

STAT 4352 - Mathematical Statistics Notes

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1 Chapter 11 - Interval Estimation

Point Estimators

θ is a unknown parameter (feature of a population)

- Ex: population mean μ
- **Fixed.**

$\hat{\theta}$ is a point estimator of θ (it is a numerical value)

- Ex: sample mean \bar{x}
- **Varies from sample to sample.**
- No guarantee of accuracy
- Must be *supplemented by* $\text{Var}(\theta)$
Standard Error $\text{SE}(\hat{\theta})$ measures how much $\hat{\theta}$ varies from sample to sample.
small SE \implies low variance thus a more reliable estimate of θ

Interval Estimators

Def: Interval Estimate

Provides a range of values that best describe the population.

Let $L = L(x)$ be the Lower Limit

$U = U(x)$ be the Upper Limit

Both L, U are Random Variables because they are functions of sample data.

Def: Confidence Level / Confidence Coefficient

Is the probability that the **interval estimate** will include population parameter θ .

- Sample means will follow the normal probability distribution for large sample sizes ($n \geq 30$)
- For small sample forces us to use the t-distribution probability distribution ($n < 30$)
- A confidence level of 95% implies that **95% of all samples would give an interval that includes θ , and only 5% of all samples would yield an erroneous interval.**
- The most frequently used confidence levels are 90%, 95%, and 99% with corresponding Z-scores 1.645, 1.96, 2.576.
- The higher the confidence level, the more strongly we believe that the value of the parameter lies within the interval.

Def: Confidence Interval

Gives plausible values for the parameter θ being estimated where degree of plausibility specified by a confidence level.

To construct an interval estimator of unknown parameter θ . We must find two statistics **L** and **U** such that:

$$P\{\mathbf{L} \leq \theta \leq \mathbf{U}\} = 1 - \alpha$$

- $P\{\mathbf{L} \leq \theta \leq \mathbf{U}\}$ **Coverage Probability**, in repeated sampling, what percent of samples or Confidence Intervals capture true θ .
- $100(1 - \alpha)$ **Confidence Interval** - for unknown fixed parameter θ .
- **L, U - Lower and Upper Bounds** - RVs because they are functions of sample data. Vary from sample to sample.
- $1 - \alpha$ **Confidence Level** (Probability) estimate will include population parameter θ .
- α **Level of Significance** Percent chance Confidence Interval will not contain population parameter θ .

Def: Coverage Probability

$P\{\mathbf{L} \leq \theta \leq \mathbf{U}\}$ Gives what % of samples or Confidence Intervals capture true θ .

Ex: Coverage Probability = 95%

Will capture θ , 95% of the time.

Will NOT capture θ , 5% of the time.

Properties of Confidence Intervals

- Confidence Intervals are not unique.
- Desirable to have $E[\text{Length of CI}]$ to be small.
- A one-sided $100(1 - \alpha)$ lower-confidence interval on θ : $L = -\infty \implies P\{L \leq \theta\} = 1 - \alpha$
- A one-sided $100(1 - \alpha)$ upper-confidence interval on θ : $U = \infty \implies P\{\theta \leq U\} = 1 - \alpha$
- If **L, U** are both finite, then we have a two sided interval.

Correctly Interpreting Confidence Intervals**Not Correct**

There is 90% probability that the true population mean is within the interval.

Correct

There is a 90% probability that any given Confidence Interval from a random sample will contain the true population mean.

How to Construct Confidence Interval Using Pivot Approach:

Suppose we have a random sample X_1, X_2, \dots, X_n from a population distribution and the parameter of interest is θ .

Given value $\alpha \in (0, 1)$. We would like to construct a $1-\alpha$ Confidence Interval using a Pivot Approach:

1. Find a variable Y , that is function of the parameter θ and data \mathbf{x} .
2. The distribution of newly created variable Y is free of θ .

In many cases:

$Y = \frac{\hat{\theta} - \theta}{SE(\hat{\theta})}$ is a pivot and the distribution of Y is symmetric about 0.

Using Pivot Approach for Two-Sided Intervals:

Find the critical points denoted $c_{\alpha/2}$ such that:

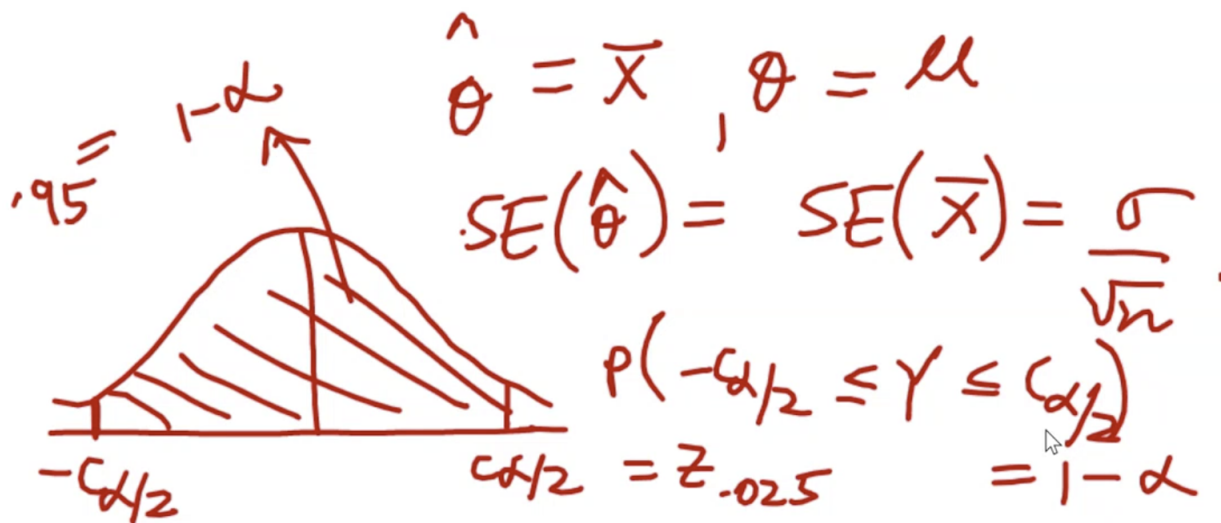
$$P\{-c_{\alpha/2} \leq Y \leq c_{\alpha/2}\} = 1 - \alpha$$

$c_{\alpha/2}$ is the upper $(\alpha / 2)100$ th percentile.

Critical points- give you the area to the right of the point.

Visualizing elements from Pivot Approach:

Let μ be parameter of interest. We can construct CI using pivot approach.



Symmetric Two-sided CI: Theorem

$\hat{\theta} \pm c_{\alpha/2}(SE(\hat{\theta}))$ is a $100(1 - \alpha)\%$ confidence interval for θ

Proof:

$$1 - \alpha = P\{-c_{\alpha/2} \leq Y \leq c_{\alpha/2}\}$$

$$= P\{-c_{\alpha/2} \leq \frac{\hat{\theta} - \theta}{SE(\hat{\theta})} \leq c_{\alpha/2}\}$$

$$= P\{\hat{\theta} - c_{\alpha/2}(SE(\hat{\theta})) \leq Y \leq \hat{\theta} + c_{\alpha/2}(SE(\hat{\theta}))\}$$

$$\implies \hat{\theta} \text{ is **within** } c_{\alpha/2}(SE(\hat{\theta})) \text{ of } \theta \text{ with **probability** } 1-\alpha$$

$c_{\alpha/2}(SE(\hat{\theta}))$ is known as *Margin of Error* (size of error in estimation)

Ex: In polls you might hear accurate with 0.02 (this is margin of error)

Asymmetric Two-sided CI(Non-symmetric distributions):

$[\hat{\theta} - c_{\alpha/2}(SE(\hat{\theta})), \hat{\theta} - c_{1-\alpha/2}(SE(\hat{\theta}))]$ is a $100(1 - \alpha)\%$ confidence interval for θ

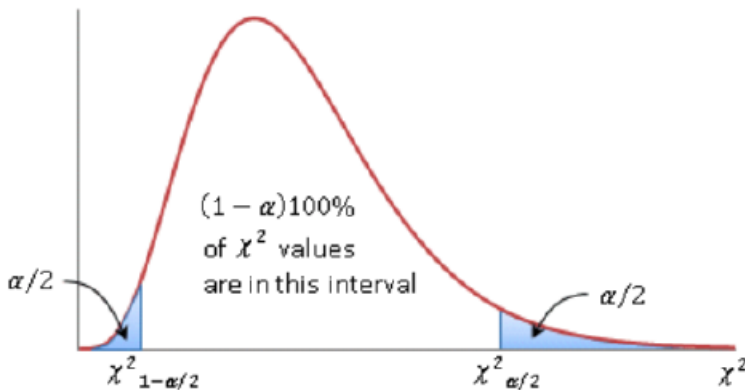
Proof:

$$1 - \alpha = P\{c_{1-\alpha/2} \leq Y \leq c_{\alpha/2}\}$$

$$= P\{c_{1-\alpha/2} \leq \frac{\hat{\theta} - \theta}{SE(\hat{\theta})} \leq c_{\alpha/2}\}$$

$$= P\{\hat{\theta} - c_{\alpha/2}(SE(\hat{\theta})) \leq \theta \leq \hat{\theta} - c_{1-\alpha/2}(SE(\hat{\theta}))\}$$

Ex: Chi-Square distribution critical points



One-sided Confidence Bound:

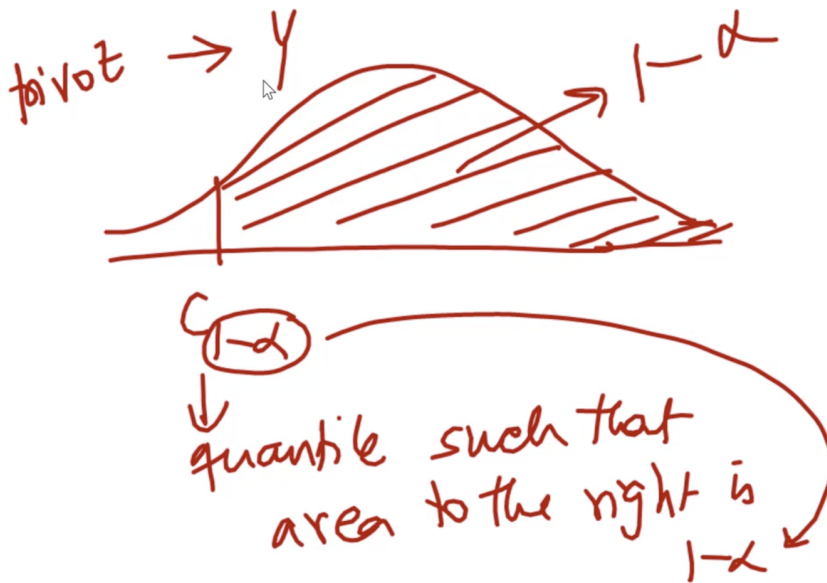
A one-sided confidence bound defines the point where a certain percentage of the population is either higher or lower than the defined point.

Upper Bound: $U = \hat{\theta} - c_{1-\alpha}(SE(\hat{\theta}))$ when $L = -\infty$

Lower Bound: $L = \hat{\theta} - c_{\alpha}(SE(\hat{\theta}))$ when $U = \infty$

Proof(Upper Bound):

Coverage probability is $1 - \alpha$.



$$\begin{aligned} 1 - \alpha &= P\{Y \geq c_{1-\alpha}\} \\ &= P\left\{\frac{\hat{\theta} - \theta}{SE(\hat{\theta})} \geq c_{1-\alpha}\right\} \\ &= P\{\theta \leq \hat{\theta} - c_{1-\alpha}(SE(\hat{\theta}))\} \end{aligned}$$

$$\implies U = \hat{\theta} - c_{1-\alpha}(SE(\hat{\theta}))$$

The lower bound can be computed in the same manner.

How to interpret a one-sided CI?

For **Lower Bound** critical region $\in [c_{\alpha}, \infty]$: We are sure parameter θ is below c_{α}

For **Upper Bound** critical region $\in [-\infty, c_{1-\alpha}]$: We are sure parameter θ is above $c_{1-\alpha}$

Choice of Sample Size for CI for Mean when Variance Known

General form of a CI:

100(1- α)% CI for θ : $\hat{\theta} \pm c_{\alpha/2} SE(\hat{\theta})$

Mean μ of a Normal population: $\bar{x} \pm Z_{\alpha/2} \left(\frac{\sigma}{\sqrt{n}} \right)$

Margin Of Error: $MOE = Z_{\alpha/2} \left(\frac{\sigma}{\sqrt{n}} \right)$

Width of CI: $2 \times MOE = 2 \cdot Z_{\alpha/2} \left(\frac{\sigma}{\sqrt{n}} \right)$

Properties of MOE:

- **As (1- α) increases MOE increases:**

The larger the CI, the larger critical points are needed.

As critical points increase, Margin Of Error: $MOE = Z_{\alpha/2} \left(\frac{\sigma}{\sqrt{n}} \right)$ increase

\implies Increase in CI will make width of CI wider

- **As σ increases MOE increases:**

Margin Of Error: $MOE = Z_{\alpha/2} \left(\frac{\sigma}{\sqrt{n}} \right)$ σ is part of numerator, if it increases it will make MOE increase too.

- **As n increases MOE decreases:**

Margin Of Error: $MOE = Z_{\alpha/2} \left(\frac{\sigma}{\sqrt{n}} \right)$ n is part of denominator, if it increases it will make MOE decrease.

Getting n from predefined MOE:

Let E_0 be a pre-specified MOE. We can find a value for n to make the following equation true.

$$\frac{Z_{\alpha/2} \cdot \sigma}{\sqrt{n}} \leq E_0 \text{ solving for } n \text{ we get: } n \geq \left(\frac{Z_{\alpha/2} \cdot \sigma}{E_0} \right)^2 \text{ (round up to nearest } n \text{.)}$$

We do this when we want to know how many observations (n) will give pre-specified margin of error - E_0

Theorem 11.1: Confidence Interval on the Mean of a Normal Distribution with known Variance

Let X be normal random variable with:

Unknown mean μ

Known variance σ^2

Suppose a random sample n , (X_1, X_2, \dots, X_n) is taken.

A $100(1-\alpha)\%$ confidence interval on μ can be obtained by considering sampling distribution of the sample mean \bar{X} .

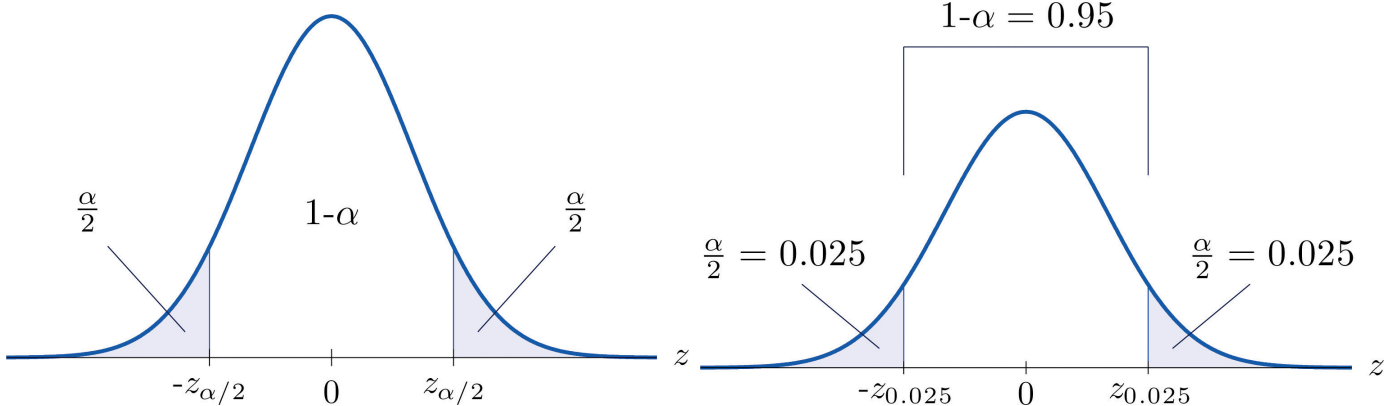
Central Limit Theorem:

$$E(\bar{X}) = \mu \text{ and } SD(\bar{X}) = \frac{\sigma}{\sqrt{n}}, \text{ so } \bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n}) \text{ as } n \rightarrow \infty$$

Let $Z = \text{Standardizing } \bar{X}$, Z will follow a Standard Normal Distribution

$$\text{Let } Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0, 1)$$

We can see from the image to the left: **Distribution of Z** and the image to the right: **Confidence Interval of 95%**



We can see that:

$$P\{-Z_{\alpha/2} \leq Z \leq Z_{\alpha/2}\} = 1 - \alpha$$

substituting Z into equation:

$$P\{-Z_{\alpha/2} \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq Z_{\alpha/2}\} = 1 - \alpha$$

isolating μ :

$$P\{\bar{X} - Z_{\alpha/2}(\sigma/\sqrt{n}) \leq \mu \leq \bar{X} + Z_{\alpha/2}(\sigma/\sqrt{n})\} = 1 - \alpha$$

Conclusion $\bar{X} \pm Z_{\alpha/2}(\sigma/\sqrt{n})$ is a $100(1-\alpha)\%$ CI for μ

Confidence Interval on the Mean of a Normal Distribution Variance Unknown and/or Small Sample:

$\bar{X} \pm t_{n-1,\alpha/2} \left(\frac{s}{\sqrt{n}} \right)$ is a $100(1 - \alpha)\%$ CI for μ

Proof:

We know that $T = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$

$$1 - \alpha = P\{-t_{n-1,\alpha/2} \leq T \leq t_{n-1,\alpha/2}\}$$

$$= P\{-t_{n-1,\alpha/2} \leq \frac{\bar{X} - \mu}{S/\sqrt{n}} \leq t_{n-1,\alpha/2}\}$$

$$= P\{\bar{X} - t_{n-1,\alpha/2}(S\sqrt{n}) \leq \mu \leq \bar{X} + t_{n-1,\alpha/2}(S\sqrt{n})\}$$

Confidence Interval on the Mean no Specific Distribution Variance Known

$\bar{X} \pm Z_{\alpha/2}(\sigma/\sqrt{n})$ is a $100(1-\alpha)\%$ CI for μ

Proof:

Assuming the sample size is large ($n \geq 30$) then by CLT:

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

The mean of any distribution **provided** n is large ($n \geq 30$) can be approximated using a Normal Distribution.

Confidence Interval on the Mean no Specific Distribution Variance Unknown

$\bar{X} \pm Z_{\alpha/2}(S/\sqrt{n})$ is a $100(1-\alpha)\%$ CI for μ

Proof:

Given the fact that S^2 is an unbiased estimator of σ^2 we can use sample variance in lieu of population variance. Also sample size is large ($n \geq 30$) then by CLT and LLN:

$$Z = \frac{\bar{X} - \mu}{S/\sqrt{n}} = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \left(\frac{\sigma}{S} \right) = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

Confidence Interval on the Proportion of a Binomial Distribution

$\hat{p} \pm Z_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ is a $100(1-\alpha)\%$ CI for p

n is random sample of size n has been taken from a large population and $X(\leq n)$ observations in this sample belong to a class of interest.

$\hat{p} = X/n$ is the point estimator of the proportion of the population that belongs to this class.

n and p are the parameters of a binomial distribution.

Proof:

The sampling distribution of \hat{p} is approximately normal with mean p and variance $p(1-p)/n$, if p is not too close to 0 or 1 and n is large.

$$Z = \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} \sim N(0, 1)$$

To construct CI on p , note that:

$$\begin{aligned} 1 - \alpha &\approx P\{-Z_{\alpha/2} \leq Z \leq Z_{\alpha/2}\} \\ &\approx P\{-Z_{\alpha/2} \leq \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} \leq Z_{\alpha/2}\} \\ &\approx P\{\hat{p} - Z_{\alpha/2} \sqrt{\frac{p(1-p)}{n}} \leq p \leq \hat{p} + Z_{\alpha/2} \sqrt{\frac{p(1-p)}{n}}\} \end{aligned}$$

Since the square root is the SE of estimator \hat{p} and also contains p in lower and upper bound.

We can replace p with \hat{p} and use Estimated SE instead of SE.

$$\approx P\{\hat{p} - Z_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq p \leq \hat{p} + Z_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\}$$

Confidence Interval on the Variance or Standard Deviation of a Normal Distribution - Mean is Unknown

$$\left[\frac{(n-1)S^2}{\chi_{n-1,\alpha/2}^2}, \frac{(n-1)S^2}{\chi_{n-1,1-\alpha/2}^2} \right] \text{ is a } 100(1-\alpha)\% \text{ CI for } \sigma^2$$

Proof:

According to theorem 8.11:

$$Y = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{\sigma^2} = \frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$$

The critical points are: $\chi_{n-1,1-\alpha/2}^2$ and $\chi_{n-1,\alpha/2}^2$

$$\begin{aligned} 1 - \alpha &= P\{\chi_{n-1,1-\alpha/2}^2 \leq \frac{(n-1)S^2}{\sigma^2} \leq \chi_{n-1,\alpha/2}^2\} \\ &= P\left[\frac{(n-1)S^2}{\chi_{n-1,\alpha/2}^2} \leq \sigma^2 \leq \frac{(n-1)S^2}{\chi_{n-1,1-\alpha/2}^2} \right] \end{aligned}$$

Confidence Interval on the Variance or Standard Deviation of a Normal Distribution - Mean is Known

$$\left[\frac{(n)S^2}{\chi_{n,\alpha/2}^2}, \frac{(n)S^2}{\chi_{n,1-\alpha/2}^2} \right] \text{ is a } 100(1-\alpha)\% \text{ CI for } \sigma^2$$

Proof:

Since μ is known then:

Sum of n, squared standard normal distributions

\implies Sum of n Chi-Square distributions with one df

$\implies \chi_n^2$

$$Y = \frac{\sum_{i=1}^n (x_i - \mu)^2}{\sigma^2} = \left(\frac{x_1 - \mu}{\sigma} \right)^2 + \dots + \left(\frac{x_n - \mu}{\sigma} \right)^2 \sim \chi_n^2$$

The critical points are: $\chi_{n,1-\alpha/2}^2$ and $\chi_{n,\alpha/2}^2$

$$\begin{aligned} 1 - \alpha &= P\{\chi_{n,1-\alpha/2}^2 \leq \frac{(n)S^2}{\sigma^2} \leq \chi_{n,\alpha/2}^2\} \\ &= P\left[\frac{(n)S^2}{\chi_{n,\alpha/2}^2} \leq \sigma^2 \leq \frac{(n)S^2}{\chi_{n,1-\alpha/2}^2} \right] \end{aligned}$$

Two-Sample Confidence Interval Estimation

Confidence Interval on the Difference between Means of Two Normal Distributions, Variances Known

In this case both means are unknown but variances are known.

$$\left[\bar{X}_1 - \bar{X}_2 \pm (Z_{\alpha/2}) \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \right] \text{ is a } 100(1-\alpha)\% \text{ CI for } \mu_1 - \mu_2$$

Proof:

Let X_1 and X_2 be two normally distributed independent random variables.

$X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$

So, $\mu_1 - \mu_2 = \bar{X}_1 - \bar{X}_2 \implies \text{SE}(\bar{X}_1 - \bar{X}_2) = \text{SD}(\bar{X}_1 - \bar{X}_2) = \sqrt{\text{Var}(\bar{X}_1 - \bar{X}_2)}$

Because $\text{Var}(A - B) = \text{Var}(A) + \text{Var}(B)$ when A,B are independent

$$\text{SE}(\bar{X}_1 - \bar{X}_2) = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \implies \boxed{\bar{X}_1 - \bar{X}_2 \sim N(\bar{X}_1 - \bar{X}_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2})}$$

Independent random samples from normal populations:

$$Z = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0, 1)$$

Now, to construct a CI:

$$\begin{aligned} 1 - \alpha &= P\{-Z_{\alpha/2} \leq Z \leq Z_{\alpha/2}\} \\ &= P\left\{-Z_{\alpha/2} \leq \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \leq Z_{\alpha/2}\right\} \\ &= P\left[(-Z_{\alpha/2})\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \leq (\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2) \leq (Z_{\alpha/2})\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right] \\ &= P\left[\bar{X}_1 - \bar{X}_2 - (Z_{\alpha/2})\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{X}_1 - \bar{X}_2 + (Z_{\alpha/2})\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}\right] \end{aligned}$$

Confidence Interval on the Difference between Means of Two Normal Distributions, Variances Unknown or Small Samples

Both means and variances are unknown. However, we can assume $\sigma_1^2 = \sigma_2^2 = \sigma^2$

$$\left[\bar{X}_1 - \bar{X}_2 \pm (t_{n_1+n_2-2, \alpha/2}) S_p \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \right] \text{ is a } 100(1-\alpha)\% \text{ CI for } \mu_1 - \mu_2$$

Proof:

Let S_1^2 and S_2^2 be sample variances of random variables X_1 and X_2 . Since both sample variances are estimates of common variance σ^2 we can obtain a pooled estimator of σ^2 .

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_2 + n_1 - 2} \sim \chi_{n_1-1}^2 + \chi_{n_2-1}^2$$

Now pivot T

$$T = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{S_p \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n_1+n_2-2}$$

Now, to construct a CI:

$$\begin{aligned} 1 - \alpha &= P\{-t_{n_1+n_2-2, \alpha/2} \leq T \leq t_{n_1+n_2-2, \alpha/2}\} \\ &= P\left\{-t_{n_1+n_2-2, \alpha/2} \leq \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{S_p \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \leq t_{n_1+n_2-2, \alpha/2}\right\} \end{aligned}$$

$$P \left[-t_{n_1+n_2-2, \alpha/2} \left(S_p \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \right) \leq (\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2) \leq t_{n_1+n_2-2, \alpha/2} \left(S_p \cdot \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \right) \right]$$

After solving for both population means:

$$P \left[\bar{X}_1 - \bar{X}_2 - (t_{n_1+n_2-2, \alpha/2}) S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{X}_1 - \bar{X}_2 + (t_{n_1+n_2-2, \alpha/2}) S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \right]$$

Confidence Interval on the Difference between Means of Two Normal Distributions, Variances Unknown and Variances Differ

Both means and variances are unknown and the variances are not equal $\sigma_1^2 \neq \sigma_2^2$. In this case we make the assumption that variances are different.

$$\left[\bar{X}_1 - \bar{X}_2 \pm (t_{v,\alpha/2}) \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}} \right] \text{ is a } 100(1-\alpha)\% \text{ CI for } \mu_1 - \mu_2$$

Proof:

From previous cases we know that:

$$\boxed{\bar{X}_1 - \bar{X}_2 \sim N\left(\bar{X}_1 - \bar{X}_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right)}$$

Now let Y be the approximate pivot:

$$T = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} \approx t_v \text{ where df } v = \frac{(S_1^2/n_1 + S_2^2/n_2)^2}{\frac{(S_1^2/n_1)^2}{n_1 + 1} + \frac{(S_2^2/n_2)^2}{n_2 + 1}}$$

$$1 - \alpha = P\{-t_{v,\alpha/2} \leq T \leq t_{v,\alpha/2}\}$$

$$= P\left\{-t_{v,\alpha/2} \leq \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} \leq t_{v,\alpha/2}\right\}$$

$$P\left[-t_{v,\alpha/2} \left(\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}\right) \leq (\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2) \leq t_{v,\alpha/2} \left(\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}\right)\right]$$

After solving for both population means:

$$P\left[\bar{X}_1 - \bar{X}_2 - (t_{v,\alpha/2})\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{X}_1 - \bar{X}_2 + (t_{v,\alpha/2})\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}\right]$$

Confidence Interval on the Ratio of Variances of Two Normal Distributions

► CI for σ_1^2/σ_2^2 .

Recall: $\frac{\chi_{n_1-1}^2/(n_1-1)}{\chi_{n_2-1}^2/(n_2-1)} \sim F_{n_1-1, n_2-1}; \quad \frac{1}{F_{n_1-1, n_2-1}} = \frac{\chi_{n_2-1}^2/(n_2-1)}{\chi_{n_1-1}^2/(n_1-1)} \sim F_{n_2-1, n_1-1}$

Pivot: $Y = \frac{\frac{(n_1-1)S_1^2}{\sigma_1^2} \cdot \frac{1}{(n_1-1)}}{\frac{(n_2-1)S_2^2}{\sigma_2^2} \cdot \frac{1}{(n_2-1)}} = \frac{S_1^2}{S_2^2} \cdot \frac{\sigma_2^2}{\sigma_1^2} \sim F_{n_1-1, n_2-1}$

$$\begin{aligned} 1 - \alpha &= P \left[F_{1-\alpha/2, n_1-1, n_2-1} \leq \frac{S_1^2}{S_2^2} \cdot \frac{\sigma_2^2}{\sigma_1^2} \leq F_{\alpha/2, n_1-1, n_2-1} \right] \\ &= P \left[F_{1-\alpha/2, n_1-1, n_2-1} \cdot \frac{S_2^2}{S_1^2} \leq \frac{\sigma_2^2}{\sigma_1^2} \leq F_{\alpha/2, n_1-1, n_2-1} \cdot \frac{S_2^2}{S_1^2} \right] \\ &= P \left[\frac{1}{F_{\alpha/2, n_1-1, n_2-1}} \cdot \frac{S_1^2}{S_2^2} \leq \frac{\sigma_1^2}{\sigma_2^2} \leq \frac{1}{F_{1-\alpha/2, n_1-1, n_2-1}} \cdot \frac{S_1^2}{S_2^2} \right] \end{aligned}$$

100(1 - α)% CI for $\frac{\sigma_1^2}{\sigma_2^2}$: $\left[\frac{1}{F_{\alpha/2, n_1-1, n_2-1}} \cdot \frac{S_1^2}{S_2^2}, \frac{1}{F_{1-\alpha/2, n_1-1, n_2-1}} \cdot \frac{S_1^2}{S_2^2} \right]$

100(1 - α)% CI for $\frac{\sigma_2^2}{\sigma_1^2}$: $\left[F_{1-\alpha/2, n_1-1, n_2-1} \cdot \frac{S_2^2}{S_1^2}, F_{\alpha/2, n_1-1, n_2-1} \cdot \frac{S_2^2}{S_1^2} \right]$

Visualizing Ration of Variances:

