STAT 4352 - Mathematical Statistics Notes

JaimeGoB

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1 Chapter 11 - Interval Estimation

Point Estimators

θ is a unknown parameter (feature of a population)

- Ex: population mean μ
- **Fixed.**

$\hat{\theta}$ is a point estimator of θ (it is a numerical value)

- Ex: sample mean $\bar{x}$
- **Varies from sample to sample.**
- No guarantee of accuracy
- Must be *supplemented by* Var(θ)

  Standard Error SE($\hat{\theta}$) measures how much $\hat{\theta}$ varies from sample to sample.

  small SE $\implies$ low variance thus a more reliable estimate of θ

Interval Estimators

Def: Interval Estimate

Provides a range of values that best describe the population.
Let L = L(x) be the Lower Limit
U = U(x) be the Upper Limit
Both L,U are Random Variables because they are functions of sample data.

Def: Confidence Level / Confidence Coefficient

Is the probability that the **interval estimate** will include population parameter θ.

- Sample means will follow the normal probability distribution for large sample sizes ($n \geq 30$)
- For small sample forces us to use the t-distribution probability distribution($n < 30$)
- A confidence level of 95% implies that **95% of all samples would give an interval that includes θ, and only 5%of all samples would yield an erroneous interval.**
- The most frequently used confidence levels are 90%, 95%, and 99% with corresponding Z-scores 1.645, 1.96, 2.576.
- The higher the confidence level, the more strongly we believe that the value of the parameter lies within the interval.

Def: Confidence Interval
Gives plausible values for the parameter θ being estimated where degree of plausibility specified by a confidence level.

To construct an interval estimator of unknown parameter θ. We must find two statistics **L** and **U** such that:

$$P\{\mathbf{L} \leq \theta \leq \mathbf{U}\} = 1 - \alpha$$

- $P\{\mathbf{L} \leq \theta \leq \mathbf{U}\}$ **Coverage Probability**, in repeated sampling, what percent of samples or Confident Intervals capture true θ.
- 100(1- α) **Confidence Interval** - for unknown fixed parameter θ.
- L,U - **Lower and Upper Bounds** - RVs because they are functions of sample data. Vary from sample to sample.
- 1-α **Confidence Level** (Probability) estimate will include population parameter θ.
- α **Level of Significance** Percent chance Confidence Interval will not contain population parameter θ.

Def: Coverage Probability
$P\{\mathbf{L} \leq \theta \leq \mathbf{U}\}$ Gives what % of samples or Confidence Intervals capture true θ.

Ex: Coverage Probability = 95%
Will capture θ, 95% of the time.
Will NOT capture θ, 5% of the time.

Properties of Confidence Intervals

- Confidence Intervals are not unique.
- Desirable to have E[Length of CI] to be small.
- A one-sided 100(1-α) lower-confidence interval on θ: L = -∞ $\implies$ $P\{L \leq \theta\} = 1-\alpha$
- A one-sided 100(1-α) upper-confidence interval on θ: $U = \infty \implies P\{\theta \leq U\} = 1-\alpha$
- If L,U are both finite, then we have a two sided interval.

Correctly Interpreting Confidence Intervals
Not Correct
There is 90% probability that the true population mean is within the interval.
Correct
There is a 90% probability that <u>any given Confidence Interval from a random sample</u> will contain the true population mean.

Theorem 11.1: Confidence Interval on the Mean of a Normal Distribution with known Variance

Let X be normal random variable with:

Unknown mean μ

Known variance σ^2

Suppose a random sample n, $(X_1, X_2, ..., X_n)$ is taken.

A 100(1-α)% confidence interval on μ can be obtained by considering sampling distribution of the sample mean $\bar{X}$.

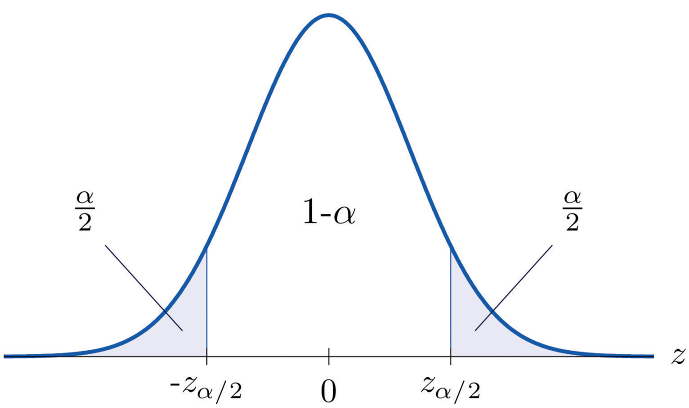
Central Limit Theorem:

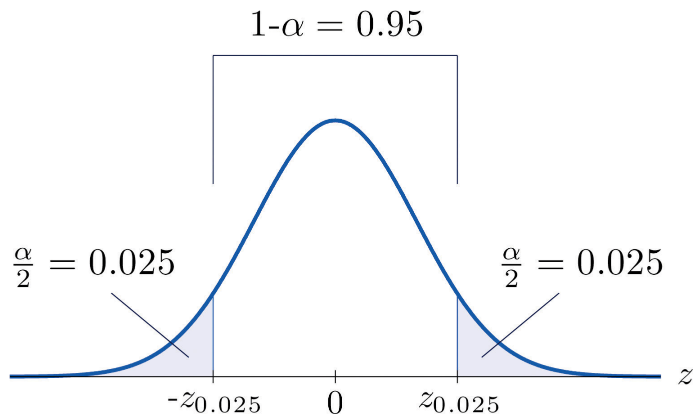
$E(\bar{X}) = \mu$ and $SD(\bar{X}) = \dfrac{\sigma}{\sqrt{n}}$, so $\bar{X} \sim \mathcal{N}(\mu, \dfrac{\sigma^2}{n})$ as $n \to \infty$

Let Z = Standardizing $\bar{X}$, Z will follow a Standard Normal Distribution

Let $Z = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0, 1)$

We can see from the image to the left: **Distribution of Z** and the image to the right: **Confidence Interval of 95%**





We can see that:

$$P\{-Z_{\alpha/2} \leq Z \leq Z_{\alpha/2}\} = 1 - \alpha$$

substituting Z into equation:

$$P\{-Z_{\alpha/2} \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq Z_{\alpha/2}\} = 1 - \alpha$$

isolating μ:

$$P\{\bar{X} - Z_{\alpha/2}(\sigma/\sqrt{n}) \leq \mu \leq \bar{X} + Z_{\alpha/2}(\sigma/\sqrt{n})\} = 1 - \alpha$$

Conclusion $\left[\bar{X} - Z_{\alpha/2}(\sigma/\sqrt{n}), \bar{X} + Z_{\alpha/2}(\sigma/\sqrt{n})\right]$ is a 100(1-α) CI for μ

How to Construct Confidence Interval Using Pivot Approach:
Suppose we have a random sample $X_1, X_2, \ldots, X_n$ from a population distribution and the parameter of interest is θ.

Given value $\alpha \in (0,1)$. We would like to construct a 1-α Confidence Interval using a Pivot Approach:

1. **Find a variable Y, that is function of the parameter θ and data x.**
2. **The distribution of newly created variable Y is free of θ.**

In many cases:

$$Y = \frac{\hat{\theta} - \theta}{SE(\hat{\theta})} \text{ is a pivot and the distribution of Y is symmetric about 0.}$$

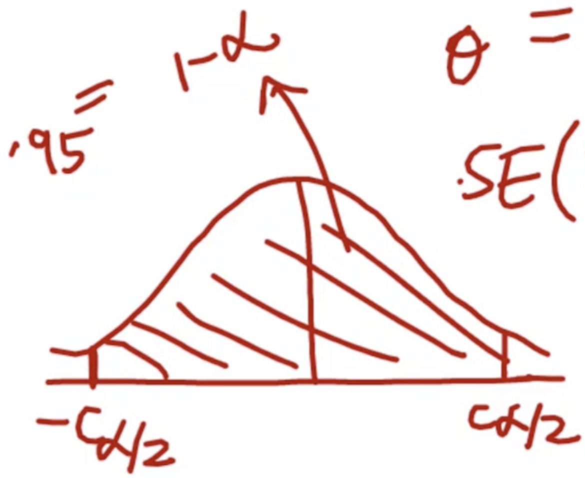
Using Pivot Approach for Two-Sided Intervals:
Find the critical points denoted $c_{\alpha/2}$ such that:

$$P\{-c_{\alpha/2} \leq Y \leq c_{\alpha/2}\} = 1 - \alpha$$

$c_{\alpha/2}$ is the upper (α / 2)100th percentile.

Critical points- give you the area to the right of the point.

Visualizing elements from Pivot Approach:
Let μ be parameter of interest. We can construct CI using pivot approach.



$\hat{\theta} = \bar{X}$, $\theta = \mu$

$SE(\hat{\theta}) = SE(\bar{X}) = \frac{\sigma}{\sqrt{n}}$.

$P(-c_{\alpha/2} \leq Y \leq c_{\alpha/2}) = 1-\alpha$

$c_{\alpha/2} = z_{.025}$

Symmetric Two-sided CI: Theorem
$\hat{\theta} \pm c_{\alpha/2}(SE(\hat{\theta}))$ is a $100(1-\alpha)\%$ confidence interval for θ

Proof:
$$1-\alpha = P\{-c_{\alpha/2} \leq Y \leq c_{\alpha/2}\}$$
$$= P\{-c_{\alpha/2} \leq \frac{\hat{\theta}-\theta}{SE(\hat{\theta})} \leq c_{\alpha/2}\}$$
$$= P\{\hat{\theta} - c_{\alpha/2}(SE(\hat{\theta})) \leq Y \leq \hat{\theta} + c_{\alpha/2}(SE(\hat{\theta}))\}$$

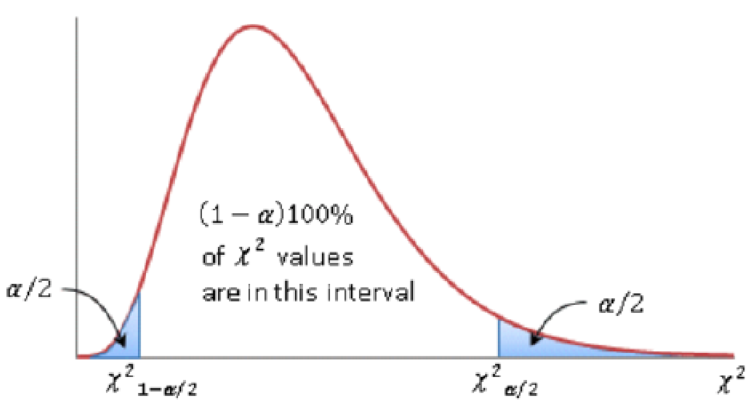
$\implies$ $\hat{\theta}$ is **within** $c_{\alpha/2}(SE(\hat{\theta}))$ of θ with **probability** 1-α

$c_{\alpha/2}(SE(\hat{\theta}))$ is known as *Margin of Error* (size of error in estimation)
Ex: In polls you might hear accurate with 0.02 (this is margin of error)

Asymmetric Two-sided CI(Non-symmetric distributions):
$[\hat{\theta} - c_{\alpha/2}(SE(\hat{\theta})), \hat{\theta} - c_{1-\alpha/2}(SE(\hat{\theta}))]$ is a $100(1-\alpha)\%$ confidence interval for θ

Proof:
$$1-\alpha = P\{c_{1-\alpha/2} \leq Y \leq c_{\alpha/2}\}$$
$$= P\{c_{1-\alpha/2} \leq \frac{\hat{\theta}-\theta}{SE(\hat{\theta})} \leq c_{\alpha/2}\}$$
$$= P\{\hat{\theta} - c_{\alpha/2}(SE(\hat{\theta})) \leq \theta \leq \hat{\theta} - c_{1-\alpha/2}(SE(\hat{\theta}))\}$$

Ex: Chi-Square distribution critical points

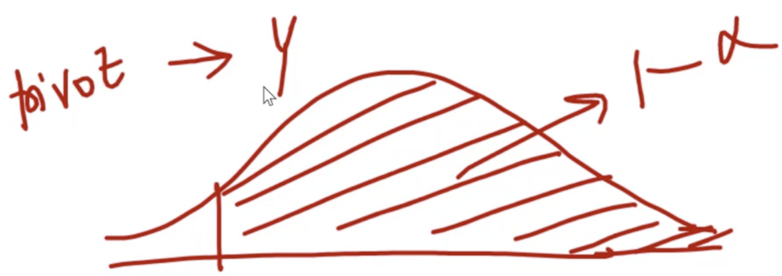


One-sided Confidence Bound:
A one-sided confidence bound defines the point where a certain percentage of the population is either higher or lower than the defined point.

Upper Bound: U = $\hat{\theta} - c_{1-\alpha}(SE(\hat{\theta}))$ when L = -∞

Lower Bound: L = $\hat{\theta} - c_{\alpha}(SE(\hat{\theta}))$ when U = ∞

Proof(Upper Bound):
Coverage probability is 1 - α.



$c_{1-\alpha}$
quantile such that area to the right is $1-\alpha$

$$1 - \alpha = P\{Y \geq c_{1-\alpha}\}$$

$$= P\{\frac{\hat{\theta} - \theta}{SE(\hat{\theta})} \geq c_{1-\alpha}\}$$

$$= P\{\theta \leq \hat{\theta} - c_{1-\alpha}(SE(\hat{\theta}))\}$$

$\implies$ U= $\hat{\theta} - c_{1-\alpha}(SE(\hat{\theta}))$

The lower bound can be computed in the same manner.

How to interpret a one-sided CI?
For **Lower Bound** critical region $\in [c_\alpha, \infty]$:We are sure parameter θ is below c_α

For **Upper Bound** critical region $\in [-\infty, c_{1-\alpha}]$:We are sure parameter θ is above $c_{1-\alpha}$

Confidence Interval on the Mean of a Normal Distribution Variance Unknown:

$\bar{X} \pm t_{\alpha/2,n-1}\left(\frac{s}{\sqrt{n}}\right)$ is a $100(1-\alpha)\%$ confidence interval for θ

Proof:

We know that $\mathrm{t}_{n-1} \sim \frac{\bar{X}-\mu}{S/\sqrt{n}}$

$$1-\alpha = P\{-t_{\alpha/2,n-1} \leq t \leq t_{\alpha/2,n-1}\}$$

$$= P\{-t_{\alpha/2,n-1} \leq \frac{\bar{X}-\mu}{S/\sqrt{n}} \leq t_{\alpha/2,n-1}\}$$

$$= P\{\bar{X} - t_{\alpha/2,n-1}(S\sqrt{n}) \leq \mu \leq \bar{X} + t_{\alpha/2,n-1}(S\sqrt{n})\}$$

Confidence Interval on the Variance or Standard Deviation of a Normal Distribution

If $X_1, X_2, \ldots, X_n$ are normally distributed and $a = \chi^2_{1-\alpha/2,n-1}$ and $b = \chi^2_{\alpha/2,n-1}$, then a $(1-\alpha)\%$ **confidence interval for the population variance** σ^2 is:

$$\left(\frac{(n-1)s^2}{b} \leq \sigma^2 \leq \frac{(n-1)s^2}{a}\right)$$

And a $(1-\alpha)\%$ **confidence interval for the population standard deviation** σ is:

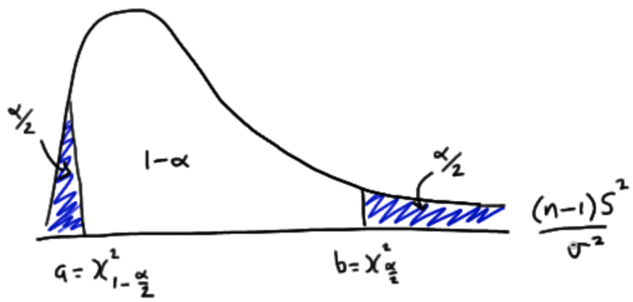
$$\left(\frac{\sqrt{(n-1)}}{\sqrt{b}}s \leq \sigma \leq \frac{\sqrt{(n-1)}}{\sqrt{a}}s\right)$$

Proof

We learned previously that if $X_1, X_2, \ldots, X_n$ are normally distributed with mean μ and population variance σ^2, then:

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

Then, using the following picture as a guide:



with ($a = \chi^2_{1-\alpha/2}$) and ($b = \chi^2_{\alpha/2}$), we can write the following probability statement:

$$P\left[a \leq \frac{(n-1)S^2}{\sigma^2} \leq b\right] = 1-\alpha$$

Now, as always it's just a matter of manipulating the quantity in the parentheses. That is:

$$a \leq \frac{(n-1)S^2}{\sigma^2} \leq b$$

Taking the reciprocal of all three terms, and thereby changing the direction of the inequalities, we get:

$$\frac{1}{a} \geq \frac{\sigma^2}{(n-1)S^2} \geq \frac{1}{b}$$

Now, multiplying through by $(n-1)S^2$, and rearranging the direction of the inequalities, we get the confidence interval for σ^2:

$$\frac{(n-1)S^2}{b} \leq \sigma^2 \leq \frac{(n-1)S^2}{a}$$

as was to be proved. And, taking the square root, we get the confidence interval for σ:

$$\frac{\sqrt{(n-1)S^2}}{\sqrt{b}} \leq \sigma \leq \frac{\sqrt{(n-1)S^2}}{\sqrt{a}}$$

as was to be proved.

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