Homework 4

● Graded

Student

Jaiden Rain Atterbury

Total Points

40 / 40 pts

Question 1

Question 1

10 / 10 pts

✔ **− 0 pts** correct

Question 2

Question 2

10 / 10 pts

✔ **− 0 pts** correct

Question 3

Question 3

10 / 10 pts

✔ **+ 5 pts** (a) P_C (2.5 points) $\begin{pmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{pmatrix}$ with explanation/work shown.

✔ **+ 5 pts** (b) P_R (2.5 points) $\begin{pmatrix} 1/9 & 2/9 & 2/9 \\ 2/9 & 4/9 & 4/9 \\ 2/9 & 4/9 & 4/9 \end{pmatrix}$ with explanation/work shown.

Question 4

Question 4

10 / 10 pts

✔ **+ 10 pts** 100% complete

Question assigned to the following page: 1

Jaiden Atterbury 04/19/2022 Homework 4

1. Matrix Construction

(a) A must be a 3x3 matrix, with 2 independent and 1 dependent columns. When multiplied by [1 1 1] the transformation must produce the $\vec{0}$ vector:

$$\begin{bmatrix} 1 & 2 & x \\ 2 & -3 & y \\ -3 & 5 & z \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} 3+x=0 & x=-3 \\ -1+y=0 & y=1 \\ 2+z=0 & z=-2 \end{matrix}$$

$$\therefore \boxed{A = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{bmatrix} \end{array}} \qquad v_3 = -1v_1 + -1v_2$$

CHECK: $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+2-3 \\ 2-3+1 \\ -3+5-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \therefore \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in Null(A)$

and since $\vec{v}_1$ and $\vec{v}_2$ are the only linearly independent columns of A $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} \in Col(A)$,

(b) Based on definition 4.1.3 which states that two subspaces of $\mathbb{R}^n$ are perpendicular/orthogonal if every vector in one subspace is orthogonal to every vector in the othersubspace. Proposition 4.1.7 tells us for any matrix $A \in \mathbb{R}^{m \times n}$, $Row(A) \perp Null(A)$, Therefore every vector in Row(A) must have a dot product of 0 with every vector in Null(A). BUT:

$$[2 \ -3 \ 5] \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 4 \neq 0$$

$\therefore$ $\boxed{\text{NOT POSSIBLE}}$

(c) By Propisition 4.1.10 $Col(A) \perp Null(A^T)$ which means every vector in $Col(A)$ dotted with every vector in $Null(A^T)$ must be the $\vec{0}$ vector:

Since $Col(C) = \{Cx : x \in \mathbb{R}^n\}$

and $Null(C^T) = \{y^T ; y^T C = 0\}$

$\therefore \begin{bmatrix}1\\1\\1\end{bmatrix} \in Col(C)$ and $\begin{bmatrix}1\\0\\0\end{bmatrix} \in Null(C^T)$

Since $[\vdots] \cdot [\vdots] = 1 \neq 0$

$\therefore$ $\boxed{\text{NOT POSSIBLE}}$

(d) Simple case of a 2x2 matrix with every row and every column orthogonal:

$$\boxed{M = \begin{bmatrix}0&0\\1&0\end{bmatrix}}$$

CHECK: $\begin{array}{c|cc} & c_1 & c_2 \\ \hline r_1 & 0 & 0 \\ r_2 & 1 & 0 \end{array}$

$\underset{c_1}{\begin{bmatrix}0\\1\end{bmatrix}} \cdot \underset{r_1^T}{\begin{bmatrix}0\\0\end{bmatrix}} = 0$ $\quad \underset{c_1}{\begin{bmatrix}0\\1\end{bmatrix}} \cdot \underset{r_2^T}{\begin{bmatrix}1\\0\end{bmatrix}} = 0$ $\quad \underset{c_2}{\begin{bmatrix}0\\0\end{bmatrix}} \cdot \underset{r_1^T}{\begin{bmatrix}0\\0\end{bmatrix}} = 0$ $\quad \underset{c_2}{\begin{bmatrix}0\\0\end{bmatrix}} \underset{r_2^T}{\begin{bmatrix}1\\0\end{bmatrix}} = 0$

Since all the dot products equal $\vec{0}$; every row is orthogonal to every column.

(e) $N\begin{bmatrix}1\\\vdots\\1\end{bmatrix} = \vec{0}$ $\qquad [1\,1\,1 \cdots 1]N = [1\,1\,1 \cdots 1]$

$\therefore \begin{bmatrix}1\\\vdots\\1\end{bmatrix} \in Null(N)$ $\qquad N^T[1\,1 \cdots] = [1\,1 \cdots 1]$

$\therefore \begin{bmatrix}1\\\vdots\\1\end{bmatrix} \in Col(N^T)$

$\hookrightarrow \begin{bmatrix}1\\\vdots\\1\end{bmatrix} \in Row(N)$

By proposition 4.1.7 $Row(N) \perp Null(N)$

BUT

$$\begin{bmatrix}1\\\vdots\\1\end{bmatrix} \cdot \begin{bmatrix}1\\\vdots\\1\end{bmatrix} = n \neq \vec{0}$$

$\therefore$ $\boxed{\text{Not possible}}$

Question assigned to the following page: 1

(e) ALSO, If $N \in \mathbb{R}^{m \times k}$ and each entry of N can be written as n_{ij} (i = rows, j = columns) then:

$\forall i \; \sum_{j=1}^{k} n_{ij} = 0$ BUT $\forall j \; \sum_{i=1}^{m} n_{ij} = 1$

$\therefore \sum_{i=1}^{m} \sum_{j=1}^{k} n_{ij} = 0$ $\therefore \sum_{j=1}^{k} \sum_{i=1}^{m} n_{ij} = n$

n can't be zero ($n \neq 0$)

$\therefore$ $\boxed{\text{NOT POSSIBLE}}$

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2. Orthogonal complements

(a) By definition 4.2.2, $S^\perp = \{\vec{x} \in \mathbb{R}^n : s^T\vec{x} = \vec{0} \ \forall s \in S\}$. Using proposition/theorem 4.2.4 if $S \in \mathbb{R}^{m \times n}$ $Row(S)^\perp = Null(S)$.

$\therefore S^\perp = null(S)$ since $\{s_1, s_2\} \in Row(S)$

$$S^\perp = null\left(\begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{cc} & n \\ m & \left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 0 \\ 1 & 3 & 3 & 2 & 0 \end{array}\right] \end{array} \sim \left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right] \begin{array}{l} x_3 = s_1 \\ x_4 = s_2 \end{array} \quad \begin{array}{l} x_2 = -s_1 + s_2 \\ x_1 = -5s_2 \end{array}$$

$\therefore$ A basis for $S^\perp$ is $\boxed{span\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 1 \\ 0 \\ 1 \end{bmatrix}\right\}}$

CHECK: By proposition 4.2.6 $\dim(S) + \dim(S^\perp) = n$ $(n=4)$

$2 + 2 = 4$ ✓

(b) $V = \left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} : x_1 + x_2 + x_3 + x_4 = 0\right\}$

$V = null([1 \ 1 \ 1 \ 1])$

$V = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \end{array}\right]$ $\begin{array}{l} x_2 = s_1 \\ x_3 = s_2 \\ x_4 = s_3 \end{array}$ $x_1 = -s_1 - s_2 - s_3$ $= span\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$

$null(V)^\perp = row(V)$ by theorem 4.2.4

$\therefore V^\perp = row(V)$

$row(V) = \boxed{span\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right\}}$

CHECK: By proposition 4.2.6 $\dim(V) + \dim(V^\perp) = n$ $(n=4)$

$3 + 1 = 4$ ✓

Question assigned to the following page: 2

(c) If $W = \{0\}$ in $\mathbb{R}^3$, then $W^\perp$ is all of $\mathbb{R}^3$

Since every vector $\vec{x} \in \mathbb{R}^3$ dotted with $\vec{0}$ is $\vec{0}$,

meaning every vector in $\mathbb{R}^3$ is orthogonal to $\vec{0}$.

$\therefore$ A basis for $W^\perp = \text{span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$.

Question assigned to the following page: 3

3. Projection Matrices

$$A = \begin{bmatrix} 3 & 6 & 6 \\ 4 & 8 & 8 \end{bmatrix} \sim \begin{bmatrix} 3 & 6 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

basis for col(A) = span $\left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\}$

basis for Row(A) = span $\left\{ \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix} \right\}$

(a) $a = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ $a^T = \begin{bmatrix} 3 & 4 \end{bmatrix}$

$P_c = \frac{aa^T}{a^Ta}$

$$a^Ta = \begin{bmatrix} 3 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 25$$

$$\frac{aa^T}{a^Ta} = \frac{1}{25} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & 4 \end{bmatrix}$$

$$P_c = \frac{1}{25} \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} = \boxed{\begin{bmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{bmatrix}}$$

(b) $a = \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix}$ $a^T = \begin{bmatrix} 3 & 6 & 6 \end{bmatrix}$

$P_R = \frac{aa^T}{a^Ta}$

$$a^Ta = \begin{bmatrix} 3 & 6 & 6 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix} = 81$$

$$\frac{aa^T}{a^Ta} = \frac{1}{81} \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix} \begin{bmatrix} 3 & 6 & 6 \end{bmatrix} = \frac{1}{81} \begin{bmatrix} 9 & 18 & 18 \\ 18 & 36 & 36 \\ 18 & 36 & 36 \end{bmatrix}$$

$$P_R = \boxed{\begin{bmatrix} 1/9 & 2/9 & 2/9 \\ 2/9 & 4/9 & 4/9 \\ 2/9 & 4/9 & 4/9 \end{bmatrix}}$$

Question assigned to the following page: 4

4. The Peterson Puzzle

(a) If J_{10} is a 10x10 matrix with all entries equal to 1, and I_{10} is a 10x10 matrix with all entries in the diagonal equal to 1. Then $J_{10} - I_{10}$ is a 10x10 matrix with all entries equal to 1 except all entries along the diagonal are equal to zero. Since each node in K_{10} is adjacent to every other node in K_{10}, meaning each node in K_{10} has 9 edges incident to it. Thus, there must be 1 entry relating to each node that is equal to zero. This happens on the entries k_{ij} where $i=j$, this ~~occurs~~ along the diagonal. This intuitively makes sense since a node can't be adjacent to itself. Since $J_{10} - I_{10}$ and K_{10} share identical entries,

$$\therefore K_{10} = J_{10} - I_{10}$$

(b) If three Peterson graphs P, Q, R tile K_{10}, this means that you can lay down three Peterson graphs on K_{10} in such a way that vertices go to vertices and each individual edge of K_{10} lies under an edge of exactly one of the three Peterson graphs. Thus, since all edges and vertrcies are covered by these three Peterson graphs when we add the adjacency matrix of each corresponding Peterson graph, (A_P, A_Q, A_R), it will have the same entries as K_{10}, and since $K_{10} = J_{10} - I_{10}$

$$\therefore A_P + A_Q + A_R = K_{10} = J_{10} - I_{10}$$

Question assigned to the following page: 4

(c) Given in the problem statement the adjacency matrix of a Peterson graph with multiplicity 5. The null space of $A_p - I_{10}$ is the same as the eigenspace of the eigenvalue 1 ($\text{null}(A_p - 1 I_{10}) = E_1(A_p)$). By proposition 5.1.7 each eigenvalue of a symmetric matrix $A_p \in \mathbb{R}^{10 \times 10}$ has $AM(\lambda) = GM(\lambda)$, since all adjacency matrices are symmetric, $\dim(E_1(A_p)) = \text{nullity}(A_p - I_{10}) = 5$

$\therefore$ $A_p - I_{10}$ has a 5 dimensional nullspace

(d) Since $\mathbb{1} = [1, 1, \cdots, 1] \in \mathbb{R}^{10}$, and $\text{Row}(A_p - I_{10})$ is $\{\vec{y}^T (A_p - I_{10}) : \vec{y}\ \mathbb{R}^{10}\}$ consider $\mathbb{1}^T (A_p - I_{10}) =$

$\begin{bmatrix} \text{sum of column 1 in } A_p - I_{10} & \cdots & \text{sum of column 10 in } A_p - I_{10} \end{bmatrix}$ thus it equals $2\mathbb{1}^t$.

so $\mathbb{1}^T = (\frac{1}{2}\mathbb{1})^T (A_p - I_{10})$, this means that $\mathbb{1}$ can be written as a linear combination of the rows of $A_p - I_{10}$ and $\therefore$ $\mathbb{1} \in \text{Row}(A_p - I_{10})$. By proposition 4.1.7 $\text{Null}(A_p - I_{10}) \perp \text{Row}(A_p - I_{10})$, since $\text{span}\{\mathbb{1}\}^\perp$ is the orthogonal complement of $\text{Row}(A_p - I_{10})$

$\therefore$ $\text{Null}(A_p - I_{10}) \subseteq \text{span}\{\mathbb{1}\}^\perp$

(e) By propisition 4.2.6, $\dim(V) + \dim(V^\perp) = n$
if $V = \text{span}\{\mathbb{1}\}$ then $\dim(V) = 1$, also $n = 10$

$\therefore$ $\dim(V) + \dim(V^\perp) = 10$

$1 + \dim(V^\perp) = 10$

$$\boxed{\dim(V^\perp) = \dim(\text{span}\{\mathbb{1}\}^\perp) = 9}$$

Question assigned to the following page: 4

(f) i. $\text{Null}(A_P - I_{10}) \subseteq \text{span}\{\mathbb{1}\}^{\perp}$: both 5 dimensional subspaces

$\text{Null}(A_Q - I_{10}) \subseteq \text{span}\{\mathbb{1}\}^{\perp}$

If $S_1 \cap S_2 = \{0\}$ then $B_{S_1} \cup B_{S_2}$ is linearly independet, equivalently If $B_{S_1} \cup B_{S_2}$ are linearly independent then $S_1 \cap S_2 \neq \vec{0}$. Since $\text{Null}(A_P - I_{10}) + \text{Null}(A_Q - I_{10}) \geq \dim \text{span}\{\mathbb{1}\}^{\perp}$ $(10 > 9)$ there must exist some dependence between the bases for each subspace and thus their intersection must contain a non-zero vector $\vec{w}$.

ii. Since $\vec{w} \in \text{null}(A_P - I_{10})$ and $\vec{w} \in \text{null}(A_Q - I_{10})$ and $\mathbb{1} \in \text{row}(A_P - I_{10})$ and $\mathbb{1} \in \text{row}(A_Q - I_{10})$ and the the fact that by proposition 4.1.7 that $\text{null}(A) \perp \text{Row}(A)$

$\therefore \mathbb{1}^T \vec{w} = \vec{0}$ (since they are orthogonal)

(g) $A_P \vec{w}$ and $A_Q \vec{w}$ equal $\vec{w}$ since $\lambda = 1$ for A_R, A_P, A_Q, thus $\vec{w}$ is an eigenvector. $\vec{w} \perp J_{10}$ since J_{10} is composed of 10 $\mathbb{1}$ vectors thus

$$A_R \vec{w} = (J_{10} - I_{10} - A_P - A_Q)\vec{w}$$

$$A_R \vec{w} = J_{10}\vec{w} - I_{10}\vec{w} - A_P\vec{w} - A_Q\vec{w}$$

$$A_R \vec{w} = 0 - \vec{w} - \vec{w} - \vec{w}$$

$$A_R \vec{w} = -3\vec{w}$$

$\therefore$ -3 is an eigenvalue of A_R

(h) In part (g) we concluded that -3 is an eigenvalue of A_R with eigenvector $\vec{w}$, BUT the eigenvalue associated with $\vec{w}$ is actually 1. With this said, and from the problem description -3 is NOT an eigenvalue of A_R. Therefore, K_{10} can NOT be tiled by 3 Peterson graphs.