Homework 6

Graded

Student

Jaiden Rain Atterbury

Total Points

40 / 40 pts

Question 1

Question 1

10 / 10 pts

✔ **− 0 pts** correct

Question 2

Question 2

10 / 10 pts

✔ **− 0 pts** correct

Question 3

Question 3

10 / 10 pts

✔ **− 0 pts** **Note that may or may not apply to you:** $(\lambda A)^T = \lambda^T A^T$ doesn't make sense if λ is a scalar. The way to make sense of this is to replace scalar multiplication by λ with *matrix* multiplication by the matrix λI.

✔ **− 0 pts** correct

Question 4

Question 4

10 / 10 pts

✔ **− 0 pts** 100% complete

Question assigned to the following page: 1

Jaiden Atterbury 05-10-2022 Homework 6

(1) Symmetric Matrices

(a) $\lambda_1 = -3$, $u_1 = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$

$\lambda_2 = 0$, $u_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$

$\lambda_3 = 3$, $u_3 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

* Since A is symmetric ($A = A^T$) by proposition 5.1.6 eigenvectors from different eigenspaces are mutually orthogonal. Since the eigenvectors of A are already orthogonal, all that must be done for these vectors to be orthonormal is to make them unit vectors.

$\|u_i\| = 3$ (all vectors u_1, u_2, u_3 have the same length).

$\frac{u_1}{\|u_i\|} = q_1 = \begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix} \rightarrow \|q_1\| = 1$

$\frac{u_2}{\|u_i\|} = q_2 = \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix} \rightarrow \|q_2\| = 1$

$\frac{u_3}{\|u_i\|} = q_3 = \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix} \rightarrow \|q_3\| = 1$

$$\boxed{\{q_1, q_2, q_3\} = \left\{ \begin{bmatrix} 1/3 \\ -2/3 \\ -2/3 \end{bmatrix} \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix} \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix} \right\}}$$

(b) $A = Q \Lambda Q^{-1} = Q \Lambda Q^T$

$$A = \underbrace{\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & 2/3 & -1/3 \\ -2/3 & -1/3 & 2/3 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 1/3 & -2/3 & -2/3 \\ 2/3 & 2/3 & -1/3 \\ 2/3 & -1/3 & 2/3 \end{bmatrix}}_{Q^{-1}/Q^T}$$

(c) $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$[\vec{x}]_B = U^{-1}\vec{x} \rightarrow [\vec{x}]_Q = Q^T\vec{x}$

$$[\vec{x}]_Q = \underbrace{\begin{bmatrix} 1/3 & -2/3 & -2/3 \\ 2/3 & 2/3 & -1/3 \\ 2/3 & -1/3 & 2/3 \end{bmatrix}}_{Q^T} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\vec{x}} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\boxed{[\vec{x}]_Q = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}}$$

Question assigned to the following page: 1

(d) $A\vec{x} = A\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1 & 0 & 2\\0 & -1 & -2\\2 & -2 & 0\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}3\\-3\\0\end{bmatrix}$

$$[A\vec{x}]_Q = Q^T A\vec{x} = \begin{bmatrix}1/3 & -2/3 & -2/3\\2/3 & 2/3 & -1/3\\2/3 & -1/3 & 2/3\end{bmatrix}\begin{bmatrix}3\\-3\\0\end{bmatrix} = \begin{bmatrix}3\\0\\3\end{bmatrix}$$

$$= \Lambda \cdot [\vec{x}]_Q = \begin{bmatrix}-3 & 0 & 0\\0 & 0 & 0\\0 & 0 & 3\end{bmatrix}\begin{bmatrix}-1\\1\\1\end{bmatrix} = \begin{bmatrix}3\\0\\3\end{bmatrix}$$

$$\boxed{[A\vec{x}]_Q = \begin{bmatrix}3\\0\\3\end{bmatrix}}$$

Question assigned to the following page: 2

(2) Quadratic forms, PSD matrices

(a) A is an $n \times n$ matrix that is symmetric, orthonormal, and positive definite

By definition 5.1.1 if $A \in \mathbb{R}^{n\times n}$ is symmetric then $A = A^T$.

By proposition 5.2.5 if $A \in \mathbb{R}^{n\times n}$ is orthonormal then $AA^T = A^TA = I_n$.

By definition 6.1.10 if $A \in \mathbb{R}^{n\times n}$ is positive definite then λ's of $A > 0$.

Since $A = A^T$ and $AA^T = I_n$, this implies $A^2 = I_n$

and since $A\vec{x} = \lambda\vec{x}$, $(A^2 - I_n)\vec{x} = \vec{0} \rightarrow (\lambda^2 - 1) = 0 \rightarrow \lambda^2 = 1 \rightarrow \lambda = \pm 1$

BUT since A is PD the only eigenvalue's of A are 1.

And by proposition 5.1.7 if $A \in \mathbb{R}^{n\times n}$ is symmetric

then $AM(\lambda) = GM(\lambda)$, thus $GM(1) = AM(1)$.

Lastly, by the Spectral Theorem, if A is symmetric then A

has an orthonormal diagonalization:

$$A = Q\Lambda Q^{-1}$$

$$A^2 = Q\Lambda Q^{-1} Q\Lambda Q^{-1} = I_n$$

$$I_n = Q\Lambda^2 Q^{-1}$$

$$Q^{-1} I_n Q = \Lambda^2$$

$I_n = \Lambda^2$ since $GM(1) = AM(1) = n$ the diagonal entries of

$$\Lambda = 1$$

$$\therefore I_n = \Lambda$$

$$A = Q I_n Q^{-1}$$

then $A = I_n$

$\therefore I_n$ is the only $n \times n$ matrix for all n, that is orthonormal, symmetric, and positive definite.

Question assigned to the following page: 2

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}$ * By definition 6.1.10, all leading principal minors of a positive definite matrix are positive.

① $\det[1] = 1$

② $\det\begin{bmatrix} 1 & 2 \\ 2 & d \end{bmatrix} = d-4 \rightarrow (d>4)$

③ $\det\begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix} = 1\det\begin{bmatrix} d & 4 \\ 4 & 5 \end{bmatrix} - 2\det\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} + 3\det\begin{bmatrix} 2 & 3 \\ d & 4 \end{bmatrix}$

$1(5d-16) - 2(10-12) + 3(8-3d)$

$5d-16+4+24-9d = 12-4d > 0 \rightarrow (3>d)$

Since $d>4$ and $d<3$, there is no value of d that makes the above matrix positive definite.

(c) $q(x,y) = x^2 - 4xy + 16y^2$

(i) $Q = \begin{bmatrix} 1 & -2 \\ -2 & 16 \end{bmatrix}$

(ii) $Q = \begin{bmatrix} 1 & -2 \\ -2 & 16 \end{bmatrix} \rightarrow \det(Q-\lambda I) = \begin{vmatrix} 1-\lambda & -2 \\ -2 & 16-\lambda \end{vmatrix} = \lambda^2 - 17\lambda + 12 = 0$

$\lambda_1 = \frac{17-\sqrt{241}}{2} \approx 0.738$

$\lambda_2 = \frac{17+\sqrt{241}}{2} \approx 16.262$

Since $\lambda_1, \lambda_2 > 0$ and Q is symmetric thus $q(x,y)$ will be nonnegative for all $(x,y) \in \mathbb{R}^2$, furthermore $q(x,y)$ is POSITIVE for all $(x,y) \in \mathbb{R}^2$ $(Q \succ 0)$.

Question assigned to the following page: 2

If $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $BB^T = \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix}$

$$\begin{bmatrix} x & y \end{bmatrix} BB^T \begin{bmatrix} x \\ y \end{bmatrix} = (a^2+b^2)x^2 + 2(ac+bd)yx + (c^2+d^2)y^2$$

$$\begin{bmatrix} x & y \end{bmatrix} Q \begin{bmatrix} x \\ y \end{bmatrix} = x^2 - 4yx + 16y^2$$

thus: $a^2+b^2 = 1$ $a = 1$ $b = 0$

$ac + bd = -2$ $c = -2$

$c^2 + d^2 = 16$ $d = \sqrt{12}$

$\therefore B = \begin{bmatrix} 1 & 0 \\ -2 & \sqrt{12} \end{bmatrix}$ $B^T = \begin{bmatrix} 1 & -2 \\ 0 & \sqrt{12} \end{bmatrix}$

$Q = BB^T$

$\downarrow$

$$\underset{Q}{\begin{bmatrix} 1 & -2 \\ -2 & 16 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 1 & 0 \\ -2 & \sqrt{12} \end{bmatrix}} \cdot \underset{B^T}{\begin{bmatrix} 1 & -2 \\ 0 & \sqrt{12} \end{bmatrix}}$$

Sum of squares: $(x-2y)^2 + (\sqrt{12}y)^2$

Question assigned to the following page: 3

(3) PSD matrices

(a) Let S represent the set of all $n \times n$ symmetric matrices

1) If $A = a_{ij}$ and $B_{ij} \in S$

then A is symmetric $a_{ij} = a_{ji}$

and B is symmetric $b_{ij} = b_{ji}$

Let $C = A + B$

since $(a_{ij} + b_{ij}) = (a_{ji} + b_{ji})$, $c_{ij} = c_{ji}$

$\therefore$ C is symmetric and $A+B \in S$

thus S is closed under addition.

2) If $A = a_{ij} \in S$ and $r \in \mathbb{R}$

then A is symmetric $a_{ij} = a_{ji}$

Let $D = rA$ where $D = ra_{ij}$

thus $ra_{ij} = ra_{ji}$ since $a_{ij} = a_{ji}$

$\therefore$ D is symmetric and $rA \in S$

thus S is closed under scalar multiplication.

3) If A = zero matrix

$A = a_{ij} = 0$ for all i and j

$\therefore$ $a_{ij} = a_{ji}$

thus $\vec{0}$ matrix $\in S$

Since S, the set of all $n \times n$ symmetric matrices, is closed under addition, scalar multiplication, and contains the zero matrix, S is a subspace of $\mathbb{R}^{n \times n}$.

Question assigned to the following page: 3

(b) For $n \times n$ matrices

(i) If $A \succeq 0$ then $x^T A x \geq 0$

if $B \succeq 0$ then $x^T B x \geq 0$

$x^T (A+B) x = x^T A x + x^T B x \geq 0$

thus $x^T (A+B) x \geq 0$

and therefore $A+B \succeq 0$.

(ii) If $A \succeq 0$ then $x^T A x \geq 0$

λ (scalar) ≥ 0

$x^T (\lambda A) x = \lambda (x^T A x) \geq 0$ (since λ and $x^T A x \geq 0$)

thus $x^T (\lambda A) x \geq 0$

and therefore $\lambda A \succeq 0$.

(c) No, the set of all $n \times n$ PSD matrices does not form a subspace in the space of all $n \times n$ symmetric matrices. Although the set of all $n \times n$ PSD matrices forms a <u>subset</u> of all $n \times n$ symmetric matrices, it does not form a subspace since the set of all $n \times n$ PSD matrices is <u>NOT</u> closed under scalar multiplication.

If $A \succ 0$ (If A is PD then it is also PSD) and $r \in \mathbb{R}$

$rA = r\lambda \vec{x}$ if we take $\lambda > 0$ and $r < 0$

then $r\lambda < 0$ and thus would no longer be a PSD matrix.

Question assigned to the following page: 4

(4) Equianguiar lines in $\mathbb{R}^d$

(a) If $i \neq j$ then $v_i^T v_j = \cos\theta$ (for a fixed angle θ)

$v_i^T v_j$ is the dot product of v_i and v_j which can be written as:

$v_i^T v_j = \|v_i\|\|v_j\| \cos\theta$ (since v_i and v_j are unit vectors $\|v_i\|$ and $\|v_j\| = 1$).

$\therefore v_i^T v_j = 1 \cdot 1 \cdot \cos\theta = \cos\theta$ (for some fixed angle θ).

(b) 3x3 symmetric matrix:

$$\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}$$

basis for 3x3 symmetric matrices:

$$a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + d\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + e\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

this set is linearly independent and spans the space of all 3x3 symmetric matrices $\therefore$ it is a basis

Since the basis of all 3 symmetric matrices includes 6 matrices (vectors), the dimension of this vector space is 6.

(c) rank one PSD matrices $v_i v_i^T$ of size 3x3

(i) The sum $\sum a_i v_i v_i^T = 0$ can be written as

$a_1 v_1 v_1^T + a_2 v_2 v_2^T + \dots + a_n v_n v_n^T = 0$

Multiplying this sum on the left by v_j^T and on the right by v_j gives

$\sum a_i v_j^T v_i v_i^T v_j = 0$ which can be written as:

$a_1 v_1^T v_1 v_1^T v_1 + a_2 v_1^T v_2 v_2^T v_1 + \dots + a_n v_1^T v_n v_n^T v_1 = 0$

which simplifies to $a_1 + a_2 \cos^2\theta + \dots + a_n \cos^2\theta = 0$

Question assigned to the following page: 4

repeating this for all v_n leads us to the expression

$$0 = a_j + \sum_{i \neq j} a_i \cos^2\theta \quad \text{(for all } n\text{)}.$$

(ii) $$M = \begin{bmatrix} 1 & \cos^2\theta & \cdots & \cos^2\theta \\ \cos^2\theta & 1 & \cdots & \cos^2\theta \\ \vdots & & \ddots & \vdots \\ \cos^2\theta & \cos^2\theta & \cdots & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\uparrow M \qquad \uparrow a \qquad \uparrow \vec{0}$$

(iii) $$(1-\cos^2\theta) I_n = \begin{bmatrix} 1-\cos^2\theta & 0 & \cdots & 0 \\ 0 & 1-\cos^2\theta & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1-\cos^2\theta \end{bmatrix}$$

$$(\cos^2\theta) J_n = \begin{bmatrix} \cos^2\theta & \cos^2\theta & \cdots & \cos^2\theta \\ \cos^2\theta & \cos^2\theta & \cdots & \cos^2\theta \\ \vdots & \vdots & \ddots & \vdots \\ \cos^2\theta & \cos^2\theta & \cdots & \cos^2\theta \end{bmatrix}$$

$$(1-\cos^2\theta) I_n + (\cos^2\theta) J_n = \begin{bmatrix} 1 & \cos^2\theta & \cdots & \cos^2\theta \\ \cos^2\theta & 1 & \cdots & \cos^2\theta \\ \vdots & \vdots & \ddots & \vdots \\ \cos^2\theta & \cos^2\theta & \cdots & 1 \end{bmatrix} = M \checkmark$$

(iv) $$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$x^T I_n x = x_1^2 + x_2^2 + \cdots + x_n^2 > 0 \quad (\text{if } \vec{x} \neq \vec{0})$$

$\therefore I_n$ is PD

$$J_n = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^T$$

$x^T J_n x = (x_1 + x_2 + \cdots + x_n)^2 \geq 0$ (if $\vec{x} \neq \vec{0}$)

$\therefore$ J_n is PSD

(V) CASE 1: $\emptyset = 0$ or multiple of 2π

if $\emptyset = 0$ or a multiple of 2π then there is only one equiangular line, since we are looking to maximize the number of equiangular lines in $\mathbb{R}^d$ ($\mathbb{R}^3$ in this case) we can assume $\emptyset$ is never 0 or a multiple of 2π.

CASE 2: $\emptyset$ is any other angle

$x = \cos\emptyset \rightarrow -1 \leq x \leq 1$

$y = \cos^2\emptyset \rightarrow 0 \leq y < 1$

$\therefore$ $1 - \cos^2\emptyset > 0$

since $\cos^2\emptyset < 1$ (since we are assuming $\emptyset \neq 0$ or $2k\pi$)

If $M = (1-\cos^2\emptyset) I_n + \cos^2\emptyset J_n$

$x^T M x = x^T((1-\cos^2\emptyset) I_n) x + x^T((\cos^2\emptyset) J_n) x$

$$\underbrace{\underset{\text{positive}}{(1-\cos^2\emptyset)} \ast \underset{\text{Positive definite}}{(x^T I_n x)}}_{PD} + \underbrace{\underset{\text{non-negative}}{(\cos^2\emptyset)} \ast \underset{\text{Positive semidefinite}}{(x^T J_n x)}}_{PSD}$$

Now all that is left to check is that PD + PSD = PD

Proof: If $A \succ 0$ and $B \succeq 0$

then $x^T(A+B)x = \underset{>0}{x^T A x} + \underset{\geq 0}{x^T B x}$ which is >0

$\therefore$ since PD + PSD = PD, M is PD

Question assigned to the following page: 4

(vi) Since M is PD, $\lambda = 0$ is not an eigenvalue of M, by the unifying theorem, M is invertible with d linearly independent columns. This means the only combination of the columns that equals the zero vector is the trivial solution itself. Therefore $\vec{a}$ is the zero vector ($a = \vec{0}$).

(vii) Since M is full rank, columns and rows are independent, and the rows of M come from the rank one psd matrices $v_i v_i^T$, we can conclude that $v_i v_i^T$ are linearly independent.

(d) In order to show that you cannot have more than $\frac{d(d+1)}{2}$ equiangular lines in $\mathbb{R}^d$ we must show that a basis for all $d \times d$ symmetric matrices has a dimension of $\frac{d(d+1)}{2}$.

Proof: As we showed in part (b) there exists an element in the basis of $d \times d$ symmetric matrices for each independent entry of the matrix (general $d \times d$ symmetric matrix)

If we let S be the space of all $d \times d$ symmetric matrices:

$$\dim(S) = \#\text{ of independent entries in a generic } d \times d \text{ matrix } A = a_{ij} \text{ where } a_{ij} = a_{ji}$$

if $a_{ij} = a_{ji}$ where $i = j$ then there are d unique entries

if $a_{ij} = a_{ji}$ where $i \neq j$ then there are $\frac{d^2 - d}{2}$ unique entries

Question assigned to the following page: 4

thus the # of independent entries of A is $d + \frac{d^2-d}{2}$

which simplifies to $\frac{d(d+1)}{2}$.

$\therefore$ A basis of all $d \times d$ symmetric matrices has a dimension of $\frac{d(d+1)}{2}$.

By part (c), if we extend $v_i v_i^T$ to rank one psd matrices of size $d \times d$ coming from equiangular lines we find these matrices to be linearly independent. Since all of these rank one matrices are in the $\frac{d(d+1)}{2}$ dimensional vector space of $d \times d$ symmetric matrices, there cannot be more than $\frac{d(d+1)}{2}$ of them. This implies there cannot be more than $\frac{d(d+1)}{2}$ equiangular lines in $\mathbb{R}^d$.