{w} \rightarrow P\vec{w} = \vec{0}$

since $P\vec{w} = \vec{0}$, $\vec{w} \in \text{Null}(P)$

$\therefore$ $\text{Col}(I-P) \subseteq \text{Null}(P)$

since $\text{Null}(P) \subseteq \text{col}(I-P)$ & $\text{col}(I-P) \subseteq \text{Null}(P)$

$\therefore$ $\text{col}(I-P) = \text{Null}(P)$

Question assigned to the following page: 2

(v) $Col(P) = Null(I-P)$

Proof: ① $\vec{v} \in Col(P)$ then $P\vec{v} = \vec{v}$

then $(I-P)\vec{v} = \vec{v} - P\vec{v} = \vec{v} - \vec{v} = \vec{0}$

since $(I-P)\vec{v} = \vec{0}$, $\vec{v} \in Null(I-P)$

$\therefore$ $Col(P) \subseteq Null(I-P)$

② $\vec{x} \in Null(I-P)$ then $(I-P)\vec{x} = \vec{0}$

then if $P = I - (I-P)$

then $P\vec{x} = \vec{x} - (I-P)\vec{x} \rightarrow P\vec{x} = \vec{x}$

also, $(I-P)\vec{x} = \vec{x} - P\vec{x} \Rightarrow P\vec{x} = \vec{x}$

since $P\vec{x} = \vec{x}$, $\vec{x} \in Col(P)$

$\therefore$ $Null(I-P) \subseteq Col(P)$

since $Null(I-P) \in Col(P)$ and $Col(P) \subseteq Null(I-P)$

$\therefore$ $Col(P) = Null(I-P)$

(vi) $Col(P) \cap Null(P) = \{\vec{0}\}$

Assume $\vec{v} \in Col(P) \cap Null(P)$

$$\left.\begin{array}{l} \vec{v} \in Col(P) \rightarrow P\vec{v} = \vec{v} \\ \vec{v} \in null(P) \rightarrow P\vec{v} = \vec{0} \end{array}\right\} \vec{0} = \vec{v}$$

$\therefore$ $\vec{v}$ must be $\vec{0}$ and only the $\vec{0}$

Question assigned to the following page: 2

(vii) By definition 4.4.2 from the Math 208 notes if a set of vectors form a basis for $\mathbb{R}^n$ then there is a way to write any vector $\vec{v}$ as a linear combination of the basis vectors:

$$\mathcal{B} = \{u_1, \cdots, u_n\} \text{ basis for } \mathbb{R}^n \text{ then:}$$

$$\vec{v} = c_1 u_1 + \cdots + c_n u_n$$

Since $\mathcal{B}_{col(P)} \cup \mathcal{B}_{null(P)}$ is linearly independent with n vectors, the union forms a basis for $\mathbb{R}^n$ thus any vector can be written as a unique (because $col(P) \cap null(P) = \{0\}$) linear combination of vectors in $col(P) = E_1$ and $Null(P) = E_0$

↑ for orthogonal projections

In this question in particular (oblique projection) we can't be sure that $col(P) \cap null(P)$ forms a basis for $\mathbb{R}^n$, but we can write $\vec{v}$ uniquely, since $col(P) \cap null(P) = \{0\}$, as the sum of vectors in $col(P)$ and $null(P)$:

$$\boxed{\underbrace{P\vec{v}}_{col(P)} + \underbrace{(I-P)\vec{v}}_{null(P)} = \vec{v}} \rightarrow \vec{v} = P\vec{v} + I\vec{v} - P\vec{v}$$

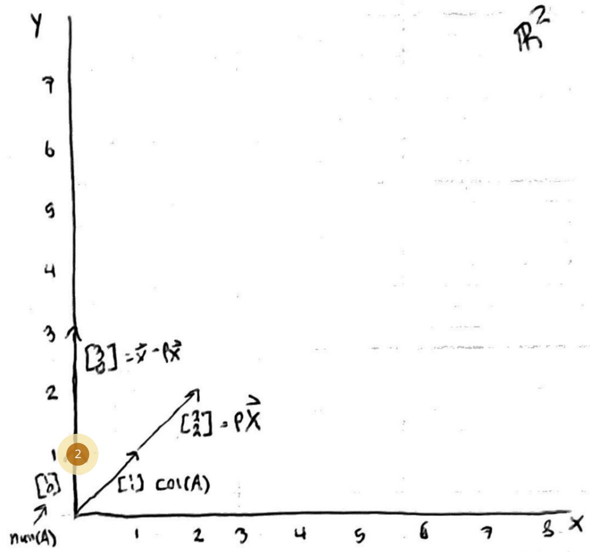
$$\vec{v} = \vec{v} \checkmark$$

$P\vec{v} \in col(P)$ $(I-P)\vec{v} \in Null(P)$
by part (i) and (iv)

Question assigned to the following page: 2

(2) Oblique Projectors

(a) (ii) and (iii)



Question assigned to the following page: 3

(3) Orthogonal Projections

(a) $P = \frac{1}{10}\begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$

(i) $P^2 = \underbrace{\begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{bmatrix}}_{P} = \frac{1}{10}\begin{bmatrix} 1/10 \\ 3/10 \end{bmatrix} + \frac{3}{10}\begin{bmatrix} 3/10 \\ 9/10 \end{bmatrix} = \begin{bmatrix} 1/10 \\ 3/10 \end{bmatrix}$

$= \frac{3}{10}\begin{bmatrix} 1/10 \\ 3/10 \end{bmatrix} + \frac{9}{10}\begin{bmatrix} 3/10 \\ 9/10 \end{bmatrix} = \begin{bmatrix} 3/10 \\ 9/10 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{bmatrix} = P \checkmark$

$P^T = \begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{bmatrix} = P \checkmark$

$\therefore$ P is an orthogonal projector.

(ii) Since all results from the previous problem apply, and Pv the shadow of v under P lives in Col(P) if $P \in \mathbb{R}^{n \times n}$ is a projector. Therefore the orthogonal projector projects onto the column space of P.

6

(iii) $P = \begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{bmatrix} \sim \begin{bmatrix} 1/10 & 3/10 \\ 0 & 0 \end{bmatrix} \rightarrow Col(P) = E_1 = span\left\{\begin{bmatrix} 1/10 \\ 3/10 \end{bmatrix}\right\} = Span\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\}$ (for graph)

$P = \begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{bmatrix} \sim \begin{bmatrix} 1/10 & 3/10 \\ 0 & 0 \end{bmatrix}$ $x_2 = 1/10\,s$, $x_1 = -3/10\,s$ $\rightarrow null(P) = E_0 = span\left\{\begin{bmatrix} -3/10 \\ 1/10 \end{bmatrix}\right\} = Span\left\{\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right\}$ (for graph)

(iv) $P\vec{v} = \underbrace{\begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 10 \\ 20 \end{bmatrix}}_{\vec{v}} = \begin{bmatrix} 1 & 6 \\ 3 & 18 \end{bmatrix} = \begin{bmatrix} 7 \\ 21 \end{bmatrix} \in Col(P)$

$\vec{v} - P\vec{v} = \begin{bmatrix} 10 \\ 20 \end{bmatrix} - \begin{bmatrix} 7 \\ 21 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ orthogonal to $P\vec{v}$/Col(P)

(b)(i) By theorem 4.2.4 for any matrix $P \in \mathbb{R}^{m \times n}$ $Col(A)^{\perp} = Null(A^T)$ and $Col(A) = Null(A^T)^{\perp}$

BUT since P is symmetric $P^T = P$

Thus $Col(P) = Null(P)^{\perp}$ and $Null(P) = Col(P)^{\perp}$

$\therefore$ Col(P) and Null(P) are Orthogonal complements

Question assigned to the following page: 3

(ii) If $P = A(A^TA)^{-1}A^T$

then $P^T = (A(A^TA)^{-1}A^T)^T$

$= A((A^TA)^{-1})^TA^T$

$= A((A^TA)^T)^{-1}A^T$ $\quad P^T = P$

↑ by given fact

$= A(A^TA)^{-1}A^T = P$ ✓

Also $P^2 = A\underbrace{(A^TA)^{-1}A^TA}_{I}(A^TA)^{-1}A^T$ $\quad P^2 = P$

$= A(A^TA)^{-1}A^T = P$ ✓

$\therefore P = A(A^TA)^{-1}A^T$ is an orthogonal projection

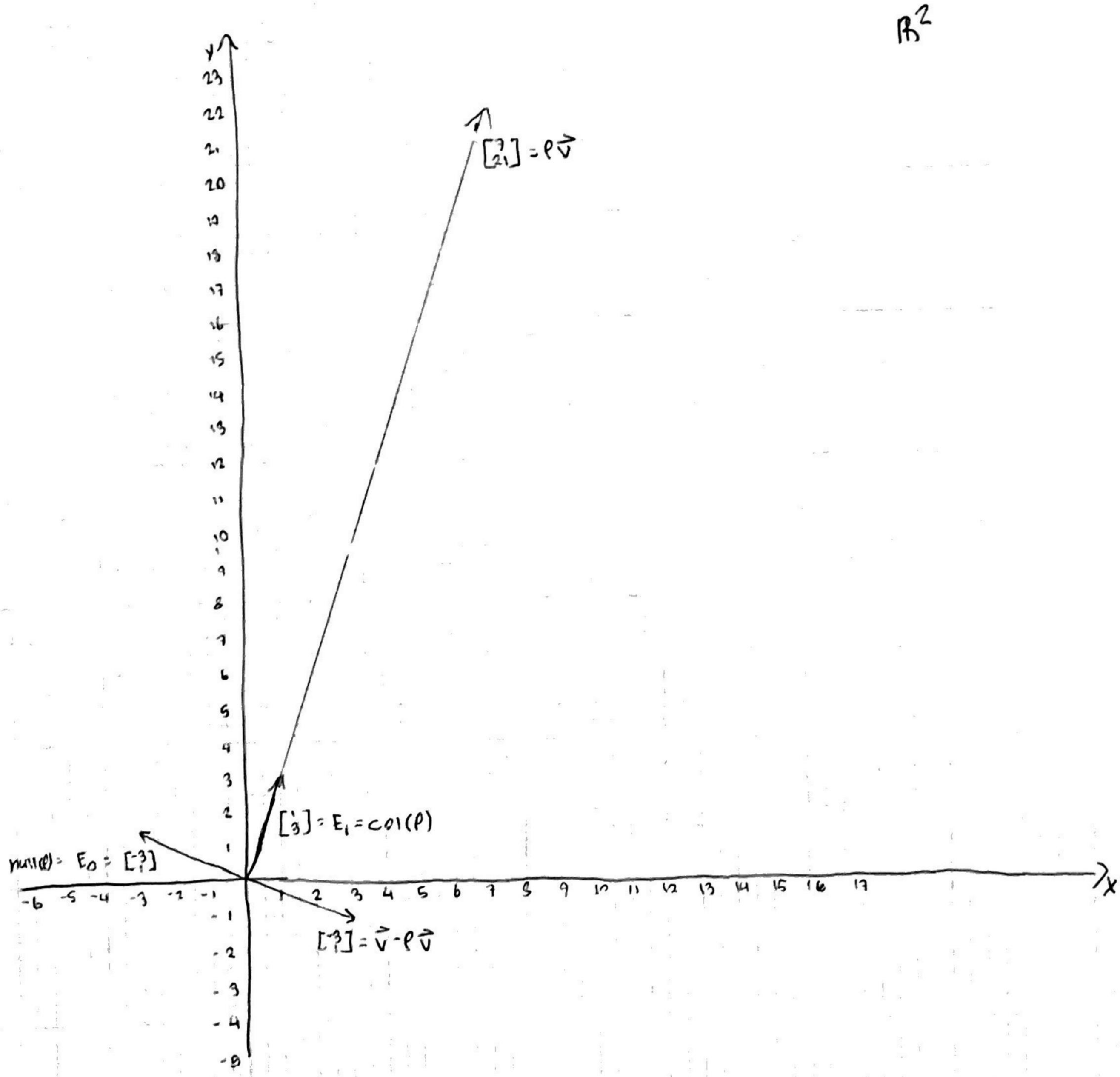
(iii) If $P = A(A^TA)^{-1}A^T$ then it projects onto Col(A) by the proof of proposition 4.3.7 from the lecture notes. Since P is a projector, then for any vector $\vec{v} \in \mathbb{R}^n$, $P\vec{v}$ is the shadow of $\vec{v}$ under P. From the problem description of problem 2 this shadow lives in Col(P). Since P projects a vector $\vec{v}$ onto Col(A) and Col(P), these two subspaces must be the same:

$\text{Col}(A(A^TA)^{-1}A^T) = \text{Col}(P) = \text{Col}(A)$

$\therefore \text{Col}(A) = \text{Col}(A(A^TA)^{-1}A^T)$

(3) Orthogonal Projections

Part (iii) and (iv)



Question assigned to the following page: 4

(4) Reflections and Projections

(a) (i) $x+y+z=0 \rightarrow [1\ 1\ 1 \mid 0]$

$\text{null}([1\ 1\ 1 \mid 0]) = \begin{matrix} x_2 = s_1 \\ x_3 = s_2 \\ x_1 = -s_1 - s_2 \end{matrix}$ ~~null(A) = basis for~~ $\Pi = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(ii) $\lambda_1 = 1$ with multiplicity 2. This eigenspace represents the 2D plane $(x+y+z=0)$ that consists of vectors unchanged by the transformation T.

$$E_{\lambda_1} = E_1(T) = \text{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\lambda_2 = -1$ with multiplicity 1. This eigenspace represents the normal line perpendicular to the plane $(x+y+z=0)$. Thus if a vector is perpendicular to the plane it will stay perpendicular, just flipped.

$$E\lambda_2 = E_{-1}(T) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(iii) $T = A\vec{x}$

$$A = \underbrace{\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}^{-1}}_{P^{-1}}$$

* diagonalizable since Eigenspaces form a basis for $\mathbb{R}^3$

$$A = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$A = \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

Question assigned to the following page: 4

(iv) $A^2 = \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$ ✓

(b) $H = I - 2uu^T \in \mathbb{R}^{n \times n}$ $\quad u^T u = 1$

(i) If H is symmetric $H = H^T$

$H^T = (I - 2uu^T)^T$

$H^T = I^T - (2uu^T)^T$

$H^T = I^T - 2(uu^T)^T$

$H^T = I - 2uu^T = H$ ✓

H is orthogonal if $H^T H = I$

(since a matrix with orthonormal columns satisfies $Q^T Q = I$, when H is symmetric $H^T = H$, $\therefore H^2 = I$)

$H^T H = (I - 2uu^T)(I - 2uu^T)$

$I - 2uu^T - 2uu^T + 4uu^T uu^T$

$I - 4uu^T + 4u\underbrace{u^T u}_{1 \text{ (since } u^T u = 1)}u^T$

$I - 4uu^T + 4uu^T$

$= I$ ✓

∴ H is both orthonormal/orthogonal and symmetric.

(ii) $Hu = \lambda u$

$Hu = Iu - 2uu^T u$

$Hu = Iu - 2u$

$Hu = -1u$

∴ u is an eigenvector of H with $\lambda_u = -1$

Question assigned to the following page: 4

(iii) $\vec{v} \in \mathbb{R}^n$ orthogonal to $\vec{u}$

$\hookrightarrow v^T u = 0$ & $u^T v = 0$

$H\vec{v} = (I - 2uu^T)\vec{v}$

$H\vec{v} = I\vec{v} - 2u\underbrace{u^T\vec{v}}_{0}$ since $u^T\vec{v} = 0$

$H\vec{v} = \vec{v}$

$\therefore$ $\vec{v}$ is an eigenvector of H with corresponding eigenvalue

$\lambda_v = 1$

(iv) $\lambda_v = 1$

If $(I - 2uu^T)\vec{v} = \vec{v}$ and $\text{span}\{u\} = E_{-1}(H)$

then $\text{span}(u) \subseteq E_1(H)$

since $(I - 2uu^T)\vec{w} = \vec{w}$

$\vec{w} - 2uu^T\vec{w} = \vec{w}$

$2uu^T\vec{w} = \vec{0}$

This shows that $\text{span}\{u\}^{\perp} = E_1(H)$

multiplicity of $\lambda_v = 1 \rightarrow \dim(\text{span}\{u\}^{\perp} = E_1(H)) = n-1$ (hyperplane)

multiplicity of $\lambda_u = -1 \rightarrow \dim(\text{span}\{u\}) = E_1(H) = 1$ (line)

H is diagonalizable by proposition 6.1.7 which states that for each eigenvalue of a symmetric matrix H, $AM(\lambda) = GM(\lambda)$ which shows H is diagonalizable since the eigenspaces form a basis for $\mathbb{R}^n$.

$$H = \underbrace{\begin{bmatrix} \underbrace{E\lambda_u}_{\text{1 vector}} & \underbrace{E\lambda_v}_{n-1 \text{ vectors}} \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} \overbrace{-1}^{\text{1 time}} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} \underbrace{E\lambda_u}_{\text{1 vector}} & \underbrace{E\lambda_1}_{n-1 \text{ vectors}} \end{bmatrix}^{-1}}_{P^{-1}}$$

In D: $\{1, \ldots, 1\}$ $n-1$ times

Question assigned to the following page: 4

(v) Geometrically, H reflects a vector across a plane (hyperplane) that goes through the origin. If a vector is already in the span$\{\vec{u}\}^{\perp}$, it will stay unchanged ($T(v)=v$). If n is orthogonal to the hyperplane, in the span$\{\vec{u}\}$, the vector will flip across the plane and remain orthogonal. ($T(n)=-n$). Lastly, if neither of these cases are true, a vector will be reflected across the hyperplane.

(vi) Yes, the transformation in part(a) and H are related as they both represent a reflection across a plane (or hyperplane in $n>3$ case).

$\vec{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\|\vec{n}\| = \sqrt{3}$

$\therefore$ $\vec{u} = \begin{bmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix}$ (reflection across plane with unit normal vector $\vec{u}$).

CHECK $I-2uu^T$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix} \begin{bmatrix} \sqrt{3}/3 & \sqrt{3}/3 & \sqrt{3}/3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -2/3 & -2/3 & -2/3 \\ -2/3 & -2/3 & -2/3 \\ -2/3 & -2/3 & -2/3 \end{bmatrix} = \begin{bmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{bmatrix} = T \text{ from part (a)} \checkmark$$

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(c) $P = I - uu^T \in \mathbb{R}^{n\times n}$ $\quad u^Tu = 1$

(i) $P = P^2$ $\qquad P^T = (I - uu^T)^T$

$P^2 = (I-uu^T)(I-uu^T)$ $\qquad = I^T - (uu^T)^T$

$= I - uu^T - uu^T + uu^Tuu^T$ $\qquad = I - uu^T = P$ ✓

$= I - 2uu^T + uu^T$

$= I - uu^T = P$ ✓

∴ P is an orthogonal projector.

(ii) Since P is a projector, the eigenvalues are 0, 1 with corresponding eigenspaces $E_1(P) = \mathrm{col}(P)$ and $E_0 = \mathrm{null}(P)$.

If we let $W = \mathrm{span}\{u\} \subseteq \mathbb{R}^n$ then

$P_W = u(u^Tu)^{-1}u^T$ $\quad$ with $\mathrm{Col}(P_W) = W$

$P_W = uu^T$ $\quad \mathrm{Null}(P_W) = W^\perp$

thus $P = I - P_{W(uu^T)}$, by question (2)(b)(iv) $I-P$ obliquely projects onto $\mathrm{Null}(P)$

So P orthogonally projects onto $\mathrm{Null}(P_W) = W^\perp$

∴ $\mathrm{Null}(P) = (W^\perp)^\perp = W$

Finaly this shows

$E_0 = \mathrm{null}(P) = \mathrm{span}\{\vec{u}\} = \text{line} = 1$ dimensional

$E_1 = \mathrm{col}(P) = \mathrm{span}\{\vec{u}\}^\perp = n$-dimensional hyperplane, $n-1$ dimensional.

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(iii) A projector P projects onto $\mathrm{col}(P)$, by problem description for problem 2, thus P projects onto $\mathrm{null}(P)^\perp$ (since P is symmetric).

$\therefore P$ projects onto $\mathrm{span}\{\vec{u}\}^\perp$

(iv) Geometrically, P projects any vector onto the subspace $\mathrm{span}\{\vec{u}\}^T$. If a vector is already in $\mathrm{span}\{\vec{u}\}^\perp$ then after applying P it will stay the same. However if a vector is orthogonal to $\mathrm{span}\{\vec{u}\}^\perp$ (or more simply in the $\mathrm{span}\{\vec{u}\}$) then after applying P it will be mapped to the zero vector. More formally the orthogonal projection b_v is the closest vector to b in $\mathrm{span}\{\vec{u}\}^\perp$.