MATH 327 Homework 7

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Note: When I say something like Theorem/Definition 1.1.1, I mean, lecture 1 page 1 part 1 of the given definition or theorem. Furthermore, when I say something like Homework 1.1.1, I mean, homework 1, exercise 1, problem 1.

Exercise 1. Let f be a continuous function on $\mathbb{R}$ such that $\lim_{x\to\infty} f(x) = L^+$ and $\lim_{x\to-\infty} f(x) = L^-$. Prove that for any real c between L^+ and L^-, there exists $x \in \mathbb{R}$ such that $f(x) = c$.

Proof: Let f be a continuous function on $\mathbb{R}$ such that $\lim_{x\to\infty} f(x) = L^+$ and $\lim_{x\to-\infty} f(x) = L^-$. Since $\lim_{x\to\infty} f(x) = L^+$, for all $\epsilon^+ > 0$, there exists an N^+, such that, if $x > N^+$, then $|f(x) - L^+| < \epsilon^+$, which implies that $L^+ - \epsilon^+ < f(x) < L^+ + \epsilon^+$. Similarly, since $\lim_{x\to-\infty} f(x) = L^-$, for all $\epsilon^- > 0$, there exists an N^-, such that, if $x < N^-$, then $|f(x) - L^-| < \epsilon^-$, which implies that $L^- - \epsilon^- < f(x) < L^- + \epsilon^-$.

Now, consider the three cases: $L^- < L^+$, $L^- = L^+$, and $L^- > L^+$. When $L^- = L^+$, the theorem doesn't hold (take the function $f(x) = e^{-x^2}$ for example), thus we only have the cases $L^- < L^+$and $L^- > L^+$. However, as noted in office hours, when $L^- > L^+$, we can work with $-f(x)$ and then we will be in the setting where $L^- < L^+$ (we can do this because the x that makes $-f(x) = -c$ is the same x that makes $f(x) = c$). Therefore, without loss of generality, we can assume that $L^- < L^+$.

Since we are assuming without loss of generality that $L^- < L^+$, then we must now pick a $c \in \mathbb{R}$, such that, c is between L^- and L^+. More concretely, we pick a $c \in \mathbb{R}$, such that, $L^- < c < L^+$. Since $L^+ > c$, this implies that $L^+ - c > 0$. Now, since $\lim_{x\to\infty} f(x) = L^+$, for $\epsilon^+ = L^+ - c > 0$, there exists an N^+, such that, if $x > N^+$, then $|f(x) - L^+| < L^+ - c$, which implies that $L^+ - L^+ + c < f(x) < L^+ + L^+ - c$, and hence we can see that $c < f(x)$. Similarly, since $L^- < c$, this implies that $0 < c - L^-$. Now, since $\lim_{x\to-\infty} f(x) = L^-$, for all $\epsilon^- = c - L^- > 0$, there exists an N^-, such that, if $x < N^-$, then $|f(x) - L^-| < \epsilon^-$, which implies that $L^- - c + L^- < f(x) < L^- + c - L^-$, and hence we can see that $f(x) < c$.

Now, pick $a \in \mathbb{R}$, such that, $a < N^-$, then from the above derivations $f(a) < L^- + \epsilon^- = c$. Similarly, pick $b \in \mathbb{R}$, such that, $b > N^+$, then from the above derivations $f(b) > L^+ - \epsilon^+ = c$. Furthermore, since f is continuous on $\mathbb{R}$, it is continuous on the closed and bounded interval $[a, b]$. Therefore, by the Intermediate Value Theorem, there exists an $x \in [a, b]$, such that, $f(x) = c$. Thus, since we assumed without loss of generality that $L^- < L^+$, it follows that, for all $c \in \mathbb{R}$ between L^- and L^+, there exists an $x \in \mathbb{R}$ such that $f(x) = c$. $\square$

Exercise 2. Prove that if f is a continuous and positive function on $[0,1]$, then there exists $\delta > 0$ such that $f(x) > \delta$ for any $x \in [0,1]$.

Proof: Let f be a continuous and positive function on $[0,1]$. Since f is continuous on a closed and bounded interval, that interval being $[0,1]$, it follows from the proof of the extreme value theorem that f is bounded. However, the above fact isn't entirely needed, since we already know that $f(x)$ is positive on $[0,1]$, and this implies that $f(x) > 0$ for all $x \in [0,1]$. Since $f(x) > 0$ for all $x \in [0,1]$, it follows that 0 is a lower bound for the set of images $\{f(x) : x \in [0,1]\}$. Therefore, by the extreme value theorem, since $\{f(x) : x \in [0,1]\}$ is bounded below, if the greatest lower bound of this set is $m > 0$, then there exists an $x_m \in [0,1]$ such that $f(x_m) = m$. Furthermore, the extreme value theorem also states that m is the minimum of f on the interval $[0,1]$, which implies that, for all x, $f(x) \geq m$. Since for all $x \in [0,1]$, we have that $f(x) \geq m > 0$, if we take $\delta = \frac{m}{2} > 0$, then since $1 > \frac{1}{2} > 0$, this implies that $m > \frac{m}{2} > 0$, and hence, for all $x \in [0,1]$, $f(x) > \delta = \frac{m}{2} > 0$. Therefore, in conclusion, if f is a continuous and positive function on $[0,1]$, then taking $\delta = \frac{m}{2}$, where m is the greatest lower bound of f on $[0,1]$, then for all $x \in [0,1]$, we have that $f(x) > \delta$. $\square$

Exercise 3

1. Let $a > 0$ be a positive real number. Is the function $f(x) = 1/x$ uniformly continuous on $[a, \infty)$? Justify your answer.

If we let $a > 0$ be a positive real number, then the function $f(x) = 1/x$ is uniformly continuous on $[a, \infty)$. We will prove this claim below.

Proof: Let $\epsilon > 0$. Choose $\delta = a^2\epsilon$, where $a > 0$. Then $\forall x, y \in [a, \infty)$ with $|x - y| < \delta$, we must show that $|f(x) - f(y)| < \epsilon$, where $f(x) = \frac{1}{x}$. Consider $f(x) - f(y)$, since $f(x) = \frac{1}{x}$, this implies that $f(y) = \frac{1}{y}$. Thus we can see that $f(x) - f(y) = \frac{1}{x} - \frac{1}{y} = \frac{y-x}{xy}$. Since $x \geq a > 0$ and $y \geq a > 0$, it follows that $|f(x) - f(y)| = \frac{|y-x|}{xy} = \frac{|x-y|}{xy}$. Again, since $x \geq a > 0$ and $y \geq a > 0$, it follows that $0 < \frac{1}{x} \leq \frac{1}{a}$ and $0 < \frac{1}{y} \leq \frac{1}{a}$, and hence $0 < \frac{1}{xy} \leq \frac{1}{a^2}$. Since we supposed that $|x - y| < \delta$ and found that $0 < \frac{1}{xy} \leq \frac{1}{a^2}$, we can see that $|f(x) - f(y)| = \frac{|x-y|}{xy} \leq \frac{|x-y|}{a^2} < \frac{\delta}{a^2} = \frac{a^2\epsilon}{a^2} = \epsilon$. Thus, we can see that $|f(x) - f(y)| < \epsilon$. Hence, in conclusion, for all $\epsilon > 0$, there exists a $\delta = a^2\epsilon$, such that, $\forall x, y \in [a, \infty)$ with $|x - y| < \delta$, then $|1/x - 1/y| < \epsilon$. Thus we have shown that $f(x) = 1/x$ is uniformly continuous on $[a, \infty)$. $\square$

2. Is $f(x) = 1/x$ uniformly continuous on $(0, \infty)$?

The function $f(x) = 1/x$ is not uniformly continuous on $(0, \infty)$. We will prove this claim below. Furthermore, since I found proofs using two different methods, I will showcase both of them.

Proof 1: Since the claim is that the function $f(x) = 1/x$ is not uniformly continuous on $(0, \infty)$, then by the negation of the definition of uniform continuity, we must show that there exists an $\epsilon > 0$, such that, for all $\delta > 0$, there exists $x, y \in (0, \infty)$ that satisfy $|x - y| < \delta$, but $|f(x) - f(y)| \geq \epsilon$. Since we only have to show that there exists a single $\epsilon > 0$ such that the above result holds, we will choose $\epsilon = 1$. Now, for all $\delta > 0$, we must show that there exists $x, y \in (0, \infty)$ that satisfies $|x - y| < \delta$, but has that $|f(x) - f(y)| \geq \epsilon$. Choose $x = \min(\delta, 1)$, and $y = \frac{x}{2}$. Hence, since $\delta > 0$ and $1 > 0$, we can see that $x > 0$ and $y > 0$, and thus $x, y \in (0, \infty)$. Furthermore, since $x = \min(\delta, 1)$, this implies that $x \leq \delta$, which means that $|x - y| = |x - \frac{x}{2}| = |\frac{x}{2}| = \frac{x}{2} \leq \frac{\delta}{2} < \delta$. Therefore, we can see that $|x - y| < \delta$. Lastly, since $f(x) = \frac{1}{x}$ and $f(y) = \frac{1}{y}$, we can see that $|f(x) - f(y)| = |\frac{1}{x} - \frac{1}{y}| = |\frac{1}{x} - \frac{2}{x}| = |\frac{-1}{x}| = \frac{1}{x}$. However, since $x = \min(\delta, 1)$, this implies that $x \leq 1$, which means that $\frac{1}{x} \geq \frac{1}{1} = 1$. Therefore, we can see that $|f(x) - f(y)| = \frac{1}{x} \geq 1$. In conclusion, we have found for $\epsilon = 1 > 0$ that, for all $\delta > 0$, there exists an $x = \min(\delta, 1)$ and $y = \frac{x}{2}$ in $(0, \infty)$, such that $|x - y| < \delta$, but $|f(x) - f(y)| \geq \epsilon = 1$. The preceding line is exactly the definition that shows that the function $f(x) = 1/x$ is not uniformly continuous on $(0, \infty)$. $\square$

Proof 2: Since the claim is that the function $f(x) = 1/x$ is not uniformly continuous on $(0, \infty)$, then by the negation of the definition of uniform continuity (sequence version), it follows that we must find two sequences u_n and v_n on $(0, \infty)$, such that, $\lim_{n\to\infty}(u_n - v_n) = 0$, but $\lim_{n\to\infty}(f(u_n) - f(v_n)) \neq 0$. Conveniently, since $I = (0, \infty)$, this implies that all of the natural numbers are in I. Thus, if we define $\forall n \in \mathbb{N}$, $u_n = \frac{1}{2n}$ and $v_n = \frac{1}{n}$, then we can see from Theorem 24.1 and Theorem 26.1 that $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{2n} = 0$ and $\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{1}{n} = 0$. Furthermore, since the limits are finite, we can use Theorem 24.2 and see that $\lim_{n\to\infty}(u_n - v_n) = \lim_{n\to\infty} u_n - \lim_{n\to\infty} v_n = 0 - 0 = 0$. Now, since $f(x) = \frac{1}{x}$, it follows that $f(u_n) = \frac{1}{1/2n} = 2n$ and $f(v_n) = \frac{1}{1/n} = n$, which implies that $f(u_n) - f(v_n) = 2n - n = n$. Therefore, using Theorem 32.1, we can see that $\lim_{n\to\infty}(f(u_n) - f(v_n)) = \lim_{n\to\infty} n = +\infty \neq 0$. In conclusion, we defined the sequences $u_n = \frac{1}{2n}$ and $v_n = \frac{1}{n}$ on $(0, \infty)$, such that $\lim_{n\to\infty}(u_n - v_n) = 0$, but $\lim_{n\to\infty}(f(u_n) - f(v_n)) \neq 0$. Therefore, by definition, the function $f(x) = 1/x$ is not uniformly continuous on $(0, \infty)$. $\square$

Exercise 4. Prove that the function $f(x) = \sqrt{x}$ is uniformly continuous on the interval $[3, \infty)$.

Since I have found two simple proofs that each use a different δ value, I will showcase them both below.

Proof 1: Let $\epsilon > 0$. Choose $\delta = 2\sqrt{3}\epsilon$. Then $\forall x, y \in [3, \infty)$ with $|x - y| < \delta$, we must show that $|f(x) - f(y)| < \epsilon$, where $f(x) = \sqrt{x}$. Consider $f(x) - f(y)$, since $f(x) = \sqrt{x}$, this implies that $f(y) = \sqrt{y}$. Thus we can see that $f(x) - f(y) = \sqrt{x} - \sqrt{y} = \frac{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}{\sqrt{x}+\sqrt{y}} = \frac{x-y}{\sqrt{x}+\sqrt{y}}$. Since $x \geq 3$ and $y \geq 3$, this implies that $\sqrt{x} \geq \sqrt{3} > 0$ and $\sqrt{y} \geq \sqrt{3} > 0$, and therefore it follows that $|f(x) - f(y)| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}}$. Furthermore, since it was shown above that $\sqrt{x} \geq \sqrt{3} > 0$ and $\sqrt{y} \geq \sqrt{3} > 0$, this implies that $\sqrt{x} + \sqrt{y} \geq \sqrt{3} + \sqrt{3} = 2\sqrt{3}$. Hence we can see that $\frac{1}{\sqrt{x}+\sqrt{y}} \leq \frac{1}{2\sqrt{3}}$. Since we supposed that $|x - y| < \delta$ and found that $\frac{1}{\sqrt{x}+\sqrt{y}} \leq \frac{1}{2\sqrt{3}}$, we can see that $|f(x) - f(y)| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \leq \frac{|x-y|}{2\sqrt{3}} < \frac{\delta}{2\sqrt{3}} = \frac{2\sqrt{3}\epsilon}{2\sqrt{3}} = \epsilon$. Thus, we can see that $|f(x) - f(y)| < \epsilon$. Hence, in conclusion, for all $\epsilon > 0$, there exists a $\delta = 2\sqrt{3}\epsilon$, such that, $\forall x, y \in [3, \infty)$ with $|x - y| < \delta$, then $|\sqrt{x} - \sqrt{y}| < \epsilon$. Thus we have shown that $f(x) = \sqrt{x}$ is uniformly continuous on $[3, \infty)$. $\square$

Proof 2: Let $\epsilon > 0$. Choose $\delta = \epsilon$. Then $\forall x, y \in [3, \infty)$ with $|x - y| < \delta$, we must show that $|f(x) - f(y)| < \epsilon$, where $f(x) = \sqrt{x}$. Consider $f(x) - f(y)$, since $f(x) = \sqrt{x}$, this implies that $f(y) = \sqrt{y}$. Thus we can see that $f(x) - f(y) = \sqrt{x} - \sqrt{y} = \frac{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}{\sqrt{x}+\sqrt{y}} = \frac{x-y}{\sqrt{x}+\sqrt{y}}$. Since $x \geq 3$ and $y \geq 3$, this implies that $\sqrt{x} \geq \sqrt{3} > 0$ and $\sqrt{y} \geq \sqrt{3} > 0$, and therefore it follows that $|f(x) - f(y)| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}}$. Furthermore, since it was shown above that $\sqrt{x} \geq \sqrt{3} > 0$ and $\sqrt{y} \geq \sqrt{3} > 0$, this implies that $\sqrt{x} + \sqrt{y} \geq \sqrt{3} + \sqrt{3} = 2\sqrt{3} > 1$. Hence we can see that $\frac{1}{\sqrt{x}+\sqrt{y}} \leq \frac{1}{2\sqrt{3}} < 1$. Since we supposed that $|x - y| < \delta$ and found that $\frac{1}{\sqrt{x}+\sqrt{y}} < 1$, we can see that $|f(x) - f(y)| = \frac{|x-y|}{\sqrt{x}+\sqrt{y}} \leq |x - y| < \delta = \epsilon$. Thus, we can see that $|f(x) - f(y)| < \epsilon$. Hence, in conclusion, for all $\epsilon > 0$, there exists a $\delta = \epsilon$, such that, $\forall x, y \in [3, \infty)$ with $|x - y| < \delta$, then $|\sqrt{x} - \sqrt{y}| < \epsilon$. Thus we have shown that $f(x) = \sqrt{x}$ is uniformly continuous on $[3, \infty)$. $\square$

Exercise 5. Let f be a uniformly continuous function on an interval I. Let g be a uniformly continuous function on an interval J. Assume that $f(I) \subset J$. Prove that the composition $g(f(x))$ is a uniformly continuous function on I. Remark: $f(I)$ is the set of all the images of elements of $I : f(I) = \{f(x), x \in I\}$.

Proof: Let $\epsilon > 0$. Since g is a uniformly continuous function on J, for all $\epsilon_g > 0$, there exists a $\delta_g > 0$, such that, for all $a, b \in J$ satisfying $|a - b| < \delta_g$, then $|g(a) - g(b)| < \epsilon_g$. Since this holds for all ϵ_g, take $\epsilon_g = \epsilon > 0$. Then, there exists a $\delta_g > 0$, such that, for all $a, b \in J$ satisfying $|a-b| < \delta_g$, then $|g(a)-g(b)| < \epsilon$. Furthermore, since f is a uniformly continuous function on I, for all $\epsilon_f > 0$, there exists a $\delta_f > 0$, such that, for all $x, y \in I$ satisfying $|x - y| < \delta_f$, then $|f(x) - f(y)| < \epsilon_f$. Since this holds for all ϵ_f, take $\epsilon_f = \delta_g > 0$. Then, there exists a $\delta_f > 0$, such that, for all $x, y \in I$ satisfying $|x - y| < \delta_f$, then $|f(x) - f(y)| < \delta_g$. However, since $f(I) \subset J$, it follows that $f(x), f(y) \in J$. Therefore, $f(x)$ and $f(y)$ are elements in J such that $|f(x) - f(y)| < \delta_g$, and hence $|g(f(x)) - g(f(y))| < \epsilon$. Therefore, for all $\epsilon > 0$, there exists a $\delta = \delta_f$, such that, for all $x, y \in I$ satisfying $|x - y| < \delta$, then $|g(f(x)) - g(f(y))| < \epsilon$. Thus we have shown that $g(f(x))$ is a uniformly continuous function on I. $\square$