STAT 4352 - Mathematical Statistics Notes

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1 Chapter 10.2 - Unbiased Estimators

Definition: 10.2 Unbiased Estimator
A statistic $\hat{\theta}$ is an ***Unbiased Estimator*** of parameter θ if and only if:

$$\mathrm{E}[\hat{\theta}] = \theta$$

That is $\hat{\theta}$ on average its value equals θ.

Definition: Bias
Bias of $\hat{\theta}$: $\mathrm{b}_n(\hat{\theta}) = \mathrm{E}[\hat{\theta}] - \theta$
When:

$$\mathrm{b}_n(\hat{\theta}) = \mathrm{E}[\hat{\theta}] - \theta = 0 \textbf{ (Unbiased Estimator)}$$
$$\mathrm{b}_n(\hat{\theta}) = \mathrm{E}[\hat{\theta}] - \theta \neq 0 \textbf{ (Biased Estimator)}$$

Definition: Asymptotically Unbiased Estimator
Based on a random sample n, from a given distribution. We say $\hat{\theta}$ is a ***Asymptotically Unbiased Estimator*** if and only if:

$$\lim_{n \to \infty} b_n(\hat{\theta}) = 0$$

Properties of Unbiased Estimators

- $\overline{x}$ is always unbiased for all distributions.
- NOT UNIQUE. (there can be multiple unbiased estimators).
  If you can have multiple unbiased estimators which one is best?
  Next desireable properties are sufficiency and low variance.
- Does not have invariance property.
  $\overline{x}$ is unbiased for $\mu \not\Rightarrow \overline{x}^2$ is unbiased for μ^2

2 Chapter 10.3 - Efficiency

How to measure accuracy of estimators?

1) Mean Absolute Error (MAE)
$\text{MAE}_\theta = \text{E}[\ |\hat{\theta} - \theta|\ ]$

2) Mean Absolute Deviation (MAD)
$\text{MAD}_\theta = \text{median}[\ |\hat{\theta} - \theta|\ ]$

3) Mean Squared Error (MSE)
$\text{MSE}_\theta = \text{E}(\hat{\theta} - \theta)^2 = \text{Var}_\theta(\hat{\theta}) + \text{Bias}^2_\theta(\hat{\theta})$

For an unbiased estimator (Bias = 0)
$\text{MSE}_\theta = \text{Var}_\theta(\hat{\theta})$

Definition: Relative Efficiency
Let $\hat{\theta}_1$ and $\hat{\theta}_2$ be unbiased estimators of θ.

$$\text{If} \quad \frac{Var_\theta(\hat{\theta}_1)}{Var_\theta(\hat{\theta}_2)} < 1$$

We can say that $\hat{\theta}_1$ is more efficient than $\hat{\theta}_2$.
You would want to pick the estimator $\hat{\theta}$ that is more efficient (lowest variance).

Efficiency Example 1: If $\dfrac{Var_\theta(\hat{\theta}_1)}{Var_\theta(\hat{\theta}_2)} = 0.50 \Longrightarrow Var_\theta(\hat{\theta}_1) = 0.5 Var_\theta(\hat{\theta}_2)$

$\hat{\theta}_1$ is 50% **MORE** efficient than $\hat{\theta}_2$

Efficiency Example 2: If $\dfrac{Var_\theta(\hat{\theta}_1)}{Var_\theta(\hat{\theta}_2)} = 1.50 \Longrightarrow Var_\theta(\hat{\theta}_1) = 1.5 Var_\theta(\hat{\theta}_2)$

$\hat{\theta}_1$ is 50% **LESS** efficient than $\hat{\theta}_2$
OR
$\hat{\theta}_2$ is 50% **MORE** efficient than $\hat{\theta}_1$

Definition: Asymptotic Relative Efficiency
Based on a random sample n, from a given distribution. We define the comparison of estimators ($\hat{\theta}_1$,$\hat{\theta}_2$) is ***Asymptotically Relative Efficiency*** when:

$$ARE = \lim_{n \to \infty} \frac{Var_\theta(\hat{\theta}_1)}{Var_\theta(\hat{\theta}_2)} < 1$$

The efficiency(gain) is reduced as sample size $n \to \infty$. For huge sample sizes both unbiased estimators are equally good. For small n one estimator is better than other.

Definition: Uniformly Minimum Variance Unbiased Estimator
An unbiased estimator $\hat{\theta}$ is ***Uniformly Minimum Variance Unbiased Estimator (UMVUE)*** for θ if it has the smallest variance in the class of all unbiased estimators for θ.

Theorem 10.2: Cramer-Rao Inequality
It is possible to obtain a lower bound on the variance of all ***unbiased estimators*** θ.

- $\hat{\theta}$ be a unbiased estimator
- f(x, θ) is the probability distribution of random variable x.
- n is a random sample size

The ***Lower Bound of Variance of an Unbiased Estimator*** is the defined by the Cramer-Rao inequality:

$$Var(\hat{\theta}) \geq \frac{1}{I(\theta)} \qquad \text{where } I(\theta) = nE\left[\left(\frac{\partial lnf(x)}{\partial \theta}\right)^2\right]$$

$I(\theta)$ is the Fisher Information in a random sample of size n and $\dfrac{\partial lnf(x)}{\partial \theta}$ is known as score function. It is the smallest possible value variance can have.

UMVUE exists when:

If $Var(\hat{\theta}) = \dfrac{1}{I(\theta)}$ It has smallest possible value for variance.

$\Longrightarrow \hat{\theta}$ ***is UMVU of*** $\hat{\theta}$

UMVUE does not exists when:

If $Var(\hat{\theta}) \neq \dfrac{1}{I(\theta)}$ ***You can't say $\hat{\theta}$ is UMVUE as lower bound is not achievable.***

3 Chapter 10.4 - Consistency

Definition: Consistency

If $\hat{\theta}$ is an estimator of θ based on a random sample of size n, we say that $\hat{\theta}$ is ***consistent (closed)*** for θ, if $\epsilon > 0$:

$$\lim_{n\to\infty} P(|\hat{\theta} - \theta| < \epsilon) = 1 \qquad \theta - targetparameter, \hat{\theta} - estimator$$

ϵ - estimator (small distance ex: 0.0001)

Consistency is an Asymptotic Property:

Error in estimation using $\hat{\theta}$ is small

$\hat{\theta}$ converges in probability to θ

When $n \to \infty$ we can be practically certain that the error made with a consistent estimator will be less than any small preassigned positive constant ϵ.

Theorem 10.3

If $\hat{\theta}$ is an unbiased estimator of the parameter θ and Var($\hat{\theta}$) $\to 0$ Bias($\hat{\theta}$) $\to 0$ as $n \to \infty$ then $\hat{\theta}$ is a consistent estimator of θ.

4 Chapter 10.5 - Sufficiency

Definition: Sufficient Principle

- Reduce data without loosing information about θ.
- Captures all information about a sample relevant to estimation of θ, that is, if all the knowledge about θ that can be gained from the individual sample values and their order can just as well be gained from the value of $\hat{\theta}$ alone.

Definition: Sufficient Estimator
The statistic $\hat{\theta}$ is a sufficient estimator of parameter θ of a given distribution ***iff*** for each value of $\hat{\theta}$ ***the conditional probability distribution or density*** of a random sample $x_1, x_2, \ldots x_n$ given $\hat{\theta} = \theta$ ***is independent of*** θ.

Sufficient property from conditional probability distribution or density when $\hat{\theta} = \theta$:

$$f(x_1, x_2, \ldots x_n ; \hat{\theta}) = \frac{f(x_1, x_2, \ldots x_n, \hat{\theta})}{g(\hat{\theta})}$$

Note: Ratio should not contain θ in order to be sufficient estimator of θ

Theorem 10.4: Factorization Theorem
The statistic $\hat{\theta}$ is a sufficient estimator of the parameter θ ***iff*** the joint probability distribution or density of the random sample can be factored so that:

$$f(x_1, x_2, \ldots x_n ; \hat{\theta}) = g(\hat{\theta}, \theta) * h(x_1, x_2, \ldots x_n)$$

where $g(\hat{\theta}, \theta)$ depends on θ and $\hat{\theta}$ and $h(x_1, x_2, \ldots x_n)$ does not depend on θ.

Using factorization you want to identify:

- g function $\Longrightarrow$ function θ
- h function $\Longrightarrow$ function without θ
  (h(x) = 1 if not present)

Properties of Sufficiency:

- Complete data is always sufficient.
- Any 1-1 function of a sufficient statistic is also sufficient.
- Good estimators should be functions of sufficient statistic.
  (a good estimator is sufficient)

5 Chapter 10.8 - Method of Maximum Likelihood

Notation

X = (X_1,X_2,...X_n) $\overset{i.i.d.}{\sim}$ f_θ(x)

Data before observed - r.v.'s with same distribution.

θ may be a vector, $\theta \in \Theta$

Θ is parameter space.

Ex: parameter space of $\mathcal{N}(\mu,\, \sigma^2)$ is $-\infty < \mu < \infty$, $-\infty < \sigma^2 < \infty$,

x = (x_1,x_2,...x_n) Data that has been observed.

Definition: Likelihood Function

Joint pdf/pmf of X considered as a function of θ keeping the data X fixed.

$$\mathcal{L}(\theta) = \prod_{n=1}^{n} f_\theta(\mathrm{x}_i) = \mathrm{f}(\ \mathrm{x}_1,\mathrm{x}_2,...\mathrm{x}_n\ ;\ \hat{\theta}) = \mathrm{f}(\mathrm{x}_1,\hat{\theta})\mathrm{f}(\mathrm{x}_2,\hat{\theta})...\mathrm{f}(\mathrm{x}_n,\hat{\theta})$$

vary θ to find value that maximizes product, this value for θ is known as MLE.

Definition: Maximum Likelihood Estimator

The Maximum Likelihood Estimator (MLE) of θ is the value θ that maximizes the Likelihood function $\mathcal{L}(\theta)$.

- Complete data is always sufficient.
- Value of θ that maximizes $\mathcal{L}(\theta)$ also maximizes $\log \mathcal{L}(\theta)/\ln \mathcal{L}(\theta)$.
- First Derivative: $\dfrac{\partial \mathcal{L}(\theta)}{\partial \theta} \implies$ Critical Points (Max/Min)
- Second Derivative: $\dfrac{\partial^2 \mathcal{L}(\theta)}{\partial \theta^2} < 0 \implies$ Maximum exits.

Log-Likelihood Function log$\mathcal{L}(\theta)$/ln$\mathcal{L}(\theta)$

Because log/ln is a monotone function, if we ***maximize log-Likelihood it is the same as maximizing Likelihood.*** The reason why because <u>$\log \mathcal{L}(\theta)/\ln \mathcal{L}(\theta)$ is used because taking the derivative is much easier.</u> When referring to log we mean $\log_e = \ln$.

- $\log(ab) = \log(a) + \log(b)$

  Ex: $f(x) = \prod_{n=1}^{n} g(x) \implies \ln f(x) = \sum_{n=1}^{n} \ln g(x)$

This property applies to both log/ln. ***The inner product can be expressed as a sum of individual elements.*** This comes super handy when taking derivatives.

Properties of a Maximum Likelihood Estimator

- $\hat{\theta}_{MLE}$ is always a function of sufficient statistics whenever they exist.
- Optimal when n is large.
- May not be good when the distribution assumptions are wrong.
- $\hat{\theta}_{MLE} \in \Theta$ (MLE is included in parameter space)
- $\hat{\theta}_{MLE}$ has invariance property:

$$\hat{\theta} \text{ is MLE for } \theta$$
$$\Longleftrightarrow$$
$$\hat{\theta}^2 \text{ is MLE for } \theta^2$$
$$\Longleftrightarrow$$
$$\ldots\ldots\ldots$$

6 Chapter number - Chapter Name

Theorem number: Theorem Name