Midterm

Graded

Student

Jaiden Rain Atterbury

Total Points

35 / 40 pts

Question 1

(no title)

13 / 15 pts

part 2

- ✔ **− 1 pt** If you think that AM(0)=GM(0), you need to explain why

part 5

- ✔ **− 1 pt** unclear /incorrect justification

1. the dimension of Null(A) is GM(0)
2. do you mean a basis with three vectors ?
3. A is symmetric, so it is diagonalizable

Question 2

(no title)

9 / 10 pts

- ✔ **− 0 pts** Part 1 is correct
- ✔ **− 1 pt** Minor error in part 2 (looks like an algebra mistake).

Question 3

(no title)

13 / 15 pts

part 3

- ✔ **− 1 pt** no clear and complete proof that your condition works
- ✔ **− 1 pt** A does not need to be square

4. This would imply m=n. Can you give a condition that works for a matrix that is not square ?
5. Then the rank could not be n
6. why ?

Spring 2022 Math 318 Midterm Exam

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- Please use the same name that appears in Canvas.
- IMPORTANT: Your exam will be scanned: DO NOT write within 1 cm of the edge. Make sure your writing is clear and dark enough.
- Write your NAME (first, last) on top of every non blank page of this exam.
- If you run out of space, continue your work on the back of the previous page and indicate clearly on the problem page that you have done so.
- Unless stated otherwise, you **MUST** justify your answers and show your work.
- Your work needs to be neat and legible.

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Problem 1. For this problem you may want to recall that the trace of a square matrix is the sum of the diagonal elements of the matrix, and it is also the sum of the matrix's eigenvalues. Consider the matrix:

$$A = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}$$

1. What is the rank of A ? No justification needed.

rank(A) = 1

2. Argue that 0 is an eigenvalue of A and that AM(0)=3.

0 is an eigenvalue of A since A is not invertible, since it is rank 1 and thus not 1-to-1 or onto. AM(0) = 3 because Null(A) has 3 elements in it.

3. A has eigenvalues 0,0,0, λ_1 . Find λ_1 . Explain your reasoning.

Since A is positive markov the dominant eigenvalue must be 1, ALSO, trace(A) = 1 and the sum of $0+0+0+\lambda_1 = \lambda_1$ which must equal 1.

4. Explain why A is a projector (the matrix associated to a projection). Is it an orthogonal projector ? Find a basis for the space V it projects onto.

A is a projector since $A^2 = A$ (and furthermore has eigenvalues 0, 1 with $E_0(A) = \text{null}(A)$ and $E_1(A) = \text{col}(A)$. A is an orthogonal projection since $A = A^T$. Basis for V: $V = \text{span}\left\{\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}\right\}$.

5. Is A similar to $B = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$? Justify your answer.

A and B are NOT similar since A is not diagonalizable, furthermore GM and AM aren't the same for both matrices meaning they don't have the same characteristic polynomial and are thus not similar.

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Problem 2. Consider the sequence defined by:

$$G_0 = 0,\ G_1 = 0,\ G_2 = 1,\ G_{k+2} = 6\,G_{k+1} - 11\,G_k + 6\,G_{k-1} \text{ for } k+2 \geq 3$$

1. Find a matrix A such that

$$\underbrace{\begin{bmatrix} G_{k+2} \\ G_{k+1} \\ G_k \end{bmatrix}}_{w_{k+1}} = A \underbrace{\begin{bmatrix} G_{k+1} \\ G_k \\ G_{k-1} \end{bmatrix}}_{w_k}.$$

$$A = \begin{bmatrix} 6 & -11 & 6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

2. Find an explicit formula for G_k. You can use the fact that the matrix A has eigenvalues 1,2,3 with eigenspaces

$$E_1 = \text{ span } (u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}),\ E_2 = \text{span}(u_2 = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}),\ E_3 = \text{span}(u_3 = \begin{bmatrix} 9 \\ 3 \\ 1 \end{bmatrix})$$

and that $= \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -1 \\ 0.5 \end{bmatrix}$. Show your work.

$$w_k = U \Lambda^k U^{-1} w_0 \qquad A^k = \begin{bmatrix} 1 & & \\ & 2^k & \\ & & 3^k \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 4 & 9 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

$$U^{-1} w_0 = \begin{bmatrix} 1/2 \\ -1 \\ 1/2 \end{bmatrix}$$

$$\begin{bmatrix} G_{k+1} \\ G_k \\ G_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & 4 & 9 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix} \begin{bmatrix} 1/2 \\ -1 \\ 1/2 \end{bmatrix}$$

$$\begin{bmatrix} G_{k+1} \\ G_k \\ G_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & 4 & 9 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ -2^k \\ \frac{1}{2}3^k \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} -4 \cdot 2^k \\ -2^{k+1} \\ -2^k \end{bmatrix} + \begin{bmatrix} 9/2 \cdot 3^k \\ 3/2 \cdot 3^k \\ \frac{1}{2} \cdot 3^k \end{bmatrix} = \begin{bmatrix} \frac{1}{2} - 4 \cdot 2^k + \frac{9}{2} \cdot 3^k \\ \frac{1}{2} - 2^{k+1} + \frac{3}{2} \cdot 3^k \\ \frac{1}{2} - 2^k + \frac{1}{2} 3^k \end{bmatrix}$$

$$\boxed{G_k = \frac{1}{2} - 2^{k+1} + \frac{3}{2} \cdot 3^k}$$

NAME (First,Last) :

Problem 3. Let $V = \text{span}\left(\begin{bmatrix}1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right)$

1. Find two different subspaces of R^4, W_1 and W_2, neither of which is a subspace of the other, and both of which are perpendicular to V, or explain why this is not possible.

$\boxed{W_1 = \text{span}\left\{\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}\right\}}$ $\boxed{W_2 = \left\{\begin{bmatrix}0\\0\\-1\\1\end{bmatrix}\right\}}$

W_1 and W_2 are linearly independent and thus aren't a subspace of each other.

$\begin{bmatrix}-1\\1\\0\\0\end{bmatrix} \cdot V = 0$ $\begin{bmatrix}0\\0\\-1\\1\end{bmatrix} \cdot V = 0$

2. Find a basis for $V^{\perp}$. Show your work (if you do some calculations to find the basis) or justify why your basis works (if you think you can easily guess a basis without calculations).

$V = \begin{bmatrix}1 & 1 & 1 & 1 & 0\\ 0 & 0 & 1 & 1 & 0\end{bmatrix}$ $x_2 = s_1$, $y_4 = s_2$, $x_3 = -s_2$, $x_1 = -s_1$

$s_1 \begin{bmatrix}-1\\1\\0\\0\end{bmatrix} + s_2 \begin{bmatrix}0\\0\\-1\\1\end{bmatrix}$

$\boxed{V^{\perp} = \text{span}\left\{\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\-1\\1\end{bmatrix}\right\}}$

3. Recall that for a matrix $A \in \mathbb{R}^{m\times n}$ of rank n, $A^T A$ is invertible. Find (1) a matrix $A \in \mathbb{R}^{m\times n}$ that has rank n and such that $A\,A^T$ is not invertible. Find (2) a condition on the rank of $A \in \mathbb{R}^{m\times n}$ that assures $A\,A^T$ is invertible and prove that if A satisfies your condition then $A\,A^T$ is invertible. (3)

(1) $A \in \mathbb{R}^{m\times n}$ rank n not invertible : $A = \begin{bmatrix}1&0\\0&0\\0&1\end{bmatrix}$ $A^T = \begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}$ $AA^T = \begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}$

which is NOT invertible.

(2) CONDITION: If $A \in \mathbb{R}^{m\times n}$ and $\text{rank}(A) = n \geq m$ then $A\,A^T$ is invertible. $(n \geq 1)$

(3) Proof: if $\text{rank}(A) = n$ and $n \geq m$ this implies that there are more columns than rows and there is at least one pivot, thus if we multiply A by A^T we will get a square matrix $(n\times n)$ that is full rank, and thus invertible.

4