Homework 7

● Graded

Student

Jaiden Rain Atterbury

Total Points

39 / 40 pts

Question 1

Question 1

10 / 10 pts

✔ **– 0 pts** correct

Question 2

Question 2

9 / 10 pts

✔ **– 1 pt** some part of answer (usually (a) or (d)) treats derivative as a map $\mathbb{R}_{\leq d} \to \mathbb{R}_{\leq d}$

Question 3

Question 3

10 / 10 pts

✔ **– 0 pts** correct

Question 4

Question 4

10 / 10 pts

✔ **– 0 pts** Complete

Question assigned to the following page: 1

Jaiden Atterbury 05/17/2022 Homework #7

(1) Graph Connectivity

(a) $D_G = \begin{bmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 2 & & & & \\ & & & 2 & & & \\ & & & & 2 & & \\ & & & & & 1 & \\ & & & & & & 1 \end{bmatrix}$ $A_G = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

$L_G = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} = D_G - A_G$

(b) $x^T L_H x = [x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7]\, L_H \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix}$

$[x_1\ x_2] \begin{bmatrix} +1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $[x_3\ x_4\ x_5] \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix}$ $[x_6\ x_7] \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_6 \\ x_7 \end{bmatrix}$

$[x_1\ x_2] \begin{bmatrix} x_1 - x_2 \\ -x_1 + x_2 \end{bmatrix}$ $[x_3\ x_4\ x_5] \begin{bmatrix} 2x_3 - x_4 - x_5 \\ -x_3 + 2x_4 - x_5 \\ -x_3 - x_4 + 2x_5 \end{bmatrix}$ $[x_6\ x_7] \begin{bmatrix} x_6 - x_7 \\ -x_6 + x_7 \end{bmatrix}$

$x_1^2 - 2x_1x_2 + x_2^2$ $2x_3^2 + 2x_4^2 + 2x_5^2 - 2x_3x_4 - 2x_3x_5 - 2x_4x_5$ $x_6^2 - 2x_6x_7 + x_7^2$

$(x_1 - x_2)^2$ $(x_3 - x_4)^2 + (x_3 - x_5)^2 + (x_4 - x_5)^2$ $(x_6 - x_7)^2$

$\therefore\ x^T L_H x =$

$\boxed{(x_1 - x_2)^2 + (x_3 - x_4)^2 + (x_3 - x_5)^2 + (x_4 - x_5)^2 + (x_6 - x_7)^2}$

Question assigned to the following page: 1

(c) From a theorem in the lecture notes (lesson 17), if G has K connected components $\lambda_H = 0$ has multiplicity K in L_G. Since L_G is symmetric $GM(0) = AM(0)$.

$\therefore$ L_G has 3 connected components so

$$E_0 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(d) Based on the above, the first eigenvalue that is going to be positive is eigenvalue 4. This comes from theorem 6.3.6 which states the eigenvalues of L_G are $0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$, since $\lambda_1, \lambda_2, \lambda_3 = 0$ represent each connected part of G, λ_4 will be the first non-zero (positive) eigenvalue since it doesn't correspond to a new connected piece and must be $\geq \lambda_3$, but since $\lambda_4 \neq 0$ it shows that $\lambda_4 > \lambda_3 = 0$.

(e) The relationship between the AM(0) and the number of connected components is simple. If a graph G has K different connected components then $AM(0) = K$. This means that for each connected piece of a graph there is an eigenvalue of 0 that corresponds to it.

Question assigned to the following page: 2

(2) Differentiation is a linear transformation

$$f(x) = a_d x^d + a_{d-1} x^{d-1} + \dots + a_1 x + a_0$$

$$\vec{f} = [a_0, a_1, \dots, a_{d-1}, a_d]^T$$

(a) $f'(x) = d\, a_d x^{d-1} + (d-1) a_{d-1} x^{d-2} + \dots + a_1 + 0$

$\vec{f}' = [0, a_1, \dots, (d-1)a_{d-1}, d a_d]^T$

(b) $D_d : \mathbb{R}[x]_{\le d} \to \mathbb{R}[x]_{\le d-1}$ $\quad f(x) \mapsto f'(x)$

1. $D_d(k f(x)) = k D_d(f(x))$?

$(k f(x))' = \frac{d}{dx}(k f(x)) = k \frac{d}{dx}(f(x)) = k f'(x)$ ✓ yes from calculus.

2. $D_d(f(x) + g(x)) = D_d(f(x)) + D_d(g(x))$

$(f(x) + g(x))' = \frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x)) = f'(x) + g'(x)$ ✓ yes from calculus.

$\therefore$ D_d is a linear transformation.

(c) monomial basis of $\mathbb{R}[x]_{\le d}$

$\mathcal{B}_d = \{1, x, x^2, \dots, x^d\}$

$D_d(1) = \frac{d}{dx}(1) = 0 \to \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$D_d(x) = \frac{d}{dx}(x) = 1 \to \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$D_d(x^2) = \frac{d}{dx}(x^2) = 2x \to \begin{bmatrix} 0 \\ 2 \\ \vdots \\ 0 \end{bmatrix}$

$\vdots$

$D_d(x^d) = \frac{d}{dx}(x^d) = d x^{d-1} \to \begin{bmatrix} 0 \\ 0 \\ \vdots \\ d \end{bmatrix}$

Question assigned to the following page: 2

(d) $\underset{n}{\mathbb{R}^{d+1}} \to \underset{m}{\mathbb{R}^{d}}$

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & d \end{bmatrix}$$

(e) $$M_5 = \begin{array}{cc} & 6 \\ 5 & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix} \end{array}$$

$\mathbb{R}^6 \to \mathbb{R}^5$

5×6

(f) $S(x) = 5x^5 - 19x^3 + 24x - 3 \in \mathbb{R}^6 / \mathbb{R}[x]_5$

$S'(x) = 25x^4 - 57x^2 + 24 \in \mathbb{R}^5 / \mathbb{R}[x]_4$

$$\begin{array}{cc} & 6 \\ 5 & \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{bmatrix} \end{array} \begin{bmatrix} -3 \\ 24 \\ 0 \\ -19 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 24 \\ 0 \\ -57 \\ 0 \\ 25 \end{bmatrix}$$

$\mathbb{R}^6 \to \mathbb{R}^5$

$\mathbb{R}[x]_{\leq 5} \to \mathbb{R}[x]_{\leq 4}$

Question assigned to the following page: 3

(3) Polynomial Interpolation

(a) $g(-2) = -3 \quad g(0)=1 \quad g(1)=0 \quad g(3)=22$

$$\begin{bmatrix} 1 & -2 & 4 & -8 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 22 \end{bmatrix}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 4 & -8 & -3 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 3 & 9 & 27 & 22 \end{array}\right] \underset{\substack{-R_1+R_2 \to R_2 \\ -R_1+R_3 \to R_3 \\ -R_1+R_4 \to R_4}}{\sim} \left[\begin{array}{cccc|c} 1 & -2 & 4 & -8 & -3 \\ 0 & 2 & -4 & 8 & 4 \\ 0 & 3 & -3 & 9 & 3 \\ 0 & 5 & 5 & 35 & 25 \end{array}\right] \underset{\substack{-\frac{3}{2}R_2+R_3 \to R_3 \\ -\frac{5}{2}R_2+R_4 \to R_4}}{\sim} \left[\begin{array}{cccc|c} 1 & -2 & 4 & -8 & -3 \\ 0 & 2 & -4 & 8 & 4 \\ 0 & 0 & 3 & -3 & -3 \\ 0 & 0 & 15 & 15 & 15 \end{array}\right]$$

$$\underset{-5R_3+R_4 = R_4}{\sim} \left[\begin{array}{cccc|c} 1 & -2 & 4 & -8 & -3 \\ 0 & 2 & -4 & 8 & 4 \\ 0 & 0 & 3 & -3 & -3 \\ 0 & 0 & 0 & 30 & 30 \end{array}\right] \quad \begin{array}{l} x_4 = 1 \\ x_3 = 0 \\ x_2 = -2 \\ x_1 = 1 \end{array}$$

$\therefore \boxed{g(x) = x^3 - 2x + 1}$

(b) $M = \begin{bmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^d \\ 1 & t_2 & t_2^2 & \cdots & t_2^d \\ 1 & t_3 & t_3^2 & \cdots & t_3^d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_m & t_m^2 & \cdots & t_m^d \end{bmatrix} \quad f = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix} \quad g = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_d \end{bmatrix}$

$f(x) = a_d x^d + \cdots + a_2 x^2 + a_1 x + a_0$

$g(x) = b_d x^d + \cdots + b_2 x^2 + b_1 x + b_0$

(i) $f(x), g(x) \in \mathbb{R}[x]_{\leq d}$

$\text{eval}(f(x)) = Mf$

$\text{eval}(g(x)) = Mg$

$\text{eval}(f(x) + g(x)) = M(f+g) = Mf + Mg = \text{eval}(f(x)) + \text{eval}(g(x))$ ✓

Question assigned to the following page: 3

(ii) $f(x) \in \mathbb{R}[x]_{\leq d}$

$\gamma \in \mathbb{R}$

$\text{eval}(f(x)) = Mf$

$\text{eval}(\gamma f(x)) = M(\gamma f) = \gamma(Mf) = \gamma\,\text{eval}(f(x))$ ✓

$\therefore$ eval is a linear transformation.

(c) 3×3 Vandermonde with t_1, t_2, t_3

$$A = \begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{bmatrix}$$

$$\det(A) = 1 \cdot \det\begin{bmatrix} t_2 & t_2^2 \\ t_3 & t_3^2 \end{bmatrix} - 1 \cdot \det\begin{bmatrix} t_1 & t_1^2 \\ t_3 & t_3^2 \end{bmatrix} + 1 \cdot \det\begin{bmatrix} t_1 & t_1^2 \\ t_2 & t_2^2 \end{bmatrix}$$

$t_3^2 t_2 - t_2^2 t_3 - (t_3^2 t_1 - t_1^2 t_3) + t_2^2 t_1 - t_1^2 t_2 =$

$t_3^2 t_2 - t_2^2 t_3 - t_3^2 t_1 + t_1^2 t_3 + t_2^2 t_1 - t_1^2 t_2 =$

$(t_3 - t_2)(t_3 - t_1)(t_2 - t_1) =$

$\boxed{(-1)^3 (t_1 - t_2)(t_1 - t_3)(t_2 - t_3)}$

Question assigned to the following page: 4

(4) Spectral Clustering

Q1 Since we are now minimizing over a larger set the minimum value of Q will become smaller than when we were minimizing over all c_A's, therefore:

$$\mu = \min_{\substack{x \in \mathbb{R}^n \\ x \neq k\mathbb{1}}} Q(x) \leq \min_{A \subseteq V} Q(c_A) \leq \varphi_G$$

thus $\mu \leq \varphi_G$

Q2 $Q(x) = Q(x + t\mathbb{1})$? $\lambda_1 = 0$ with eigenvector $\mathbb{1}$

$$Q(x+t\mathbb{1}) = n\frac{(x+t\mathbb{1})^T L_G (x+t\mathbb{1})}{(x+t\mathbb{1})^T L_{K_n} (x+t\mathbb{1})}$$

$\rightarrow x^T L_G x + \underbrace{t x^T L_G \mathbb{1}}_{0} + \underbrace{t\mathbb{1}^T L_G x}_{0} + \underbrace{t^2 \mathbb{1}^T L_G \mathbb{1}}_{0} = x^T L_G x$

$\rightarrow x^T L_{K_n} x + \underbrace{t x^T L_{K_n} \mathbb{1}}_{0} + \underbrace{t\mathbb{1}^T L_{K_n} x}_{0} + \underbrace{t^2 \mathbb{1}^T L_{K_n} \mathbb{1}}_{0} = x^T L_{K_n} x$

$$= n\frac{x^T L_G x}{x^T L_{K_n} x} = Q(x) \checkmark$$

Q3 Let $V = \text{span}\{\mathbb{1}\}$

$x \in \mathbb{R}^n \rightarrow x = \underbrace{t\mathbb{1}}_{\text{proj}_V x \,\in\, \text{span}(\mathbb{1})} + \underbrace{e}_{e \perp \mathbb{1} / \in \text{span}\{\mathbb{1}\}^T}$

$Q(x) = Q(e + t\mathbb{1})$

$Q(x) = Q(e)$

$\therefore x \in \text{span}\{\mathbb{1}\}^T$

$x \perp \mathbb{1} \rightarrow \mathbb{1}^T x = 0$

thus $\mu = \min_{\substack{x \in \mathbb{R}^n \\ x \neq k\mathbb{1} \\ x \perp \mathbb{1}}} Q(x)$

Question assigned to the following page: 4

Q_4 (i) $L_G = D_G - A_G \rightarrow L_{K_n} = D_{K_n} - A_{K_n}$

If K_n is a complete graph of n components, the degree of vertex i, d_i, is $n-1$ for every vertex i:

$$D_{K_n} = \begin{bmatrix} n-1 & 0 & 0 & \cdots & 0 \\ 0 & n-1 & 0 & \cdots & 0 \\ 0 & 0 & n-1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n-1 \end{bmatrix}$$

and since K_n is complete the adjacency matrix will be all ones except on the diagonal:

$$A_{K_n} = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}$$

and thus the Laplacian matrix of K_n is:

$$L_{K_n} = \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & -1 \end{bmatrix}$$

$$nI_n = \begin{bmatrix} n & 0 & 0 & \cdots & 0 \\ 0 & n & 0 & \cdots & 0 \\ 0 & 0 & n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad J_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

$$nI_n - J_n = \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -1 & n-1 & -1 & \cdots & -1 \\ -1 & -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & n-1 \end{bmatrix} = L_{K_n} \checkmark$$

Question assigned to the following page: 4

(ii) $\mathbb{1}^T = [1\ 1\ 1\ \cdots\ 1]$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_4 \end{bmatrix}$

$\mathbb{1}^T x = x_1 + x_2 + x_3 + \cdots + x_n = 0$

$$J_n = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} + x_n \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} x_1+x_2+x_3+\cdots+x_n \\ x_1+x_2+x_3+\cdots+x_n \\ x_1+x_2+x_3+\cdots+x_n \\ \vdots \\ x_1+x_2+x_3+\cdots+x_n \end{bmatrix} = \vec{0}$$

(iii) $Q(x)$ denominator $= x^T L_{K_n} x \rightarrow x^T(nI_n - J_n)x = x^T(nI_n)x - \underbrace{x^T(J_n)x}_{0} = n x^T I_n x$

$= n x^T x$

$= n\|x\|^2$

(iv) for any matrix M $\quad (kx)^T M (kx) = k^2 x^T M x$

thus $Q(\alpha x) = n \cdot \dfrac{(\alpha x)^T L_G (\alpha x)}{(\alpha x)^T L_{K_n} (\alpha x)} = n \cdot \dfrac{\alpha^2}{\alpha^2} \dfrac{x^T L_G x}{x^T L_{K_n} x} = Q(x)$

$\therefore Q(x) = Q(\alpha x)$ for any non-zero $\alpha \in \mathbb{R}$.

(v) Since $Q(x) = Q(\alpha x)$, Q is determined by direction and not length, thus we can take x as a unit vector.

$\hookrightarrow n\|x\|^2$ when $\|x\| = 1$ is n

$\therefore x^T L_{K_n} x = n$

$\therefore Q(x) = \frac{n}{n} \cdot x^T L_G x = x^T L_G x$

thus $\mu = \min\limits_{\substack{x \in \mathbb{R}^n \\ x \neq k\mathbb{1} \\ \|x\| = 1 \\ x \perp \mathbb{1}}} x^T L_G x$

Question assigned to the following page: 4

Q5(i) L_G is a connected graph with eigenvalues:

$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$

with a set of corresponding orthonormal eigenvectors

$\{v_1, v_2, v_3 \cdots v_n\}$

$\vec{x} \in span\{\mathbb{1}\}^\perp$ since $\vec{x} \perp \mathbb{1}$

$dim(span(\mathbb{1})^\perp) = n-1$ since $dim(span(\mathbb{1})) = 1$ and by proposition 4.2.6 $dim(V) + dim(V^\perp) = n$

since $dim(V) = 1$

$dim(V^\perp) = n - dim(V) = n-1$

The set $\{v_2 \cdots v_n\} \subseteq span\{\mathbb{1}\}^\perp$

since $v_1 = \mathbb{1}$ and eigenvectors of symmetric matrices are mutually orthogonal by proposition 5.1.6. Furthermore, the dimension of $\{v_2 \cdots v_n\}$ is $n-1$ so

$\{v_2 \cdots v_n\} = span\{\mathbb{1}\}^\perp$

Since $\vec{x} \in span\{\mathbb{1}\}^\perp$ it can be written as a linear combination of the vectors in the set:

$\therefore \vec{x} = \alpha_2 v_2 + \cdots + \alpha_n v_n$

(ii) $x = \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n = \Sigma_{i=2}^n \alpha_i v_i$

since $||x|| = 1$:

$\cancel{1} = ||x^T x|| = (\alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n)^T (\alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n)$

$= \alpha_2^2 + 0 + \cdots + 0$

$0 + \alpha_3^2 + \cdots + 0$ → since $v_i^T v_i = 1$ ($||v_i|| = 1$ since $\{v_1 \cdots v_n\}$ is an orthonormal set)

$0 + 0 + \cdots + \alpha_n^2$ $\quad v_i^T v_j = 0$ $i \neq j$

$= \alpha_2^2 + \alpha_3^2 + \cdots + \alpha_n^2 = \Sigma_{i=2}^n \alpha_i^2 = 1$

Question assigned to the following page: 4

(iii) $(\alpha_2 v_2 + \alpha_3 v_3 + \cdots \alpha_n v_n)^T L_G (\alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n)$

$= \alpha_2 v_2^T L_G \alpha_2 v_2 + \alpha_2 v_2^T L_G \alpha_3 v_3 + \cdots + \alpha_2 v_2^T L_G \alpha_n v_n$

$= \alpha_2^2 \lambda_2 \underbrace{v_2^T v_2}_{1} + \alpha_2 \alpha_3 \lambda_3 \underbrace{v_2^T v_3}_{0} + \cdots + \alpha_2 \alpha_n \lambda_n \underbrace{v_2^T v_n}_{0}$

$= \alpha_2^2 \lambda_2 + 0 + \cdots + 0$

repeating this calculation for $\alpha_3 v_3$ and $\alpha_n v_n$ we get

$\alpha_2^2 \lambda_2 + \alpha_3^2 \lambda_3 + \cdots + \alpha_n^2 \lambda_n$ ✓

(iv) since $\lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_n$

and $x^T L_G x = \alpha_2^2 \lambda_2 + \alpha_3^2 \lambda_3 + \cdots + \alpha_n^2 \lambda_n$

we can say:

$$\alpha_2^2 \lambda_2 + \alpha_3^2 \lambda_3 + \cdots + \alpha_n^2 \lambda_n \geq \alpha_2^2 \lambda_2 + \alpha_3^2 \lambda_2 + \cdots + \alpha_n^2 \lambda_2$$

$$\geq \left(\sum_{i=2}^{n} (\alpha_i^2)\right) \lambda_2$$

$\therefore \min_{\substack{x \in \mathbb{R}^n \\ \|x\|=1 \\ x \perp \mathbb{1}}} x^T L_G x = \mu \geq \lambda_2$

(v) $\|v_2\| = 1$

$L_G v_2 = \lambda_2 v_2$

$\therefore v_2^T L_G v_2 = v_2^T \lambda_2 v_2 = \lambda_2 \|v_2\|^2 = \lambda_2$

thus $\mu = \min_{\substack{x \in \mathbb{R}^n \\ \|x\|=1 \\ x \perp \mathbb{1}}} x^T L_G x \leq \lambda_2$

(vi) since $\lambda_2 \leq \mu \leq \lambda_2$

$\mu = \lambda_2$