Final

Graded

Student

Jaiden Rain Atterbury

Total Points

65 / 65 pts

Question 1

Problem 1

12 / 12 pts

- ✔ **+ 3 pts** (1) Correct: $x_1^2 + 2x_2^2 + 3x_3^2 - 2x_1x_3$
- ✔ **+ 3 pts** (2) A is PD (or "both") with reasoning.
- ✔ **+ 3 pts** (i) "True" with reasoning.
- ✔ **+ 3 pts** (ii) "True" with reasoning.
- **+ 0 pts** *(1) No relevant work.
- **+ 2 pts** *(1) Minor error(s)
- **+ 2 pts** *(2) Argued that A is PSD, but did not conclude that A is actually PD.
- **+ 1 pt** *(2) Incorrect/incomplete, but some work is shown.
- **+ 1 pt** *(2) More explanation needed.
- **+ 1 pt** *(i) More explanation needed.
- **+ 1 pt** *(ii) Incorrect or incomplete answer but good attempt at explanation.
- **+ 0 pts** (ii) Incorrect answer with no explanation.

Question 2

Problem 2

10 / 10 pts

- ✔ **+ 5 pts** (a) Correct drawing (explanation not necessary, but helps for partial credit if incorrect.)
- ✔ **+ 5 pts** (b) Correct ("not possible") with complete explanation.
- **+ 3 pts** *(b) Correct (or correct-ish), but incomplete explanation.
- **+ 1 pt** *(b) Incomplete or incorrect answer with little work shown.
- **+ 0 pts** Question not answered

Question 3

Problem 3

10 / 10 pts

✔ **– 0 pts** Correct

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rows/columns of V/U do not have unit lenght

– 2 pts the rows/columns of V/U form an orthogonal set , but not orthonormal.

– 0 pts If you write A as $U\Sigma V^T$ where U has for columns unit vectors in the directions of the semiaxes, Σ has 5,3,2 along the diagonal , the rows of V form an orthogonal but not orthonormal set, then if, say the first row of one of your V is r_1^T and has length l, and the first column of U is u_1 then $U\Sigma V^T \frac{1}{l} r_1 = l * 5u_1$, which is outside of the desired ellipsoid.

Σ not correct

– 2 pts one of your matrices maps the unit sphere inside the given ellipsoid, not onto

– 2 pts The lenghts of the semiaxes are not correct

– 4 pts Both your matrices send unit vectors outside the given ellipsoid

– 8 pts No work

– 6 pts I could not understand the rational for writing the two matrices. Neither seems to work.

– 4 pts one matrix missing or not correct

– 2 pts this matrix maps the unit shpere outside the given ellipsoid

– 4 pts This matrix does not seem to work: check its SVD

– 2 pts you should swap your U and V

– 5 pts good start, several details not correct or missing

– 8 pts little correct work

– 2 pts Your matrices should have rank 3

– 2 pts the rows of V should be pairwise orthogonal

– 6 pts Good start, several details not correct or missing

– 7 pts little correct work

– 2 pts your matrix V is not orthogonal

– 6 pts Click here to replace this description.

Question 4

Problem 4

12 / 12 pts

✔ **− 0 pts** Correct

Checking that S is a subspace of $R^{3\times3}$

- **− 1 pt** You have shown that the 3x3 symmetric matrices form a subspace of the space of all 3x3 matrices, what about having 0s along the diagonal ?
- **− 1 pt** You also have to check kA and A + B are symmetric
- **− 1 pt** If A and B are in S, is A+B in S?
- **− 1 pt** Yhe second condition involves scalar multiplication, not matrix multiplication
- **− 0.5 pts** Minor mistake
- **− 1 pt** Give a general argument, not examples
- **− 1 pt** You should add more details

Giving a basis for S

- **− 0 pts** Basis for S
- **− 4 pts** Your basis should be a list of matrices, or you should explain how you identify a matrix with a vector.
- **− 3 pts** Some matrices in your basis are not in S
- **− 3 pts** The matrices in your basis are linearly independent, but they do not Span S
- **− 4 pts** No correct basis given

Justification that B is a basis

- **− 0 pts** Can you show why B is linearly independent ?

- **− 9 pts** Little correct work
- **− 12 pts** No work

Question 5

Problem 5

12 / 12 pts

✔ **− 0 pts** Correct

− 1 pt part 1. Incorrect/incomplete justification

− 2 pts part 1: U is not a square matrix, so we do not have the full SVD

− 1 pt Part 2. A has 3 columns since the right singular vector are in R^3

− 2 pts part 2. Wrong dimensions

− 1 pt part 2 . A has 4 rows : u vectors are in R^4

− 1 pt part 3: instructions say to assume $\sigma_3 = 0$, so the rank is 2

− 2 pts part 3 . rank =2, number of non zero singular vectors

− 2 pts part 4 : wrong number of vectors in the basis

− 2 pts Part 4. the first two columns of U form a basis for col(A)

− 2 pts part 5 You need the first right singular vector

− 1 pt part 5: right singular vectors are rows of V, not columns

− 2 pts Part 6: the projection is $\vec{0}$

− 1 pt part 6: the projection is $\vec{0}$

Question 6

Problem 6

9 / 9 pts

✔ **+ 6 pts** (a) Correct (work not required to be shown) $\begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (transpose of this also works)

+ 5 pts *(a) Correct values, but elements of the matrix are in the wrong order.

+ 4 pts *(a) Correct transformations identified, but matrix is written incorrectly.

+ 3 pts *(a) Matrix A not written out correctly, but good work toward finding A is shown.

+ 1 pt *(a) Some work shown but question not answered.

✔ **+ 3 pts** (b) Correct; transformation is fourth derivative. Some form of work or explanation is expected.

+ 2 pts *(b) Correct answer but insufficient reasoning.

+ 2 pts *(b) Incorrect answer but thoughtful explanation.

+ 1 pt *(b) Incorrect answer with explanation.

+ 0 pts *(a) A incorrect without explanation.

+ 0 pts *(b) Insufficient/incorrect answer without explanation.

+ 0 pts *(b) not answered

Spring 2022 Math 318 Final Exam

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- Please use the same name that appears in Canvas.
- IMPORTANT: Your exam will be scanned: DO NOT write within 1 cm of the edge. Make sure your writing is clear and dark enough.
- Write your NAME (first, last) on top of every odd page of this exam.
- If you run out of space, continue your work on the back of the last page and indicate clearly on the problem page that you have done so.
- Unless stated otherwise, you MUST justify your answers and show your work.
- Your work needs to be neat and legible.

Problem 1.Let $A \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$

1. Write $x^T Ax$ as a quadratic polynomial $Q(x_1, x_2, x_3)$.

$x^T A x = x_1^2 + 2x_2^2 + 3x_3^2 - 2x_3 x_1$

2. Is A PSD (positive semi-definite) or PD (positive definite) ? Justify your answer.

since $x^T A x = x_1^2 - 2x_3 x_1 + x_3^2 + 2x_3^2 + 2x_2^2$
$= (x_1 - x_3)^2 + 2x_3^2 + 2x_2^2$ if $x \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ then
$x^T A x > 0$ and is thus PD (positive) definite.

Answer True or False, with justification:

i) If A PD then A^3 is PD.

If A is PD then all eigenvalues of A are positive,
$A^3 \vec{x} = \lambda^3 x$, since λ^3 are the eigenvalues of A^3
and the eigen values of A are positive, the eigenvalues
of A^3 must be positive.
$\therefore A^3$ is PD, true

ii) If A is PSD and B is PD, then A+B is PD.

If A is PSD $x^T A x \geq 0$ if B is PD $x^T B x > 0$

$x^T (A+B) x = \underbrace{x^T A x}_{\geq 0} + \underbrace{x^T B x}_{> 0}$

since $A \geq 0$ and $B > 0$ then A+B must be > 0
$\therefore$ A + B is PD, TRUE

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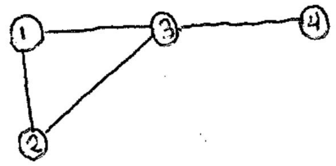
Problem 2. The matrix $L_G = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \end{array}$ is the Laplacian of some undirected graph G.

1. Draw G.



2. Find an eigenvector v of L_G for $\lambda = 0$ that is not a scalar multiple of $\mathbb{1}$ (the vector all of whose entries are equal to 1), or explain why this is not possible.

This is not possible since λ_2 of L_G is greater than zero since G is connected. Thus $\lambda = 0$ has AM and GM multiplicity of 1, and the only eigenvector is the $\mathbb{1}$ vector, corresponding to $\lambda = 0$.

↳ $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_4$

Problem 3. Find matrices A and B for two different linear transformations T_1 and T_2 from $\mathbb{R}^3$ to $\mathbb{R}^3$ that map the unit sphere S^2 in $\mathbb{R}^3$ onto the hyperellipse in $\mathbb{R}^3$ that has the following semiaxes:

1. semiaxis 1 has length 5 and it is in the direction of the vector $[1, 1, 1]^T$
2. semiaxis 2 has length 3 and it is in the direction of the vector $[1, -1, 0]^T$
3. semiaxis 3 has length 2 and it is in the direction of the vector $[1, 1, -2]^T$

$AV_i = \sigma_i u_i$

If you write A and B as products of other matrices, you do not need to multiply out.

$\det(A) = 30$

A is full rank

ignore ~~A and B are PD so their orthogonal diagonalization is the same as their SVD. Since r=3 and all singular values / eigenvalues of AA^T/A^TA are positive~~

$$A = \underbrace{\begin{bmatrix} \sqrt{3}/3 & \sqrt{2}/2 & \sqrt{5}/5 \\ \sqrt{3}/3 & -\sqrt{2}/2 & \sqrt{5}/5 \\ \sqrt{3}/3 & 0 & -2\sqrt{5}/5 \end{bmatrix}}_{U} * \overbrace{\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}}^{\Sigma} \times \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{V^T}$$

1. $\sigma_1 = 5$ $\quad x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \rightarrow u_1 = \begin{bmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix}$
2. $\sigma_2 = 3$ $\quad x_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \rightarrow u_2 = \begin{bmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \\ 0 \end{bmatrix}$
3. $\sigma_3 = 2$ $\quad x_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \rightarrow u_3 = \begin{bmatrix} \sqrt{5}/5 \\ \sqrt{5}/5 \\ -2\sqrt{5}/5 \end{bmatrix}$

↳ let $V^T = I_n$ then

$AV_i = \sigma_i u_i$

$A = \sigma_i u_i$

$$A = \begin{bmatrix} 5\sqrt{3}/3 & 3\sqrt{2}/2 & 2\sqrt{5}/5 \\ 5\sqrt{3}/3 & -3\sqrt{2}/2 & 2\sqrt{5}/5 \\ 5\sqrt{3}/3 & 0 & -4\sqrt{5}/5 \end{bmatrix}$$

~~$B = \begin{bmatrix} \sqrt{3}/3 & \sqrt{2}/2 & \sqrt{5}/5 \\ \sqrt{3}/3 & -\sqrt{2}/2 & \sqrt{5}/5 \\ \sqrt{3}/3 & 0 & -2\sqrt{5}/5 \end{bmatrix} * \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} * \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$~~

let $V^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$$B = \begin{bmatrix} 2\sqrt{5}/5 & 3\sqrt{2}/2 & 5\sqrt{3}/3 \\ 2\sqrt{5}/5 & -3\sqrt{2}/2 & 5\sqrt{3}/3 \\ -4\sqrt{5}/5 & 0 & 5\sqrt{3}/3 \end{bmatrix}$$

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Problem 4. Show that the set S of all 3x3 symmetric matrices with 0 's along the diagonal is a subspace of $\mathbb{R}^3$, the space of all 3x3 matrices. Find a basis B for S. Remember to justify why B is a basis.

1. The 3x3 0 matrix is in the set S since if $A=0$ $A=A^T$, plus the all zero matrix has zeros along the diagonal

2. If $A \in S$ and $r \in \mathbb{R}$ then $rA \in S$:

$$A = \begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix} \quad rA = \begin{bmatrix} 0 & ra & rb \\ ra & 0 & rc \\ rb & rc & 0 \end{bmatrix} \quad (rA)^T = \begin{bmatrix} 0 & ra & rb \\ ra & 0 & rc \\ rb & rc & 0 \end{bmatrix}$$

$\therefore$ $rA \in S$ since $(rA) = (rA)^T$ and rA has zeros along the diagonal

3. If $A, B \in S$ then $A+B \in S$;

$$A = \begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & d & e \\ d & 0 & f \\ e & f & 0 \end{bmatrix} \quad A+B = \begin{bmatrix} 0 & a+d & b+e \\ a+d & 0 & c+f \\ b+e & c+f & 0 \end{bmatrix} \quad (A+B)^T = \begin{bmatrix} 0 & a+d & b+e \\ a+d & 0 & c+f \\ b+e & c+f & 0 \end{bmatrix}$$

$\therefore$ $A+B \in S$ since $(A+B) = (A+B)^T$ and $A+B$ has zeros along the diagonal

Since S includes the zero matrix, and is closed under addition and scalar multiplication

$\therefore$ S is a subspace of $\mathbb{R}^{3\times 3}/\mathbb{R}^3$.

Basis: $B_S = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}$

since $\dim(B_S) = 3$ and $\dim(\mathbb{R}^{3\times 3}) = 3$

a basis for $\mathbb{R}^{3\times 3}$

$\therefore$ B_S spans $\mathbb{R}^{3\times 3}$

$$a\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}$$ which only equals $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

if $a, b, c = 0$

$\therefore$ B_S is linearly independent, and thus B_S is a basis

Problem 5. On input a certain matrix A, the command SVD(A) on a certain computer program, has produced the following singular value decomposition of A: (I have approximated to two decimal digits):

$$A = \begin{array}{c} m\times r \\ 4\times 3 \\ \begin{pmatrix} -0.91 & -0.18 & 0.34 \\ 0 & -0.37 & -0.5 \\ -0.37 & 0 & -0.78 \\ 0.18 & -0.91 & 0.13 \end{pmatrix} \\ \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \end{array} \begin{array}{c} r\times r \\ 3\times 3 \\ \begin{pmatrix} 6.71 & 0 & 0 \\ 0 & 3.87 & 0 \\ 0 & 0 & 1.76\times 10^{-16} \end{pmatrix} \end{array} \begin{array}{c} r\times n \\ 3\times 3 \\ \begin{pmatrix} -0.8 & 0.41 & 0.41 \\ 0 & -0.71 & 0.71 \\ -0.58 & -0.58 & -0.58 \end{pmatrix} \end{array} \begin{array}{c} \\ \\ v_1 \\ v_2 \\ v_3 \end{array}$$

Note: the program thinks that the third singular value $\sigma_3 = 1.76\times 10^{-16} \neq 0$ but we want to think as 1.6×10^{-16} as some approximation error to 0, so in answering the questions below assume $\sigma_3 = 0$.

1. Is the program giving us a full SVD ? Explain how you know.

   No, in the full SVD the U and V^T matrices are square and the Σ matrix is size $m\times n$ but by the command SVD(A) we see U is not square and Σ is not $m\times n$. Thus the program does not give the full SVD.

2. What are the dimensions of A ? Fill the blanks below. No justification needed.

   A has 4 rows.

   A has 3 columns.

   A is $\in \mathbb{R}^{4\times 3}$

3. What is the rank of A ? Explain how you know.

   Since A has 2 non-zero singular values (since $\sigma_3 = 0$) the rank of A is 2.

4. Find a basis for the column space of A. Write down numeric vectors.

   basis for Col(A) = span$\{u_1, u_2\}$ where $u_1 = \begin{bmatrix} -0.91 \\ 0 \\ -0.37 \\ 0.18 \end{bmatrix}$ $u_2 = \begin{bmatrix} -0.18 \\ -0.37 \\ 0 \\ -0.91 \end{bmatrix}$

5. The best fit 1 dimensional space to the rows of A is V_1=span(w). What is w ? (Write down a numeric vector).

   $w = v_1$ where $w = \begin{bmatrix} -0.8 \\ 0.41 \\ 0.41 \end{bmatrix}$

6. What is the projection of the second row of A on V_1 ? Explain how you calculated it.

   Proj_{V_1}(second row of A)

   $A_1 = \sigma_1 u_1 v_1^T = 6.71 \begin{bmatrix} -0.91 \\ -8.7 \\ 0.18 \end{bmatrix} [-0.8 \ \ 0.41 \ \ 0.41]$; second row = [0 0 0]

   Since the rows of A_1 are the rows of the projection of the rows of A onto V_1.

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Problem 6. Consider $T : \mathbb{R}[x]_{\leq 3} \to \mathbb{R}[x]_{\leq 3}$ defined by $T(p(x)) = p''(x)$ (the second derivative of p).

1. Find the matrix A of T.

$p(x) = a + bx + cx^2 + dx^3$

$p'(x) = b + 2c + 3dx^2$

$p''(x) = 2 + 6dx$

$$\underset{4\times 4}{A} = \left[T\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \;\; T\begin{bmatrix}0\\1\\0\\0\end{bmatrix} \;\; T\begin{bmatrix}0\\0\\1\\0\end{bmatrix} , \; T\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \right]$$

$T\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = 1 \qquad T(1) = f''(1) = 0 \to \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

$T\begin{bmatrix}0\\1\\0\\0\end{bmatrix} = x \qquad T(x) = f''(x) = 0 \to \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$

$T\begin{bmatrix}0\\0\\1\\0\end{bmatrix} = x^2 \qquad T(x) = f''(x^2) = 2 \to \begin{bmatrix}2\\0\\0\\0\end{bmatrix}$

$T\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = x^3 \qquad T(x) = f''(x^3) = 6x \to \begin{bmatrix}0\\6\\0\\0\end{bmatrix}$

$$\boxed{A = \begin{bmatrix}0&0&2&0\\0&0&0&6\\0&0&0&0\\0&0&0&0\end{bmatrix}}$$

2. A^2 is the matrix of a linear transformation $S : \mathbb{R}[x]_{\leq 3} \to \mathbb{R}[x]_{\leq 3}$. Calculating $S(p)$ corresponds to performing what calculus operation on the polynomial p ?

$A \in \mathbb{R}^{4\times 4}$

$$A = \begin{bmatrix}0&0&2&0\\0&0&0&6\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$A^2 = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

Calculating $S(p)$ is the same as calculating $p''''(x)$ on a degree 3 polynomial.