Homework 2

Graded

Student

Jaiden Rain Atterbury

Total Points

36.5 / 40 pts

Question 1

Question 1

7.5 / 10 pts

(a)(ii)

- ✓ **– 0.5 pts** sign mistake

(b)(i)

- ✓ **– 1 pt** proved only for 2x2 case

(b)(ii)

- ✓ **– 1 pt** used commutativity incorrectly

1 Careful here.

Question 2

Question 2

9 / 10 pts

(a)

- ✓ **– 1 pt** minor error

2 What has this proved?

Question 3

Question 3

10 / 10 pts

- ✓ **– 0 pts** correct

Question 4

Question 4

10 / 10 pts

- ✓ **+ 10 pts** 100% complete

Question assigned to the following page: 1

Jaiden Atterbury 4/5/2022 Homework 2.

(1)a. If A is diagonalizable there exists an invertible matrix U and a matrix Λ such that $A = U\Lambda U^{-1}$.

By proposition 5.2.5 $\det(AB) = \det(A)\det(B)$

& By proposition 5.2.6 $\det(A^{-1}) = \frac{1}{\det(A)}$

$\therefore$ $\det(A) = \det(U)\det(\Lambda)\frac{1}{\det(U)}$

$\det(A) = \det(\Lambda)$

Since Λ is the matrix of the eigenvalues of A:

$$\det(A) = \det\begin{vmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \ \lambda_n \end{vmatrix} \rightarrow \det(A) = \lambda_1\lambda_2\cdots\lambda_n \checkmark$$

This makes since because A and Λ have "the same action" if we are willing to change basis.

(ii) $\det(A - tI) = (-1)^n(t-\lambda_1)(t-\lambda_2)\cdots(t-\lambda_n)$

let $t=0$

$\det(A) = (-1)^n(-\lambda_1)(-\lambda_2)\cdots(-\lambda_n)$

$\det(A) = (-1)^n \lambda_1\lambda_2\cdots\lambda_n$

$\det(A) = \lambda_1\lambda_2\cdots\lambda_n \checkmark$

(b) (i) $\underset{Q}{\begin{bmatrix} q_{11} & \cdots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \cdots & q_{nn} \end{bmatrix}} \quad \underset{P}{\begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nn} \end{bmatrix}}$

$QP = \begin{bmatrix} p_{11}q_{11} + p_{n1}q_{1n} & p_{1n}q_{11} + p_{nn}q_{1n} \\ p_{11}q_{n1} + p_{n1}q_{nn} & p_{1n}q_{n1} + p_{nn}q_{nn} \end{bmatrix}$ $p_{11}\begin{bmatrix} q_{11} \\ q_{n1} \end{bmatrix} + p_{n1}\begin{bmatrix} q_{1n} \\ q_{nn} \end{bmatrix} \leftarrow 1^{st}$ column

$p_{1n}\begin{bmatrix} q_{11} \\ q_{n1} \end{bmatrix} + p_{nn}\begin{bmatrix} q_{1n} \\ q_{nn} \end{bmatrix} \leftarrow 2^{nd}$ column

$\text{trace}(QP) = (p_{11}q_{11} + p_{n1}q_{1n} + p_{1n}q_{n1} + p_{nn}q_{nn})$

Question assigned to the following page: 1

$$PQ = \begin{bmatrix} q_{11}p_{11} + q_{n1}p_{1n} & q_{1n}p_{11} + q_{nn}p_{1n} \\ q_{11}p_{n1} + q_{n1}p_{nn} & q_{1n}p_{n1} + q_{nn}p_{nn} \end{bmatrix}$$

$q_{11}\begin{bmatrix} p_{11} \\ p_{n1} \end{bmatrix} + q_{n1}\begin{bmatrix} p_{1n} \\ p_{nn} \end{bmatrix} \rightarrow 1^{st}$ column

$q_{1n}\begin{bmatrix} p_{11} \\ p_{n1} \end{bmatrix} + q_{nn}\begin{bmatrix} p_{1n} \\ p_{nn} \end{bmatrix} \rightarrow 2^{nd}$ column

$\text{trace}(PQ) = (q_{11}p_{11} + q_{n1}p_{1n} + q_{1n}p_{n1} + q_{nn}p_{nn})$

becawse $(q_{11}p_{11} + q_{n1}p_{1n} + q_{1n}p_{n1} + q_{nn}p_{nn}) = (p_{11}q_{11} + p_{n1}q_{1n} + p_{1n}q_{n1} + p_{nn}q_{nn})$

↳ all that is different is the ordering of p and q

∴ they are the same

and ∴ $\text{trace}(QP) = \text{trace}(PQ)$ ✓

(ii) In the last problem we learned that the trace of a product of matrices is commutative, with that said, if A is diagonalizable then we can write A as:

$$A = P\Lambda P^{-1}$$

$\text{trace}(A) = \text{trace}(P\Lambda P^{-1})$

$\text{trace}(A) = \text{trace}(PP^{-1}\Lambda)$

$\text{trace}(A) = \text{trace}(\Lambda)$

since $\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ $\text{trace}(A) = \lambda_1 + \cdots \lambda_n$ ✓

(iii) $P(\lambda) = (-1)^3(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$

$= -1(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$

$= -1(\lambda^2 - \lambda_1\lambda - \lambda_2\lambda + \lambda_1\lambda_2)(\lambda-\lambda_3)$

$= -1(\lambda^3 - \lambda_1\lambda^2 - \lambda_2\lambda^2 + \lambda_1\lambda_2\lambda - \lambda_3\lambda^2 + \lambda_1\lambda_3\lambda + \lambda_2\lambda_3\lambda - \lambda_1\lambda_2\lambda_3)$

$= -\lambda^3 + (\lambda_1+\lambda_2+\lambda_3)\lambda^2 - (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\lambda + (\lambda_1\lambda_2\lambda_3)$

Question assigned to the following page: 1

(iii) continued

$P(\lambda) = \det(A - \lambda I)$

$$A - \lambda I \quad \begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ a_{21} & a_{22}-\lambda & a_{23} \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix} = B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$\det(A-\lambda I) = \underbrace{b_{11} b_{22} b_{33}}_{123} - \underbrace{b_{11} b_{23} b_{32}}_{132} - \underbrace{b_{12} b_{21} b_{33}}_{213} + \underbrace{b_{12} b_{23} b_{31}}_{231} + \underbrace{b_{13} b_{21} b_{32}}_{312} - \underbrace{b_{13} b_{22} b_{31}}_{321}$$

123

$(a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)$

$a_{11} a_{22} - a_{11}\lambda - a_{22}\lambda + \lambda^2 (a_{33} - \lambda)$

$a_{11} a_{22} a_{33} - a_{11} a_{33} \lambda - a_{22} a_{33} \lambda + a_{33}\lambda^2 - a_{11} a_{22} \lambda + a_{11}\lambda^2 + a_{22}\lambda^2 - \lambda^3$

$-\lambda^3 + (a_{11} + a_{22} + a_{33})\lambda^2 - (a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33})\lambda + (a_{11}a_{22}a_{33})$

132

$-(a_{11}-\lambda)(a_{23})(a_{32})$

$(a_{11}a_{23} - a_{23}\lambda)(a_{32})$

$-a_{11}a_{23}a_{32} + a_{23}a_{32}\lambda$

213

$-(a_{12})(a_{21})(a_{33}-\lambda)$

$-(a_{12}a_{21}a_{33}) + a_{12}a_{21}\lambda$

231

$a_{12}a_{23}a_{31}$

312

$a_{13}a_{21}a_{32}$

321

$-(a_{13})(a_{22}-\lambda)(a_{31})$

$-(a_{13}a_{31}a_{22}) + (a_{13}a_{31})\lambda$

Question assigned to the following page: 1

$$P(\lambda) = -\lambda^3 + (a_{11} + a_{22} + a_{33})\lambda^2 - (a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} + a_{23}a_{32} + a_{12}a_{21} + a_{13}a_{31})\lambda$$
$$+ (a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{31}a_{22})$$

since $(a_{11} + a_{22} + a_{33})\lambda^2 = (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2$

$\therefore \text{trace}(A) = \lambda_1 + \lambda_2 + \lambda_3$

Question assigned to the following page: 2

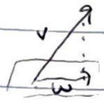
(2) Projections

(a) In general when $A^2 = A$ we assume A "behaves like a Projection. If $A^2 = A$, and $Av = \lambda v$ / $A^2 v = \lambda^2 v$ then:

$$\lambda^2 = \lambda$$

$$\lambda(\lambda - 1) = 0$$

$$\lambda\text{'s of } A = 1, 0$$



$\text{proj}(v) = w$

$$Pv = w$$

$$P(Pv)w$$

$$Pv \neq 0$$

$$Pw = w \quad \text{w/ } \lambda = 1 \text{ and eigenvector } \neq \vec{0}$$

To show that $E_1(A) = Col(A)$ we need to show that $E_1(A) \subseteq Col(A)$ and $Col(A) \subseteq E_1(A)$.

1. Assume $\vec{v} \in E_1(A)$ i.e $(P\vec{v} = \vec{v})$

   if $P = [c_1 \cdots c_n]$ and $\vec{v} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

   then $\vec{v} = c_1 x_1 + \cdots + c_2 x_2$

   $\therefore$ $\vec{v}$ is a linear combination of the columns of P, which also means $\vec{v} \in \text{Span}\{c_1 \cdots c_n\}$

   $\therefore$ $E_1(A) \subseteq Col(A)$

   True for all matricies

2. Assume $\vec{w} \in Col(A)$ i.e $(P\vec{v} : \vec{v} \in \mathbb{R}^n)$

if $P = [c_1 \cdots c_n]$ and $\vec{v} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

then $w = c_1 x_1 + \cdots + c_n x_n$

$$P\vec{v} = \vec{w}$$

then $\left.\begin{array}{l} PPv = Pw \\ P^2 v = w \\ Pv = w \\ w = w \end{array}\right\}$ by $A^2 = A$ property

$\therefore$ $Col(A) \subseteq E_1(A)$

Question assigned to the following page: 2

Since $E_1(A) \subseteq col(A)$ and $col(A) \subseteq E_1(A)$, $E_1(A) = col(A)$. ✓

(b) If A is invertible then $A^{-1}A^2 = A^{-1}A$, which equals $A = I_n$ (only invertible matrix that satisfies $A^2 = A$).

↳ $\lambda = 1$ is the only eigenvalue and $col(A) = \mathbb{R}^n$

thus $col(A)/E_1(A)$ = the vectors in $\mathbb{R}^n$, since $E_1(A)$ forms a basis of $\mathbb{R}^n$, A is digonalizable.

$$A = \underset{P}{\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}} \underset{D}{\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}} \underset{P^{-1}}{\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}} = I_n$$

$$A = I_n \cdot I_n \cdot I_n = I_n$$

(c) If A is not invertible 0 is an eigenvalue, also since A is non-zero 1 is the other eigenvalue of A.

$$\begin{cases} E_1(A) = col(A) \\ E_0(A) = null(A) \end{cases}$$

forms basis of $\mathbb{R}^n$ ∴ diagonalizable

$\dim(col(A)/E_1(A)) = y$

$\dim(null(A)/E_0(A)) = n-y$

since $0 \leq rank(A) \leq n$

$col(A) = \{v_1, v_2, \cdots, v_y\}$

$null(A) = \{x_1, x_2, \cdots, x_{n-y}\}$

$$A = \underset{P}{[v_1, v_2, \cdots, v_y, x_1, x_2, \cdots x_{n-y}]} \underset{D}{\begin{bmatrix} \underset{y \text{ times}}{1} & & \\ & & \underset{n-y \text{ times}}{0} \end{bmatrix}} \underset{P^{-1}}{[v_1, v_2, \cdots, v_y, x_1, x_2, \cdots, x_{n-y}]}$$

Question assigned to the following page: 3

(3) Simultaneous Diagonalization

(a) $S_4 = \{U\Lambda U^{-1} \in \mathbb{R}^{4\times 4} : \Lambda \text{ diagonal matrix}\}$

1. $A \in S_4 \to A = U\Lambda_1 U^{-1}$ or $U(A+B)U^{-1} = U\Lambda_1 U^{-1} + U\Lambda_2 U^{-1} = A+B$

$B \in S_4 \to B = U\Lambda_2 U^{-1}$ $\Lambda_1\ \Lambda_2$

$A+B = U\Lambda_1 U^{-1} + U\Lambda_2 U^{-1}$ → since U and U^{-1} are the same for both

$= U(\Lambda_1 + \Lambda_2)U^{-1}$

let $\Lambda_1 + \Lambda_2 = \Lambda_3$ → Λ_3 is still a diagonal matrix

↳ since Λ_3 is a diagonal matrix and U and U^{-1} diagonalize it, $A+B \in S_4$, and ∴ S_4 is closed under addition.

2. $A \in S_4 \to A = U\Lambda_1 U^{-1}$

$r \in \mathbb{R}$

rA has corresponding diagonal matrix $r\Lambda$ since $rU\Lambda U^{-1} \to Ur\Lambda U^{-1}$

∴ $rA \to Ur\Lambda_1 U^{-1}$

let $r\Lambda_1 = \Lambda_2 \to \Lambda_2$ is still a diagonal matrix

then $rA \to U\Lambda_2 U^{-1}$

↳ since Λ_2 is a diagonal matrix and U and U^{-1} diagonalize it: $rA \in S_4$, and ∴ S_4 is closed under scalar multiplication.

It is also important to note that $rA = rU\Lambda_1 U^{-1}$ which is just a scaling of the diagonalization, which is enough to show $rA \in S_4$.

3. 0 matrix has $\Lambda_1 = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}$

$U\Lambda_1 U^{-1} = U\begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix}U^{-1} = 0$

∴ $0 \in S_4$

Question assigned to the following page: 3

- Since S_4 is closed under addition, scalar multiplication, and contains the zero matrix, S_4 is a subspace of $\mathbb{R}^{4\times4}$.

(b) $S_I = \{I \Lambda I^{-1} \in \mathbb{R}^4 : \Lambda \text{ is a diagonal matrix}\}$

$S_I = [^{1} {}_{1} {}_{1} {}_{1}] \Lambda [^{1} {}_{1} {}_{1} {}_{1}]$

↳ $S_I = \Lambda \in \mathbb{R}^4$

$\therefore$ S_I: Subset of matricies that are diagonalized by $[^{1} {}_{1} {}_{1} {}_{1}]$, this is only the case when a matrix is diagonal, $\begin{bmatrix} a & & & \\ & b & & \\ & & c & \\ & & & d \end{bmatrix}$, therefore S_I is all 4×4 diagonal matricies.

(c) $AB = U\Lambda_1 U^{-1} U \Lambda_2 U^{-1} = U \Lambda_1 \Lambda_2 U^{-1}$ $\quad A = U\Lambda_1 U^{-1}$ $\quad B = U \Lambda_2 U^{-1}$

$BA = U\Lambda_2 U^{-1} U \Lambda_1 U^{-1} = U \Lambda_2 \Lambda_1 U^{-1}$

since Λ_1 and Λ_2 are diagonal matricies $\Lambda_1 \Lambda_2 = \Lambda_2 \Lambda_1$

↳ PROOF: $A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}$ $\quad B = \begin{bmatrix} b_{11} & 0 \\ 0 & b_{22} \end{bmatrix}$

$AB = \begin{bmatrix} a_{11}b_{11} & 0 \\ 0 & a_{22}b_{22} \end{bmatrix}$ $\quad BA = \begin{bmatrix} b_{11}a_{11} & 0 \\ 0 & b_{22}a_{22} \end{bmatrix}$

Since the entries in AB and BA are identical $AB = BA$ when two matricies are diagonal.

$\therefore$ $U\Lambda_1\Lambda_2 U^{-1} = U\Lambda_2\Lambda_1 U^{-1}$, this shows $AB = BA$.

Question assigned to the following page: 4

(4) How to find a triangle in a graph

(a) $\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$

(b) $B = A^2$

$\begin{bmatrix} 2 & 0 & 0 & 2 & 1 \\ 0 & 2 & 2 & 0 & 1 \\ 0 & 2 & 3 & 1 & 1 \\ 2 & 0 & 1 & 3 & 1 \\ 1 & 1 & 1 & 1 & 2 \end{bmatrix}$

* b_{ij} is found by multiplying the dot product vectors in column^T_i (same as row_i) and column_j

* b_{23} is found by the dot product of $\underset{3}{[1\ 0\ 0\ 1\ 1]^T}$ and $\underset{\text{row } 2}{[1\ 0\ 0\ 1\ 0]^T} = 2$

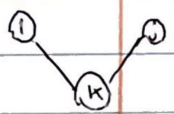
(c) $a_{ik}a_{kj} = 1$ when k is a common neighbor of i and j

CHECK: $a_{12}a_{24} = 1 \cdot 1 = 1$ (2 is a common neighbor)

$a_{13}a_{32} = 1 \cdot 0 = 0$ (3 is NOT a common neighbor)

EXPLANATION: a_{ik} says there is an edge between i and k (if $a_{ik} = 1$)

a_{kj} says there is an edge between j and k (if $a_{kj} = 1$)



if both are yes than both will have an entry value of 1 and $1 \cdot 1 = 1$ this means there is an edge between "k and j" and "i and k", therefore i and j are seperated by a mutual neighbor k. If there isn't a connection between "i and k" or "k and j" one of the entry values will be zero meaing $1 \cdot 0 = 0$ and there is no mutual connection.

Question assigned to the following page: 4

(d) b_{ij} represents how many ways/paths node i can get to node j in two steps/ does i have a common/mutual connection/neighbor. For example b_{11} represents how node 1 can get to node 1 in two steps, 1 to 2 to 1 and 1 to 3 to 1. but b_{12} represents how node 1 can get to node 2 in two steps but there is no mutual neighbor so its matrix entry is 0.

(e) i, j, k form a triangle iff i and j are neighbors and k is a mutual neighbor
↳ both a_{ij} and b_{ij} must be nonzero
↳ in particular $a_{ij} = 1$ (i and j are neighbors), and $b_{ij} > 0$ (at least one mutual neighbor to i and j).

(f) If $a_{ij} = 1$ then the two indices are neighbors
If $b_{ij} > 0$ there exists at least one mutual neighbor

ALGORITHM: If $(a_{ij})(b_{ij}) > 0$ then there exists a triangle with the edge between i and j, (at least 1 triangle, if $a_{ij} b_{ij} = 2$ then it's the edge of 2).

Justification: This should hold for any combination of i, j of 3, 4, 5 since it forms a triangle.

This holds true for large graphs

$(a_{34})(b_{34}) = (a_{43})(b_{43}) = (1)(1) > 0$
$(a_{35})(b_{35}) = (a_{53})(b_{53}) = (1)(1) = 1 > 0$
$(a_{45})(b_{45}) = (a_{54})(b_{54}) = (1)(1) = 1 > 0$

all of these i, j are sides of the same triangle since there are 3 sides of a triangle.

$(a_{12})(b_{12}) = (a_{21})(b_{21}) = (1)(0) = 0$ → Not an edge of a triangle but we expect this.