

STAT 4352 - Mathematical Statistics Notes

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1 Chapter 11 - Interval Estimation

Point Estimators

θ is a unknown parameter (feature of a population)

- Ex: population mean μ
- **Fixed.**

$\hat{\theta}$ is a point estimator of θ (it is a numerical value)

- Ex: sample mean \bar{x}
- **Varies from sample to sample.**
- No guarantee of accuracy
- Must be *supplemented by* $\text{Var}(\theta)$
Standard Error $\text{SE}(\hat{\theta})$ measures how much $\hat{\theta}$ varies from sample to sample.
small SE \implies low variance thus a more reliable estimate of θ

Interval Estimators

Def: Interval Estimate

Provides a range of values that best describe the population.

Let $L = L(x)$ be the Lower Limit

$U = U(x)$ be the Upper Limit

Both L, U are Random Variables because they are functions of sample data.

Def: Confidence Level / Confidence Coefficient

Is the probability that the **interval estimate** will include population parameter θ .

- Sample means will follow the normal probability distribution for large sample sizes ($n \geq 30$)
- For small sample forces us to use the t-distribution probability distribution ($n < 30$)
- A confidence level of 95% implies that **95% of all samples would give an interval that includes θ , and only 5% of all samples would yield an erroneous interval.**
- The most frequently used confidence levels are 90%, 95%, and 99% with corresponding Z-scores 1.645, 1.96, 2.576.
- The higher the confidence level, the more strongly we believe that the value of the parameter lies within the interval.

Def: Confidence Interval

Gives plausible values for the parameter θ being estimated where degree of plausibility specified by a confidence level.

To construct an interval estimator of unknown parameter θ . We must find two statistics **L** and **U** such that:

$$P\{\mathbf{L} \leq \theta \leq \mathbf{U}\} = 1 - \alpha$$

- $P\{\mathbf{L} \leq \theta \leq \mathbf{U}\}$ **Coverage Probability**, in repeated sampling, what percent of samples or Confidence Intervals capture true θ .
- $100(1 - \alpha)$ **Confidence Interval** - for unknown fixed parameter θ .
- **L, U - Lower and Upper Bounds** - RVs because they are functions of sample data. Vary from sample to sample.
- $1 - \alpha$ **Confidence Level** (Probability) estimate will include population parameter θ .
- α **Level of Significance** Percent chance Confidence Interval will not contain population parameter θ .

Def: Coverage Probability

$P\{\mathbf{L} \leq \theta \leq \mathbf{U}\}$ Gives what % of samples or Confidence Intervals capture true θ .

Ex: Coverage Probability = 95%

Will capture θ , 95% of the time.

Will NOT capture θ , 5% of the time.

Properties of Confidence Intervals

- Confidence Intervals are not unique.
- Desirable to have $E[\text{Length of CI}]$ to be small.
- A one-sided $100(1 - \alpha)$ lower-confidence interval on θ : $L = -\infty \implies P\{L \leq \theta\} = 1 - \alpha$
- A one-sided $100(1 - \alpha)$ upper-confidence interval on θ : $U = \infty \implies P\{\theta \leq U\} = 1 - \alpha$
- If **L, U** are both finite, then we have a two sided interval.

Correctly Interpreting Confidence Intervals**Not Correct**

There is 90% probability that the true population mean is within the interval.

Correct

There is a 90% probability that any given Confidence Interval from a random sample will contain the true population mean.

Theorem 11.1: Confidence Interval on the Mean of a Normal Distribution with known Variance

Let X be normal random variable with:

Unknown mean μ

Known variance σ^2

Suppose a random sample n , (X_1, X_2, \dots, X_n) is taken.

A $100(1-\alpha)\%$ confidence interval on μ can be obtained by considering sampling distribution of the sample mean \bar{X} .

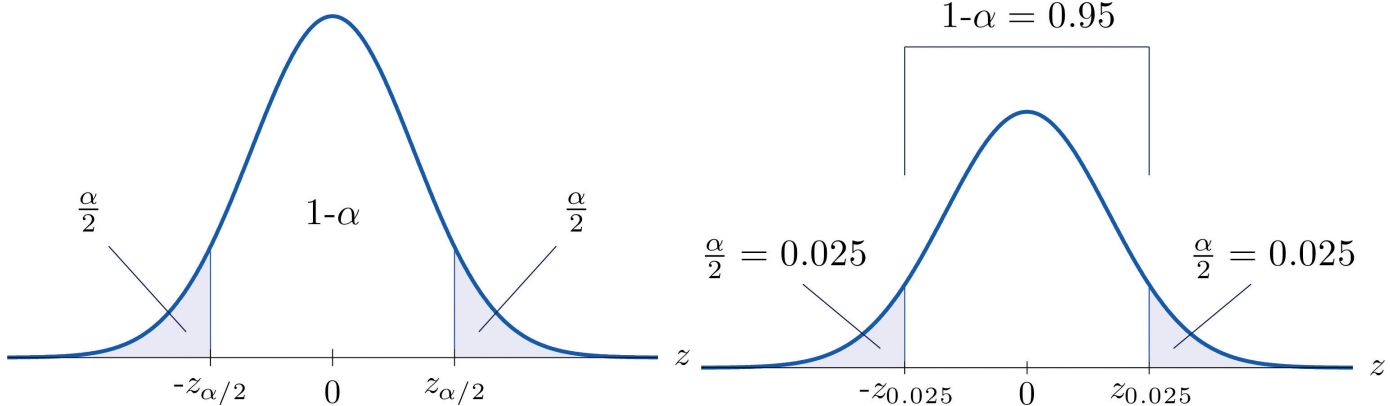
Central Limit Theorem:

$$E(\bar{X}) = \mu \text{ and } SD(\bar{X}) = \frac{\sigma}{\sqrt{n}}, \text{ so } \bar{X} \sim \mathcal{N}(\mu, \frac{\sigma^2}{n}) \text{ as } n \rightarrow \infty$$

Let $Z = \text{Standardizing } \bar{X}$, Z will follow a Standard Normal Distribution

$$\text{Let } Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \mathcal{N}(0, 1)$$

We can see from the image to the left: **Distribution of Z** and the image to the right: **Confidence Interval of 95%**



We can see that:

$$P\{-Z_{\alpha/2} \leq Z \leq Z_{\alpha/2}\} = 1 - \alpha$$

substituting Z into equation:

$$P\{-Z_{\alpha/2} \leq \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \leq Z_{\alpha/2}\} = 1 - \alpha$$

isolating μ :

$$P\{\bar{X} - Z_{\alpha/2}(\sigma/\sqrt{n}) \leq \mu \leq \bar{X} + Z_{\alpha/2}(\sigma/\sqrt{n})\} = 1 - \alpha$$

Conclusion $[\bar{X} - Z_{\alpha/2}(\sigma/\sqrt{n}), \bar{X} + Z_{\alpha/2}(\sigma/\sqrt{n})]$ is a $100(1-\alpha)$ CI for μ

How to Construct Confidence Interval Using Pivot Approach:

Suppose we have a random sample X_1, X_2, \dots, X_n from a population distribution and the parameter of interest is θ .

Given value $\alpha \in (0, 1)$. We would like to construct a $1-\alpha$ Confidence Interval using a Pivot Approach:

1. Find a variable Y , that is function of the parameter θ and data \mathbf{x} .
2. The distribution of newly created variable Y is free of θ .

In many cases:

$Y = \frac{\hat{\theta} - \theta}{SE(\hat{\theta})}$ is a pivot and the distribution of Y is symmetric about 0.

Using Pivot Approach for Two-Sided Intervals:

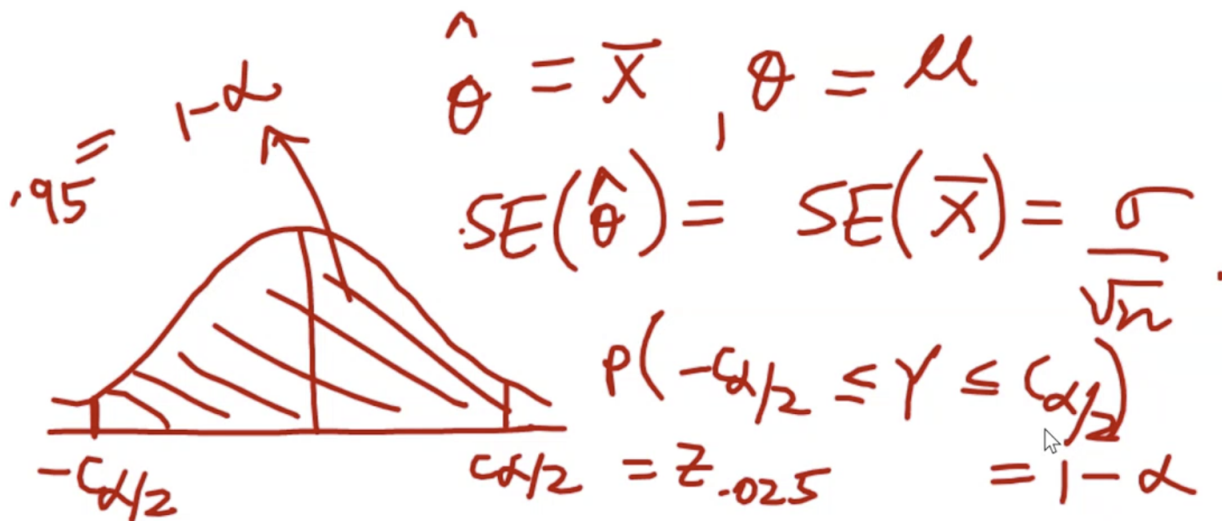
Find the cut points(quantiles) $c_{\alpha/2}$ such that:

$$P\{-c_{\alpha/2} \leq Y \leq c_{\alpha/2}\} = 1 - \alpha$$

$c_{\alpha/2}$ is the upper $(\alpha / 2)100$ th percentile.

Visualizing elements from Pivot Approach:

Let μ be parameter of interest. We can construct CI using pivot approach.



Symmetric Two-sided CI: Theorem

$\hat{\theta} \pm c_{\alpha/2}(SE(\hat{\theta}))$ is a $100(1 - \alpha)\%$ confidence interval for θ

Proof:

$$1 - \alpha = P\{-c_{\alpha/2} \leq Y \leq c_{\alpha/2}\}$$

$$= P\left\{-c_{\alpha/2} \leq \frac{\hat{\theta} - \theta}{SE(\hat{\theta})} \leq c_{\alpha/2}\right\}$$

$$= P\{\hat{\theta} - c_{\alpha/2}(SE(\hat{\theta})) \leq Y \leq \hat{\theta} + c_{\alpha/2}(SE(\hat{\theta}))\}$$

$$\implies \hat{\theta} \text{ is within } c_{\alpha/2}(SE(\hat{\theta})) \text{ of } \theta \text{ with probability } 1-\alpha$$

$c_{\alpha/2}(SE(\hat{\theta}))$ is known as *Margin of Error* (size of error in estimation)

Ex: In polls you might hear accurate with 0.02 (this is margin of error)

Asymmetric Two-sided CI(Non-symmetric distributions):

$\hat{\theta} \pm c_{\alpha/2}(SE(\hat{\theta}))$ is a $100(1 - \alpha)\%$ confidence interval for θ CHANGE THIS!!!!!!!!!!!!!!

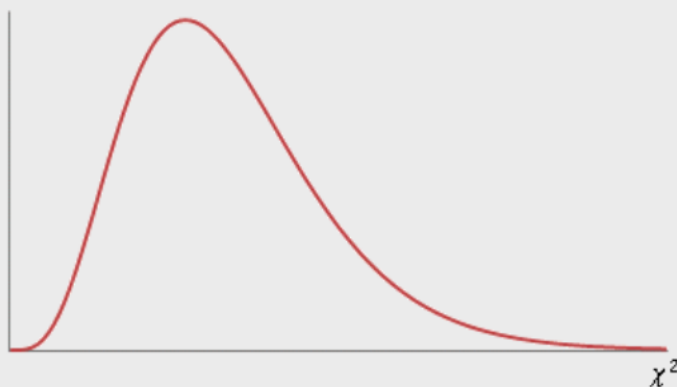
Example:

The Chi-Square (χ^2) distribution

If a simple random sample size n is obtained from a normally distributed population with mean μ and standard deviation σ , then

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2}$$

has a **chi-square distribution** with $n-1$ degrees of freedom.



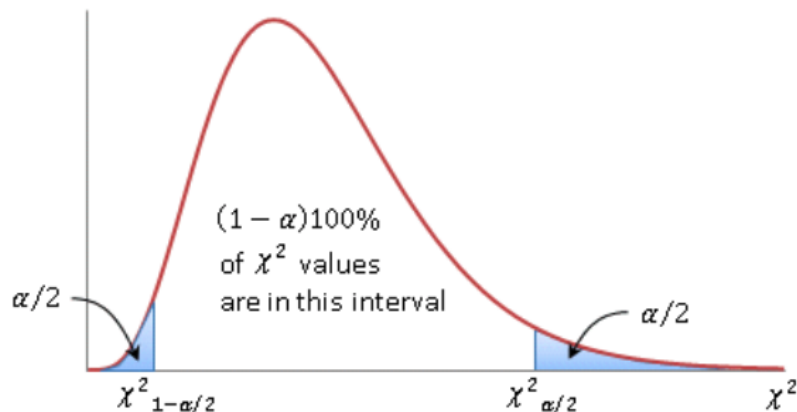
Properties of the χ^2 distribution

1. It is *not* symmetric.
2. The shape depends on the degrees of freedom.
3. As the number of degrees of freedom increases, the distribution becomes more symmetric.
4. $\chi^2 \geq 0$

Constructing Confidence Intervals about σ^2 and σ

Now that we have the basics of the distribution of the variable χ^2 , we can work on constructing a formula for the confidence interval.

From the distribution shape on the previous page, we know that $(1 - \alpha)100\%$ of the χ^2 values will be between the two critical values shown below.



This gives us the following inequality:

$$\chi^2_{1-\alpha/2} < \frac{(n-1)s^2}{\sigma^2} < \chi^2_{\alpha/2}$$

If we solve the inequality for σ^2 , we get the formula for the confidence interval:

A **$(1-\alpha)100\%$ confidence interval for σ^2** is

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}$$

Note: The sample *must* be taken from a normally distributed population.

Note #2: If a confidence interval for σ is desired, we can take the square root of each part.

Notes:

Chi-Square takes on non-negative values.

Since the distribution is not symmetrical(asymmetrical) you cannot just take plus,minus cut points. We can see from the image above that the **area to the right of cut off points** are:

- Lower bound critical point : $1-\alpha/2$
- Upper bound critical point: $\alpha/2$