MATH 327 Homework 3

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Note: When I say something like Theorem/Definition 1.1.1, I mean, lecture 1 page 1 part 1 of the given definition or theorem. Furthermore, when I say something like Homework 1.1.1, I mean, homework 1, exercise 1, problem 1.

Exercise 1. Prove using the definition of the limit that $\lim_{n\to\infty} \frac{2n+1}{n+2} + \sqrt{\frac{n}{n+3}} = 3$.

Proof:
Let $\epsilon > 0$ and let $N = \max\{\frac{6}{\epsilon} - 2, \frac{6}{\epsilon} - 3\}$. Then $n > N$ implies that $n > \frac{6}{\epsilon} - 2$ and $n > \frac{6}{\epsilon} - 3$. We will start with $n > \frac{6}{\epsilon} - 2$, and then move onto $n > \frac{6}{\epsilon} - 3$. Since $n > N$ implies that $n > \frac{6}{\epsilon} - 2$, we can manipulate the inequality as follows

$$n > \frac{6}{\epsilon} - 2 \implies n + 2 > \frac{6}{\epsilon} \implies \epsilon > \frac{6}{n+2} \implies \frac{\epsilon}{2} > \frac{3}{n+2}.$$

Since $\frac{-3}{n+2} < 0$, $\forall n$, it follows that $|\frac{-3}{n+2}| = \frac{3}{n+2}$, and hence we can see that $\frac{\epsilon}{2} > |\frac{-3}{n+2}|$. We can further manipulate the inequality as follows

$$\frac{\epsilon}{2} > |\frac{-3}{n+2}| \implies \frac{\epsilon}{2} > |\frac{2n+1-2n-4}{n+2}| \implies \frac{\epsilon}{2} > |\frac{2n+1}{n+2} - \frac{2n+4}{n+2}| \implies \frac{\epsilon}{2} > |\frac{2n+1}{n+2} - \frac{2(n+2)}{n+2}|.$$

Therefore, we can see that $\frac{\epsilon}{2} > |\frac{2n+1}{n+2} - 2|$. Moving on to the next part of the proof, since $n > N$ also implies that $n > \frac{6}{\epsilon} - 3$ we can manipulate the inequality as follows

$$n > \frac{6}{\epsilon} - 3 \implies n + 3 > \frac{6}{\epsilon} \implies \epsilon > \frac{6}{n+3} \implies \frac{\epsilon}{2} > \frac{3}{n+3} \implies \frac{\epsilon}{2} > \frac{n+3-n}{n+3}$$
$$\implies \frac{\epsilon}{2} > \frac{n+3}{n+3} - \frac{n}{n+3} \implies \frac{\epsilon}{2} > 1 - \frac{n}{n+3} \implies \frac{\epsilon}{2} > \frac{1 - \frac{n}{n+3}}{1}.$$

Note that, since $\sqrt{\frac{n}{n+3}} > 0$, $\forall n$, it follows that $1 + \sqrt{\frac{n}{n+3}} > \sqrt{\frac{n}{n+3}}$, $\forall n$. Furthermore, since $1 + \sqrt{\frac{n}{n+3}} > \sqrt{\frac{n}{n+3}}$, $\forall n$, it follows that $\frac{1-\frac{n}{n+3}}{1} > \frac{1-\frac{n}{n+3}}{1+\sqrt{\frac{n}{n+3}}}$. The previous result implies that

$$\frac{\epsilon}{2} > \frac{1 - \frac{n}{n+3}}{1 + \sqrt{\frac{n}{n+3}}} \implies \frac{\epsilon}{2} > \frac{\left(1 - \sqrt{\frac{n}{n+3}}\right)\left(1 + \sqrt{\frac{n}{n+3}}\right)}{1 + \sqrt{\frac{n}{n+3}}} \implies \frac{\epsilon}{2} > 1 - \sqrt{\frac{n}{n+3}}$$

Since $\sqrt{\frac{n}{n+3}} - 1 < 0$, $\forall n$, it follows that $|\sqrt{\frac{n}{n+3}} - 1| = 1 - \sqrt{\frac{n}{n+3}}$, and hence we can see that $\frac{\epsilon}{2} > |\sqrt{\frac{n}{n+3}} - 1|$. Therefore, based on previously computed results, we can see that

$$\epsilon = \frac{\epsilon}{2} + \frac{\epsilon}{2} > |\frac{2n+1}{n+2} - 2| + |\sqrt{\frac{n}{n+3}} - 1|.$$

Using Theorem 2.7.3 (the triangle inequality), the above line can be extended to the following

$$\epsilon = \frac{\epsilon}{2} + \frac{\epsilon}{2} > |\frac{2n+1}{n+2} - 2| + |\sqrt{\frac{n}{n+3}} - 1| \geq |\frac{2n+1}{n+2} - 2 + \sqrt{\frac{n}{n+3}} - 1| = |\frac{2n+1}{n+2} + \sqrt{\frac{n}{n+3}} - 3|.$$

Therefore, $\forall \epsilon > 0$, $\exists N = \max\{\frac{6}{\epsilon} - 2, \frac{6}{\epsilon} - 3\}$, such that, if $n > N$, then $|\frac{2n+1}{n+2} + \sqrt{\frac{n}{n+3}} - 3| < \epsilon$. Thus we have shown that $\frac{2n+1}{n+2} + \sqrt{\frac{n}{n+3}}$ is a convergent sequence to 3, and hence it follows that $\lim_{n\to\infty} \frac{2n+1}{n+2} + \sqrt{\frac{n}{n+3}} = 3$. $\square$

Exercise 2. Let u_n be a convergent sequence (u_n has a finite limit as n goes to ∞). Prove that $|u_n|$ is bounded.

Since I have came up with two very simple ways to prove this theorem I will showcase both of them.

Proof 1:
Since u_n is a convergent sequence, by Homework 2.5.1, it follows that $|u_n|$ is convergent. Furthermore, since $|u_n|$ is convergent, by Theorem 6.5.2, it follows that $|u_n|$ is a bounded sequence. $\square$

Proof 2:
Since u_n is a convergent sequence, by Theorem 6.5.2, it follows that u_n is a bounded sequence. Therefore, $\exists m, M$, such that, $m \leq u_n \leq M$, $\forall n$. Furthermore, since $m \leq u_n \leq M$, by Homework 1.3.1, it follows that $|u_n| \leq \max(|m|, |M|)$. Furthermore, by the definition of the absolute value it follows that $0 \leq |u_n|$. Thus we can see that $0 \leq |u_n| \leq \max(|m|, |M|)$, $\forall n$. Therefore u_n is bounded above by $\max(|m|, |M|)$, and bounded below by 0. Hence we have shown that, when u_n is convergent, $|u_n|$ is bounded. $\square$

Exercise 3. Let a_n and b_n be two convergent sequences. Let A and B be their respective limits.

- Assume that $B = 0$, prove, using the definition of the limit, that $\lim_{n\to\infty} a_n b_n = AB = 0$.

Proof:
Let $\epsilon > 0$. Since a_n is a convergent sequence, by Homework 3.2 (the previous exercise), it follows that $|a_n|$ is bounded. Therefore, $\exists M > 0$, such that, $|an| \leq M$ for all n. Furthermore, since b_n is convergent to $B = 0$, there exists an N_B such that

$$n > N_B \implies |b_n - 0| = |b_n| < \frac{\epsilon}{M}.$$

Now, if $N = N_B$, then $n > N$ implies that

$$|a_n b_n - AB| = |a_n b_n - A(0)| = |a_n b_n|.$$

By Theorem 2.7.2, it follows that $|a_n b_n| = |a_n||b_n|$. Since $|a_n| \leq M$ and $|b_n| < \frac{\epsilon}{M}$, by Theorem 2.5.7, it follows that $|an||b_n| < M \cdot \frac{\epsilon}{M} = \epsilon$. Therefore, $\forall \epsilon > 0$, $\exists N = N_B$, such that, if $n > N$, then $|a_n b_n - AB| = |a_n b_n - A(0)| = |anb_n| < \epsilon$. Thus we have shown that, if a_n and b_n are two convergent sequences, and $B = 0$, then $a_n b_n$ is also a convergent sequence to $AB = 0$, and hence it follows that $\lim_{n\to\infty} a_n b_n = AB = 0$. $\square$

- Assume that $B \neq 0$, prove, using the definition of the limit, that $\lim_{n\to\infty} a_n b_n = AB$.

Since I have came up with two ways to prove this theorem I will showcase both of them.

Proof 1:
Let $\epsilon > 0$. Since a_n is a convergent sequence, by Homework 3.2 (the previous exercise), it follows that $|a_n|$ is bounded. Therefore, $\exists M > 0$, such that, $|a_n| \leq M$ for all n. Furthermore, since a_n is convergent to A, there exists an N_A such that

$$n > N_A \implies |a_n - A| < \frac{\epsilon}{2(|B| + 1)}.$$

Also, since b_n is convergent to B, there exists an N_B such that

$$n > N_B \implies |b_n - B| < \frac{\epsilon}{2M}.$$

Now, if $N > \max(N_A, N_B)$ and we use the fact that $a_n b_n - AB = a_n(b_n - B) + B(a_n - A)$, then $n > N$ implies that

$$|a_n b_n - AB| = |a_n(b_n - B) + B(a_n - A)| \leq |a_n(b_n - B)| + |B(a_n - A)| = |a_n||b_n - B| + |B||a_n - A|.$$

Where the first inequality is the result of Theorem 2.7.3 (the triangle inequality), and the second inequality is the result of Theorem 2.7.2. Since $|a_n| \leq M$ and $|b_n - B| < \frac{\epsilon}{2M}$, by Theorem 2.5.7, it follows that $|a_n||b_n - B| < M \cdot \frac{\epsilon}{2M} = \frac{\epsilon}{2}$. Similarly, since $0 \leq |B|$ and $|a_n - A| < \frac{\epsilon}{2(|B|+1)}$, using Theorem 2.5.7 twice, it follows that $|B||a_n - A| < |B| \cdot \frac{\epsilon}{2(|B|+1)} < |B| \cdot \frac{\epsilon}{2|B|} = \frac{\epsilon}{2}$. Thus, since $|a_n||b_n - B| < \frac{\epsilon}{2}$ and $|B||a_n - A| < \frac{\epsilon}{2}$, by Theorem 2.5.6 it follows that $|a_n||b_n - B| + |B||a_n - A| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Therefore, $\forall \epsilon > 0$, $\exists N = \max(N_A, N_B)$, such that, if $n > N$, then $|a_n b_n - AB| < \epsilon$. Thus we have shown that, if a_n and b_n are two convergent sequences, then $a_n b_n$ is also a convergent sequence to AB, and hence it follows that $\lim_{n\to\infty} a_n b_n = AB$. $\square$

We will now move onto the second proof. This is done below.

Proof 2:
Let $\epsilon > 0$. Since a_n is a convergent sequence, by Homework 3.2 (the previous exercise), it follows that $|a_n|$ is bounded. Therefore, $\exists M > 0$, such that, $|a_n| \leq M$ for all n. In particular, if we use the Archimedian law of real numbers with $n = 1$, since $M > 0$, we can choose this M such that $|B| < M$. Furthermore, since a_n is convergent to A, there exists an N_A such that

$$n > N_A \implies |a_n - A| < \frac{\epsilon}{2M}.$$

Also, since b_n is convergent to B, there exists an N_B such that

$$n > N_B \implies |b_n - B| < \frac{\epsilon}{2M}.$$

Now, if $N > \max(N_A, N_B)$ and we use the fact that $a_n b_n - AB = a_n(b_n - B) + B(a_n - A)$, then $n > N$ implies that

$$|a_n b_n - AB| = |a_n(b_n - B) + B(a_n - A)| \leq |a_n(b_n - B)| + |B(a_n - A)| = |a_n||b_n - B| + |B||a_n - A|.$$

Where the first inequality is the result of Theorem 2.7.3 (the triangle inequality), and the second inequality is the result of Theorem 2.7.2. Since $|a_n| \leq M$ and $|b_n - B| < \frac{\epsilon}{2M}$, by Theorem 2.5.7, it follows that $|a_n||b_n - B| < M \cdot \frac{\epsilon}{2M} = \frac{\epsilon}{2}$. Similarly, since $|B| < M$ and $|a_n - A| < \frac{\epsilon}{2M}$, by Theorem 2.5.7, it follows that $|B||a_n - A| < M \cdot \frac{\epsilon}{2M} = \frac{\epsilon}{2}$. Thus, since $|a_n||b_n - B| < \frac{\epsilon}{2}$ and $|B||a_n - A| < \frac{\epsilon}{2}$, by Theorem 2.5.6 it follows that $|a_n||b_n - B| + |B||a_n - A| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Therefore, $\forall \epsilon > 0$, $\exists N = \max(N_A, N_B)$, such that, if $n > N$, then $|a_n b_n - AB| < \epsilon$. Thus we have shown that, if a_n and b_n are two convergent sequences, then $a_n b_n$ is also a convergent sequence to AB, and hence it follows that $\lim_{n\to\infty} a_n b_n = AB$. $\square$

Exercise 4. Given a convergent sequence a_n, prove $|a_n|$ is convergent.

This theorem was proven to be true in Homework 2.5.1. With that being said, I will simply display the proof from that homework.

Proof:
Let a_n be any convergent sequence. Since a_n is convergent this means that $\exists L$ such that, $\forall \epsilon > 0$, $\exists N$, such that, if $n > N$, then $|a_n - L| < \epsilon$. Since $a_n, L \in \mathbb{R}$ and the real numbers are totally ordered, it follows by Theorem 3.7.4 (the second theorem I proved in the appendix of Homework 2), that $||a_n| - |L|| \leq |a_n - L|$. Since $|a_n - L| < \epsilon$, it follows that $||a_n| - |L|| \leq |a_n - L| < \epsilon$. Thus, $\forall \epsilon > 0$, $\exists N$, such that, if $n > N$, then $||a_n| - |L|| < \epsilon$, which means that $|a_n|$ converges to $|L|$. Therefore for any sequence a_n, if a_n is convergent, then $|a_n|$ is convergent. $\square$

Exercise 5. Given two sequences a_n and b_n such that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = \infty$

- Prove that $\lim_{n\to\infty} a_n + b_n = \infty$.

Proof:
Let $A \in \mathbb{R}$. Since a_n is convergent to L, there exists an N_A such that

$$n > N_A \implies |a_n - L| < 327 \implies L - 327 < a_n$$

Also, since $\lim_{n\to\infty} b_n = \infty$, there exists an N_B such that

$$n > N_B \implies b_n > A - L + 327.$$

Since $L - 327 < a_n$ and $A - L + 327 < b_n$, by Theorem 2.5.6 it follows that $L - 327 + A - L + 327 < a_n + b_n$. Now, if $N = \max(N_A, N_B)$, then $n > N$ implies that

$$a_n + b_n > L - 327 + A - L + 327 = A$$

Therefore, $\forall A \in \mathbb{R}$, $\exists N = \max(N_A, N_B)$, such that, if $n > N$, then $a_n + b_n > A$. Thus we have shown that $a_n + b_n$ diverges to ∞, and hence it follows that $\lim_{n\to\infty} a_n + b_n = \infty$. $\square$

Note that, instead of 327, we could've chosen any number greater than 0 and the proof still would've worked.

- Prove that $\lim_{n\to\infty} \frac{1}{b_n} = 0$.

Proof:
Let $\epsilon > 0$. Since $\lim_{n\to\infty} b_n = \infty$, $\exists N$, such that, if $n > N$, then $b_n > \frac{1}{\epsilon}$. Since $b_n > \frac{1}{\epsilon}$, it follows that $\frac{1}{b_n} < \epsilon$. Furthermore, since $b_n > \frac{1}{\epsilon} > 0$, this implies that $b_n > 0$, and thus $b_n = |b_n|$. Therefore we can rewrite $\frac{1}{b_n} < \epsilon$ as $\frac{1}{|b_n|} < \epsilon$. However, notice that $\frac{1}{|b_n|} = |\frac{1}{b_n}| = |\frac{1}{b_n} - 0| < \epsilon$. Therefore, $\forall \epsilon > 0$, $\exists N$, such that, if $n > N$, then $|\frac{1}{b_n} - 0| < \epsilon$. Thus we have shown that, if $\lim_{n\to\infty} b_n = \infty$, then $\frac{1}{b_n}$ is a convergent sequence to 0, and hence it follows that $\lim_{n\to\infty} \frac{1}{b_n} = 0$. $\square$