Homework 3

Graded

Student

Jaiden Rain Atterbury

Total Points

39.5 / 40 pts

Question 1

Question 1

10 / 10 pts

✔ **– 0 pts** correct

Question 2

Question 2

9.5 / 10 pts

(c) 3 points

✔ **– 0.5 pts** minor error

✔ **– 0 pts** **Note, the easiest solution to part (a) is as follows:** using the fact that the columns of A sum to 1, we have that $\mathbf{1}^T A = \mathbf{1}^T$. Multiply by u on the right to get $\mathbf{1}^T Au = \mathbf{1}^T u$. The left-hand side is the sum of the entries of Au, while the right-hand side is the sum of the entries of u, so we are done. (I've marked this rubric entry for everyone, whether or not you used this approach.)

1. What you've written is incorrect.
2. It's not enough to know that the sum of the entries is 3.

Question 3

Question 3

10 / 10 pts

✔ **– 0 pts** Correct Markov matrix and page rank vector, and work is explained or shown.

Question 4

Question 4

10 / 10 pts

✔ **– 0 pts** correct

3. "If" implies you are assuming this as an extra condition, but this is always true.

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Jaiden Atterbury 4/12/2022 Homework #3

(1) Recursive Sequence

$\{G_K\} \rightarrow G_{K+2} = \frac{1}{2}G_{K+1} + \frac{1}{2}G_K$

$\underbrace{\begin{bmatrix} G_{K+2} \\ G_{K+1} \end{bmatrix}}_{G_{K+1}} = A \underbrace{\begin{bmatrix} G_{K+1} \\ G_K \end{bmatrix}}_{G_K}$

(a) Rule: $G_{K+2} = \frac{1}{2}G_{K+1} + \frac{1}{2}G_K$ $\rightarrow \boxed{A = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}} \rightarrow G_{K+1} = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix} G_K$

$G_{K+1} = G_{K+1}$

(b) $A - \lambda I = \begin{bmatrix} 1/2-\lambda & 1/2 \\ 1 & -\lambda \end{bmatrix}$ $\det(A-\lambda I) = 0$

$\lambda^2 - \frac{1}{2}\lambda - \frac{1}{2} = 0$

$(\lambda - 1)(\lambda + \frac{1}{2}) = 0 \rightarrow \boxed{\lambda = 1, -\frac{1}{2}}$

$\text{null}(A - 1I) = \begin{bmatrix} -1/2 & 1/2 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} -1/2 & 1/2 \\ 0 & 0 \end{bmatrix}$ $x_2 = s$, $x_1 = s$ $\boxed{u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

$\text{null}(A + \frac{1}{2}I) = \begin{bmatrix} 1 & 1/2 \\ 1 & 1/2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}$ $x_2 = s$, $x_1 = -\frac{1}{2}s$ $\boxed{u_2 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}}$

(c) $G_K = \begin{bmatrix} G_{K+1} \\ G_K \end{bmatrix}$ $G_0 = \begin{bmatrix} G_1 \\ G_0 \end{bmatrix}$ $\rightarrow G_K = A^K G_0$

$\lim_{K\to\infty} A^K = ?$ $\rightarrow \lim_{K\to\infty} A^K = U \lim_{K\to\infty} \Lambda^K U^{-1}$

↑ diagonalization of A^K

$U = \begin{bmatrix} 1 & -1/2 \\ 1 & 1 \end{bmatrix}$ $\Lambda^K = \begin{bmatrix} 1^K & 0 \\ 0 & (-\frac{1}{2})^K \end{bmatrix}$ $U^{-1} = \frac{2}{3}\begin{bmatrix} 1 & 1/2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ -2/3 & 2/3 \end{bmatrix}$

$\lim_{K\to\infty} A^K = \begin{bmatrix} 1 & -1/2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2/3 & 1/3 \\ -2/3 & 2/3 \end{bmatrix} = \boxed{\begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}}$

Question assigned to the following page: 1

(d) $\begin{bmatrix} G_{K+1} \\ G_K \end{bmatrix} = \lim_{n \to \infty} A^K w_0$ $\quad w_0 = \begin{bmatrix} G_1 \\ G_0 \end{bmatrix} \to \begin{matrix} G_0 = 0 \\ G_1 = 1 \end{matrix}$

$$\begin{bmatrix} G_{K+1} \\ G_K \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} G_{K+1} \\ G_K \end{bmatrix} = \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix}$$

$$\boxed{\therefore\ G_K = 2/3 \text{ as } n \to \infty}$$

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(2) Markov Matrices

(a) If A is markov then $\mathbb{1}^t A = \mathbb{1}^t$, $\mathbb{1}^t A\vec{u}$ is therefore equal to $\mathbb{1}^t\vec{u}$. Since $\mathbb{1}^t$ dotted with any vector is just the sum of the vector components, then if the sum of the components of $\vec{u}$ is α then $\mathbb{1}^t\vec{u}$ will also sum to α. Thus:

$$\mathbb{1}^t A = \mathbb{1}^t$$
$$\mathbb{1}^t A\vec{u} = \mathbb{1}^t\vec{u}$$
then $\mathbb{1}^t A\vec{u} = \alpha$

$\therefore$ The sum of the components of $A\vec{u}$ is also α.

Also, since A and $\vec{u}$ are both non-negative there is no possible way $A\vec{u}$ can be NOT non-negative.

(b) $\vec{V} \neq \vec{0}$

$A \geq 0$

$\lambda \neq 1$ and $\lambda < 1$ (since A is non-negative markov)

$\vec{V} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$

$\alpha = v_1 + \cdots + v_n$

If $A\vec{V} = \lambda\vec{V}$ then $A\vec{V} = \alpha$ according to part (a), $\lambda\vec{v}$ equals:

$$\mathbb{1}^t\lambda\vec{V} = [1, 1, \cdots 1]\,\lambda \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$
$$= \lambda v_1 + \cdots + \lambda v_n$$
$$= \lambda(v_1 + \cdots + v_n)$$
$$= \lambda\alpha$$

$\therefore A\vec{v} = \lambda\vec{v} \Rightarrow \alpha = \lambda\alpha \Rightarrow \alpha(1-\lambda) = 0$

$\hookrightarrow \boxed{\alpha \text{ must be } 0 \text{ since } \lambda \neq 1}$

Question assigned to the following page: 2

2x2 example:

$$A=\begin{bmatrix}1/2 & 1\\ 1/2 & 0\end{bmatrix} \rightarrow A-\lambda I=\begin{bmatrix}1/2-\lambda & 1\\ 1/2 & -\lambda\end{bmatrix} \rightarrow \det(A-\lambda I)=0$$

$$\lambda^2-\tfrac{1}{2}\lambda-\tfrac{1}{2}=0$$

$$\lambda=-\tfrac{1}{2},1$$

$$\text{null}(A-I)=\begin{bmatrix}-1/2 & 1\\ 1/2 & -1\end{bmatrix}\sim\begin{bmatrix}-1/2 & 1\\ 0 & 0\end{bmatrix}\begin{matrix}x_2=s\\ x_1=2s\end{matrix} \qquad u_1=\begin{bmatrix}2\\1\end{bmatrix}$$

$$\text{null}(A+\tfrac{1}{2}I)=\begin{bmatrix}1 & 1\\ 1/2 & 1/2\end{bmatrix}\sim\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\begin{matrix}x_2=s\\ x_1=-s\end{matrix} \qquad u_2=\begin{bmatrix}-1\\1\end{bmatrix}$$

$\therefore\ v=\begin{bmatrix}-1\\1\end{bmatrix}$, $\lambda=-\frac{1}{2}\neq 1$; and $\alpha=0$

↑ non-zero

(c) $\begin{bmatrix}.7 & .1 & .2\\ .2 & .7 & -\\ - & - & -\end{bmatrix}$ must be $\begin{bmatrix}.7 & .1 & .2\\ .2 & .7 & .1\\ .1 & .2 & .7\end{bmatrix}$

With dominant eigenvalue $\lambda_A=1$.

- EXPLANATION: I started off by noticing that a_{31} must be .1 and a_{32} must be .2 in order for the columns to add up to 1. Given that the dominant eigenvector $\vec{v}$ was $(1,1,1)^T$, using what I solved in part (a) I knew that $\mathbb{1}^t A\vec{v}$ must equal the sum of $\vec{v}$, which in this case was 3. First multiplying A by $\vec{v}$ we get the vector $[1, .9+a_{23}, .3+a_{33}]^T$. Since $\mathbb{1}^t \cdot A\vec{v}$ must equal 3 we see that $a_{23}=.1$ and $a_{33}=.7$. To get the dominant eigenvalue we use the fact that u_A is the only eigenvector such that $u_A>0$ IF $\lambda_A=1$, all other eigenvalues of this positive markov matrix must be less than λ_A with eigenvectors

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(c) Continued

that have at least one negative component (theorem 3.2.4).

Another way to see $\lambda_A = 1$ is by noticing $AV = \begin{bmatrix}1\\1\end{bmatrix}$, using $A\vec{v} = \lambda v$ we see $\begin{bmatrix}1\\1\end{bmatrix} = \lambda \begin{bmatrix}1\\1\end{bmatrix}$, which can only be satisfied by $\lambda = 1$.

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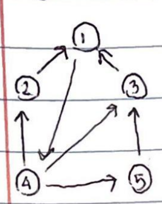
(3) Page Rank



Rule: teleport probability → 1/4
link probability → 3/4

$$A = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & \frac14\cdot\frac15=\frac{1}{20} & \frac14\cdot\frac15+\frac34=\frac{16}{20} & \frac14\cdot\frac15+\frac34=\frac{16}{20} & \frac14\cdot\frac15=\frac{1}{20} & \frac14\cdot\frac15=\frac{1}{20} \\ 2 & \frac14\cdot\frac15=\frac{1}{20} & \frac14\cdot\frac15=\frac{1}{20} & \frac14\cdot\frac15=\frac{1}{20} & \frac14\cdot\frac15+\frac34\cdot\frac13=6/20 & \frac14\cdot\frac15=\frac{1}{20} \\ 3 & \frac14\cdot\frac15=\frac{1}{20} & \frac14\cdot\frac15=\frac{1}{20} & \frac14\cdot\frac15=\frac{1}{20} & \frac14\cdot\frac15+\frac34\cdot\frac13=6/20 & \frac14\cdot\frac15+\frac34=\frac{16}{20} \\ 4 & \frac14\cdot\frac15+\frac34=\frac{16}{20} & \frac14\cdot\frac15=\frac{1}{20} & \frac14\cdot\frac15=\frac{1}{20} & \frac14\cdot\frac15=\frac{1}{20} & \frac14\cdot\frac15=\frac{1}{20} \\ 5 & 1/4\cdot1/5=1/20 & \frac14\cdot\frac15=\frac{1}{20} & \frac14\cdot\frac15=\frac{1}{20} & \frac14\cdot\frac15+\frac34\cdot\frac13=\frac{6}{20} & \frac14\cdot\frac15=\frac{1}{20} \end{array}$$

$$A = \begin{bmatrix} 1/20 & 16/20 & 16/20 & 1/20 & 1/20 \\ 1/20 & 1/20 & 1/20 & 6/20 & 1/20 \\ 1/20 & 1/20 & 1/20 & 6/20 & 16/20 \\ 16/20 & 1/20 & 1/20 & 1/20 & 1/20 \\ 1/20 & 1/20 & 1/20 & 6/20 & 1/20 \end{bmatrix}$$

A is positive markov

Eigenvalues: $\lambda_1 = -0.33$ $\quad \lambda_2 = 0$ $\quad \lambda_3 = 1$ $\quad \lambda_4 = -0.21 - 0.62i$ $\quad \lambda_5 = -0.21 + 0.62i$

eigenvectors: $u_1 = \begin{bmatrix} 0.59 \\ 1 \\ -1.26 \\ -1.33 \\ 1 \end{bmatrix}$ $u_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $u_3 = \begin{bmatrix} 2.43 \\ 1 \\ 1.75 \\ 2.29 \\ 1 \end{bmatrix}$ $u_4 = \begin{bmatrix} -1.79+1.37i \\ 1 \\ 0.63+1.09i \\ -0.84-2.47i \\ 1 \end{bmatrix}$ $u_5 = \begin{bmatrix} -1.79+1.37i \\ 1 \\ 0.63-1.09i \\ -0.84+2.47i \\ 1 \end{bmatrix}$

- Since A is positive and markov the dominant eigenvalue is 1 and all other eigenvalues are less than 1 therefore $\lim_{K\to\infty} w_K = \lim_{K\to\infty} c_1(1)^K u_A$ since all of the other terms approach 0 when K approaches ∞. This is why the page rank vector is u_A (dominant eigenvector).

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$$u_A = \begin{bmatrix} 2.49 \\ 1 \\ 1.75 \\ 2.29 \\ 1 \end{bmatrix}$$

$2.49 + 1 + 1.75 + 2.29 + 1 = 8.53$

$$u_A = \begin{bmatrix} 2.49/8.53 \\ 1/8.53 \\ 1.75/8.53 \\ 2.29/8.53 \\ 1/8.53 \end{bmatrix} \rightarrow \boxed{\text{page rank vector} = \begin{bmatrix} 0.29 \\ 0.12 \\ 0.21 \\ 0.27 \\ 0.12 \end{bmatrix} \approx \begin{bmatrix} 0.3 \\ 0.1 \\ 0.2 \\ 0.3 \\ 0.1 \end{bmatrix}}$$

Question assigned to the following page: 4

(4) Products of Matricies

(a) (i) $AB = A[b_1, b_2, \cdots, b_n] = [Ab_1 \; Ab_2 \cdots Ab_n]$

$Col(A) = \{A\vec{x} : \vec{x} \in \mathbb{R}^n\}$ thus $A\vec{x}$ is a linear combination of the columns of A

The columns of AB are a linear combination of the columns of A

meaning that Col(AB) are contained in Col(A).

$\boxed{\therefore Col(AB) \subseteq Col(A)}$

If B = 0 matrix then we can't say col(AB) = col(A), $\therefore$ it is not true in general.

(ii) $AB = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix} B = \begin{bmatrix} a_1^T B \\ \vdots \\ a_m^T B \end{bmatrix}$

$Row(B) = \{\vec{y}^T B : \vec{y} \in \mathbb{R}^n\}$ thus $\vec{y}^T B$ is a linear combination of the rows of B

The rows of AB are a linear combination of the rows of B

meaning that Row(AB) is contained in Row(B).

$\therefore \boxed{Row(AB) \subseteq Row(B)}$

If A = 0 matrix then we can't say Row(AB) = Row(B) $\therefore$ it is not true in general.

(iii) Since $col(AB) \subseteq col(A)$ | since $row(AB) \subseteq row(B)$

$rank(AB) = dim(col(AB))$ | if $rank(AB) = dim(row(AB))$

ALSO

and if $rank(A) = dim(col(A))$ | and if $rank(B) = dim(row(B))$

then $rank(AB) \leq rank(A)$ | then $rank(AB) \leq rank(B)$

Since rank(AB) is $\leq$ to both rank(A) and rank(B)

it must be $\leq$ to the lower of these two values

$\therefore rank(AB) \leq min\{rank(A), rank(B)\}$

(iv) Given $A \in \mathbb{R}^{n \times k}$, assume the columns of A are linearly dependent, A^T is thus $\in \mathbb{R}^{k \times n}$ and $A^T A$ is $\in \mathbb{R}^{k \times k}$. since the columns of A are dependent $rank(A) < K$ then:

$rank(A^T A) \leq min(rank(A^T), rank(A)) \rightarrow$ since $rank(A) < k$

$\boxed{rank(A^T A) < K \rightarrow \therefore A^T A \text{ is not invertible}}$

(b) If $AB=0$ then the columns of B are in the NullSpace of A.

This is true because $AB = [Ab_1 \cdots Ab_n] = \vec{0}$ is a linear combination of the columns of A. Thus every $Ab_i = 0$ which means b_i must be in $null(A)$.

$$\boxed{\text{Thus } col(B) \subseteq Null(A)}$$

We can't say $col(B) = Null(A)$ in general because if B is the zero matrix $col(B) = \{\vec{0}\}$ and unless A is the zero matrix $Null(A)$ will have more than the zero vector in its subspace.

(c) $A \in \mathbb{R}^{3\times 3}$ & $B \in \mathbb{R}^{3\times 3}$, $AB = 0$; $rank(A)$ & $rank(B) = 2$

based on part (b): $\dim(Null(A)) \geq \dim(col(B))$

since $\quad \dim(rank(A)) + \dim(null(A)) = 3$

$\quad \dim(null(A)) = 3 - \dim(rank(A))$

then $\quad 3 - \dim(rank(A)) \geq \dim(rank(B))$

thus $\quad 3 \geq \dim(rank(A)) + \dim(rank(B))$

if $\quad rank(A)$ & $rank(B) = 2$

then $\quad 3 \geq 4$ (NOT POSSIBLE)

$$\boxed{\therefore \text{ A and B cannot be } 3\times 3 \text{ matrices of rank } 2}$$

(d) $A \in \mathbb{R}^{3\times 4}$ and $B \in \mathbb{R}^{4\times 5}$ and $AB = 0$

Since $\quad col(B) \subseteq null(A)$ when $AB = 0$ & $rank(B) = \dim(col(B))$

$\dim(null(A)) \geq \dim(col(B))$

$\dim(rank(A)) + \dim(null(A)) = 4$

$\dim(rank(B)) + \dim(null(B)) = 5$

thus $\quad \dim(null(A)) = 4 - rank A \geq rank B$

$$\boxed{\therefore 4 \geq rank A + rank B}$$