MATH 327 Homework 2

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Note: When I say something like Theorem/Definition 1.1.1, I mean, lecture 1 page 1 part 1 of the given definition or theorem. Furthermore, Homework 1.1.1 means homework 1, exercise 1, problem 1.

Exercise 1. Let y be a non-negative real number ($0 \leq y$). Prove that if $y \leq \frac{1}{n}$ for every $n \in \mathbb{N}$, then $y = 0$.

Proof: (Proof by contradiction) Let $y \in \mathbb{R}$ such that $0 \leq y$. Assume for the sake of contradiction that $y \neq 0$. This implies that $0 < y$. Since y is a positive real number, by the Archimedian law of real numbers, there exists a positive integer $N \in \mathbb{N}$, such that $1 < Ny$. Since $N \in \mathbb{N}$ implies that $N > 0$, by Theorem 2.5.8, $N^{-1} = \frac{1}{N} > 0$. Furthermore, letting $a = 1$, $b = Ny$, and $c = N^{-1}$, by Definition 2.3.5, it follows that $y > \frac{1}{N}$. Since $y \leq \frac{1}{n}$ for every $n \in \mathbb{N}$, it must also be true for $n = N$. This means that for $n = N$, it follows that $y \leq \frac{1}{N}$. However, we just said that $y > \frac{1}{N}$. Thus we have reached a contradiction, and our assumption that $y \neq 0$ was wrong. Therefore we have shown that if $y \leq \frac{1}{n}$ for every $n \in \mathbb{N}$, then $y = 0$. $\square$

Exercise 2. Let A and B be two subsets of real numbers. Assume that the sets are both non-empty and both bounded above. Let $Sup(A)$ be the least upper bound of A and $Sup(B)$ be the least upper bound of B.

1. Prove that $max(Sup(a), Sup(b))$ is the least upper bound of $A \cup B$.

Proof: Since A and B are two non-empty subsets of real numbers that are also bounded above, using the first proof in the appendix, it follows that $A \cup B$ is also non-empty and bounded above. Since $A \cup B$ is bounded above, by the continuity axiom, it follows that $A \cup B$ has a least upper bound. We will denote the least upper bound of $A \cup B$ as $Sup(A \cup B)$. In order to show that $Sup(A \cup B) = max(Sup(a), Sup(b))$, we will split the proof into two parts. These two parts consist of showing that $Sup(A \cup B) \leq max(Sup(a), Sup(b))$, and showing that $Sup(A \cup B) \geq max(Sup(a), Sup(b))$.

Step 1: $\mathbf{Sup(A \cup B) \leq max(Sup(a), Sup(b))}$:
Let $x \in A \cup B$. In order to show that $Sup(A \cup B) \leq max(Sup(a), Sup(b))$, we will use the definition of the union of two sets and notice that $x \in A$ or $x \in B$. Thus there are two cases we must check, those being: $x \in A$ and $x \notin B$, as well as $x \notin A$ and $x \in B$. We do not need to check the case where $x \in A$ and $x \in B$ because in this case the validation of the previous two cases implies this case holds as well.

Case 1: $\mathbf{x \in A}$
If $x \in A$ and $x \notin B$, then it follows by the definition of an upper bound that $x \leq Sup(A)$. Furthermore, we know from the definition of a maximum value that $Sup(A) \leq max(Sup(A), Sup(B))$. Therefore we can write $x \leq Sup(A) \leq max(Sup(A), Sup(B))$, which implies that $x \leq max(Sup(A), Sup(B))$. In this case since we started with the assumption that $x \in A \cup B$, it follows that $max(Sup(A), Sup(B))$ is an upper bound for $A \cup B$ in this case.

Case 2: $\mathbf{x \in B}$
If $x \in B$ and $x \notin A$, then it follows by the definition of an upper bound that $x \leq Sup(B)$. Furthermore, we know from the definition of a maximum value that $Sup(B) \leq max(Sup(A), Sup(B))$. Therefore we can write $x \leq Sup(B) \leq max(Sup(A), Sup(B))$, which implies that $x \leq max(Sup(A), Sup(B))$. In this case since we started with the assumption that $x \in A \cup B$, it follows that $max(Sup(A), Sup(B))$ is an upper bound for $A \cup B$ in this case.

Therefore, for all elements $x \in A \cup B$, regardless of if x is an element of A, B, or both, $max(Sup(A), Sup(B))$ is greater than x. Thus $max(Sup(A), Sup(B))$ is an upper for $A \cup B$. However, we already know that $Sup(A \cup B)$ is the least upper bound, therefore it follows that $Sup(A \cup B) \leq max(Sup(A), Sup(B))$.

Step 2: $\mathbf{Sup(A \cup B) \geq max(Sup(a), Sup(b))}$:
In order to show that, $Sup(A \cup B) \geq max(Sup(a), Sup(b))$, we will use the fact that $A \subseteq A \cup B$ and $B \subseteq A \cup B$. Since $A \subseteq A \cup B$, it follows that $A \cup B$ could have more elements than A alone. If it doesn't, then $A = A \cup B$, and hence it follows that $Sup(A) = Sup(A \cup B)$. However in the case that $A \neq A \cup B$, then the extra terms in $A \cup B$ could either be smaller than or equal to or greater than $max(A)$. In the former case, it would follow that $Sup(A) = Sup(A \cup B)$. However, in the latter case, it would follow that $Sup(A) < Sup(A \cup B)$. Putting these cases together we can see that $Sup(A) \leq Sup(A \cup B)$. The same reasoning leads us to the fact that $Sup(B) \leq Sup(A \cup B)$. Since $Sup(A) \leq Sup(A \cup B)$ and $Sup(B) \leq Sup(A \cup B)$, by the definition of a maximum, it follows that $max(Sup(A), Sup(B)) \leq Sup(A \cup B)$.

Therefore, since $Sup(A \cup B) \leq max(Sup(a), Sup(b))$ and $Sup(A \cup B) \geq max(Sup(a), Sup(b))$, by Definition 2.3.2, it follows that $Sup(A \cup B) = max(Sup(a), Sup(b))$. Therefore, we have shown that $max(Sup(a), Sup(b))$ is the least upper bound of $A \cup B$. $\square$

2. Is it true that the least upper bound of $A \cap B$ is $min(Sup(A), Sup(b))$? Either prove the result or find a counterexample.

This statement is false, and to disprove it we will find a counterexample. This is done below.

Proof: Before we get into a concrete counterexample, note that for any case where $A \cap B$ is the empty set, it follows that this theorem doesn't hold since $Sup(A \cap B)$ does not exist in that case. We will now show that, even if $A \cap B$ is non-empty, if $Sup(A)$ and $Sup(B)$ aren't in both sets, then the theorem will not hold.

Let $A = \{1, 2, 9\}$ and $B = \{1, 2, 8\}$. This implies that $Sup(A) = 9$ and $Sup(B) = 8$, respectively. Thus, $min(Sup(A), Sup(B)) = min(8, 9) = 8$. Furthermore, since $A = \{1, 2, 9\}$ and $B = \{1, 2, 8\}$, it follows that $A \cap B = \{1, 2\}$. This implies that $Sup(A \cap B) = max(1, 2) = 2$. Since $min(Sup(A), Sup(B)) = 8$ and $Sup(A \cap B) = 2$, it follows that the least upper bound of $A \cap B$ is not $min(Sup(A), Sup(b))$. Thus we have shown, even if $A \cap B$ is not empty, the above theorem does not hold for all sets A and B. $\square$

Exercise 3. Prove using the definition of the limit that $\lim_{n\to\infty} \frac{n^2+1}{2n-1} = \infty$.

Discussion: Consider any $A \in \mathbb{R}$. We need to determine how large n must be to guarantee that $\frac{n^2+1}{2n-1} > A$. The idea is to bound the fraction $\frac{n^2+1}{2n-1}$ below by some multiple of $\frac{n^2}{n} = n$. Since $n^2+1 > n^2$ and $2n-1 < 2n$ for all $n \in \mathbb{N}$, we have $\frac{n^2+1}{2n-1} > \frac{n^2}{2n} = \frac{n}{2}$. Thus it suffices to arrange for $\frac{1}{2}n > A \implies n > 2A$.

Proof: Let $A \in \mathbb{R}$ and let $N = 2A$. Then $n > N$ implies $n > 2A$, and hence $\frac{n}{2} > A$. Notice that $\frac{n^2}{2n} = \frac{n}{2} > A$. Furthermore, since $n^2+1 > n^2$ and $2n-1 < 2n$, we have $\frac{n^2+1}{2n-1} > \frac{n^2}{2n}$, for all $n \in \mathbb{N}$. Therefore, for all $A \in \mathbb{R}$, it follows that $\exists N = 2A$, such that $n > N = 2A$ implies $\frac{n^2+1}{2n-1} > \frac{n^2}{2n} = \frac{n}{2} > A$. Therefore $\lim_{n\to\infty} \frac{n^2+1}{2n-1} = \infty$. $\square$

Exercise 4. Prove using the definition of the limit that $\lim_{n\to\infty} \frac{1+n}{1-2n} = \frac{-1}{2}$.

Discussion: For each $\epsilon > 0$, we need to decide how big n must be to guarantee $|\frac{1+n}{1-2n} - \frac{-1}{2}| = |\frac{1+n}{1-2n} + \frac{1}{2}| < \epsilon$. Thus we want $|\frac{2+2n+1-2n}{2(1-2n)}| < \epsilon$, which is equivalent to saying $|\frac{3}{2(1-2n)}| < \epsilon$. Since $2(1-2n) < 0$ for all positive integers n, we must multiply by -1 to remove the absolute value symbol and obtain $\frac{-3}{2(1-2n)} = \frac{3}{2(2n-1)} < \epsilon$. We will now manipulate the inequality further to "solve" for n:

$$\frac{3}{2(2n-1)} < \epsilon \implies \frac{3}{2\epsilon} < 2n - 1 \implies \frac{3}{2\epsilon} + 1 < 2n \implies \frac{3}{4\epsilon} + \frac{1}{2} < n$$

Since our steps are reversible, we will make $N = \frac{3}{4\epsilon} + \frac{1}{2}$.

Proof: Let $\epsilon > 0$ and let $N = \frac{3}{4\epsilon} + \frac{1}{2}$. Then $n > N$ implies that $n > \frac{3}{4\epsilon} + \frac{1}{2}$, hence $2n > \frac{3}{2\epsilon} + 1$, and thus $2n - 1 > \frac{3}{2\epsilon}$, which implies that $\frac{3}{2(2n-1)} < \epsilon$. Also, notice that $\frac{3}{2(2n-1)} = \frac{-3}{2(1-2n)}$ Furthermore, since $\frac{-3}{2(1-2n)} > 0$ for all $n \in \mathbb{N}$, it follows from Definition 2.6 that $\frac{-3}{2(1-2n)} = |\frac{-3}{2(1-2n)}|$ Using the fact that $|-1| = 1$ and Definition 2.7.2, we can say that

$$|\frac{-3}{2(1-2n)}| = |-1| * |\frac{3}{2(1-2n)}| = |\frac{3}{2(1-2n)}|$$

Consequently, if we notice that

$$|\frac{3}{2(1-2n)}| = |\frac{2+2n+1-2n}{2(1-2n)}| = |\frac{2+2n}{2(1-2n)} + \frac{1-2n}{2(1-2n)}| = |\frac{1+n}{1-2n} + \frac{1}{2}| = |\frac{1+n}{1-2n} - \frac{-1}{2}|$$

we can see that $|\frac{1+n}{1-2n} - \frac{-1}{2}| < \epsilon$. Therefore, for all $\epsilon > 0$, $\exists N = \frac{3}{4\epsilon} + \frac{1}{2}$, such that $n > N$ implies $|\frac{1+n}{1-2n} - \frac{-1}{2}| < \epsilon$. Therefore, $\lim_{n\to\infty} \frac{1+n}{1-2n} = \frac{-1}{2}$. $\square$

Exercise 5.

1. Prove that for any sequence u_n, if u_n is convergent, then $|u_n|$ is convergent.

Proof: Let u_n be any convergent sequence. Since u_n is convergent this means that $\exists L$ such that, $\forall \epsilon > 0$, $\exists N$ such that, if $n > N$, then $|u_n - L| < \epsilon$. Since $u_n, L \in \mathbb{R}$ and the real numbers are totally ordered, it follows by Theorem 3.7.4 (the second theorem proved in the appendix), that $||u_n| - |L|| \leq |u_n - L|$. Since $|u_n - L| < \epsilon$, it follows that $||u_n| - |L|| \leq |u_n - L| < \epsilon$. Thus, $\forall \epsilon > 0$, $\exists N$ such that, if $n > N$, then $||u_n| - |L|| < \epsilon$, which means that $|u_n|$ converges to $|L|$. Therefore for any sequence u_n, if u_n is convergent, then $|u_n$ is convergent. $\square$

2. Consider the converse: for any sequence u_n, if $|u_n|$ is convergent, then u_n is convergent. Is the converse true? Prove your claim.

After considering the converse, I believe that the converse is false. To see this, we will look at the inverse (also known as the contrapositive of the converse). The inverse states: "for any sequence u_n, if u_n is divergent, then $|u_n|$ is divergent." To prove this statement wrong (and thus prove the converse wrong), we will provide a proof by counterexample.

Proof: (Proof by counterexample) Let $u_n = (-1)^n$. As proven in Lecture 6, $u_n = (-1)^n$ is divergent. In particular, $\lim_{n\to\infty}(-1)^n$ does not exist (it bounces from 1 to -1 infinitely). However, if we consider $|u_n| = |(-1)^n|$, then considering an even natural number, we can see that $|u_{2n}| = |(-1)^{2n}| = |1| = 1$. Furthermore, considering an odd natural number, we can see that $|u_{2n+1}| = |(-1)^{2n+1}| = |-1| = 1$. Therefore, it follows that for all $n \in \mathbb{N}$, $|u_n| = |(-1)^n| = 1$. Lastly, as proven in Lecture 7 (and further emphasized in Lecture 8), $\lim_{n\to\infty} 1 = 1$. Therefore, $|u_n|$ converges to 1. Thus we have shown that the statement, "for any sequence u_n, if u_n is divergent, then $|u_n|$ is divergent," is false. Thus, since the inverse is false, the converse must also be false since the two are logically equivalent. Therefore, we have shown that, "for any sequence u_n, if $|u_n|$ is convergent, then u_n is convergent," is not a true statement. $\square$

Appendix.

In this section, theorems that we didn't prove in class will be proven so that they can be used in the exercises.

1. Let A and B be two subsets of real numbers. Assume that the sets are both non-empty and both bounded above. Let $Sup(A)$ be the least upper bound of A and $Sup(B)$ be the least upper bound of B. Prove that $A \cup B$ is non-empty and bounded above.

Proof: We will start by showing that $A \cup B$ is non-empty. Since A is non-empty, let $x \in A$. Since $x \in A$, it follows that $x \in A$ or $x \in B$. Therefore, from the definition of the union $x \in A \cup B$. This logic works for any element in B as well. Thus we have shown that $A \cup B$ is non-empty when A and B are non-empty (this result also holds if only one of the sets were non-empty).

We will now show that $A \cup B$ is bounded above. Since A and B are both bounded above, by the continuity axiom, they both have a least upper bound which we have called $Sup(A)$ and $Sup(B)$, respectively. We will focus on two cases. These two cases are: $Sup(A) \leq Sup(B)$, and $Sup(A) > Sup(B)$.

Case 1: $\mathbf{Sup(A) \leq Sup(B)}$: For every element $a \in A$, by the definition of an upper bound, $a \leq Sup(A)$. However, we have assumed that $Sup(A) \leq Sup(B)$, therefore we can see that $a \leq Sup(A) \leq Sup(B)$, which implies $a \leq Sup(B)$. Also, $\forall b \in B$, by the definition of an upper bound, $b \leq Sup(B)$. Therefore $Sup(B)$ is an upper bound for both the sets A and B.

Case 2: $\mathbf{Sup(A) > Sup(B)}$: For every element $b \in B$, by the definition of an upper bound, $b \leq Sup(B)$. However, we have assumed that $Sup(A) > Sup(B)$, therefore we can see that $b \leq Sup(B) < Sup(A)$, which implies $b < Sup(A)$. Also, $\forall a \in A$, by the definition of an upper bound, $a \leq Sup(A)$. Therefore $Sup(A)$ is an upper bound for both the sets A and B.

Now, let $x \in A \cup B$. By the definition of the union, $x \in A$ or $x \in B$. Thus, from the above derivations, either $x \leq Sup(A)$ or $x < Sup(B)$. Thus, either $Sup(A)$ or $Sup(B)$ is an upper bound for $A \cup B$. Therefore, by the definition of an upper bound, $A \cup B$ is bounded above by either $Sup(A)$ or $Sup(B)$. Thus we have shown that $A \cup B$ is non-empty and bounded above. $\square$

Appendix.
In this section, theorems that we didn't prove in class will be proven so that they can be used in the exercises.

2. For any $x, y \in \mathbb{R}$, it follows that $||a| - |b|| \leq |a - b|$.

Proof: Let $x, y \in \mathbb{R}$. By the triangle identity we have the following inequalities

$$|y| = |x + y - x| \leq |x| + |y - x|$$
$$|x| = |y + x - y| \leq |y| + |x - y|$$

By Theorem 2.7.2, we have that $|y - x| = |(-1)(x - y)| = |-1||x - y| = |x - y|$. This implies that $|x| + |y - x| = |x| + |x - y|$. Furthermore, subtracting $|x|$ and $|y|$ to the other side of the inequality and using Definition 2.3.4 we obtain the following inequalities

$$|y| - |x| \leq |x - y|$$
$$|x| - |y| \leq |x - y|$$

Noticing that $|y| - |x|$ is equivalent to $-(|x| - |y|)$, we can again rewrite the inequalities as

$$-(|x| - |y|) \leq |x - y|$$
$$|x| - |y| \leq |x - y|$$

Since $-(|x| - |y|) \leq |x - y| \implies |x| - |y| \geq -|x - y|$ and $|x| - |y| \leq |x - y|$, it follows that $-|x - y| \leq |x| - |y| \leq |x - y|$. Therefore, by Homework 1.3.2, it follows that $||x| - |y|| \leq |x - y|$. $\square$