Note

These are *most of* the definitions and theorems for the Graph Theory section of the course. They are from memory as practice for the exam (hence the most of). It's probably a good idea to make sure you know these before the exam.

1 Introduction To Graph Theory

Def: Graphs are isomorphic if there exists a bijection $f : V(G_1) \to V(G_2)$ such that $f(u)$ and $f(v)$ are adjacent *if and only if* u and v are adjacent in G.

Thm $\sum_{v \in V(G)} \deg(v) = 2|E(G)|$ (Handshake Lemma)

Thm Number of vertices of odd degree is even.

Thm Average degree of a vertex is $\frac{2|E(G)|}{|V(G)|}$.

Def Complete Graph ($\mathcal{K}_n$) has all vertices adjacent to each other.

Def Complete Bipartite Graph ($\mathcal{K}_{m,n}$) has all vertices in A adjacent to all vertices in B where $|A| = m, |B| = n$.

Def A subgraph of G has a vertex set that is a subset U of $V(G)$ and whose edge set is a subset of those edges of G with both vertices in U.

Def A walk is an alternating sequence of vertices and edges. The length of a walk is indicated by the number of edges. A walk is closed if $v_0 = v_n$

Def A path is a walk with all vertices unique (and thus distinct edges).

Def A trail is a walk with no repeated edges

Thm If there exists a walk from x to y, then there is a path from x to y.

Def A cycle is a subgraph with n distinct vertices and n distinct edges (where $n \geq 1$). Or, also known as a connected regular graph of degree 2.

Thm If every vertex has degree at least 2, then G contains a cycle.

Def Girth of a graph G is the length of the smallest cycle.

Def Spanning cycle is known as a Hamilton cycle (visits every vertex exactly once).

Def A graph is connected if for each two vertices x, y, there is a path from x to y.

Thm If for each vertex w in G there is a path from v to w in G, then G is connected.

Def A component of G is a subgraph C of G such that a) C is connected b) No subgraph that properly contains X is connected.

Def A cut induced by X is the set of edges that have exactly one end in X.

Thm A graph is not connected *if and only if* there exists a proper non-empty subset X of $V(G)$ such that the cut induced by X is empty.

Def Eulerian circuit of a graph G is a closed walk that contains every edge of G once.

Thm G has an Eulerian circuit *if and only if* every vertex has even degree.

Def An edge e is a bridge if $G - e$ has more components than G.

Thm An edge e is a bridge of a graph *if and only if* it is not contained in a cycle of G.

Cor If there are two distinct paths from u to v, G contains a cycle.

2 Trees

Def A tree is a connected graph with no cycles.

Def A forest is a graph with no cycles.

Lemma If u and v are in T, there exists a unique uv-path in T.

Lemma Every edge in a tree is a bridge.

Thm If T is a tree, $|E(T)| = |V(T)| - 1$

Cor If G is a forest with k components, $|E(G)| = |V(G)| - k$.

Def A leaf in a tree is a vertex of degree 1.

Thm A tree with at least 2 vertices has at least two leaves.

Def A spanning subgraph that is a tree is a spanning tree. They have the fewest edges while remaining connected.

Thm A graph G is connected *if and only if* it has a spanning tree.

Cor If G is connected, with p vertices and $q = p - 1$ edges, then G is a tree.

Thm If T is a spanning tree of G and e is an edge not in T, then $T + e$ contains exactly one cycle C. If e' is any edge in C, then $T + e - e'$ is also a spanning tree of G.

Thm If T is a spanning tree of G and e is an edge in T, then $T - e$ has 2 components. If e' is in the cut induced by one of the components, then $T - e + e'$ is also a spanning tree of G.

Thm An odd cycle is not bipartite.

Thm A graph is bipartite *if and only if* it has no odd cycles.

3 Planar Graphs

Def A graph is planar if no edges intersect.

Thm Given a planar embedding of a connected graph G with faces $f_1, \ldots, f_s$, then $\sum_{i=1}^{s} \deg(f_i) = 2|E(G)|$ (Faceshaking Lemma). Note that, a bridge contributes 2 to one face, but a non-bridge edge contributes 1 to one face, and 1 to another face.

Cor If a connected planar graph G has f faces, the average degree of a face is $\frac{2|E(G)|}{f}$

Thm Let G is a connected graph with p vertices and q edges. If G has a planar embedding with f faces, then $p + f - q = 2$.

Def A platonic solid is a graph where all faces have the same degree and all vertices have the same degree.

Thm There are exactly five platonic solids (See below Lemma).

Lemma (vertex degree, face degree) $\rightarrow \{(3,3), (3,4), (4,3), (5,3), (3,5)\}$

Thm If G contains a cycle, then in a planar embedding of G, the boundary of each face contains a cycle.

Thm Let G be a planar embedding with p vertices, q edges. If each face has degree $\geq d^*$, then $(d^* - 2)q \leq d^*(p - 2)$.

Thm In a planar graph with $p \geq 3$ vertices and q edges, $q \leq 3p - 6$.

Cor $\mathcal{K}_5$ is not planar.

Cor A planar graph has a vertex of degree at most five.

Thm In a bipartite planar graph with $p \geq 3$ vertices and q edges, $q \leq 2p - 4$.

Lemma $\mathcal{K}_{3,3}$ is not planar.

Def An edge subdivision takes an edge and replaces it with a path of length 1 or more.

Thm A graph is not planar *if and only if* it has a subgraph that is an edge subdivision of $\mathcal{K}_5$ or $\mathcal{K}_{3,3}$,

Def A graph with $k-$colouring has adjacent vertices as different colours (and can be done with a total of k colours).

Thm A graph is $2-$colourable *if and only if* it is bipartite.

Thm $\mathcal{K}_n$ is $n-$colourable, and not k-colourable for any $k < n$.

Thm Every planar graph is $6-$colourable.

Thm Every planar graph is $5-$colourable.

Thm Every planar graph is $4-$colourable.

4 Matchings

Def A matching in a graph is a set of M edges such that no two edges have a common end.

Def A vertex is saturated by M if the vertex is incident with an edge in M.

Def We are often interested in finding a maximum matching.

Def A perfect matching has size $\frac{p}{2}$ since it saturates every vertex. Every perfect matching is a maximum matching,

Def A path is an alternating path with respect to M if it alternates between M and not M.

Def A path is augmented if it joins two distinct vertices that are not saturated by M. Augmented paths always have odd length.

Lemma If M has an augmenting path, it is not a maximum matching.

Def A cover of graph G is a set C of vertices such that every edge of G has at least one end in C.

Lemma If M is a matching of G and C is a cover of G, then $|M| \leq |C|$.

Lemma If M is a matching and C is a cover and $|M| = |C|$, then M is a maximum matching and C is a minimal cover.

Thm In a bipartite graph, the maximum size of a matching is the minimum size of a cover.

XY Construction

G is bipartite (A, B), M is a matching of G.

X_0 = set of vertices in A not saturated by M. Y_0 = set of vertices in B unsaturated. Z = set of vertices in G that are joined to a vertex in X_0 by an alternating path.

$X = A \cap Z$ $Y = B \cap Z$

If $Y \cap Y_0 = \emptyset$, M is a maximum matching and $C = Y \cup (A \setminus X)$

- No edge of G from X to $B \setminus Y$.
- $C = Y \cup (A \setminus X)$ is a cover.
- No edge of M from Y to $A \setminus X$.
- $|M| = |C| - |U|$ where U is the set of unsaturated vertices in Y
- Augmenting path to each vertex in U.

(There's also the Bipartite Matching Algorithm. Pretty much the same thing, just faster).

Thm A bipartite graph G with bipartition A, B has a matching saturating every vertex in A, *if and only if* every subset of D of A satisfies $|N(D)| \geq |D|$

Def Let $D \leq V(G)$. The neighbour set of D is the set of all vertices adjacent to at least one vertex in D. $N(D) = \{v \in V(G) : u \in D, uv \in E(G)\}$

Cor A bipartite graph G with bipartition A, B has a perfect matching *if and only if* $|A| = |B|$ and every subset D of A satisfies $|N(D)| \geq |D|$.

Thm If G is a $k-$regular bipartite graph with $k \geq 1$, then G has a perfect matching.

Def A graph with an edge $k-$colouring has edges incident with a vertex assigned different colours.

Thm A bipartite graph with maximum degree Δ has an edge $\Delta-$colouring.

Lemma Let G be a bipartite graph having at least one edge. Then G has a matching saturating each vertex of maximum degree.

Tips

- $p \iff q$ statements are powerful. Definitely remember those.
- Induction is useful and usually very straightforward (colourings, trees etc.)
- When in doubt, contradiction (usually longest path or odd number of odd degrees).