GAMES OF STRATEGY

SECOND EDITION

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Contents

PART FOUR
Applications to Specific Strategic Situations

PART ONE

Introduction and General Principles

1

Basic Ideas and Examples

ALL INTRODUCTORY TEXTBOOKS begin by attempting to convince the student readers that the subject is of great importance in the world and therefore merits their attention. The physical sciences and engineering claim to be the basis of modern technology and therefore of modern life; the social sciences discuss big issues of governance—for example, democracy and taxation; the humanities claim that they revive your soul after it has been deadened by exposure to the physical and social sciences and to engineering. Where does the subject games of strategy, often called game theory, fit into this picture, and why should you study it?

We offer a practical motivation much more individual and closer to your personal concerns than most other subjects. You play games of strategy all the time: with your parents, siblings, friends, and enemies, and even with your professors. You have probably acquired a lot of instinctive expertise, and we hope you will recognize in what follows some of the lessons that you have already learned. We will build on this experience, systematize it, and develop it to the point where you will be able to improve your strategic skills and use them more methodically. Opportunities for such uses will appear throughout the rest of your life; you will go on playing such games with your employers, employees, spouses, children, and even strangers.

Not that the subject lacks wider importance. Similar games are played in business, politics, diplomacy, and wars—in fact, whenever people interact to strike mutually agreeable deals or to resolve conflicts. Being able to recognize such games will enrich your understanding of the world around you and will make you a better participant in all its affairs.

It will also have a more immediate payoff in your study of many other subjects. Economics and business courses already use a great deal of game-theoretic thinking. Political science is rapidly catching up. Biology has been importantly influenced by the concepts of evolutionary games and has in turn exported these ideas to economics. Psychology and philosophy also interact with the study of games of strategy. Game theory has become a provider of concepts and techniques of analysis for many disciplines, one might say all disciplines except those dealing with completely inanimate objects.

1 WHAT IS A GAME OF STRATEGY?

The word *game* may convey an impression that the subject is frivolous or unimportant in the larger scheme of things—that it deals with trivial pursuits such as gambling and sports when the world is full of more weighty matters such as war and business and your education, career, and relationships. Actually, games of strategy are not "just a game"; all of these weighty matters are instances of games, and game theory helps us understand them all. But it will not hurt to start with gambling or sports.

Most games include chance, skill, and strategy in varying proportions. Playing double or nothing on the toss of a coin is a game of pure chance, unless you have exceptional skill in doctoring or tossing coins. A hundred-yard dash is a game of pure skill, although some chance elements can creep in; for example, a runner may simply have a slightly off day for no clear reason.

Strategy is a skill of a different kind. In the context of sports, it is a part of the mental skill needed to play well; it is the calculation of how best to use your physical skill. For example, in tennis, you develop physical skill by practicing your serves (first serves hard and flat, second serves with spin or kick) and passing shots (hard, low, and accurate). The strategic skill is knowing where to put your serve (wide, or on the T) or passing shot (crosscourt, or down the line). In football, you develop such physical skills as blocking and tackling, running and catching, and throwing. Then the coach, knowing the physical skills of his own team and those of the opposing team, calls the plays that best exploit his team's skills and the other team's weaknesses. The coach's calculation constitutes the strategy. The physical game of football is played on the gridiron by jocks; the strategic game is played in the offices and on the sidelines by coaches and by nerdy assistants.

A hundred-yard dash is a matter of exercising your physical skill as best you can; it offers no opportunities to observe and react to what other runners in the race are doing and therefore no scope for strategy. Longer races do entail strategy—whether you should lead to set the pace, how soon before the finish you should try to break away, and so on.

Strategic thinking is essentially about your interactions with others: someone else is also doing similar thinking at the same time and about the same situation. Your opponents in a marathon may try to frustrate or facilitate your attempts to lead, as they think best suits their interests. Your opponent in tennis tries to guess where you will put your serve or passing shot; the opposing coach in football calls the play that will best counter what he thinks you will call. Of course, just as you must take into account what the other player is thinking, he is taking into account what you are thinking. Game theory is the analysis, or science, if you like, of such interactive decision making.

When you think carefully before you act—when you are aware of your objectives or preferences and of any limitations or constraints on your actions and choose your actions in a calculated way to do the best according to your own criteria—you are said to be behaving rationally. Game theory adds another dimension to rational behavior—namely, interaction with other equally rational decision makers. In other words, game theory is the science of rational behavior in interactive situations.

We do not claim that game theory will teach you the secrets of perfect play or ensure that you will never lose. For one thing, your opponent can read the same book, and both of you cannot win all the time. More importantly, many games are complex and subtle enough, and most actual situations include enough idiosyncratic or chance elements, that game theory cannot hope to offer surefire recipes for action. What it does is to provide some general principles for thinking about strategic interactions. You have to supplement these ideas and some methods of calculation with many details specific to your situation before you can devise a successful strategy for it. Good strategists mix the science of game theory with their own experience; one might say that game playing is as much art as science. We will develop the general ideas of the science but will also point out its limitations and tell you when the art is more important.

You may think that you have already acquired the art from your experience or instinct, but you will find the study of the science useful nonetheless. The science systematizes many general principles that are common to several contexts or applications. Without general principles, you would have to figure out from scratch each new situation that requires strategic thinking. That would be especially difficult to do in new areas of application—for example, if you learned your art by playing games against parents and siblings and must now practice strategy against business competitors. The general principles of game theory provide you with a ready reference point. With this foundation in place, you can proceed much more quickly and confidently to acquire and add the situation-specific features or elements of the art to your thinking and action.

2 SOME EXAMPLES AND STORIES OF STRATEGIC GAMES

With the aims announced in Section 1, we will begin by offering you some simple examples, many of them taken from situations that you have probably encountered in your own lives, where strategy is of the essence. In each case we will point out the crucial strategic principle. Each of these principles will be discussed more fully in a later chapter, and after each example we will tell you where the details can be found. But don't jump to them right away; for a while, just read all the examples to get a preliminary idea of the whole scope of strategy and of strategic games.

A. Which Passing Shot?

Tennis at its best consists of memorable duels between top players: John McEnroe versus Ivan Lendl, Pete Sampras versus Andre Agassi, and Martina Navratilova versus Chris Evert. Picture the 1983 U.S. Open final between Evert and Navratilova.[1] Navratilova at the net has just volleyed to Evert on the baseline. Evert is about to hit a passing shot. Should she go down the line or crosscourt? And should Navratilova expect a down-the-line shot and lean slightly that way or expect a crosscourt shot and lean the other way?

Conventional wisdom favors the down-the-line shot. The ball has a shorter distance to travel to the net, so the other player has less time to react. But this does not mean that Evert should use that shot all of the time. If she did, Navratilova would confidently come to expect it and prepare for it, and the shot would not be so successful. To improve the success of the down-the-line passing shot, Evert has to use the crosscourt shot often enough to keep Navratilova guessing on any single instance.

Similarly in football, with a yard to go on third down, a run up the middle is the percentage play—that is, the one used most often—but the offense must throw a pass occasionally in such situations "to keep the defense honest."

Thus the most important general principle of such situations is not what Evert *should* do but what she *should not* do: she should not do the same thing all the time or systematically. If she did, then Navratilova would learn to cover that, and Evert's chances of success would fall.

Not doing any one thing systematically means more than not playing the same shot in every situation of this kind. Evert should not even mechanically

[1]Chris Evert won her first title at the U.S. Open in 1975. Navratilova claimed her first title in the 1983 final.

switch back and forth between the two shots. Navratilova would spot and exploit this *pattern* or indeed any other detectable system. Evert must make the choice on each particular occasion *at random* to prevent this guessing.

This general idea of "mixing one's plays" is well known, even to sports commentators on television. But there is more to the idea, and these further aspects require analysis in greater depth. Why is down-the-line the percentage shot? Should one play it 80% of the time or 90% or 99%? Does it make any difference if the occasion is particularly big; for example, does one throw that pass on third down in the regular season but not in the Super Bowl? In actual practice, just how does one mix one's plays? What happens when a third possibility (the lob) is introduced? We will examine and answer such questions in Chapters 7 and 8.

The movie *The Princess Bride* (1987) illustrates the same idea in the "battle of wits" between the hero (Westley) and a villain (Vizzini). Westley is to poison one of two wineglasses out of Vizzini's sight, and Vizzini is to decide who will drink from which glass. Vizzini goes through a number of convoluted arguments as to why Westley should poison one glass. But all of the arguments are innately contradictory, because Westley can anticipate Vizzini's logic and choose to put the poison in the other glass. Conversely, if Westley uses any specific logic or system to choose one glass, Vizzini can anticipate that and drink from the other glass, leaving Westley to drink from the poisoned one. Thus, Westley's strategy has to be random or unsystematic.

The scene illustrates something else as well. In the film, Vizzini loses the game and with it his life. But it turns out that Westley had poisoned both glasses; over the last several years, he had built up immunity to the poison. So Vizzini was actually playing the game under a fatal information disadvantage. Players can sometimes cope with such asymmetries of information; Chapter 9 examines when and how they can do so.

B. The GPA Rat Race

You are enrolled in a course that is graded on a curve. No matter how well you do in absolute terms, only 40% of the students will get As, and only 40% will get Bs. Therefore you must work hard, not just in absolute terms, but relative to how hard your classmates (actually, "class enemies" seems a more fitting term in this context) work. All of you recognize this, and after the first lecture you hold an impromptu meeting in which all students agree not to work too hard. As weeks pass by, the temptation to get an edge on the rest of the class by working just that little bit harder becomes overwhelming. After all, the others are not able to observe your work in any detail; nor do they have any real hold over you. And the benefits of an improvement in your grade point average are substantial. So you hit the library more often and stay up a little longer.

The trouble is, everyone else is doing the same. Therefore your grade is no better than it would have been if you and everyone else had abided by the agreement. The only difference is that all of you have spent more time working than you would have liked.

This is an example of the prisoners' dilemma. In the original story, two suspects are being separately interrogated and invited to confess. One of them, say A, is told, "If the other suspect, B, does not confess, then you can cut a very good deal for yourself by confessing. But if B does confess, then you would do well to confess, too; otherwise the court will be especially tough on you. So you should confess no matter what the other does." B is told to confess, with the use of similar reasoning. Faced with this choice, both A and B confess. But it would have been better for both if neither had confessed, because the police had no really compelling evidence against them.

Your situation is similar. If the others slacken, then you can get a much better grade by working hard; if the others work hard, then you had better do the same or else you will get a very bad grade. You may even think that the label "prisoner" is very fitting for a group of students trapped in a required course.

There is a prisoners' dilemma for professors and schools, too. Each professor can make his course look good or attractive by grading it slightly more liberally, and each school can place its students in better jobs or attract better applicants by grading all of its courses a little more liberally. Of course, when all do this, none has any advantage over the others; the only result is rampant grade inflation, which compresses the spectrum of grades and therefore makes it difficult to distinguish abilities.

People often think that in every game there must be a winner and a loser. The prisoners' dilemma is different—both or all players can come out losers. People play (and lose) such games every day, and the losses can range from minor inconveniences to potential disasters. Spectators at a sports event stand up to get a better view but, when all stand, no one has a better view than when they were all sitting. Superpowers acquire more weapons to get an edge over their rivals but, when both do so, the balance of power is unchanged; all that has happened is that both have spent economic resources that they could have used for better purposes, and the risk of accidental war has escalated. The magnitude of the potential cost of such games to all players makes it important to understand the ways in which mutually beneficial cooperation can be achieved and sustained. All of Chapter 11 deals with the study of this game.

Just as the prisoners' dilemma is potentially a lose-lose game, there are win-win games, too. International trade is an example; when each country produces more of what it can do relatively best, all share in the fruits of this international division of labor. But successful bargaining about the division of the pie is needed if the full potential of trade is to be realized. The same applies to many other bargaining situations. We will study these in Chapter 17.

C. "We Can't Take the Exam, Because We Had a Flat Tire"

Here is a story, probably apocryphal, that circulates on the undergraduate e-mail networks; each of us independently received it from our students:

> There were two friends taking Chemistry at Duke. Both had done pretty well on all of the quizzes, the labs, and the midterm, so that going into the final they each had a solid A. They were so confident the weekend before the final that they decided to go to a party at the University of Virginia. The party was so good that they overslept all day Sunday, and got back too late to study for the Chemistry final that was scheduled for Monday morning. Rather than take the final unprepared, they went to the professor with a sob story. They said they each had gone up to UVA and had planned to come back in good time to study for the final but had had a flat tire on the way back. Because they didn't have a spare, they had spent most of the night looking for help. Now they were really too tired, so could they please have a makeup final the next day? The professor thought it over and agreed.
>
> The two studied all of Monday evening and came well prepared on Tuesday morning. The professor placed them in separate rooms and handed the test to each. The first question on the first page, worth 10 points, was very easy. Each of them wrote a good answer, and greatly relieved, turned the page. It had just one question, worth 90 points. It was: "Which tire?"

The story has two important strategic lessons for future party goers. The first is to recognize that the professor may be an intelligent game player. He may suspect some trickery on the part of the students and may use some device to catch them. Given their excuse, the question was the likeliest such device. They should have foreseen it and prepared their answer in advance. This idea that one should look ahead to future moves in the game and then reason backward to calculate one's best current action is a very general principle of strategy, which we will elaborate on in Chapter 3. We will also use it, most notably, in Chapter 10.

But it may not be possible to foresee all such professorial countertricks; after all, professors have much more experience of seeing through students' excuses than students have of making up such excuses. If the pair are unprepared, can they independently produce a mutually consistent lie? If each picks a tire at random, the chances are only 25% that the two will pick the same one. (Why?) Can they do better?

You may think that the front tire on the passenger side is the one most likely to suffer a flat, because a nail or a shard of glass is more likely to lie closer to the side of the road than the middle and the front tire on that side will encounter it first. You may think this is good logic, but that is not enough to make it a good choice. What matters is not the logic of the choice but making the same choice as your friend does. Therefore you have to think about whether your friend would use the same logic and would consider that choice equally obvious. But even that is not the end of the chain of reasoning. Would your friend think that the choice would be equally obvious to you? And so on. The point is not whether a choice is obvious or logical, but whether it is obvious to the other that it is obvious to you that it is obvious to the other. . . . In other words, what is needed is a convergence of expectations about what should be chosen in such circumstances. Such a commonly expected strategy on which the players can successfully coordinate is called a focal point.

There is nothing general or intrinsic to the structure of all such games that creates such convergence. In some games, a focal point may exist because of chance circumstances about the labeling of strategies or because of some experience or knowledge shared by the players. For example, if the passenger's front side of a car were for some reason called the Duke's side, then two Duke students would be very likely to choose it without any need for explicit prior understanding. Or, if the driver's front side of all cars were painted orange (for safety, to be easily visible to oncoming cars), then two Princeton students would be very likely to choose that tire, because orange is the Princeton color. But without some such clue, tacit coordination might not be possible at all.

We will study focal points in more detail in Chapter 4. Here in closing we merely point out that when asked in classrooms, more than 50% of students choose the driver's front side. They are generally unable to explain why, except to say that it seems the obvious choice.

D. Why Are Professors So Mean?

Many professors have inflexible rules not to give makeup exams and never to accept late submission of problem sets or term papers. Students think the professors must be really hardhearted to behave in this way. The true strategic reason is often exactly the opposite. Most professors are kindhearted and would like to give their students every reasonable break and accept any reasonable excuse. The trouble lies in judging what is reasonable. It is hard to distinguish between similar excuses and almost impossible to verify their truth. The professor knows that on each occasion he will end up by giving the student the benefit of the doubt. But the professor also knows that this is a slippery slope. As the students come to know that the professor is a soft touch, they will procrastinate more and produce ever-flimsier excuses. Deadlines will cease to mean anything, and examinations will become a chaotic mix of postponements and makeup tests.

Often the only way to avoid this slippery slope is to refuse to take even the first step down it. Refusal to accept any excuses at all is the only realistic alternative to accepting them all. By making an advance commitment to the "no excuses" strategy, the professor avoids the temptation to give in to all.

But how can a softhearted professor maintain such a hardhearted commitment? He must find some way to make a refusal firm and credible. The simplest way is to hide behind an administrative procedure or university-wide policy. "I wish I could accept your excuse, but the university won't let me" not only puts the professor in a nicer light, but removes the temptation by genuinely leaving him no choice in the matter. Of course, the rules may be made by the same collectivity of professors as hides behind them but, once made, no individual professor can unmake the rule in any particular instance.

If the university does not provide such a general shield, then the professor can try to make up commitment devices of his own. For example, he can make a clear and firm announcement of the policy at the beginning of the course. Any time an individual student asks for an exception, he can invoke a fairness principle, saying, "If I do this for you, I would have to do it for everyone." Or the professor can acquire a reputation for toughness by acting tough a few times. This may be an unpleasant thing for him to do and it may run against his true inclination, but it helps in the long run over his whole career. If a professor is believed to be tough, few students will try excuses on him, so he will actually suffer less pain in denying them.

We will study commitments, and related strategies, such as threats and promises, in considerable detail in Chapter 10.

E. Roommates and Families on the Brink

You are sharing an apartment with one or more other students. You notice that the apartment is nearly out of dishwasher detergent, paper towels, cereal, beer, and other items. You have an agreement to share the actual expenses, but the trip to the store takes time. Do you spend your time or do you hope that someone else will spend his, leaving you more time to study or relax? Do you go and buy the soap or stay in and watch TV to catch up on the soap operas?[2]

In many situations of this kind, the waiting game goes on for quite a while before someone who is really impatient for one of the items (usually beer) gives in and spends the time for the shopping trip. Things may deteriorate to the point of serious quarrels or even breakups among the roommates.

This game of strategy can be viewed from two perspectives. In one, each of the roommates is regarded as having a simple binary choice—to do the shopping or not. The best outcome for you is where someone else does the shopping

[2]This example comes from Michael Grunwald's "At Home" column, "A Game of Chicken," in the *Boston Globe Magazine*, April 28, 1996.

and you stay at home; the worst is where you do the shopping while the others get to use their time better. If both do the shopping (unknown to each other, on the way home from school or work), there is unnecessary duplication and perhaps some waste of perishables; if neither does, there can be serious inconvenience or even disaster if the toilet paper runs out at a crucial time.

This is analogous to the game of chicken that used to be played by American teenagers. Two of them drove their cars toward each other. The first to swerve to avoid a collision was the loser (chicken); the one who kept driving straight was the winner. We will analyze the game of chicken further in Chapter 4 and in Chapters 8 and 13.

A more interesting dynamic perspective on the same situation regards it as a "war of attrition," where each roommate tries to wait out the others, hoping that someone else's patience will run out first. In the meantime, the risk escalates that the apartment will run out of something critical, leading to serious inconvenience or a blowup. Each player lets the risk escalate to the point of his own tolerance; the one revealed to have the least tolerance loses. Each sees how close to the brink of disaster the others will let the situation go. Hence the name "brinkmanship" for this strategy and this game. It is a dynamic version of chicken, offering richer and more interesting possibilities.

One of us (Dixit) was privileged to observe a brilliant example of brinkmanship at a dinner party one Saturday evening. Before dinner, the company was sitting in the living room when the host's fifteen-year-old daughter appeared at the door and said, "Bye, Dad." The father asked, "Where are you going?" and the daughter replied, "Out." After a pause that was only a couple of seconds but seemed much longer, the host said, "All right, bye."

Your strategic observer of this scene was left thinking how it might have gone differently. The host might have asked, "With whom?" and the daughter might have replied, "Friends." The father could have refused permission unless the daughter told him exactly where and with whom she would be. One or the other might have capitulated at some such later stage of this exchange or it could have led to a blowup.

This was a risky game for both the father and the daughter to play. The daughter might have been punished or humiliated in front of strangers; an argument could have ruined the father's evening with his friends. Each had to judge how far to push the process, without being fully sure whether and when the other might give in or whether there would be an unpleasant scene. The risk of an explosion would increase as the father tried harder to force the daughter to answer and as she defied each successive demand.

In this respect the game played by the father and the daughter was just like that between a union and a company's management who are negotiating a labor contract or between two superpowers who are encroaching on each other's sphere of influence in the world. Neither side can be fully sure of the other's intentions, so each side explores them through a succession of small incremental steps, each of which escalates the risk of mutual disaster. The daughter in our story was exploring previously untested limits of her freedom; the father was exploring previously untested—and perhaps unclear even to himself—limits of his authority.

This was an example of brinkmanship, a game of escalating mutual risk, par excellence. Such games can end in one of two ways. In the first way, one of the players reaches the limit of his own tolerance for risk and concedes. (The father in our story conceded quickly, at the very first step. Other fathers might be more successful strict disciplinarians, and their daughters might not even initiate a game like this.) In the second way, before either has conceded, the risk that they both fear comes about, and the blowup (the strike or the war) occurs. The feud in our host's family ended "happily"; although the father conceded and the daughter won, a blowup would have been much worse for both.

We will analyze the strategy of brinkmanship more fully in Chapter 10; in Chapter 14, we will examine a particularly important instance of it—namely, the Cuban missile crisis of 1962.

F. The Dating Game

When you are dating, you want to show off the best attributes of your personality to your date and to conceal the worst ones. Of course, you cannot hope to conceal them forever if the relationship progresses; but you are resolved to improve or hope that by that stage the other person will accept the bad things about you with the good ones. And you know that the relationship will not progress at all unless you make a good first impression; you won't get a second chance to do so.

Of course, you want to find out everything, good and bad, about the other person. But you know that, if the other is as good at the dating game as you are, he or she will similarly try to show the best side and hide the worst. You will think through the situation more carefully and try to figure out which signs of good qualities are real and which ones can easily be put on for the sake of making a good impression. Even the worst slob can easily appear well groomed for a big date; ingrained habits of courtesy and manners that are revealed in a hundred minor details may be harder to simulate for a whole evening. Flowers are relatively cheap; more expensive gifts may have value, not for intrinsic reasons, but as credible evidence of how much the other person is willing to sacrifice for you. And the "currency" in which the gift is given may have different significance, depending on the context; from a millionaire, a diamond may be worth less in this regard than the act of giving up valuable time for your company or time spent on some activity at your request.

You should also recognize that your date will similarly scrutinize your actions for their information content. Therefore you should take actions that are credible signals of your true good qualities, and not just the ones that anyone can imitate.

This is important not just on a first date; revealing, concealing, and eliciting information about the other person's deepest intentions remain important throughout a relationship. Here is a story to illustrate that.

Once upon a time in New York City there lived a man and a woman who had separate rent-controlled apartments, but their relationship had reached the point at which they were using only one of them. The woman suggested to the man that they give up the other apartment. The man, an economist, explained to her a fundamental principle of his subject: it is always better to have more choice available. The probability of their splitting up might be small but, given even a small risk, it would be useful to retain the second low-rent apartment. The woman took this very badly and promptly ended the relationship!

Economists who hear this story say that it just confirms their principle that greater choice is better. But strategic thinking offers a very different and more compelling explanation. The woman was not sure of the man's commitment to the relationship, and her suggestion was a brilliant strategic device to elicit the truth. Words are cheap; anyone can say, "I love you." If the man had put his property where his mouth was and had given up his rent-controlled apartment, that would have been concrete evidence of his love. The fact that he refused to do so constituted hard evidence of the opposite, and the woman did right to end the relationship.

These are examples, designed to appeal to your immediate experience, of a very important class of games—namely, those where the real strategic issue is manipulation of information. Strategies that convey good information about yourself are called signals; strategies that induce others to act in ways that will credibly reveal their private information, good or bad, are called screening devices. Thus the woman's suggestion of giving up one of the apartments was a screening device, which put the man in the situation of offering to give up his apartment or else revealing his lack of commitment. We will study games of information, as well as signaling and screening, in Chapter 9.

3 OUR STRATEGY FOR STUDYING GAMES OF STRATEGY

We have chosen several examples that relate to your experiences as amateur strategists in real life to illustrate some basic concepts of strategic thinking and strategic games. We could continue, building a whole stock of dozens of similar stories. The hope would be that, when you face an actual strategic situation, you might recognize a parallel with one of these stories, which would help you decide the appropriate strategy for your own situation. This is the *case study* approach taken by most business schools. It offers a concrete and memorable vehicle for the underlying concepts. However, each new strategic situation typically consists of a unique combination of so many variables that an intolerably large stock of cases is needed to cover all of them.

An alternative approach focuses on the general principles behind the examples and so constructs a *theory* of strategic action—namely, formal game theory. The hope here is that, facing an actual strategic situation, you might recognize which principle or principles apply to it. This is the route taken by the more academic disciplines, such as economics and political science. A drawback to this approach is that the theory is presented in a very abstract and mathematical manner, without enough cases or examples. This makes it difficult for most beginners to understand or remember the theory and to connect the theory with reality afterward.

But knowing some general theory has an overwhelming compensating advantage. It gives you a deeper understanding of games and of *why* they have the outcomes they do. This helps you play better than you would if you merely read some cases and knew the recipes for *how* to play some specific games. With the knowledge of why, you can think through new and unexpected situations where a mechanical follower of a "how" recipe would be lost. A world champion of checkers, Tom Wiswell, has expressed this beautifully: "The player who knows how will usually draw, the player who knows why will usually win."[3] This is not to be taken literally for all games; some games may be hopeless situations for one of the players no matter how knowledgable he may be. But the statement contains the germ of an important general truth—knowing why gives you an advantage beyond what you can get if you merely know how. For example, knowing the why of a game can help you foresee a hopeless situation and avoid getting into such a game in the first place.

Therefore we will take an intermediate route that combines some of the advantages of both approaches—case studies (how) and theory (why). We will organize the subject around its general principles, generally one in each of the chapters to follow. Therefore you don't have to figure them out on your own from the cases. But we will develop the general principles through illustrative cases rather than abstractly, so the context and scope of each idea will be clear and evident. In other words, we will focus on theory but build it up through cases, not abstractly.

Of course, such an approach requires some compromises of its own. Most importantly, you should remember that each of our examples serves the purpose of conveying some general idea or principle of game theory. Therefore we

[3]Quoted in Victor Niederhoffer, *The Education of a Speculator* (New York: Wiley, 1997), p. 169. We thank Austin Jaffe of Pennsylvania State University for bringing this aphorism to our attention.

will leave out many details of each case that are incidental to the principle at stake. If some examples seem somewhat artificial, please bear with us; we have generally considered the omitted details and left them out for good reasons.

A word of reassurance. Although the examples that motivate the development of our conceptual or theoretical frameworks are deliberately selected for that purpose (even at the cost of leaving out some other features of reality), once the theory has been constructed, we pay a lot of attention to its connection with reality. Throughout the book, we examine factual and experimental evidence in regard to how well the theory explains reality. The frequent answer—very well in some respects and less well in others—should give you cautious confidence in using the theory and should be a spur to contributing to the formulation of better theories. In appropriate places, we examine in great detail how institutions evolve in practice to solve some problems pointed out by the theories; note in particular the discussion in Chapter 11 of how prisoners' dilemmas arise and are solved in reality and a similar discussion of more general collective-action problems in Chapter 12. Finally, in Chapter 14 we will examine the use of brinkmanship in the Cuban missile crisis. Such theory-based case studies, which take rich factual details of a situation and subject them to an equally detailed theoretical analysis, are becoming common in such diverse fields as business studies, political science, and economic history; we hope our original study of an important episode in the diplomatic and military areas will give you an interesting introduction to this genre.

To pursue our approach in which examples lead to general theories that are then tested against reality and used to interpret reality, we must first identify the general principles that serve to organize the discussion. We will do so in Chapter 2 by classifying or dichotomizing games along several key dimensions of different strategic matters or concepts. Along each dimension, we will identify two extreme pure types. For example, one such dimension concerns the order of moves, and the two pure types are those in which the players take turns making moves (sequential games) and those in which all players act at once (simultaneous games). Actual games rarely correspond to exactly one of these conceptual categories; most partake of some features of each extreme type. But each game can be located in our classification by considering which concepts or dimensions bear on it and how it mixes the two pure types in each dimension. To decide how to act in a specific situation, one then combines in appropriate ways the lessons learned for the pure types.

Once this general framework has been constructed in Chapter 2, the chapters that follow will build on it, developing several general ideas and principles for each player's strategic choice and the interaction of all players' strategies in games.

2

How to Think About Strategic Games

CHAPTER 1 GAVE SOME simple examples of strategic games and strategic thinking. In this chapter, we begin a more systematic and analytical approach to the subject. We choose some crucial conceptual categories or dimensions, in each of which there is a dichotomy of types of strategic interactions. For example, one such dimension concerns the timing of the players' actions, and the two pure types are games where the players act in strict turns (sequential moves) and where they act at the same time (simultaneous moves). We consider some matters that arise in thinking about each pure type in this dichotomy, as well as in similar dichotomies, with respect to other matters, such as whether the game is played only once or repeatedly and what the players know about each other.

In the chapters that follow, we will examine each of these categories or dimensions in more detail and show how the analysis can be used in several specific applications. Of course, most actual applications are not of a pure type but rather a mixture. Moreover, in each application, two or more of the categories have some relevance. The lessons learned from the study of the pure types must therefore be combined in appropriate ways. We will show how to do this by using the context of our applications.

In this chapter, we state some basic concepts and terminology—such as strategies, payoffs, and equilibrium—that are used in the analysis and briefly describe solution methods. We also provide a brief discussion of the uses of game theory and an overview of the structure of the remainder of the book.

1 DECISIONS VERSUS GAMES

When a person (or team or firm or government) decides how to act in dealings with other people (or teams or firms or governments), there must be some cross-effect of their actions; what one does must affect the outcome for the other. When George Pickett (of Pickett's Charge at the battle of Gettysburg) was asked to explain the Confederacy's defeat in the Civil War, he responded, "I think the Yankees had something to do with it."[1]

For the interaction to become a strategic game, however, we need something more—namely, the participants' mutual awareness of this cross-effect. What the other person does affects you; if you know this, you can react to his actions, or take advance actions to forestall the bad effects his future actions may have on you and to facilitate any good effects, or even take advance actions so as to alter his future reactions to your advantage. If you know that the other person knows that what you do affects him, you know that he will be taking similar actions. And so on. It is this mutual awareness of the cross-effects of actions and the actions taken as a result of this awareness that constitute the most interesting aspects of strategy.

This distinction is captured by reserving the label **strategic games** (or sometimes just **games,** because we are not concerned with other types of games, such as those of pure chance or pure skill) for interactions between mutually aware players and **decisions** for action situations where each person can choose without concern for reaction or response from others. If Robert E. Lee (who ordered Pickett to lead the ill-fated Pickett's Charge) had thought that the Yankees had been weakened by his earlier artillery barrage to the point that they no longer had any ability to resist, his choice to attack would have been a decision; if he was aware that the Yankees were prepared and waiting for his attack, then the choice became a part of a (deadly) game. The simple rule is that unless there are two or more players, each of whom responds to what others do (or what each thinks the others might do), it is not a game.

Strategic games arise most prominently in head-to-head confrontations of two participants: the arms race between the United States and the Soviet Union from the 1950s through the 1980s; wage negotiations between General Motors and the United Auto Workers; or a Super Bowl matchup between two "pirates," the Tampa Bay Buccaneers and the Oakland Raiders. In contrast, interactions among a large number of participants seem less prone to the issues raised by mutual awareness. Because each farmer's output is an insignificant part of the

[1] James M. McPherson, "American Victory, American Defeat," in *Why the Confederacy Lost,* ed. Gabor S. Boritt (New York: Oxford University Press, 1993), p. 19.

whole nation's or the world's output, the decision of one farmer to grow more or less corn has almost no effect on the market price, and not much appears to hinge on thinking of agriculture as a strategic game. This was indeed the view prevalent in economics for many years. A few confrontations between large companies, as in the U.S. auto market that was once dominated by GM, Ford, and Chrysler, were usefully thought of as strategic games, but most economic interactions were supposed to be governed by the impersonal forces of supply and demand.

In fact, game theory has a much greater scope. Many situations that start out as impersonal markets with thousands of participants turn into strategic interactions of two or just a few. This happens for one of two broad classes of reasons—mutual commitments or private information.

Consider commitment first. When you are contemplating building a house, you can choose one of several dozen contractors in your area; the contractor can similarly choose from several potential customers. There appears to be an impersonal market. Once each side has made a choice, however, the customer pays an initial installment, and the builder buys some materials for the plan of this particular house. The two become tied to each other, separately from the market. Their relationship becomes *bilateral*. The builder can try to get away with a somewhat sloppy job or can procrastinate, and the client can try to delay payment of the next installment. Strategy enters the picture. Their initial contract has to anticipate their individual incentives and specify a schedule of installments of payments that are tied to successive steps in the completion of the project. Even then, some adjustments have to be made after the fact, and these adjustments bring in new elements of strategy.

Next, consider private information. Thousands of farmers seek to borrow money for their initial expenditures on machinery, seed, fertilizer, and so forth, and hundreds of banks exist to lend to them. Yet the market for such loans is not impersonal. A borrower with good farming skills who puts in a lot of effort will be more likely to be successful and will repay the loan; a less-skilled or lazy borrower may fail at farming and default on the loan. The risk of default is highly personalized. It is not a vague entity called "the market" that defaults, but individual borrowers who do so. Therefore each bank will have to view its lending relation with each individual borrower as a separate game. It will seek collateral from each borrower or will investigate each borrower's creditworthiness. The farmer will look for ways to convince the bank of his quality as a borrower; the bank will look for effective ways to ascertain the truth of the farmer's claims.

Similarly, an insurance company will make some efforts to determine the health of individual applicants and will check for any evidence of arson when a claim for a fire is made; an employer will inquire into the qualifications of individual employees and monitor their performance. More generally, when participants in a transaction possess some private information bearing on the outcome, each bilateral deal becomes a game of strategy, even though the larger picture may have thousands of very similar deals going on.

To sum up, when each participant is significant in the interaction, either because each is a large player to start with or because commitments or private information narrow the scope of the relationship to a point where each is an important player *within* the relationship, we must think of the interaction as a strategic game. Such situations are the rule rather than the exception in business, in politics, and even in social interactions. Therefore the study of strategic games forms an important part of all fields that analyze these matters.

2 CLASSIFYING GAMES

Games of strategy arise in many different contexts and accordingly have many different features that require study. This task can be simplified by grouping these features into a few categories or dimensions, along each of which we can identify two pure types of games and then recognize any actual game as a mixture of the pure types. We develop this classification by asking a few questions that will be pertinent for thinking about the actual game that you are playing or studying.

A. Are the Moves in the Game Sequential or Simultaneous?

Moves in chess are sequential: White moves first, then Black, then White again, and so on. In contrast, participants in an auction for an oil-drilling lease or a part of the airwave spectrum make their bids simultaneously, in ignorance of competitors' bids. Most actual games combine aspects of both. In a race to research and develop a new product, the firms act simultaneously, but each competitor has partial information about the others' progress and can respond. During one play in football, the opposing offensive and defensive coaches simultaneously send out teams with the expectation of carrying out certain plays but, after seeing how the defense has set up, the quarterback can change the play at the line of scrimmage or call a time-out so that the coach can change the play.

The distinction between **sequential** and **simultaneous moves** is important because the two types of games require different types of interactive thinking. In a sequential-move game, each player must think: if I do this, how will my opponent react? Your current move is governed by your calculation of its *future* consequences. With simultaneous moves, you have the trickier task of trying to figure out what your opponent is going to do *right now*. But you must recognize that, in making his own calculation, the opponent is also trying to figure out your current move, while at the same time recognizing that you are doing the same with him. . . . Both of you have to think your way out of this circle.

In the next three chapters, we will study the two pure cases. In Chapter 3, we examine sequential-move games, where you must look ahead to act now; in Chapters 4 and 5, the subject is simultaneous-move games, where you must square the circle of "He thinks that I think that he thinks . . . " In each case, we will devise some simple tools for such thinking—trees and payoff tables—and obtain some simple rules to guide actions.

The study of sequential games also tells us when it is an advantage to move first and when second. Roughly speaking, this depends on the relative importance of commitment and flexibility in the game in question. For example, the game of economic competition among rival firms in a market has a first-mover advantage if one firm, by making a firm commitment to compete aggressively, can get its rivals to back off. But, in political competition, a candidate who has taken a firm stand on an issue may give his rivals a clear focus for their attack ads, and the game has a second-mover advantage.

Knowledge of the balance of these considerations can also help you devise ways to manipulate the order of moves to your own advantage. That in turn leads to the study of strategic moves, such as threats and promises, which we will take up in Chapter 10.

B. Are the Players' Interests in Total Conflict or Is There Some Commonality?

In simple games such as chess or football, there is a winner and a loser. One player's gain is the other's loss. Similarly, in gambling games, one player's winnings are the others' losses, so the total is zero. This motivates the name *zero-sum games* for such situations. More generally, the idea is that the players' interests are in complete conflict. Such conflict arises when players are dividing up any fixed amount of possible gain, whether it be measured in yards, dollars, acres, or scoops of ice cream. Because the available gain need not always be exactly zero, the term *constant-sum game* is often substituted for zero-sum; we will use the two interchangeably.

Most economic and social games are not zero-sum. Trade, or economic activity more generally, offers scope for deals that benefit everyone. Joint ventures can combine the participants' different skills and generate synergy to produce more than the sum of what they could have produced separately. But the interests are not completely aligned either; the partners can cooperate to create a larger total pie, but they will clash when it comes to deciding how to split this pie among them.

Even wars and strikes are not zero-sum games. A nuclear war is the most striking example of a situation where there can be only losers, but the concept is far older. Pyrrhus, the king of Epirus, defeated the Romans at Heraclea in

280 B.C. but at such great cost to his own army that he exclaimed: "Another such victory and we are lost." Hence the phrase "Pyrrhic victory." In the 1980s, at the height of the frenzy of business takeovers, the battles among rival bidders led to such costly escalation that the successful bidder's victory was often similarly Pyrrhic.

Most games in reality have this tension between conflict and cooperation, and many of the most interesting analyses in game theory come from the need to handle it. The players' attempts to resolve their conflict—distribution of territory or profit—are influenced by the knowledge that, if they fail to agree, the outcome will be bad for all of them. One side's threat of a war or a strike is its attempt to frighten the other side into conceding its demands.

Even when a game is constant-sum for all players, when there are three (or more) players, we have the possibility that two of them will cooperate at the expense of the third; this leads to the study of alliances and coalitions. We will examine and illustrate these ideas later, especially in Chapter 17 on bargaining.

C. Is the Game Played Once or Repeatedly, and with the Same or Changing Opponents?

A game played just once is in some respects simpler and in others more complicated than one with a longer interaction. You can think about a one-shot game without worrying about its repercussions on other games you might play in the future against the same person or against others who might hear of your actions in this one. Therefore actions in one-shot games are more likely to be unscrupulous or ruthless. For example, an automobile repair shop is much more likely to overcharge a passing motorist than a regular customer.

In one-shot encounters, each player doesn't know much about the others; for example, what their capabilities and priorities are, whether they are good at calculating their best strategies or have any weaknesses that can be exploited, and so on. Therefore in one-shot games, secrecy or surprise are likely to be important components of good strategy.

Games with ongoing relationships require the opposite considerations. You have an opportunity to build a reputation (for toughness, fairness, honesty, reliability, and so forth, depending on the circumstances) and to find out more about your opponent. The players together can better exploit mutually beneficial prospects by arranging to divide the spoils over time (taking turns to "win") or to punish a cheater in future plays (an eye-for-an-eye or tit-for-tat). We will consider these possibilities in Chapter 11 on the prisoners' dilemma.

More generally, a game may be zero-sum in the short run but have scope for mutual benefit in the long run. For example, each football team likes to win, but they all recognize that close competition generates more spectator interest, which benefits all teams in the long run. That is why they agree to a drafting scheme where teams get to pick players in reverse order of their current standing, thereby reducing the inequality of talent. In long-distance races, the runners or cyclists often develop a lot of cooperation; two or more of them can help one another by taking turns to follow in one another's slipstream. Near the end of the race, the cooperation collapses as all of them dash for the finish line.

Here is a useful rule of thumb for your own strategic actions in life. In a game that has some conflict and some scope for cooperation, you will often think up a great strategy for winning big and grinding a rival into dust but have a nagging worry at the back of your mind that you are behaving like the worst 1980s yuppie. In such a situation, the chances are that the game has a repeated or ongoing aspect that you have overlooked. Your aggressive strategy may gain you a short-run advantage, but its long-run side effects will cost you even more. Therefore you should dig deeper and recognize the cooperative element and then alter your strategy accordingly. You will be surprised how often niceness, integrity, and the golden rule of doing to others as you would have them do to you turn out to be not just old nostrums, but good strategies as well, when you consider the whole complex of games that you will be playing in the course of your life.

D. Do the Players Have Full or Equal Information?

In chess, each player knows exactly the current situation and all the moves that led to it, and each knows that the other aims to win. This situation is exceptional; in most other games, the players face some limitation of information. Such limitations come in two kinds. First, a player may not know all the information that is pertinent for the choice that he has to make at every point in the game. This includes uncertainty about relevant external circumstances—for example, the weather—as well as the prior and contemporaneous actions of other players. Such situations are said to have **imperfect information.** We will develop the theory of games with contemporaneous actions in Chapter 4 and methods for making choices under uncertainty in the appendix to Chapter 7 and in Chapter 9.

Trickier strategic situations arise when one player knows more than another does; they are called situations of **incomplete** or, better, **asymmetric information.** In such situations, the players' attempts to infer, conceal, or sometimes convey their private information become an important part of the game and the strategies. In bridge or poker, each player has only partial knowledge of the cards held by the others. Their actions (bidding and play in bridge, the number of cards taken and the betting behavior in poker) give information to opponents. Each player tries to manipulate his actions to mislead the opponents (and, in bridge, to inform his partner truthfully), but in doing so each must be aware that the opponents know this and that they will use strategic thinking to interpret that player's actions.

You may think that, if you have superior information, you should always conceal it from other players. But that is not true. For example, suppose you are the CEO of a pharmaceutical firm that is engaged in an R&D competition to develop a new drug. If your scientists make a discovery that is a big step forward, you may want to let your competitors know, in the hope that they will give up their own searches. In war, each side wants to keep its tactics and troop deployments secret; but, in diplomacy, if your intentions are peaceful, then you desperately want other countries to know and believe this fact.

The general principle here is that you want to release your information selectively. You want to reveal the good information (the kind that will draw responses from the other players that work to your advantage) and conceal the bad (the kind that may work to your disadvantage).

This raises a problem. Your opponents in a strategic game are purposive rational players and know that you are one, too. They will recognize your incentive to exaggerate or even to lie. Therefore they are not going to accept your unsupported declarations about your progress or capabilities. They can be convinced only by objective evidence or by actions that are credible proof of your information. Such actions on the part of the more-informed player are called **signals,** and strategies that use them are called **signaling.** Conversely, the less-informed party can create situations in which the more-informed player will have to take some action that credibly reveals his information; such strategies are called **screening,** and the methods they use are called **screening devices.** The word *screening* is used here in the sense of testing in order to sift or separate, not in the sense of concealing. Recall that, in the dating game in Section 2.F of Chapter 1, the woman was screening the man to test his commitment to their relationship, and her suggestion that the pair give up one of their two rent-controlled apartments was the screening device. If the man had been committed to the relationship, he might have acted first and volunteered to give up his apartment; this action would have been a signal of his commitment.

Now we see how, when different players have different information, the manipulation of information itself becomes a game, perhaps more important than the game that will be played after the information stage. Such information games are ubiquitous, and to play them well is essential for success in life. We will study more games of this kind in greater detail in Chapter 9.

E. Are the Rules of the Game Fixed or Manipulable?

The rules of chess, card games, or sports are given, and every player must follow them, no matter how arbitrary or strange they seem. But, in games of business, politics, and ordinary life, the players can make their own rules to a greater or lesser extent. For example, in the home, parents constantly try to make the rules, and children constantly look for ways to manipulate or circumvent those rules. In legislatures, rules for the progress of a bill (including the order in which amendments and main motions are voted on) are fixed, but the game that sets the agenda—which amendments are brought to vote first—can be manipulated; that is where political skill and power have the most scope, and we will address these matters in detail in Chapter 15.

In such situations, the real game is the "pregame" where rules are made, and your strategic skill must be deployed at that point. The actual playing out of the subsequent game can be more mechanical; you could even delegate it to someone else. However, if you "sleep" through the pregame, you might find that you have lost the game before it ever began. For many years, American firms ignored the rise of foreign competition in just this way and ultimately paid the price. Others, such as oil magnate John D. Rockefeller, Sr., adopted the strategy of limiting their participation to games in which they could also participate in making the rules.[2]

The distinction between changing rules and acting within the chosen rules will be most important for us in our study of strategic moves, such as threats and promises. Questions of how you can make your own threats and promises credible or how you can reduce the credibility of your opponent's threats basically have to do with a pregame that entails manipulating the rules of the subsequent game in which the promises or threats may have to be carried out. More generally, the strategic moves that we will study in Chapter 10 are essentially ploys for such manipulation of rules.

But, if the pregame of rule manipulation is the real game, what fixes the rules of the pregame? Usually these pregame rules depend on some hard facts related to the players' innate abilities. In business competition, one firm can take preemptive actions that alter subsequent games between it and its rivals; for example, it can expand its factory or advertise in a way that twists the results of subsequent price competition more favorably to itself. Which firm can do this best or most easily depends on which one has the managerial or organizational resources to make the investments or to launch the advertising campaigns.

Players may also be unsure of their rivals' abilities. This often makes the pregame one of unequal information, requiring more subtle strategies and occasionally resulting in some big surprises. We will comment on all these matters in the appropriate places in the chapters that follow.

F. Are Agreements to Cooperate Enforceable?

We saw that most strategic interactions consist of a mixture of conflict and common interest. Then there is a case to be made that all participants should get together and reach an agreement about what everyone should do, balancing their

[2]For more on the methods used in Rockefeller's rise to power, see Ron Chernow, *Titan* (New York: Random House, 1998).

mutual interest in maximizing the total benefit and their conflicting interests in the division of gains. Such negotiations can take several rounds, in which agreements are made on a tentative basis, better alternatives are explored, and the deal is finalized only when no group of players can find anything better. The concept of the core in Chapter 18 embodies such a process and its outcome. However, even after the completion of such a process, additional difficulties often arise in putting the final agreement into practice. For instance, all the players must perform, in the end, the actions that were stipulated for them in the agreement. When all others do what they are supposed to do, any one participant can typically get a better outcome for himself by doing something different. And, if each one suspects that the others may cheat in this way, he would be foolish to adhere to his stipulated cooperative action.

Agreements to cooperate can succeed if all players act immediately and in the presence of the whole group, but agreements with such immediate implementation are quite rare. More often the participants disperse after the agreement has been reached and then take their actions in private. Still, if these actions are observable to the others and a third party—for example, a court of law—can enforce compliance, then the agreement of joint action can prevail.

However, in many other instances individual actions are neither directly observable nor enforceable by external forces. Without enforceability, agreements will stand only if it is in all participants' individual interests to abide by them. Games among sovereign countries are of this kind, as are many games with private information or games where the actions are either outside the law or too trivial or too costly to enforce in a court of law. In fact, games where agreements for joint action are not enforceable constitute a vast majority of strategic interactions.

Game theory uses a special terminology to capture the distinction between situations in which agreements are enforceable and those in which they are not. Games in which joint-action agreements are enforceable are called **cooperative,** and those in which such enforcement is not possible, and individual participants must be allowed to act in their own interests, are called **noncooperative.** This has become standard terminology, but it is somewhat unfortunate because it gives the impression that the former will produce cooperative outcomes and the latter will not. In fact, individual action can be compatible with the achievement of a lot of mutual gain, especially in repeated interactions. The important distinction is that in so-called noncooperative games, cooperation will emerge only if it is in the participants' separate and individual interests to continue to take the prescribed actions. This emergence of cooperative outcomes from noncooperative behavior is one of the most interesting findings of game theory, and we will develop the idea in Chapters 11, 12, and 13.

We will adhere to the standard usage, but emphasize that the terms *cooperative* and *noncooperative* refer to the way in which actions are implemented or enforced—collectively in the former mode and individually in the latter—and not to the nature of the outcomes.

As we said earlier, most games in practice do not have adequate mechanisms for external enforcement of joint-action agreements. Therefore most of our analytical development will proceed in the noncooperative mode. The few exceptions include the discussion of bargaining in Chapter 17 and a brief treatment of markets and competition in Chapter 18.

3 SOME TERMINOLOGY AND BACKGROUND ASSUMPTIONS

When one thinks about a strategic game, the logical place to begin is by specifying its structure. This includes all the strategies available to all the players, their information, and their objectives. The first two aspects will differ from one game to another along the dimensions discussed in the preceding section, and one must locate one's particular game within that framework. The objectives raise some new and interesting considerations. Here, we consider aspects of all these matters.

A. Strategies

Strategies are simply the choices available to the players, but even this basic notion requires some further study and elaboration. If a game has purely simultaneous moves made only once, then each player's strategy is just the action taken on that single occasion. But, if a game has sequential moves, then the actions of a player who moves later in the game can respond to what other players (or he himself) have done at earlier points. Therefore each such player must make a complete plan of action, for example: "If the other does A, then I will do X but, if the other does B, then I will do Y." This complete plan of action constitutes the strategy in such a game.

There is a very simple test to determine whether your strategy is complete. It should specify how you would play the game in such full detail—describing your action in every contingency—that, if you were to write it all down, hand it to someone else, and go on vacation, this other person acting as your representative could play the game just as you would have played it. He would know what to do on each occasion that could conceivably arise in the course of play, without ever needing to disturb your vacation for instructions on how to deal with some situation that you had not foreseen.

This test will become clearer in Chapter 3, when we develop and apply it in some specific contexts. For now, you should simply remember that a strategy is a complete plan of action.

This notion is similar to the common usage of the word *strategy* to denote a longer-term or larger-scale plan of action, as distinct from tactics that pertain to a shorter term or a smaller scale. For example, the military makes strategic plans for a war or a large-scale battle, while tactics for a smaller skirmish or a particular theater of battle are often left to be devised by lower-level officers to suit the local conditions. But game theory does not use the term *tactics* at all. The term *strategy* covers all the situations, meaning a complete plan of action when necessary and meaning a single move if that is all that is needed in the particular game being studied.

The word *strategy* is also commonly used to describe a person's decisions over a fairly long time span and a sequence of choices, even though there is no game in our sense of purposive and aware interaction with other people. Thus you have probably already chosen a career strategy. When you start earning an income, you will make saving and investment strategies and eventually plan a retirement strategy. This usage of the term *strategy* has the same sense as ours—a plan for a succession of actions in response to evolving circumstances. The only difference is that we are reserving it for a situation—namely, a game—where the circumstances evolve because of actions taken by other purposive players.

B. Payoffs

When asked what a player's objective in a game is, most newcomers to strategic thinking respond that it is "to win"; but matters are not always so simple. Sometimes the margin of victory matters; for example, in R&D competition, if your product is only slightly better than the nearest rival's, your patent may be more open to challenge. Sometimes there may be smaller prizes for several participants, so winning isn't everything. Most importantly, very few games of strategy are purely zero-sum or win-lose; they combine some common interest and some conflict among the players. Thinking about such mixed-motive games requires more refined calculations than the simple dichotomy of winning and losing—for example, comparisons of the gains from cooperating versus cheating.

We will give each player a complete numerical scale with which to compare all logically conceivable outcomes of the game, corresponding to each available combination of choices of strategies by all the players. The number associated with each possible outcome will be called that player's **payoff** for that outcome. Higher payoff numbers attach to outcomes that are better in this player's rating system.

Sometimes the payoffs will be simple numerical ratings of the outcomes, the worst labeled 1, the next worst 2, and so on, all the way to the best. In other games, there may be more natural numerical scales—for example, money income or profit for firms, viewer-share ratings for television networks, and so on. In many situations, the payoff numbers are only educated guesses; then we should do some sensitivity tests by checking that the results of our analysis do not change significantly if we vary these guesses within some reasonable margin of error.

Two important points about the payoffs need to be understood clearly. First, the payoffs for one player capture everything in the outcomes of the game that he cares about. In particular, the player need not be selfish, but his concern about others should be already included in his numerical payoff scale. Second, we will suppose that, if the player faces a random prospect of outcomes, then the number associated with this prospect is the average of the payoffs associated with each component outcome, each weighted by its probability. Thus, if in one player's ranking, outcome A has payoff 0 and outcome B has payoff 100, then the prospect of a 75% probability of A and a 25% probability of B should have the payoff $0.75 \times 0 + 0.25 \times 100 = 25$. This is often called the **expected payoff** from the random prospect. The word *expected* has a special connotation in the jargon of probability theory. It does not mean what you think you will get or expect to get; it is the mathematical or probabilistic or statistical expectation, meaning an average of all possible outcomes, where each is given a weight proportional to its probability.

The second point creates a potential difficulty. Consider a game where players get or lose money and payoffs are measured simply in money amounts. In reference to the preceding example, if a player has a 75% chance of getting nothing and a 25% chance of getting \$100, then the expected payoff as calculated in that example is \$25. That is also the payoff that the player would get from a simple nonrandom outcome of \$25. In other words, in this way of calculating payoffs, a person should be indifferent to whether he receives \$25 for sure or faces a risky prospect of which the average is \$25. One would think that most people would be averse to risk, preferring a sure \$25 to a gamble that yields only \$25 on the average.

A very simple modification of our payoff calculation gets around this difficulty. We measure payoffs not in money sums but by using a nonlinear rescaling of the dollar amounts. This is called the expected utility approach, and we will present it in detail in the Appendix to Chapter 7. For now, please take our word that incorporating differing attitudes toward risk into our framework is a manageable task. Almost all of game theory is based on the expected utility approach, and it is indeed very useful, although not without flaws. We will adopt it in this book, but we also indicate some of the difficulties that it leaves unresolved, with the use of a simple example in Chapter 8, Section 3.

C. Rationality

Each player's aim in the game will be to achieve as high a payoff for himself as possible. But how good is each player at pursuing this aim? This question is not about whether and how other players pursuing their own interests will impede him; that is in the very nature of a game of strategic interaction. We mean how good is each player at calculating the strategy that is in his own best interests and at following this strategy in the actual course of play.

Much of game theory assumes that players are perfect calculators and flawless followers of their best strategies. This is the assumption of **rational behavior.** Observe the precise sense in which the term *rational* is being used. It means that each has a consistent set of rankings (values or payoffs) over all the logically possible outcomes and calculates the strategy that best serves these interests. Thus rationality has two essential ingredients: complete knowledge of one's own interests, and flawless calculation of what actions will best serve those interests.

It is equally important to understand what is *not* included in this concept of rational behavior. It does not mean that players are selfish; a player may rate highly the well-being of some other and incorporate this high rating into his payoffs. It does not mean that players are short-run thinkers; in fact, calculation of future consequences is an important part of strategic thinking, and actions that seem irrational from the immediate perspective may have valuable long-term strategic roles. Most importantly, being rational does not mean having the same value system as other players, or sensible people, or ethical or moral people would use; it means merely pursuing one's own value system consistently. Therefore, when one player carries out an analysis of how other players will respond (in a game with sequential moves) or of the successive rounds of thinking about thinking (in a game with simultaneous moves), he must recognize that the other players calculate the consequences of their choices by using their own value or rating system. You must not impute your own value systems or standards of rationality to others and assume that they would act as you would in that situation. Thus many "experts" commenting on the Persian Gulf conflict in late 1990 predicted that Saddam Hussein would back down "because he is rational"; they failed to recognize that Saddam's value system was different from the one held by most Western governments and by the Western experts.

In general, each player does not really know the other players' value systems; this is part of the reason why in reality many games have incomplete and asymmetric information. In such games, trying to find out the values of others and trying to conceal or convey one's own become important components of strategy.

Game theory assumes that all players are rational. How good is this assumption, and therefore how good is the theory that employs it? At one level, it is obvious that the assumption cannot be literally true. People often don't even have full advance knowledge of their own value systems; they don't think ahead about how they would rank hypothetical alternatives and then remember these rankings until they are actually confronted with a concrete choice. Therefore they find it very difficult to perform the logical feat of tracing all possible consequences of their and other players' conceivable strategic choices and ranking the outcomes in advance in order to choose which strategy to follow. Even if they knew their preferences, the calculation would remain far from easy. Most games in real life are very complex, and most real players are limited in their thinking and computational abilities. In games such as chess, it is known that the calculation for the best strategy can be performed in a finite number of steps, but no one has succeeded in performing it, and good play remains largely an art.

The assumption of rationality may be closer to reality when the players are regulars who play the game quite often. Then they benefit from having experienced the different possible outcomes. They understand how the strategic choices of various players lead to the outcomes and how well or badly they themselves fare. Then we as analysts of the game can hope that their choices, even if not made with full and conscious computations, closely approximate the results of such computations. We can think of the players as implicitly choosing the optimal strategy or behaving as if they were perfect calculators. We will offer some experimental evidence in Chapter 5 that the experience of playing the game generates more rational behavior.

The advantage of making a complete calculation of your best strategy, taking into account the corresponding calculations of a similar strategically calculating rival, is that then you are not making mistakes that the rival can exploit. In many actual situations, you may have specific knowledge of the way in which the other players fall short of this standard of rationality, and you can exploit this in devising your own strategy. We will say something about such calculations, but very often this is a part of the "art" of game playing, not easily codifiable in rules to be followed. You must always beware of the danger that the others are merely pretending to have poor skills or strategy, losing small sums through bad play and hoping that you will then raise the stakes, when they can raise the level of their play and exploit your gullibility. When this risk is real, the safer advice to a player may be to assume that the rivals are perfect and rational calculators and to choose his own best response to them. In other words, one should play to the opponents' capabilities instead of their limitations.

D. Common Knowledge of Rules

We suppose that, at some level, the players have a common understanding of the rules of the game. In a *Peanuts* cartoon, Lucy thought that body checking was allowed in golf and decked Charlie Brown just as he was about to take his swing. We do not allow this.

The qualification "at some level" is important. We saw how the rules of the immediate game could be manipulated. But this merely admits that there is another game being played at a deeper level—namely, where the players choose the rules of the superficial game. Then the question is whether the rules of this deeper game are fixed. For example, in the legislative context, what are the rules of the agenda-setting game? They may be that the committee chairs have the power. Then how are the committees and their chairs elected? And so on. At some basic level, the rules are fixed by the constitution, by the technology of campaigning, or by general social norms of behavior. We ask that all players recognize the given rules of this basic game, and that is the focus of the analysis. Of course, that is an ideal; in practice, we may not be able to proceed to a deep enough level of analysis.

Strictly speaking, the rules of the game consist of (1) the list of players, (2) the strategies available to each player, (3) the payoffs of each player for all possible combinations of strategies pursued by all the players, and (4) the assumption that each player is a rational maximizer.

Game theory cannot properly analyze a situation where one player does not know whether another player is participating in the game, what the entire sets of actions available to the other players are from which they can choose, what their value systems are, or whether they are conscious maximizers of their own payoffs. But, in actual strategic interactions, some of the biggest gains are to be made by taking advantage of the element of surprise and doing something that your rivals never thought you capable of. Several vivid examples can be found in historic military conflicts. For example, in 1967 Israel launched a preemptive attack that destroyed the Egyptian air force on the ground; in 1973 it was Egypt's turn to create a surprise by launching a tank attack across the Suez Canal.

It would seem, then, that the strict definition of game theory leaves out a very important aspect of strategic behavior, but in fact matters are not that bad. The theory can be reformulated so that each player attaches some small probability to the situation where such dramatically different strategies are available to the other players. Of course, each player knows his own set of available strategies. Therefore the game becomes one of asymmetric information and can be handled by using the methods developed in Chapter 9.

The concept of common knowledge itself requires some explanation. For some fact or situation X to be common knowledge between two people, A and B, it is not enough for each of them separately to know X. Each should also know that the other knows X; otherwise, for example, A might think that B does not know X and might act under this misapprehension in the midst of a game. But then A should also know that B knows that A knows X, and the other way around, otherwise A might mistakenly try to exploit B's supposed ignorance of A's knowledge. Of course, it doesn't even stop there. A should know that B knows that A knows that B knows, and so on ad infinitum. Philosophers have a lot of fun exploring the fine points of this infinite regress and the intellectual paradoxes that it can generate. For us, the general notion that the players have a common understanding of the rules of their game will suffice.

E. Equilibrium

Finally, what happens when rational players' strategies interact? Our answer will generally be in the framework of **equilibrium.** This simply means that each player is using the strategy that is the best response to the strategies of the other players. We will develop game-theoretic concepts of equilibrium in Chapters 3 through 8 and then use them in subsequent chapters.

Equilibrium does not mean that things don't change; in sequential-move games the players' strategies are the complete plans of action and reaction, and the position evolves all the time as the successive moves are made and responded to. Nor does equilibrium mean that everything is for the best; the interaction of rational strategic choices by all players can lead to bad outcomes for all, as in the prisoners' dilemma. But we will generally find that the idea of equilibrium is a useful descriptive tool and organizing concept for our analysis. We will consider this idea in greater detail later, in connection with specific equilibrium concepts. We will also see how the concept of equilibrium can be augmented or modified to remove some of its flaws and to incorporate behavior that falls short of full calculating rationality.

Just as the rational behavior of individual players can be the result of experience in playing the game, the fitting of their choices into an overall equilibrium can come about after some plays that involve trial and error and nonequilibrium outcomes. We will look at this matter in Chapter 5.

Defining an equilibrium is not hard; actually finding an equilibrium in a particular game—that is, solving the game—can be a lot harder. Throughout this book we will solve many simple games in which there are two or three players, each of them having two or three strategies or one move each in turn. Many people believe this to be the limit of the reach of game theory and therefore believe that the theory is useless for the more complex games that take place in reality. That is not true.

Humans are severely limited in their speed of calculation and in their patience for performing long calculations. Therefore humans can easily solve only the simple games with two or three players and strategies. But computers are very good at speedy and lengthy calculations. Many games that are far beyond the power of human calculators are easy games for computers. The level of complexity that exists in many games in business and politics is already within the powers of computers. Even in games such as chess that are far too complex to solve completely, computers have reached a level of ability comparable to that of the best humans; we consider chess in more detail in Chapter 3.

Computer programs for solving quite complex games exist, and more are appearing rapidly. *Mathematica* and similar program packages contain routines for finding mixed-strategy equilibria in simultaneous-move games. Gambit, a National Science Foundation project led by Professors Richard D. McKelvey of the California Institute of Technology and Andrew McLennan of the University of Minnesota, is producing a comprehensive set of routines for finding equilibria in sequential- and simultaneous-move games, in pure and mixed strategies, and with varying degrees of uncertainty and incomplete information. We will refer to this project again in several places in the next several chapters. The biggest advantage of the project is that its programs are in the public domain and can easily be obtained from its Web site with the URL *http://hss.Caltech.edu/~gambit/Gambit.html.*

Why then do we set up and solve several simple games in detail in this book? The reason is that understanding the concepts is an important prerequisite for making good use of the mechanical solutions that computers can deliver, and understanding comes from doing simple cases yourself. This is exactly how you learned and now use arithmetic. You came to understand the ideas of addition, subtraction, multiplication, and division by doing many simple problems mentally or using paper and pencil. With this grasp of basic concepts, you can now use calculators and computers to do far more complicated sums than you would ever have the time or patience to do manually. If you did not understand the concepts, you would make errors in using calculators; for example you might solve $3 + 4 \times 5$ by grouping additions and multiplications incorrectly as $(3 + 4) \times 5 = 35$ instead of correctly as $3 + (4 \times 5) = 23$.

Thus the first step of understanding the concepts and tools is essential. Without it, you would never learn to set up correctly the games that you ask the computer to solve. You would not be able to inspect the solution with any feeling for whether it was reasonable and, if it was not, would not be able to go back to your original specification, improve it, and solve it again until the specification and the calculation correctly capture the strategic situation that you want to study. Therefore please pay serious attention to the simple examples that we solve and the drill exercises that we ask you to solve, especially in Chapters 3 through 8.

F. Dynamics and Evolutionary Games

The theory of games based on assumptions of rationality and equilibrium has proved very useful, but it would be a mistake to rely on it totally. When games are played by novices who do not have the necessary experience to perform the calculations to choose their best strategies, explicitly or implicitly, their choices, and therefore the outcome of the game, can differ significantly from the predictions of analysis based on the concept of equilibrium.

However, we should not abandon all notions of good choice; we should recognize the fact that even poor calculators are motivated to do better for their own sakes and will learn from experience and by observing others. We should allow for a dynamic process in which strategies that proved to be better in previous plays of the game are more likely to be chosen in later plays.

The **evolutionary** approach to games does just this. It is derived from the idea of evolution in biology. Any individual animal's genes strongly influence its behavior. Some behaviors succeed better in the prevailing environment, in the sense that the animals exhibiting those behaviors are more likely to reproduce successfully and pass their genes to their progeny. An evolutionary stable state, relative to a given environment, is the ultimate outcome of this process over several generations.

The analogy in games would be to suppose that strategies are not chosen by conscious rational maximizers, but instead that each player comes to the game with a particular strategy "hardwired" or "programmed" in. The players then confront other players who may be programmed to play the same or different strategies. The payoffs to all the players in such games are then obtained. The strategies that fare better—in the sense that the players programmed to play them get higher payoffs in the games—multiply faster, while the strategies that fare worse decline. In biology, the mechanism of this growth or decay is purely genetic transmission through reproduction. In the context of strategic games in business and society, the mechanism is much more likely to be social or cultural—observation and imitation, teaching and learning, greater availability of capital for the more successful ventures, and so on.

The object of study is the dynamics of this process. Does it converge to an evolutionary stable state? Does just one strategy prevail over all others in the end or can a few strategies coexist? Interestingly, in many games the evolutionary stable limit is the same as the equilibrium that would result if the players were consciously rational calculators. Therefore the evolutionary approach gives us a backdoor justification for equilibrium analysis.

The concept of evolutionary games has thus imported biological ideas into game theory; there has been an influence in the opposite direction, too. Biologists have recognized that significant parts of animal behavior consist of strategic interactions with other animals. Members of a given species compete with one another for space or mates; members of different species relate to one another as predators and prey along a food chain. The payoff in such games in turn contributes to reproductive success and therefore to biological evolution. Just as game theory has benefited by importing ideas from biological evolution for its analysis of choice and dynamics, biology has benefited by importing game-theoretic ideas of strategies and payoffs for its characterization of basic interactions between animals. We have truly an instance of synergy or symbiosis. We provide an introduction to the study of evolutionary games in Chapter 13.

G. Observation and Experiment

All of Section 3 to this point has concerned how to think about games or how to analyze strategic interactions. This constitutes theory. This book will give an extremely simple level of theory, developed through cases and illustrations instead of formal mathematics or theorems, but it will be theory just the same. All theory should relate to reality in two ways. Reality should help structure the theory, and reality should provide a check on the results of the theory.

We can find out the reality of strategic interactions in two ways: (1) by observing them as they occur naturally and (2) by conducting special experiments that help us pin down the effects of particular conditions. Both methods have been used, and we will mention several examples of each in the proper contexts.

Many people have studied strategic interactions—the participants' behavior and the outcomes—under experimental conditions, in classrooms among "captive" players, or in special laboratories with volunteers. Auctions, bargaining, prisoners' dilemmas, and several other games have been studied in this way. The results are a mixture. Some conclusions of the theoretical analysis are borne out; for example, in games of buying and selling, the participants generally settle quickly on the economic equilibrium. In other contexts, the outcomes differ significantly from the theoretical predictions; for example, prisoners' dilemmas and bargaining games show more cooperation than theory based on the assumption of selfish maximizing behavior would lead us to expect, while auctions show some gross overbidding.

At several points in the chapters that follow, we will review the knowledge that has been gained by observation and experiments, discuss how it relates to the theory, and consider what reinterpretations, extensions, and modifications of the theory have been made or should be made in the light of this knowledge.

4 THE USES OF GAME THEORY

We began Chapter 1 by saying that games of strategy are everywhere—in your personal and working life; in the functioning of the economy, society, and polity around you; in sports and other serious pursuits; in war; and in peace. This should be motivation enough to study such games systematically, and that is what game theory is about, but your study can be better directed if you have a clearer idea of just how you can put game theory to use. We suggest a threefold method.

The first use is in *explanation*. Many events and outcomes prompt us to ask: Why did that happen? When the situation requires the interaction of decision makers with different aims, game theory often supplies the key to understanding the situation. For example, cutthroat competition in business is the result of the rivals being trapped in a prisoners' dilemma. At several points in the book we will mention actual cases where game theory helps us to understand how and why the events unfolded as they did. This includes the detailed case study of the Cuban missile crisis from the perspective of game theory.

The other two uses evolve naturally from the first. The second is in *prediction*. When looking ahead to situations where multiple decision makers will interact strategically, we can use game theory to foresee what actions they will take and what outcomes will result. Of course, prediction for a particular context depends on its details, but we will prepare you to use prediction by analyzing several broad classes of games that arise in many applications.

The third use is in *advice* or *prescription*: we can act in the service of one participant in the future interaction and tell him which strategies are likely to yield good results and which ones are likely to lead to disaster. Once again such work is context specific, and we equip you with several general principles and techniques and show you how to apply them to some general types of contexts. For example, in Chapters 7 and 8 we will show how to mix moves, in Chapter 10 we will examine how to make your commitments, threats, and promises credible, and in Chapter 11 we will examine alternative ways of overcoming prisoners' dilemmas.

The theory is far from perfect in performing any of the three functions. To explain an outcome, one must first have a correct understanding of the motives and behavior of the participants. As we saw earlier, most of game theory takes a specific approach to these matters—namely, the framework of rational choice of individual players and the equilibrium of their interaction. Actual players and interactions in a game might not conform to this framework. But the proof of the pudding is in the eating. Game-theoretic analysis has greatly improved our understanding of many phenomena, as reading this book should convince you. The theory continues to evolve and improve as the result of ongoing research. This book will equip you with the basics so that you can more easily learn and profit from the new advances as they appear.

When explaining a past event, we can often use historical records to get a good idea of the motives and the behavior of the players in the game. When attempting prediction or advice, there is the additional problem of determining what motives will drive the players' actions, what informational or other limitations they will face, and sometimes even who the players will be. Most importantly, if game-theoretic analysis assumes that the other player is a rational maximizer of his own objectives when in fact he is unable to do the calculations or is a clueless person acting at random, the advice based on that analysis may prove wrong. This risk is reduced as more and more players recognize the importance of strategic interaction and think through their strategic choices or get expert advice on the matter, but some risk remains. Even then, the systematic thinking made possible by the framework of game theory helps keep the errors down to this irreducible minimum, by eliminating the errors that arise from faulty logical thinking about the strategic interaction. Also, game theory can take into account many kinds of uncertainty and incomplete information, including that about the strategic possibilities and rationality of the opponent. We will consider a few examples in the chapters to come.

5 THE STRUCTURE OF THE CHAPTERS TO FOLLOW

In this chapter we introduced several considerations that arise in almost every game in reality. To understand or predict the outcome of any game, we must know in greater detail all of these ideas. We also introduced some basic concepts that will prove useful in such analysis. However, trying to cope with all of the concepts at once merely leads to confusion and a failure to grasp any of them. Therefore we will build up the theory one concept at a time. We will develop the appropriate technique for analyzing that concept and illustrate it.

In the first group of chapters, from Chapters 3 to 8, we will construct and illustrate the most important of these concepts and techniques. We will examine purely sequential-move games in Chapter 3 and introduce the techniques—game trees and rollback reasoning—that are used to analyze and solve such games. In Chapters 4 and 5, we will turn to games with simultaneous moves and develop for them another set of concepts—payoff tables, dominance, and Nash equilibrium. Both chapters will focus on games where players use pure strategies; in Chapter 4, we will restrict players to a finite set of pure strategies and, in Chapter 5, we will allow strategies that are continuous variables. Chapter 5 will also examine some deeper conceptual matters and a prominent alternative to Nash equilibrium—namely, rationalizability. In Chapter 6, we will show how games that have some sequential moves and some simultaneous moves can be studied by combining the techniques developed in Chapters 3 through 5. In Chapter 7, we will turn to simultaneous-move zero-sum games that require the use of randomization or mixed strategies. Chapter 8 will then examine the role of mixed strategies in nonzero-sum games and will develop a little general theory of mixed strategies.

The ideas and techniques developed in Chapters 3 through 8 are the most basic ones: (1) correct forward-looking reasoning for sequential-move games and (2) equilibrium strategies—pure and mixed—for simultaneous-move games. Equipped with these concepts and tools, we can apply them to study some broad classes of games and strategies in Chapters 9 through 13.

Chapter 9 studies what happens in games when players are subject to uncertainty or when they have asymmetric information. We will examine strategies for coping with risk and even for using risk strategically. We will also study the important strategies of signaling and screening that are used for manipulating and eliciting information. In Chapter 10, we will continue to examine the role of player manipulation in games as we consider strategies that players use to manipulate the rules of a game, by seizing a first-mover advantage and making a strategic move. Such moves are of three kinds—

commitments, threats, and promises. In each case, credibility is essential to the success of the move, and we will outline some ways of making such moves credible.

In Chapter 11, we will move on to study the most well known game of them all—the prisoners' dilemma. We will study whether and how cooperation can be sustained, most importantly in a repeated or ongoing relationship. Then, in Chapter 12, we will turn to situations where large populations, rather than pairs or small groups of players, interact strategically, games that concern problems of collective action. Each person's actions have an effect—in some instances beneficial, in others, harmful—on the others. The outcomes are generally not the best from the aggregate perspective of the society as a whole. We will clarify the nature of these outcomes and describe some simple policies that can lead to better outcomes.

All these theories and applications are based on the supposition that the players in a game fully understand the nature of the game and deploy calculated strategies that best serve their objectives in the game. Such rationally optimal behavior is sometimes too demanding of information and calculating power to be believable as a good description of how people really act. Therefore Chapter 13 will look at games from a very different perspective. Here, the players are not calculating and do not pursue optimal strategies. Instead, each player is tied, as if genetically preordained, to a particular strategy. The population is diverse, and different players have different predetermined strategies. When players from such a population meet and act out their strategies, which strategies perform better? And, if the more successful strategies proliferate better in the population, whether through inheritance or imitation, then what will the eventual structure of the population look like? It turns out that such evolutionary dynamics often favor exactly those strategies that would be used by rational optimizing players. Thus our study of evolutionary games lends useful indirect support to the theories of optimal strategic choice and equilibrium that we will have studied in the previous chapters.

In the final group, Chapters 14 through 18, we will take up specific applications to situations of strategic interactions. Here, we will use as needed the ideas and methods from all the earlier chapters. We will start with a particularly interesting dynamic version of a threat, known as the strategy of brinkmanship. We will elucidate its nature in Chapter 14 and apply the idea to study the Cuban missile crisis of 1962. Chapter 15 is about voting in committees and elections. We will look at the variety of voting rules available and some paradoxical results that can arise. In addition, we will address the potential for strategic behavior not only by voters but also by candidates in a variety of election types.

Chapters 16 through 18 will look at mechanisms for the allocation of valuable economic resources: Chapter 16 will treat auctions, Chapter 17 will consider bargaining processes, and Chapter 18 will look at markets. In our discussion of auctions, we will emphasize the roles of information and attitudes toward risk in the formulation of optimal strategies for both bidders and sellers. We will also take the opportunity to apply the theory to the newest type of auctions, those that take place online. Chapter 17 will present bargaining in both cooperative and noncooperative settings. Finally, Chapter 18 will consider games of market exchange, building on some of the concepts used in bargaining theory and including some theory of the core.

All of these chapters together provide a lot of material; how might readers or teachers with more specialized interests choose from it? Chapters 3, 4, 6, and 7 constitute the core theoretical ideas that are needed throughout the rest of the book. Chapters 10 and 11 are likewise important for the general classes of games and strategies considered therein. Beyond that, there is a lot from which to pick and choose. Chapters 5 and 8 consider some more advanced topics and go somewhat deeper into theory and mathematics. These chapters will appeal to those with more scientific and quantitative backgrounds and interests, but those who come from the social sciences or humanities and have less quantitative background can omit them without loss of continuity. Chapter 9 deals with an important topic in that most games in practice have incomplete and asymmetric information, and the players' attempts to manipulate information is a critical aspect of many strategic interactions. However, the concepts and techniques for analyzing information games are inherently somewhat more complex. Therefore some readers and teachers may choose to study just the examples that convey the basic ideas of signaling and screening and leave out the rest. We have placed this chapter early in Part Three, however, in view of the importance of the subject. Chapters 12 and 13 both look at games with large numbers of players. In Chapter 12, the focus is on social interactions; in Chapter 13, the focus is on evolutionary biology. The topics in Chapter 13 will be of greatest interest to those with interests in biology, but similar themes are emerging in the social sciences, and students from that background should aim to get the gist of the ideas even if they skip the details. Chapters 14 and 15 present topics from political science—international diplomacy and elections, respectively—and Chapters 16 through 18 cover topics from economics—auctions, bargaining, and markets. Those teaching courses with more specialized audiences may choose a subset and indeed expand on the ideas considered therein.

Whether you come from mathematics, biology, economics, politics, other sciences, or from history or sociology, the theory and examples of strategic games will stimulate and challenge your intellect. We urge you to enjoy the subject even as you are studying or teaching it.

SUMMARY

Strategic *games* situations are distinguished from individual decision-making situations by the presence of significant interactions among the players. Games can be classified according to a variety of categories including the timing of play, the common or conflicting interests of players, the number of times an interaction occurs, the amount of information available to the players, the type of rules, and the feasibility of coordinated action.

Learning the terminology for a game's structure is crucial for analysis. Players have *strategies* that lead to different *outcomes* with different associated *payoffs*. Payoffs incorporate everything that is important to a player about a game and are calculated by using probabilistic averages or *expectations* if outcomes are random or include some risk. *Rationality*, or consistent behavior, is assumed of all players, who must also be aware of all of the relevant rules of conduct. *Equilibrium* arises when all players use strategies that are best responses to others' strategies; some classes of games allow learning from experience and the study of dynamic movements toward equilibrium. The study of behavior in actual game situations provides additional information about the performance of the theory.

Game theory may be used for explanation, prediction, or prescription in various circumstances. Although not perfect in any of these roles, the theory continues to evolve; the importance of strategic interaction and strategic thinking has also become more widely understood and accepted.

KEY TERMS[3]

asymmetric information (23)
cooperative game (26)
decision (18)
equilibrium (33)
evolutionary game (35)
expected payoff (29)
game (18)
imperfect information (23)
incomplete information (23)
noncooperative game (26)
payoff (28)
rational behavior (30)
screening (24)
screening device (24)
sequential moves (20)
signal (24)
signaling (24)
simultaneous moves (20)
strategic game (18)
strategies (27)

[3]The number in parentheses after each key term is the page on which that term is defined or discussed.

EXERCISES

1. Determine which of the following situations describe games and which describe decisions. In each case, indicate what specific features of the situation caused you to classify it as you did.
   (a) A group of grocery shoppers choosing what flavor of yogurt to purchase
   (b) A pair of teenage girls choosing dresses for their prom
   (c) A college student considering what type of postgraduate education to pursue
   (d) Microsoft and Netscape choosing prices for their Internet browsers
   (e) A state gubernatorial candidate picking a running mate

2. Consider the strategic games described below. In each case, state how you would classify the game according to the six dimensions outlined in the text: whether moves are sequential or simultaneous, whether the game is zero-sum, whether the game is repeated, whether there is imperfect or incomplete (asymmetric) information, whether the rules are fixed, and whether cooperative agreements are possible. If you do not have enough information to classify a game in a particular dimension, explain why not.
   (a) *Rock–Paper–Scissors:* On the count of three, each player makes the shape of one of the three items with his hand. Rock beats scissors, scissors beats paper, and paper beats rock.
   (b) *Roll-call voting:* Voters cast their votes orally as their names are called. In a two-candidate election, the candidate with the most votes wins.
   (c) *Sealed-bid auction:* Bidders seal their bids in envelopes for a bottle of wine. The highest bidder wins the item and pays his bid.

3. "A game player would never prefer an outcome in which every player gets a little profit to an outcome in which he gets all the available profit." Is this statement true or false? Explain why in one or two sentences.

4. You and a rival are engaged in a game in which there are three possible outcomes: you win, your rival wins (you lose), and the two of you tie. You get a payoff of 50 if you win, a payoff of 20 if you tie, and a payoff of zero if you lose. What is your expected payoff in each of the following situations:
   (a) There is a 50% chance that the game ends in a tie, but only a 10% chance that you win. (There is thus a 40% chance that you lose.)
   (b) There is a 50–50 chance that you win or lose. There are no ties.
   (c) There is an 80% chance that you lose and a 10% chance that you win or that you tie.

5. Explain the difference between game theory's use as a predictive tool and its use as a prescriptive tool. In what types of real-world settings might these two uses be most important?

PART TWO

Concepts and Techniques

3

Games with Sequential Moves

Sequential-move games entail strategic situations in which there is a strict order of play. Players take turns making their moves, and they know what players who have gone before them have done. To play well in such a game, participants must use a particular type of interactive thinking. Each player must consider: If I make this move, how will my opponent respond? Whenever actions are taken, players need to think about how their current actions will influence future actions, both for their rivals and for themselves. Players thus decide their current moves on the basis of calculations of future consequences.

Most actual games combine aspects of both sequential- and simultaneous-move situations. But the concepts and methods of analysis are more easily understood if they are first developed separately for the two pure cases. Therefore in this chapter we study purely sequential games. Chapters 4 and 5 deal with purely simultaneous games, and Chapter 6 and parts of Chapters 7 and 8 show how to combine the two types of analysis in more realistic mixed situations. The analysis presented here can be used whenever a game includes sequential decision making. Analysis of sequential games also provides information about when it is to a player's advantage to move first and when it is better to move second. Players can then devise ways, called *strategic moves*, to manipulate the order of play to their advantage. The analysis of such moves is the focus of Chapter 10.

1 GAME TREES

We begin by developing a graphical technique for displaying and analyzing sequential-move games, called a **game tree.** This tree is referred to as the **extensive form** of a game. It shows all the component parts of the game that we introduced in Chapter 2: players, actions, and payoffs.

You have probably come across **decision trees** in other contexts. Such trees show all the successive decision points, or nodes, for a single decision maker in a neutral environment. Decision trees also include branches corresponding to the available choices emerging from each node. Game trees are just joint decision trees for all of the players in a game. The trees illustrate all of the possible actions that can be taken by all of the players and indicate all of the possible outcomes from the game.

A. Nodes, Branches, and Paths of Play

Figure 3.1 shows the tree for a particular sequential game. We do not supply a story for this game, because we want to omit circumstantial details and to help you focus on general concepts. Our game has four players: Ann, Bob, Chris, and Deb. The rules of the game give the first move to Ann; this is shown at the leftmost point, or **node,** which is called the **initial node** or **root** of the game tree. At this node, which may also be called an **action** or **decision node,** Ann has two choices available to her. Ann's possible choices are labeled "Stop" and "Go" (remember that these labels are abstract and have no necessary significance) and are shown as **branches** emerging from the initial node.

If Ann chooses "Stop," then it will be Bob's turn to move. At his action node, he has three available choices labeled 1, 2, and 3. If Ann chooses "Go," then Chris gets the next move, with choices "Risky" and "Safe." Other nodes and branches follow successively and, rather than list them all in words, we draw your attention to a few prominent features.

If Ann chooses "Stop" and then Bob chooses 1, Ann gets another turn, with new choices, "Up" and "Down." It is quite common in actual sequential-move games for a player to get to move several times and to have her available moves differ at different turns. In chess, for example, two players make alternate moves; each move changes the board and therefore the available moves at subsequent turns.

B. Uncertainty and "Nature's Moves"

If Ann chooses "Go" and then Chris chooses "Risky," something happens at random—a fair coin is tossed and the outcome of the game is determined by whether that coin comes up "heads" or "tails." This aspect of the game is han-

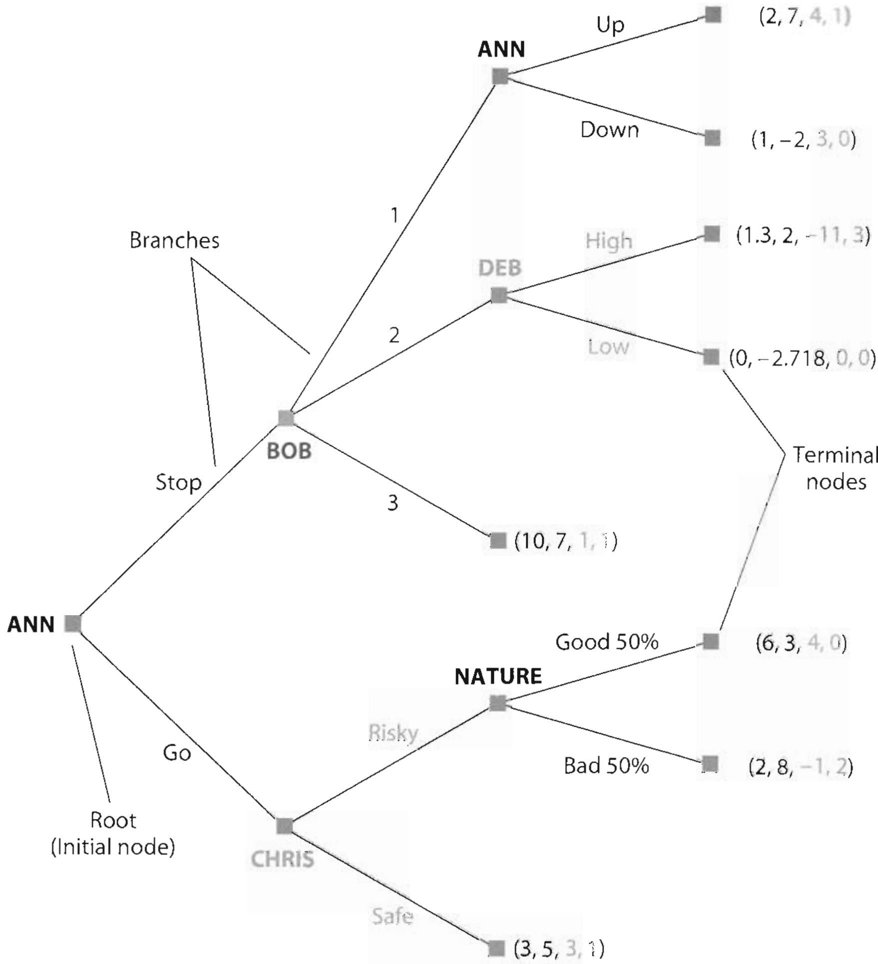


FIGURE 3.1 An Illustrative Game Tree

dled in the tree by introducing an outside player called "Nature." Control over the random event is ceded to the player known as Nature who chooses, as it were, one of two branches each with 50% probability. The probabilities here are fixed by the type of random event, a coin toss, but could vary in other circumstances; for example, with the throw of a die, Nature could specify six possible outcomes each with $16\frac{2}{3}$% probability. Use of the player Nature allows us to introduce uncertainty in a game and gives us a mechanism to allow things to happen that are outside the control of any of the actual players.

You can trace a number of different paths through the game tree by following successive branches. In Figure 3.1, each path leads you to an end point of the game after a finite number of moves. An end point is not a necessary feature of all games; some may in principle go on forever. But most applications that we will consider are finite games.

C. Outcomes and Payoffs

At the last node along each path, called a **terminal node,** no player has another move. (Note that terminal nodes are thus distinguished from *action* nodes.) Instead, we show the outcome of that particular sequence of actions, as measured by the payoffs of the players. For our four players, we list the payoffs in the order (Ann, Bob, Chris, Deb). It is important to specify which payoff belongs to which player. The usual convention is to list payoffs in the order in which the players make the moves. But this method may sometimes be ambiguous; in our example, it is not clear whether Bob or Chris should be said to have the second move. Thus we have used the alphabetical order. Further, we have color-coded everything so that Ann's name, choices and payoffs are all in black, Bob's in red, Chris's in grey, and Deb's in pink. When drawing trees for any games that you analyze, you can choose any specific convention you like, but you should state and explain it clearly for the reader.

The payoffs are numerical, and generally for each player a higher number means a better outcome. Thus, for Ann, the outcome of the bottommost path (payoff 3) is better than that of the topmost path (payoff 2) in Figure 3.1. But there is no necessary comparability across players. Thus there is no necessary sense in which, at the end of the topmost path, Bob (payoff 7) does better than Ann (payoff 2). Sometimes, if payoffs are dollar amounts, for example, such interpersonal comparisons may be meaningful.

Players use information about payoffs when deciding among the various actions available to them. The inclusion of a random event (a choice made by Nature) means that players need to determine what they get on average when Nature moves. For example, if Ann chooses "Go" at the game's first move, Chris may then choose "Risky," giving rise to the coin toss and Nature's "choice" of "Good" or "Bad." In this situation, Ann could anticipate a payoff of 6 half the time and a payoff of 2 half the time, or a statistical average or *expected payoff* of $4 = (0.5 \times 6) + (0.5 \times 2)$.

D. Strategies

Finally, we use the tree in Figure 3.1 to explain the concept of a strategy. A single action taken by a player at a node is called a **move.** But players can, do, and should make plans for the succession of moves that they expect to make in all of the various eventualities that might arise in the course of a game. Such a plan of action is called a strategy.

In this tree, Bob, Chris, and Deb each get to move at most once; Chris, for example, gets a move only if Ann chooses "Go" on her first move. For them, there is no distinction between a move and a strategy. We can qualify the move by specifying the contingency in which it gets made; thus, a strategy for Bob might be, "Choose 1 if Ann has chosen Stop." But Ann has two opportunities to move, so her strategy needs a fuller specification. One strategy for her is, "Choose Stop, and then if Bob chooses 1, choose Down."

In more complex games such as chess, where there are long sequences of moves with many choices available at each, descriptions of strategies get very complicated; we consider this aspect in more detail later in this chapter. But the general principle for constructing strategies is simple, except for one peculiarity. If Ann chooses "Go" on her first move, she never gets to make a second move. Should a strategy in which she chooses "Go" also specify what she would do in the hypothetical case in which she somehow found herself at the node of her second move? Your first instinct may be to say *no,* but formal game theory says *yes,* and for two reasons.

First, Ann's choice of "Go" at the first move may be influenced by her consideration of what she would have to do at her second move if she were to choose "Stop" originally instead. For example, if she chooses "Stop," Bob may then choose 1; then Ann gets a second move and her best choice would be "Up," giving her a payoff of 2. If she chooses "Go" on her first move, Chris would choose "Safe" (because his payoff of 3 from "Safe" is better than his expected payoff of 1.5 from "Risky"), and that outcome would yield Ann a payoff of 3. To make this thought process clearer, we state Ann's strategy as, "Choose Go at the first move, and choose Up if the next move arises."

The second reason for this seemingly pedantic specification of strategies has to do with the stability of equilibrium. When considering stability, we ask what would happen if players' choices were subjected to small disturbances. One such disturbance is that players make small mistakes. If choices are made by pressing a key, for example, Ann may intend to press the "Go" key, but there is a small probability that her hand may tremble and she may press the "Stop" key instead. In such a setting, it is important to specify how Ann will follow up when she discovers her error because Bob chooses 1 and it is Ann's turn to move again. More advanced levels of game theory require such stability analyses, and we want to prepare you for that by insisting on your specifying strategies as such complete plans of action right from the beginning.

E. Tree Construction

Now we sum up the general concepts illustrated by the tree of Figure 3.1. Game trees consist of nodes and branches. Nodes are connected to one another by the branches and come in two types. The first node type is called a decision node. Each decision node is associated with the player who chooses an action at that node; every tree has one decision node that is the game's initial node, the starting point of the game. The second type of node is called a terminal node. Each terminal node has associated with it a set of outcomes for the players taking part in the game; these outcomes are the payoffs received by each player if the game has followed the branches that lead to this particular terminal node.

The branches of a game tree represent the possible actions that can be taken from any decision node. Each branch leads from a decision node on the tree either to another decision node, generally for a different player, or to a terminal node. The tree must account for all of the possible choices that could be made by a player at each node; so some game trees include branches associated with the choice "do nothing." There must be at least one branch leading from each decision node, but there is no maximum. Every decision node can have only one branch leading to it, however.

Game trees are often drawn from left to right across a page. However, game trees can be drawn in any orientation that best suits the game at hand: bottom up, sideways, top down, or even radially outward from a center. The tree is a metaphor, and the important feature is the idea of successive branching, as decisions are made at the tree nodes.

2 SOLVING GAMES BY USING TREES

We illustrate the use of trees in finding equilibrium outcomes of sequential-move games in a very simple context that many of you have probably confronted—whether to smoke. This situation and many other similar one-player strategic situations can be described as games if we recognize that future choices are actually made by a different player. That player is one's future self who will be subject to different influences and will have different views about the ideal outcome of the game.

Take, for example, a teenager named Carmen who is deciding whether to smoke. First, she has to decide whether to try smoking at all. If she does try it, she has the further decision of whether to continue. We illustrate this example as a simple decision in the tree of Figure 3.2.

The nodes and the branches are labeled with Carmen's available choices, but we need to explain the payoffs. Choose the outcome of never smoking at all as the standard of reference, and call its payoff 0. There is no special significance to the number zero in this context; all that matters for comparing outcomes, and thus for Carmen's decision, is whether this payoff is bigger or smaller than the others. Suppose Carmen best likes the outcome in which she tries smoking for a while but does not continue. The reason may be that she just likes to have experienced many things first-hand or so that she can more convincingly be able to say "I have been there and know it to be a bad situation" when she tries in the future to dissuade her children from smoking. Give this outcome the payoff 1. The outcome in which she tries smoking and then contin-

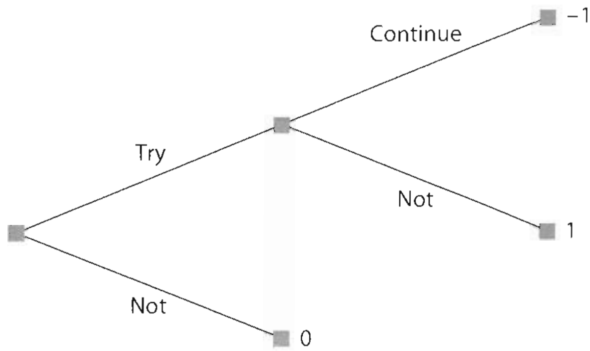


FIGURE 3.2 The Smoking Decision

ues is the worst. Leaving aside the long-term health hazards, there are immediate problems—her hair and clothes will smell bad, and her friends will avoid her. Give this outcome the payoff -1. Carmen's best choice then seems clear—she should try smoking but she should not continue.

However, this analysis ignores the problem of addiction. Once Carmen has tried smoking for a while, she becomes a different person with different tastes, as well as different payoffs. The decision of whether to continue will be made, not by "Today's Carmen" with today's assessment of outcomes as shown in Figure 3.2, but by a different "Future Carmen" with a different ranking of the alternatives then available. When she makes her choice today, she has to look ahead to this consequence and factor it into her current decision, which she should make on the basis of her current preferences. In other words, the choice problem concerning smoking is not really a decision in the sense explained in Chapter 2—a choice made by a single person in a neutral environment—but a game in the technical sense also explained in Chapter 2, where the other player is Carmen's future self with her own distinct preferences. When Today's Carmen makes her decision, she has to play against her future self.

We convert the decision tree of Figure 3.2 into a game tree in Figure 3.3, by distinguishing between the two players who make the choices at the two nodes. At the initial node, "Today's Carmen" decides whether to try smoking. If her decision is to try, then the addicted "Future Carmen" comes into being and chooses whether to continue. We show the healthy red-blooded Today's Carmen, her actions, and her payoffs in red, and the addicted Future Carmen, her actions, and her payoffs in black, the color that her lungs have become. The payoffs of Today's Carmen are as before. But Future Carmen will enjoy continuation and will suffer terrible withdrawal symptoms if she does not continue. Let Future Carmen's payoff from "continue" be $+1$ and that from "not" be -1.

Given the preferences of the addicted Future Carmen, she will choose "continue" at her decision node. Today's Carmen should look ahead to this prospect and fold it into her current decision, recognizing that the choice to try smoking

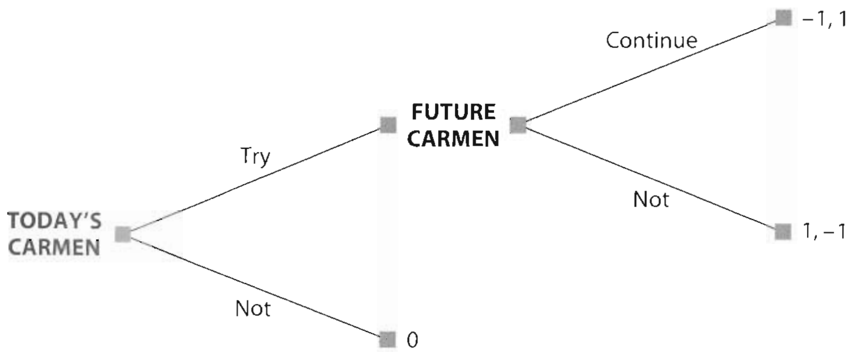


FIGURE 3.3 The Smoking Game

will inevitably lead to continuation. Even though Today's Carmen does not want continuation to happen given her preferences today, she will not be able to implement her currently preferred choice at the future time, because a different Carmen with different preferences will make that choice. So Today's Carmen should foresee that the choice "Try" will lead to "Continue" and get her the payoff -1 as judged by her today, whereas the choice "Don't Try" will get her the payoff 0. So she should choose the latter.

This argument is shown more formally and with greater visual effect in Figure 3.4. In Figure 3.4a, we cut off, or **prune,** the branch "Not" emerging from the second node. This pruning corresponds to the fact that Future Carmen, who makes the choice at that node, will not choose the action associated with that branch, given her preferences as shown in black.

The tree that remains has two branches emerging from the first node where Today's Carmen makes her choice; each of these branches now leads directly to a terminal node. The pruning allows Today's Carmen to forecast completely the eventual consequence of each of her choices. "Try" will be followed by "Continue" and yield a payoff -1, as measured in the preferences of Today's Carmen, while "Not" will yield 0. Carmen's choice today should then be "Not" rather than "Try." Therefore we can prune the "Try" branch emerging from the first node (along with its foreseeable continuation). This pruning is done in Figure 3.4b. The tree shown there is now "fully pruned," leaving only one branch emerging from the initial node and leading to a terminal node. Following the only remaining path through the tree shows what will happen in the game when all players make their best choices with correct forecasting of all future consequences.

In pruning the tree in Figure 3.4, we crossed out the branches not chosen. Another equivalent but alternative way of showing player choices is to "highlight" the branches that *are* chosen. To do so, you can place check marks or arrowheads on these branches or show them as thicker lines. Any one method will do; Figure 3.5 shows them all. You can choose whether to prune or to highlight,

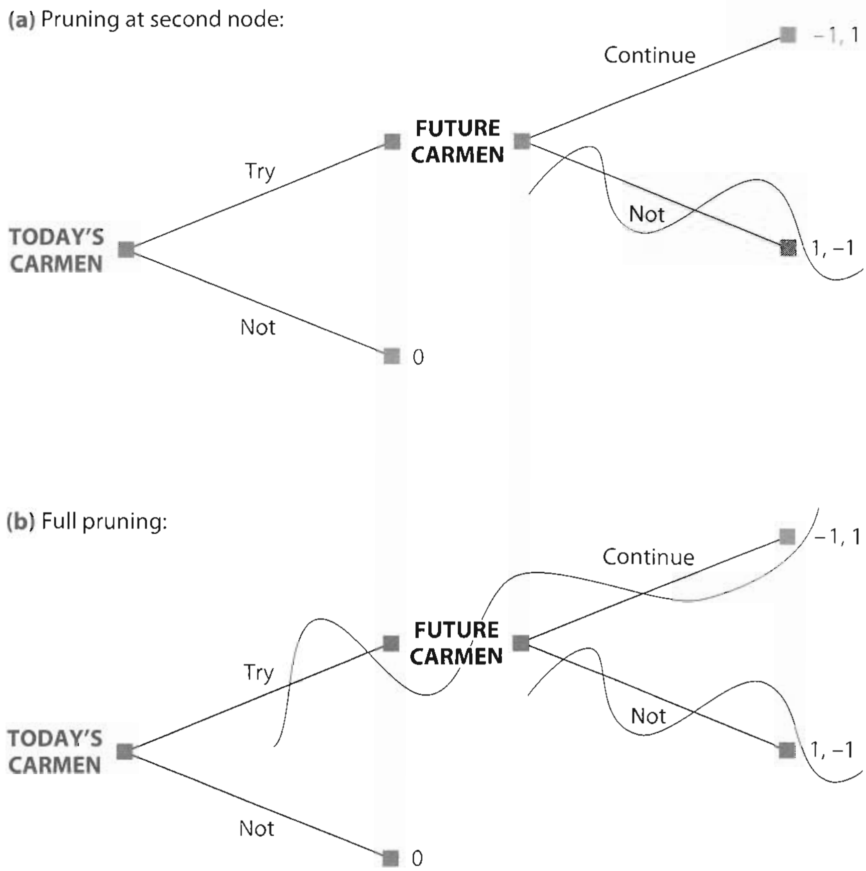


FIGURE 3.4 Pruning the Tree of the Smoking Game

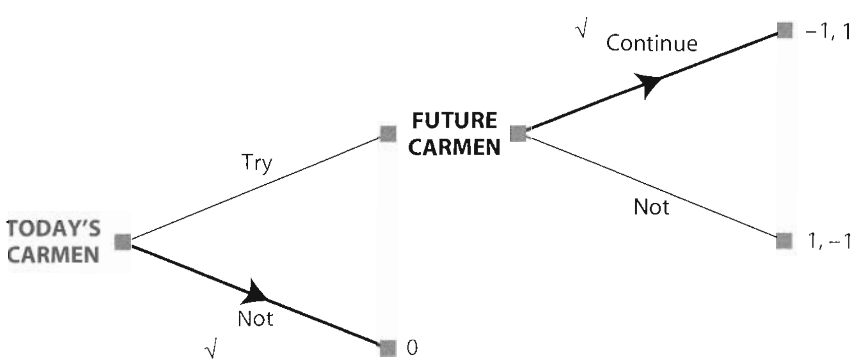


FIGURE 3.5 Showing Branch Selection on the Tree of the Smoking Game

but the latter, especially in its arrowhead form, has some advantages. First, it produces a cleaner picture. Second, the mess of the pruning picture sometimes does not clearly show the order in which various branches were cut. For example, in Figure 3.4b, a reader may get confused and incorrectly think that the "Continue" branch at the second node was cut first and that the "Try" branch at the first node followed by the "Not continue" branch at the second node were cut next. Finally, and most importantly, the arrowheads show the outcome of the sequence of optimal choices most visibly as a continuous link of arrows from the initial node to a terminal node. Therefore, in subsequent diagrams of this type, we generally use arrows instead of pruning. When you draw game trees, you should practice showing both methods for a while; when you are comfortable with trees, you can choose either to suit your taste.

No matter how you display your thinking in a game tree, the logic of the analysis is the same and important. You must start your analysis by considering those action nodes that lead directly to terminal nodes. The optimal choices for a player moving at such a node can be found immediately by comparing her payoffs at the relevant terminal nodes. With the use of these end-of-game choices to forecast consequences of earlier actions, the choices at nodes just preceding the final decision nodes can be determined. Then the same can be done for the nodes before them, and so on. By working backward along the tree in this way, you can solve the whole game.

This method of looking ahead and reasoning back to determine behavior in sequential-move games is known as **rollback.** As the name suggests, using rollback requires starting to think about what will happen at all the terminal nodes and literally "rolling back" through the tree to the initial node as you do your analysis. Because this reasoning requires working backward one step at a time, the method is also called **backward induction.** We use the term rollback because it is simpler and becoming more widely used, but other sources on game theory will use the older term, backward induction. Just remember that the two are equivalent.

When all players choose their optimal strategies found by doing rollback analysis, we call this set of strategies the **rollback equilibrium** of the game; the outcome that arises from playing these strategies is the *rollback equilibrium outcome*. Game theory predicts this outcome as the equilibrium of a sequential game when all players are rational calculators in pursuit of their respective best payoffs. Later in this chapter, we address how well this prediction is borne out in practice. For now, you should know that all finite sequential-move games presented in this book have at least one rollback equilibrium. In fact, most have exactly one. Only in exceptional cases where a player gets equal payoffs from two or more different sets of moves, and is therefore indifferent between them, will games have more than one rollback equilibrium.

In the smoking game, the rollback equilibrium is where Today's Carmen chooses the strategy "Not" and Future Carmen chooses the strategy "Continue."

When Today's Carmen takes her optimal action, the addicted Future Carmen does not come into being at all and therefore gets no actual opportunity to move. But Future Carmen's shadowy presence and the strategy that she would choose if Today's Carmen chose "Try" and gave her an opportunity to move are important parts of the game. In fact, they are instrumental in determining the optimal move for Today's Carmen.

We introduced the ideas of the game tree and rollback analysis in a very simple example, where the solution was obvious from verbal argument. Now we proceed to use the ideas in successively more complex situations, where verbal analysis becomes harder to conduct and the visual analysis with the use of the tree becomes more important.

3 ADDING MORE PLAYERS

The techniques developed in Section 2 in the simplest setting of two players and two moves can be readily extended. The trees get more complex, with more branches, nodes, and levels, but the basic concepts and the method of rollback remain unchanged. In this section, we consider a game with three players, each of whom has two choices; with slight variations, this game reappears in many subsequent chapters.

The three players, Emily, Nina, and Talia, all live on the same small street. Each has been asked to contribute toward the creation of a flower garden at the intersection of their small street with the main highway. The ultimate size and splendor of the garden depends on how many of them contribute. Furthermore, although each player is happy to have the garden—and happier as its size and splendor increase—each is reluctant to contribute because of the cost that she must incur to do so.

Suppose that, if two or all three contribute, there will be sufficient resources for the initial planting and subsequent maintenance of the garden; it will then be quite attractive and pleasant. However, if one or none contribute, it will be too sparse and poorly maintained to be pleasant. From each player's perspective, there are thus four distinguishable outcomes:

- She does not contribute, both of the others do (pleasant garden, saves cost of own contribution)
- She contributes, and one or both of the others do (pleasant garden, incurs cost of contribution)
- She does not contribute, only one or neither of the others does (sparse garden, saves cost of own contribution)
- She contributes, but neither of the others does (sparse garden, incurs cost of own contribution)

Of these outcomes, the one listed at the top is clearly the best and the one listed at the bottom is clearly the worst. We want higher payoff numbers to indicate outcomes that are more highly regarded; so we give the top outcome the payoff 4 and the bottom one the payoff 1. (Sometimes payoffs are associated with an outcome's rank order; so, with four outcomes, 1 would be best and 4 worst, and smaller numbers would denote more preferred outcomes. When reading, you should carefully note which convention the author is using; when writing, you should carefully state which convention you are using.)

There is some ambiguity about the two middle outcomes. Let us suppose that each player regards a pleasant garden more highly than her own contribution. Then the outcome listed second gets payoff 3, and the outcome listed third gets payoff 2.

Suppose the players move sequentially. Emily has the first move, and chooses whether to contribute. Then, after observing what Emily has chosen, Nina makes her choice between contributing and not contributing. Finally, having observed what Emily and Nina have chosen, Talia makes a similar choice.[1]

Figure 3.6 shows the tree for this game. We have labeled the action nodes for easy reference. Emily moves at the initial node, *a*, and the branches corre-

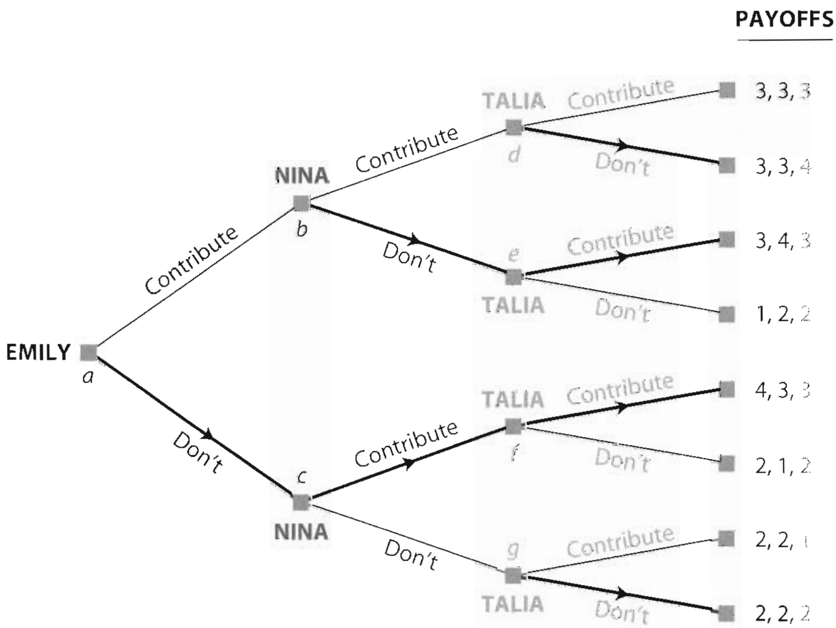


FIGURE 3.6 The Street Garden Game

[1]In later chapters, we vary the rules of this game—the order of moves and payoffs—and examine how such variation changes the outcomes.

sponding to her two choices, Contribute and Don't, respectively, lead to nodes *b* and *c*. At each of these nodes, Nina gets to move and to choose between Contribute and Don't. Her choices lead to nodes *d, e, f,* and *g,* at each of which Talia gets to move. Her choices lead to eight terminal nodes, where we show the payoffs in the order (Emily, Nina, Talia).[2] For example, if Emily contributes, then Nina does not, and finally Talia does, then the garden is pleasant, and the two contributors get payoffs 3 each, while the noncontributor gets her top outcome with payoff 4; in this case, the payoff list is (3, 4, 3).

To apply rollback analysis to this game, we begin with the action nodes that come immediately before the terminal nodes—namely, *d, e, f,* and *g.* Talia moves at each of these nodes. At *d,* she faces the situation where both Emily and Nina have contributed. The garden is already assured to be pleasant; so, if Talia chooses Don't, she gets her best outcome, 4, whereas, if she chooses Contribute, she gets the next best, 3. Her preferred choice at this node is Don't. We show this preference both by thickening the branch for Don't and by adding an arrowhead; either one would suffice to illustrate Talia's choice. At node *e,* Emily has contributed and Nina has not; so Talia's contribution is crucial for a pleasant garden. Talia gets the payoff 3 if she chooses Contribute and 2 if she chooses Don't. Her preferred choice at *e* is Contribute. You can check Talia's choices at the other two nodes similarly.

Now we roll back the analysis to the preceding stage—namely, nodes *b* and *c,* where it is Nina's turn to choose. At *b,* Emily has contributed. Nina's reasoning now goes as follows: "If I choose Contribute, that will take the game to node *d,* where I know that Talia will choose Don't, and my payoff will be 3. (The garden will be pleasant, but I will have incurred the cost of my contribution.) If I choose Don't, the game will go to node *e,* where I know that Talia will choose Contribute, and I will get a payoff of 4. (The garden will be pleasant, and I will have saved the cost of my contribution.) Therefore I should choose Don't." Similar reasoning shows that at *c,* Nina will choose Contribute.

Finally, consider Emily's choice at the initial node, *a.* She can foresee the subsequent choices of both Nina and Talia. Emily knows that, if she chooses Contribute, these later choices will be Don't for Nina and Contribute for Talia. With two contributors, the garden will be pleasant but Emily will have incurred a cost; so her payoff will be 3. If Emily chooses Don't, then the subsequent choices will both be Contribute, and, with a pleasant garden and no cost of her own contribution, Emily's payoff will be 4. So her preferred choice at *a* is Don't.

The result of rollback analysis for this street garden game is now easily summarized. Emily will choose Don't, then Nina will choose Contribute, and finally

[2]Recall from the discussion of the general tree in Section 1 that the usual convention for sequential-move games is to list payoffs in the order in which the players move; however, in case of ambiguity or simply for clarity, it is good practice to specify the order explicitly.

Talia will choose Contribute. These choices trace a particular **path of play** through the tree—along the lower branch from the initial node, *a*, and then along the upper branches at each of the two subsequent nodes reached, *c* and *f*. In Figure 3.6, the path of play is easily seen as the continuous sequence of arrowheads joined tail to tip from the initial node to the terminal node fifth from the top of the tree. The payoffs that accrue to the players are shown at this terminal node.

Rollback analysis is simple and appealing. Here, we emphasize some features that emerge from it. First, notice that the **equilibrium path of play** of a sequential-move game misses most of the branches and nodes. Calculating the best actions that would be taken if these other nodes were reached, however, is an important part of determining the ultimate equilibrium. Choices made early in the game are affected by players' expectations of what would happen if they chose to do something other than their best actions and by what would happen if any opposing player chose to do something other than what was best for her. These expectations, based on predicted actions at out-of-equilibrium nodes (nodes associated with branches pruned in the process of rollback), keep players choosing optimal actions at each node. For instance, Emily's optimal choice of Don't at the first move is governed by the knowledge that, if she chose Contribute, then Nina would choose Don't, followed by Talia choosing Contribute; this sequence would give Emily the payoff 3, instead of the 4 that she can get by choosing Don't at the first move.

The rollback equilibrium gives a complete statement of all this analysis by specifying the optimal *strategy* for each player. Recall that a strategy is a complete plan of action. Emily moves first and has just two choices; so her strategy is quite simple and is effectively the same thing as her move. But Nina, moving second, acts at one of two nodes, at one if Emily has chosen Contribute and at the other if Emily has chosen Don't. Nina's complete plan of action has to specify what she would do in either case. One such plan, or strategy, might be "choose Contribute if Emily has chosen Contribute, choose Don't if Emily has chosen Don't." We know from our rollback analysis that Nina will not choose this strategy, but our interest at this point is in describing all the available strategies from which Nina can choose within the rules of the game. We can abbreviate and write C for Continue and D for Don't; then this strategy can be written as "C if Emily chooses C so that the game is at node *b*, D if Emily chooses D so that the game is at node *c*," or, more simply, "C at *b*, D at *c*," or even "CD" if the circumstances in which each of the stated actions is taken are evident or previously explained. Now it is easy to see that, because Nina has two choices available at each of the two nodes where she might be acting, she has available to her four plans, or strategies—"C at *b*, C at *c*," "C at *b*, D at *c*," "D at *b*, C at *c*," and "D at *b*, D at *c*," or "CC," "CD," "DC," and "DD." Of these strategies, the rollback analysis and the arrows at nodes *b* and *c* of Figure 3.6 show that her optimal strategy is "DC."

Matters are even more complicated for Talia. When her turn comes, the history of play can, according to the rules of the game, be any one of four possibilities. Talia's turn to act comes at one of four nodes in the tree, one after Emily has chosen C and Nina has chosen C (node *d*), the second after Emily's C and Nina's D (node *e*), the third after Emily's D and Nina's C (node *f*), and the fourth after both Emily and Nina choose D (node *g*). Each of Talia's strategies, or complete plans of action, must specify one of her two actions for each of these four scenarios, or one of her two actions at each of her four possible action nodes. With four nodes at which to specify an action and with two actions from which to choose at each node, there are 2 times 2 times 2 times 2, or 16 possible combinations of actions. So Talia has available to her 16 possible strategies. One of them could be written as

"C at *d*, D at *e*, D at *f*, C at *g*" or "CDDC" for short,

where we have fixed the order of the four scenarios (the histories of moves by Emily and Nina) in the order of nodes *d*, *e*, *f*, and *g*. Then, with the use of the same abbreviation, the full list of 16 strategies available to Talia is

CCCC, CCCD, CCDC, CCDD, CDCC, CDCD, CDDC, CDDD,
DCCC, DCCD, DCDC, DCDD, DDCC, DDCD, DDDC, DDDD.

Of these strategies, the rollback analysis of Figure 3.6 and the arrows at nodes *d*, *e*, *f*, and *g* show that Talia's optimal strategy is DCCD.

Now we can express the findings of our rollback analysis by stating the strategy choices of each player—Emily chooses D from the two strategies available to her, Nina chooses DC from the four strategies available to her, and Talia chooses DCCD from the sixteen strategies available to her. When each player looks ahead in the tree to forecast the eventual outcomes of her current choices, she is calculating the optimal strategies of the other players. This configuration of strategies, D for Emily, DC for Nina, and DCCD for Talia, then constitutes the rollback equilibrium of the game.

We can put together the optimal strategies of the players to find the actual path of play that will result in the rollback equilibrium. Emily will begin by choosing D. Nina, following her strategy DC, chooses the action C in response to Emily's D. (Remember that Nina's DC means "choose D if Emily has played C, and choose C if Emily has played D.") According to the convention that we have adopted, Talia's actual action after Emily's D and then Nina's C—from node *f*—is the third letter in the four-letter specification of her strategies. Because Talia's optimal strategy is DCCD, her action along the path of play is C. Thus the actual path of play consists of Emily playing D, followed successively by Nina and Talia playing C.

To sum up, we have three distinct concepts:

1. The lists of available strategies for each player. The list, especially for later players, may be very long, because their actions in situations corresponding to all conceivable preceding moves by other players must be specified.
2. The optimal strategy, or complete plan of action for each player. This strategy must specify the player's best choices at each node where the rules of the game specify that she moves, even though many of these nodes will never be reached in the actual path of play. This specification is in effect the preceding movers' forecasting of what would happen if they took different actions and is therefore an important part of their calculation of their own best actions at the earlier nodes. The optimal strategies of all players together yield the rollback equilibrium.
3. The actual path of play in the rollback equilibrium, found by putting together the optimal strategies for all the players.

4 ORDER ADVANTAGES

In the rollback equilibrium of the street-garden game, Emily gets her best outcome (payoff 4), because she can take advantage of the opportunity to make the first move. When she chooses not to contribute, she puts the onus on the other two players—each can get her next-best outcome if and only if both of them choose to contribute. Most casual thinkers about strategic games have the preconception that such **first-mover advantage** should exist in all games. However, that is not the case. It is easy to think of games in which an opportunity to move second is an advantage. Consider the strategic interaction between two firms that sell similar merchandise from catalogs—say, Land's End and L. L. Bean. If one firm had to release its catalog first, and then the second firm could see what prices the first had set before printing its own catalog, then the second mover could just undercut its rival on all items and gain a tremendous competitive edge.

First-mover advantage comes from the ability to commit oneself to an advantageous position and to force the other players to adapt to it; **second-mover advantage** comes from the flexibility to adapt oneself to the others' choices. Whether commitment or flexibility is more important in a specific game depends on its particular configuration of strategies and payoffs; no generally valid rule can be laid down. We will come across examples of both kinds of advantages throughout this book. The general point that there need not be first-mover advantage, a point that runs against much common perception, is so important that we felt it necessary to emphasize at the outset.

When a game has a first- or second-mover advantage, each player may try to manipulate the order of play so as to secure for herself the advantageous position. Tactics for such manipulation are strategic moves, which we consider in Chapter 10.

5 ADDING MORE MOVES

We saw in Section 3 that adding more players increases the complexity of the analysis of sequential-play games. In this section, we consider another type of complexity that arises from adding additional moves to the game. We can do so most simply in a two-person game by allowing players to alternate moves more than once. Then the tree is enlarged in the same fashion that a multiple-player game tree would be, but later moves in the tree are made by the players who have made decisions earlier in the same game.

Many common games, such as tic-tac-toe, checkers, and chess, are two-person strategic games with such alternating sequential moves. The use of game trees and rollback should allow us to "solve" such games—to determine the rollback equilibrium outcome and the equilibrium strategies leading to that outcome. Unfortunately, as the complexity of the game grows and as strategies become more and more intricate, the search for an optimal solution becomes more and more difficult as well. In such cases, when manual solution is no longer really feasible, computer routines such as Gambit, mentioned in Chapter 2, become useful.

A. Tic-Tac-Toe

Start with the most simple of the three examples mentioned in the preceding paragraph, tic-tac-toe, and consider an easier-than-usual version in which two players (X and O) each try to be the first to get two of their symbols to fill any row, column, or diagonal of a two-by-two game board. The first player has four possible positions in which to put her X. The second player then has three possible actions at each of four decision nodes. When the first player gets to her second turn, she has two possible actions at each of 12 (4 times 3) decision nodes. As Figure 3.7 shows, even this mini-game of tic-tac-toe has a very complex game tree. This tree is actually not too complex, because the game is guaranteed to end after the first player moves a second time; but there are still 24 terminal nodes to consider.

We show this tree merely as an illustration of how complex game trees can become in even simple (or simplified) games. As it turns out, using rollback on the mini-game of tic-tac-toe leads us quickly to an equilibrium. Rollback shows

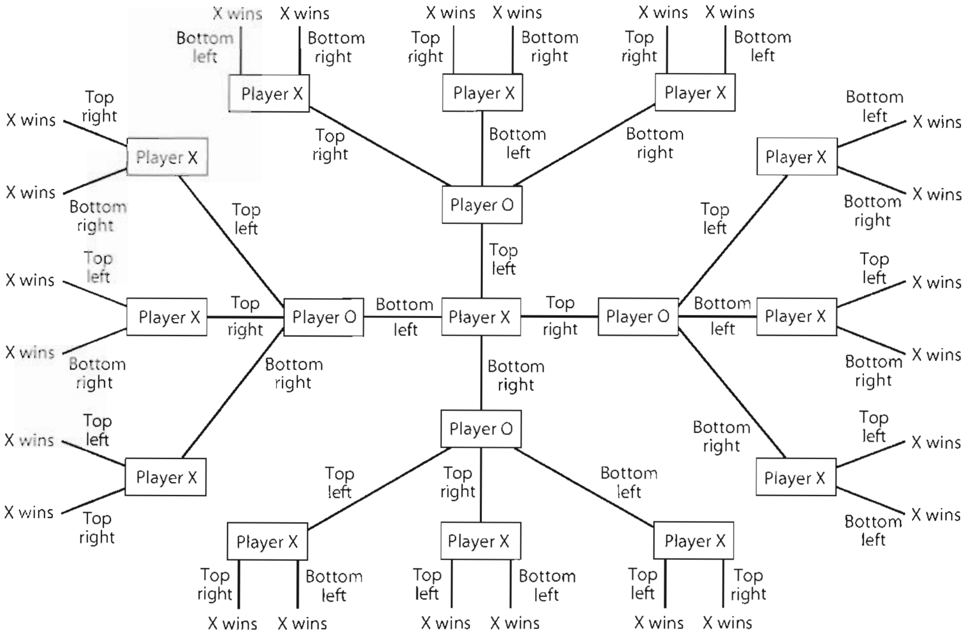


FIGURE 3.7 The Complex Tree for Simple Two-by-Two Tic-Tac-Toe

that all of the choices for the first player at her second move lead to the same outcome. There is no optimal action; any move is as good as any other move. Thus when the second player makes her first move, she also sees that each possible move yields the same outcome and she, too, is indifferent among her three choices at each of her four decision nodes. Finally, the same is true for the first player on her first move; any choice is as good as any other, so she is guaranteed to win the game.

Although this version of tic-tac-toe has an interesting tree, its solution is not as interesting. The first player always wins; so choices made by either player cannot affect the ultimate outcome. Most of us are more familiar with the three-by-three version of tic-tac-toe. To illustrate that version with a game tree, we would have to show that the first player has nine possible actions at the initial node, the second player has eight possible actions at each of nine decision nodes, and then the first player, on her second turn, has seven possible actions at each of 8 times 9 = 72 nodes, while the second player, on her second turn, has six possible actions at each of 7 times 8 times 9 = 504 nodes. This pattern continues until eventually the tree stops branching so rapidly because certain combinations of moves lead to a win for one player and the game ends. But no win is possible until at least the fifth move. Drawing the complete tree for this game requires a very large piece of paper or very tiny handwriting.

Most of you know, however, how to achieve at worst a tie when you play three-by-three tic-tac-toe. So there is a simple solution to this game that can be found by rollback, and a learned strategic thinker can reduce the complexity of the game considerably in the quest for such a solution. It turns out that, as in the two-by-two version, many of the possible paths through the game tree are strategically identical. Of the nine possible initial moves, there are only three types; you put your X in either a corner position (of which there are four possibilities), a side position (of which there are also four possibilities), or the (one) middle position. Using this method to simplify the tree can help reduce the complexity of the problem and lead you to a description of an optimal rollback equilibrium strategy. Specifically, we could show that the player who moves second can always guarantee at least a tie with an appropriate first move and then by continually blocking the first player's attempts to get three symbols in a row.[3]

B. Chess

When we consider more complicated games, such as checkers and especially chess, finding a complete solution becomes much more difficult. Students of checkers claim, but have not proved, that the second player can always guarantee a tie. Those who have studied and are still studying chess have even less to report, despite the fact that chess is the quintessential game of sequential moves and therefore technically amenable to full rollback analysis.

In chess, White opens with a move, Black responds with one, and so on, in turns. All the moves are visible to the other player, and nothing is left to chance, as it would be in card games that include shuffling and dealing. Therefore the "purest" kind of strategic reasoning in chess is to look ahead to the consequences of your move in just the way explained earlier. An example of such reasoning might be: "If I move that pawn now, my opponent will bring up her knight and threaten my rook. I should protect the square to which the knight wants to move with my bishop, before I move the pawn." Moreover, a chess game must end in a finite number of moves. The rules of chess declare that a game is drawn if a given position on the board is repeated three times in the course of play. Because there are a finite number of ways to place the 32 (or fewer after captures) pieces on 64 squares, a game could not go on infinitely long without running up against this rule.

[3]If the first player puts her first symbol in the middle position, the second player must put her first symbol in a corner position. Then the second player can guarantee a tie by taking the third position in any row, column, or diagonal that the first player tries to fill. If the first player goes to a corner or a side position first, the second player can guarantee a tie by going to the middle first and then following the same blocking technique. Note that, if the first player picks a corner, the second player picks the middle, and the first player then picks the corner opposite from her original play, then the second player must not pick one of the remaining corners if she is to ensure at least a tie.

Therefore in principle chess is amenable to full rollback analysis. We should know what will happen in a rollback equilibrium where both players use the optimal strategies: a win for White, a win for Black, or a draw. Then why has such analysis not been carried out? Why do people continue to play, using different strategies and getting different results in different matchups?

The answer is that, for all its simplicity of rules, chess is a bewilderingly complex game. From the initial set position of the pieces, White can open with any one of 20 moves,[4] and Black can respond with any of 20. Therefore 20 branches emerge from the first node of the tree, each leading to a second node from each of which 20 more branches emerge. There are already 400 branches, each leading to a node from which many more branches emerge. For a while, at each turn the number of available choices actually increases to a number much larger than 20. And a typical game goes on for 40 or more moves by each player. The total number of possible moves in chess has been estimated to be 10^{120}, or a "one" with 120 zeros after it. A supercomputer a thousand times as fast as your PC, making a billion calculations a second, would need approximately 3 times 10^{103} years to check out all these moves.[5] Astronomers offer us much less time before the sun turns into a red giant and swallows the earth. Recently, a physicist carried out the thought experiment of regarding the whole universe as a gigantic computer. If all the energy and order in the universe were dedicated to computation, how many calculations could have been made since the universe began in the big bang?[6] His answer is about 10^{120}. So coincidentally, if all the energy and order in the universe since its creation had been used to solve the game of chess by rollback, we would only now be on the verge of completing the task.

There is clearly no hope of ever drawing the complete tree for chess. Even in the smaller game starting from any specific position, the remaining tree becomes far too complex in just five or six moves. Looking ahead even that far is possible only because just a little experience enables a player to recognize that several of the logically possible moves are clearly bad and therefore to dismiss them from consideration.

The general point is that, although a game may be amenable in principle to a complete solution by rollback, its complete tree may be too complex to permit such solution in practice. Faced with such a situation, what is a player to do? We

[4]He can move one of eight pawns forward either one square or two or he can move one of the two knights in one of two ways (to the squares labeled rook-3 or bishop-3).

[5]This would have to be done only once because, after the game has been solved, anyone can use the solution and no one will actually need to play. Everyone will know whether White has a win or whether Black can force a draw. Players will toss to decide who gets which color. They will then know the outcome, shake hands, and go home.

[6]"The Physics of Information," *The Economist*, June 6, 2002.

can learn a lot about this by reviewing the history of attempts to program computers to play chess.

When computers first started to prove their usefulness for complex calculations in science and business, many mathematicians and computer scientists thought that a chess-playing computer program would soon beat the world champion. It took a lot longer, even though computer technology improved dramatically while human thought progressed much more slowly. Finally, in December 1992, a German chess program called Fritz2 beat world champion Gary Kasparov in some blitz (high-speed) games. Under regular rules, where each player gets $2\frac{1}{2}$ hours to make 40 moves, humans retained greater superiority for longer. A team sponsored by IBM put a lot of effort and resources into the development of a specialized chess-playing computer and its associated software. In February 1996, this package, called Deep Blue, was pitted against Gary Kasparov in a best-of-six series. Deep Blue caused a sensation by winning the first game, but Kasparov quickly figured out its weaknesses, improved his counterstrategies, and won the series handily. In the next 15 months, the IBM team improved Deep Blue's hardware and software, and the resulting Deeper Blue beat Kasparov in another best-of-six series in May 1997.

To sum up, computers have progressed in a combination of slow patches and some rapid spurts, while humans have held some superiority but have not been able to improve sufficiently fast to keep ahead. Closer examination reveals that the two use quite different approaches to thinking through the very complex game tree of chess.

If you cannot look ahead to the end of the whole game, how about looking part of the way—say, 5 or 10 moves—and working back from there? The game need not end within this limited horizon; that is, the nodes that you reach after 5 or 10 moves will not generally be terminal nodes. Only terminal nodes have payoffs specified by the rules of the game—nodes from which you can work backward. Therefore you must attach payoffs to the nonterminal nodes at the end of your limited look-ahead, but you cannot know exactly what those payoffs should be. You need some indirect way of assigning plausible payoffs to nonterminal nodes, because you are not able to explicitly roll back from a full look-ahead. A rule that assigns such payoffs is called an **intermediate valuation function.**

In chess, humans and computer programs both use such partial look-ahead in conjunction with an intermediate valuation function. The typical method is as follows. Each piece is assigned a value—say, 1 for a pawn, 4 for a bishop, and 9 for a queen. Positional and combinational advantages are assigned their own values. For example, two rooks that are aligned in support of each other are more valuable than they would be when isolated from each other. The sum of all the numerical values attached to pieces and their combinations in a position is the intermediate value of that position. Quantification and trade-off between different good and bad aspects of the position are made not on the basis of an explicit backward calculation but from the whole chess-playing community's experience of play in past games starting from such positions or patterns. Then a move is judged by the value of the position to which it is expected to lead after an explicit forward-looking calculation for a certain number—say, five or six—of moves.

The evaluation of intermediate positions has progressed farthest with respect to chess openings—that is, the first dozen or so moves of a game. Each opening can lead to any one of a vast multitude of further moves and positions, but experience enables players to sum up certain openings as being more or less likely to favor one player or the other. This knowledge has been written down in massive books of openings, and all top players and computer programs remember and use this information.

At the end stages of a game, when only a few pieces are left on the board, backward reasoning on its own is often simple enough to be doable and complete enough to give the full answer. For example, some endings are known or easy to figure out: if one player has the king and a rook while the other has only her king, then the first will win; if one player has her king and a bishop and the other has only her king, then the second player can achieve a draw. Farther away from the end, however, the position that would be reached at the end of your five- or six-move horizon is no simpler than the current position—it may be even more complex—and players have to use indirect methods to evaluate it.

The hardest stage is the midgame, where positions have evolved into a level of complexity that will not simplify within a few moves. Looking ahead 5 or even 10 moves will not give you a sure conclusion from which to reason back. If you could spend a lot of computing effort to improve your look-ahead ability from five moves to six, it is not clear that this effort would always improve your performance. You may do well in situations where disaster would have struck on precisely the sixth move, but at other times after six moves you may end up in a more complex situation that you cannot evaluate. In other words, to find a good move in a midgame position, it may be better to have a good intermediate valuation function than to be able to calculate another few moves farther ahead.

This is where the art of chess playing comes into its own. The best human players develop an intuition or instinct that enables them to sniff out good opportunities and avoid subtle traps in a way that computer programs find hard to match. Computer scientists have found it generally very difficult to teach their machines the skills of pattern recognition that humans acquire and use instinctively—for example, recognizing faces and associating them with names. The art of the midgame in chess also is an exercise in recognizing and evaluating patterns in the same, still mysterious way. This is where Kasparov has his greatest advantage over Fritz2 or Deep Blue. It also explains why computer programs do better against humans at blitz or limited-time games: a human does not have the time to marshal his art of the midgame.

In other words, the best human players have subtle "chess knowledge," based on experience or the ability to recognize patterns, which endows them with a better intermediate valuation function. Computers have the advantage when it comes to raw or brute-force calculation. For a while, chess programs were designed to imitate human thinking. This was the approach of the artificial intelligence community. But it did not work well. Chess programmers gradually found that it was better to "let computers be computers" and use their computational power to look ahead to more moves in less time. Now, although both human and computer players use a mixture of look-ahead and intermediate valuation, they use them in different proportions: humans do not look as many moves ahead but have better intermediate valuations based on knowledge; computers have less sophisticated valuation functions but look ahead farther by using their superior computational powers.

Recently, chess computers have begun to acquire more knowledge. When modifying Deep Blue in 1996 and 1997, IBM enlisted the help of human experts to improve the intermediate valuation function in its software. These consultants played repeatedly against the machine, noted its weaknesses, and suggested how the valuation function should be modified to correct the flaws. An episode of this process is particularly instructive.[7] Grandmaster Joel Benjamin "noticed that whenever Deep Blue had a pawn exchange that could open a file for a rook, it would always make the exchange earlier than was strategically correct. Deep Blue had to understand that the rook was already well placed, and that it didn't have to make the exchange right away. It should award evaluation points for the rook being there even before the file is opened." In other words, Deep Blue's intermediate valuation function knew that a well-positioned rook with an open file was good, so it immediately made the captures that opened the file and created this situation. But it failed to recognize that the mere positioning of the rook had "option value"; once the rook was there, it could exercise the option of making the pawn exchange that opens up the file at any time. If the pawn was serving some other good purpose where it was, then the actual exchange would be best postponed. This subtle kind of thinking results from long experience and an awareness of complex interconnections among the pieces on the board; it is easier for humans to acquire, often without even knowing that they have it, than it is to codify into the formal steps of a computer program.

If humans can gradually make explicit their subtle knowledge and transmit it to computers, what hope is there for human players who do not get reciprocal help from computers? At times in their 1997 encounter, Kasparov was amazed by the human or even superhuman quality of Deep Blue's play. He even attributed one of the computer's moves to "the hand of God." And matters can only

[7]This account is taken from "What Deep Blue Learned in Chess School," by Bruce Weber, *The New York Times*, May 18, 1997.

get worse: the brute-force calculating power of computers is increasing rapidly while they are simultaneously, but more slowly, gaining some of the subtlety that constitutes the advantage of humans.

Finally, will the time come when the machine is able to spot its own weaknesses and modify its own intermediate valuation function? In other words, can the machine learn from its own experience and acquire chess knowledge as humans do? If that ever happens, will such programs spread to other areas of strategic interaction, such as business? And then what will it mean to be human anyway?

We leave such questions to philosophers, who have thousands of years of experience in discussing the really big questions about life, the universe, and everything, and not finding any answers to any of them. We point out only that the more powerful computers are at the disposal of human users. And, after all, humans can do one thing that computers cannot do: a human can pull the plug or switch off the power to a computer, but computers have no such ability to terminate humans.

The abstract theory of chess says that it is a finite game that can be solved by rollback. The practice of chess requires a lot of "art" based on experience, intuition, and subtle judgment. Is this bad news for the use of rollback in sequential-move games? We think not. It is true that theory does not take us all the way to an answer. But it does take us a long way; looking ahead a few moves constitutes an important part of the approach that mixes brute-force calculation of moves with a knowledge-based assessment of intermediate positions. And, as computational power increases, the role played by brute-force calculation, and therefore the scope of the rollback theory, will also increase.

We can sum up the matter of complexity in the following way. For really simple games, we can find the rollback equilibrium by verbal reasoning without having to draw the game tree explicitly. For truly complex games, such as chess, we can draw only a small part of the game tree, and we must use a combination of two methods: (1) calculation based on the logic of rollback and (2) rules of thumb for valuing intermediate positions on the basis of experience. For games having an intermediate range of complexity, verbal reasoning is too hard but a complete tree can be drawn and used for rollback. Sometimes, we may enlist the aid of a computer to draw and analyze a moderately complicated game tree.

Thankfully, most of the strategic games that we encounter in economics, politics, sports, business, and daily life are far less complex than chess. The games may have a number of players who move a number of times; they may even have a large number of players or a large number of moves. But we have a chance at being able to draw a reasonable-looking tree for those games that are sequential in nature. The logic of rollback remains valid, and it is also often the case that, once you understand the idea of rollback, you can carry out the necessary logical thinking and solve the game without explicitly drawing a tree. Moreover, it is precisely at this intermediate level of difficulty, between the simple examples that we solved explicitly in this chapter and the insoluble cases such as chess, that computer software such as Gambit is most likely to be useful; this is indeed fortunate for the prospect of applying the theory to solve many games in practice.

6 EVIDENCE CONCERNING ROLLBACK

How well do actual participants in sequential-move games perform the calculations of rollback reasoning? There is very little systematic evidence, but classroom and research experiments with some games have yielded outcomes that appear to counter the predictions of the theory. Some of these experiments and their outcomes have interesting implications for the strategic analysis of sequential-move games.

For instance, many experimenters have had subjects play a single-round bargaining game in which two players, designated A and B, are chosen from a class or a group of volunteers. The experimenter provides a dollar (or some known total), which can be divided between them according to the following procedure. Player A proposes a split—for example, "75 to me, 25 to B." If player B accepts this proposal, the dollar is divided as proposed by A. If B rejects the proposal, neither player gets anything.

Rollback in this case predicts that B should accept any sum, no matter how small, because the alternative is even worse—namely, zero—and, foreseeing this, A should propose "99 to me, 1 to B." This particular outcome almost never happens. Most players assigned the A role propose a much more equal split. In fact, 50–50 is the single most common proposal. Furthermore, most players assigned the B role turn down proposals that leave them 25% or less of the total and walk away with nothing; some reject proposals that would give them 40% of the pie.[8]

Many game theorists remain unpersuaded that these findings undermine the theory. They counter with some variant of the following argument. "The sums are so small as to make the whole thing trivial in the players' minds. The B players lose 25 or 40 cents, which is almost nothing, and perhaps gain some private satisfaction that they walked away from a humiliatingly small award. If the total were a thousand dollars, so that 25% of it amounted to real money, the B

[8]Read Richard H. Thaler, "Anomalies: The Ultimate Game," *Journal of Economic Perspectives*, vol. 2, no. 4 (fall 1988), pp. 195–206, and Douglas D. Davis and Charles A. Holt, *Experimental Economics* (Princeton: Princeton University Press, 1993), pp. 263–269, for a detailed account of this game and related ones.

players would accept." But this argument does not seem to be valid. Experiments with much larger stakes show similar results. The findings from experiments conducted in Indonesia, with sums that were small in dollars but amounted to as much as three months' earnings for the participants, showed no clear tendency on the part of the A players to make less equal offers, although the B players tended to accept somewhat smaller shares as the total increased; similar experiments conducted in the Slovak Republic found the behavior of inexperienced players unaffected by large changes in payoffs.[9]

The participants in these experiments typically have no prior knowledge of game theory and no special computational abilities. But the game is extremely simple; surely even the most naive player can see through the reasoning, and answers to direct questions after the experiment generally show that most participants do. The results show not so much the failure of rollback as the theorist's error in supposing that each player cares only about her own money earnings. Most societies instill in their members a strong sense of fairness, which then causes most A players to offer 50–50 or something close and the B players to reject anything that is grossly unfair. This argument is supported by the observation that even in a most drastic variant called the *dictator game*, where the A player decides on the split and the B player has no choice at all, many As give significant shares to the Bs.[10]

Some experiments have been conducted to determine the extent to which fairness plays into behavior in ultimatum and dictator games. The findings show that changes in the information available to players and to the experimenter have a significant effect on the final outcome. In particular, when the experimental design is changed so that not even the experimenter can identify who proposed (or accepted) the split, the extent of sharing drops noticeably. Evidence from the new field of "neuroeconomics" includes similar results. Alan Sanfey and colleagues conducted experiments of the ultimatum game in which they took MRI readings of the subjects' brains as they made their choices. They found "that activity in a region [of the brain] well known for its involvement in negative emotion" was stimulated in the responders (B players) when they rejected "unfair" (less than 50:50) offers. Thus deep instincts or emotions of

[9]The results of the Indonesian experiment are reported in Lisa Cameron, "Raising the Stakes in the Ultimatum Game: Experimental Evidence from Indonesia," *Economic Inquiry*, vol. 37, no. 1 (January 1999), pp. 47–59. Slonim and Roth report results similar to Cameron's, but also found that offers (in all rounds of play) were rejected less often as the payoffs were raised. See Robert Slonim and Alvin Roth, "Learning in High Stakes Ultimatum Games: An Experiment in the Slovak Republic," *Econometrica*, vol. 66, no. 3 (May 1998), pp. 569–596.

[10]One could argue that this social norm of fairness may actually have value in the ongoing evolutionary game being played by the whole of society. Players who are concerned with fairness reduce transaction costs and the costs of fights that can be beneficial to society in the long run. These matters will be discussed in Chapters 11 and 12.

anger and disgust seem to be implicated in these rejections. They also found that "unfair" (less than 50:50) offers were rejected less often when responders knew that the offerer was a computer than when they knew that the offerer was human.[11]

Another experimental game with similarly paradoxical outcomes goes as follows. Two players are chosen and designated A and B. The experimenter puts a dime on the table. Player A can take it or pass. If A takes the dime, the game is over, with A getting the 10 cents and B getting nothing. If A passes, the experimenter adds a dime, and now B has the choice of taking the 20 cents or passing. The turns alternate, and the pile of money grows until reaching some limit—say, a dollar—that is known in advance by both players.

We show the tree for this game in Figure 3.8. Because of the appearance of the tree, this type of game is often called the *centipede game*. You may not even need the tree to use rollback on this game. Player B is sure to take the dollar at the last stage; so A should take the 90 cents at the penultimate stage, and so on. Thus A should take the very first dime and end the game.

However, in most classroom or experimental settings, such games go on for at least a few rounds. Remarkably, by behaving "irrationally," the players as a group make more money than they would if they followed the logic of backward reasoning. Sometimes A does better and sometimes B, but sometimes they even solve this conflict or bargaining problem. In a classroom experiment that one of us (Dixit) conducted, one such game went all the way to the end. Player B collected the dollar, and quite voluntarily gave 50 cents to player A. Dixit asked A, "Did you two conspire? Is B a friend of yours?" and A replied, "No, we didn't even know each other before. But he is a friend now."

Once again, what is revealed is not that players cannot calculate and use game-theoretic logic but that their value systems and payoffs are different from those attributed to them by the theorist who predicted that the game should end with A taking the dime on the first step. We will come across some similar

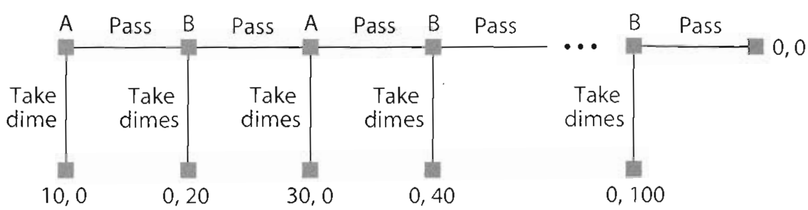


FIGURE 3.8 The Centipede Game

[11]See Alan Sanfey, James Rilling, Jessica Aronson, Leigh Nystrom, and Jonathan Cohen, "The Neural Basis of Economic Decision-Making in the Ultimatum Game," *Science*, vol. 300 (June 13, 2003), pp. 1755–1758.

evidence of cooperation that seems to contradict rollback reasoning when we look at finitely repeated prisoners' dilemma games in Chapter 11.[12]

The examples discussed here seem to indicate that apparent violations of strategic logic can be explained by recognizing that people do not care merely about their own money payoffs, but internalize concepts such as fairness. But not all observed plays, contrary to the precepts of rollback, have some such explanation. People do fail to look ahead far enough, and they do fail to draw the appropriate conclusions from attempts to look ahead. For example, when issuers of credit cards offer favorable initial interest rates or no fees for the first year, many people fall for them without realizing that they may have to pay much more later. Therefore the game-theoretic analysis of rollback and rollback equilibria serves an advisory or prescriptive role as much as it does a descriptive role. People equipped with the theory of rollback are in a position to make better strategic decisions and get higher payoffs, no matter what is included in their payoff calculations. And game theorists can use their expertise to give valuable advice to those who are placed in complex strategic situations but lack the skill to determine their own best strategies.

7 STRATEGIES IN THE SURVIVOR© GAME

The examples in the preceding sections were deliberately constructed to illustrate and elucidate basic concepts such as nodes, branches, moves, and strategies, as well as the technique of rollback. Now we show how all of them can be applied, by considering a real-life (or at least "reality-TV-life") situation.

In the summer of 2000, CBS television broadcast the first of the series of Survivor shows, which became an instant hit and helped launch the whole new genre of "reality TV." Leaving aside many complex details and some earlier stages not relevant for our purpose, the concept was as follows. A group of contestants, called a "tribe," was put on an uninhabited island and left largely to fend for themselves for food and shelter. Every 3 days they had to vote one of themselves out of the tribe. The person who had the most votes cast against him or her at a meeting of the remaining players (called the "tribal council") was the victim of the day. However, before each meeting of the tribal council, the survivors up to that point competed in a game of physical or mental skill that was devised by the producers of the game for that occasion, and the winner of this competition, called a "challenge," was immune from being voted off at the following meeting. Also, one could not vote against oneself. Finally,

[12]Once again, one wonders what would happen if the sum added at each step were a thousand dollars instead of a dime.

when two people were left, the seven who had been voted off most recently returned as a "jury" to pick one of the two remaining survivors as the million-dollar winner of the whole game.

The strategic problems facing all contestants were: (1) to be generally regarded as a productive contributor to the tribe's search for food and other tasks of survival, but to do so without being regarded as too strong a competitor and therefore a target for elimination, (2) to form alliances to secure blocks of votes to protect oneself from being voted off, (3) to betray these alliances when the numbers got too small and one had to vote against someone, but (4) to do so without seriously losing popularity with the other players, who would ultimately have the power of the vote on the jury.

We pick up the story when just three contestants were left: Rudy, Kelly, and Rich. Of them, Rudy was the oldest contestant, an honest and blunt person who was very popular with the contestants who had been previously voted off. It was generally agreed that, if he was one of the last two, then he would be voted the million-dollar winner. So it was in the interests of both Kelly and Rich that they should face each other, rather than face Rudy, in the final vote. But neither wanted to be seen as instrumental in voting off Rudy. With just three contestants left, the winner of the immunity challenge is effectively decisive in the cast-off vote, because the other two must vote against each other. Thus the jury would know who was responsible for voting off Rudy and, given his popularity, would regard the act of voting him off with disfavor. The person doing so would harm his or her chances in the final vote. This was especially a problem for Rich, because he was known to have an alliance with Rudy.

The immunity challenge was one of stamina; each contestant had to stand on an awkward support and lean to hold one hand in contact with a totem on a central pole, called the "immunity idol." Anyone whose hand lost contact with the idol, even for an instant, lost the challenge; the one to hold on longest was the winner.

An hour and a half into the challenge, Rich figured out that his best strategy was to deliberately lose this immunity challenge. Then, if Rudy won immunity, he would maintain his alliance and keep Rich—Rudy was known to be a man who always kept his word. Rich would lose the final vote to Rudy in this case, but that would make him no worse off than if he won the challenge and kept Rudy. If Kelly won immunity, the much more likely outcome, then it would be in her interest to vote off Rudy—she would have at least some chance against Rich but zero against Rudy. Then Rich's chances of winning were quite good. Whereas, if Rich himself held on, won immunity, and then voted off Rudy, his chances against Kelly would be decreased by the fact that he voted off Rudy.

So Rich deliberately stepped off, and later explained his reasons quite clearly to the camera. His calculation was borne out. Kelly won that challenge and voted off Rudy. And, in the final jury vote between Rich and Kelly, Rich won by one vote.

Rich's thinking was essentially a rollback analysis along a game tree. He did this analysis instinctively, without drawing the tree, while standing awkwardly and holding on to the immunity idol, but it took him an hour and a half to come to his conclusion. With all due credit to Rich, we show the tree explicitly, and can reach the answer faster.

Figure 3.9 shows the tree. You can see that it is much more complex than the trees encountered in earlier sections. It has more branches and moves; in addition, there are uncertain outcomes, and the chances of winning or losing in various alternative situations have to be estimated instead of being known precisely. But you will see how we can make reasonable assumptions about these chances and proceed with the analysis.

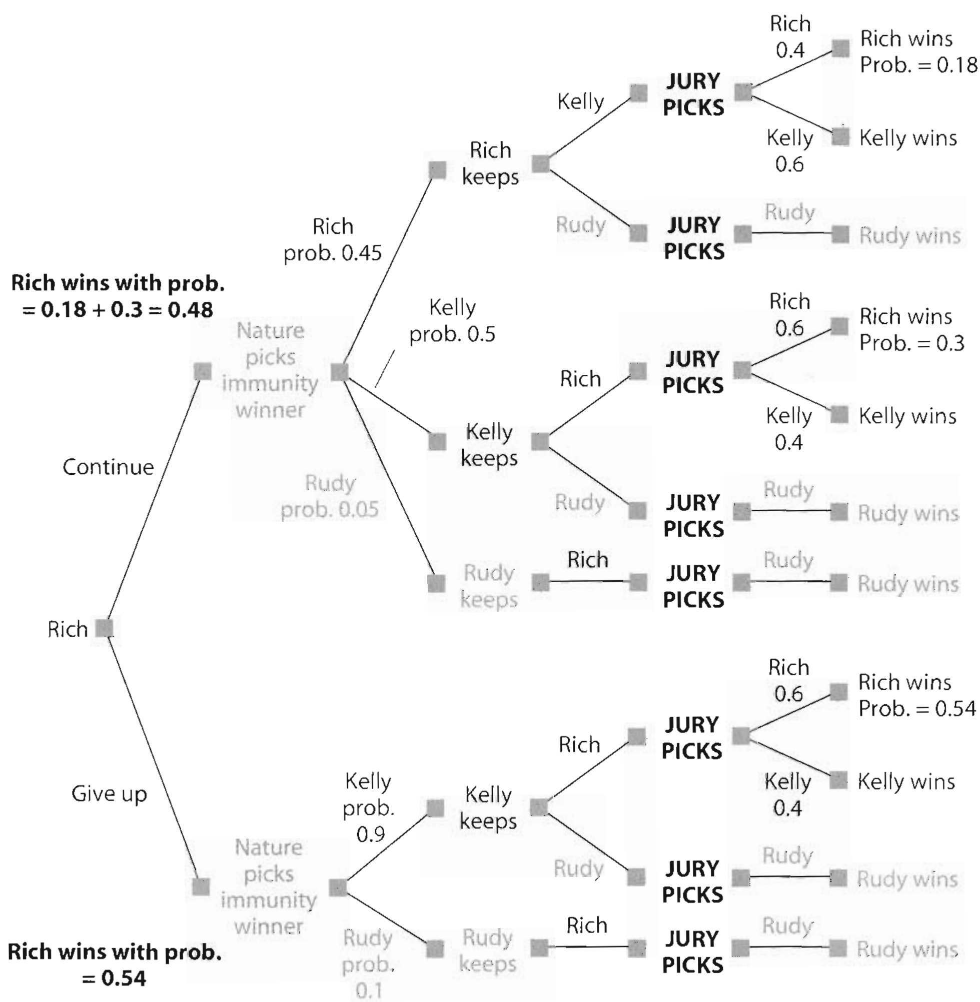


FIGURE 3.9 Survivor Immunity Game Tree

At the initial node, Rich decides whether to continue or to give up in the immunity challenge. In either case, the winner of the challenge cannot be forecast with certainty; this is indicated in the tree by letting "Nature" make the choice, as we did with the coin-toss situation in Figure 3.1. If Rich continues, Nature chooses the winner from the three contestants. We don't know the actual probabilities, but we will assume particular values for exposition and point out the crucial assumptions. The supposition is that Kelly has a lot of stamina and that Rudy, being the oldest, is not likely to win. So we posit the following probabilities of a win when Rich chooses to continue: 0.5 (50%) for Kelly, 0.45 for Rich, and only 0.05 for Rudy. If Rich gives up on the challenge, Nature picks the winner of the immunity challenge randomly between the two who remain; in this case, we assume that Kelly wins with probability 0.9 and Rudy with probability 0.1.

The rest of the tree follows from each of the three possible winners of the challenge. If Rudy wins, he keeps Rich as he promised, and the jury votes Rudy the winner.[13] If Rich wins immunity, he has to decide whether to keep Kelly or Rudy. If he keeps Rudy, the jury votes for Rudy. If he keeps Kelly, it is not certain whom the jury chooses. We assume that Rich alienates some jurors by turning on Rudy and that, despite being better liked than Kelly, he gets the jury's vote in this situation only with probability 0.4. Similarly, if Kelly wins immunity, she can either keep Rudy and lose the jury's vote, or keep Rich. If she keeps Rich, his probability of winning the jury's vote is higher, at 0.6, because in this case he is both better liked by the jury and hasn't voted off Rudy.

What about the players' actual payoffs? We can safely assume that both Rich and Kelly want to maximize the probability of his or her emerging as the ultimate winner of the $1 million. Rudy similarly wants to get the prize, but keeping his word to Rich is paramount. With these preferences of the various players in mind, Rich can now do rollback analysis along the tree to determine his own initial choice.

Rich knows that, if he wins the immunity challenge (the uppermost path after his own first move and Nature's move), he will have to keep Kelly to have a 40% chance of eventual victory; keeping Rudy at this stage would mean a zero probability of eventual victory. Rich can also calculate that, if Kelly wins the immunity challenge (which occurs once in each of the upper and lower halves of the tree), she will choose to keep him for similar reasons, and then the probability of his eventual victory will be 0.6.

[13]Technically, Rudy faces a choice between keeping Rich or Kelly at the action node after he wins the immunity challenge. Because everyone placed zero probability on his choosing Kelly (owing to the Rich–Rudy alliance), we illustrate only Rudy's choice of Rich. The jury, similarly, has a choice between Rich and Rudy at the last action node along this branch of play. Again, the foregone conclusion is that Rudy wins in this case.

What are Rich's chances as he calculates them at the initial node? If Rich chooses Give Up at the initial node, then there is only one way for him to emerge as the eventual winner—if Kelly wins immunity (probability 0.9), if she then keeps Rich (probability 1), and if the jury votes for Rich (probability 0.6). Because all three things need to happen for Rich to win, his overall probability of victory is the product of the three probabilities—namely, $0.9 \times 1 \times 0.6 = 0.54$.[14] If Rich chooses Continue at the initial node, then there are two ways in which he can win. First, he wins the game if he wins the immunity challenge (probability 0.45), if he then eliminates Rudy (probability 1), and if he still wins the jury's vote against Kelly (probability 0.4); the total probability of winning in this way is $0.45 \times 0.4 = 0.18$. Second, he wins the game if Kelly wins the challenge (probability 0.5), if she eliminates Rudy (probability 1), and if Rich gets the jury's vote (probability 0.6); total probability here is $0.5 \times 0.6 = 0.3$. Rich's overall probability of eventual victory if he chooses Continue is the sum of the probabilities of these two paths to victory—namely, $0.18 + 0.3 = 0.48$.

Rich can now compare his probability of winning the million dollars when he chooses Give Up (0.54) with his probability of winning when he chooses Continue (0.48). Given the assumed values of the various probabilities in the tree, Rich has a better chance of victory if he gives up. Thus, Give Up is his optimal strategy. Although this result is based on assuming specific numbers for the probabilities, Give Up remains Rich's optimal strategy as long as (1) Kelly is very likely to win the immunity challenge once Rich gives up and (2) Rich wins the jury's final vote more often when Kelly has voted out Rudy than when Rich has done so.[15]

This example serves several purposes. Most importantly, it shows how a complex tree, with much uncertainty and missing information about precise probabilities, can still be solved by using rollback analysis. We hope this gives you some confidence in using the method and some training in converting a somewhat loose verbal account into a more precise logical argument. You might counter that Rich did this reasoning without drawing any trees. But knowing the system or general framework greatly simplifies the task even in new and unfamiliar circumstances. Therefore it is definitely worth the effort to acquire the systematic skill.

A second purpose is to illustrate the seemingly paradoxical strategy of "losing to win." Another instance of this strategy can be found in some sporting competitions that are held in two rounds, such as the soccer World Cup. The first round is played on a league basis in several groups of four teams each. The top two teams from each group then go to the second round where they play

[14]Readers who need instruction or a refresher course in the rules for combining probabilities will find a quick tutorial in the Appendix to Chapter 7.

[15]Readers who can handle the algebra of probabilities can solve this game by using more general symbols instead of specific numbers for the probabilities, as in Exercise 10 of this chapter.

others chosen according to a prespecified pattern; for example, the top-ranked team in group A meets the second-ranked team in group B, and so on. In such a situation, it may be good strategy for a team to lose one of its first-round matches if this loss causes it to be ranked second in its group; that ranking might earn it a subsequent match against a team that, for some particular reason, it is more likely to beat than the team that it would meet if it had placed first in its group in the first round.

SUMMARY

Sequential-move games require players to consider the future consequences of their current moves before choosing their actions. Analysis of pure sequential-move games generally requires the creation of a *game tree*. The tree is made up of *nodes* and *branches* that show all of the possible actions available to each player at each of her opportunities to move, as well as the payoffs associated with all possible outcomes of the game. Strategies for each player are complete plans that describe actions at each of the player's decision nodes contingent on all possible combinations of actions made by players who acted at earlier nodes. The equilibrium concept employed in sequential-move games is that of *rollback equilibrium,* in which players' equilibrium strategies are found by looking ahead to subsequent nodes and the actions that would be taken there and by using these forecasts to calculate one's current best action. This process is known as *rollback,* or *backward induction.*

Different types of games entail advantages for different players, such as *first-mover advantages.* The inclusion of many players or many moves enlarges the game tree of a sequential-move game but does not change the solution process. In some cases, drawing the full tree for a particular game may require more space or time than is feasible. Such games can often be solved by identifying strategic similarities between actions that reduces the size of the tree or by simple logical thinking.

Chess, the ultimate strategic sequential-move game, is theoretically amenable to solution by rollback. The complexity of the game, however, makes it effectively impossible to draw a full game tree and to write out a complete solution. In actual play, elements of both art (identification of patterns and of opportunities versus peril) and science (forward-looking calculations of the possible outcomes arising from certain moves) have a role in determining player moves. Human chess players still maintain an advantage in the use of knowledge and instinct, but computers perform the raw calculations necessary for true rollback much faster and more accurately than humans. As computers increase in power, a more complete solution of chess using rollback comes closer to becoming a reality.

Tests of the theory of sequential-move games seem to suggest that actual play shows the irrationality of the players or the failure of the theory to adequately predict behavior. The counterargument points out the complexity of actual preferences for different possible outcomes and the usefulness of strategic theory for identifying optimal actions when actual preferences are known.

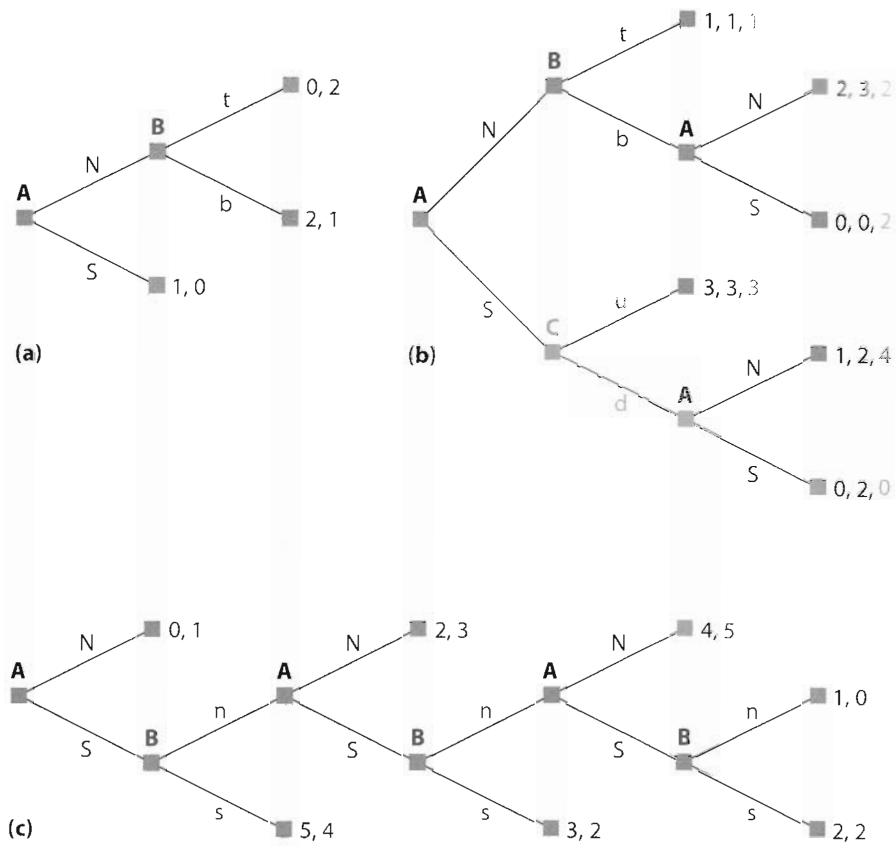
KEY TERMS

action node (46)
backward induction (54)
branch (46)
decision node (46)
decision tree (46)
equilibrium path of play (58)
extensive form (46)
first-mover advantage (60)
game tree (46)
initial node (46)
intermediate valuation function (65)
move (48)
node (46)
path of play (58)
prune (52)
rollback (54)
rollback equilibrium (54)
root (46)
second-mover advantage (59)
terminal node (48)

EXERCISES

1. Suppose two players, First and Second, take part in a sequential-move game. First moves first, Second moves second, and each player moves only once.
   (a) Draw a game tree for a game in which First has two possible actions (Up or Down) at each node and Second has three possible actions (Top, Middle, or Bottom) at each node. How many of each node type—decision and terminal—are there?
   (b) Draw a game tree for a game in which First and Second each have three possible actions (Sit, Stand, or Jump) at each node. How many of the two node types are there?
   (c) Draw a game tree for a game in which First has four possible actions (North, South, East, or West) at each node and Second has two possible actions (Stay or Go) at each node. How many of the two node types are there?

2. Use rollback to find equilibria for the following games:



In each case, how many pure strategies (complete plans of action) are available to each player? For each game, write out all of the pure strategies for each player.

3. For each of the games illustrated in Exercise 2, identify the equilibrium outcome and the complete equilibrium strategy for each player.

4. "In a sequential-move game, the player who moves first is sure to win." Is this statement true or false? State the reason for your answer in a few brief sentences, and give an example of a game that illustrates your answer.

5. Consider the rivalry between Airbus and Boeing to develop a new commercial jet aircraft. Suppose Boeing is ahead in the development process and Airbus is considering whether to enter the competition. If Airbus stays out, it earns zero profit, whereas Boeing enjoys a monopoly and earns a profit of \$1 billion. If Airbus decides to enter and develop the rival airplane, then Boeing has to decide whether to accommodate Airbus peaceably or to wage a price war. In the event of peaceful competition, each firm will make a

profit of $300 million. If there is a price war, each will lose $100 million because the prices of airplanes will fall so low that neither firm will be able to recoup its development costs.

Draw the tree for this game. Find the rollback equilibrium and describe the firms' equilibrium strategies.

6. The centipede game illustrated in Figure 3.8 can be solved by using rollback without drawing the complete game tree. In the version of the game discussed in the text, players A and B alternately had the opportunity to claim or pass a growing pile of dimes (to a maximum of 10) placed on the table. Suppose the rules of the games were changed so that:
   (a) Player A gets a nickel reward every time she passes, giving her opponent another turn. Find the rollback equilibrium strategies for each player.
   (b) Two rounds of the game are played with the same two players, A and B. In the first round, A may not keep more than five dimes, and B may not keep more than nine. Find the rollback equilibrium strategies for each player.
   (c) Two rounds of the game are played with the same two players, A and B. In the first round, A may not keep more than five dimes, and B may not keep more than four. Find the rollback equilibrium strategies for each player.

7. Two players, Amy and Beth, take turns choosing numbers; Amy goes first. On her turn, a player may choose any number between 1 and 10, inclusive, and this number is added to a running total. When the running total of both players' choices reaches 100, the game ends. Consider two alternative endings: (i) the player whose choice of number takes the total to exactly 100 is the winner and (ii) the player whose choice of number causes this total to equal or exceed 100 is the loser. For each case, answer the following questions:
   (a) Who will win the game?
   (b) What are the optimal strategies (complete plans of action) for each player?

8. Consider three major department stores—Big Giant, Titan, and Frieda's—contemplating opening a branch in one of two new Boston-area shopping malls. Urban Mall is located close to the large and rich population center of the area; it is relatively small and can accommodate at most two department stores as "anchors" for the mall. Rural Mall is farther out in a rural and relatively poorer area; it can accommodate as many as three anchor stores. None of the three stores wants to have its store in both malls because there is sufficient overlap of customers between the malls that locating in both would just be competing with oneself. Each store prefers to be in a mall with one or more other department stores than to be alone in the same mall, because a mall with multiple department stores will attract sufficiently many more total customers that each store's profit will be higher. Further, each store prefers Urban Mall to Rural Mall because of the richer customer base. Each store

must choose between trying to get a space in Urban Mall (knowing that, if the attempt fails, they will try for a space in Rural Mall) and trying to get a space in Rural Mall directly (without even attempting to get into Urban Mall).

In this case, the stores rank the five possible outcomes as follows: 5 (best), in Urban Mall with one other department store; 4, in Rural Mall with one or two other department stores; 3, alone in Urban Mall; 2, alone in Rural Mall; and 1 (worst), alone in Rural Mall after having attempted to get into Urban Mall and failed, by which time other nondepartment stores have signed up the best anchor locations in Rural Mall.

The three stores are sufficiently different in their managerial structures that they experience different lags in doing the paperwork required to request an expansion space in a new mall. Frieda's moves quickly, followed by Big Giant, and finally by Titan, which is the least efficient in readying a location plan. When all three have made their requests, the malls decide which stores to let in. Because of the name recognition that both Big Giant and Titan have with the potential customers, a mall would take either (or both) of those stores before it took Frieda's. Thus, Frieda's does not get one of the two spaces in Urban Mall if all three stores request those spaces; this is true even though Frieda's moves first.

(a) Draw the game tree for this mall location game.

(b) Illustrate the rollback pruning process on your game tree and use the pruned tree to find the rollback equilibrium. Describe the equilibrium by using the (complete) strategies employed by each department store. What are the payoffs to each store at the rollback equilibrium outcome?

9. There are two distinct proposals, A and B, being debated in Washington. The Congress likes proposal A, and the president likes proposal B. The proposals are not mutually exclusive; either or both or neither may become law. Thus there are four possible outcomes, and the rankings of the two sides are as follows, where a larger number represents a more favored outcome.

Outcome	Congress	President
A becomes law	4	1
B becomes law	1	4
Both A and B become law	3	3
Neither (status quo prevails)	2	2

(a) The moves in the game are as follows. First, the Congress decides whether to pass a bill and whether it is to contain A or B or both. Then the president decides whether to sign or veto the bill. Congress does not have enough votes to override a veto. Draw a tree for this game and find the rollback equilibrium.

(b) Now suppose the rules of the game are changed in only one respect: the president is given the extra power of a line-item veto. Thus, if the Congress passes a bill containing both A and B, the president may choose not only to sign or veto the bill as a whole, but also to veto just one of the two items. Show the new tree and find the rollback equilibrium.

(c) Explain intuitively why the difference between the two equilibria arises.

10. Consider the Survivor game tree illustrated in Figure 3.9. Suppose that, unlike in Figure 3.9, you want to use only general values for the various probabilities. In particular, suppose that the probability of winning the immunity challenge when Rich chooses Continue is x for Rich, y for Kelly, and $1 - x - y$ for Rudy; similarly, the probability of winning when Rich chooses Give Up is z for Kelly and $1 - z$ for Rudy. Further, suppose that Rich's chance of being picked by the jury is p if he has won immunity and has voted off Rudy; his chance of being picked is q if Kelly has won immunity and has voted off Rudy. Continue to assume that, if Rudy wins immunity, he keeps Rich with probability 1, and that Rudy wins the game with probability 1 if he ends up in the final two.

(a) What is the algebraic formula for the probability, in terms of p, q, x, and y, that Rich wins the million dollars if he chooses Continue? What is the probability that he wins if he chooses Give Up? Can you determine Rich's optimal strategy with only this level of information?

(b) The discussion in Section 3.7 suggests that Give Up is optimal for Rich as long as (i) Kelly is very likely to win the immunity challenge once Rich gives up and (ii) Rich wins the jury's final vote more often when Kelly has voted out Rudy than when Rich has done so. Write out expressions entailing the general probabilities (p, q, x, y) that summarize these two conditions.

(c) Suppose that the two conditions from part **b** hold. Prove that Rich's optimal strategy is Give Up.

4

Simultaneous-Move Games with Pure Strategies I: Discrete Strategies

RECALL FROM CHAPTER 2 that games are said to have simultaneous moves if players must move without knowledge of what their rivals have chosen to do. It is obviously so if players choose their actions at exactly the same time. A game is also simultaneous when players choose their actions in isolation, with no information about what other players have done or will do, even if the choices are made at different hours of the clock. (For this reason, simultaneous-move games are often referred to as games of *imperfect information* or *imperfect knowledge.*) This chapter focuses on games that have such purely simultaneous interactions among players. We consider a variety of types of simultaneous games, introduce a solution concept called Nash equilibrium for these games, and study games with one equilibrium, many equilibria, or no equilibrium at all.

Many familiar strategic situations can be described as simultaneous-move games. The various producers of television sets, stereos, or automobiles make decisions about product design and features without knowing what rival firms are doing about their own products. Voters in U.S. elections simultaneously cast their individual votes; no voter knows what the others have done when she makes her own decision. The interaction between a soccer goalie and an opposing striker during a penalty kick requires both players to make their decisions simultaneously—the goalie cannot afford to wait until the ball has actually been kicked to decide which way to go, because then it would be far too late.

When a player in a simultaneous move game chooses her action, she obviously does so without any knowledge of the choices made by other players. She also cannot look ahead to how they will react to her choice, because they act simultaneously and do not know what she is choosing. Rather, each player must figure out what others are doing when what the others are doing is figuring out what this player is doing. This circularity makes the analysis of simultaneous-move games somewhat more intricate than that of sequential-move games, but the analysis is not difficult. In this chapter, we develop a simple concept of equilibrium for such games that has considerable explanatory and predictive power.

1 DEPICTING SIMULTANEOUS-MOVE GAMES WITH DISCRETE STRATEGIES

In Chapters 2 and 3, we emphasized that a strategy is a complete plan of action. But, in a purely simultaneous move game, each player can have at most one opportunity to act (although that action may have many component parts); if a player had multiple opportunities to act, that would be an element of sequentiality. Therefore there is no real distinction between strategy and action in simultaneous-move games, and the terms are often used as synonyms in this context. There is only one complication. A strategy can be a probabilistic choice from the basic actions initially specified. For example, in sports, a player or team may deliberately randomize its choice of action to keep the opponent guessing. Such probabilistic strategies are called **mixed strategies,** and we consider them in Chapters 7 and 8. In this chapter, we confine our attention to the basic initially specified actions, which are called **pure strategies.**

In many games, each player has available to her a finite number of discrete pure strategies—for example, Dribble, Pass, or Shoot in basketball. In other games, each player's pure strategy can be any number from a continuous range—for example, the price charged by a firm.[1] This distinction makes no difference to the general concept of equilibrium in simultaneous-move games, but the ideas are more easily conveyed with discrete strategies; solution of games with continuous strategies needs slightly more advanced tools. Therefore, in this chapter, we restrict the analysis to the simpler case of discrete pure strategies and take up continuously variable strategies in Chapter 5.

Simultaneous-move games with discrete strategies are most often depicted with the use of a **game table** (also called a **game matrix** or **payoff table**). The table is called the **normal form** or the **strategic form** of the game. Games with any number of players can be illustrated by using a game table, but its dimen-

[1]In fact, prices must be denominated in the minimum unit of coinage—for example, whole cents—and can therefore take on only a finite number of discrete values. But this unit is usually so small that it makes more sense to think of the price as a continuous variable.

sion must equal the number of players. For a two-player game, the table is two-dimensional and appears similar to a spreadsheet. The row and column headings of the table are the strategies available to the first and second players, respectively. The size of the table, then, is determined by the numbers of strategies available to the players. Each cell within the table lists the payoffs to all players that arise under the configuration of strategies that placed players into that cell. Games with three players require three-dimensional tables; we consider them later in this chapter.

We illustrate the concept of a payoff table for a simple game in Figure 4.1. The game here has no special interpretation; so we can develop the concepts without the distraction of a "story." The players are named Row and Column. Row has four choices (strategies or actions) labeled Top, High, Low, and Bottom; Column has three choices labeled Left, Middle, and Right. Each selection of Row and Column generates a potential outcome of the game. Payoffs associated with each outcome are shown in the cell corresponding to that row and that column. By convention, of the two payoff numbers, the first is Row's payoff and the second is Column's. For example, if Row chooses High and Column chooses Right, the payoffs are 6 to Row and 4 to Column. For additional convenience, we show everything pertaining to Row—player name, strategies, and payoffs—in black, and everything pertaining to Column in red.

Remember that, in some games, most notably in sports contexts, the interests of the two sides are exactly the opposite of each other. Then, for each combination of the players' choices, the payoffs of one can be obtained by reversing the sign of the payoffs to the other. As noted in Chapter 2, we call these **zero-sum** (or **constant-sum**) games.

In zero-sum games, we can simplify the game table by showing the payoffs of just one player, generally the Row player. Those of the Column player are left implicit. Figure 4.2 shows an example of this shorthand notation for a very simplified version of a single play in (American) football. The team on the offense is attempting to move the ball forward to improve its chances of kicking a field

		COLUMN		
		Left	Middle	Right
ROW	Top	3, 1	2, 3	10, 2
	High	4, 5	3, 0	6, 4
	Low	2, 2	5, 4	12, 3
	Bottom	5, 6	4, 5	9, 7

FIGURE 4.1 Representing a Simultaneous-Move Game in a Table

		DEFENSE		
		Run	Pass	Blitz
OFFENSE	Run	2	5	13
	Short Pass	6	5.6	10.5
	Medium Pass	6	4.5	1
	Long Pass	10	3	−2

FIGURE 4.2 Representing a Zero-Sum Simultaneous-Move Game in a Table

goal. It has four possible strategies: a run and one of three different-length passes (short, medium, and long). The defense can adopt one of three strategies to try to keep the offense at bay: a run defense, a pass defense, or a blitz of the quarterback. The game is zero-sum; the offense tries to gain yardage while the defense tries to prevent it from doing so. Suppose we have enough information about the underlying strengths of the two teams to work out the probabilities of completing different plays and to determine the average gain in yardage that could be expected under each combination of strategies. For example, when Offense chooses the Medium Pass and Defense counters with its Pass defense, we estimate Offense's payoff to be 4.5 (yards).[2] Defense's payoff is −4.5 (yards), but this number is not explicitly shown in the table. The other cells similarly show Offense's payoff with Defense's payoff implicit and equal to the negative of whatever Offense receives.

2 NASH EQUILIBRIUM

To analyze simultaneous games, we need to consider how players choose their actions. Return to the game table in Figure 4.1. Focus on one specific outcome—namely, the one where Row chooses Low and Column chooses Middle; payoffs there are 5 to Row and 4 to Column. Each player wants to pick an action

[2]Here is how the payoffs for this case were constructed. When Offense chooses the Medium Pass and Defense counters with its Pass defense, our estimate is that with probability 50% the pass will be completed for a gain of 15 yards, with probability 40% the pass will fall incomplete (0 yards), and with probability 10% the pass will be intercepted with a loss of 30 yards; this makes an average of $0.5 \times 15 + 0.4 \times 0 + 0.1 \times (-30) = 4.5$ yards. The numbers in the table were constructed by a small panel of expert neighbors and friends convened by Dixit on one fall Sunday afternoon. They received a liquid consultancy fee.

that yields her the highest payoff, and in this outcome each indeed makes such a choice, given what her opponent chooses. Given that Row is choosing Low, can Column do any better by choosing something other than Middle? No, because Left would give her the payoff 2, and Right would give her 3, neither of which is better than the 4 she gets from Middle. Thus Middle is Column's **best response** to Row's choice of Low. Conversely, given that Column is choosing Middle, can Row do better by choosing something other than Low? Again no, because the payoffs from switching to Top (2), High (3), or Bottom (4) would all be no better than what Row gets with Low (5). Thus Low is Row's best response to Column's choice of Middle.

The two choices, Low for Row and Middle for Column, have the property that each is the chooser's best response to the other's action. If they were making these choices, neither would want to switch to anything different *on her own.* By the definition of a noncooperative game, the players are making their choices independently; therefore such unilateral changes are all that each player can contemplate. Because neither wants to make such a change, it is natural to call this state of affairs an equilibrium. This is exactly the concept of Nash equilibrium.

To state it a little more formally, a **Nash Equilibrium**[3] in a game is a list of strategies, one for each player, such that no player can get a better payoff by switching to some other strategy that is available to her while all the other players adhere to the strategies specified for them in the list.

A. Some Further Explanation of the Concept of Nash Equilibrium

To understand the concept of Nash equilibrium better, we take another look at the game in Figure 4.1. Consider now a cell other than (Low, Middle)—say, the one where Row chooses High and Column chooses Left. Can this be a Nash equilibrium? No, because, if Column is choosing Left, Row does better to choose Bottom and get the payoff 5 rather than to choose High, which gives her only 4. Similarly, (Bottom, Left) is not a Nash equilibrium, because Column can do better by switching to Right, thereby improving her payoff from 6 to 9.

The definition of Nash equilibrium does not require equilibrium choices to be strictly better than other available choices. Figure 4.3 is the same as Figure 4.1

[3]This concept is named for the mathematician and economist John Nash, who developed it in his doctoral dissertation at Princeton in 1949. Nash also proposed a solution to cooperative games, which we consider in Chapter 17. He shared the 1994 Nobel Prize in economics with two other game theorists, Reinhard Selten and John Harsanyi; we will treat some aspects of their work in Chapters 9 and 10. Sylvia Nasar's biography of Nash, *A Beautiful Mind* (New York: Simon & Schuster, 1998), was the (loose) basis for a movie starring Russell Crowe. Unfortunately, the movie's attempt to explain the concept of Nash equilibrium is hopelessly wrong. We explain this failure in Exercise 12 of this chapter and in Exercise 9 of Chapter 8.

		COLUMN		
		Left	Middle	Right
Row	Top	3, 1	2, 3	10, 2
	High	4, 5	3, 0	6, 4
	Low	2, 2	5, 4	12, 3
	Bottom	5, 6	5, 5	9, 7

FIGURE 4.3 Variation on Game of Figure 4.1 with a Tie in Payoffs

except that Row's payoff from (Bottom, Middle) is changed to 5, the same as that from (Low, Middle). It is still true that, given Column's choice of Middle, Row *could not do any better* than she does when choosing Low. So neither player has a reason to change her action when the outcome is (Low, Middle), and that qualifies it for a Nash equilibrium.[4]

More importantly, a Nash equilibrium does not have to be jointly best for the players. In Figure 4.1, the strategy pair (Bottom, Right) gives payoffs (9, 7), which are better for both players than the (5, 4) of the Nash equilibrium. However, playing independently, they cannot sustain (Bottom, Right). Given that Column plays Right, Row would want to deviate from Bottom to Low and get 12 instead of 9. Getting the jointly better payoffs of (9, 7) would require cooperative action that made such "cheating" impossible. We examine this type of behavior later in this chapter and in more detail in Chapter 11. For now, we merely point out the fact that a Nash equilibrium may not be in the joint interests of the players.

To reinforce the concept of Nash equilibrium, look at the football game of Figure 4.2. If the Defense is choosing the Pass defense, then the best choice for the Offense is Short Pass (payoff of 5.6 versus 5, 4.5, or 3). Conversely, if the Offense is choosing the Short Pass, then the Defense's best choice is the Pass defense—it holds the Offense down to 5.6 yards, whereas the Run defense and the Blitz would be expected to concede 6 and 10.5 yards, respectively. (Remember that the entries in each cell of a zero-sum game are the Row player's payoffs; therefore the best choice for the Column player is the one that yields the smallest number, not the largest.) In this game, the strategy combination (Short Pass, Pass defense) is a Nash equilibrium, and the resulting payoff to the Offense is 5.6 yards.

[4]But note that (Bottom, Middle) with the payoffs of (5, 5) is not itself a Nash equilibrium. If Row was choosing Bottom, Column's own best choice would not be Middle; she could do better by choosing Right. In fact, you can check all the other cells in the table to verify that none of them can be a Nash equilibrium.

How does one find Nash equilibria in games? One can always check every cell to see if the strategies that generate it satisfy the definition of a Nash equilibrium. Such **cell-by-cell-inspection** or **enumeration** is foolproof, but tedious, and unmanageable except in simple games or unless one is using a good computer program for finding equilibria. Luckily, there are many other methods, applicable to special types of games, that not only find Nash equilibria more quickly when they apply, but also give us a better understanding of the process of thinking by which beliefs and then choices are formed. We develop several such methods in later sections.

B. Nash Equilibrium As a System of Beliefs and Choices

Before we proceed with further study and use of the Nash equilibrium concept, we should try to clarify something that may have bothered some of you. We said that in a Nash equilibrium each player chooses her "best response" to the other's choice. But the two choices are made simultaneously. How can one *respond* to something that has not yet happened, at least when one does not *know* what has happened?

People play simultaneous-move games all the time and do make choices. To do so, they must find a substitute for actual knowledge or observation of the others' actions. Players could make blind guesses and hope that they turn out to be inspired ones, but luckily there are more systematic ways to try to figure out what the others are doing. One method is experience and observation—if the players play this game or similar games with similar players all the time, they may develop a pretty good idea of what the others do. Then choices that are not best will be unlikely to persist for long. Another method is the logical process of thinking through the others' thinking. You put yourself in the position of other players and think what they are thinking, which of course includes their putting themselves in your position and thinking what you are thinking. The logic seems circular, but there are several ways of breaking into the circle, and we demonstrate these ways by using specific examples in the sections that follow. Nash equilibrium can be thought of as a culmination of this process of thinking about thinking, where each player has correctly figured out the others' choice.

Whether by observation or logical deduction or some other method, you the game player acquire some notion of what the others are choosing in simultaneous-move games. It is not easy to find a word to describe the process or its outcome. It is not anticipation, nor is it forecasting, because the others' actions do not lie in the future but occur simultaneously with your own. The word most frequently used by game theorists is **belief**. This word is not perfect either, because it seems to connote more confidence or certainty than is intended; in fact, in Chapter 5, we allow for the possibility that beliefs are held with some uncertainty. But for lack of a better word, it will have to suffice.

Now think of Nash equilibrium in this light. We defined it as a configuration of strategies such that each player's strategy is her best response to that of the others. If she does not know the actual choices of the others but has beliefs about them, in Nash equilibrium those beliefs would have to be correct—the others' actual actions should be just what you believe them to be. Thus we can define Nash equilibrium in an alternative and equivalent way: it is a set of strategies, one for each player, such that (1) each player has correct beliefs about the strategies of the others and (2) the strategy of each is the best for herself, given her beliefs about the strategies of the others.

This way of thinking about Nash equilibrium has two advantages. First, the concept of "best response" is no longer logically flawed. Each player is choosing her best response, not to the as yet unobserved actions of the others, but only to her own already formed beliefs about their actions. Second, in Chapters 7 and 8, where we allow mixed strategies, the randomness in one player's strategy may be better interpreted as uncertainty in the other players' beliefs about this player's action. For now, we proceed by using both interpretations of Nash equilibrium in parallel.

You might think that formation of correct beliefs and calculation of best responses is too daunting a task for mere humans. We discuss some criticisms of this kind, as well as empirical and experimental evidence concerning Nash equilibrium, in Chapter 5 for pure strategies and Chapters 7 and 8 for mixed strategies. For now, we simply say that the proof of the pudding is in the eating. We develop and illustrate the Nash equilibrium concept by applying it. We hope that seeing it in use will prove a better way to understand its strengths and drawbacks than would an abstract discussion at this point.

3 DOMINANCE

Some games have a special property that one strategy is uniformly better than or worse than another. When this is the case, it provides one way in which the search for Nash equilibrium and its interpretation can be simplified.

The well-known game of the **prisoners' dilemma** illustrates this concept well. Consider a story line of the type that appears regularly in the television program *NYPD Blue*. Suppose that a husband and wife have been brought to the precinct under the suspicion that they were conspirators in the murder of a young woman. Detectives Sipowicz and Clark place the suspects in separate detention rooms and interrogate them one at a time. There is little concrete evidence linking the pair to the murder, although there is some evidence that they were involved in kidnapping the victim. The detectives explain to each suspect that they are both looking at jail time for the kidnapping charge, probably 3 years, even if there is no confession from either of them. In addition, the husband and wife are told individually that the detectives "know" what happened and "know" how one had been coerced by the other to participate in the crime; it is implied that jail time for a solitary confessor will be significantly reduced if the whole story is committed to paper. (In a scene from virtually every episode, the yellow legal pad and a pencil are produced and placed on the table at this point.) Finally, they are told that, if both confess, jail terms could be negotiated down but not as much as they would be if there were one confession and one denial.

Both husband and wife are then players in a two-person, simultaneous-move game in which each has to choose between confessing and not confessing to the crime of murder. They both know that no confession leaves them each with a 3-year jail sentence for involvement with the kidnapping. They also know that, if one of them confesses, he or she will get a short sentence of 1 year for cooperating with the police, while the other will go to jail for a minimum of 25 years. If both confess, they figure that they can negotiate for jail terms of 10 years each.

The choices and outcomes for this game are summarized by the game table in Figure 4.4. The strategies Confess and Deny can also be called Defect and Cooperate to capture their roles in the relationship between the *two players;* thus Defect means to defect from any tacit arrangement with the spouse, and Cooperate means to take the action that helps the spouse (not cooperate with the cops).

Payoffs here are the lengths of the jail sentences associated with each outcome; so low numbers are better for each player. In that sense, this example differs from most of the games that we analyze in which large payoffs are good rather than bad. We take this opportunity to alert you that "large is good" is not always true. When payoff numbers indicate players' rankings of outcomes, people often use 1 for the best alternative and successively higher numbers for successively worse ones. Also, in the table for a zero-sum game that shows only one player's bigger-is-better payoffs, smaller numbers are better for the other. In the prisoners' dilemma here, smaller numbers are better for both. Thus, if you ever write a payoff table where large numbers are bad, you should alert the reader by pointing it out clearly. And, when reading someone else's example, be aware of the possibility.

		WIFE	
		Confess (Defect)	Deny (Cooperate)
HUSBAND	Confess (Defect)	10 yr, 10 yr	1 yr, 25 yr
	Deny (Cooperate)	25 yr, 1 yr	3 yr, 3 yr

FIGURE 4.4 Prisoners' Dilemma

Now consider the prisoners' dilemma game in Figure 4.4 from the husband's perspective. He has to think about what the wife will choose. Suppose he believes that she will confess. Then his best choice is to confess; he gets a sentence of only 10 years, when denial would have meant 25 years. What if he believes the wife will deny? Again, his own best choice is to confess; he gets only 1 year instead of the 3 that his own denial would bring in this case. Thus, in this special game, Confess is better than Deny for the husband *regardless of his belief about the wife's choice.* We say that, for the husband, the strategy Confess is a **dominant strategy** or that the strategy Deny is a **dominated strategy.** Equivalently, we could say that the strategy Confess *dominates* the strategy Deny or that the strategy Deny is *dominated* by the strategy Confess.

If an action is clearly best for a player, no matter what the others might be doing, then there is compelling reason to think that a rational player would choose it. And, if an action is clearly bad for a player, no matter what the others might be doing, then there is equally compelling reason to think that a rational player would avoid it. Therefore dominance, when it exists, provides a compelling basis for the theory of solutions to simultaneous-move games.

A. Both Players Have Dominant Strategies

In the preceding prisoners' dilemma, dominance should lead the husband to choose Confess. Exactly the same logic applies to the wife's choice. Her own strategy Confess dominates her own strategy Deny; so she also should choose Confess. Therefore (Confess, Confess) is the outcome predicted for this game. Note that it is a Nash equilibrium. (In fact it is the only Nash equilibrium.) Each player is choosing his or her own best strategy.

In this special game, the best choice for each is independent of whether their beliefs about the other are correct—that is the meaning of dominance—but each, attributing to the other the same rationality as he or she practices, should be able to form correct beliefs. And the actual action of each is the best response to the actual action of the other. Note that the fact that Confess dominates Deny for both players is completely independent of whether they are actually guilty, as in many episodes of *NYPD Blue,* or are being framed, as happened in the movie *LA Confidential.* It only depends on the pattern of payoffs dictated by the various sentence lengths.

Any game with the same general payoff pattern as that illustrated in Figure 4.4 is given the generic label "prisoners' dilemma." More specifically, a prisoners' dilemma has three essential features. First, each player has two strategies: to cooperate with one's rival (deny any involvement in the crime, in our example) or to defect from cooperation (confess to the crime, here). Second, each player also has a dominant strategy (to confess or to defect from cooperation). Finally, the dominance solution equilibrium is worse for both players than the nonequilibrium situation in which each plays the dominated strategy (to cooperate with rivals).

Games of this type are particularly important in the study of game theory for two reasons. The first is that the payoff structure associated with the prisoners' dilemma arises in many quite varied strategic situations in economic, social, political, and even biological competitions. This wide-ranging applicability makes it an important game to study and to understand from a strategic standpoint. The whole of Chapter 11 and sections in several other chapters deal with its study.

The second reason that prisoners' dilemma games are integral to any discussion of games of strategy is the somewhat curious nature of the equilibrium outcome achieved in such games. Both players follow conventional wisdom in choosing their dominant strategies, but the resulting equilibrium outcome yields them payoffs that are lower than they could have achieved if they had each chosen their dominated strategies. Thus the equilibrium outcome in the prisoners' dilemma is actually a bad outcome for the players. They could find another outcome that they both prefer to the equilibrium outcome; the problem is how to guarantee that someone will not cheat. This particular feature of the prisoners' dilemma has received considerable attention from game theorists who have asked an obvious question: What can players in a prisoners' dilemma do to achieve the better outcome? We leave this question to the reader momentarily, as we continue the discussion of simultaneous games, but return to it in detail in Chapter 11.

B. One Player Has a Dominant Strategy

When a rational player has a dominant strategy, she will use it, and the other player can safely believe this. In the prisoners' dilemma, it applied to both players. In some other games, it applies only to one of them. If you are playing in a game in which you do not have a dominant strategy but your opponent does, you can assume that she will use her dominant strategy and so you can choose your equilibrium action (your best response) accordingly.

We illustrate this case by using a game frequently played between the Congress, which is responsible for fiscal policy (taxes and government expenditures), and the Federal Reserve (Fed), which is in charge of monetary policy (primarily, interest rates).[5] Simplifying the game to its essential features, the Congress's fiscal policy can have either a balanced budget or a deficit, and the Fed can set interest rates either high or low. In reality, the game is not clearly

[5]Similar games are played in many other countries with central banks that have operational independence in the choice of monetary policy. Fiscal policies may be chosen by different political entities—the executive or the legislature—in different countries.

simultaneous; nor is who has the first move obvious if choices are sequential. We consider the simultaneous-move version here, and in Chapter 6 study how the outcomes differ for different rules of the game.

Almost everyone wants lower taxes. But there is no shortage of good claims on government funds: defense, education, health care, and so on. There are also various politically powerful special interest groups—including farmers and industries hurt by foreign competition—who want government subsidies. Therefore the Congress is under constant pressure both to lower taxes and to increase spending. But such behavior runs the budget into deficit, which can lead to higher inflation. The Fed's primary task is to prevent inflation. However, it also faces political pressure for lower interest rates from many important groups, especially homeowners who benefit from lower mortgage rates. Lower interest rates lead to higher demand for automobiles, housing, and capital investment by firms, and all this demand can cause higher inflation. The Fed is generally happy to lower interest rates but only so long as inflation is not a threat. And there is less threat of inflation when the government's budget is in balance. With all this in mind, we construct the payoff matrix for this game in Figure 4.5.

Congress likes best (payoff 4) the outcome with a budget deficit and low interest rates. This pleases all the immediate political constituents. It may entail trouble for the future, but political time horizons are short. For the same reason, Congress likes worst (payoff 1) the outcome with a balanced budget and high interest rates. Of the other two outcomes, it prefers (payoff 3) the outcome with a balanced budget and low interest rates; this outcome pleases the important home-owning middle classes and, with low interest rates, less expenditure is needed to service the government debt, so the balanced budget still has room for many other items of expenditure or for tax cuts.

The Fed likes worst (payoff 1) the outcome with a budget deficit and low interest rates, because this combination is the most inflationary. It likes best (payoff 4) the outcome with a balanced budget and low interest rates, because this combination can sustain a high level of economic activity without much risk of inflation. Comparing the other two outcomes with high interest rates, the Fed prefers the one with budget balance because it reduces the risk of inflation.

		FEDERAL RESERVE	
		Low interest rates	High interest rates
CONGRESS	Budget balance	3, 4	1, 3
	Budget deficit	4, 1	2, 2

FIGURE 4.5 Game of Fiscal and Monetary Policies

We look now for dominant strategies in this game. The Fed does better by choosing low interest rates if it believes that the Congress is opting for budget balance (Fed's payoff 4 rather than 3), but it does better choosing high interest rates if it believes that the Congress is choosing to run a budget deficit (Fed's payoff 2 rather than 1). The Fed, then, does not have a dominant strategy. But the Congress does. If the Congress believes that the Fed is choosing low interest rates, it does better for itself by choosing a budget deficit rather than budget balance (Congress's payoff 4 instead of 3). If the Congress believes that the Fed is choosing high interest rates, again it does better for itself by choosing a budget deficit rather than budget balance (Congress's payoff 2 instead of 1). Choosing to run a budget deficit is then Congress's dominant strategy.

The choice for the Congress in the game is now clear. No matter what it believes the Fed is doing, the Congress will choose to run a budget deficit. The Fed can now take this choice into account when making its own decision. The Fed should believe that the Congress will choose its dominant strategy (budget deficit) and choose the best strategy for itself, given this belief. That means that the Fed should choose high interest rates.

In this outcome, each side gets payoff 2. But an inspection of Figure 4.5 shows that, just like in the prisoners' dilemma, there is another outcome—namely, a balanced budget and low interest rates—that can give both players higher payoffs—namely, 3 for the Congress and 4 for the Fed. Why is that outcome not achievable as an equilibrium? The problem is that Congress would be tempted to deviate from its stated strategy and sneakily run a budget deficit. The Fed, knowing this temptation and that it would then get its worst outcome (payoff 1), deviates also to its high interest rate strategy. In Chapters 6 and 10, we consider how the two sides can get around this difficulty to achieve their mutually preferred outcome. But we should note that, in most countries and at many times, the two policy authorities are indeed stuck in the bad outcome; the fiscal policy is too loose, and the monetary policy has to be tightened to keep inflation down.

C. Successive Elimination of Dominated Strategies

The games considered so far have had only two pure strategies available to each player. In such games, if one strategy is dominant, the other is dominated; so choosing the dominant strategy is equivalent to eliminating the dominated one. In larger games, some of a player's strategies may be dominated even though no single strategy dominates all of the others. If players find themselves in a game of this type, they may be able to reach an equilibrium by removing dominated strategies from consideration as possible choices. Removing dominated strategies reduces the size of the game, and then the "new" game may have another dominated strategy, for the same player or for her opponent, that can also be removed. Or the "new" game may even have a dominant strategy for one of the players. **Successive or iterated elimination of dominated strategies** uses this process of removal of dominated strategies and reduction in the size of a game until no further reductions can be made. If this process ends in a unique outcome, then the game is said to be **dominance solvable;** that outcome is the Nash equilibrium of the game, and the strategies that yield it are the equilibrium strategies for each player.

We can use the game of Figure 4.1 to provide an example of this process. Consider first Row's strategies. If any one of Row's strategies always provides worse payoffs for Row than another of her strategies, then that strategy is dominated and can be eliminated from consideration for Row's equilibrium choice. Here, the only dominated strategy for Row is High, which is dominated by Bottom; if Column plays Left, Row gets 5 from Bottom and only 4 from High; if Column plays Middle, Row gets 4 from Bottom and only 3 from High; and, if Column plays Right, Row gets 9 from Bottom and only 6 from High. So we can eliminate High. We now turn to Column's choices to see if any of them can be eliminated. We find that Column's Left is now dominated by Right (with similar reasoning, $1 < 2$, $2 < 3$, and $6 < 7$). Note that we could not say this before Row's High was eliminated; against Row's High, Column would get 5 from Left but only 4 from Right. Thus the first step of eliminating Row's High makes possible the second step of eliminating Column's Left. Then, within the remaining set of strategies (Top, Low, and Bottom for Row, and Middle and Right for Column), Row's Top and Bottom are both dominated by his Low. When Row is left with only Low, Column chooses his best response—namely, Middle.

The game is thus dominance solvable, and the outcome is (Low, Middle) with payoffs (5, 4). We identified this outcome as a Nash equilibrium when we first illustrated that concept by using this game. Now we see in better detail the thought process of the players that leads to the formation of correct beliefs. A rational Row will not choose High. A rational Column will recognize this, and thinking about how her various strategies perform for her against Row's remaining strategies, will not choose Left. In turn, Row will recognize this, and therefore will not choose either Top or Bottom. Finally, Column will see through all this, and choose Middle.

Other games may not be dominance solvable or successive elimination of dominated strategies may not yield a unique outcome. Even in such cases, some elimination may reduce the size of the game and make it easier to solve by using one or more of the techniques described in the following sections. Thus eliminating dominated strategies can be a useful step toward solving a large simultaneous-play game, even when their elimination does not completely solve the game.

Thus far in our consideration of iterated elimination of dominated strategies, all the payoff comparisons have been unambiguous. What if there are some ties? Consider the variation on the preceding game that is shown in Figure 4.3. In that version of the game, High (for Row) and Left (for Column) also are eliminated. And, at the next step, Low still dominates Top. But the dominance of Low over Bottom is now less clear-cut. The two strategies give Row equal payoffs when played against Column's Middle, although Low does give Row a higher payoff than Bottom when played against Column's Right. We say that, from Row's perspective at this point, Low *weakly* dominates Bottom. In contrast, Low *strictly* dominates Top, because it gives strictly higher payoffs than does Top when played against both of Column's strategies, Middle and Right, under consideration at this point.

We give a more precise definition of the distinction between strict and weak dominance in the Appendix to this chapter. Here, though, we provide a word of warning. Successive elimination of weakly dominated strategies can get rid of some Nash equilibria.

Consider the game illustrated in Figure 4.6. For Row, Up is weakly dominated by Down; if Column plays Left, then Row gets a better payoff by playing Down than by playing Up, and, if Column plays Right, then Row gets the same payoff from her two strategies. Similarly, for Column, Right weakly dominates Left. Dominance solvability then tells us that (Down, Right) is a Nash equilibrium. That is true, but (Down, Left) and (Up, Right) also are Nash equilibria. Consider (Down, Left). When Row is playing Down, Column cannot improve her payoff by switching to Right, and, when Column is playing Left, Row's best response is clearly to play Down. A similar reasoning verifies that (Up, Right) also is a Nash equilibrium.

Therefore, if you use weak dominance to eliminate some strategies, it is a good idea to make a quick cell-by-cell check to see if you have missed any other equilibria. The iterated dominance solution seems to be a reasonable outcome to predict as the likely Nash equilibrium of this simultaneous-play game, but it is also important to consider the significance of multiple equilibria as well as of

		COLUMN	
		Left	Right
ROW	Up	0, 0	1, 1
	Down	1, 1	1, 1

FIGURE 4.6 Elimination of Weakly Dominated Strategies

the other equilibria themselves. We address these issues in later chapters, taking up a discussion of multiple equilibria in Chapter 5 and the interconnections between sequential- and simultaneous-move games in Chapter 6.

4 BEST-RESPONSE ANALYSIS

Many simultaneous-move games have no dominant strategies and no dominated strategies. Others may have one or several dominated strategies, but iterated elimination of dominated strategies will not yield a unique outcome. In such cases, we need a next step in the process of finding a solution to the game. We are still looking for a Nash equilibrium in which every player does the best she can, given the actions of the other player(s), but we must now rely on more subtle strategic thinking than the simple elimination of dominated strategies requires.

Here, we develop another systematic method for finding Nash equilibria that will prove very useful in later analysis. We begin without imposing a requirement of correctness of beliefs. We take each player's perspective in turn and ask the following question: For each of the choices that the other player(s) might be making, what is the best choice for this player? Thus we find the best responses of each player to all available strategies of the others. Mathematically, we find each player's best-response strategy, depending on, or as a function of, the other players' available strategies.

Return to the game of Figure 4.1, reproduced as Figure 4.7, and consider Row first. If Column chooses Left, Row's best response is Bottom, yielding 5. We show this best response by circling that payoff in the game table. If Column chooses Middle, Row's best response is Low (also yielding 5). And, if Column chooses Right, Row's best choice is again Low (now yielding 12). As before, we show Row's best choices by circling the appropriate payoffs. Similarly, Column's best re-

		COLUMN		
		Left	Middle	Right
ROW	Top	3, 1	2, 3	10, 2
	High	4, 5	3, 0	6, 4
	Low	2, 2	5, 4	12, 3
	Bottom	5, 6	4, 5	9, 7

FIGURE 4.7 Best Response Analysis

sponses are shown by circling her payoffs 3 (Middle as best response to Row's Top), 5 (Left to High), 4 (Middle to Low), and 7 (Right to Bottom).[6] We see that one cell—namely, (Low, Middle)—has both its payoffs circled. Therefore the strategies Low for Row and Middle for Column are simultaneously best responses to each other. We have found the Nash equilibrium of this game. (Again.)

Best-response analysis is a comprehensive way of locating all possible Nash equilibria of a game. You should improve your understanding of it by trying it out on the other games that have been used in this chapter. The cases of dominance are of particular interest. If Row has a dominant strategy, that same strategy is her best response to all of Column's strategies; therefore her best responses are all lined up horizontally in the same row. Similarly, if Column has a dominant strategy, her best responses are all lined up vertically in the same column. You should see for yourself how the Nash equilibria of the preceding prisoners' dilemma and Congress–Fed games emerge from such a drawing.

There will be some games for which best-response analysis does not find a Nash equilibrium, just as dominance solvability sometimes fails. But in this case we can say something more specific than can be said when dominance fails. When best-response analysis of a discrete strategy game does not find a Nash equilibrium, then the game has no equilibrium in pure strategies. We address games of this type in Section 8 of this chapter. In Chapter 5, we extend best-response analysis to games where the players' strategies are continuous variables—for example, prices or advertising expenditures. There, we construct best-response *curves* to help us find Nash equilibria, and we see that such games are less likely—by virtue of the continuity of strategy choices—to have no equilibrium.

5 THE MINIMAX METHOD FOR ZERO-SUM GAMES

For zero-sum games, an alternative to best-response analysis works by using the special logic of strict conflict that exists in such games. This approach, the **minimax method,** works only for zero-sum games and relies on a thought process that accounts for the fact that outcomes that are good for one player

[6]Alternatively and equivalently, one could mark in some way the choices that are *not* made. For example, in Figure 4.3, Row will not choose Top, High, or Bottom as responses to Column's Right; one could show this by drawing slashes through Row's payoffs in these cases, respectively, 10, 6, and 9. When this is done for all strategies of both players, (Low, Middle) has both of its payoffs unslashed; it is then the Nash equilibrium of the game. The alternatives of circling choices that are made and slashing choices that are not made stand in a conceptually similar relation to each other, as do the alternatives of showing chosen branches by arrows and pruning unchosen branches for sequential-move games. We prefer the first alternative in each case, because the resulting picture looks cleaner and tells the story better.

are, by definition, bad for the other. In this method, each player is assumed to choose her strategy by thinking: "Would this be the best choice for me, even if the other player found out that I was playing it?" She must then consider her opponent's best response to her chosen strategy. But in a zero-sum game, that best response is the worst one for her. In other words, each player believes that her opponent will choose an action that yields her the worst possible consequences of each of her own actions. Then acting on those beliefs she should choose the action that leads to the least-bad outcome.

This logic may seem extremely pessimistic, but it still relies on a type of best-response calculation and it is appropriate for finding the equilibrium of a zero-sum game. In equilibrium, each player is choosing her own best response, given her beliefs about what the other will do. In anticipating such best responses, each player will expect to receive the worst payoff associated with each action and will choose her own action accordingly. She is thus choosing her best payoff from among the set of worst payoffs.

Suppose the payoff table shows the row player's payoffs, and Row wants the outcome to be a cell with as high a number as possible. Then Column wants the outcome to be a cell with as low a number as possible. Using the pessimistic logic just described, Row figures that, for each of her rows, Column will choose the column with the lowest number in that row. Therefore Row should choose the row that gives her the highest among these lowest numbers, or the maximum among the minima—the **maximin** for short. Similarly, Column reckons that, for each of her columns, Row will choose the row with the largest number in that column. Then Column should choose the column with the smallest number among these largest ones, or the minimum among the maxima—the **minimax.** If Row's maximin value and Column's minimax value are in the same cell of the game table, then that outcome is a Nash equilibrium of the zero-sum game. This method of finding equilibria in zero-sum games should be called the maximin-minimax method, but it is called simply the *minimax method* for short. It will lead you to a Nash equilibrium in pure strategies if one exists.

To illustrate the minimax method, we use the football example of Figure 4.2. We already know the Nash equilibrium for that game, but now we obtain it by using the minimax method. We reproduce the game table in Figure 4.8, adding information that pertains to the minimax argument.

Begin by finding the lowest number in each row (the offense's worst payoff from each strategy) and the highest number in each column (the defense's worst payoff from each strategy). The offense's worst payoff from Run is 2; its worst payoff from Short Pass, 5.6; its worst payoff from Medium Pass, 1; and its worst payoff from Long Pass, −2. We write the minimum for each row at the far right of that row. The defense's worst payoff from Run is 10; its worst payoff from Pass, 5.6; and its worst payoff from Blitz, 13. We write the maximum for each column at the bottom of that column.

		DEFENSE			
		Run	Pass	Blitz	
OFFENSE	Run	2	5	13	min = 2
	Short Pass	6	5.6	10.5	min = 5.6
	Medium Pass	6	4.5	1	min = 1
	Long Pass	10	3	−2	min = −2
		max = 10	max = 5.6	max = 13	

FIGURE 4.8 The Minimax Method

The next step is to find the best of each player's worst possible outcomes, the largest row minimum and the smallest column maximum. The largest of the row minima is 5.6; so the offense can ensure itself a gain of 5.6 yards by playing the Short Pass; this is its maximin. The lowest of the column maxima is 5.6; so the defense can be sure of holding the offense down to a gain of 5.6 yards by deploying its Pass defense. This is the defense's minimax.

Looking at these two strategy choices, we see that the maximin and minimax values are found in the same cell of the game table. Thus the offense's maximin strategy is its best response to the defense's minimax and vice versa; we have found the Nash equilibrium of this game. That equilibrium entails the offense attempting a Short Pass while the defense defends against a Pass. A total of 5.6 yards will be gained by the offense (and given up by the defense).

The minimax method may fail to find an equilibrium in some zero-sum games. If so, then our conclusion is similar to that when best-response analysis fails: the game has no Nash equilibrium in pure strategies. We address this matter later in this chapter and examine mixed strategy equilibria in Chapters 7 and 8. And, to repeat, the minimax method cannot be applied to non-zero-sum games. In such games, your opponent's best is not necessarily your worst. Therefore the pessimistic assumption that leads you to choose the strategy that makes your minimum payoff as large as possible may not be your best strategy.

6 THREE PLAYERS

So far, we have analyzed only games between two players. All of the methods of analysis that have been discussed, however, can be used to find the pure-strategy Nash equilibria of any simultaneous-play game among any number of players. When a game is played by more than two players, each of whom has a relatively small number of pure strategies, the analysis can be done with a game table, as we did in the first five sections of this chapter.

In Chapter 3, we described a game among three players, each of whom had two pure strategies. The three players, Emily, Nina, and Talia, had to choose whether to contribute toward the creation of a flower garden for their small street. We assumed there that the garden when all three contributed was no better than when only two contributed and that a garden with just one contributor was so sparse that it was as bad as no garden at all. Now, let us suppose instead that the three players make their choices simultaneously and that there is a somewhat richer variety of possible outcomes and payoffs. In particular, the size and splendor of the garden will now differ according to the exact number of contributors; three contributors will produce the largest and best garden, two contributors will produce a medium garden, and one contributor will produce a small garden.

Suppose Emily is contemplating the possible outcomes of the street-garden game. There are six possibilities to consider. Emily can choose either to contribute or not to contribute when both Nina and Talia contribute or when neither of them contributes or when just one of them contributes. From her perspective, the best possible outcome, with a rating of 6, would be to take advantage of her good-hearted neighbors and to have both Nina and Talia contribute while she does not. Emily could then enjoy a medium-sized garden without putting up her own hard-earned cash. If both of the others contribute and Emily also contributes, she gets to enjoy a large, very splendid garden but at the cost of her own contribution; she rates this outcome second-best, or 5.

At the other end of the spectrum are the outcomes that arise when neither Nina nor Talia contributes to the garden. If that is the case, Emily would again prefer not to contribute, because she would foot the bill for a public garden that everyone could enjoy; she would rather have the flowers in her own yard. Thus, when neither other player is contributing, Emily ranks the outcome in which she contributes as a 1 and the outcome in which she does not as a 2.

In between these cases are the situations in which either Nina or Talia contributes to the flower garden but not both. When one of them contributes, Emily knows that she can enjoy a small garden without contributing; she also feels that the cost of her contribution outweighs the increase in benefit that she gets from being able to increase the size of the garden. Thus she ranks the outcome in which she does not contribute, but still enjoys the small garden, as a 4 and the outcome in which she does contribute, to provide a medium garden, as a 3. Because Nina and Talia have the same views as Emily on the costs and benefits of contributions and garden size, each of them orders the different outcomes in the same way—the worst outcome being the one in which each contributes and the other two do not, and so on.

TALIA chooses:

Contribute

		NINA	
		Contribute	Don't
EMILY	Contribute	5, 5, 5	3, 6, 3
	Don't	6, 3, 3	4, 4, 1

Don't Contribute

		NINA	
		Contribute	Don't
EMILY	Contribute	3, 3, 6	1, 4, 4
	Don't	4, 1, 4	2, 2, 2

FIGURE 4.9 Street Garden Game

If all three women decide whether to contribute to the garden without knowing what their neighbors will do, we have a three-person simultaneous-move game. To find the Nash equilibrium of the game, we then need a game table. For a three-player game, the table must be three-dimensional and the third player's strategies correspond to the new dimension. The easiest way to add a third dimension to a two-dimensional game table is to add pages. The first page of the table shows payoffs for the third player's first strategy, the second page shows payoffs for the third player's second strategy, and so on.

We show the three-dimensional table for the street-garden game in Figure 4.9. It has two rows for Emily's two strategies, two columns for Nina's two strategies, and two pages for Talia's two strategies. We show the pages side by side so that you can see everything at the same time. In each cell, payoffs are listed for the row player first, the column player second, and the page player third; in this case, the order is Emily, Nina, Talia.

Our first test should be to determine whether there are dominant strategies for any of the players. In one-page game tables, we found this test to be simple; we just compared the outcomes associated with one of a player's strategies with the outcomes associated with another of her strategies. In practice this comparison required, for the row player, a simple check within columns of the single page of the table and vice versa for the column player. Here, we must check in both pages of the table to determine whether any player has a dominant strategy.

For Emily, we compare the two rows of both pages of the table and note that, when Talia contributes, Emily has a dominant strategy not to contribute, and, when Talia does not contribute, Emily also has a dominant strategy not to contribute. Thus the best thing for Emily to do, regardless of what either of the other players does, is not to contribute. Similarly, we see that Nina's dominant strategy—in both pages of the table—is not to contribute. When we check for a dominant strategy for Talia, we have to be a bit more careful. We must compare outcomes that keep Emily's and Nina's behavior constant, checking

Talia's payoffs from choosing Contribute versus Don't. That is, we compare cells across pages of the table—the top-left cell in the first page (on the left) with the top-left cell in the second page (on the right), and so on. As for the first two players, this process indicates that Talia also has a dominant strategy not to contribute.

Each player in this game has a dominant strategy, which must therefore be her equilibrium pure strategy. The Nash equilibrium of the street-garden game entails all three players choosing not to contribute to the street garden and getting their second-worst payoffs; the garden is not planted, but no one has to contribute either.

Notice that this game is yet another example of a prisoners' dilemma. There is a unique Nash equilibrium in which all players receive a payoff of 2. Yet there is another outcome in the game—in which all three neighbors contribute to the garden—that for all three players yields higher payoffs of 5. Even though it would be beneficial to each of them for all to pitch in to build the garden, no one has the individual incentive to do so. As a result, gardens of this type are either not planted at all or paid for through tax dollars—because the town government can require its citizens to pay such taxes. In Chapter 12, we will encounter more such dilemmas of collective action and study some methods for resolving them.

The Nash equilibrium of the game can also be found using the cell-by-cell inspection method. For example, consider another cell in Figure 4.9—say, the one where Emily and Nina contribute but Talia does not, with the payoffs (3, 3, 6). When Emily considers changing her strategy, as the row player she can change only the row position of the game's outcome. Emily can move the outcome only from a given cell in a given row, column, and page to another cell in a different row but the same column and same page of the table. If she does that in this instance, she improves her payoff from 3 to 4. Similarly, Nina can change only the column position of the outcome, moving it to a cell in another column but in the same row and same page of the table. Doing so improves Nina's payoff from 3 to 4. Finally, Talia can change only the page position of the game's outcome. She can move the outcome to a different page, but the row and column positions must remain the same. Doing so would worsen Talia's payoff from 6 to 5. Because at least one player can do better by unilaterally changing her strategy, the cell that we examined cannot be the outcome of a Nash equilibrium.

We can also use the best-response method, as shown in Figure 4.10, by drawing circles around the best responses, as in Figure 4.7. Because each player has Don't as her dominant strategy, all of Emily's best responses are on her Don't rows, all of Nina's on her Don't columns, and all of Talia's on her Don't page. The cell at the bottom right has all three best responses; therefore it gives us the Nash equilibrium.

TALIA chooses:

Contribute

		NINA	
		Contribute	Don't
EMILY	Contribute	5, 5, 5	3, 6, 3
	Don't	6, 3, 3	4, 4, 1

Don't Contribute

		NINA	
		Contribute	Don't
EMILY	Contribute	3, 3, 6	1, 4, 4
	Don't	4, 1, 4	2, 2, 2

FIGURE 4.10 Best-Response Analysis in the Street-Garden Game

7 MULTIPLE EQUILIBRIA IN PURE STRATEGIES

Each of the games considered in preceding sections has had a unique pure-strategy Nash equilibrium. In general, however, games need not have unique Nash equilibria. We illustrate this result by using a class of games that have many applications. As a group, they may be labeled **coordination games.** The players in such games have some (but not always completely) common interests. But, because they act independently (by virtue of the nature of noncooperative games), the coordination of actions needed to achieve a jointly preferred outcome is problematic.

To illustrate this idea, picture two undergraduates, Harry and Sally, who meet in their college library. They are attracted to each other and would like to continue the conversation but have to go off to their separate classes. They arrange to meet for coffee after the classes are over at 4:30. Sitting separately in class, each realizes that in the excitement they forgot to fix the place to meet. There are two possible choices, Starbucks and Local Latte. Unfortunately, these locations are on opposite sides of the large campus; so it is not possible to try both. And Harry and Sally have not exchanged pager numbers; so they can't send messages. What should each do?

Figure 4.11 illustrates this situation as a game and shows the payoff matrix. Each player has two choices—Starbucks and Local Latte. The payoffs for each are 1 if they meet and 0 if they do not. Cell-by-cell inspection shows at once that the game has two Nash equilibria, one where both choose Starbucks and the other where both choose Local Latte. It is important for both that they achieve one of the equilibria, but which one is immaterial because the two yield equal payoffs. All that matters is that they coordinate on the same action; it does not matter which action. That is why the game is said to be one of **pure coordination.**

		SALLY	
		Starbucks	Local Latte
HARRY	Starbucks	1, 1	0, 0
	Local Latte	0, 0	1, 1

FIGURE 4.11 Pure Coordination

But will they coordinate successfully? Or will they end up in different cafés, each thinking that the other has let him or her down? Alas, that risk exists. Harry might think that Sally will go to Starbucks because she said something about the class to which she was going and that class is on the Starbucks side of the campus. But Sally may have the opposite belief about what Harry will do. When there are multiple Nash equilibria, if the players are to select one successfully, they need some way to coordinate their beliefs or expectations about each other's actions.

The situation is similar to that of the heroes of the "Which tire?" game in Chapter 1, where we labeled the coordination device a **focal point.** In the present context, one of the two cafés may be generally known as the student hangout. But it is not enough that Harry knows this to be the case. He must know that Sally knows, and that she knows that he knows, and so on. In other words, their expectations must *converge* on the focal point. Otherwise Harry might be doubtful about where Sally will go because he does not know what she is thinking about where he will go; and similar doubts may arise at the third or fourth or higher level of thinking about thinking.

When one of us (Dixit) posed this question to students in his class, the freshmen generally chose Starbucks and the juniors and seniors generally chose the local café in the campus student center. These responses are understandable—freshmen, who have not been on campus long, focus their expectations on a nationwide chain that is known to everyone, whereas juniors and seniors have acquired the local habits, which they now regard as superior, and expect their peers to believe likewise.

If one café had an orange decor and the other a crimson decor, then in Princeton the former may serve as a focal point because orange is the Princeton color, whereas at Harvard crimson may be focal for the same reason. If one person is a Princeton student and the other a Harvard student, they may fail to meet at all, either because each thinks that his or her color "should" get priority or because each thinks that the other will be inflexible and so tries to accommodate him or her. More generally, whether players in coordination games can find a focal point depends on their having some commonly known point of contact, whether historical, cultural, or linguistic.

		SALLY	
		Starbucks	Local Latte
HARRY	Starbucks	1, 1	0, 0
	Local Latte	0, 0	2, 2

FIGURE 4.12 Assurance

Now change the game payoffs a little. The behavior of juniors and seniors suggests that our pair may not be quite indifferent about which café they both choose. The coffee may be better at one or the ambiance better at one. Or they may want to choose the one that is not the general student hangout, to avoid the risk of running into former boyfriends or girlfriends. Suppose they both prefer Local Latte; so the payoff of each is 2 when they meet there versus 1 when they meet at Starbucks. The new payoff matrix is shown in Figure 4.12.

Again, there are two Nash equilibria. But, in this version of the game, each prefers the equilibrium where both choose Local Latte. Unfortunately, their mere liking of that outcome is not guaranteed to bring it about. First of all (and as always in our analysis), the payoffs have to be common knowledge—both have to know the entire payoff matrix, both have to know that both know, and so on. Such detailed knowledge about the game can arise if the two discussed and agreed on the relative merits of the two cafés but simply forgot to settle definitely to meet at Local Latte. Even then, Harry might think that Sally has some other reason for choosing Starbucks, or think that she thinks that he does, and so on. Without genuine **convergence of expectations** about actions, they may choose the worse equilibrium or, worse still, they may fail to coordinate actions and get 0 each.

To repeat, players in the game illustrated in Figure 4.12 can get the preferred equilibrium outcome only if each has enough certainty or assurance that the other is choosing the appropriate action. For this reason, such games are called **assurance games.**

In many real-life situations of this kind, such assurance is easily obtained, given even a small amount of communication between the players. Their interests are perfectly aligned; if one of them says to the other, "I am going to Local Latte," the other has no reason to doubt the truth of this statement and will follow to get the mutually preferred outcome. That is why we had to construct the story with the two students isolated in different classes with no means of communication. If the players' interests conflict, truthful communication becomes more problematic. We examine this problem further when we consider strategic manipulation of information in games in Chapter 9.

In larger groups, communication can be achieved by scheduling meetings or by making announcements. These devices work only if everyone knows that everyone else is paying attention to them, because successful coordination requires the desired equilibrium to be a focal point. The players' expectations must converge on it; everyone should know that everyone knows that . . . everyone is choosing it. Many social institutions and arrangements serve this role. Meetings where the participants sit in a circle facing inward ensure that everyone sees everyone else paying attention. Advertisements during the Super Bowl, especially when they are proclaimed in advance as major attractions, ensure each viewer that many others are viewing them also. That makes such ads especially attractive to companies making products that are more desirable for any one buyer when many others are buying them, too; such products include those produced by the computer, telecommunication, and Internet industries.[7]

Now introduce another complication to the café-choice game. Both players want to meet but prefer different cafés. So Harry might get a payoff of 2 and Sally a payoff of 1 from meeting at Starbucks, and the other way around from meeting at Local Latte. This payoff matrix is shown in Figure 4.13.

This game is called the **battle of the sexes.** The name derives from the story concocted for this payoff structure by game theorists in the sexist 1950s. A husband and wife were supposed to choose between going to a boxing match and a ballet, and (presumably for evolutionary genetic reasons) the husband was supposed to prefer the boxing match and the wife the ballet. The name has stuck and we will keep it, but our example should make it clear that it has no necessary sexist connotations.

What will happen in this game? There are still two Nash equilibria. If Harry believes that Sally will choose Starbucks, it is best for him to do likewise, and the other way around. For similar reasons, Local Latte also is a Nash equilibrium. To achieve either of these equilibria and avoid the outcomes where the two go to different cafés, the players need a focal point, or convergence of ex-

		SALLY	
		Starbucks	Local Latte
HARRY	Starbucks	2, 1	0, 0
	Local Latte	0, 0	1, 2

FIGURE 4.13 Battle of the Sexes

[7]Michael Chwe develops this theme in *Rational Ritual: Culture, Coordination, and Common Knowledge* (Princeton: Princeton University Press, 2001).

pectations, exactly as in the pure-coordination and assurance games. But the risk of coordination failure is greater in the battle of the sexes. The players are initially in quite symmetric situations, but each of the two Nash equilibria gives them asymmetric payoffs; and their preferences between the two outcomes are in conflict. Harry prefers the outcome where they meet in Starbucks, and Sally prefers to meet in Local Latte. They must find some way of breaking the symmetry.

In an attempt to achieve his or her preferred equilibrium, each player may try to act tough and follow the strategy leading to the better equilibrium. In Chapter 10, we consider in detail such advance devices, called strategic moves, that players in such games can adopt to try to achieve their preferred outcomes. Or each may try to be nice, leading to the unfortunate situation where Harry goes to Local Latte because he wants to please Sally, only to find that she has chosen to please him and gone to Starbucks, like the couple choosing Christmas presents for each other in O. Henry's short story titled *The Gift of the Magi*. Alternatively, if the game is repeated, successful coordination may be negotiated and maintained as an equilibrium. For example, the two can arrange to alternate between the cafés. In Chapter 11, we examine such tacit cooperation in repeated games in the context of a prisoners' dilemma.

Our final example in this section is a slightly different kind of coordination game. In this game, the players want to avoid, not choose, actions with the same labels. Further, the consequences of one kind of coordination failure are far more drastic than those of the other kind.

The story comes from a game that was supposedly played by American teenagers in the 1950s. Two teenagers take their cars to opposite ends of Main Street, Middle-of-Nowhere, USA, at midnight and start to drive toward each other. The one who swerves to prevent a collision is the "chicken," and the one who keeps going straight is the winner. If both maintain a straight course, there is a collision in which both cars are damaged and both players injured.[8]

The payoffs for **chicken** depend on how negatively one rates the "bad" outcome—being hurt and damaging your car in this case—against being labeled chicken. As long as words hurt less than crunching metal, a reasonable payoff

[8]A slight variant was made famous by the 1955 James Dean movie *Rebel Without a Cause*. There, two players drove their cars in parallel, very fast, toward a cliff. The first to jump out of his car before it went over the cliff was the chicken. The other, if he left too late, risked going over the cliff in his car to his death. The characters in the film referred to this as a "chicky game." In the mid-1960s, the British philosopher Bertrand Russell and other peace activists used this game as an analogy for the nuclear arms race between the United States and the U.S.S.R., and the game theorist Anatole Rapoport gave a formal game-theoretic statement. Other game theorists have chosen to interpret the arms race as a prisoners' dilemma or as an assurance game. For a review and interesting discussion, see Barry O'Neill, "Game Theory Models of Peace and War," in *The Handbook of Game Theory*, vol. 2, ed. Robert J. Aumann and Sergiu Hart (Amsterdam: North Holland, 1994), pp. 995–1053.

		DEAN	
		Swerve (Chicken)	Straight (Tough)
JAMES	Swerve (Chicken)	0, 0	−1, 1
	Straight (Tough)	1, −1	−2, −2

FIGURE 4.14 Chicken

table for the 1950s version of chicken is found in Figure 4.14. Each player most prefers to win, having the other be chicken, and each least prefers the crash of the two cars. In between these two extremes, it is better to have your rival be chicken with you (to save face) than to be chicken by yourself.

This story has four essential features that define any game of chicken. First, each player has one strategy that is the "tough" strategy and one that is the "weak" strategy. Second, there are two pure-strategy Nash equilibria. These are the outcomes in which exactly one of the players is chicken, or weak. Third, each player strictly prefers that equilibrium in which the other player chooses chicken, or weak. Fourth, the payoffs when both players are tough are very bad for both players. In games such as this one, the real game becomes a test of how to achieve one's preferred equilibrium.

We are now back in a situation similar to that discussed for the battle-of-the-sexes game. One expects most real-life chicken games to be even worse as battles than most battles of the sexes—the benefit of winning is larger, as is the cost of the crash, and so all the problems of conflict of interest and asymmetry between the players are aggravated. Each player will want to try to influence the outcome. It may be the case that one player will try to create an aura of toughness that everyone recognizes so as to intimidate all rivals.[9] Another possibility is to come up with some other way to convince your rival that you will not be chicken, by making a visible and irreversible commitment to going straight. (In Chapter 10, we consider just how to make such commitment moves.) In addition, both players also want to try to prevent the bad (crash) outcome if at all possible.

As with the battle of the sexes, if the game is repeated, tacit coordination is a better route to a solution. That is, if the teenagers played the game every Saturday night at midnight, they would have the benefit of knowing that the game

[9]Why would a potential rival play chicken against someone with a reputation for never giving in? The problem is that participation in chicken, as in lawsuits, is not really voluntary. Put another way, choosing whether to play chicken is itself a game of chicken. As Thomas Schelling says, "If you are publicly invited to play chicken and say you would rather not, then you have just played [and lost]" (*Arms and Influence*, Yale University Press, 1965, p. 118).

had both a history and a future when deciding their equilibrium strategies. In such a situation, they might logically choose to alternate between the two equilibria, taking turns being the winner every other week. (But, if the others found out about this deal, both players would lose face.)

There is one final point, arising from these coordination games, that must be addressed. The concept of Nash equilibrium requires each player to have the correct belief about the other's choice of strategy. When we look for Nash equilibria in pure strategies, the concept requires each to be confident about the other's choice. But our analysis of coordination games shows that thinking about the other's choice in such games is fraught with uncertainty. How can we incorporate such uncertainty in our analysis? In Chapter 5, we introduce the concept of a mixed strategy, where actual choices are made randomly among the available actions. This approach generalizes the concept of Nash equilibrium to situations where the players may be unsure about each other's actions.

8 NO EQUILIBRIUM IN PURE STRATEGIES

Each of the games considered so far has had at least one Nash equilibrium in pure strategies. Some of these games, such as those in Section 7, had more than one equilibrium, while games in earlier sections had exactly one. Unfortunately, not all games that we come across in the study of strategy and game theory will have such easily definable outcomes in which players always choose one particular action as an equilibrium strategy. In this section, we look at games in which there is not even one pure-strategy Nash equilibrium—games in which none of the players would consistently choose one strategy as that player's equilibrium action.

A simple example of a game with no equilibrium in pure strategies is that of a single point in a tennis match. Imagine a match between the two all-time best women players—Martina Navratilova and Chris Evert.[10] Navratilova at the net has just volleyed a ball to Evert on the baseline, and Evert is about to attempt a passing shot. She can try to send the ball either down the line (DL; a hard, straight shot) or crosscourt (CC; a softer, diagonal shot). Navratilova must

[10]For those among you who remember only the latest phenom who shines for a couple of years and then burns out, here are some amazing facts about these two, who were at the top levels of the game for almost two decades and ran a memorable rivalry all that time. Navratilova was a left-handed serve-and-volley player. In grand slam tournaments, she won 18 singles titles, 31 doubles, and 7 mixed doubles. In all tournaments, she won 167, a record. Evert, a right-handed baseliner, had a record win–loss percentage (90% wins) in her career and 150 titles, of which 18 were for singles in grand slam tournaments. She probably invented (and certainly popularized) the two-handed backhand that is now so common. From 1973 to 1988, the two played each other 80 times, and Navratilova ended up with a slight edge, 43–37.

likewise prepare to cover one side or the other. Each player is aware that she must not give any indication of her planned action to her opponent, knowing that such information will be used against her. Navratilova would move to cover the side to which Evert is planning to hit or Evert would hit to the side that Navratilova is not planning to cover. Both must act in a fraction of a second, and both are equally good at concealing their intentions until the last possible moment; therefore their actions are effectively simultaneous, and we can analyze the point as a two-player simultaneous-move game.

Payoffs in this tennis-point game are given by the fraction of times a player wins the point in any particular combination of passing shot and covering play. Given that a down-the-line passing shot is stronger than a crosscourt shot and that Evert is more likely to win the point when Navratilova moves to cover the wrong side of the court, we can work out a reasonable set of payoffs. Suppose Evert is successful with a down-the-line passing shot 80% of the time if Navratilova covers crosscourt; she is successful with the down-the-line shot only 50% of the time if Navratilova covers down the line. Similarly, Evert is successful with her crosscourt passing shot 90% of the time if Navratilova covers down the line. This success rate is higher than when Navratilova covers crosscourt, in which case Evert wins only 20% of the time.

Clearly, the fraction of times that Navratilova wins this tennis point is just the difference between 100% and the fraction of time that Evert wins. Thus the game is zero-sum (more precisely, constant-sum, because the two payoffs sum to 100), and we can represent all the necessary information in the payoff table with just the payoff to Evert in each cell. Figure 4.15 shows the payoff table and the fraction of time that Evert wins the point against Navratilova in each of the four possible combinations of their strategy choices.

The rules for solving simultaneous-move games tell us to look first for dominant or dominated strategies and then to try minimax (in that this is a zero-sum game) or use cell-by-cell inspection to find a Nash equilibrium. It is a useful exercise to verify that no dominant strategies exist here. Going on to cell-by-cell inspection, we start with the choice of DL for both players. From that outcome, Evert can improve her success from 50% to 90% by choosing CC instead. But then Navratilova can hold Evert down to 20% by choosing CC. After this, Evert

		NAVRATILOVA	
		DL	CC
EVERT	DL	50	80
	CC	90	20

FIGURE 4.15 No Equilibrium in Pure Strategies

can raise her success again to 80% by making her shot DL, and Navratilova in turn can do better with DL. In every cell, one player always wants to change her play, and we cycle through the table endlessly without finding an equilibrium.

An important message is contained in the absence of a Nash equilibrium in this game and similar ones. What is important in games of this type is not what players should do, but what players should *not* do. In particular, each player should neither always nor systematically pick the same shot when faced with this situation. If either player engages in any determinate behavior of that type, the other can take advantage of it. (So, if Evert consistently went crosscourt with her passing shot, Navratilova would learn to cover crosscourt every time and would thereby reduce Evert's chances of success with her crosscourt shot.) The most reasonable thing for players to do here is to act somewhat unsystematically, hoping for the element of surprise in defeating their opponents. An unsystematic approach entails choosing each strategy part of the time. (Evert should be using her weaker shot with enough frequency to guarantee that Navratilova cannot predict which shot will come her way. She should not, however, use the two shots in any set pattern, because that, too, would cause her to lose the element of surprise.) This approach, in which players randomize their actions, is known as mixing strategies and is the focus of Chapters 7 and 8. The game illustrated in Figure 4.15 may not have an equilibrium in pure strategies, but it can still be solved by looking for an equilibrium in mixed strategies, as we do in Chapter 7, Section 1.

SUMMARY

In simultaneous-move games, players make their strategy choices without knowledge of the choices being made by other players. Such games are illustrated by *game tables,* where cells show payoffs to each player and the dimensionality of the table equals the number of players. Two-person *zero-sum games* may be illustrated in shorthand with only one player's payoff in each cell of the game table.

Nash equilibrium is the solution concept used to solve simultaneous-move games; such an equilibrium consists of a set of strategies, one for each player, such that each player has chosen her best response to the other's choice. Nash equilibrium can also be defined as a set of strategies such that each player has correct *beliefs* about the others' strategies and strategies are best for each player given beliefs about other's strategies. Nash equilibria can be found by using *cell-by-cell inspection,* through a search for *dominant strategies,* by *successive elimination of dominated strategies,* or with *best-response analysis.* Zero-sum games can also be solved by using the *minimax method.*

There are many classes of simultaneous games. *Prisoners' dilemma* games appear in many contexts. Coordination games, such as *assurance, chicken,* and *battle of the sexes,* have multiple equilibria, and the solution of such games requires players to achieve coordination by some means. If a game has no equilibrium in *pure strategies*, we must look for an equilibrium in *mixed strategies*, the analysis of which is presented in Chapters 7 and 8.

KEY TERMS

assurance game (107)
battle of the sexes (108)
belief (89)
best response (87)
best-response analysis (99)
cell-by-cell inspection (89)
chicken (109)
constant-sum game (85)
convergence of expectations (107)
coordination game (105)
dominance solvable (96)
dominant strategy (92)
dominated strategy (92)
enumeration (89)
focal point (106)
game matrix (84)
game table (84)
iterated elimination of dominated strategies (96)
maximin (100)
minimax (100)
minimax method (99)
mixed strategy (84)
Nash equilibrium (87)
normal form (84)
payoff table (84)
prisoners' dilemma (90)
pure coordination game (105)
pure strategy (84)
strategic form (84)
successive elimination of dominated strategies (96)
zero-sum game (85)

EXERCISES

1. "If a player has a dominant strategy in a simultaneous-move game, then she is sure to get her best possible outcome." True or false? Explain and give an example of a game that illustrates your answer.

2. Find all Nash equilibria in pure strategies for the zero-sum games in the following tables by checking for dominant strategies and using iterated dominance. Verify your answers by using the minimax method.

(a)

		COLUMN	
		Left	Right
ROW	Up	1	4
	Down	2	3

(b)

		COLUMN	
		Left	Right
ROW	Up	1	2
	Down	4	3

(c)

		COLUMN		
		Left	Middle	Right
ROW	Up	5	3	1
	Straight	6	2	1
	Down	1	0	0

(d)

		COLUMN		
		Left	Middle	Right
ROW	Up	5	3	2
	Straight	6	4	3
	Down	1	6	0

3. Find all Nash equilibria in pure strategies in the following non-zero-sum games. Describe the steps that you used in finding the equilibria.

(a)

		COLUMN	
		Left	Right
ROW	Up	2, 4	1, 0
	Down	6, 5	4, 2

(b)

		COLUMN	
		Left	Right
ROW	Up	1, 1	0, 1
	Down	1, 0	1, 1

(c)

		COLUMN		
		Left	Middle	Right
ROW	Up	0, 1	9, 0	2, 3
	Straight	5, 9	7, 3	1, 7
	Down	7, 5	10, 10	3, 5

(d)

		COLUMN		
		West	Center	East
ROW	North	2, 3	8, 2	10, 6
	Up	3, 0	4, 5	6, 4
	Down	5, 4	6, 1	2, 5
	South	4, 5	2, 3	5, 2

4. Check the following game for dominance solvability. Find a Nash equilibrium.

		COLUMN		
		Left	Middle	Right
ROW	Up	4, 3	2, 7	0, 4
	Down	5, 5	5, −1	−4, −2

5. Find all of the pure-strategy Nash equilibria for the following game. Describe the process that you used to find the equilibria. Use this game to ex-

plain why it is important to describe an equilibrium by using the strategies employed by the players, not merely by the payoffs received in equilibrium.

		COLUMN		
		Left	Center	Right
ROW	Up	1, 2	2, 1	1, 0
	Level	0, 5	1, 2	7, 4
	Down	−1, 1	3, 0	5, 2

6. The game known as the *battle of the Bismarck Sea* (named for that part of the southwestern Pacific Ocean separating the Bismarck Archipelago from Papua-New Guinea) summarizes a well-known game actually played in a naval engagement between the United States and Japan in World War II. In 1943, a Japanese admiral was ordered to move a convoy of ships to New Guinea; he had to choose between a rainy northern route and a sunnier southern route, both of which required 3 days sailing time. The Americans knew that the convoy would sail and wanted to send bombers after it, but they did not know which route it would take. The Americans had to send reconnaissance planes to scout for the convoy, but they had only enough reconnaissance planes to explore one route at a time. Both the Japanese and the Americans had to make their decisions with no knowledge of the plans being made by the other side.

   If the convoy was on the route explored by the Americans first, they could send bombers right away; if not, they lost a day of bombing. Poor weather on the northern route would also hamper bombing. If the Americans explored the northern route and found the Japanese right away, they could expect only 2 (of 3) good bombing days; if they explored the northern route and found that the Japanese had gone south, they could also expect 2 days of bombing. If the Americans chose to explore the southern route first, they could expect 3 full days of bombing if they found the Japanese right away but only 1 day of bombing if they found that the Japanese had gone north.

   (a) Illustrate this game in a game table.

   (b) Identify any dominant strategies in the game and solve for the Nash equilibrium.

7. An old lady is looking for help crossing the street. Only one person is needed to help her; more are okay but no better than one. You and I are the two people in the vicinity who can help; we have to choose simultaneously

whether to do so. Each of us will get pleasure worth a 3 from her success (no matter who helps her). But each one who goes to help will bear a cost of 1, this being the value of our time taken up in helping. Set this up as a game. Write the payoff table, and find all pure-strategy Nash equilibria.

8. Two players, Jack and Jill, are put in separate rooms. Then each is told the rules of the game. Each is to pick one of six letters; G, K, L, Q, R, or W. If the two happen to choose the same letter, both get prizes as follows:

Letter	G	K	L	Q	R	W
Jack's prize	3	2	6	3	4	5
Jill's prize	6	5	4	3	2	1

If they choose different letters, each gets 0. This whole schedule is revealed to both players, and both are told that both know the schedules, and so on.

(a) Draw the table for this game. What are the Nash equilibria in pure strategies?

(b) Can one of the equilibria be a focal point? Which one? Why?

9. Suppose two players, A and B, select from three different numbers, 1, 2, and 3. Both players get dollar prizes if their choices match, as indicated in the following table.

		B		
		1	2	3
A	1	10, 10	0, 0	0, 0
	2	0, 0	15, 15	0, 0
	3	0, 0	0, 0	15, 15

(a) What are the Nash equilibria of this game? Which, if any, is likely to emerge as the (focal) outcome? Explain.

(b) Consider a slightly changed game in which the choices are again just numbers but the two cells with (15, 15) in the table become (25, 25). What is the expected (average) payoff to each player if each flips a coin to decide whether to play 2 or 3? Is this better than focusing on both choosing 1 as a focal equilibrium? How should you account for the risk that A might do one thing while B does the other?

10. In Chapter 3, the three gardeners, Emily, Nina, and Talia, play a sequential version of the street-garden game in which there are four distinguishable outcomes (rather than the six different outcomes specified in the example of Section 4.6). For each player, the four outcomes are:
    (i) player does not contribute, both of the others do (pleasant garden, saves cost of own contribution)
    (ii) player contributes, and one or both of the others do (pleasant garden, incurs cost of contribution)
    (iii) player does not contribute, only one or neither of the others does (sparse garden, saves cost of own contribution)
    (iv) player contributes, but neither of the others does (sparse garden, incurs cost of own contribution)

    Of them, outcome i is the best (payoff 4) and outcome iv is the worst (payoff 1). If each player regards a pleasant garden more highly than her own contribution, then outcome ii gets payoff 3 and outcome iii gets payoff 2.
    (a) Suppose that the gardeners play this game simultaneously, deciding whether to contribute to the street garden without knowing what choices the others will make. Draw the three-player game table for this version of the game.
    (b) Find all of the Nash equilibria in this game.
    (c) How might this simultaneous version of the street-garden game be played out in reality?

11. Consider a game in which there is a prize worth \$30. There are three contestants, A, B, and C. Each can buy a ticket worth \$15 or \$30 or not buy a ticket at all. They make these choices simultaneously and independently. Then, knowing the ticket-purchase decisions, the game organizer awards the prize. If no one has bought a ticket, the prize is not awarded. Otherwise, the prize is awarded to the buyer of the highest-cost ticket if there is only one such player or is split equally between two or three if there are ties among the highest-cost ticket buyers. Show this game in strategic form. Find all pure-strategy Nash equilibria.

12. In the film *A Beautiful Mind,* John Nash and three of his graduate school colleagues find themselves faced with a dilemma while at a bar. There are four brunettes and a single blonde available for them to approach. Each young man wants to approach and win the attention of one of the young women. The payoff to each of winning the blonde is 10; the payoff of winning a brunette is 5; the payoff from ending up with no girl is 0. The catch is that, if two or more young men go for the blonde, she rejects all of them and then the brunettes also reject the men because they don't want to be second

choice. Thus, each player gets a payoff of 10 only if he is the sole suitor for the blonde.

(a) First consider a simpler situation where there are only two young men, instead of four. (There are two brunettes and one blonde, but these women merely respond in the manner just described and are not active players in the game.) Show the playoff table for the game, and find all of the pure-strategy Nash equilibria of the game.

(b) Now show the (three-dimensional) table for the case in which there are three young men (and three brunettes and one blonde who are not active players). Again, find all of the Nash equilibria of the game.

(c) Use your results to parts **a** and **b** to generalize your analysis to the case in which there are four young men and then to the case in which there is some arbitrary number, n, of young men. Do not attempt to write down an n-dimensional payoff table. Merely find the payoff to one player when k of the others choose Blonde and $(n-k-1)$ choose Brunette, for $k = 0, 1, \ldots (n-1)$. Can the outcome specified in the movie as the Nash equilibrium of the game—that all of the young men choose to go for brunettes—ever be a true Nash equilibrium of the game?

Appendix: Some General Definitions

We introduced the concepts of dominance and Nash equilibrium in this chapter by using numerical examples. Although this approach may suffice for many of you, some will benefit from knowing more precise definitions. For this purpose,

		COLUMN		
		C1	C2	C3
ROW	R1	R11, C11	R12, C12	R13, C13
	R2	R21, C21	R22, C22	R23, C23
	R3	R31, C31	R32, C32	R33, C33
	R4	R41, C41	R42, C42	R43, C43

FIGURE 4A.1 A General Game

consider a general game with players Row and Column. Suppose Row has four available strategies, labeled R1, R2, R3, and R4, and Column has three, labeled C1, C2, and C3. This set of strategies will suffice to convey the necessary ideas; more strategies provide no additional concepts.

The payoff table for this Row versus Column game is shown in Figure 4A.1. In this general setting, we do not use specific numerical payoffs; instead we use general algebraic entities. The algebraic symbol for each payoff consists of one letter followed by two numbers. The letter indicates the player whose payoff it is, the first of the two numbers indicates Row's strategy and the second number indicates Column's strategy. For example, in the shaded cell, which results when Row chooses R3 and Column chooses C2, Row's payoff is R32 and Column's payoff is C32.

Suppose that this combination of strategies is a Nash equilibrium. What does that imply about the payoffs? R3 should be Row's best choice, given her correct belief that Column is choosing C2. Therefore R32 should be the best among the payoffs that Row can get by choosing various strategies while Column is choosing C2. That is,

$$R32 \geq R12, \quad R32 \geq R22, \quad \text{and} \quad R32 \geq R42.$$

Note that, in this definition, we allow ties; therefore we use the weak inequality $\geq$ rather than the strict inequality $>$. Similarly, for C2 to be Column's best choice, given her correct belief that Row is choosing R3, we need

$$C32 \geq C31, \quad \text{and} \quad C32 \geq C33.$$

When we test a particular cell to see if it is a Nash equilibrium, we should test all such inequalities. That is, Row's payoff in this cell should be the largest (in the weak sense) of all her payoffs in this column of the game table, and Column's payoff in this cell should be the largest (again in the weak sense) of all her payoffs in this row of the game table.

We can use the same table to define dominance. Suppose Column's C2 dominates C1. Then, no matter what Row chooses, Column gets a higher payoff by playing C2 than by playing C1. Now it becomes necessary to distinguish between two cases. We say that C2 *strictly dominates* C1 if in each of these cases the payoff from C2 is strictly higher than that from C1, that is,

$$C12 > C11, \quad C22 > C21, \quad C32 > C31, \quad \text{and} \quad C42 > C41.$$

We say that C2 *weakly dominates* C1 if in each of these cases the payoff from C2 is at least as high as that from C1. In at least one case, it must be strictly higher; otherwise, C2 and C1 would be completely equivalent. And, in at least one case, it must be equal: otherwise, there would be strict dominance. Thus we require

$$C12 \geq C11, \quad C22 \geq C21, \quad C32 \geq C31,$$
$$\text{and} \quad C42 \geq C41, \quad \text{with at least one} > \text{and at least one} =$$

For example, in the game of Figure 4.1, for Row, the strategy Bottom strictly dominates High, (because 5 > 4, 4 > 3, and 9 > 6). But, in Figure 4.3, after Column's Left has been eliminated, Low only weakly dominates Bottom (because 5 = 5 and 12 > 9).

5

Simultaneous-Move Games with Pure Strategies II: Continuous Strategies and III: Discussion and Evidence

THE DISCUSSION OF SIMULTANEOUS-MOVE GAMES in Chapter 4 focused on games in which each player had a discrete set of actions from which to choose. Discrete strategy games of this type include sporting contests in which a small number of well-defined plays can be used in a given situation—soccer penalty kicks, in which the kicker can choose to go high or low, to a corner or the center, for example. Other examples include coordination and prisoners' dilemma games in which players have only two or three available strategies. Such games are amenable to analysis with the use of a game table, at least for situations with a reasonable number of players and available actions.

Many simultaneous-move games differ from those considered so far; they entail players choosing strategies from a wide range of possibilities. Games in which manufacturers choose prices for their products, philanthropists choose charitable contribution amounts, or contractors choose project bid levels are examples in which players have a virtually infinite set of choices. Technically, prices and other dollar amounts do have a minimum unit, such as a cent, and so there is actually only a finite and discrete set of price strategies. But in practice the unit is so small that it is simpler and better to regard such choices as continuously variable real numbers. When players have such a large range of actions available, game tables become virtually useless as analytical tools; they become too unwieldy to be of practical use. For these games we need a different solution technique. We present the analytical tools for handling such **continuous strategy** games in the first part of this chapter.

This chapter also takes up some broader matters relevant to behavior in simultaneous-move games and to the concept of Nash equilibrium. We review the empirical evidence on Nash equilibrium play that has been collected both from the laboratory and from real-life situations. We also present some theoretical criticisms of the Nash equilibrium concept and rebuttals of these criticisms. You will see that game-theoretic predictions are often a reasonable starting point for understanding actual behavior, with some caveats, such as the level of player.

1 PURE STRATEGIES THAT ARE CONTINUOUS VARIABLES

In Chapter 4 we developed the method of best-response analysis for finding all pure-strategy Nash equilibria of simultaneous-move games. Now we extend that method to games in which each player has available a continuous range of choices—for example, firms setting prices of their products. To calculate best responses in this type of game, we find, for each possible value of one firm's price, the value of the other firm's price that is best for it (maximizes its payoff). The continuity of the sets of strategies allows us to show these best responses as curves in a graph, with each player's price (or any other continuous strategy) on one of the axes. In such an illustration, the Nash equilibrium of the game occurs where the two curves meet. We develop this idea and technique by using two stories.

A. Price Competition

Our first story is set in a small town, Yuppie Haven, that has two restaurants, Xavier's Tapas Bar and Yvonne's Bistro. To keep the story simple, we suppose that each place has a set menu. Xavier and Yvonne have to set the prices of their respective menus. Prices are their strategic choices in the game of competing with each other; each bistro's goal is to set price to maximize profit, the payoff in this game. We suppose that they must get their menus printed separately without knowing the other's price; so the game has simultaneous moves.[1] Because prices can take any value within an (almost) infinite range, we start with general or algebraic symbols for them. We then find **best-response rules** that we use to solve the game and to determine equilibrium prices. Let us call Xavier's price P_x and Yvonne's price P_y.

[1]In reality, the competition extends over time; so each can observe the other's past choices. This repetition of the game introduces new considerations, which we cover in Chapter 11.

In setting its price, each restaurant has to calculate the consequences for its profit. To keep things relatively simple, we put the two restaurants in a very symmetric relationship, but readers with a little more mathematical skill can do a similar analysis by using much more general numbers or even algebraic symbols. Suppose the cost of serving each customer is \$8 for each restaurateur. Suppose further that experience or market surveys have shown that, when Xavier's price is P_x and Yvonne's price is P_y, the number of their respective customers, respectively Q_x and Q_y (measured in hundreds per month), are given by the equations[2]

$$Q_x = 44 - 2P_x + P_y,$$
$$Q_y = 44 - 2P_y + P_x.$$

The key idea in these equations is that, if one restaurant raises its price by \$1 (say, Yvonne increases P_y by \$1), its sales will go down by 200 per month (Q_y changes by -2) and those of the other restaurant will go up by 100 per month (Q_x changes by 1). Presumably, 100 of Yvonne's customers switch to Xavier's and another 100 stay at home.

Xavier's profit per week (in hundreds of dollars per week), call it B_x (the B stands for "bottom line"), is given by the product of the net revenue per customer (price less cost or $P_x - 8$) and the number of customers served:

$$B_x = (P_x - 8)\ Q_x = (P_x - 8)\ (44 - 2\ P_x + P_y)$$

Xavier sets his price P_x to maximize this payoff. Doing so for each possible level of Yvonne's price P_y gives us Xavier's best-response rule; we can then graph it.

There are two ways to find the P_x that maximizes B_x for each P_y. First, we can multiply out and rearrange the terms on the right-hand side of the preceding expression for B_x:

$$\begin{aligned} B_x &= -\ 8(44 + P_y) + (16 + 44 + P_y)P_x - 2(P_x)^2 \\ &= -\ 8(44 + P_y) + 2(15 + 0.25P_y)^2 - 2(15 + 0.25P_y - P_x)^2 \end{aligned}$$

In the formula on the second line, only the last term includes P_x. The term appears with a negative sign, so what appears after the negative sign should be made as small as possible in order to make the whole expression (Xavier's profit) as large as possible. But that last term is a square, and therefore its smallest value is zero, which is achieved when

$$P_x = 15 + 0.25P_y.$$

[2]Readers who know some economics will recognize that the equations linking quantities to prices are demand functions for the two products X and Y. The quantity demanded of each product is decreasing in its own price (demands are downward sloping) and increasing in the price of the other product (the two are substitutes).

This equation solves for the value of P_x that maximizes Xavier's profit, given a particular value of Yvonne's price, P_y. In other words, it is exactly what we want, the rule for Xavier's best response.

The other method to derive a best-response rule is to use calculus. Use the first line of the preceding rearrangement of the expression for B_x, and take its derivative with respect to P_x (holding the other restaurant's price, P_y, fixed):

$$\frac{dB_x}{dP_x} = (60 + P_y) - 4P_x$$

The first-order condition for P_x to maximize B_x is that this derivative should be zero. Setting it equal to zero and solving for P_x gives the same equation as the preceding one for the best-response rule. (The second-order condition is $d^2B_x / dP_x^2 < 0$, which is satisfied because the second-order derivative is just -4.)

Yvonne's best-response rule can be found similarly by using either method. Because the costs and sales of the two restaurants are entirely symmetric, the equation is obviously going to be

$$P_y = 15 + 0.25P_x.$$

Both rules are used in the same way to develop best-response graphs. If Xavier sets a price of 16, for example, then Yvonne plugs this value into her best-response rule to find $P_y = 15 + 0.25(16) = 19$; similarly, Xavier's best response to Yvonne's $P_y = 16$ is $P_x = 19$, and each restaurant's best response to the other's price of 4 is 16, that to 8 is 17, and so on.

Figure 5.1 shows the graphs of these two best-response relations. Owing to the special features of our example—namely, the linear relation between quan-

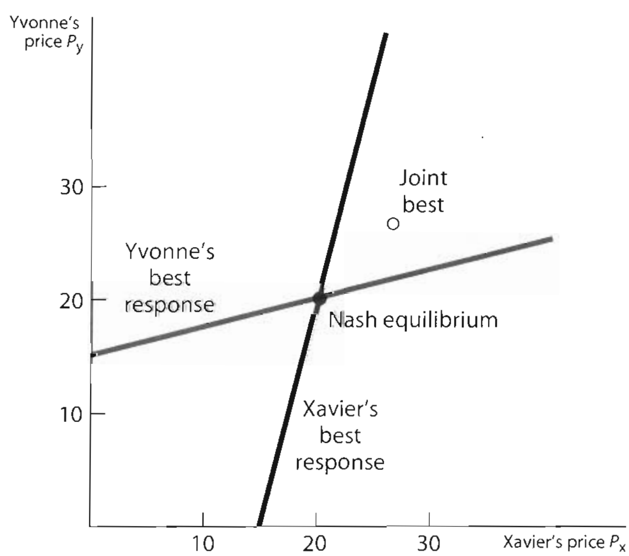


FIGURE 5.1 Best-Response Curves and Equilibrium in the Restaurant Pricing Game

tity sold and prices charged, and the constant cost of producing each meal—each of the two **best-response curves** is a straight line. For other specifications of demands and costs, the curves can be other than straight, but the method of obtaining them is the same: namely, first holding one restaurant's price (say, P_y) fixed and finding the value of the other's price (say, P_x) that maximizes the second restaurant's profit, and then the other way around.

The point of intersection of the two best-response curves is the Nash equilibrium of the pricing game between the two restaurants. That point represents the set of prices, one for each firm, that are best responses to each other. The specific values for each restaurant's pricing strategy in equilibrium can be found algebraically by solving the two best-response rules jointly for P_x and P_y. We deliberately chose our example to make the equations linear, and the solution is easy. In this case, we simply substitute the expression for P_x into the expression for P_y to find

$$P_y = 15 + 0.25P_x = 15 + 0.25(15 + 0.25P_y) = 18.75 + 0.0625\,P_y.$$

This last equation simplifies to $P_y = 20$. Given the symmetry of the problem, it is simple to determine that $P_x = 20$ also.[3] Thus, in equilibrium, each restaurant charges \$20 for its menu and makes a profit of \$12 on each of the 2,400 customers [2,400 = (44−2 × 20 + 20) hundred] that it serves each month, for a total profit of \$28,800 per month.

Our main purpose here is to illustrate how the Nash equilibrium can be found in a game where the strategies are continuous variables, such as prices. But it is interesting to take a further look into this example and explain some economics of firms' pricing strategies and profits.

Begin by observing that each best-response curve slopes upward. Specifically, when one restaurant raises its price by \$1, the other's best response is to raise its own price by 0.25, or 25 cents. When one restaurant raises its price, some of its customers switch to the other restaurant, and its rival can then best profit from them by raising its price part of the way. Thus a restaurant that raises its price is helping to increase the other's profit. In Nash equilibrium, where each restaurant chooses its price independently and out of concern for its own profit, it does not take into account this benefit that it conveys to the other. Could they get together and cooperatively agree to raise their prices, thereby raising both profits? Yes. Suppose the two restaurant charged \$24 each. Then each would make a profit of \$16 on each of the 2,000 customers [2,000 = (44 − 2 × 24 + 24) hundred] that it would serve each month, for a total profit of \$32,000.

[3]Without this symmetry, the two best-response equations will be different but, given our other specifications, still linear. So it is not much harder to solve the nonsymmetric case.

This pricing game is exactly like the prisoners' dilemma game presented in Chapter 4, but now the strategies are continuous variables. Each of the husband–wife couple in the story in Chapter 4 was tempted to cheat the other and confess to the police; but, when they both did so, both ended up with longer prison sentences (worse outcomes). In the same way, the more profitable price of \$24 is not a Nash equilibrium. The separate calculations of the two restaurants will lead them to undercut such a price. Suppose that Yvonne somehow starts by charging \$24. Using the best-response formula, we see that Xavier will then charge $15 + 0.25 \times 24 = 21$. Then Yvonne will come back with her best response to that: $15 + 0.25 \times 21 = 20.25$. Continuing this process, the prices of both will converge toward the Nash equilibrium price of \$20.

But what price is jointly best for the two restaurants? Given the symmetry, suppose both charge the same price P. Then the profit of each will be

$$B_x = B_y = (P - 8)(44 - 2P + P) = (P - 8)(44 - P).$$

The two can choose P to maximize this expression. Using either of the preceding methods, we can easily see that the solution is $P = 26$. The resulting profit for each restaurant is \$32,400 per month.

In the jargon of economics, such collusion to raise prices to the jointly optimal level is called a *cartel*. The high prices hurt consumers, and regulatory agencies of the U.S. government often try to prevent the formation of cartels and to make firms compete with one another. Explicit collusion over price is illegal, but it may be possible to maintain tacit collusion in a repeated prisoners' dilemma; we examine such repeated games in Chapter 11.[4]

Collusion need not always lead to higher prices. In the preceding example, if one restaurant lowers its price, its sales increase, in part because it draws some customers away from its rival because the products (meals) of the two restaurants are *substitutes* for each other. In other contexts, two firms may be selling products that are *complements* to each other—for example, hardware and software. In that case, if one firm lowers its price, the sales of both firms increase. In a Nash equilibrium, where the firms act independently, they do not take into account the benefit that a lower price of each would convey on the other. Therefore they keep prices higher than they would if they were able to coordinate their actions. Allowing them to cooperate would lead to lower prices and thus be beneficial to the consumers as well.

[4]Firms do try to achieve explicit collusion when they think they can get away with it. An entertaining and instructive story of one such episode is in *The Informant*, by Kurt Eichenwald (New York: Broadway Books, 2000).

B. Political Campaign Advertising

Our second example is one drawn from politics. It requires just a little more mathematics than we normally use, but we explain the intuition behind the calculations in words and with a graph.

Consider an election contested by two parties or candidates. Each is trying to win votes away from the other by advertising—either positive ads that highlight the good things about oneself or negative, attack ads that emphasize the bad things about the opponent. To keep matters simple, suppose the voters start entirely ignorant and unconcerned and are moved solely by the ads. (Many people would claim that this is a pretty accurate description of U.S. politics, but more advanced analyses in political science do recognize that there are informed and strategic voters. We address the behavior of such voters in detail in Chapter 15.) Even more simply, suppose the vote share of a party equals its share of the total campaign advertising that is done. Call the parties or candidates L and R; when L spends $\$x$ million on advertising and R spends $\$y$ million, L will get a share $x/(x + y)$ of the votes and R will get $y/(x + y)$. Once again, readers who get interested in this application can find more general treatments in specialized political science writings.

Raising money to pay for these ads includes a cost: money to send letters and make phone calls; time and effort of the candidates, party leaders, and activists; the future political payoff to large contributors; and possible future political costs if these payoffs are exposed and lead to scandals. For simplicity of analysis, let us suppose all these costs are proportional to the direct campaign expenditures x and y. Specifically, let us suppose that party L's payoff is measured by its vote share minus its advertising expenditure, $x/(x + y) - x$. Similarly party R's payoff is $y/(x + y) - y$.

Now we can find the best responses. Because we cannot do so without calculus, we derive the formula mathematically and then explain its general meaning intuitively, in words. For a given strategy x of party L, party R chooses y to maximize its payoff. The calculus first-order condition is found by holding x fixed and setting the derivative of $y/(x + y) - y$ with respect to y equal to zero. It is $x/(x + y)^2 - 1 = 0$, or $y = \sqrt{x} - x$. Figure 5.2 shows its graph and that of the analogous best-response function of party L—namely, $x = \sqrt{y} - y$.

Look at the best-response curve of party R. As the value of party L's x increases, party R's y increases for a while and then decreases. If the other party is advertising very little, then one's own ads have a high reward in the form of votes, and it pays to respond to a small increase in the other's expenditures by spending more oneself to compete harder. But, if the other party already has a massive expenditure, then one's own ads get only a small return in relation to their cost, so it is better to respond to the other's increase by scaling back.

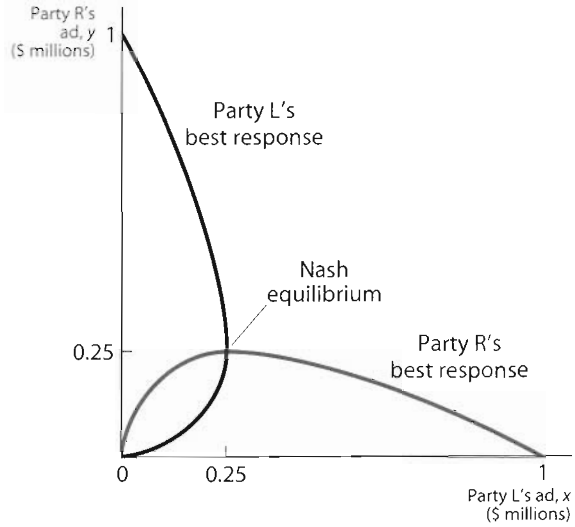


FIGURE 5.2 Best Responses and Nash Equilibrium in the Campaign Advertising Game

As it happens, the two parties' best-response curves intersect at their peak points. Again, some algebraic manipulation of the equations for the two curves yields us exact values for the equilibrium values of x and y. You should verify that here x and y are each equal to $1/4$, or \$250,000. (This is presumably a little local election.)

As in the pricing game, we have a prisoners' dilemma. If both parties cut back on their ads in equal proportions, their vote shares would be entirely unaffected, but both would save on their expenditures and so both would have a larger payoff. Unlike a producers' cartel for substitute products that keeps prices high and hurts consumers, but like a producers' cartel for complements that leads to lower prices, a politicians' cartel to advertise less would probably benefit voters and society. We could all benefit from finding ways to resolve this particular prisoners' dilemma. In fact, Congress has been trying to do just that for several years and has imposed some partial curbs, but political competition seems too fierce to permit a full or lasting resolution.

What if the parties are not symmetrically situated? Two kinds of asymmetries can arise. One party (say, R) may be able to advertise at a lower cost, because it has favored access to the media. Or R's advertising dollars may be more effective than L's; for example, L's vote share may be $x/(x + 2y)$, while R's is $2y/(x + 2y)$.

In the first of these cases, R exploits its cheaper access to advertising by choosing a higher level of expenditures y for any given x for party L; that is, R's best-response curve in Figure 5.2 shifts upward. The Nash equilibrium shifts to the northwest along L's unchanged best-response curve. Thus R ends up advertising more, and L less, than before. It is as if the advantaged party uses its muscle and the disadvantaged party gives up to some extent in the face of this adversity.

In the second case, both parties' best-response curves shift in more complex ways. The outcome is that both spend equal amounts, but less than the 1/4 that they spent in the symmetric case. In our example where R's dollars are twice as effective as L's, it turns out that their common expenditure level is 2/9. (Thus the symmetric case is the one of most intense competition.) When R's spending is more effective, it is also true that the best-response curves are asymmetric in such a way that the new Nash equilibrium, rather than being at the peak points of the two best-response curves, is on the downward part of L's best-response curve and on the upward part of R's best-response curve. That is to say, although both parties spend the same dollar amount, the favored party R spends more than the amount that would bring forth the maximum response from party L, and the underdog party L spends less than the amount that would bring forth the maximum response from party R. We include an optional exercise (Exercise 10) in this chapter that lets the mathematically advanced students derive these results.

2 EMPIRICAL EVIDENCE CONCERNING NASH EQUILIBRIUM

In Chapter 3, when we considered empirical evidence on sequential-move games and rollback, we said that the evidence came from observations on games actually played in the world, as well as games deliberately constructed for testing the theory in the laboratory or the classroom. We pointed out the different merits and drawbacks of the two methods. Similar issues arise in securing and interpreting the evidence on simultaneous-move games.

Real-world games are played by experienced players for substantial stakes, who therefore have the knowledge and the incentives to play good strategies. But these situations include many factors beyond those considered in the theory. Therefore, if the data do not bear out the predictions of the theory, we cannot tell whether the theory is wrong or whether some other factor is having an effect that overwhelms the strategic considerations.

Laboratory experiments can control for the other factors and therefore provide cleaner tests. But they bring in inexperienced players and provide them meager time and incentives to learn the game and play it well. Confronted with a new game, most of us would flounder and try things out at random. Thus the first several plays of the game in an experimental setting may represent this learning phase and not the equilibrium that experienced players would learn to play. Some experiments do control for inexperience and learning by discarding several initial plays from their data. But the learning phase may last longer than the one morning or one afternoon that is the typical limit of laboratory sessions.

A. Laboratory and Classroom Experiments

Researchers have conducted numerous laboratory experiments in the past three decades to test how people act when placed in certain interactive strategic situations. In particular, do they play Nash equilibrium strategies? Reviewing this work, Douglas Davis and Charles Holt[5] conclude that, in relatively simple single-move games with a unique Nash equilibrium, that outcome "has considerable drawing power . . . after some repetitions with different partners." But in more complex or repeated situations or when coordination is required because there are multiple Nash equilibria or when the calculations required for finding a Nash equilibrium are more complex, the theory's success is more mixed. We briefly consider the performance of Nash equilibrium in such circumstances.

I. CHOOSING AMONG MULTIPLE EQUILIBRIA When there are multiple equilibria, players generally fail to coordinate unless they have some common cultural background (and this fact is common knowledge among them) that is needed for locating focal points. Thomas Schelling and David Kreps report on several experiments of coordination games.[6] Kreps played the following game between pairs of his students. One student was assigned Boston, and the other was assigned San Francisco. Each was given a list of nine other U.S. cities—Atlanta, Chicago, Dallas, Denver, Houston, Los Angeles, New York, Philadelphia, and Seattle—and asked to choose a subset of these cities. The two chose simultaneously and independently. If their choices divided up the nine cities completely and without any overlap between them, both got a prize. Otherwise, neither got anything. This game has numerous (512) Nash equilibria in pure strategies. But, when both players were Americans or long-time U.S. residents, more than 80% of the time they chose the division geographically; the student assigned Boston chose all the cities east of the Mississippi, and the student assigned San Francisco chose those west of the Mississippi. Such coordination was much less likely when one or both students were non-U.S. residents. In such pairs the choices were sometimes made alphabetically, but even then there was no clear dividing point.

II. REVELATION OF INNATE ALTRUISM OR PUBLIC-SPIRITEDNESS IN EXPERIMENTS One respect in which the behavior of players in some experimental situations does not often conform to the experimenter's predicted Nash equilibrium is that people seem to "err" on the side of niceness or fairness. Thus in prisoners' dilemma games,

[5]Douglas D. Davis and Charles A. Holt, *Experimental Economics* (Princeton: Princeton University Press, 1993).

[6]Thomas Schelling, *The Strategy of Conflict* (New York: Oxford University Press, 1960), pp. 54–58; David Kreps, *A Course in Microeconomic Theory* (Princeton: Princeton University Press, 1990), pp. 302–303, 414–415.

we observe "too much" cooperation and, in bargaining games, people concede "too much" to opponents. But the reason may not be any failure to calculate or to learn to play Nash equilibrium. It may instead be that the players' payoffs are different from those assumed by the experimenter.

As with observations of naturally occurring games, participants in experimental situations also may know some complexities of the situation better than the experimenter knows them. For example, the possibility of repetition or a separate ongoing relationship with the other player may affect their choices in this game. Or the players' value systems may have internalized some social norms of niceness and fairness that have proved useful in the larger social context and that therefore carry over to their behavior in the experimental game.[7] These observations do not show any deficiency of the Nash equilibrium concept itself. However, they do warn us against using the concept under naive or mistaken assumptions about people's payoffs; it might be a mistake, for example, to assume that players are always driven by the selfish pursuit of money.

III. LEARNING FROM EXPERIENCE One game, often used in classrooms or laboratories, asks each participant to choose a number between 0 and 100. Typically, the players are handed cards on which to write their names and a choice, so this game is a simultaneous-move game. When the cards are collected, the average of the numbers is calculated. The person whose choice is closest to a specified fraction—say half—of the average is the winner. The rules of the game (this whole procedure) are announced in advance.

The Nash equilibrium of this game is for everyone to choose 0. In fact the game is dominance solvable. Even if everyone chooses 100, half of the average can never exceed 50; so, for each player, any choice above 50 is dominated by 50.[8] But all players should rationally figure this out, so the average can never exceed 50 and half of it can never exceed 25, and so any choice above 25 is dominated by 25. The iteration goes on until only 0 is left.

However, when a group actually plays this game for the first time, the winner is typically a player who has chosen a number just a little less than 25. This outcome seems to suggest that the winner assumes that everyone else will choose randomly (so their average is 50) and then chooses her own best response to that. The outcome is quite far from the Nash equilibrium.

[7]The distinguished game theorist Jörgen Weibull argues this position in detail in "Testing Game Theory," Working Paper EFI 382, Stockholm School of Economics, 2000.

[8]If you factor in your own choice, the calculation is strengthened. Suppose there are N players. In the "worst-case scenario" where all the other $(N-1)$ players choose 100 and you choose x, the average is $[x + (N-1)100]/N$. Then your best choice is half of this; so $x = [x + (N-1)100]/(2N)$, or $x = 100(N-1)/(2N-1)$. If $N = 10$, then $x = 900/19 = 47$ (approximately). So any choice above 47 is dominated by 47. The same reasoning applies to the successive rounds.

What happens if the game is repeated? Our experience in classroom trials has been that the winning choice falls rapidly in successive plays. In Skeath's class, half the class played the game first while the other half watched, then the other half played, and finally the whole class played. In Dixit's class, the game was played by different groups of 10 students at a time. By the third round, the winner's choice was usually as low as 2 or 3.

How should one interpret this result? Critics would say that, unless the exact Nash equilibrium is reached, the theory is refuted. Indeed, they would argue, if you have good reason to believe that other players will not play their Nash equilibrium strategies, then your best choice is not your Nash equilibrium strategy either. If you can figure out how others will deviate from their Nash equilibrium strategies, then you should play your best response to what you believe they are choosing. Others would argue that theories in social science can never hope for the kind of precise prediction that we expect in sciences such as physics and chemistry. If the observed outcomes are close to the Nash equilibrium, that is a vindication of the theory. In this case, the experiment not only produces such a vindication, but illustrates the process by which people gather experience and learn to play strategies close to Nash equilibrium. We sympathize with this latter viewpoint.

Interestingly, we have found that people learn somewhat faster by observing others play a game than while they play it themselves. This may be because, as observers, they are free to focus on the game as a whole and think about it analytically. Players' brains are occupied with the task of making their own choices and they are less able to take the broader perspective.

We should clarify the concept of gaining experience by playing the game. The quotation from Davis and Holt at the start of this section spoke of "repetitions with different partners." In other words, experience should be gained by playing the game frequently, but with different opponents each time. However, for any learning process to generate outcomes increasingly closer to the Nash equilibrium, the whole *population* of learners needs to be stable. If novices keep appearing on the scene and trying new experimental strategies, then the original group may unlearn what they had learned by playing against one another.

If a game is played repeatedly between two players or even among the same small group of known players, then any pair is likely to play each other repeatedly. In such a situation, the whole repeated game becomes a game in its own right. It can have very different Nash equilibria from those that simply repeat the Nash equilibrium of a single play. For example, tacit cooperation may emerge in repeated prisoners' dilemmas, owing to the expectation that any temporary gain from cheating will be more than offset by the subsequent loss of trust. If games are repeated in this way, then learning about them must come from playing whole sets of the repetitions frequently, against different partners each time.

B. Real-World Evidence

The predictions of game theory have been subjected to real-world empirical evidence in two distinct ways. One is to see if the theory can explain some observed phenomena in general terms. The other is to test statistically some implications of the theory against data. We briefly consider each in turn.

I. EXPLANATORY POWER The first approach uses game-theoretic reasoning to explain phenomena that are observed in reality, of which there are numerous successful examples. One of the earliest was in the area of international relations. Thomas Schelling pioneered the use of game theory to explain phenomena such as the escalation of arms races, even between countries that have no intention of attacking each other, and the credibility of deterrent threats. Subsequent applications in this area have included the questions of when and how a country can credibly signal its resolve in diplomatic negotiation or in the face of a potential war. Game theory began to be used systematically in economics and business in the mid-1970s, and such applications continue to proliferate. We have space for only a couple of prominent examples.

The theory has helped us to understand when and how the established firms in an industry can make credible commitments to deter new competition—for example, to wage a destructive price war against any new entrant. The prisoners' dilemma game, in its one-time and repeated forms, has helped us to understand what kinds of industries will see fierce competition and exhibit low prices and what kinds will sustain tacit agreements to keep prices and profits high. More recently, game theory has become the tool of choice for the study of political systems and institutions within a country as well as for cross-country comparisons. For example, game theory has shown how voting and agenda setting in committees and elections can be strategically manipulated in pursuit of one's ultimate objectives. In this introductory book, we can develop only a few elementary examples of this kind. We already saw an example (price competition) in this chapter. More examples appear later, including a case study of the Cuban missile crisis and analyses of auctions, voting, and bargaining.[9]

[9]For those who would like to see more applications, here are some suggested sources. Thomas Schelling's *Strategy of Conflict* (New York: Oxford University Press, 1960) and *Arms and Influence* (New Haven: Yale University Press, 1966) are still required reading for all students of game theory. The classic textbook on game-theoretic treatment of industries is Jean Tirole, *Industrial Organization* (Cambridge: MIT Press, 1988). In political science, an early classic is William Riker, *Liberalism Against Populism* (San Francisco: W. H. Freeman, 1982). For surveys of more recent work, see several articles in *The Handbook of Game Theory*, ed. Robert J. Aumann and Sergiu Hart (Amsterdam: North-Holland, 1992, 1994, 2002), particularly Barry O'Neill, "Game Theory Models of Peace and War," in volume 2, and Kyle Bagwell and Asher Wolinsky, "Game Theory and Industrial Organization," and Jeffrey Banks, "Strategic Aspects of Political Systems," in volume 3.

Some critics remain unpersuaded by these successful applications of the theory. They claim that the same understanding of these phenomena can be obtained without using game theory, by basing one's analysis on previously known general principles of economics, political science, and so on. In one sense they are right. A few of these analyses existed before game theory came along. For example, the equilibrium of the interaction between two price-setting firms, which we developed in Section 1 of this chapter, was known in economics for more than a hundred years; one can think of Nash equilibrium as just a general formulation of that equilibrium concept for all games. Some theories of strategic voting date to the 18th century, and some notions of credibility can be found in history as far back as Thucydides' *Peloponnesian War*. However, what game theory does is to unify all these applications and thereby facilitate the development of new ones.

In the past 30 years, several new ideas and applications have been identified. For example, we now understand how different forms of auctions (English and Dutch, sealed bid and open outcry) lead to differences in bidding strategies and in the seller's revenues. We understand how the existence of a second-strike capability reduces the fear of surprise attack. And we understand how governments can successfully manipulate fiscal and monetary policies to improve their chances of reelection even when the voters are sophisticated and aware of such attempts. If these examples were all amenable to previously known approaches, they would have been discovered long ago.

II. STATISTICAL TESTING The second approach to examining empirical evidence is quantitative and statistical. The general procedure in this work is to assume that Nash equilibrium prevails and to derive the implications of this assumption in the form of equations linking various magnitudes—the players' choices and outcomes—that may be observable in the situation being studied. These equations can then be estimated by using real data. In industrial economics, firms compete by choosing their quantities and prices as illustrated in the examples in this chapter; they also have other strategic choices at their disposal, including product quality, investment, R & D, and so on. While the choice of quantities or prices may be studied in a static context (at a given time), games of strategic competition in investment or R & D are dynamic. Numerous studies of both kinds of interactions have been carried out.[10] This work has produced encour-

[10] A survey of static studies of prices and quantity competition is "Empirical Studies of Industries with Market Power," by Timothy F. Bresnahan, in *Handbook of Industrial Organization*, ed. Richard L. Schmalansee and Robert D. Willig, vol. 2 (Amsterdam: North-Holland, 1989). A general method for dynamic studies is developed in "A Framework for Applied Dynamic Analysis in Industrial Organization," by Ariel Pakes, National Bureau of Economic Research Working Paper No. 8024 (Cambridge, MA: NBER, December 2000).

aging results. Game-theoretic models, based on the Nash equilibrium concept and its dynamic generalizations, fit the data for many major industries, such as automobile manufacturers, reasonably well and give us a better understanding of the determinants of competition than the older analysis, which assumed perfect competition and estimated supply-and-demand curves.

In politics, the votes on various issues within legislatures are the outcome of the legislators' strategic interaction. The equilibrium of this game depends on the legislators' underlying preferences. Detailed voting records in the U.S. Congress are public information. On the basis of the relation between preferences and voting in a Nash equilibrium, these data can be used to infer the legislators' preferences. This method has been used with remarkable success by Keith Poole and Howard Rosenthal.[11] They find that U.S. politics can be adequately summarized by conflicts over issues in a two-dimensional space, one representing economic inequality and the other racial inequality.

Pankaj Ghemawat, a professor at the Harvard Business School, has developed a mixed mode of quantitative analysis, using case studies of individual firms or industries and statistical analysis of larger data samples.[12] His game-theoretic models are remarkably successful in improving our understanding of several initially puzzling business decisions on pricing, capacity, innovation, and so on. However, his work also brings out the need to construct models that have sufficiently rich detail to do justice to the circumstances of the firms or industries being analyzed. In a general and introductory textbook such as this one, we lack the space and eschew the more advanced techniques that are needed to construct such models. But we will set you on the way to further study that will bring these methods within your grasp. And in Chapter 14 we develop one such theory-based case study from the field of international politics to illustrate the method.

III. A REAL-WORLD EXAMPLE OF LEARNING We conclude by offering an interesting illustration of equilibrium and the learning process in a real-world game. The setting is outside the laboratory or classroom, where people play the game frequently and the stakes are high, creating strong motivation and good opportunities to learn. Stephen Jay Gould discovered this beautiful example.[13] Through most of the 20th century, the best batting averages recorded in a baseball season have been declining. In particular, the number of instances of a player averaging .400 or better used to be much more frequent than they are now. Devotees of baseball

[11]Keith Poole and Howard Rosenthal, "Patterns of Congressional Voting," *American Journal of Political Science*, vol. 35, no. 1 (February 1991), pp. 228–278, and *Congress: A Political-Economic History of Roll Call Voting* (New York: Oxford University Press, 1996).

[12]Pankaj Ghemawat, *Games Businesses Play: Cases and Models* (Cambridge: MIT Press, 1997).

[13]"Losing the Edge," in *The Flamingo's Smile* (New York: Norton, 1985), pp. 215–229.

history often explain this decline by invoking nostalgia: "There were giants in those days." A moment's thought should make one wonder why there were no corresponding pitching giants who would keep batting averages low. But Gould demolishes such arguments in a more systematic way. He points out that we should look at all batting averages, not just the top ones. The worst batting averages are not as bad as they used to be; there are also many fewer .150 hitters in the major leagues than there used to be. He argues that this overall decrease in *variation* is a standardization or stabilization effect:

> When baseball was very young, styles of play had not become sufficiently regular to foil the antics of the very best. Wee Willie Keeler could "hit 'em where they ain't" (and compile an average of .432 in 1897) because fielders didn't yet know where they should be. Slowly, players moved toward *optimal* methods of positioning, fielding, pitching, and batting—and variation inevitably declined. The best [players] now met an opposition too finely honed to its own perfection to permit the extremes of achievement that characterized a more casual age [emphasis added].

In other words, through a succession of adjustments of strategies to counter one another, the system settled down into its (Nash) equilibrium.

Gould marshals decades of hitting statistics to demonstrate that such a decrease in variation did indeed occur, except for occasional "blips." And indeed the blips confirm his thesis, because they occur soon after an equilibrium is disturbed by an externally imposed change. Whenever the rules of the game are altered (the strike zone is enlarged or reduced, the pitching mound is lowered, or new teams and many new players enter when an expansion takes place) or the technology changes (a livelier ball is used or perhaps, in the future, aluminum bats are allowed), the preceding system of mutual best responses is thrown out of equilibrium. Variation increases for a while as players experiment, and some succeed while others fail. Finally a new equilibrium is attained, and variation goes down again. That is exactly what we should expect in the framework of learning and adjustment to a Nash equilibrium.

We take up the evidence concerning mixed strategies in Chapter 8 and the evidence for some specific games or types of games—for example, the prisoners' dilemma, bargaining, and auctions—at appropriate points in later chapters. For now, the experimental and empirical evidence that we have presented should make you cautiously optimistic about using Nash equilibrium as a first approach or as the point of departure for your analysis. On the whole, we believe you should have considerable confidence in using the Nash equilibrium concept when the game in question is played frequently by players from a reasonably stable population and under relatively unchanging rules and conditions. When the game is new or is played just once and the players are inexperienced, you should use the equilibrium concept more cautiously and should not be surprised if the outcome that you observe is not the equilibrium that you calculate. But, even then, your first step in the analysis should be to look for a Nash equilibrium; then you can judge whether it seems a plausible outcome and, if not, proceed to the further step of asking why not. Often the reason will be your misunderstanding of the players' objectives, not the players' failure to play the game correctly, given their true objectives.

3 CRITICAL DISCUSSION OF THE NASH EQUILIBRIUM CONCEPT

In addition to the critiques lodged against the Nash equilibrium concept by those who have examined the empirical evidence, there have also been theoretical criticisms of the concept. In this section, we briefly review some such criticisms and some rebuttals, in each case by using an example.[14] Some of the criticisms are mutually contradictory, and some can be countered by thinking of the games themselves in a better way. Others tell us that the Nash equilibrium concept by itself is not enough and suggest some augmentations or relaxations of it that have better properties. We develop one such alternative here and point to some others that appear in later chapters. We believe our presentation will leave you with renewed but cautious confidence in using the Nash equilibrium concept. But some serious doubts remain unresolved, indicating that game theory is not yet a settled science. Even this should give encouragement, not the opposite, to budding game theorists, because it shows that there is a lot of room for new thinking and new research in the subject. A totally settled science would be a dead science.

We begin by considering the basic appeal of the Nash equilibrium concept. Most of the games in this book are noncooperative, in the sense that every player takes her action independently. Therefore it seems natural to suppose that, if her action is not the best according to her own value system (payoff scale), given what everyone else does, then she will change it. In other words, it is appealing to suppose that every player's action will be the best response to the actions of all the others. Nash equilibrium has just this property of "simultaneous best responses"; indeed, that is its very definition. In any purported final outcome that is not a Nash equilibrium, at least one player could have done better by switching to a different action.

This consideration leads eminent game theorist Roger Myerson to rebut those criticisms of the Nash equilibrium that are based on the intuitive appeal of playing a different strategy. His rebuttal simply shifts the burden of proof onto the

[14] David M. Kreps, *Game Theory and Economic Modelling* (Oxford: Clarendon Press, 1990) gives an excellent in-depth discussion.

critic. "When asked why players in a game should behave as in some Nash equilibrium," he says, "my favorite response is to ask 'Why not?' and to let the challenger specify what he thinks the players should do. If this specification is not a Nash equilibrium, then . . . we can show that it would destroy its own validity if the players believed it to be an accurate description of each other's behavior."[15]

A. The Treatment of Risk in Nash Equilibrium

Some critics argue that the Nash equilibrium concept does not pay due attention to risk. In some games, people might find strategies different from their Nash equilibrium strategies to be safer and may therefore choose those strategies. We offer two examples of this kind. The first is due to John Morgan, an economics professor at the University of California, Berkeley; Figure 5.3 shows the game table.

Cell-by-cell inspection quickly reveals that this game has a unique Nash equilibrium—namely, (A, A), yielding the payoffs (2, 2). But you may think, as did several participants in an experiment conducted by Morgan, that playing C has a lot of appeal, for the following reasons. It *guarantees* you the same payoff as you would get in the Nash equilibrium—namely, 2; whereas, if you play your Nash equilibrium strategy A, you will get a 2 only so long as the other player also plays A. Why take that chance? What is more, if you think the other player might use this rationale for playing C, then you would be making a serious mistake by playing A; you would get only a 0 when you could have gotten your guaranteed 2 by playing C.

Myerson would respond, "Not so fast. If you really believe that the other player would think thus and would play C, then you should play B to get the payoff 3. And if you think the other person would think thus and so would play B, then your best response to B is A. And if you think the other would figure this out too, you should be playing your best response to A, namely A. Back to the Nash

		COLUMN		
		A	B	C
ROW	A	2, 2	3, 1	0, 2
	B	1, 3	2, 2	3, 2
	C	2, 0	2, 3	2, 2

FIGURE 5.3 A Game with a Questionable Nash Equilibrium

[15]Roger Myerson, *Game Theory* (Cambridge: Harvard University Press, 1991), p. 106.

equilibrium!" As you can see, criticizing Nash equilibrium and rebutting the criticisms is itself something of an intellectual game, and quite a fascinating one.

The second example, due to David Kreps, is even more dramatic. The payoff matrix is in Figure 5.4. Before doing any theoretical analysis of this game, you should pretend that you are actually playing the game and that you are player A. Which of your two actions would you choose?

Keep in mind your answer to the preceding question and let us proceed to analyze the game. If we start by looking for dominant strategies, we see that player A has no dominant strategy but player B does. Playing Left guarantees B a payoff of 10, no matter what A does, versus the 9.9 that is gained by playing Right (also no matter what A does). Thus, player B should play Left. Given that player B is going to go Left, player A does better to go Down. The unique pure-strategy Nash equilibrium of this game is (Down, Left); each player achieves a payoff of 10 at this outcome.

The problem that arises here is that many people (but not all) would not, as player A, choose to play Down. (What did you choose?) This is true for those who have been students of game theory for years as well as for those who have never heard of the subject. If A has *any* doubts about *either* B's payoffs *or* B's rationality, then it is a lot safer for A to play Up than to play her Nash equilibrium strategy of Down. What if A thought the payoff table was as illustrated in Figure 5.4 but in reality B's payoffs were the reverse—the 9.9s went with Left and the 10s went with Right? What if the 9.9s were only an approximation and the exact payoffs were actually 10.1? What if B was a player with a substantially different value system or was not a truly rational player who might choose the "wrong" action just for fun? Obviously, our assumptions of perfect information and rationality can really be crucial to the analysis that we use in the study of strategy. Doubts of players can alter equilibria from those that we would normally predict and can call the reasonableness of the Nash equilibrium concept into question.

However, the real problem with many such examples is not that the Nash equilibrium concept is inappropriate but that the examples choose to use it in an inappropriately simplistic way. In this example, if there are any doubts about B's payoffs, then this fact should be made an integral part of the analysis. If A does not know B's payoffs, the game is one of asymmetric information, and we

		B	
		Left	Right
A	Up	9, 10	8, 9.9
	Down	10, 10	−1000, 9.9

FIGURE 5.4 Disastrous Nash Equilibrium?

do not develop the general techniques for studying such games until Chapter 9. But this particular example is a relatively simple game of that kind, and we can figure out its equilibrium very easily.

Suppose A thinks there is a probability p that B's payoffs from Left and Right are the reverse of those shown in Figure 5.4; so $(1 - p)$ is the probability that B's payoffs are as stated in that figure. Because A must take her action without knowing which is the case, she must choose her strategy to be "best on the average." In this game the calculation is simple, because in each case B has a dominant strategy; the only problem for A is that in the two different cases different strategies are dominant for B. With probability $(1 - p)$, B's dominant strategy is Left (the case shown in the figure) and, with probability p, it is Right (the opposite case). Therefore if A chooses Up, then with probability $(1 - p)$ he will meet B playing Left and so get a payoff of 9; with probability p, he will meet B playing Right and so get a payoff of 8. Thus A's statistical or probability-weighted average payoff from playing Up is $9(1 - p) + 8p$. Similarly, A's statistical average payoff from playing Down is $10(1 - p) - 1000p$. Therefore it is better for A to choose Up if

$$9(1 - p) + 8p > 10(1 - p) - 1000p, \quad \text{or} \quad p > 1/1009.$$

Thus, even if there is only a very slight chance that B's payoffs are the opposite of those in Figure 5.4, it is optimal for A to play Up. In this case, analysis based on rational behavior, when done correctly, contradicts neither the intuitive suspicion nor the experimental evidence after all.

In the preceding calculation, we supposed that, facing an uncertain prospect of payoffs, player A would calculate the statistical average payoffs from her different actions and would choose that action which yields her the highest statistical average payoff. This implicit assumption, though it serves the purpose in this example, is not without its own problems. For example, it implies that a person faced with two situations, one having a 50-50 chance of winning or losing \$10 and the other having a 50-50 chance of winning \$10,001 and losing \$10,000, should choose the second situation, because it yields a statistical average winning of 50 cents ($\frac{1}{2} \times 10{,}001 - \frac{1}{2} \times 10{,}000$), whereas the first yields 0 ($\frac{1}{2} \times 10 - \frac{1}{2} \times 10$). But most people would think that the second situation carries a much bigger risk and would therefore prefer the first situation. This difficulty is quite easy to resolve. In the Appendix to Chapter 7, we show how the construction of a scale of payoffs that is suitably nonlinear in money amounts enables the decision maker to allow for risk as well as return.

B. Multiplicity of Nash Equilibria

Another criticism of the Nash equilibrium concept is based on the observation that many games have multiple Nash equilibria. Thus, the argument goes, the concept fails to pin down outcomes of games sufficiently precisely to give unique predictions. This argument does not automatically require us to abandon the Nash equilibrium concept. It merely says that the concept is not enough by itself and must be supplemented by some other consideration that selects the one equilibrium with better claims to be the outcome than the rest.

In Chapter 4, we studied many games of coordination with multiple equilibria. From among these equilibria, the players may be able to select one as a focal point if they have some common social, cultural, or historical knowledge or if the game has some deliberate or accidental features that enable their expectations to converge. Here is a very extreme example of multiplicity of Nash equilibria in a coordination game. Two players are asked to write down, simultaneously and independently, the share that each wants of a total prize of \$100. If the amounts that they write down add up to \$100 or less, each player is given what she wrote. If the two add up to more than \$100, neither gets anything. For any x, one player writing x and the other writing $(100 - x)$ is a Nash equilibrium. Thus the game has an (almost) infinite range of Nash equilibria. But in practice, 50:50 emerges as a focal point. This social norm of equality or fairness seems so deeply ingrained as to be almost an instinct; players who choose 50 say that it is the obvious answer. To be a true focal point, not only should it be obvious to each, but everyone should know that it is obvious to each, and everyone should know that . . . ; in other words, its obviousness should be common knowledge. That need not always be the case, as we see when we consider a situation in which one player is a woman from an enlightened and egalitarian society who believes that 50:50 is obvious and the other is a man from a patriarchal society who believes it is obvious that, in any matter of division, a man should get three times as much as a woman. Then each will do what is obvious to her or him, and they will end up with nothing, because neither's obvious solution is obvious as common knowledge to both.

The existence of focal points is often a matter of coincidence, and creating them where none exist is basically an art that requires a lot of attention to the historical and cultural context of a game and not merely its mathematical description. This bothers many game theorists, who would prefer the outcome to depend only on an abstract specification of a game—players and their strategies should be identified by numbers without any external associations. We disagree. We think that historical and cultural contexts are just as important to a game as its purely mathematical description, and, if such context helps in selecting a unique outcome from multiple Nash equilibria, that is all to the better.

In Chapter 6, we will see that sequential-move games can have multiple Nash equilibria. There, we introduce the requirement of *credibility* that enables us to select a particular equilibrium; it turns out that this one is in fact the rollback equilibrium of Chapter 3. In more complex games with information asymmetries or additional complications, other restrictions called **refinements** have been developed to identify and rule out Nash equilibria that are unreasonable in some way. In Chapter 9, we consider one such refinement process that selects an outcome called a *Bayesian perfect equilibrium*. The motivation for each refinement is often specific to a particular type of game. A refinement stipulates how players update their information when they observe what moves other players made or failed to make. Each such stipulation is often perfectly reasonable in its context, and in many games it is not difficult to eliminate most of the Nash equilibria and therefore to narrow down the ambiguity in prediction.

The opposite of the criticism that some games may have too many Nash equilibria is that some games may have none at all. We saw an example of this in Section 4.8 and said that, by extending the concept of strategy to random mixtures, Nash equilibrium could be restored. In Chapters 7 and 8 we explain and consider Nash equilibria in mixed strategies. In higher reaches of game theory, there are more esoteric examples of games that have no Nash equilibrium in mixed strategies either. However, this added complication is not relevant for the types of analysis and applications that we deal with in this book; so we do not attempt to address it here.

C. Requirements of Rationality for Nash Equilibrium

Remember that Nash equilibrium can be regarded as a system of the strategy choices of each player and the belief that each player holds about the other players' choices. In equilibrium, (1) the choice of each should give her the best payoff given her belief about the others' choices, and (2) the belief of each player should be correct—that is, her actual choices should be the same as what this player believes them to be. These seem to be natural expressions of the requirements of the mutual consistency of individual rationality. If all players have common knowledge that they are all rational, how can any one of them rationally believe something about others' choices that would be inconsistent with a rational response to her own actions?

To begin to address this question, we consider the three-by-three game in Figure 5.5. Cell-by-cell inspection quickly reveals that it has only one Nash equilibrium—namely, (R2, C2), leading to payoffs (3, 3). In this equilibrium, Row plays R2 because she believes that Column is playing C2. Why does she believe this? Because she knows Column to be rational, Row must simultaneously believe that Column believes that Row is choosing R2, because C2 would not be Column's best choice if she believed Row would be playing either R1 or R3. Thus, the claim goes, in any rational process of formation of beliefs and responses, beliefs would have to be correct.

The trouble with this argument is that it stops after one round of thinking about beliefs. If we allow it to go far enough, we can justify other choice combinations. We can, for example, rationally justify Row's choice of R1. To do so, we note that R1 is Row's best choice if she believes Column is choosing C3.

		COLUMN		
		C1	C2	C3
ROW	R1	0, 7	2, 5	7, 0
	R2	5, 2	3, 3	5, 2
	R3	7, 0	2, 5	0, 7

FIGURE 5.5 Justifying Choices by Chains of Beliefs and Responses

Why does she believe this? Because she believes that Column believes that Row is playing R3. Row justifies this belief by thinking that Column believes that Row believes that Column is playing C1, believing that Row is playing R1, believing in turn . . . This is a chain of beliefs, each link of which is perfectly rational.

Thus rationality alone does not justify Nash equilibrium. There are more sophisticated arguments of this kind that do justify a special form of Nash equilibrium in which players can condition their strategies on a publicly observable randomization device. But we leave that to more advanced treatments. In the next section, we develop a simpler concept that captures what is logically implied by the players' common knowledge of their rationality alone.

4 RATIONALIZABILITY

What strategy choices in games can be justified on the basis of rationality alone? In the matrix of Figure 5.5, we can justify any pair of strategies, one for each player, by using the same type of logic as that used in Section 3.C. In other words, we can justify any one of the nine logically conceivable combinations. Thus rationality alone does not give us any power to narrow down or predict outcomes at all. Is this a general feature of all games? No. For example, if a strategy is dominated, rationality alone can rule it out of consideration. And, when players recognize that other players, being rational, will not play dominated strategies, iterated elimination of dominated strategies can be performed on the basis of common knowledge of rationality. Is this the best that can be done? No. Some more ruling out of strategies can be done, by using a property slightly stronger than being dominated in pure strategies. This property identifies strategies that are **never a best response.** The set of strategies that survive elimination on this ground are called **rationalizable,** and the concept itself is known as **rationalizability.**

A. Applying the Concept of Rationalizability

Consider the game in Figure 5.6, which is the same as Figure 5.5 but with an additional strategy for each player.[16] We just indicated that nine of the strategy combinations that pick one of the first three strategies for each of the players can be justified by a chain of beliefs about each other's beliefs. That remains true in this enlarged matrix. But can R4 and C4 be justified in this way?

Could Row ever believe that Column would play C4? Such a belief would have to be justified by Column's beliefs about Row's choice. What might Column believe about Row's choice that would make C4 Column's best response? Nothing. If Column believes that Row would play R1, then Column's best choice is C1. If Column believes that Row will play R2, then Column's best choice is C2. If Column believes that Row will play R3, then C3 is Column's best choice. And, if Column believes that Row will play R4, then C1 and C3 are tied for her best choice. Thus C4 is never a best response for Column.[17] This means that Row, knowing Column to be rational, can never attribute to Column any belief about Row's choice that would justify Column's choice of C4. Therefore Row should never believe that Column would choose C4.

Note that, although C4 is never a best response, it is not dominated by any of C1, C2, and C3. For Column, C4 does better than C1 against Row's R3, better than C2 against Row's R4, and better than C3 against Row's R1. If a strategy *is* dominated, it also can never be a best response. Thus "never a best response" is a more general concept than "dominated." Eliminating strategies that are

		COLUMN			
		C1	C2	C3	C4
ROW	R1	0, 7	2, 5	7, 0	0, 1
	R2	5, 2	3, 3	5, 2	0, 1
	R3	7, 0	2, 5	0, 7	0, 1
	R4	0, 0	0, −2	0, 0	10, −1

FIGURE 5.6 Rationalizable Strategies

[16]This example is taken from the original article that developed the concept of rationalizability. See Douglas Bernheim, "Rationalizable Strategic Behavior," *Econometrica*, vol. 52, no. 4 (July 1984), pp. 1007–1028. See also Andreu Mas-Colell, Michael Whinston, and Jerry Green, *Microeconomic Theory* (New York: Oxford University Press, 1995), pp. 242–245.

[17]Note that in each case the best choice is strictly better than C4 for Column. Thus C4 is never even tied for a best response. We can distinguish between weak and strong senses of never being a best response just as we distinguished between weak and strong dominance. Here, we have the strong sense.

never a best response may be possible even when eliminating dominated strategies is not. So eliminating strategies that are never a best response can narrow down the set of possible outcomes more than can elimination of dominated strategies.[18]

The elimination of "never best response" strategies can also be carried out iteratively. Because a rational Row can never believe that a rational Column will play C4, a rational Column should foresee this. Because R4 is Row's best response only against C4, Column should never believe that Row will play R4. Thus R4 and C4 can never figure in the set of rationalizable stratgies. The concept of rationalizability does allow us to narrow down the set of possible outcomes of this game to this extent.

If a game has a Nash equilibrium, it is rationalizable and in fact can be sustained by a simple one-round system of beliefs, as we saw in Section 3.C. But more generally, even if a game does not have a Nash equilibrium, it may have rationalizable outcomes. Consider the two-by-two game obtained from Figure 5.5 or Figure 5.6 by retaining just the strategies R1 and R3 for Row and C1 and C3 for Column. It is easy to see that it has no Nash equilibrium in pure strategies. But all four outcomes are rationalizable with the use of exactly the chain of beliefs, constructed earlier, that went around and around these strategies.

Thus the concept of rationalizability provides a possible way of solving games that do not have a Nash equilibrium. And more importantly, the concept tells us how far we can narrow down the possibilities in a game on the basis of rationality alone.

B. Rationalizability Leading to Nash Equilibrium

In some games, iterated elimination of never-best-response strategies narrows things down all the way to Nash equilibrium. This result is useful because in these games we can strengthen the case for Nash equilibrium by arguing that it follows purely from the players' rational thinking about each other's thinking. Interestingly, one class of games that can be solved in this way is very important in economics. This class consists of competition between firms that choose the quantities that they produce, knowing that the total quantity that is put on the market will determine the price.

We illustrate a game of this type in the context of a small coastal town. It has two fishing boats that go out every evening and return the following morning to put their night's catch on the market. The game is played out in an era before modern refrigeration; so all the fish has to be sold and eaten the same day. Fish are

[18]In Chapter 8, we will see that in two-player games, a strategy that is never a best response can be dominated by a *mixture* of the other strategies. Therefore, in two-player games that allow mixed strategies, the two kinds of elimination become equivalent.

quite plentiful in the ocean near the town; so the owner of each boat can decide how much to catch each night. But each knows that, if the total that is brought to the market is too large, the glut of fish will mean a low price and low profits.

Specifically, we suppose that, if one boat brings X barrels and the other brings Y barrels of fish to the market, the price P (measured in ducats per barrel) will be $P = 60 - (X + Y)$. We also suppose that the two boats and their crews are somewhat different in their fishing efficiency. Fishing costs the first boat 30 ducats per barrel and the second boat 36 ducats per barrel.

Now we can write down the profits of the two boat owners, U and V, in terms of their strategies X and Y:

$$U = [(60 - X - Y) - 30]X = (30 - Y)X - X^2$$
$$V = [(60 - X - Y) - 36]Y = (24 - X)Y - Y^2$$

With these payoff expressions, we construct best-response curves and find the Nash equilibrium.

The first boat's best response X should maximize U for each given value of the other boat's Y. With the use of calculus, this means that we should differentiate U with respect to X, holding Y fixed, and set the derivative equal to zero, which gives

$$(30 - Y) - 2X = 0; \quad \text{so} \quad X = 15 - Y/2.$$

The noncalculus approach would be to complete the square in the expression for U:

$$U = [(30 - Y)/2]^2 - [(30 - Y)/2 - X]^2$$

and observe that X appears only in the second squared term on the right-hand side. Because this term has a negative sign, it should be made as small as possible—namely, zero—to make U as large as possible. This gives $(30 - Y)/2 - X = 0$, or $X = 15 - Y/2$.

Similarly, the best-response equation of the second boat is found by choosing Y to maximize V for each fixed X, yielding

$$(24 - X) - 2Y = 0; \quad \text{so} \quad Y = 12 - X/2.$$

The Nash equilibrium is found by solving the two best-response equations jointly for X and Y, which is easy to do. So we just state the results:[19] Quantities are $X = 12$ and $Y = 6$; price is $P = 42$; and profits are $U = 144$ and $V = 36$.

[19]Although they are incidental to our purpose, some interesting properties of the solution are worth pointing out. The quantities differ because the costs differ; the more efficient (lower-cost) boat gets to sell more. The cost and quantity differences together imply even bigger differences in the resulting profits. The cost advantage of the first boat over the second is only 20%, but it makes four times as much profit as the second boat.

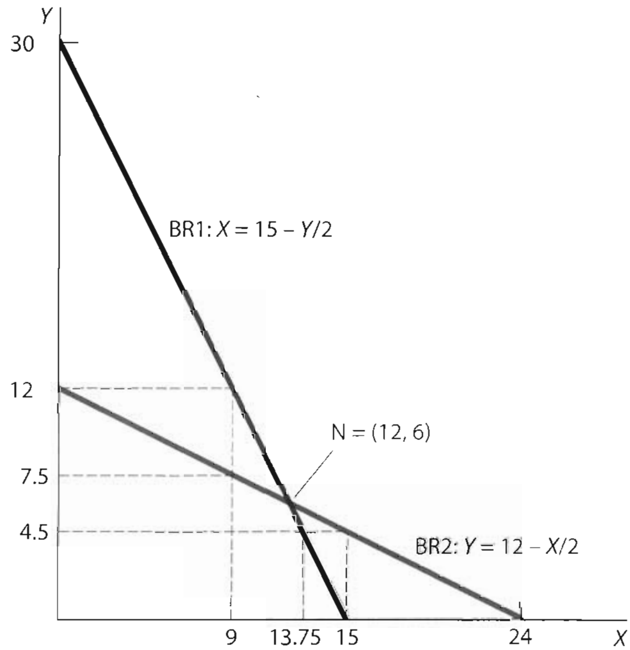


FIGURE 5.7 Nash Equilibrium Through Rationalizability

Figure 5.7 shows the two fishermen's best-response curves (labeled BR1 and BR2 with the equations displayed) and the Nash equilibrium (labeled N with its coordinates displayed) at the intersection of the two curves. Figure 5.7 also shows how the players' beliefs about each other's choices can be narrowed down by iteratively eliminating strategies that are never best responses.

What values of Y can the first owner rationally believe the second owner will choose? That depends on what the second owner thinks the first owner will produce. But no matter what this might be, the whole range of the second owner's best responses is between 0 and 12. So the first owner cannot rationally believe that the second owner will choose anything else; all negative choices of Y (obviously) and all choices of Y greater than 12 (less obviously) are eliminated. Similarly, the second owner cannot rationally think that the first owner will produce anything less than 0 or greater than 15.

Now take this to the second round. When the first owner has restricted the second owner's choices of Y to the range between 0 and 12, her own choices of X are restricted to the range of best responses to Y's range. The best response to $Y = 0$ is $X = 15$, and the best response to $Y = 12$ is $X = 15 - 12/2 = 9$. Because BR1 has a negative slope throughout, the whole range of X allowed at this round of thinking is between 9 and 15. Similarly, the second owner's choice of Y is restricted to the range of best responses to X between 0 and 15—namely, values between $Y = 12$ and $Y = 12 - 15/2 = 4.5$. Figure 5.7 shows these restricted ranges on the axes.

The third round of thinking narrows the ranges further. Because X must be at least 9 and BR2 has a negative slope, Y can be at most the best response to 9—namely, $Y = 12 - 9/2 = 7.5$. In the second round, Y was already shown to be at least 4.5. Thus Y is now restricted to be between 4.5 and 7.5. Similarly, because Y must be at least 4.5, X can be at most $15 - 4.5/2 = 13.75$. In the second round, X was shown to be at least 9; so now it is restricted to the range from 9 to 13.75.

This succession of rounds can be carried on as far as you like, but it is already evident that the successive narrowing of the two ranges is converging on the Nash equilibrium, $X = 12$ and $Y = 6$. Thus the Nash equilibrium is the only outcome that survives the iterated elimination of strategies that are never best responses.[20] This process will work for any game that has a unique Nash equilibrium at the intersection of downward-sloping best-response curves.[21]

This argument should be carefully distinguished from an older one based on a succession of best responses. The old reasoning proceeded as follows. Start at any strategy for one of the players—say, $X = 18$. Then the best response of the other is $Y = 12 - 18/2 = 3$. The best response of X to $Y = 3$ is $X = 15 - 3/2 = 13.5$. In turn, the best response of Y to $X = 13.5$ is $12 - 13.5/2 = 5.25$. Then, in its turn, the best X against this Y is $X = 15 - 5.25/2 = 12.375$. And so on.

The chain of best responses in the old argument also converges to the Nash equilibrium. But the argument is flawed. The game is played once with simultaneous moves. It is not possible for one player to respond to what the other player has chosen, then have the first player respond back again, and so on. If such dynamics of actual play were allowed, would the players not foresee that the other is going to respond and so do something different in the first place?

The rationalizability argument is different. It clearly incorporates the fact that the game is played only once and with simultaneous moves. All the thinking regarding the chain of best responses is done in advance, and all the successive rounds of thinking and responding are purely conceptual. Players are not responding to actual choices but are merely calculating those choices that will never be made. The dynamics are purely in the minds of the players.

[20] This example can also be solved by iteratively eliminating dominated strategies, but proving dominance is harder and needs more calculus, whereas the never-best-response property is obvious from Figure 5.7; so we use the simpler argument.

[21] A similar argument works with upward-sloping best-response curves, such as those in the pricing game of Figure 5.1, for narrowing the range of best responses starting at low prices. Narrowing from the higher end is possible only if there is some obvious starting point. This starting point might be a very high price that can never be exceeded for some externally enforced reason—if, for example, people simply do not have the money to pay prices beyond a certain level.

SUMMARY

When players in a simultaneous-move game have a continuous range of actions to choose, best-response analysis yields mathematical *best-response rules* that can be solved simultaneously to obtain Nash equilibrium strategy choices. The best-response rules can be shown on a diagram in which the intersection of the two curves represents the Nash equilibrium. Firms choosing price or quantity from a large range of possible values and political parties choosing campaign advertising expenditure levels are examples of games with *continuous strategies.*

The results of laboratory tests of the Nash equilibrium concept show that a common cultural background is essential for coordinating in games with multiple equilibria. Repeated play of some games shows that players can learn from experience and begin to choose strategies that approach Nash equilibrium choices. Further, predicted equilibria are accurate only when the experimenters' assumptions match the true preferences of players. Real-world applications of game theory have helped economists and political scientists, in particular, to understand important consumer, firm, voter, legislature, and government behaviors.

Theoretical criticisms of the Nash equilibrium concept have argued that the concept does not adequately account for risk, that it is of limited use because many games have multiple equilibria, and that it cannot be justified on the basis of rationality alone. In many cases, a better description of the game and its payoff structure or a refinement of the Nash equilibrium concept can lead to better predictions or fewer potential equilibria. The concept of *rationalizability* relies on the elimination of strategies that are *never a best response* to obtain a set of *rationalizable* outcomes. When a game has a Nash equilibrium, that outcome will be rationalizable; but rationalizability also allows one to predict equilibrium outcomes in games that have no Nash equilibria.

KEY TERMS

best-response curves (127)
best-response rule (124)
continuous strategy (124)
never a best response (145)
rationalizability (145)
rationalizable (145)
refinement (143)

EXERCISES

1. In the political campaign advertising game in Section 1.B, party L chooses an advertising budget, x (millions of dollars), and party R similarly chooses a budget, y (millions of dollars). We showed there that the best-response rules

in that game are $y = \sqrt{x} - x$, for party R, and $x = \sqrt{y} - y$ for party L. Use these best-response rules to verify that the Nash equilibrium advertising budgets are $x = y = 1/4$, or \$250,000.

2. The bistro game of Figure 5.1 defines demand functions for Xavier's (Q_x) and Yvonne's (Q_y) as $Q_x = 44 - 2\,P_x + P_y$, and $Q_y = 44 - 2\,P_y + P_x$. Profits for each firm depend, in addition, on their costs of serving each customer. Suppose, here, that Yvonne's is able to reduce its costs to \$6 per customer by redistributing the serving tasks and laying off several servers; Xavier's continues to incur a cost of \$8 per customer.
   (a) Recalculate the best-response rules and the Nash equilibrium prices for the two firms, given the change in the cost conditions.
   (b) Graph the two best-response curves and describe the differences between your graph and Figure 5.1. In particular, which curve has moved and by how much? Can you account for the changes in the diagram?

3. Yuppietown has two food stores, La Boulangerie, which sells bread, and La Fromagerie, which sells cheese. It costs \$1 to make a loaf of bread and \$2 to make a pound of cheese. If La Boulangerie's price is P_1 dollars per loaf of bread and La Fromagerie's price is P_2 dollars per pound of cheese, their respective weekly sales, Q_1 thousand loaves of bread and Q_2 thousand pounds of cheese, are given by the following equations:

$$Q_1 = 10 - P_1 - 0.5P_2, \quad Q_2 = 12 - 0.5P_1 - P_2.$$

   (a) Find the two stores' best-response rules, illustrate the best-response curves, and find the Nash equilibrium prices in this game.
   (b) Suppose that the two stores collude and set prices jointly to maximize the sum of their profits. Find the joint profit-maximizing prices for the stores.
   (c) Provide a short intuitive explanation for the differences between the Nash equilibrium prices and those that maximize joint profit. Why is joint profit maximization not a Nash equilibrium?
   (d) In this problem, bread and cheese are mutual *complements*. They are often consumed together; that is why a drop in the price of one increases the sales of the other. The products in our bistro example in Section 1.A are *substitutes* for each other. How does this difference explain the differences between your findings for the best-response rules, the Nash equilibrium prices, and the joint profit-maximizing prices in this question, and the corresponding entities in the bistro example in the text?

5. Two carts selling coconut milk (from the coconut) are located at 0 and 1, 1 mile apart on the beach in Rio de Janeiro. (They are the only two coconut

milk carts on the beach.) The carts—Cart 0 and Cart 1—charge prices p_0 and p_1, respectively, for each coconut. Their customers are the beach goers uniformly distributed along the beach between 0 and 1. Each beach goer will purchase one coconut milk in the course of her day at the beach and, in addition to the price, each will incur a transport cost of 0.5 times d^2, where d is the distance (in miles) from her beach blanket to the coconut cart. In this system, Cart 0 sells to all of the beach goers located between 0 and x, and Cart 1 sells to all of the beach goers located between x and 1, where x is the location of the beach goer who pays the same total price if she goes to 0 or 1. Location x is then defined by the expression

$$p_0 + 0.5x^2 = p_1 + 0.5(1 - x)^2.$$

The two carts will set their prices to maximize their bottom-line profit figures, B; profits are determined by revenue (the cart's price times its number of customers) and cost (the carts each incur a cost of \$0.25 per coconut times the number of coconuts sold).

(a) Determine the expression for the number of customers served at each cart. (Recall that Cart 0 gets the customers between 0 and x, or just x, while Cart 1 gets the customers between x and 1, or $1 - x$.)

(b) Write out profit functions for the two carts and find the two best-response rules for their prices.

(c) Graph the best-response rules, and then calculate (and show on your graph) the Nash equilibrium price level for coconuts on the beach.

6. The game illustrated in Figure 5.3 has a unique Nash equilibrium in pure strategies. However, all nine outcomes in that game are rationalizable. Confirm this assertion, explaining your reasoning for each outcome.

7. The game illustrated in Figure 5.4 has a unique Nash equilibrium in pure strategies. Find that Nash equilibrium, and then show that it is also the unique rationalizable outcome in that game.

8. Section 4.B describes a fishing game played in a small coastal town. When the response rules for the two boats have been derived, rationalizability can be used to justify the Nash equilibrium in the game. In the description in the text, we take the process of narrowing down strategies that can never be best responses through three rounds. By the third round, we know that X (the number of barrels of fish brought home by boat 1) must be at least 9, and that Y (the number of barrels of fish brought home by boat 2) must be at least 4.5. The narrowing process in that round restricted X to the range between 9 and 13.75 while restricting Y to the range between 4.5 and 7.5. Take this process of narrowing through one additional (fourth) round and show the reduced ranges of X and Y that are obtained at the end of the round.

9. Nash equilibrium through rationalizability can be achieved in games with upward-sloping best-response curves if the rounds of eliminating never-best-response strategies begin with the smallest possible values. Consider the pricing game between Xavier's Tapas Bar and Yvonne's Bistro that is illustrated in Figure 5.1. Use Figure 5.1 and the best-response rules from which it is derived to begin rationalizing the Nash equilibrium in that game. Start with the lowest possible prices for the two firms and describe (at least) two rounds of narrowing the set of rationalizable prices toward the Nash equilibrium.

10. **Optional, requires calculus** Recall the political campaign advertising example from Section 1.B concerning parties L and R. In that example, when L spends $\$x$ million on advertising and R spends $\$y$ million, L gets a share $x/(x + y)$ of the votes and R gets $y/(x + y)$. We also mentioned that two types of asymmetries can arise between the parties in that model. One party—say, R—may be able to advertise at a lower cost or R's advertising dollars may be more effective in generating votes than L's. To allow for both possibilities, we can write the payoff functions of the two parties as

$$V_L = \frac{x}{x + ky} - x \quad \text{and} \quad V_R = \frac{ky}{x + ky} - cy.$$

These payoff functions show that R has an advantage in the relative effectiveness of its ads when k is high and that R has an advantage in the cost of its ads when c is low.

(a) Use the payoff functions to derive the best-response functions for R (which chooses y) and L (which chooses x).

(b) Use your calculator or your computer to graph these best-response functions when $k = 1$ and $c = 1$. Compare the graph with the one for the case in which $k = 1$ and $c = 0.8$. What is the effect of having an advantage in the cost of advertising?

(c) Compare the graph from part b, when $k = 1$ and $c = 1$ with the one for the case in which $k = 2$ and $c = 1$. What is the effect of having an advantage in the effectiveness of advertising dollars?

(d) Solve the best-response functions that you found in part a, jointly for x and y, to show that the campaign advertising expenditures in Nash equilibrium are

$$x = \frac{ck}{(c + k)^2} \quad \text{and} \quad y = \frac{k}{(c + k)^2}.$$

(e) Let $k = 1$ in the equilibrium spending level equations and show how the two equilibrium spending levels vary with changes in c. The let $c = 1$ and show how the two equilibrium spending levels vary with changes in k. Do your answers support the effects that you observed in parts b and c of this exercise?

6

Combining Sequential and Simultaneous Moves

IN CHAPTER 3 we considered games of purely sequential moves; Chapters 4 and 5 dealt with games of purely simultaneous moves. We developed concepts and techniques of analysis appropriate to the pure game types—trees and rollback equilibrium for sequential moves, payoff tables and Nash equilibrium for simultaneous moves. In reality, however, many strategic situations contain elements of both types of interaction. Also, although we used game trees (extensive forms) as the sole method of illustrating sequential-move games and game tables (strategic forms) as the sole method of illustrating simultaneous-move games, we can use either form for any type of game.

In this chapter, we examine many of these possibilities. We begin by showing how games that combine sequential and simultaneous moves can be solved by combining trees and payoff tables and by combining rollback and Nash equilibrium analysis in appropriate ways. Then we consider the effects of changing the nature of the interaction in a particular game. Specifically, we look at the effects of changing the rules of a game to convert sequential play into simultaneous play and vice versa and of changing the order of moves in sequential play. This topic gives us an opportunity to compare the equilibria found by using the concept of rollback, in a sequential-move game, with those found by using the Nash equilibrium concept, in the simultaneous version of the same game. From this comparison, we extend the concept of Nash equilibria to sequential-play games. It turns out that the rollback equilibrium is a special case, usually called a refinement, of these Nash equilibria.

1 GAMES WITH BOTH SIMULTANEOUS AND SEQUENTIAL MOVES

As mentioned several times thus far, most real games that you will encounter will be made up of numerous smaller components. Each of these components may entail simultaneous play or sequential play; so the full game requires you to be familiar with both. The most obvious examples of strategic interactions containing both sequential and simultaneous parts are those between two (or more) players over an extended period of time. You may play a number of different simultaneous-play games against your roommate, for example, in the course of a week; your play, as well as hers, in previous situations is important in determining how each of you decide to act in the next "round." Also, many sporting events, interactions between competing firms in an industry, and political relationships are sequentially linked series of simultaneous-move games. Such games are analyzed by combining the tools presented in Chapter 3 (trees and rollback) and in Chapters 4 and 5 (payoff tables and Nash equilibria).[1] The only difference is that the actual analysis becomes more complicated as the number of moves and interactions increases.

A. Two-Stage Games and Subgames

Our main illustrative example for such situations includes two would-be telecom giants, CrossTalk and GlobalDialog. Each can choose whether to invest $10 billion in the purchase of a fiberoptic network. They make their investment decisions simultaneously. If neither chooses to make the investment, that is the end of the game. If one invests and the other does not, then the investor has to make a pricing decision for its telecom services. It can choose either a high price, which will attract 60 million customers, from each of whom it will make an operating profit of $400, or a low price, which will attract 80 million customers, from each of whom it will make an operating profit of $200. If both firms acquire fiberoptic networks and enter the market, then their pricing choices become a second simultaneous-move game. Each can choose either the high or the low price. If both choose the high price, they will split the total market equally; so each will get 30 million customers and an operating profit of $400 from each. If both choose the low price, again they will split the total market equally; so each will get 40 million customers and an operating profit of $200 from each. If one chooses the high price and the other the low price, then the

[1] Sometimes the simultaneous part of the game will have equilibria in mixed strategies, when the tools we develop in Chapters 7 and 8 will be required. We mention this possibility in this chapter where relevant and give you an opportunity to use such methods in exercises for the later chapters.

low-price firm will get all the 80 million customers at that price, and the high-price firm will get nothing.

The interaction between CrossTalk and GlobalDialog forms a two-stage game. Of the four combinations of the simultaneous-move choices at the first (investment) stage, one ends the game, two lead to a second-stage (pricing) decision by just one player, and the fourth leads to a simultaneous-move (pricing) game at the second stage. We show this game pictorially in Figure 6.1.

Regarded as a whole, Figure 6.1 illustrates a game tree, but one that is more complex than the trees in Chapter 3. You can think of it as an elaborate "tree house" with multiple levels. The levels are shown in different parts of the same two-dimensional figure, as if you are looking down at the tree from a helicopter positioned directly above it.

The first-stage game is represented by the payoff table in the top-left quadrant of Figure 6.1. You can think of it as the first floor of the tree house. It has four "rooms." The room in the northwest corner corresponds to the "Don't invest" first-stage moves of both firms. If the firms' decisions take the game to this room, there are no further choices to be made; so we can think of it being like a terminal node of a tree in Chapter 3 and show the payoffs in the cell of the table;

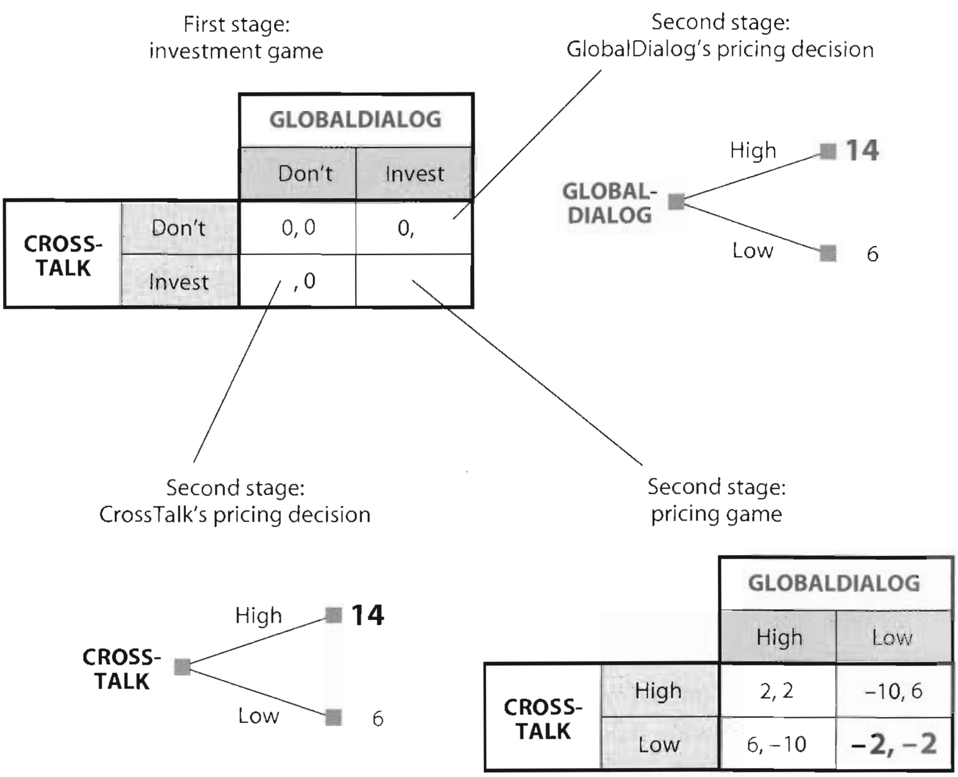


FIGURE 6.1 A Two-Stage Game Combining Sequential and Simultaneous Moves

both firms get 0. However, all of the other combinations of actions for the two firms lead to rooms that lead to further choices; so we cannot yet show the payoffs in those cells. Instead, we show branches leading to the second floor. The northeast and southwest rooms show only the payoff to the firm that has not invested; the branches leading from each of these rooms take us to single-firm pricing decisions in the second stage. The southeast room leads to a multiroomed second-floor structure within the tree house, which represents the second-stage pricing game that is played if both firms have invested in the first stage. This second-floor structure has four rooms corresponding to the four combinations of the two firms' pricing moves.

All of the second-floor branches and rooms are like terminal nodes of a game tree, so we can show the payoffs in each case. Payoffs here consist of each firm's operating profits minus the previous investment costs; payoff values are written in billions of dollars.

Consider the branch leading to the southwest corner of Figure 6.1. The game arrives in that corner if CrossTalk is the only firm that has invested. Then, if it chooses the high price, its operating profit is $400 $\times$ 60 million = $24 billion; after subtracting the $10 billion investment cost, its payoff is $14 billion, which we write as 14. In the same corner, if CrossTalk chooses the low price, then its operating profit is $200 $\times$ 80 million = $16 billion, yielding the payoff 6 after accounting for its original investment. In this situation, GlobalDialog's payoff is 0, as shown in the southwest room of the first floor of our tree. Similar calculations for the case in which GlobalDialog is the only firm to invest give us the payoffs shown in the northeast corner of Figure 6.1; again, the payoff of 0 for CrossTalk is shown in the northeast room of the first-stage game table.

If both firms invest, both play the second-stage pricing game illustrated in the southeast corner of the figure. When both choose the high price in the second stage, each gets operating profit of $400 $\times$ 30 million (half of the market), or $12 billion; after subtracting the $10 billion investment cost, each is left with a net profit of $2 billion, or a payoff of 2. If both firms choose the low price in the second stage, each gets operating profit of $200 $\times$ 40 million = $8 billion, and, after subtracting the $10 billion investment cost, each is left with a net loss of $2 billion, or a payoff of −2. Finally, if one firm charges the high price and the other firm the low price, then the low-price firm has operating profit of $200 $\times$ 80 million = $16 billion, leading to the payoff 6, while the high-price firm gets no operating profit and simply loses its $10 billion investment, for a payoff of −10.

As with any multistage game in Chapter 3, we must solve this game backward, starting with the second-stage game. In the two single-firm decision problems, we see at once that the high-price policy yields the higher payoff. We highlight this by showing that payoff in a larger-size type.

The second-stage pricing game has to be solved by using methods developed in Chapter 4. It is immediately evident, however, that this game is a prisoners' dilemma. Low is the dominant strategy for each firm; so the outcome is the room in the southeast corner of the second-stage game table; each firm gets payoff −2.[2] Again, we show these payoffs in a larger type size to highlight the fact that they are the payoffs obtained in the second-stage equilibrium.

Rollback now tells us that each first-stage configuration of moves should be evaluated by looking ahead to the equilibrium of the second-stage game (or the optimum second-stage decision) and the resulting payoffs. We can therefore substitute the payoffs that we have just calculated into the previously empty or partly empty rooms on the first floor of our tree house. This substitution gives us a first floor with known payoffs, shown in Figure 6.2.

Now we can use the methods of Chapter 4 to solve this simultaneous-move game. You should immediately recognize the game in Figure 6.2 as a chicken game. It has two Nash equilibria, each of which entails one firm choosing Invest and the other choosing Don't. The firm that invests makes a huge profit; so each firm prefers the equilibrium in which it is the investor while the other firm stays out. In Chapter 4, we briefly discussed the ways in which one of the two equilibria might get selected. We also pointed out the possibility that each firm might try to get its preferred outcome, with the result that both of them invest and both lose money. Indeed, this is what seems to have happened in the real-life play of this game. In Chapter 8, we investigate this type of game further, showing that it has a third Nash equilibrium, in mixed strategies.

Analysis of Figure 6.2 shows that the first-stage game in our example does not have a unique Nash equilibrium. This problem is not too serious, because we can leave the solution ambiguous to the extent that was done in the preceding paragraph. Matters would be worse if the second-stage game did not have a unique equilibrium. Then it would be essential to specify the precise process by which an outcome gets selected so that we could figure out the second-stage payoffs and use them to roll back to the first stage.

The second-stage pricing game shown in the table in the bottom-right quadrant of Figure 6.1 is one part of the complete two-stage game. However, it

		GLOBALDIALOG	
		Don't	Invest
CROSSTALK	Don't	0, 0	0, 14
	Invest	14, 0	−2, −2

FIGURE 6.2 Stage One Investment Game (After Substituting Rolled-Back Payoffs from the Equilibrium of the Second Stage)

[2]As is usual in a prisoners' dilemma, if the firms could successfully collude and charge high prices, both could get the higher payoff of 2. But this outcome is not an equilibrium, because each firm is tempted to cheat to try to get the much higher payoff of 6.

is also a full-fledged game in its own right, with a fully specified structure of players, strategies, and payoffs. To bring out this dual nature more explicitly, it is called a **subgame** of the full game.

More generally, a subgame is the part of a multimove game that begins at a particular node of the original game. The tree for a subgame is then just that part of the tree for the full game that takes this node as its root, or initial, node. A multimove game has as many subgames as it has decision nodes.

B. Configurations of Multistage Games

In the multilevel game illustrated in Figure 6.1, each stage consists of a sequential-move game. However, that may not always be the case. Simultaneous and sequential components may be mixed and matched in any way. We give two more examples to clarify this point and to reinforce the ideas introduced in the preceding section.

The first example is a slight variation of the CrossTalk–GlobalDialog game. Suppose one of the firms—say, GlobalDialog—has already made the $10 billion investment in the fiberoptic network. CrossTalk knows of this investment and now has to decide whether to make its own investment. If CrossTalk does not invest, then GlobalDialog will have a simple pricing decision to make. If CrossTalk invests, then the two firms will play the second-stage pricing game already described. The tree for this multistage game has conventional branches at the initial node and has a simultaneous-move subgame starting at one of the nodes to which these initial branches lead. The complete tree is shown in Figure 6.3.

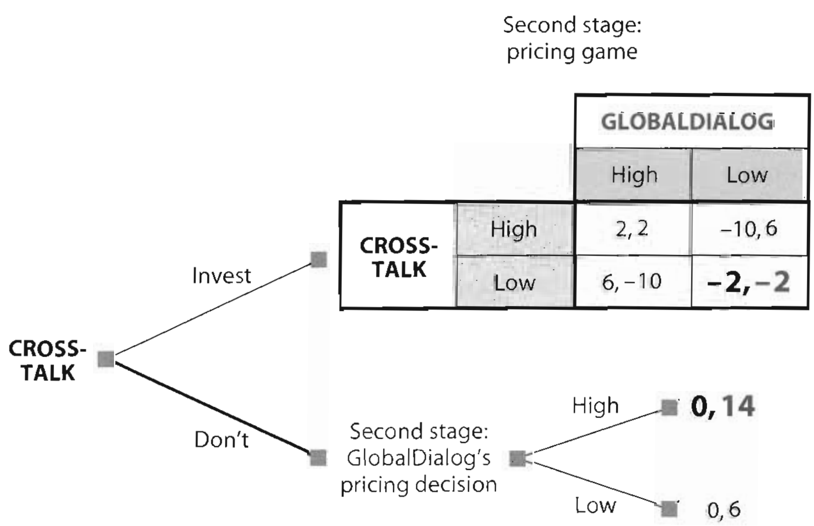


FIGURE 6.3 Two-Stage Game When One Firm Has Already Invested

When the tree has been set up, it is easy to analyze the game. We show the rollback analysis in Figure 6.3 by using large type for the equilibrium payoffs that result from the second-stage game or decision and a thicker branch for CrossTalk's first-stage choice. In words, CrossTalk figures out that, if it invests, the ensuing prisoners' dilemma of pricing will leave it with payoff −2, whereas staying out will get it 0. Thus it prefers the latter. GlobalDialog gets 14 instead of the −2 that it would have gotten if CrossTalk had invested, but CrossTalk's concern is to maximize its own payoff and not to ruin GlobalDialog deliberately.

This analysis does raise the possibility, though, that GlobalDialog may try to get its investment done quickly before CrossTalk makes its decision so as to ensure its most preferred outcome from the full game. And CrossTalk may try to beat GlobalDialog to the punch in the same way. In Chapter 10, we study some methods, called strategic moves, that may enable players to secure such advantages.

Our second example comes from football. Before each play, the coach for the offense chooses the play that his team will run; simultaneously, the coach for the defense sends his team out with instructions on how they should align themselves to counter the offense. Thus these moves are simultaneous. Suppose the offense has just two alternatives, a safe play and a risky play, and the defense may align itself to counter either of them. If the offense has planned to run the risky play and the quarterback sees the defensive alignment that will counter it, he can change the play at the line of scrimmage. And the defense, hearing the change, can respond by changing its own alignment. Thus we have a simultaneous-move game at the first stage, and one of the combination of choices of moves at this stage leads to a sequential-move subgame. Figure 6.4 shows the complete tree.

This is a zero-sum game, and we show only the offense's payoffs, measured in the number of yards that they expect to gain. The safe play gets 2 yards, even if the defense is ready for it; if the defense is not ready for it, the safe play does not do much better, gaining 6 yards. The risky play, if it catches the defense unready to cover it, gains 30 yards. But, if the defense is ready for the risky play, the offense loses 10 yards. We show this payoff of −10 for the offense at the terminal node where the offense does not change the play. If the offense changes the play (back to safe), the payoffs are 2 if the defense responds and 6 if it does not; these payoffs are the same as those that arise when the offense plans the safe play from the start.

We show the chosen branches in the sequential subgame as thick lines in Figure 6.4. It is easy to see that, if the offense changes its play, the defense will respond to keep the offense's gain to 2 rather than 6 and that the offense should change the play to get 2 rather than −10. Rolling back, we should put the resulting payoff, 2, in the bottom-right cell of the simultaneous-move game of the first stage. Then we see that this game has no Nash equilibrium in pure strategies.

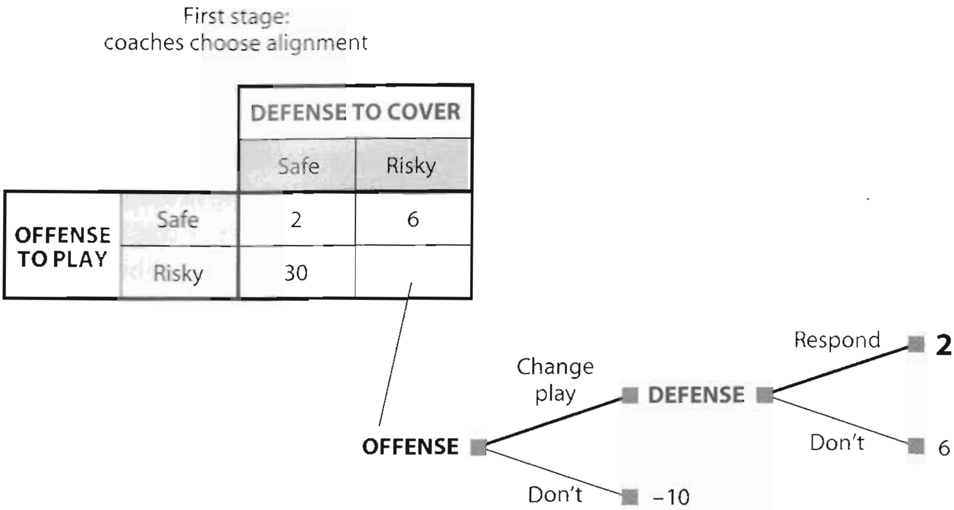


FIGURE 6.4 Simultaneous-Move First Stage Followed by Sequential Moves

The reason is the same as that in the tennis game of Chapter 4, Section 8; one player (defense) wants to match the moves (align to counter the play that the offense is choosing) while the other (offense) wants to unmatch moves (catch the defense in the wrong alignment). In Chapter 7, we show how to calculate the mixed-strategy equilibrium of such a game. It turns out that the offense should choose the risky play with probability 1/8, or 12.5%.

2 CHANGING THE ORDER OF MOVES IN A GAME

The games considered in preceding chapters were presented as either sequential or simultaneous in nature. We used the appropriate tools of analysis to predict equilibria in each type of game. In Section 1 of this chapter, we discussed games with elements of both sequential and simultaneous play. These games required both sets of tools to find solutions. But what about games that could be played either sequentially or simultaneously? How would changing the play of a particular game and thus changing the appropriate tools of analysis alter the expected outcomes?

The task of turning a sequential-play game into a simultaneous one requires changing only the timing or observability with which players make their choices of moves. Sequential-move games become simultaneous if the players cannot observe moves made by their rivals before making their own choices. In that case, we would analyze the game by searching for a Nash equilibrium rather than for a rollback equilibrium. Conversely, a simultaneous-move game could become sequential if one player were able to observe the other's move before choosing her own.

Any changes to the rules of the game can also change its outcomes. Here, we illustrate a variety of possibilities that arise owing to changes in different types of games.

A. Changing Simultaneous-Move Games into Sequential-Move Games

I. NO CHANGE IN OUTCOME Certain games have the same outcomes in the equilibria of both simultaneous and sequential versions and regardless of the order of play in the sequential-play game. This result generally arises only when both or all players have dominant strategies. We show that it holds for the prisoners' dilemma.

Consider the prisoners' dilemma game of Chapter 4, in which a husband and wife are being questioned regarding their roles in a crime. The simultaneous version of that game, reproduced in Figure 6.5a, can be redrawn as either of the sequential-play games shown in Figure 6.5b and c. As in Figure 4.4, the payoff numbers indicate years in jail; so low numbers are better than high ones. In Figure 6.5b, Husband chooses his strategy before Wife does; so she knows what he has chosen before making her own choice; in Figure 6.5c the roles are reversed.

The Nash equilibrium of the prisoners' dilemma game in Figure 6.5a is for both players to confess (or to defect from cooperating with the other). Using rollback to solve the sequential versions of the game, illustrated in Figure 6.5b and c, we see that the second player does best to confess if the first has confessed (10 rather than 25 years in jail) and the second player also does best to confess if the first has denied (1 year rather than 3 years of jail). Given these choices by the second player, the first player does best to confess (10 rather than 25 years in jail). The equilibrium entails 10 years of jail for both players regardless of which player moves first. Thus, the equilibrium is the same in all three versions of this game.

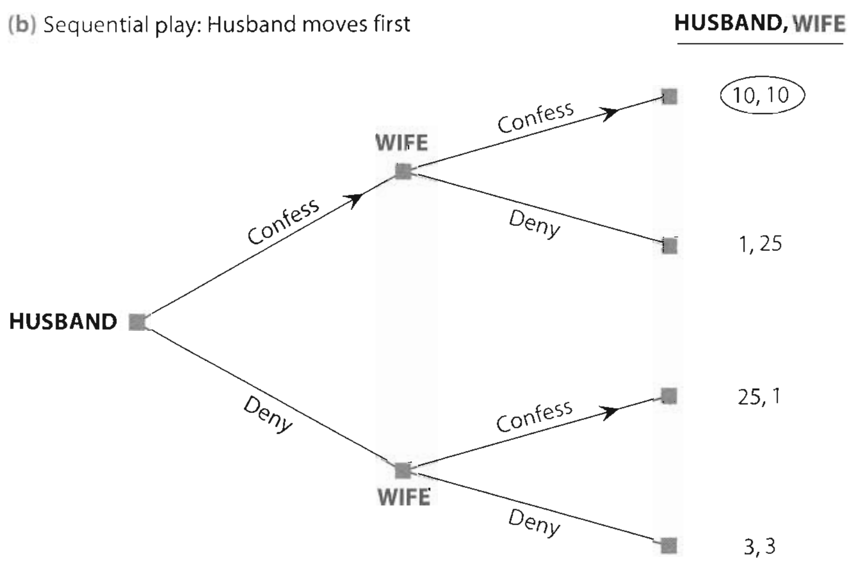
II. FIRST-MOVER ADVANTAGE A first-mover advantage may emerge when the rules of a game are changed from simultaneous to sequential play. At a minimum, if the simultaneous-move version has multiple equilibria, the sequential-move version enables the first mover to choose his preferred outcome. We illustrate such a situation with the use of chicken, the game in which two teenagers drive toward each other in their cars, both determined not to swerve. We reproduce the strategic form of Figure 4.14 in Figure 6.6a and two extensive forms, one for each possible ordering of play, in Figure 6.6b and c.

Under simultaneous play, the two outcomes in which one player swerves (is "chicken") and the others goes straight (is "tough") are both pure-strategy Nash

(a) Simultaneous play

		WIFE	
		Confess (Defect)	Deny (Cooperate)
HUSBAND	Confess (Defect)	10 yr, 10 yr	1 yr, 25 yr
	Deny (Cooperate)	25 yr, 1 yr	3 yr, 3 yr

(b) Sequential play: Husband moves first



Sequential moves: Wife moves first

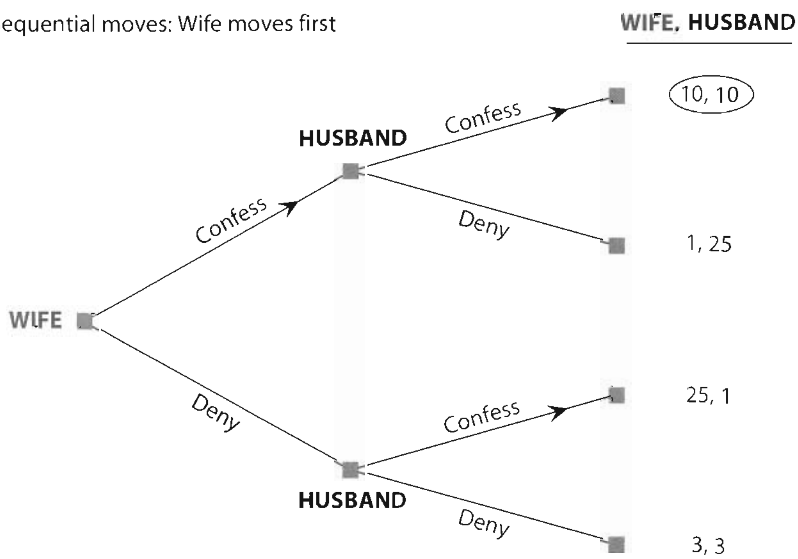


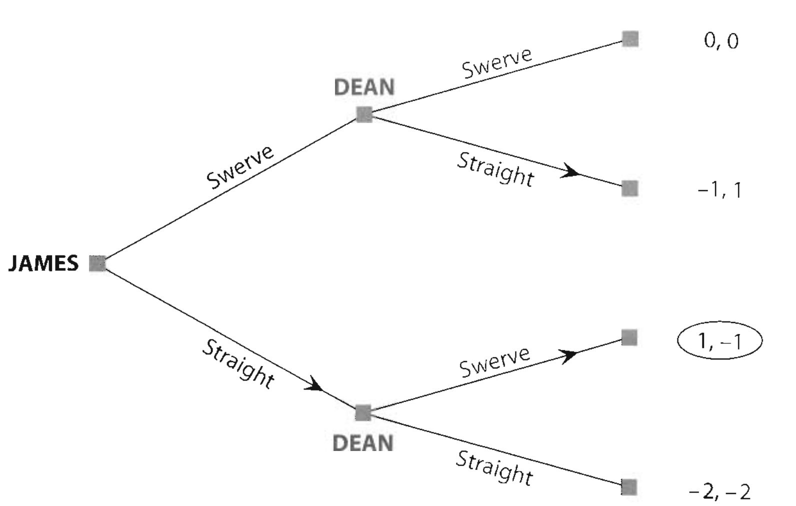
FIGURE 6.5 Three Versions of the Prisoners' Dilemma Game

(a) Simultaneous play

		DEAN	
		Swerve (Chicken)	Straight (Tough)
JAMES	Swerve (Chicken)	0, 0	−1, 1
	Straight (Tough)	1, −1	−2, −2

(b) Sequential play: James moves first

JAMES, DEAN



(c) Sequential play: Dean moves first

DEAN, JAMES

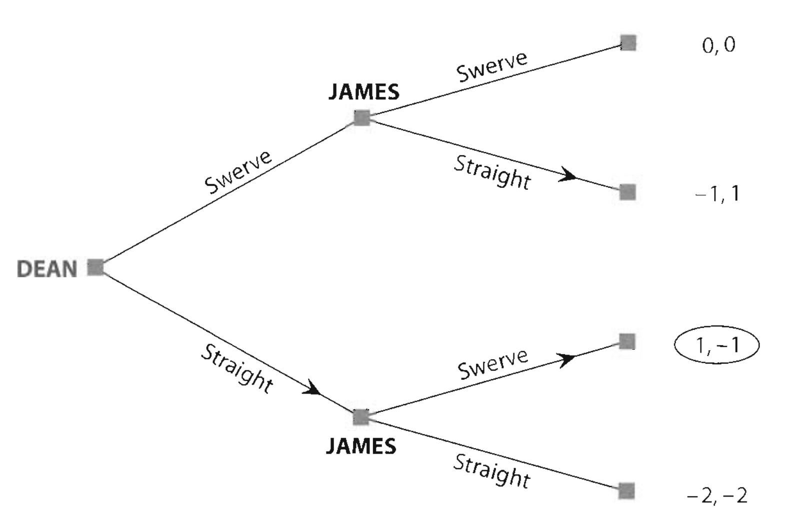


FIGURE 6.6 Chicken in Simultaneous- and Sequential-Play Versions

equilibria. Without specification of some historical, cultural, or other convention, neither has a claim to be a focal point. Our analysis in Chapter 4 suggested that coordinated play could help the players in this game, perhaps through an agreement to alternate between the two equilibria.

When we alter the rules of the game to allow one of the players the opportunity to move first, there are no longer two equilibria. Rather, we see that the second mover's equilibrium strategy is to choose the action opposite that chosen by the first mover. Rollback then shows that the first mover's equilibrium strategy is Straight. We see in Figure 6.6b and c that allowing one person to move first and to be observed making the move results in a single rollback equilibrium in which the first mover gets a payoff of 1, while the second mover gets a payoff of -1. The actual play of the game becomes almost irrelevant under such rules, which may make the sequential version uninteresting to many observers. Although teenagers might not want to play such a game with the rule change, the strategic consequences of the change are significant.

III. SECOND-MOVER ADVANTAGE In other games, a second-mover advantage may emerge when simultaneous play is changed into sequential play. This can be illustrated using the tennis game of Chapter 4. Recall that, in that game, Evert is planning the location of her return while Navratilova considers where to cover. The version considered earlier assumed that both players were skilled at disguising their intended moves until the very last moment so that they moved at essentially the same time. If Evert's movement as she goes to hit the ball belies her shot intentions, however, then Navratilova can react and move second in the game. In the same way, if Navratilova leans toward the side that she intends to cover before Evert actually hits her return, then Evert is the second mover. Figure 6.7 shows all three possibilities. The simultaneous-move version is Figure 4.15 reproduced as Figure 6.7a; the two orderings of the sequential-play game are Figure 6.7b and c.

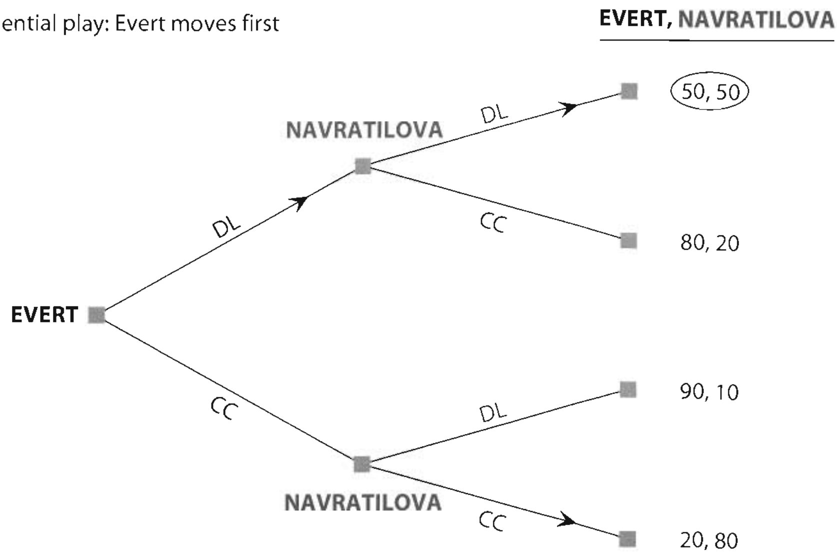
The simultaneous-play version of this game has no equilibrium in pure strategies. In each ordering of the sequential version, however, there is a unique rollback equilibrium outcome; the equilibrium differs, depending on who moves first. If Evert moves first, then Navratilova chooses to cover whichever direction Evert chooses and Evert opts for a down-the-line shot. Each player is expected to win the point half the time in this equilibrium. If the order is reversed, Evert chooses to send her shot in the opposite direction from that which Navratilova covers; so Navratilova should move to cover crosscourt. In this case, Evert is expected to win the point 80% of the time. The second mover does better by being able to respond optimally to the opponent's move.

We return to the simultaneous version of this game in Chapter 7. There we show that it does have a Nash equilibrium in mixed strategies. In that equilibrium, Evert succeeds on average 62% of the time. Her success rate in the mixed-

(a) Simultaneous play

		NAVRATILOVA	
		DL	CC
EVERT	DL	50	80
	CC	90	20

(b) Sequential play: Evert moves first



(c) Sequential play: Navratilova moves first

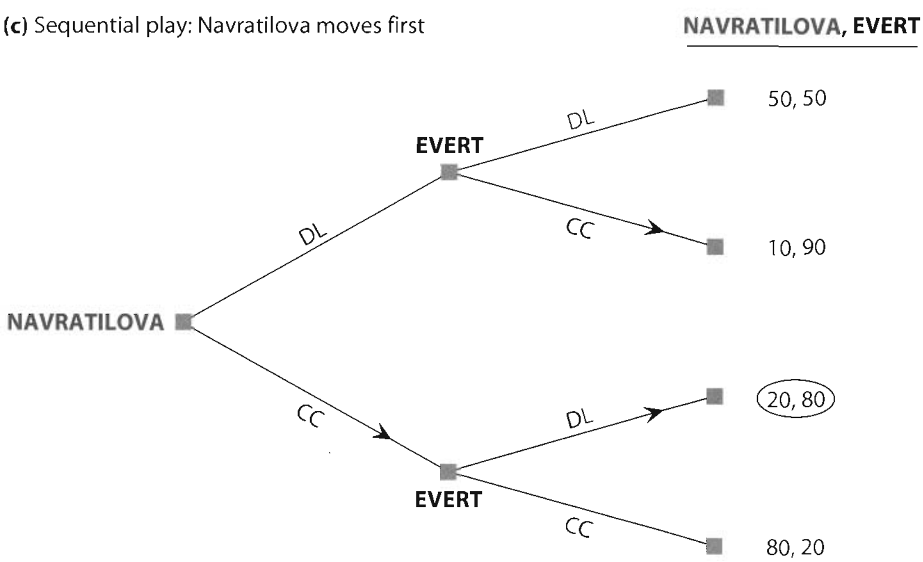


FIGURE 6.7 Tennis Game in Simultaneous- and Sequential-Play Versions

strategy equilibrium of the simultaneous game is thus better than the 50% that she gets by moving first but is worse than the 80% that she gets by moving second, in the two sequential-move versions.

IV. BOTH PLAYERS MAY DO BETTER That a game may have a first-mover or a second-mover advantage, which is suppressed when moves have to be simultaneous but emerges when an order of moves is imposed, is quite intuitive. Somewhat more surprising is the possibility that both players may do better under one set of rules of play than under another. We illustrate this possibility by using the game of monetary and fiscal policies played by the Federal Reserve and the Congress. In Chapter 4, we studied this game with simultaneous moves; we reproduce the payoff table (Figure 4.5) as Figure 6.8a and show the two sequential-move versions as Figure 6.8b and c. For brevity, we write the strategies as Balance and Deficit instead of Budget Balance and Budget Deficit for the Congress and as High and Low instead of High Interest Rates and Low Interest Rates for the Fed.

In the simultaneous-move version, the Congress has a dominant strategy (Deficit), and the Fed, knowing this, chooses High, yielding payoffs of 2 to both players. Almost the same thing happens in the sequential version where the Fed moves first. The Fed foresees that, for each choice it might make, the Congress will respond with Deficit. Then High is the better choice for Fed, yielding 2 instead of 1.

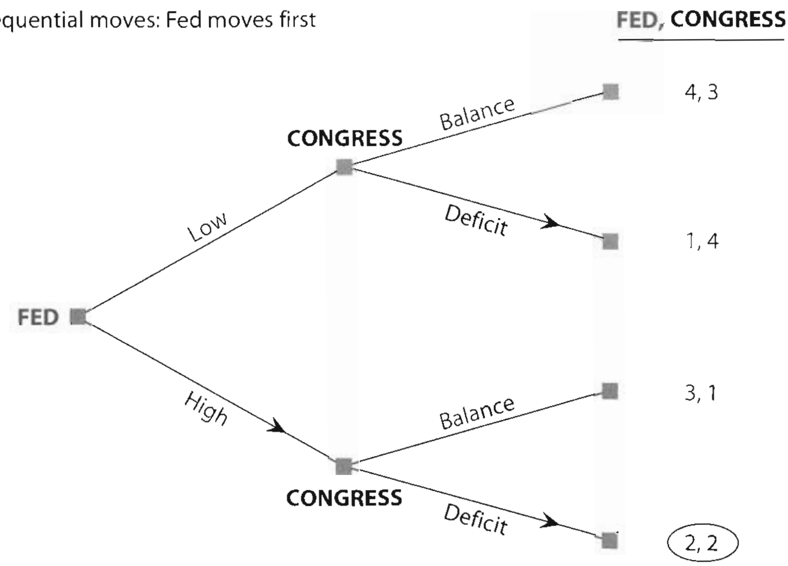
But the sequential-move version where the Congress moves first is different. Now the Congress foresees that, if it chooses Deficit, the Fed will respond with High, whereas, if it chooses Balance, the Fed will respond with Low. Of these two developments, the Congress prefers the latter, where it gets payoff 3 instead of 2. Therefore the rollback equilibrium with this order of moves is for the Congress to choose a balanced budget and the Fed to respond with low interest rates. The resulting payoffs, 3 for the Congress and 4 for the Fed, are better for both players than those of the other two versions.

The difference between the two outcomes is even more surprising because the better outcome obtained in Figure 6.8c results from the Congress choosing Balance, which is its dominated strategy in Figure 6.8a. To resolve the apparent paradox, one must understand more precisely the meaning of dominance. For Deficit to be a dominant strategy, it must be better than Balance from the Congress's perspective for *each given* choice of the Fed. This type of comparison between Deficit and Balance is relevant in the simultaneous-move game because there the Congress must make a decision without knowing the Fed's choice. The Congress must think through, or formulate a belief about, the Fed's action, and choose its best response to that. In our example, this best response is always Deficit for the Congress. The concept of dominance is also relevant with sequential moves if the Congress moves second, because then it knows what the

(a) Simultaneous moves

		FEDERAL RESERVE	
		Low interest rates	High interest rates
CONGRESS	Budget balance	3, 4	1, 3
	Budget deficit	4, 1	2, 2

(b) Sequential moves: Fed moves first



(c) Sequential moves: Congress moves first

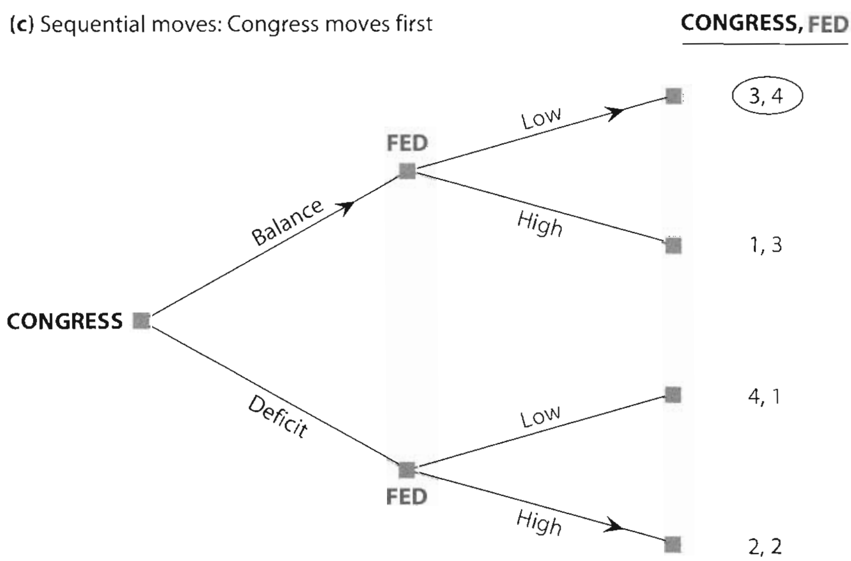


FIGURE 6.8 Three Versions of the Monetary–Fiscal Policy Game

Fed has already done and merely picks its best response, which is always Deficit. However, if the Congress moves first, it cannot take the Fed's choice as *given*. Instead, it must recognize how the Fed's second move will be affected by its own first move. Here it knows that the Fed will respond to Deficit with High and to Balance with Low. The Congress is then left to choose between these two alternatives; its most preferred outcome of Deficit and Low becomes irrelevant because it is precluded by the Fed's response.

The idea that dominance may cease to be a relevant concept for the first mover reemerges in Chapter 10. There we consider the possibility that one player or the other may deliberately change the rules of a game to become the first mover. Players can alter the outcome of the game in their favor in this way.

Suppose that the two players in our current example could choose the order of moves in the game. In this case, they would agree that the Congress should move first. Indeed, when budget deficits and inflation threaten, the chairs of the Federal Reserve in testimony before various Congressional committees often offer such deals; they promise to respond to fiscal discipline by lowering interest rates. But it is often not enough to make a verbal deal with the other player. The technical requirements of a first move—namely, that it be observable to the second mover and not reversible thereafter—must be satisfied. In the context of macroeconomic policies, it is fortunate that the legislative process of fiscal policy in the United States is both very visible and very slow, whereas monetary policy can be changed quite quickly in a meeting of the Federal Reserve Board. Therefore the sequential play where the Congress moves first and the Fed moves second is quite realistic.

B. Other Changes in the Order of Moves

The preceding section presented various examples in which the rules of the game were changed from simultaneous play to sequential play. We saw how and why such rule changes can change the outcome of a game. The same examples also serve to show what happens if the rules are changed in the opposite direction, from sequential to simultaneous moves. Thus, if a first- or a second-mover advantage exists with sequential play, it can be lost under simultaneous play. And, if a specific order benefits both players, then losing the order can hurt both.

The same examples also show us what happens if the rules are changed to reverse the order of play while keeping the sequential nature of a game unchanged. If there is a first-mover or a second-mover advantage, then the player who shifts from moving first to moving second may benefit or lose accordingly, with the opposite change for the other player. And, if one order is in the common interests of both, then an externally imposed change of order can benefit or hurt them both.

3 CHANGE IN THE METHOD OF ANALYSIS

Game trees are the natural way to display sequential-move games, and payoff tables the natural representation of simultaneous-move games. However, each technique can be adapted to the other type of game. Here we show how to translate the information contained in one illustration to an illustration of the other type. In the process, we develop some new ideas that will prove useful in subsequent analysis of games.

A. Illustrating Simultaneous-Move Games by Using Trees

Consider the game of the passing shot in tennis as originally described in Chapter 4, where the action is so quick that moves are truly simultaneous, as shown in Figure 6.7a. But suppose we want to show the game in extensive form—that is, by using a tree. We show how this can be done in Figure 6.9.

To draw the tree in the figure, we must choose one player—say, Evert—to make her choice at the initial node of the tree. The branches for her two choices, DL and CC, then end in two nodes at each of which Navratilova makes her choices. However, because the moves are actually simultaneous, Navratilova must choose without knowing what Evert has picked. That is, she must choose without knowing whether she is at the node following Evert's DL branch or the one following Evert's CC branch. Our tree diagram must in some way show this lack of information on Navratilova's part.

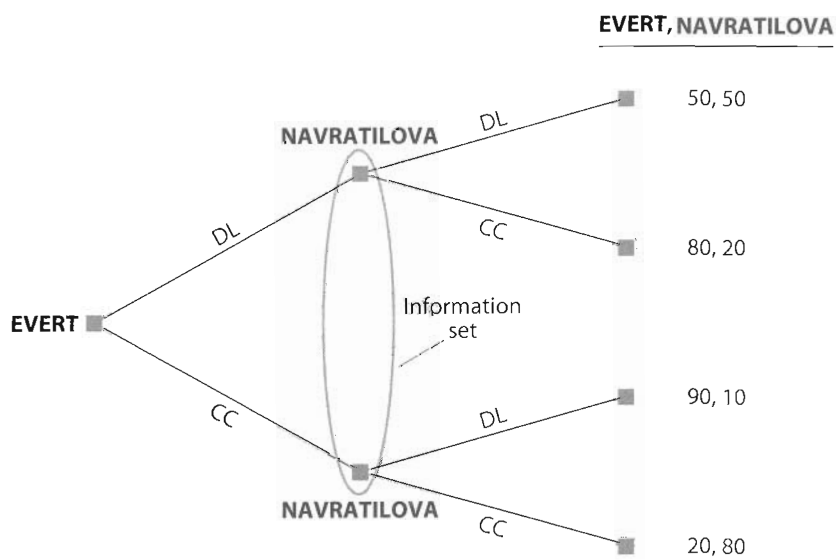


FIGURE 6.9 Simultaneous-Move Tennis Game Shown in Extensive Form

We illustrate Navratilova's uncertainty about the node from which her decision is being made by drawing an oval to surround the two relevant nodes. (An alternative is to connect them by a dotted line; a dotted line is used to distinguish it from the solid lines that represent the branches of the tree.) The nodes within this oval or balloon are called an **information set** for the player who moves there. Such a set indicates the presence of imperfect information for the player; she cannot distinguish between the nodes in the set, given her available information (because she cannot observe the row player's move before making her own). As such, her strategy choice from within a single information set must specify the same move at all the nodes contained in it. That is, Navratilova must choose either DL at both the nodes in this information set or CC at both of them. She cannot choose DL at one and CC at the other, as she could in Figure 6.7b, where the game had sequential moves and she moved second.

Accordingly, we must adapt our definition of strategy. In Chapter 3, we defined a strategy as a complete plan of action, specifying the move that a player would make at each *node* where the rules of the game specified that it was her turn to move. We should now more accurately redefine a strategy as a complete plan of action, specifying the move that a player would make at each *information set* at whose nodes the rules of the game specify that it is her turn to move.

The concept of an information set is also relevant when a player is uncertain about some external circumstances of the game, rather than about another player's moves. For example, a farmer planting a crop may be uncertain about the forces of climate, such as el Niño, that are in motion and will affect his outcomes. We can show this outside player, usually called Nature, as making the first move to choose alternatives (no, weak, or strong el Niño), with probabilities that are known from experience. Then we enclose the three nodes into an information set and have the farmer make his choice, constraining him to take the same action at all three nodes.

Although this representation is conceptually simple, it does not provide any simpler way of solving the game. Therefore we use it only occasionally, where it conveys some point more simply. Some examples of game illustrations using information sets can be found later in Chapters 9 and 14.

B. Showing and Analyzing Sequential-Move Games in Strategic Form

Consider now the sequential-move game of monetary and fiscal policy from Figure 6.8c, in which the Congress has the first move. Suppose we want to show it in normal or strategic form—that is, by using a payoff table. The rows and the columns of the table are the strategies of the two players. We must therefore begin by specifying the strategies.

For the Congress, the first mover, listing its strategies is easy. There are just two moves—Balance and Deficit—and they are also the two strategies. For the second mover, matters are more complex. Remember that a strategy is a complete plan of action, specifying the moves to be made at each node where it is a player's turn to move. Because the Fed gets to move at two nodes (and because we are supposing that this game actually has sequential moves and so the two nodes are not confounded into one information set) and can choose either Low or High at each node, there are four combinations of its choice patterns. These combinations are (1) Low if Balance, High if Deficit (we write this as "L if B, H if D" for short); (2) High if Balance, Low if Deficit ("H if B, L if D" for short); (3) Low always; and (4) High always.

We show the resulting two-by-four payoff matrix in Figure 6.10. The last two columns are no different from those for the two-by-two payoff matrix for the game under simultaneous-move rules (Figure 6.8a). This is because, if the Fed is choosing a strategy in which it makes the same move always, it is just as if the Fed were moving without taking into account what the Congress had done; it is as if their moves were simultaneous. But calculation of the payoffs for the first two columns, where the Fed's second move does depend on the Congress's first move, needs some care.

To illustrate, consider the cell in the first row and the second column. Here the Congress is choosing Balance, and the Fed is choosing "H if B, L if D." Given Congress's choice, the Fed's actual choice under this strategy is High. Then the payoffs are those for the Balance and High combination—namely, 1 for Congress and 3 for the Fed.

Cell-by-cell inspection quickly shows that the game has two pure-strategy Nash equilibria, which we show by shading the cells gray. One is in the top-left cell, where the Congress's strategy is Balance and the Fed's is "L if B, H if D," and so the Fed's actual choice is L. This outcome is just the rollback equilibrium of the sequential-move game. But there is another Nash equilibrium in the bottom-right cell, where the Congress chooses Deficit and the Fed chooses "High always." As always in a Nash equilibrium, neither player has a clear reason to deviate from the strategies that lead to this outcome. The Congress would do worse by switching to Balance, and the Fed could do no better by switching to any of its other three strategies, although it could do just as well with "L if B, H if D."

		FED			
		L if B, H if D	H if B, L if D	Low always	High always
CONGRESS	Balance	3, 4	1, 3	3, 4	1, 3
	Deficit	2, 2	4, 1	4, 1	2, 2

FIGURE 6.10 Sequential-Move Game of Monetary and Fiscal Policy in Strategic Form

The sequential-move game, when analyzed in its extensive form, produced just one rollback equilibrium. But when analyzed in its normal or strategic form, it has two Nash equilibria. What is going on?

The answer lies in the different nature of the logic of Nash and rollback analyses. Nash equilibrium requires that neither player have a reason to deviate, given the strategy of the other player. However, rollback does not take the strategies of later movers as given. Instead, it asks what would be optimal to do if the opportunity to move actually arises.

In our example, the Fed's strategy of "High always" does not satisfy the criterion of being optimal if the opportunity to move actually arises. If the Congress chose Low, then High is indeed the Fed's optimal response. However, if the Congress chose Balance and the Fed had to respond, it would want to choose Low, not High. So "High always" does not describe the Fed's optimal response in all possible configurations of play and cannot be a rollback equilibrium. But the logic of Nash equilibrium does not impose such a test, instead regarding the Fed's "High always" as a strategy that the Congress could legitimately take as given. If it does so, then Deficit is the Congress's best response. And, conversely, "High always" is one best response of the Fed to the Congress's Deficit (although it is tied with "L if B, H if D"). Thus the pair of strategies "Deficit" and "High always" are mutual best responses and constitute a Nash equilibrium, although they do not constitute a rollback equilibrium.

We can therefore think of rollback as a further test, supplementing the requirements of a Nash equilibrium and helping to select from among multiple Nash equilibria of the strategic form. In other words, it is a refinement of the Nash equilibrium concept.

To state this idea somewhat more precisely, recall the concept of a subgame. At any one node of the full game tree, we can think of the part of the game that begins there as a subgame. In fact, as successive players make their choices, the play of the game moves along a succession of nodes, and each move can be thought of as starting a subgame. The equilibrium derived by using rollback corresponds to one particular succession of choices in each subgame and gives rise to one particular path of play. Certainly, other paths of play are consistent with the rules of the game. We call these other paths **off-equilibrium paths,** and we call any subgames that arise along these paths **off-equilibrium subgames,** for short.

With this terminology, we can now say that the equilibrium path of play is itself determined by the players' expectations of what would happen if they chose a different action—if they moved the game to an off-equilibrium path and started an off-equilibrium subgame. Rollback requires that all players make their best choices in *every* subgame of the larger game, whether or not the subgame lies along the path to the ultimate equilibrium outcome.

Strategies are complete plans of action. Thus a player's strategy must specify what she will do in each eventuality, or each and every node of the game, whether on or off the equilibrium path, where it is her turn to act. When one such node arrives, only the plan of action starting there—namely, the part of the full strategy that pertains to the subgame starting at that node—is pertinent. This part is called the **continuation** of the strategy for that subgame. Rollback requires that the equilibrium strategy be such that its continuation in every subgame is optimal for the player whose turn it is to act at that node, whether or not the node and the subgame lie on the equilibrium path of play.

Return to the monetary policy game with the Congress moving first, and consider the second Nash equilibrium that arises in its strategic form. Here the path of play is for the Congress to choose Deficit and the Fed to choose High. On the equilibrium path, High is indeed the Fed's best response to Deficit. The Congress's choice of Low would be the start of an off-equilibrium path. It leads to a node where a rather trivial subgame starts—namely, a decision by the Fed. The Fed's purported equilibrium strategy "High always" asks it to choose High in this subgame. But that is not optimal; this second equilibrium is specifying a nonoptimal choice for an off-equilibrium subgame.

In contrast, the equilibrium path of play for the Nash equilibrium in the upper-left corner of Figure 6.10 is for the Congress to choose Balance and the Fed to follow with Low. The Fed is responding optimally on the equilibrium path. The off-equilibrium path would have the Congress choosing Deficit, and the Fed, given its strategy of "L if B, H if D," would follow with High. It is optimal for the Fed to respond to Deficit with High, so the strategy remains optimal off the equilibrium path, too.

The requirement that continuation of a strategy remain optimal under all circumstances is important because the equilibrium path itself is the result of players' thinking strategically about what would happen if they did something different. A later player may try to achieve an outcome that she would prefer by threatening the first mover that certain actions would be met with dire responses or by promising that certain other actions would be met with nice responses. But the first mover will be skeptical of the **credibility** of such threats and promises. The only way to remove that doubt is to check if the stated responses would actually be optimal if the need arose. If the responses are not optimal, then the threats or promises are not credible, and the responses would not be observed along the equilibrium path of play.

The equilibrium found by using rollback is called a **subgame-perfect equilibrium (SPE).** It is a set of strategies (complete plans of action), one for each player, such that, at every node of the game tree, whether or not the node lies along the equilibrium path of play, the continuation of the same strategy in the subgame starting at that node is optimal for the player who takes the action there. More simply, an SPE requires players to use strategies that constitute a Nash equilibrium in every subgame of the larger game.

In fact, as a rule, in games with finite trees and perfect information, where players can observe every previous action taken by all other players so that there are no multiple nodes enclosed in one information set, rollback finds the unique (except for trivial and exceptional cases of ties) subgame-perfect equilibrium of the game. Consider: if you look at any subgame that begins at the last decision node for the last player who moves, the best choice for that player is the one that gives her the highest payoff. But that is precisely the action chosen with the use of rollback. As players move backward through the game tree, rollback eliminates all unreasonable strategies, including incredible threats or promises, so that the collection of actions ultimately selected is the SPE. Therefore, for the purposes of this book, subgame perfectness is just a fancy name for rollback. At more advanced levels of game theory, where games include complex information structures and information sets, subgame perfectness becomes a richer notion.

4 THREE-PLAYER GAMES

We have restricted the discussion so far in this chapter to games with two players and two moves each. But the same methods also work for some larger and more general examples. We now illustrate this by using the street-garden game of Chapter 3. Specifically, we (1) change the rules of the game from sequential to simultaneous moves and then (2) keep the moves sequential but show and analyze the game in its strategic form. First we reproduce the tree of that sequential-move game (Figure 3.6) as Figure 6.11 here and remind you of the rollback equilibrium.

The equilibrium strategy of the first mover (Emily) is simply a move, "Don't contribute." The second mover chooses from among four possible strategies (choice of two responses at each of two nodes) and chooses the strategy "Don't contribute (D) if Emily has chosen her Contribute, and Contribute (C) if Emily has chosen her Don't contribute," or, more simply, "D if C, C if D," or even more simply "DC." Talia has 16 available strategies (choice of two responses at each of four nodes), and her equilibrium strategy is "D following Emily's C and Nina's C, C following their CD, C following their DC, and D following their DD," or "DCCD" for short.

Remember, too, the reason for these choices. The first mover has the opportunity to choose Don't, knowing that the other two will recognize that the nice garden won't be forthcoming unless they contribute and that they like the nice garden sufficiently strongly that they will contribute.

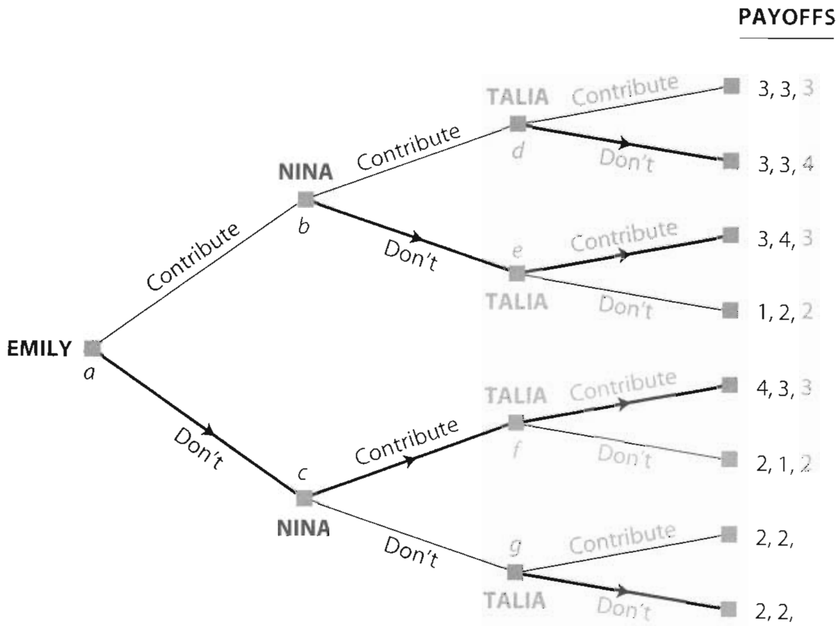


FIGURE 6.11 The Street-Garden Game with Sequential Moves

Now we change the rules of the game to make it a simultaneous-move game. (In Chapter 4, we solved a simultaneous-move version with somewhat different payoffs; here we keep the payoffs the same as in Chapter 3.) The payoff matrix is in Figure 6.12. Cell-by-cell inspection shows very easily that there are four Nash equilibria.

In three of the Nash equilibria of the simultaneous-move game, two players contribute, while the third does not. These equilibria are similar to the rollback equilibrium of the sequential-move game. In fact, each one corresponds to the rollback equilibrium of the sequential game with a particular order of play. Further, any given order of play in the sequential-move version of this game leads to the same simultaneous-move payoff table.

TALIA chooses:

Contribute

		NINA	
		Contribute	Don't
EMILY	Contribute	3, 3, 3	3, 4, 3
	Don't	4, 3, 3	2, 2, 1

Don't Contribute

		NINA	
		Contribute	Don't
EMILY	Contribute	3, 3, 4	1, 2, 2
	Don't	2, 1, 2	2, 2, 2

FIGURE 6.12 The Street-Garden Game with Simultaneous Moves

But there is also a fourth Nash equilibrium here, where no one contributes. *Given* the specified strategies of the other two—namely, Don't contribute—any one player is powerless to bring about the nice garden and therefore chooses not to contribute as well. Thus, in the change from sequential to simultaneous moves, the first-mover advantage has been lost. Multiple equilibria arise, only one of which retains the original first mover's high payoff.

Next we return to the sequential-move version—Emily first, Nina second, and Talia third—but show the game in its normal or strategic form. In the sequential-move game, Emily has 2 pure strategies, Nina has 4, and Talia has 16; so this means constructing a payoff table that is 2 by 4 by 16. With the use of the same conventions as we used for three-player tables in Chapter 4, this particular game would require a table with 16 "pages" of two-by-four payoff tables. That would look too messy; so we opt instead for a reshuffling of the players. Let Talia be the row player, Nina be the column player, and Emily be the page player. Then "all" that is required to illustrate this game is the 16 by 4 by 2 game table shown in Figure 6.13. The order of payoffs still corresponds to our earlier convention in that they are listed row, column, page player; in our example, that means the payoffs are now listed in the order Talia, Nina, and Emily.

As in the monetary–fiscal policy game between the Fed and the Congress, there are multiple Nash equilibria in the simultaneous street-garden game. (In Exercise 6, we ask you to find them all.) But there is only one subgame-perfect equilibrium, corresponding to the rollback equilibrium found in Figure 6.12. Although cell-by-cell inspection finds all of the Nash equilibria, iterated elimination of dominated strategies can reduce the number of reaonable equilibria for us here. This process works because elimination identifies those strategies that include noncredible components (such as "High always" for the Fed in Section 3.B). As it turns out, such elimination can take us all the way to the unique subgame-perfect equilibrium.

In Figure 6.13, we start with Talia and eliminate all of her (weakly) dominated strategies. This step eliminates all but the strategy listed in the eleventh row of the table, DCCD, which we have already identified as Talia's rollback equilibrium strategy. Elimination can continue with Nina, for whom we must compare outcomes from strategies across both pages of the table. To compare her CC to CD, for example, we look at the payoffs associated with CC in *both pages* of the table and compare these payoffs with the similarly identified payoffs for CD. For Nina, the elimination process leaves only her strategy DC; again, this is the rollback equilibrium strategy found for her above. Finally, Emily has only to compare the two remaining cells associated with her choice of Don't and Contribute; she gets the highest payoff when she chooses Don't and so makes that choice. As before, we have identified her rollback equilibrium strategy.

The unique subgame-perfect outcome in the game table in Figure 6.13 thus corresponds to the cell associated with the rollback equilibrium strategies for each player. Note that the process of iterated elimination that leads us to this

	EMILY							
	Contribute				Don't			
	NINA				NINA			
TALIA	CC	CD	DC	DD	CC	CD	DC	DD
CCCC	3, 3, 3	3, 3, 3	3, 4, 3	3, 4, 3	3, 3, 4	1, 2, 2	3, 3, 4	1, 2, 2
CCCD	3, 3, 3	3, 3, 3	3, 4, 3	3, 4, 3	3, 3, 4	2, 2, 2	3, 3, 4	2, 2, 2
CCDC	3, 3, 3	3, 3, 3	3, 4, 3	3, 4, 3	2, 1, 2	1, 2, 2	2, 1, 2	1, 2, 2
CDCC	3, 3, 3	3, 3, 3	2, 2, 1	2, 2, 1	3, 3, 4	1, 2, 2	3, 3, 4	1, 2, 2
DCCC	4, 3, 3	4, 3, 3	3, 4, 3	3, 4, 3	3, 3, 4	1, 2, 2	3, 3, 4	1, 2, 2
CCDD	3, 3, 3	3, 3, 3	3, 4, 3	3, 4, 3	2, 1, 2	2, 2, 2	2, 1, 2	2, 2, 2
CDDC	3, 3, 3	3, 3, 3	2, 2, 1	2, 2, 1	2, 1, 2	1, 2, 2	2, 1, 2	1, 2, 2
DDCC	4, 3, 3	4, 3, 3	2, 2, 1	2, 2, 1	3, 3, 4	1, 2, 2	3, 3, 4	1, 2, 2
CDCD	3, 3, 3	3, 3, 3	2, 2, 1	2, 2, 1	3, 3, 4	2, 2, 2	3, 3, 4	2, 2, 2
DCDC	4, 3, 3	4, 3, 3	3, 4, 3	3, 4, 3	2, 1, 2	1, 2, 2	2, 1, 2	1, 2, 2
DCCD	4, 3, 3	4, 3, 3	3, 4, 3	3, 4, 3	3, 3, 4	2, 2, 2	3, 3, 4	2, 2, 2
CDDD	3, 3, 3	3, 3, 3	2, 2, 1	2, 2, 1	2, 1, 2	2, 2, 2	2, 1, 2	2, 2, 2
DCDD	4, 3, 3	4, 3, 3	3, 4, 3	3, 4, 3	2, 1, 2	2, 2, 2	2, 1, 2	2, 2, 2
DDCD	4, 3, 3	4, 3, 3	2, 2, 1	2, 2, 1	3, 3, 4	2, 2, 2	3, 3, 4	2, 2, 2
DDDC	4, 3, 3	4, 3, 3	2, 2, 1	2, 2, 1	2, 1, 2	1, 2, 2	2, 1, 2	1, 2, 2
DDDD	4, 3, 3	4, 3, 3	2, 2, 1	2, 2, 1	2, 1, 2	2, 2, 2	2, 1, 2	2, 2, 2

FIGURE 6.13 Street-Garden Game in Strategic Form

subgame-perfect equilibrium is carried out by considering the players in reverse order of the actual play of the game. This order conforms to the order in which player actions are considered in rollback analysis and therefore allows us to eliminate exactly those strategies, for each player, that are not consistent with rollback. In so doing, we eliminate all of the Nash equilibria that are not subgame perfect.

SUMMARY

Many games include multiple components, some of which entail simultaneous play and others of which entail sequential play. In *two-stage* (and multistage) games, a "tree house" can be used to illustrate the game; this construction allows the identification of the different stages of play and the ways in which those stages are linked together. Full-fledged games that arise in later stages of play are called *subgames* of the full game.

Changing the rules of a game to alter the timing of moves may or may not alter the equilibrium outcome of a game. Simultaneous-move games that are changed to make moves sequential may have the same outcome (if both players have dominant strategies), may have a first-mover or second-mover advantage, or may lead to an outcome in which both players are better off. The sequential version of a simultaneous game will generally have a unique rollback equilibrium even if the simultaneous version has no equilibrium or multiple equilibria. Similarly, a sequential-move game that has a unique rollback equilibrium may have several Nash equilibria when the rules are changed to make the game a simultaneous-move game.

Simultaneous-move games can be illustrated in a game tree by collecting decision nodes in *information sets* when players make decisions without knowing at which specific node they find themselves. Similarly, sequential-move games can be illustrated by using a game table; in this case, each player's full set of strategies must be carefully identified. Solving a sequential-move game from its strategic form may lead to many possible Nash equilibria. The number of potential equilibria can be reduced by using the criteria of *credibility* to eliminate some strategies as possible equilibrium strategies. This process leads to the *subgame-perfect equilibrium (SPE)* of the sequential-move game. These solution processes also work for games with additional players.

KEY TERMS

continuation (175)
credibility (175)
information set (172)
off-equilibrium paths (174)
off-equilibrium subgames (174)
subgame (160)
subgame-perfect equilibrium (SPE) (175)

EXERCISES

1. Consider the two-player sequential-move game illustrated in Exercise 2a in Chapter 3. Reexpress that game in strategic (table) form. Find all of the pure-strategy Nash equilibria in the game. If there are multiple equilibria, indicate which one is subgame-perfect, and, for those equilibria that are not subgame perfect, identify the reason (the source of the lack of credibility).

2. Consider the Airbus–Boeing game in Exercise 5 in Chapter 3. Show that game in strategic form and locate all of the Nash equilibria. Which one of the equilibria is subgame perfect? For those equilibria that are not subgame perfect, identify the source of the lack of credibility.

3. Consider a game in which there are two players, A and B. Player A moves first and chooses either Up or Down. If Up, the game is over, and each player gets a payoff of 2. If A moves Down, then B gets a turn and chooses between Left and Right. If Left, both players get 0; if Right, A gets 3 and B gets 1.
   (a) Draw the tree for this game, and find the subgame-perfect equilibrium.
   (b) Show this sequential-play game in strategic form, and find all the Nash equilibria. Which is or are subgame perfect and which is or are not? If any are not, explain why.
   (c) What method of solution could be used to find the subgame-perfect equilibrium from the strategic form of the game?

4. Consider the cola industry, in which Coke and Pepsi are the two dominant firms. (To keep the analysis simple, just forget about all the others.) The market size is $8 billion. Each firm can choose whether to advertise. Advertising costs $1 billion for each firm that chooses to do so. If one firm advertises and the other doesn't, then the former captures the whole market. If both firms advertise, they split the market 50:50 and pay for the advertising. If neither advertises, they split the market 50:50 but without the expense of advertising.
   (a) Write down the payoff table for this game, and find the equilibrium when the two firms move simultaneously.
   (b) Write down the game tree for this game (assume that it is played sequentially), with Coke moving first and Pepsi following.
   (c) Is either equilibrium in parts a and b best from the joint perspective of Coke and Pepsi? How could the two firms do better?

5. Along a stretch of a beach are 500 children in five clusters of 100 each. (Label the clusters A, B, C, D, and E in that order.) Two ice-cream vendors are deciding simultaneously where to locate. They must choose the exact location of one of the clusters.

   If there is a vendor in a cluster, all 100 children in that cluster will buy an ice cream. For clusters without a vendor, 50 of the 100 children are willing to walk to a vendor who is one cluster away, only 20 are willing to walk to a vendor two clusters away, and none are willing to walk the distance of three or more clusters. The ice cream melts quickly; so the walkers cannot buy for the nonwalkers.

   If the two vendors choose the same cluster, each will get a 50% share of the total demand for ice cream. If they choose different clusters, then those children (locals or walkers) for whom one vendor is closer than the other will go to the closer one, and those for whom the two are equidistant will split 50% each.

Each vendor seeks to maximize her sales.

(a) Construct the five-by-five payoff table for their location game; the entries stated here will give you a start and a check on your calculations:

If both vendors choose to locate at A, each sells 85 units.

If the first vendor chooses B and the second chooses C, the first sells 150 and the second sells 170.

If the first vendor chooses E and the second chooses B, the first sells 150 and the second sells 200.

(b) Eliminate dominated strategies as far as possible.

(c) In the remaining table, locate all pure-strategy Nash equilibria.

(d) If the game is altered to one with sequential moves, where the first vendor chooses her location first and the second vendor follows, what are the locations and the sales that result from the subgame-perfect equilibrium? How does the change in the timing of moves here help players resolve the coordination problem in part c?

6. The street-garden game analyzed in Section 4 of this chapter has a 16 by 4 by 2 game table when the sequential-move version of the game is expressed in strategic form, as seen in Figure 6.13. There are *many* Nash equilibria that can be found in this table.

(a) Use cell-by-cell inspection to find all of the Nash equilibria in the table in Figure 6.13.

(b) Identify the subgame-perfect equilibrium from among your set of all Nash equilibria. Other equilibrium outcomes look identical with the subgame perfect one—they entail the same payoffs for each of the three players—but they arise after different combinations of strategies. Explain how this can happen. Describe the credibility problems that arise in the non-subgame-perfect equilibria.

7. Recall the mall location game, from Exercise 8 in Chapter 3. That three-player sequential game has a game tree that is similar to the one for the street-garden game, shown in Figure 6.13.

(a) Draw the tree for the mall location game, and then illustrate the game in strategic form.

(b) Find all of the pure-strategy Nash equilibria in the game.

(c) Use iterated dominance to find the subgame-perfect equilibrium.

(d) Now assume that the game is played sequentially but with a different order of play: Big Giant, then Titan, then Frieda's. Draw the payoff table and find the rollback equilibrium of the game. How and why is this equilibrium different from the one found in the original version of the game?

8. The rules of the mall location game, analyzed in Exercise 7, specify that, when all three stores request space in Urban Mall, the two bigger (more prestigious) stores get the available spaces. The original version of the game also specifies that the firms move sequentially in requesting mall space.

   (a) Suppose that the three firms make their location requests simultaneously. Draw the payoff table for this version of the game and find all of the Nash equilibria. Which equilibrium is subgame perfect and why?

   Now suppose that when all three stores simultaneously request Urban Mall, the two spaces are allocated by lottery, giving each store an equal chance of getting into Urban Mall. With such a system, each would have a two-thirds probability (or a 66.67% chance) of getting into Urban Mall when all three had requested space there.

   (b) Draw the game table for this new version of the simultaneous-play mall location game. Find all of the Nash equilibria of the game. Identify the subgame-perfect equilibrium.

   (c) Compare and contrast your answers to part b with the equilibria found in part a. Do you get the same Nash equilibria? Why? What equilibrium do you think is focal in this lottery version of the game?

9. Two French aristocrats, Chevalier Chagrin and Marquis de Renard, fight a duel. Each has a pistol loaded with one bullet. They start 10 steps apart and walk toward each other at the same pace, 1 step at a time. After each step, either may fire his gun. When one shoots, the probability of scoring a hit depends on the distance; after k steps it is $k/5$, and so it rises from 0.2 after the first step to 1 (certainty) after 5 steps, at which point they are right up against each other. If one player fires and misses while the other has yet to fire, the walk must continue even though the bulletless one now faces certain death; this rule is dictated by the code of the aristocracy. Each gets a payoff of -1 if he himself is killed and 1 if the other is killed. If neither or both are killed, each gets 0.

   This is a game with five sequential steps and simultaneous moves (shoot or not shoot) at each step. Find the rollback (subgame-perfect) equilibrium of this game.

   Hint: Begin at step 5, when they are right up against each other. Set up the two-by-two table of the simultaneous-move game at this step, and find its Nash equilibrium. Now move back to step 4, where the probability of scoring a hit is 4/5, or 0.8, for each. Set up the two-by-two table of the simultaneous-move game at this step, correctly specifying in the appropriate cell what happens in the future. For example, if one shoots and misses while the other does not shoot, then the other will wait until step 5 and score a sure hit; if neither shoots, then the game will go to the next step, for which you have already found the equilibrium. Using all this information, find the payoffs in the two-by-two table of step 4, and find the Nash equilibrium at this step. Work backward in the same way through the rest of the steps to find the Nash equilibrium strategies of the full game.

10. Think of an example of business competition that is similar in structure to the duel in Exercise 9.

7

Simultaneous-Move Games with Mixed Strategies I: Zero-Sum Games

IN OUR EARLIER STUDY of simultaneous-move games in Chapter 4, we came across one class of games that the solution methods described there could not solve; in fact, games in that class have no Nash equilibria in pure strategies. To predict outcomes for such games, we need an extension of our concepts of strategies and equilibria. This is to be found in the randomization of moves, which is the focus of this chapter and the next.

The need for randomized moves in the play of a game usually arises when one player prefers a coincidence of actions, while her rival prefers to avoid it. Such direct conflict between players arises in zero-sum games, and we concentrate on these games in this chapter. Games in which randomized moves can be beneficial to players and in which they can help resolve the problem of a nonexistent pure-strategy equilibrium include military conflicts as well as most sporting contests.

We develop the ideas by using the tennis-point game from the end of Chapter 4. This game is zero-sum; the interests of the two tennis players are purely in mutual conflict. Evert wants to hit her passing shot to whichever side—down the line (DL) or crosscourt (CC)—is not covered by Navratilova, while Navratilova wants to cover the side to which Evert hits her shot. In Chapter 4, we pointed out that, in such a situation, each player would want to keep the other guessing, which could be done by acting unsystematically or randomly. We explore this idea more completely in this chapter and in Chapter 8.

1 WHAT IS A MIXED STRATEGY?

When players choose to act unsystematically, they pick from among their pure strategies in some random way. In the tennis-point game, Navratilova and Evert each choose from two initially given pure strategies, DL and CC. We call a random mixture between these two pure strategies a mixed strategy.

There is a whole continuous range of such mixed strategies. At one extreme, DL could be chosen with probability 1 (for sure), meaning that CC is never chosen (probability 0); this "mixture" is just the pure strategy DL. At the other extreme, DL could be chosen with probability 0 and CC with probability 1; this "mixture" is the same as pure CC. In between is the whole set of possibilities: DL chosen with probability 75% (0.75) and CC with probability 25% (0.25) or both chosen with probabilities 50% (0.5) each; DL with probability 1/3 (33.33 . . .%) and CC with probability 2/3 (66.66 . . .%); and so on.[1] We can algebraically indicate the whole range of possible mixtures available to a particular player by saying that DL may be chosen with probability p and CC with probability $(1 - p)$, where p can be any real number ranging from 0 to 1.

The payoffs from a mixed strategy are defined as the corresponding probability-weighted averages of the payoffs from its constituent pure strategies. For example, in the tennis game of Section 4.8, against Navratilova's DL, Evert's payoff from DL is 50 and from CC is 90. Therefore the payoff of Evert's mixture (0.75 DL, 0.25 CC) against Navratilova's DL is $0.75 \times 50 + 0.25 \times 90 = 37.4 + 22.5 = 60$. This is Evert's **expected payoff** from this particular mixed strategy.[2]

You may have noticed that mixed strategies are very similar to the continuous strategies studied in Chapter 5. In fact, they are just special kinds of continuously variable strategies. With the possibility of mixing made available, each player can now choose from among all conceivable mixtures of the basic or

[1]When a chance event has just two possible outcomes, people often speak of the odds in favor of or against one of the outcomes. If the two possible outcomes are labeled A and B, and the probability of A is p so that, if B is $(1 - p)$, then the ratio $p/(1 - p)$ gives the odds in favor of A, and the reverse ratio $(1 - p)/p$ gives the odds against A. Thus, when Evert chooses CC with probability 0.25 (25%), the odds against her choosing CC are 3 to 1, and the odds in favor of it are 1 to 3. This terminology is often used in betting contexts; so those of you who misspent your youth in that way will be more familiar with it. However, this usage does not readily extend to situations when there are three or more possible outcomes; so we avoid its use here.

[2]Game theory assumes that players will calculate and try to maximize their expected payoffs when probabilistic mixtures of strategies or outcomes are included. We consider this further in the Appendix to this chapter, but for now we proceed to use it, with just one important note. The word "expected" in "expected payoff" is a technical term from probability and statistics. It merely denotes a probability-weighted average. It does not mean this is the payoff that the player should expect in the sense of regarding it as her right or entitlement.

pure strategies initially specified (which, as we just saw, include pure strategies as extreme special cases).

The notion of Nash equilibrium also extends easily to include mixed strategies. Nash equilibrium is defined as a list of mixed strategies, one for each player, such that the choice of each is her best choice, in the sense of yielding the highest expected payoff for her, given the mixed strategies of the others. The method of best-response analysis can be used to find the best mixture probability (the variable p described earlier) of each player as a function of the mixture probability of the other. These functional relationships can be shown in a graph with the two players' mixture probabilities on the axes, and Nash equilibria will be found at the points of intersection of the two best-response curves. Allowing for mixed strategies in a game solves the problem of possible nonexistence of Nash equilibrium, which we encountered for pure strategies, automatically and almost entirely. Nash's celebrated theorem shows that, under very general circumstances (which are broad enough to cover all the games that we meet in this book and many more besides), a Nash equilibrium in mixed strategies exists.

At this broadest level, incorporating mixed strategies into our analysis does not entail anything different from the general theory of continuous strategies developed in Chapter 5. However, the special case of mixed strategies does bring with it several special conceptual as well as methodological matters, and therefore deserves separate study. First, best-response curves for mixed strategies have some very special and peculiar features, which spill over into some equally special features of mixed strategy equilibria. Second, interpretation of mixed strategy equilibria is very different in zero-sum games from that in non-zero-sum games. Third, our view of Nash equilibrium as a system of self-confirming beliefs about other players' choices and one's own best responses to those beliefs requires some reinterpretation in the context of mixed strategies. Finally, putting mixed strategies into practice is tricky. In view of all these potential problems, empirical evidence about mixing and mixed-strategy equilibria should be examined with special care. In this chapter and the next, we take up all these matters. But we begin a little mechanically, by using best-response analysis to find mixed-strategy Nash equilibria in the tennis example that brought to light the need for mixing in the first place.

2 UNCERTAIN ACTIONS: MIXING MOVES TO KEEP THE OPPONENT GUESSING

We begin with the tennis example of Section 4.8, which did not have a Nash equilibrium in pure strategies. We show how the extension to mixed strategies remedies this deficiency, and we interpret the resulting equilibrium as one in which each player keeps the other guessing.

A. Best-Response Analysis

Mixed strategies are a special kind of continuously variable strategy. Thus to solve a game in which players have access to mixed strategies, we can use the solution method that we developed for finding Nash equilibria in games with continuous strategies—namely, best-response analysis.

First we need to describe the full range of mixed strategies available to each player. Evert has two pure strategies, DL and CC, and in principle she can mix them with any probabilities. Let the algebraic variable p denote the probability that she plays DL. Then $(1 - p)$ must be the probability that she plays CC. Here p can range from 0 to 1. The two extremes of this range of p correspond to the special cases of the two pure strategies: $p = 1$ means that Evert plays pure DL, and $p = 0$ means that she plays pure CC. We call this general mixed strategy Evert's p-mix for short. Similarly, Navratilova's general mixed strategy has a probability q with which she defends DL, and probability $(1 - q)$ with which she defends CC, where q ranges from 0 to 1; call this Navratilova's q-mix for short.

We reproduce in Figure 7.1 the payoff matrix of Figure 4.15. Even though this is a constant-sum game, we now show the payoffs for the two players separately, because it is more intuitive to think of each player as trying to get a higher payoff for herself. We also add a third strategy for each player, representing a mixture between the two pure strategies. Because we do not yet know what the probabilities in the equilibrium mixtures will be, we must use general algebraic symbols p and q for the two players.

To understand how the expected payoff expressions in the p-mix row and the q-mix column are computed, consider the case where Evert plays CC and Navratilova plays her q-mix. Evert's payoff from her CC against the DL part of Navratilova's q-mix is 90; that against the CC part is 20. Therefore Evert's expected payoff from her CC against Navratilova's mixture, in which DL gets probability q and CC gets probability $(1 - q)$, is $90 \times q + 20 \times (1 - q)$. This is the

		NAVRATILOVA		
		DL	CC	q-mix
EVERT	DL	50, 50	80, 20	$50q + 80(1-q)$, $50q + 20(1-q)$
	CC	90, 10	20, 80	$90q + 20(1-q)$, $10q + 80(1-q)$
	p-mix	$50p + 90(1-p)$, $50p + 10(1-p)$	$80p + 20(1-p)$, $20p + 80(1-p)$	

FIGURE 7.1 Expected Payoffs for General Mixtures in the Tennis Point

expression (in black) for Evert's payoff in the cell (CC, q-mix). Navratilova's payoffs when Evert's CC is matched with Navratilova's DL and CC are 10 and 80, respectively. Therefore Navratilova's expected payoff when Evert's CC is matched against her q-mix is $10 \times q + 80 \times (1 - q)$. This is the expression (in red for Navratilova's payoff in the cell (CC, q-mix). The same method of calculation applies to the other three cells where one player's mix meets one of the other's pure strategies. The algebraic expression for the cell where mix meets mix is more complicated, but we don't need it and so we leave that cell blank.

In equilibrium, each player chooses a mixture (p or q) that is her best response to the other's mixed strategy (q or p). As in Chapter 5, we can find this equilibrium most easily by developing best-response rules for the two players. Thus Navratilova's best-response rule must show her best q for each value of Evert's p, and Evert's best-response rule must show her best p for each value of Navratilova's q. We can then depict the best-response relations in a graph with p on one axis and q on the other, where each variable ranges from 0 to 1, to find the Nash equilibrium values of p and q.

We start by calculating a best-response rule for Navratilova. To do so, we construct a diagram where we illustrate Navratilova's possible responses to each value of Evert's p and then identify the *best* response in each case. This diagram is shown in Figure 7.2. The horizontal axis shows the full range of p from 0 to 1 in Evert's p-mix, and the vertical axis shows Navratilova's expected payoffs from each of her pure strategies, when played against Evert's p-mix.

Navratilova's expected payoff from her DL against Evert's p-mix is $50p + 10(1 - p)$. This is a linear function of p, and its graph is the rising straight line in Figure 7.2; it starts at 10 when $p = 0$ and reaches 50 when $p = 1$. Navratilova's expected payoff from her CC against Evert's p-mix is $20p + 80(1 - p)$. This is the falling straight line in Figure 7.2, which starts at 80 when $p = 0$ and reaches 20 when $p = 1$. Navratilova's expected payoff from any mixture of DL and CC is just an average of the payoffs from the two pure strategies; therefore it lies somewhere between those two, or somewhere between the two expected payoff lines in Figure 7.2.

It is useful to calculate the value of p at the point of intersection. That p satisfies $50p + 10(1 - p) = 20p + 80(1 - p)$, or $p = 0.7$. The common expected payoff for this value of p is $50 \times 0.7 + 10 \times 0.3 = 35 + 3 = 38$, calculated by using the line for DL, or $20 \times 0.7 + 80 \times 0.3 = 14 + 24 = 38$, calculated using the line for CC.

We want to identify the strategy that is best for Navratilova against each possible value of Evert's p. This is just the strategy that yields her the highest expected payoff in each case. For $p = 0$, for example, the diagram in Figure 7.2 shows that Navratilova gets a higher payoff when she plays her CC rather than DL. If Navratilova plays some mixture of CC and DL, her payoff is the corresponding average of her payoffs with pure CC and pure DL. If her payoff from

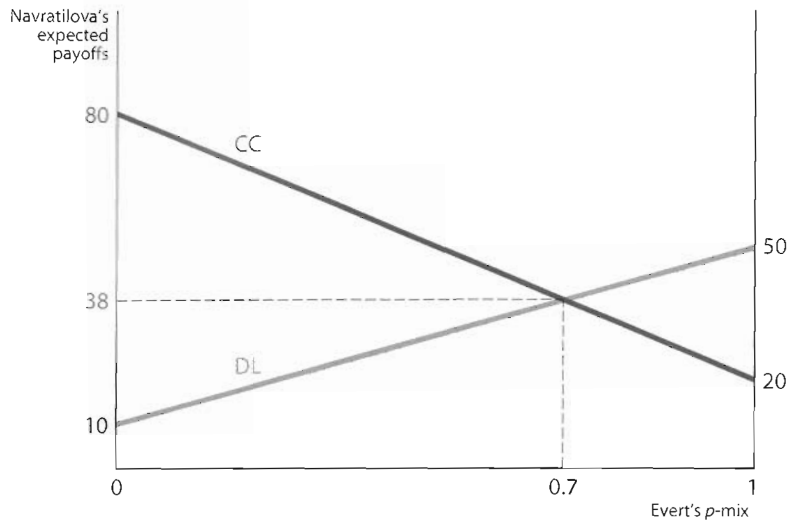


FIGURE 7.2 Calculation of Navratilova's Best Response

pure CC is higher than her payoff from pure DL, then her payoff from pure CC is also higher than that from any such average. Therefore her pure CC is not only better for Navratilova than her pure DL, but also her best choice among all her strategies, pure and mixed.

This continues to be true for all values of p to the left of the intersection point in the diagram. Thus, for all values of p to the left of the point of intersection, pure CC is better than DL for Navratilova when played against Evert's p-mix and *also* better than any mixed strategy of her own. Pure CC is Navratilova's best response for this range of values of p. Similarly, to the right of the point of intersection, pure DL is her best response.[3]

Against Evert's specific p-mix at the point of intersection, Navratilova does equally well (expected payoff 38) from her CC and DL. Because the expected payoff associated with a mixture of her CC and DL lies between the CC and DL lines, it follows that she also does equally well with any q-mix of her own. Thus Navratilova's best response to Evert's choice of $p = 0.7$ can be any q in the entire range from 0 to 1.

We can summarize Navratilova's best-response rule as follows:

If $p < 0.7$, choose pure CC ($q = 1$).
If $p = 0.7$, all values of q in the range from 0 to 1 are equal best responses.
If $p > 0.7$, choose pure DL ($q = 0$).

[3]If, in some numerical problem you were trying to solve, the expected payoff lines for the pure strategies did not intersect, that would indicate that one pure strategy was best for all of the opponent's mixtures. Then this player's best response would always be that pure strategy.

To see the intuition behind this rule, note that p is the probability with which Evert hits her passing shot down the line. Navratilova wants to prepare for Evert's shot. Therefore, if p is low (Evert is more likely to go crosscourt), Navratilova does better to defend crosscourt. If p is high (Evert is more likely to go down the line), Navratilova does better to defend down the line. For a critical value of p in between, the two choices are equally good for Navratilova. This break-even point is not $p = 0.5$, because the consequences (measured by the payoff numbers of the various combinations) are not symmetric for the two choices.

A simple algebraic calculation reinforces the visual analysis based on Figure 7.2. Against Evert's general p-mix, CC is better than DL (and therefore better than any q-mix) for Navratilova if

$$20p + 80(1 - p) > 50p + 10(1 - p),$$
$$\text{or} \quad 70(1 - p) > 30, \quad \text{or} \quad 70 > 100p, \quad \text{or} \quad p < 0.7.$$

DL is better than CC (and therefore better than any q-mix) if the opposite inequalities hold ($p > 0.7$). And DL and CC (and all q-mixes) are equally good if the two expressions are equal ($p = 0.7$).

So far, we have done all of the analysis in terms of Navratilova's own payoffs, because that is easier to understand at first approach. But this is a constant-sum game, and the game matrix often shows only the row player's payoffs. In this game, Evert is the row player and her payoffs are what we would ordinarily see in the table. Could we find Navratilova's best response by using Evert's payoffs? Yes, we just have to remember that in the constant-sum game Navratilova does better when Evert gets low payoffs.

Figure 7.3 puts Evert's p-mix on the horizontal axis as in Figure 7.2, but Evert's own expected payoffs are on the vertical axis. We show two lines again, one corresponding to Navratilova's choice of pure CC and the other for her choice of pure DL. Here, the CC line is rising from left to right. This corresponds to the idea that Evert's expected payoff rises as she uses her DL shot more often (higher p) against Navratilova covering purely CC. Similarly, the DL line is falling from left to right, as does Evert's expected payoff from using DL more often against an opponent always covering for that shot. We see that the point of intersection of the expected payoff lines is again at $p = 0.7$. To the left of this p value, the CC line is lower than the DL line. Therefore CC is better than DL for Navratilova when p is low; it entails a lower expected payoff for Evert. To the right of $p = 0.7$, the DL line is lower than the CC line; so DL is better than CC for Navratilova. And the expected payoff at the point of intersection is 62 for Evert, which is just 100 minus the 38 that we found for Navratilova in the analysis based on her payoffs. Thus our results are the same as we found by using Navratilova's own payoffs; only the basis of the analysis is slightly different. You should get comfortable with both methods of analysis in zero-sum games.

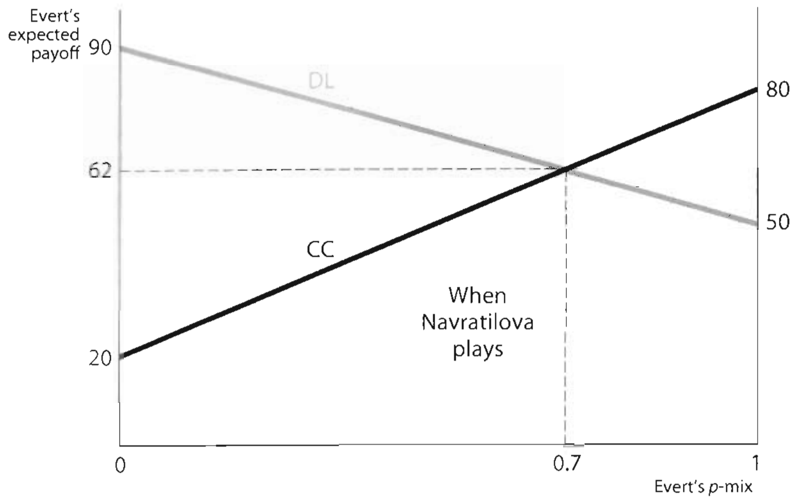


FIGURE 7.3 Navratilova's Best-Response Calculation by Using Evert's Payoffs

Evert's best-response choice of p against Navratilova's q-mixes can be found by using exactly the same reasoning. We leave the analysis to you. The result—Evert's best-response rule—is

If $q < 0.6$, choose pure DL ($p = 1$).
If $q = 0.6$, all values of p from 0 to 1 are equal best responses.
If $q > 0.6$, choose pure CC ($p = 0$).

To understand the intuition, observe that q is the probability of Navratilova defending down the line. Evert wants to hit her passing shot to the side that Navratilova is not defending. Therefore, when q is low, Evert does better to go down the line and, when q is high, she does better to go crosscourt.

We now have two best-response curves that we can graph, as is done in Figure 7.4. The left- hand panel shows Navratilova's best response in her choice of q to Evert's choices of p. Therefore p is on the horizontal axis (as it was in Figure 7.2), and q is on the vertical axis. We know that, for $p < 0.7$, Navratilova does better by choosing pure CC ($q = 0$); this segment of her best-response curve is the horizontal solid (red) line along the bottom edge of the graph. For p > 0.7, Navratilova does better by choosing pure DL ($q = 1$); this segment of her best-response curve is the horizontal solid line along the top edge of the graph. For $p = 0.7$, Navratilova does equally well with all of her choices, pure and mixed, and so any value of q between 0 and 1 (inclusive) is a best response; the vertical solid line in the graph at $p = 0.7$ shows her best responses for this choice of Evert. Navratilova's best responses are then shown by the three separate line segments joined end to end and shown in red. As we did for general continuous strategies in Chapter 5, we call this construction Navratilova's *best-response*

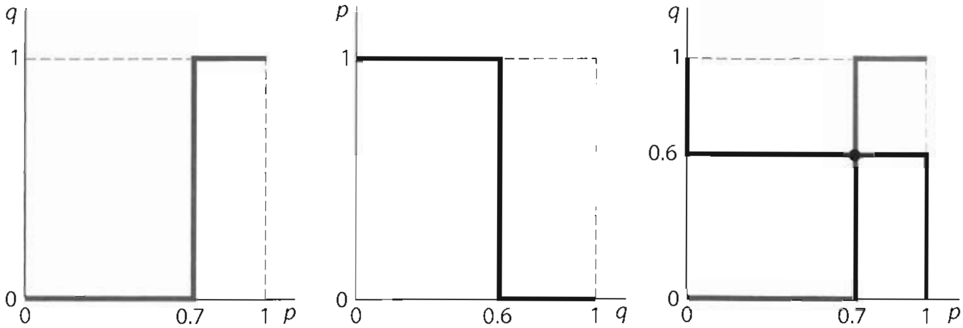


FIGURE 7.4 Best-Response Curves and Nash Equilibrium

curve. It is conceptually the same as the best-response curves that we drew in Chapter 5; the only difference is that, for mixed strategies, the curve has this special shape. Because it actually consists of three straight line segments, "curve" is a misnomer, but it is the standard general terminology in this context.

The middle panel of Figure 7.4 shows Evert's best-response curve. Here, Evert's best p values are determined in relation to Navratilova's various possible choices for q; so q is on the horizontal axis and p on the vertical axis. For $q < 0.6$, Evert does better playing pure DL ($p = 1$); for $q > 0.6$, she does better with pure CC ($p = 0$); for $q = 0.6$, she does equally well with all choices, pure or mixed. The thick black curve, consisting of three line segments joined end to end, is Evert's best response curve.

The right-hand panel in Figure 7.4 combines the other two panels, by reflecting the middle graph across the diagonal (45° line) so that p is on the horizontal axis and q on the vertical axis and then superimposing this graph on the left-hand graph. Now the red and black curves meet at the point $p = 0.7$ and $q = 0.6$. Here each player's mixture choice is a best response to the other's choice, so we see clearly the derivation of our Nash equilibrium in mixed strategies.

This picture also shows that the best-response curves do not have any other common points. Thus the mixed-strategy equilibrium in this example is the unique Nash equilibrium in the game. What is more, this representation includes pure strategies as special cases corresponding to extreme values of p and q. So we can also see that the best-response curves do not have any points in common at any of the sides of the square where each value of p and q equals either 0 or 1; thus we have another way to verify that the game does not have any pure-strategy equilibria.

Observe that each player's best response is a pure strategy for almost all values of her opponent's mixture. Thus Navratilova's best response is pure CC for all of Evert's choices of $p < 0.7$, and it is pure DL for all of Evert's choices of $p > 0.7$. Only for the one crucial value $p = 0.7$ is Navratilova's best response a mixed strategy, as is represented by the vertical part of her three-segment best-response curve in Figure 7.4. Similarly, only for the one crucial value $q = 0.6$ of Navratilova's mixture is Evert's best response a mixed strategy—namely, the horizontal segment of her best-response curve in Figure 7.4. But these seemingly exceptional or rare strategies are just the ones that emerge in the equilibrium.

These special values of p and q have an important feature in common. Evert's equilibrium p is where Navratilova's best-response curve has its vertical segment; so Navratilova is indifferent among all her strategies, pure and mixed. Navratilova's equilibrium q is where Evert's best-response curve has its horizontal segment; so Evert is indifferent among all her strategies, pure and mixed. Thus *each* player's equilibrium mixture is such that *the other* player is indifferent among all her mixes. We call this the **opponent's indifference property**.

The best-response-curve method thus provides a very complete analysis of the game. Like the cell-by-cell inspection method, which examines all the cells of a pure-strategy game, the best-response-curve method is the one to use when one wants to locate all of the equilibria, whether in pure or mixed strategies, that a game might have. The best-response-curve diagram can show both types of equilibria in the same place. (It could also be used to show equilibria in which one player uses a mixed strategy and the other player uses a pure strategy, although the diagram shows that such hybrids can be ruled out except in some very exceptional cases.)

If you had reasons to believe that a game would have just one mixed-strategy equilibrium, you could still find it directly without bothering to calculate and graph best-response curves to find their intersection. The opponent's indifference property specifies the set of equations to solve.[4] Evert's equilibrium $p = 0.7$ equates Navratilova's expected payoffs from DL and CC against Evert's p-mix. In other words, p solves the algebraic equation $50p + 10(1 - p) = 20p + 80(1 - p)$. Similarly, Navratilova's equilibrium $q = 0.6$ comes from equating Evert's expected payoffs from DL or CC against Navratilova's q-mix; it is the solution to $50q + 80(1 - q) = 90q + 20(1 - q)$. Later in this chapter and again in Chapter 8, we show how these methods must be modified when a player has three or more pure strategies.

B. The Minimax Method

Because this is a zero-sum game, when one player is indifferent, it follows that the other is also. Therefore in this special situation we can think of the opponent's indifference as one's own indifference. In a zero-sum game, each player's equilibrium mixture probability is such that she is indifferent about which of

[4]Sometimes this method will yield solutions in which some probabilities are either negative or greater than 1. That is an indication that the game in question has an equilibrium in pure strategies. You should go back to the basic payoff matrix and look for it.

her opponent's pure strategies the opponent chooses. This makes sense because, in the zero-sum context, if the opponent saw any positive benefit to choosing a particular pure strategy, that could work only to one's own disadvantage. The intuition here is that in zero-sum games players deliberately mix their moves to prevent the opponent from taking advantage of their predictability. We develop this idea by extending the minimax method of Chapter 4 to handle mixed strategies.

In Figure 7.5, we show the payoff table of the game with pure strategies alone but augmented to show row minima and column maxima as in Figure 4.8. Evert's minimum payoffs (success values) for each of her strategies are shown at the far right of the table: 50 for the DL row and 20 for the CC row. The maximum of these minima, or Evert's maximin, is 50. This is the best that Evert can guarantee for herself by using pure strategies, knowing that Navratilova responds in her own best interests. To achieve this guaranteed payoff, Evert would play DL.

At the foot of each column, we show the maximum of Evert's success percentages attainable in that column. This value represents the worst payoff that Navratilova can get if she plays the pure strategy corresponding to that column. The smallest of these maxima, or the minimax, is 80. This is the best payoff that Navratilova can guarantee for herself by using pure strategies, knowing that Evert responds in her own best interests. To achieve this, Navratilova would play CC.

The maximin and minimax values for the two players are not the same: Evert's maximin success percentage (50) is less than Navratilova's minimax (80). As we explained in Chapter 4, this shows that the game has no equilibrium in the (pure) strategies available to the players in this analysis. In this game, each player can achieve a better outcome—a higher maximin value for Evert and a lower minimax value for Navratilova—by choosing a suitable random mixture of DL and CC.

Let us expand the sets of strategies available to players to include randomization of moves or mixed strategies. We do the analysis here from Evert's perspective; that for Navratilova is similar. First, in Figure 7.6 we expand the payoff table of Figure 7.5 by adding a row for Evert's p-mix, where she plays DL with

		NAVRATILOVA		
		DL	CC	
EVERT	DL	50	80	min = 50
	CC	90	20	min = 20
		max = 90	max = 80	

FIGURE 7.5 Minimax Analysis of the Tennis Point with Pure Strategies

		NAVRATILOVA		
		DL	CC	
EVERT	DL	50	80	min = 50
	CC	90	20	min = 20
	p-mix	$50p + 90(1 - p)$	$80p + 20(1 - p)$	min = ?

FIGURE 7.6 Minimax Analysis of the Tennis Point with Evert's Mixed Strategies

probability p and CC with probability $(1 - p)$. We show the expected payoffs of this mix against each of Navratilova's pure strategies, DL and CC. The row minimum for the p-mix is then just the smaller of these two expressions. But which expression is smaller depends on the value of p; so we have to graph the two against p. This graph is shown in Figure 7.7.

When Navratilova plays DL, Evert's p-mix yields her an expected payoff of $50p + 90(1 - p)$; this is the downward-sloping line in Figure 7.7. Evert's payoff from her p-mix against CC is $80p + 20(1 - p)$; this is the upward-sloping line in Figure 7.7. The two lines intersect when

$$50p + 90(1 - p) = 80p + 20(1 - p),$$
$$\text{or} \quad 70(1 - p) = 30p, \quad \text{or} \quad 100p = 70, \quad \text{or} \quad p = 0.7.$$

The common expected payoff at the point of intersection is found by plugging $p = 0.7$ into either of these expressions, and it equals 62.

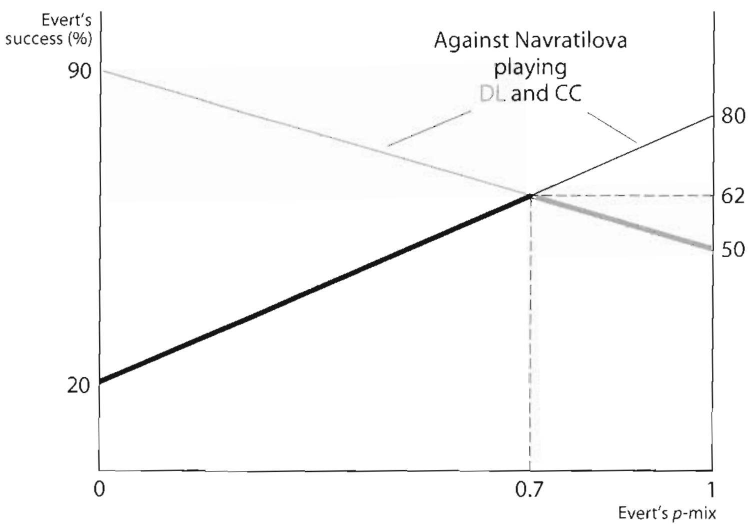


FIGURE 7.7 Evert's Maximin Choice of p

For each possible p-mix, Evert should expect Navratilova to respond, in equilibrium, with the action that is best for her. Because the game is zero-sum, the action that is best for Navratilova is the action that is worst for Evert. Thus, Evert should expect Navratilova always to make the choice that corresponds to the lower of the two lines in the graph for any value of p. Given the graph that we have drawn, Evert should expect Navratilova to cover CC when Evert's DL percentage in her mix is less than 70 (when $p < 0.7$) and to cover DL when Evert's DL percentage is more than 70 (when $p > 0.7$). If Evert is mixing exactly 70% down the line and 30% crosscourt, then Navratilova does equally well with either of her pure-strategy choices.

The lower segments of the DL and CC lines in Figure 7.7 have been made thicker to highlight the lowest success percentage to which Navratilova can hold Evert under each possible version of Evert's p-mix. This thick, inverted-V-shaped graph thus shows the minimum values of Evert's success percentage in relation to her choice of p in her mixed strategy. The whole inverted-V graph is what should be given as the minimum in the cell at the extreme right of the p-mix row in Figure 7.6. Indeed, having done this, we could even omit the first two rows that correspond to Evert's two pure strategies because they are included in the graph. The point at the extreme left, where $p = 0$, corresponds to Evert's pure choice of CC and yields 20% success. The point at the extreme right, where $p = 1$, corresponds to the pure choice of DL and yields 50% success.

Most interestingly, there are values of p in between 0 and 1—representing true mixed strategies for Evert—where she has success percentages above 50. This outcome illustrates the advantage of keeping Navratilova guessing. If $p = 0.6$, for example, Navratilova does best by playing her CC. This choice keeps Evert down to 20% success only for the 40% of the time that Evert plays CC; the rest of the time, 60%, Evert is playing DL and gets 80%. The **expected value** for Evert is $80 \times 0.6 + 20 \times 0.4 = 48 + 8 = 56$, which is more than the maximin of 50 that Evert could achieve by choosing one of her two pure strategies. Thus we see that a mix of 60% DL and 40% CC is better for Evert than either of her two pure strategies. But is it her best mix?

Figure 7.7 immediately reveals to us Evert's best choice of mix: it is at the peak formed by the inverted-V-shaped thicker lines. That peak represents the maximum of her minima from all values of p and is at the intersection of the DL and CC lines. We already calculated that p is 70% at this intersection. Thus, when Evert chooses p to be 70%, she has a success percentage of 62 no matter what Navratilova does. For any other p-mix Evert might choose, Navratilova would respond with whichever one of her pure strategies would hold Evert down to a success rate of less than 62%, along either of the thicker lines that slope away from the peak. Thus Evert's mix of 70% DL and 30% CC is the only one that cannot be exploited by Navratilova to her own advantage (and therefore to Evert's disadvantage).

To sum up, suitable mixing enables Evert to raise her maximin from 50 to 62. A similar analysis done from Navratilova's perspective shows that suitable mixing (choice of q) enables her to lower her minimax. We see from Figure 7.1 that Navratilova's q-mix against Evert's DL results in Evert's expected payoff $50q + 80(1 - q)$, while against Evert's DL the result is $90q + 20(1 - q)$. Navratilova's minimax is where these two are equal, that is, $40q = 60(1 - q)$, or $100q = 60$, or $q = 0.6$, and the resulting expected payoff to Evert is $50 \times 0.6 + 80 \times 0.4 = 30 + 32 = 62$. Thus Navratilova's best mixing enables her to lower her minimax from 80 to 62. When the two best mixes are pitted against each other, Evert's maximin equals Navratilova's minimax, and we have a Nash equilibrium in mixed strategies.

The equality of maximin and minimax for optimal mixes is a general property of zero-sum games and was proved by John von Neumann and Oskar Morgenstern in 1945. But conceptually even more basic is the observation that mixing enables each player to achieve a better outcome. This is the precise mathematical formulation of the general idea that, if a zero-sum game has no pure-strategy Nash equilibrium, then each player would be ill-advised to stick to any pure action systematically and would do better to keep the other player guessing.

3 NASH EQUILIBRIUM AS A SYSTEM OF BELIEFS AND RESPONSES

When the moves in a game are simultaneous, a player cannot respond to the other's actual choice. Instead, each takes her best action in the light of what she thinks the other might be choosing at that instant. In Chapter 4, we called such thinking a player's *belief* about the other's strategy choice. We then interpreted Nash equilibrium as a configuration where such beliefs are correct, so each chooses her best response to the actual actions of the other. This concept proved useful for understanding the structures and outcomes of many important types of games, most notably zero-sum games and minimax strategies, dominance and the prisoners' dilemma, focal points and various coordination games, as well as chicken.

However, in Chapter 4 we considered only pure-strategy Nash equilibria. Therefore a hidden assumption went almost unremarked—namely, that each player was sure or confident in her belief that the other would choose a particular pure strategy. Now that we are considering more general mixed strategies, the concept of belief requires a corresponding reinterpretation.

Players may be unsure about what others might be doing. In the coordination game in Chapter 5, in which Harry wanted to meet Sally, he might be unsure whether she would go to Starbucks or Local Latte, and his belief might be that there was a 50-50 chance that she would go to either one. And, in the tennis example, Evert might recognize that Navratilova was trying to keep her (Evert) guessing and therefore be unsure of which of Navratilova's available actions she would play.

It is important however to distinguish between being unsure and having incorrect beliefs. For example, in the Nash equilibrium of the tennis example in Section 2 of this chapter, Navratilova cannot be sure of what Evert is choosing. But she can still have correct beliefs about Evert's mixture—namely, about the probabilities with which Evert chooses between her two pure strategies. Having correct beliefs about mixed actions means knowing or calculating or guessing the correct probabilities with which the other player chooses from among her underlying basic or pure actions. In the equilibrium of our example, it turned out that Evert's equilibrium mixture was 70% DL and 30% CC. If Navratilova believes that Evert will play DL with 70% probability and CC with 30% probability, then her belief, although uncertain, will be correct in equilibrium.

Thus we have an alternative and mathematically equivalent way to define Nash equilibrium in terms of beliefs: each player forms beliefs about the probabilities of the mixture that the other is choosing and chooses her own best response to this. A Nash equilibrium in mixed strategies occurs when the beliefs are correct, in the sense just explained.

In Chapter 8, we consider mixed strategies and their Nash equilibria in non-zero-sum games. In such games, there is no general reason why the other player's pursuit of her own interests should work against your interests. Therefore it is not in general the case that you would want to conceal your intentions from the other player, and there is no general argument in favor of keeping the other player guessing. However, because moves are simultaneous, each player may still be subjectively unsure of what action the other is taking and therefore may have uncertain beliefs, which in turn lead her to be unsure about how she should act. This can lead to mixed-strategy equilibria, and their interpretation in terms of subjectively uncertain but correct beliefs proves particularly important.

4 MIXING WHEN ONE PLAYER HAS THREE OR MORE PURE STRATEGIES

The discussion of mixed strategies to this point has been confined to games in which each player has only two pure strategies, as well as mixes between them. In many strategic situations, each player has available a larger number of pure strategies, and we should be ready to calculate equilibrium mixes for those cases as well. However, these calculations get complicated quite quickly. For truly complex games, we would turn to a computer to find the mixed-strategy equilibrium. But, for some small games, it is possible to calculate equilibria by hand quite easily. The calculation process gives us a better understanding of how the equilibrium works than can be obtained just from looking at a computer-generated solution. Therefore in this section and the next one we solve some larger games.

Here we consider zero-sum games in which one of the players has only two pure strategies, whereas the other has more. In such games, we find that the player who has three (or more) pure strategies typically uses only two of them in equilibrium. The others do not figure in her mix; they get zero probabilities. We must determine which ones are used and which ones are not.[5]

Our example is that of the tennis-point game augmented by giving Evert a third type of return. In addition to going down the line or crosscourt, she now can consider using a lob (a slower but higher and longer return). The equilibrium depends on the payoffs of the lob against each of Navratilova's two defensive stances. We begin with the case that is most likely to arise and then consider a coincidental or exceptional case.

A. A General Case

Evert now has three pure strategies in her repertoire: DL, CC, and Lob. We leave Navratilova with just two pure strategies, Cover DL or Cover CC. The payoff table for this new game can be obtained by adding a Lob row to the table in Figure 7.1. The result is shown in Figure 7.8. Now that you are more familiar with

		NAVRATILOVA		
		DL	CC	q-mix
EVERT	DL	50	80	$50q + 80(1 - q)$
	CC	90	20	$90q + 20(1 - q)$
	Lob	70	60	$70q + 60(1 - q)$
	p-mix	$50p_1 + 90p_2 + 70(1 - p_1 - p_2)$	$80p_1 + 20p_2 + 60(1 - p_1 - p_2)$	

FIGURE 7.8 Payoff Table for Tennis Point with Lob

[5]Even when a player has only two pure strategies, she may not use one of them in equilibrium. The other player then generally finds one of her strategies to be better against the one that the first player does use. In other words, the equilibrium "mixtures" collapse to the special case of pure strategies. But, when one or both players have three or more strategies, we can have a genuinely mixed-strategy equilibrium where some of the pure strategies go unused.

mixing in constant-sum games, we show only Evert's payoffs, and ask you to remember that Navratilova chooses her strategies so as to achieve smaller expected payoffs for Evert.

The payoffs in the first three rows of the table are straightforward. When Evert uses her pure strategies DL and CC, her payoffs against Navratilova's pure strategies or the q-mix are exactly as in Figure 7.1. The third row also is analogous. When Evert uses Lob, we assume that her success percentages against Navratilova's DL and CC are, respectively, 70% and 60%. When Navratilova uses her q-mix, using DL a fraction q of the time and CC a fraction $(1 - q)$ of the time, Evert's expected payoff from Lob is $70q + 60(1 - q)$; therefore that is the entry in the cell where Evert's Lob meets Navratilova's q-mix.

The really new feature here is the last row of the table. Evert now has three pure strategies, so she must now consider three different actions in her mix. The mix cannot be described by just one number p. Rather, we suppose that Evert plays DL with probability p_1 and CC with probability p_2, leaving Lob to get the remaining probability, $1 - p_1 - p_2$. Thus we need two numbers, p_1 and p_2, to define Evert's p-mix. Each of them, being a probability, must be between 0 and 1. Moreover, the two must add to something no more than 1; that is, they must satisfy the condition $p_1 + p_2 \leq 1$, because the probability $(1 - p_1 - p_2)$ of using Lob must be nonnegative.

Using this characterization of Evert's p-mix then, we see that her expected payoff, when Navratilova plays her pure strategy DL, is given by $50p_1 + 90p_2 + 70(1 - p_1 - p_2)$. This is the entry in the first cell of the last row of the table in Figure 7.8. Evert's expected payoff from using her p-mix against Navratilova's CC is similarly $80p_1 + 20p_2 + 60(1 - p_1 - p_2)$. We do not show the expression for the payoff of mix against mix, because it is too long and we do not need it for our calculations.

Technically, before we begin looking for a mixed-strategy equilibrium, we should verify that there is no pure-strategy equilibrium. This is easy to do, however, so we leave it to you and turn to mixed strategies.

The key to solving for a mixed-strategy equilibrium in a constant-sum game of this type is to use the minimax method from the perspective of the player who has just two pure strategies; here that player is Navratilova. This approach works because Navratilova's mixture can be specified by using just one variable—namely, the single probability (q) used to define her mixed strategy. That probability, of choosing DL in this case, fully specifies her mixed strategy; after q is known, the probability of choosing CC is simply $(1 - q)$.

Figure 7.9 shows Evert's expected payoffs (success percentages) from playing DL, CC, and Lob as the q in Navratilova's q-mix varies over its full range from 0 to 1. These graphs are just those of the expressions in the right-hand column of Figure 7.8. Given the usual worst-case assumption that is appropriate in zero-sum games, Navratilova's calculation of her minimax strategy is as follows.

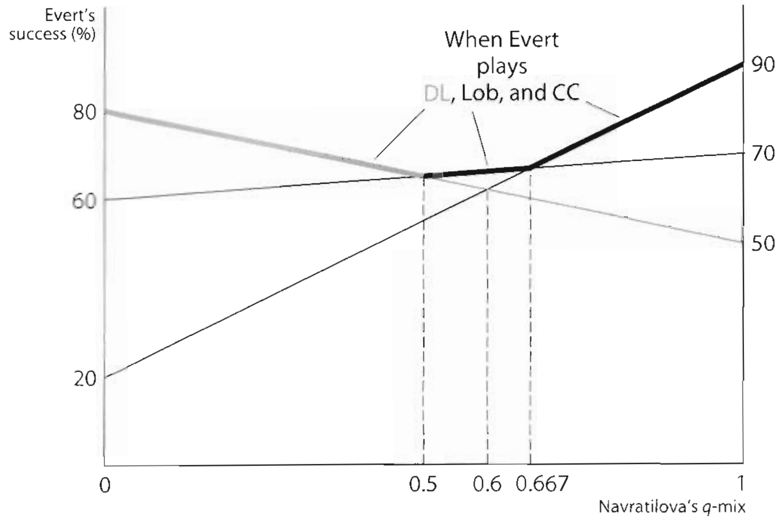


FIGURE 7.9 Diagrammatic Solution for Navratilova's q-Mix

For each q, if Navratilova were to choose that q-mix in equilibrium, Evert's best response would be to choose the strategy that gives her (Evert) the highest payoff. Evert's best response, which is also the worst-case scenario for Navratilova, is then shown on the highest of the three lines at that q. We show this set of worst-case outcomes with the thicker lines in Figure 7.9; these outcomes are formed from the *upper envelope* of the three payoff lines. Navratilova's optimal choice of q will make Evert's payoff as low as possible (thereby making her own as large as possible) from this set of worst-case outcomes.

To be more precise about Navratilova's optimal choice of q, we must calculate the coordinates of the kink points in the line showing her worst-case outcomes. The value of q at the left-most kink in this line makes Evert indifferent between DL and Lob. That q must equate the two payoffs from DL and Lob when used against the q-mix. Setting those two expressions equal gives us $50q + 80(1 - q) = 70q + 60(1 - q)$ and $q = 20/40 = 1/2$. This first kink is thus at $q = 0.5$, or 50%. Evert's expected payoff at this point is $50 \times 0.5 + 80 \times 0.5 = 70 \times 0.5 + 60 \times 0.5 = 65$. At the second (right-most) kink, Evert is indifferent between CC and Lob. Thus the q value at this kink is the one that equates the CC and Lob payoff expressions. Setting $90q + 20(1 - q) = 70q + 60(1 - q)$, we find $q = 40/60$; the right-most kink is at $q = 0.667$, or 66.7%. Here, Evert's expected payoff is $90 \times 0.667 + 20 \times 0.333 = 70 \times 0.667 + 60 \times 0.333 = 66.67$.

Now we can explicitly describe Evert's best responses to Navratilova's different choices of q. Evert's best response is DL when $q < 0.5$, CC when $q > 0.667$, and Lob when $0.5 < q < 0.667$. As usual, Evert's best response is pure for most values of q. When $q = 0.5$, Evert is indifferent between DL and Lob and therefore equally indifferent between those two pure strategies and any mixture of them. When $q = 0.667$, she is indifferent between CC and Lob and therefore equally indifferent between those two pure strategies and any mixture of them.

Figure 7.9 also shows that, of all the worst-case scenarios for Navratilova, the best (or least bad) occurs at the left kink, where $q = 0.5$ and Evert's expected payoff is 65. The thick line shows all the maxima (for each q) and this point represents the minimum among them; this is Navratilova's minimax. At this point, Evert achieves the smallest of the payoffs associated with choosing her best response to each q that Navratilova might pick. Therefore, at $q = 0.5$, Navratilova achieves the largest of her worst-case payoffs and she should choose this q in equilibrium.

When Navratilova chooses $q = 0.5$, Evert is indifferent between DL and Lob, and either of these choices gives her a better payoff than does CC. Therefore Evert will not use CC at all in equilibrium. CC will be an unused strategy in her equilibrium mix.

Now we can proceed with the equilibrium analysis as if this were a game with just two pure strategies for each player: DL and CC for Navratilova, and DL and Lob for Evert. We are back in familiar territory. Therefore we leave the calculation to you and just tell you the result. Evert's optimal mixture in this game entails her using DL with probability 0.25 and Lob with probability 0.75. Evert's expected payoff from this mixture, taken against Navratilova's DL and CC, respectively, is

$$50 \times 0.25 + 70 \times 0.75 = 80 \times 0.25 + 60 \times 0.75 = 65.$$

This payoff is Evert's maximin value, and it equals Navratilova's minimax, in conformity with the general result on mixed-strategy Nash equilibrium in zero-sum games.

We could not have started our analysis with this two-by-two game, because we did not know in advance which of her three strategies Evert would not use. But we can be confident that in the general case there will be one such strategy. When the three expected payoff lines take the most general positions, they intersect pair by pair rather than all crossing at a single point. Then the upper envelope has the shape that we see in Figure 7.9. Its lowest point is defined by the intersection of the payoff lines associated with two of the three strategies. The payoff from the third strategy lies below the intersection at this point, so the player choosing among the three strategies does not use that third one.

B. Exceptional Cases

The positions and intersections of the three lines of Figure 7.9 depend on the payoffs specified for the pure strategies. We chose the payoffs for that particular game to show a general configuration of the lines. But, if the payoffs stand in very specific relationships to each other, we can get some special configurations.

First, if Evert's payoffs from Lob against Navratilova's DL and CC are equal, then the line for Lob is horizontal, and a whole range of q-values achieve Navratilova's minimax. For example, if the two payoffs in the Lob row of the table in Figure 7.8 are 70 each, then it is easy to calculate that the left kink in a revised Figure 7.9 would be at $q = 1/3$ and the right kink at $q = 5/7$. For any q in the range from 1/3 to 5/7, Evert's best response is Lob, and we get an unusual equilibrium in which Evert plays a pure strategy and Navratilova mixes.

Second, if Evert's payoffs from Lob against Navaratilova's DL and CC are lower than those of Figure 7.9 by just the right amounts (or those of the other two strategies are higher by just the right amounts), all three lines can meet in one point. For example, if the payoffs of Evert's Lob are 66 and 56 against Navratilova's DL and CC, respectively, instead of 70 and 60, then for $q = 0.6$ Evert's expected payoff from the Lob becomes $66 \times 0.6 + 56 \times 0.4 = 39.6 + 22.6 = 62$, the same as that from DL and CC when $q = 0.6$. Then Evert is indifferent between all three of her strategies when $q = 0.6$ and is equally willing to mix among them.

In this special case, Evert's equilibrium mixture probabilities are not fully determinate. Rather, a whole range of mixtures, including some where all three strategies are used, can do the job of keeping Navratilova indifferent between her DL and CC and therefore willing to mix. However, to achieve her minimax, Navratilova must use the mixture with $q = 0.6$. If she does not, Evert's best response will be to switch to one of her pure strategies, and this will work to Navratilova's detriment. We do not dwell on the determination of the precise range over which Evert's equilibrium mixtures can vary, because this case can only arise for exceptional combinations of the payoff numbers and is therefore relatively unimportant.

C. Case of Domination by a Mixed Strategy

What if Evert's payoffs from using her Lob against Navratilova's DL and CC are even lower than the values that make all three lines intersect in one point? Figure 7.10 illustrates such a payoff matrix.

When we graph the expected payoff lines from Evert's three pure strategies against Navratilova's choice of q, we now find that Lob has shifted down from its position in Figure 7.9. Figure 7.11 shows the new configuration of lines and the determination of Navratilova's minimax. The calculations that give us the positions of the three lines labeled DL, CC, and Lob and their intersections follow the same procedures as before; so we omit the details. The line labeled Mix is explained soon.

It is clear that, with these numbers, Lob is not a very good strategy for Evert. In fact, the line showing the payoffs from Lob lies everywhere below either DL or CC—and so below the upper envelope of those lines—as well as below the

		NAVRATILOVA	
		DL	CC
EVERT	DL	50	80
	CC	90	20
	Lob	75	30

FIGURE 7.10 Tennis Point When Lob Is Never a Best Response

point of intersection of the DL and CC lines. Thus, for each q in Navratilova's mix, at least one of the pure strategies DL or CC gives Evert a higher payoff than does Lob. Figure 7.11 show that, if $q < 0.667$, DL is a better response for Evert than Lob; if $q > 0.4$, CC is better than Lob; and, when $0.4 < q < 0.667$, both DL and CC are better than Lob. In other words, Lob is *never the best response* for Evert.

However, it is also true that Lob is not dominated by either DL or CC. If $q < 0.4$, Lob does better than CC, whereas, if $q > 0.667$, Lob does better than DL. Thus it appears that Lob cannot be eliminated from Evert's consideration by using dominance reasoning.

But, now that we are allowing mixed strategies, we can consider mixtures of DL and CC as well. Can one such mixture dominate Lob? Yes. Consider a mixture between DL and CC with respective probabilities p and $(1 - p)$. Its payoff against Navratilova's pure DL is $50p + 90(1 - p)$, and its payoff against Navratilova's

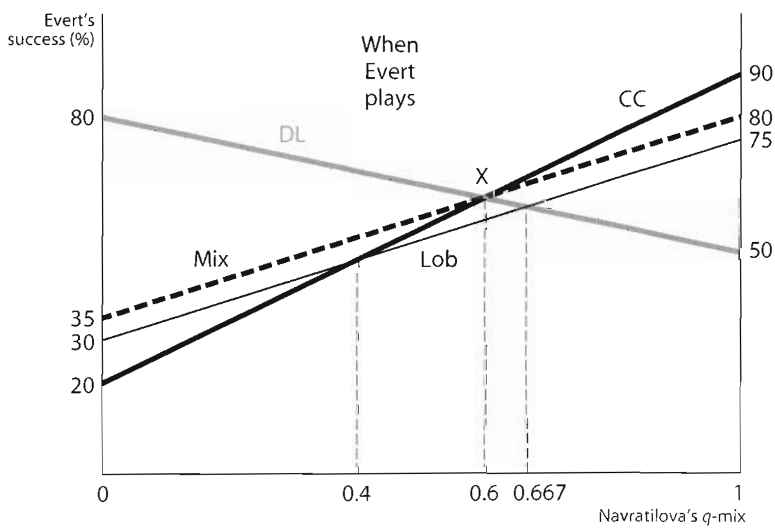


FIGURE 7.11 Domination by a Mixed Strategy

pure CC is $80p + 20(1 - p)$. Therefore its payoff against Navratilova's q-mix is $[50p + 90(1 - p)]\,q + [80p + 20(1 - p)](1 - q) = [50q + 80(1 - q)]p + [90q + 20(1 - q)](1 - p)$. This expression represents an average of the heights of the DL and CC lines, with proportions p and $(1 - p)$.

In other words, Evert's expected payoff from her p-mix between DL and CC against Navratilova's q-mix is a straight line that averages the two separate DL and CC lines. Naturally, the average line must pass through the point of intersection of the two separate DL and CC lines; if, for some q, the two pure strategies DL and CC give equal payoffs, so must any mixture of them.

If $p = 1$ in Evert's mixture, she plays pure DL and the line for her mixture is just the DL line; if $p = 0$, the mixture line is the CC line. As p ranges from 0 to 1, the line for the p-mix rotates from the pure CC to the pure DL line, always passing through the point of intersection at $q = 0.6$. Among all such lines corresponding to different values of p, one will be parallel to the line for Lob. This line is shown in Figure 7.11 as the fourth (dashed) line labeled Mix. Because the line for Lob lies below the point of intersection of DL and CC, it also lies entirely below this parallel line representing a particular mixture of DL and CC. Then, no matter what q may be, Lob yields a lower expected payoff than this particular mixed strategy. In other words, Lob is dominated by this mixed strategy.

It remains to find the probabilities that define the mixture that dominates Lob for Evert. By construction, the Mix line is parallel to the Lob line and passes through the point at which $q = 0.6$ and the expected payoff is 62. Then it is easy to calculate the vertical coordinates of the Mix line. We use the fact that the slope of Lob $= (75 - 30)/(1 - 0) = 45$. Then the vertical coordinate of the Mix line when $q = 0$ is given by v in the equation $45 = (62 - v)/(0.6 - 0)$, or $v = 35$; similarly, when $q = 1$, the vertical coordinate solves $45 = (v - 62)/(1 - 0.6)$, which gives us $v = 80$. For any value of q between 0 and 1, the vertical coordinate of the Mix line is then $80 \times q + 35 \times (1 - q)$. Now compare this result with the expression for the general p-mix line just derived. For the two to be consistent, the part of that p-mix expression that multiplied q—$50p + 90(1 - p)$—must equal 80, and $20\,p + 80(1 - p)$ must equal 35. Both of them imply that $p = 0.75$.

Thus Evert's mixture of DL with probability 0.75 and CC with probability 0.25 yields the Mix line drawn in Figure 7.11. This mixed strategy gives her a better expected payoff than does her Lob, for each and every value of Navratilova's q. In other words, this particular mixture dominates Lob from Evert's perspective.[6]

[6]We constructed the parallel line to guarantee dominance. Other lines through the point of intersection of the DL and CC lines also can dominate Lob so long as their slopes are not too different from that of the Lob line.

Now we have shown that, if a strategy is never a best response, then we can find a mixed strategy that dominates it.[7] In the process, we have expanded the scope of the concept of dominance to include domination by mixed strategies. The converse also is true; if a strategy is dominated by another strategy, albeit a mixed one, it can never be a best response to any of the other player's strategies. We can then use all of the other concepts associated with dominance that were developed in Chapter 4, but now allowing for mixed strategies also. We can do successive or iterated elimination of strategies, pure or mixed, that are dominated by other strategies, pure or mixed. If the process leaves just one strategy for each player, the game is dominance solvable and we have found a Nash equilibrium. More typically, the process only narrows down the range of strategies. In Chapter 5, we defined as *rationalizable* the set of strategies that remain after iterated elimination of pure strategies that are never best responses. Now we see that in two-player games we can think of rationalizable strategies as the set that survives after doing all possible iterated elimination of strategies that are dominated by other pure or mixed strategies.[8]

5 MIXING WHEN BOTH PLAYERS HAVE THREE STRATEGIES

As we saw in our two-by-three strategy example in the preceding section, a player mixing among three pure strategies can choose the probabilities of any two of them independently (so long as they are nonnegative and add up to no more than 1); then the probability of the third must equal 1 minus the sum of the probabilities of the other two. Thus we need two variables to specify a mix.[9] When both players have three strategies, we cannot find a mixed-strategy equilibrium without doing two-variable algebra. In many cases, such algebra is still manageable.

[7]In the example of Section 5.4 (Figure 5.6), we saw that Column's strategy C4 is never a best response but it is not dominated by any of the pure strategies C1, C2, or C3. Now we know that we can look for domination by a mixed strategy. In that game, it is easy to see that C4 is (strictly) dominated by an equal-probability mixture of C1 and C3.

[8]This equivalence between "never a best response" and "dominated by a mixed strategy" works fine in two-player games, but an added complication arises in many-player games. Consider a game with three players, A, B, and C. One of A's strategies—say, A1—may never be a best response to any pure strategies or independently mixed strategies of B and C, but A1 may fail to be dominated by any other pure or mixed strategy for A. However, if A1 is never a best response to any pure strategies or arbitrarily correlated mixed strategies of B and C, then it must be dominated by another of A's pure or mixed strategies. A complete treatment of this requires more advanced game theory; so we merely mention it. See Andreu Mas-Colell, Michael Whinston, and Jerry R. Green, *Microeconomic Theory* (New York: Oxford University Press, 1995), pp. 242–245 and 262–263.

[9]More generally, if a player has N pure strategies, then her mix has $(N - 1)$ independent variables, or "degrees of freedom of choice."

A. Full Mixture of All Strategies

Consider a simplified representation of a penalty kick in soccer. Suppose the kicker has just three pure strategies: kick to the left, right, or center. Then she can mix among these strategies, with probabilities denoted by p_L, p_R, and p_C, respectively. Any two of them can be taken to be the independent variables and the third expressed in terms of them. If p_L and p_R are made the two independent choice variables, then $p_C = 1 - p_L - p_R$. The goalie also has three pure strategies—namely, move to the kicker's left (the goalie's own right), move to the kicker's right, or continue to stand in the center—and can mix among them with probabilities q_L, q_R, and q_C, two of which can be chosen independently.

Best-response analysis also works in this context. The goalie would choose her two independent variables, say (q_L, q_R), as her best response to the kicker's two, (p_L, p_R), and vice versa. The Nash equilibrium occurs when each is choosing her best response to the other's mix. However, because each is choosing two magnitudes in response to the other's two, we can't use best-response graphs, because they would have to be in four dimensions.

Instead, we use the principle of the opponent's indifference, which enables us to focus on one player's mixture probabilities. They should be such that the other is indifferent among all the pure strategies that constitute her mixture. This gives us a set of equations that can be solved for the mixture probabilities. In the soccer example, the kicker's (p_L, p_R) would satisfy two equations expressing the requirement that the goalie's expected payoff from using her left should equal that from using her right and that the goalie's expected payoff from using her right should equal that from using her center. (Then the equality of expected payoffs from left and center follows automatically and is not a separate equation.) With more pure strategies, the number of the probabilities to be solved for and the number of equations that they must satisfy also increase.

Figure 7.12 shows the payoff matrix with the kicker as the Row player and the goalie as the Column player. Because the kicker wants to maximize the percentage probabilities that she successfully scores a goal and the goalie wants to

		GOALIE		
		Left	Center	Right
KICKER	Left	45	90	90
	Center	85	0	85
	Right	95	95	60

FIGURE 7.12 Mixing in Soccer Penalty Kick When All Pure Strategies Are Used

minimize the probability that she lets the goal through, this is a zero-sum game. We show the payoffs from the kicker's (the Row player's) perspective. For example, if the kicker kicks to her left while the goalie moves to the kicker's left (the top-left-corner cell), we suppose that the kicker still succeeds 45% of the time. But if the kicker kicks to her right and the goalie goes to the kicker's left, then the kicker has a 90% chance of scoring; we suppose a 10% probability that she might kick wide or too high. If the kicker kicks to her right (bottom row of the matrix), her probabilities of success are 95% if the goalie guesses wrong and 60% if the goalie guesses correctly. You can experiment with different payoff numbers that you think might be more appropriate.

It is easy to verify that the game has no equilibrium in pure strategies. So suppose the kicker is mixing with probabilities p_L, p_R, and $p_C = 1 - p_L - p_R$. Against each of the goalie's pure strategies, this mixture yields the following payoffs:

$$
\begin{aligned}
&\text{Left:} && 45p_L + 85p_C + 95p_R = 45p_L + 85(1 - p_L - p_R) + 95p_R \\
&\text{Center:} && 90p_L + 0p_C + 95p_R = 90p_L + 95p_R \\
&\text{Right:} && 90p_L + 85p_C + 60p_R = 90p_L + 85(1 - p_L - p_R) + 60p_R
\end{aligned}
$$

The goalie wants these numbers to be as small as possible. But, in a mixed strategy equilibrium, the kicker's mixture must be such that the goalie is indifferent between her pure strategies. Therefore all three of these expressions must be equal in equilibrium.

Equating the Left and Right expressions and simplifying, we have $45p_L = 35p_R$, or $p_R = (9/7)p_L$. Next, equate the Center and Right expressions and simplify, by using the link between p_L and p_R just obtained. This gives

$$
\begin{aligned}
&90p_L + 95(9p_L/7) = 90p_L + 85[1 - p_L - (9p_L/7)] + 60(9p_L/7), \\
&\text{or} \quad [85 + 120(9/7)]\, p_L = 85, \\
&\text{which yields } p_L = 0.355.
\end{aligned}
$$

Then we get $p_R = 0.355(9/7) = 0.457$, and finally $p_C = 1 - 0.355 - 0.457 = 0.188$. The kicker's payoff from this mixture against any of the goalie's pure strategies and therefore against any mixture of them can then be calculated by using any of the preceding three payoff lines; the result is 75.4.

The goalie's mixture probabilities can be found by writing down and solving the equations for the kicker's indifference among her three pure strategies against the goalie's mixture. We do this in detail for a slight variant of the same game in Section 5.B; so we omit the details here and just give you the answer: $q_L = 0.325$, $q_R = 0.561$, and $q_C = 0.113$. The kicker's payoff from any of her pure strategies when played against the goalie's equilibrium mixture is 75.4. Note that this payoff is the same as the number found when calculating the kicker's mix; this is just the maximin = minimax property of zero-sum games.

Now we can interpret the findings. The kicker does better with her pure Right than her pure Left, both when the goalie guesses correctly (60 > 45) and when she guesses incorrectly (95 > 90). (Presumably the kicker is left-footed and can kick harder to her right.) Therefore the kicker chooses her Right with greater probability and, to counter that, the goalie chooses Right with the highest probability, too. However, the kicker should not and does not choose her pure-strategy Right; if she did so, the goalie would then choose her own pure-strategy Right, too, and the kicker's payoff would be only 60, less than the 75.4 that she gets in equilibrium.

B. Equilibrium Mixtures with Some Strategies Unused

In the preceding equilibrium, the probabilities of using Center in the mix are quite low for each player. The (Center, Center) combination would result in a sure save and the kicker would get a really low payoff—namely, 0. Therefore the kicker puts a low probability on this choice. But then the goalie also should put a low probability on it, concentrating on countering the kicker's more likely choices. The payoff numbers in the game of Figure 7.12 are such that the kicker should not avoid Center altogether. If she did, then so would the goalie, and the game would reduce to one with just two basic pure strategies, Left and Right, for each player. With the right combination of payoffs, however, such an outcome could arise in equilibrium.

We show such a variant of the soccer game in Figure 7.13. The only difference in payoffs between this variant and the original game of Figure 7.12 is that the kicker's payoffs from Center have been lowered even farther, from 85 to 70. Let us try to calculate the equilibrium here by using the same methods as in Section 5.A. This time we do it from the goalie's perspective; we try to find her mixture probabilities q_L, q_R, and q_C, by using the condition that the kicker should be indifferent among all three of her pure strategies when played against this mixture.

The kicker's payoffs from her pure strategies are

Left: $45q_L + 90q_C + 90q_R = 45q_L + 90(1 - q_L - q_R) + 90q_R = 45q_L + 90(1 - q_L)$
Center: $70q_L + 0q_C + 70q_R = 70q_L + 70q_R$
Right: $95q_L + 95q_C + 60q_R = 95q_L + 95(1 - q_L - q_R) + 60\,q_R = 95(1 - q_R) + 60q_R$

		GOALIE		
		Left	Center	Right
KICKER	Left	45	90	90
	Center	70	0	70
	Right	95	95	60

FIGURE 7.13 Mixing in Soccer Penalty Kick When Not All Pure Strategies Are Used

Equating the Left and Right expressions and simplifying, we have $90 - 45q_L = 95 - 35q_R$, or $35q_R = 5 + 45q_L$. Next, equate the Left and Center expressions and simplify to get $90 - 45q_L = 70q_L + 70q_R$, or $115q_L + 70q_R = 90$. Substituting for q_R from the first of these equations (after multiplying through by 2 to get $70q_R = 10 + 90q_L$) into the second yields $205q_L = 80$, or $q_L = 0.390$. Then using this value for q_L in either of the equations gives $q_R = 0.644$. Finally, we use both of these values to obtain $q_C = 1 - 0.390 - 0.644 = -0.034$. Because probabilities cannot be negative, something has obviously gone wrong.

To understand what happens in this example, start by noting that Center is now a poorer strategy for the kicker than it was in the original version of the game, where her probability of choosing it was already quite low. But the concept of the opponent's indifference, expressed in the equations that led to the solution, means that the kicker has to be kept willing to use this poor strategy. That can happen only if the goalie is using her best counter to the kicker's Center—namely, the goalie's own Center—sufficiently infrequently. And in this example that logic has to be carried so far that the goalie's probability of Center has to become negative.

As pure algebra, the solution that we derived may be fine, but it violates the requirement of probability theory and real-life randomization that probabilities be nonnegative. The best that can be done in reality is to push the goalie's probability of choosing Center as low as possible—namely, to zero. But that leaves the kicker unwilling to use her own Center. In other words, we get a situation in which each player is not using one of her pure strategies in her mixture, that is, using it with zero probability.

Can there then be an equilibrium in which each player is mixing between her two remaining strategies—namely, Left and Right? If we regard this reduced two-by-two game in its own right, we can easily find its mixed-strategy equilibrium. With all the practice that you have had so far, it is safe to leave the details to you and to state the result:

Kicker's mixture probabilities: $p_L = 0.4375$, $p_R = 0.5625$;
Goalie's mixture probabilities: $q_L = 0.3750$, $q_R = 0.6250$;
Kicker's expected payoff (success percentage): 73.13.

We found this result by simply removing the two players' Center strategies from consideration on intuitive grounds. But we must check that it is a genuine equilibrium of the full three-by-three game. That is, we must check that neither player finds it desirable to bring in her third strategy, given the mixture of two strategies chosen by the other player.

When the goalie is choosing this particular mixture, the kicker's payoff from pure Center is $0.375 \times 70 + 0.625 \times 70 = 70$. This payoff is less than the 73.13 that she gets from either of her pure Left or pure Right or any mixture between the two; so the kicker does not want to bring her Center strategy into play. When the kicker is choosing the two-strategy mixture with the preceding probabilities, her payoff against the goalie's pure Center is $0.4375 \times 90 + 0.5625 \times 95 = 92.8$. This number is higher than the 73.13 that the kicker would get against the goalie's pure Left or pure Right or any mixture of the two and is therefore worse for the goalie. Thus the goalie does not want to bring her Center strategy into play either. The equilibrium that we found for the two-by-two game is indeed an equilibrium of the three-by-three game.

To allow for the possibility that some strategies may go unused in an equilibrium mixture, we must modify or extend the "opponent's indifference" principle. Each player's equilibrium mix should be such that the other player is indifferent among all the strategies *that are actually used in her equilibrium mix.* The other player is not indifferent between these and her unused strategies; she prefers the ones used to the ones unused. In other words, against the opponent's equilibrium mix, all of the strategies used in your own equilibrium mix should give you the same expected payoff, which in turn should be higher than what you would get from any of your unused strategies. This is called the principle of *complementary slackness;* we consider it in greater generality in Chapter 8, where the reason for this strange-sounding name will become clearer.

6 EVIDENCE ON MIXING IN ZERO-SUM GAMES

Researchers who perform laboratory experiments are generally dismissive of mixed strategies. As Douglas Davis and Charles Holt note, "Subjects in experiments are rarely (if ever) observed flipping coins, and when told *ex post* that the equilibrium involves randomization, subjects have expressed surprise and skepticism."[10] When the predicted equilibrium entails mixing two or more pure strategies, experimental results do show some subjects in the group pursuing one of the pure strategies and others pursuing another, but this does not constitute true mixing by an individual player. When subjects play zero-sum games repeatedly, individual players often choose different pure strategies over time. But they seem to mistake alternation for randomization; that is, they switch their choices more often than true randomization would require.

Real life seems ahead of the laboratory, as the following story illustrates.[11] In Malaya in the late 1940s, the British army escorted convoys of food trucks to protect the trucks from Communist terrorist attacks. The terrorists could either

[10]Douglas D. Davis and Charles A. Holt, *Experimental Economics* (Princeton: Princeton University Press, 1993), p. 99.

[11]R. S. Beresford and M. H. Peston, "A Mixed Strategy in Action," *Operations Research,* vol. 6, no. 4 (December 1955), pp. 173–176.

launch a large-scale attack or create a smaller sniping incident intended to frighten the truck drivers and keep them from serving again. The British escort could be either concentrated or dispersed in the convoy. For the army, concentration was better to counter a full-scale attack, and dispersal was better against sniping. For the terrorists, a full-scale attack was better if the army escort was dispersed, and sniping was better if the escort was concentrated. This zero-sum game has only a mixed-strategy equilibrium. The escort commander, who had never heard of game theory, made his decision as follows. Each morning, as the convoy was forming, he took a blade of grass and concealed it in one of his hands, holding both hands behind his back. Then he asked one of his troops to guess which hand held the blade, and he chose the form of the convoy according to whether the man guessed correctly. While the precise payoff numbers are difficult to judge and therefore we cannot say whether 50-50 was the right mixture, the officer had correctly figured out the need for true randomization and its method.

The best evidence in support of mixed strategies in zero-sum games comes from sports, especially from professional sports, where the players accumulate a great deal of experience of such games and their intrinsic desires to win are buttressed by large financial gains from winning.

Mark Walker and John Wooders examined the serve-and-return play of top-level players at Wimbledon.[12] They model this interaction as a game with two players, the server and the receiver, in which each player has two pure strategies. The server can serve to the receiver's forehand or backhand, and the receiver can guess to which side the serve will go and move that way. Because serves are so fast at the top levels of men's singles, the receiver cannot react after observing the actual direction of the serve; rather, the receiver must move in anticipation of the serve's direction. This game then has simultaneous moves. Further, because the receiver wants to guess correctly and the server wants to wrong-foot the receiver, this interaction has a mixed-strategy equilibrium.

If the tennis players are using their equilibrium mixtures in the serve-and-return game, the server should win the point with the same probability whether he serves to the receiver's forehand or backhand. An actual tennis match contains a hundred or more points played by the same two players; thus there is enough data to test whether this implication holds. Walker and Wooders tabulated the results of serves in 10 matches. Each match contains four kinds of serve-and-return combinations: A serving to B and vice versa, combined with service from the right or the left side of the court (Deuce or Ad side). Thus they had data on 40 serving situations and found that in 39 of them the

[12]Mark Walker and John Wooders, "Minimax Play at Wimbledon," *American Economic Review*, vol. 91, no. 5 (December 2001), pp. 1521–1538.

server's success rates with forehand and backhand serves were equal to within acceptable limits of statistical error.

The top-level players must have had enough general experience playing the game, as well as particular experience playing against the specific opponents, to have learned the general principle of mixing and the correct proportions to mix against the specific opponents. However, there was one respect in which the servers' choices departed from true mixing. To achieve the necessary unpredictability, there should be no pattern of any kind in a sequence of serves: the choice of side for each serve should be independent of what has gone before. As we said in reference to the practice of mixed strategies, players can alternate too much, not realizing that alternation is a pattern just as much as doing the same action repeatedly a few times would be a pattern. And, indeed, the data show that the tennis servers alternated too much. But the data also indicate that this departure from true mixing was not enough to be picked up and exploited by the opponents.

In another study of mixing strategies in sports contests, Pierre-André Chiappori, Timothy Groseclose, and Steven Levitt analyzed penalty kicks in soccer.[13] They collected a large data set from European league competitions to look at the game played between kicker and goalie, as we did in Section 5. Each kicker–goalie combination had too few data points for statistical use; so they examined what the implications of the theory would be for aggregate data coming from different kickers and different goalies. This statistical work is quite advanced, but its results are simple: the predictions of the theory are borne out by the data. Indeed, in their tracing of the sequence of choices of each kicker and goalie, they do not even find too much alternation. Thus these findings suggest that behavior in soccer penalty kicks is even closer to true mixing than behavior in the tennis serve-and-return game.

7 HOW TO USE MIXED STRATEGIES IN PRACTICE

Finally, we note some important things to remember when finding or using a mixed strategy in a zero-sum game. First, to use a mixed strategy effectively in such a game, a player needs to do more than calculate the equilibrium percentages with which to use each of her actions. Indeed, in our tennis-point game, Evert cannot simply play DL five-eighths of the time and CC three-eighths of the time by mechanically rotating five shots down the line and three shots

[13]Pierre-André Chiappori, Timothy Groseclose, and Steven Levitt, "Testing Mixed Strategy Equilibria When Players Are Heterogeneous: The Case of Penalty Kicks in Soccer," *American Economic Review*, vol. 92, no. 4 (September 2002), pp. 1138–1151.

cross-court. Why not? Because mixing your strategies is supposed to help you benefit from the element of surprise against your opponent. If you use a recognizable pattern of plays, your opponent is sure to discover it and exploit it to her advantage.

The lack of a pattern means that, after any history of choices, the probability of choosing DL or CC on the next turn is the same as it always was. If a run of several successive DLs happens by chance, there is no sense in which CC is now "due" on the next turn. Many people in practice mistakenly think otherwise, and therefore they alternate their choices too much compared with what a truly random sequence of choices would require; they produce too few runs of identical successive choices. However, detecting a pattern from observed actions is a tricky statistical exercise, which the opponents may not be able to perform while playing the game. As we saw in Section 6, Walker and Wooders, in their analysis of data from grand slam tennis finals, found that servers alternated their serves too much, but receivers were not able to detect this departure from true randomization.

However, a good strategist should not rely on the opponent's errors. To make sure that your mixed strategy works effectively, you need a truly random pattern of actions on each play of the game. You may want to rely on a computer's ability to generate random numbers for you, for example, from which you can determine your appropriate choice of action. If the computer generates numbers between 1 and 100 and you want to mix pure strategies A and B in a 60:40 split, you may decide to play A for any random number between 1 and 60 and to play B for any random number between 61 and 100. Similarly, you could employ a device like the color-coded spinners provided in many childrens' games. For the same 60:40 mixture, you would color 60% of the circle on the spinner in blue, for example, and 40% in red; the first 216 degrees of the circle would be blue, the remaining 144 red. Then you would spin the spinner arrow and play A if it landed in blue but B if it landed in red. The second hand on a watch can provide the same type of device, but it is important that your watch not be so accurate and synchronized that your opponent can use the same watch and figure out what you are going to do.

The importance of avoiding a predictable system of randomization is clearest in ongoing interactions of a zero-sum nature. Because of the diametrically opposed interests of the players in such games, your opponent always benefits from exploiting your choice of action to the greatest degree possible. Thus, if you play the same game against each other on a regular basis, she will always be on the look out for ways to break the code that you are using to randomize your moves. If she can do so, she has a chance to improve her payoffs in future plays of the game. Mixing is still justified in single-meet (sometimes called one-shot) zero-sum games because the benefit of "tactical surprise" remains important.

Finally, players must understand and accept the fact that the use of mixed strategies guards you against exploitation and gives the best possible expected

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payoff but that it is only a probabilistic average. On particular occasions, you can get poor outcomes: for example, the long pass on third down with a yard to go, intended to keep the defense honest, may fail on any specific occasion. If you use a mixed strategy in a situation in which you are responsible to a higher authority, however, you may need to plan ahead for this possibility. You may need to justify your use of such a strategy ahead of time to your coach or your boss, for example. They need to understand why you have adopted your mixture and why you expect it to yield you the best possible payoff, on average, even though it might yield an occasional low payoff as well. Even such advance planning may not work to protect your "reputation," though, and you should prepare yourself for criticism in the face of a bad outcome.

SUMMARY

Zero-sum games in which one player prefers a coincidence of actions that the other prefers the opposite often have no Nash equilibrium in pure strategies. In these games, each player wants to be unpredictable and so uses a mixed strategy that specifies a probability distribution over his set of pure strategies. Mixed strategies are a special case of continuous strategies but have additional matters that deserve separate study.

Best-response analysis can be used to solve for mixed-strategy equilibria. The best-response-curve diagram can be used to show all mixed-strategy as well as pure-strategy equilibria of a game. The *opponent's indifference* property of mixed-strategy equilibria indicates that each player's equilibrium mixture is such that the other player is indifferent among all her mixes. Minimax analysis also can be used to find players' equilibrium mixtures. Mixed-strategy equilibria can be interpreted as outcomes in which all players have correct beliefs about the probabilities with which the other player chooses from among her underlying pure actions.

When one or both players have three (or more) strategies, equilibrium mixed strategies may put positive probability on all pure strategies or may include only a subset of the pure strategies. If one player has three strategies and the other has only two, the player with three available pure strategies will generally use only two in her equilibrium mix. Equilibrium mixtures may also be indeterminate.

Evidence on mixed-strategy play in the laboratory is mixed, but there is evidence of the use of equilibrium mixes in other more realistic settings. Sports examples, including tennis serves and soccer penalty kicks, provide significant evidence of mixing in actual play. And, in actual play, players should remember that good use of a mixed strategy requires that there be no predictable system of randomization.

KEY TERMS

expected payoff (186)
expected value (197)
opponent's indifference property (194)

EXERCISES

1. "When a game has a mixed-strategy equilibrium, a player's equilibrium mixture is designed to yield her the same expected payoff when used against each of the other player's pure strategies." True or false? Explain and give an example of a game that illustrates your answer.

2. In Figures 7.2 and 7.3 of the text, you saw how to derive Navratilova's best-response rule in the tennis-point game in two different ways. Draw figures similar to Figures 7.2 and 7.3 to show how to derive Evert's best-response rule in the same two ways. Use the algebraic approach to finding the best-response rule to verify the equilibrium value of p for Evert and the payoff that she receives in the mixed-strategy equilibrium.

3. Find Nash equilibria in mixed strategies for the following games. In each case, verify your answer by using a second method to recalculate the equilibrium mixtures for each player.

(a)

		COLUMN	
		Left	Right
ROW	Up	4	-1
	Down	1	2

(b)

		COLUMN	
		Left	Right
ROW	Up	3	2
	Down	1	4

4. Many of you will be familiar with the children's game Rock–Scissors–Paper. In Rock–Scissors–Paper, two people simultaneously choose either "Rock," "Scissors," or "Paper," usually by putting their hands into the shape of one of the three choices. The game is scored as follows. A person choosing Scissors beats

a person choosing Paper (because scissors cut paper). A person choosing Paper beats a person choosing Rock (because paper covers rock). A person choosing Rock beats a person choosing Scissors (because rock breaks scissors). If two players choose the same object, they tie. Suppose that each individual play of the game is worth 10 points. The following matrix shows the possible outcomes in the game:

		PLAYER 2		
		Rock	Scissors	Paper
PLAYER 1	Rock	0	10	−10
	Scissors	−10	0	10
	Paper	10	−10	0

(a) Suppose that Player 2 announced that she would use a mixture in which her probability of choosing Rock would be 40%, her probability of choosing Scissors would be 30%, and her probability of Paper, 30%. What is Player 1's best response to this strategy choice of Player 2? Explain why your answer makes sense, given your knowledge of mixed strategies.

(b) Find the mixed-strategy equilibrium of this Rock–Scissors–Paper game.

5. Consider a slightly different version of Rock–Scissors–Paper in which Player 1 has an advantage. If Player 1 picks Rock and Player 2 picks Scissors, Player 1 wins 20 points from Player 2 (rather than 10). The new payoff matrix is:

		PLAYER 2		
		Rock	Scissors	Paper
PLAYER 1	Rock	0	20	−10
	Scissors	−10	0	10
	Paper	10	−10	0

(a) What is the mixed-strategy equilibrium in this version of the game?

(b) Compare your answer here with your answer for the mixed-strategy equilibrium in Exercise 4. How can you explain the differences in the equilibrium strategy choices?

6. In baseball, pitchers and batters have conflicting goals. Pitchers want to get the ball *past* batters but batters want to *connect* with pitched balls. The following table shows the payoffs for a hypothetical game between a pitcher and a batter. The payoffs take into account the fact that fastballs, when anticipated and hit by a batter, yield better hits (more bases) than do similarly anticipated and hit curve balls.

		PITCHER	
		Throw curve	Throw fastball
BATTER	Anticipate curve	2	−1
	Anticipate fastball	−1	4

(a) Verify that there is no pure-strategy equilibrium in this game, and then find the mixed-strategy equilibrium.
(b) Draw the best-response curves for both players, and indicate the mixed-strategy equilibrium on your diagram.
(c) Can the pitcher improve his expected payoff in the mixed-strategy equilibrium of this game by slowing down his fastball (thereby making it more similar to a curve ball)? Assume that slowing down the pitched fastball changes the payoff to the hitter in the "anticipate fastball/throw fastball" cell from 4 to 3. Explain carefully how you determine the answer here and show your work. Also explain *why* slowing the fastball can or cannot improve the pitcher's expected payoff in the game.

7. Lucy offers to play the following game with Charlie: "Let us show pennies to each other, each choosing either heads or tails. If we both show heads, I pay you \$3. If we both show tails, I pay you \$1. If the two don't match, you pay me \$2." Charlie reasons as follows. "The probability of both heads is 1/4, in which case I get \$3. The probability of both tails is 1/4, in which case I get \$1. The probability of no match is 1/2, and in that case I pay \$2. So it is a fair game." Is he right? If not, (a) why not, and (b) what is Lucy's expected profit from the game?

8. Consider the following zero-sum game:

		COLUMN	
		Left	Right
ROW	Up	0	A
	Down	B	C

The entries are the Row player's payoffs, and the numbers A, B, and C are all positive. What other relations between these numbers (for example, $A < B < C$) must be valid for each of the following cases to arise?
(a) At least one of the players has a dominant strategy.
(b) Neither player has a dominant strategy, but there is a Nash equilibrium in pure strategies.
(c) There is no Nash equilibrium in pure strategies, but there is one in mixed strategies.
(d) Given that case **c** holds, write a formula for Row's probability of choosing Up. Call this probability p, and write it as a function of A, B, and C.

9. Suppose that the soccer penalty kick game of Section 5 in this chapter is expanded to include a total of six distinct strategies for the kicker: to shoot high and to the left (HL), low and to the left (LL), high and in the center (HC), low and in the center (LC), high right (HR), and low right (LR). The goalkeeper continues to have three strategies: to move to the kicker's left (L) or right (R) or to stay in the center (C). The kicker's success percentages are shown in the following table.

		GOALIE		
		L	C	R
KICKER	HL	0.50	0.85	0.85
	LL	0.40	0.95	0.95
	HC	0.85	0	0.85
	LC	0.70	0	0.70
	HR	0.85	0.85	0.50
	LR	0.95	0.95	0.40

These payoffs incorporate the following information. Shooting high runs some risk of missing the goal even if the goalie goes the wrong way (hence $0.85 < 0.95$). If the goalie guesses correctly, she has a better chance of collecting or deflecting a low shot than a high one (hence $0.40 < 0.50$). And, if the shot is to the center while the goalie goes to one side, she has a better chance of using her feet to deflect a low shot than a high one (hence $0.70 < 0.85$).

Verify that, in the mixed-strategy equilibrium of this game, the goalkeeper uses L and R 42.2% of the time each, and C 15.6% of the time, while the kicker uses LL and LR 37.8% of the time each, and HC 24.4% of the time. Also verify that the payoff to the kicker in equilibrium is 71.8 (percentage rate of success). (This conforms well with the observation of three or four successes out of every five attempts in recent soccer World Cup penalty shootouts.)

Note that, to verify the claim, all that you have to do is to take the given numbers and show that they satisfy the conditions of equilibrium. That is, against the opponent's equilibrium mix, all of the strategies used in a player's own equilibrium mix should give him the same expected payoff, which in turn should be higher than what he would get from any of his unused strategies.

10. Recall the duel game between Renard and Chagrin in Chapter 6, Exercise 9. Remember that the duelists start 10 steps apart and walk toward each other at the same pace, 1 step at a time, and either may fire his gun after each

step. When one duelist shoots, the probability of scoring a hit depends on the distance; after k steps, it is $k/5$. Each gets a payoff of -1 if he himself is killed and 1 if the other duelist is killed. If neither or both are killed, each gets zero. Now, however, suppose that the duelists have guns with silencers. If one duelist fires and misses, the other does not know that this has happened and cannot follow the strategy of then holding his fire until the final step to get a sure shot. Each must formulate a strategy at the outset that is not conditional on the other's intermediate actions. Thus we have a simultaneous-move game, with strategies of the form "Shoot after n steps if still alive." Each player has five such strategies corresponding to the five steps that can be taken toward each other in the duel.

(a) Show that the five-by-five payoff table for this game is as follows:

		CHAGRIN				
		1	2	3	4	5
RENARD	1	0	−0.12	−0.28	−0.44	−0.6
	2	0.12	0	0.04	−0.08	−0.2
	3	0.28	−0.04	0	0.28	0.2
	4	0.44	0.08	−0.28	0	0.6
	5	0.6	0.2	−0.2	−0.6	0

(b) Verify that there is a mixed-strategy Nash equilibrium with the following strategy for each player: strategies 1 and 4 are unused, whereas strategies 2, 3, and 5 are used in the proportions 5/11, 5/11, and 1/11, respectively. (That is, check that these numbers satisfy the conditions for Nash equilibrium in mixed strategies in a zero-sum game.)

(c) What is the expected payoff for each player in equilibrium?

Appendix: Probability and Expected Utility

To calculate the expected payoffs and mixed-strategy equilibria of games in this chapter, we had to do some simple manipulation of probabilities. Some simple rules govern calculations involving probabilities. Many of you may be familiar with them, but we give a brief statement and explanation of the basics here by way of reminder or remediation, as appropriate. We also state how to calculate expected values of random numerical values.

We also consider the expected utility approach to calculating expected payoffs. When the outcomes of your action in a particular game are not certain, either because your opponent is mixing strategies or because of some uncertainty in nature, you may not want to maximize your expected monetary payoff as we have generally assumed in our analysis to this point; rather, you may want to give some attention to the *riskiness* of the payoffs. As mentioned in Chapter 2, such situations can be handled by using the expected values (which are probability-weighted averages) of an appropriate nonlinear rescaling of the monetary payoffs. We offer here a brief discussion of how this can be done.

You should certainly read this material but, to get real knowledge and mastery of it, the best thing to do is to use it. The chapters to come, especially Chapters 8, 9, and 13, will give you plenty of opportunity for practice.

1 THE BASIC ALGEBRA OF PROBABILITIES

The basic intuition about the probability of an event comes from thinking about the frequency with which this event occurs by chance among a larger set of possibilities. Usually any one element of this larger set is just as likely to occur by chance as any other, so finding the probability of the event in which we are interested is simply a matter of counting the elements corresponding to that event and dividing by the total number of elements in the whole large set.[1]

In any standard deck of 52 playing cards, for instance, there are four suits (clubs, diamonds, hearts, and spades) and 13 cards in each suit (ace through 10 and the face cards—jack, queen, king). We can ask a variety of questions about the likelihood that a card of a particular suit or value or suit *and* value might be

[1]When we say "by chance," we simply mean that a systematic order cannot be detected in the outcome or that it cannot be determined by using available scientific methods of prediction and calculation. Actually, the motions of coins and dice are fully determined by laws of physics, and highly skilled people can manipulate decks of cards but, for all practical purposes, coin tosses, rolls of dice, or card shuffles are devices of chance that can be used to generate random outcomes. However, randomness can be harder to achieve than you think. For example, a perfect shuffle, where a deck of cards is divided exactly in half and then interleaved by dropping cards one at a time alternately from each, may seem a good way to destroy the initial order of the deck. But Cornell mathematician Persi Diaconis has shown that, after eight of the shuffles, the original order is fully restored. For slightly imperfect shuffles that people carry out in reality, he finds that some order persists through six, but randomness suddenly appears on the seventh! See "How to Win at Poker, and Other Science Lessons," *The Economist*, October 12, 1996. For an interesting discussion of such topics, see Deborah J. Bennett, *Randomness* (Cambridge: Harvard University Press, 1998), chaps. 6–9.

drawn from this deck of cards: How likely are we to draw a spade? How likely are we to draw a black card? How likely are we to draw a 10? How likely are we to draw the queen of spades? and so on. We would need to know something about the calculation and manipulation of probabilities to answer such questions. If we had two decks of cards, one with blue backs and one with green backs, we could ask even more complex questions ("How likely are we to draw one card from each deck and have them both be the jack of diamonds?"), but we would still use the algebra of probabilities to answer them.

In general, a **probability** measures the likelihood of a particular event or set of events occurring. The likelihood that you draw a spade from a deck of cards is just the probability of the event "drawing a spade." Here the large set has 52 elements—the total number of equally likely possibilities—and the event "drawing a spade" corresponds to a subset of 13 particular elements. Thus you have 13 chances out of the 52 to get a spade, which makes the probability of getting a spade in a single draw equal to $13/52 = 1/4 = 25\%$. To see this another way, consider the fact that there are four suits of 13 cards each; so your chance of drawing a card from any particular suit is one out of four, or 25%. If you made a large number of such draws (each time from a complete deck), then out of 52 times you will not always draw exactly 13 spades; by chance you may draw a few more or a few less. But the chance averages out over different such occasions—over different sets of 52 draws. Then the probability of 25% is the average of the frequencies of spades drawn in a large number of observations.[2]

The algebra of probabilities simply develops such ideas in general terms and obtains formulas that you can then apply mechanically instead of having to do the thinking from scratch every time. We will organize our discussion of these probability formulas around the types of questions that one might ask when drawing cards from a standard deck (or two: blue backed and green backed).[3] This method will allow us to provide both specific and general formulas for you to use later. You can use the card-drawing analogy to help you reason out other questions about probabilities that you encounter in other contexts. One other point to note: In ordinary language, it is customary to write probabilities as percentages, but the algebra requires that they be written as fractions or decimals; thus instead of 25% the mathematics works with 13/52 or 0.25. We will use one or the other, depending on the occasion; be aware that they mean the same thing.

[2]Bennett, *Randomness*, chaps. 4 and 5, offers several examples of such calculations of probabilities.

[3]If you want a more detailed exposition of the following addition and multiplication rules, as well as more exercises to practice these rules, we recommend David Freeman, Robert Pisani, and Robert Purves, *Statistics*, 3rd ed. (New York: Norton, 1998), chaps. 13 and 14.

A. The Addition Rule

The first questions that we ask are: If we were to draw one card from the blue deck, how likely are we to draw a spade? And how likely are we to draw a card that is not a spade? We already know that the probability of drawing a spade is 25% because we determined that earlier. But what is the probability of drawing a card that is not a spade? It is the same likelihood of drawing a club or a diamond or a heart instead of a spade. It should be clear that the probability in question should be larger than any of the individual probabilities of which it is formed; in fact, the probability is 13/52 (clubs) + 13/52 (diamonds) + 13/52 (hearts) = 0.75. The *or* in our verbal interpretation of the question is the clue that the probabilities should be added together, because we want to know the chances of drawing a card from any of those three suits.

We could more easily have found our answer to the second question by noting that not getting a spade is what happens the other 75% of the time. Thus the probability of drawing "not a spade" is 75% (100% − 25%) or, more formally, $1 - 0.25 = 0.75$. As is often the case with probability calculations, the same result can be obtained here by two different routes, entailing different ways of thinking about the event for which we are trying to find the probability. We will see other examples of this later in this Appendix, where it will become clear that the different methods of calculation can sometimes require vastly different amounts of effort. As you develop experience, you will discover and remember the easy ways or shortcuts. In the meantime, be comforted that each of the different routes, when correctly followed, leads to the same final answer.

To generalize our preceding calculation, we note that, if you divide the set of events, X, in which you are interested into some number of subsets, $Y, Z, \ldots,$ none of which overlap (in mathematical terminology, such subsets are said to be **disjoint**), then the probabilities of each subset occurring must sum to the probability of the full set of events; if that full set of events includes all possible outcomes, then its probability is 1. In other words, if the occurrence of X requires the occurrence of *any one* of several disjoint $Y, Z, \ldots,$ then the probability of X is the sum of the separate probabilities of $Y, Z, \ldots$ Using Prob(X) to denote the probability that X occurs and remembering the caveats on X (that it requires any one of $Y, Z, \ldots$) and on $Y, Z, \ldots$ (that they must be disjoint), we can write the **addition rule** in mathematical notation as $\text{Prob}(X) = \text{Prob}(Y) + \text{Prob}(Z) + \cdots$.

EXERCISE Use the addition rule to find the probability of drawing two cards, one from each deck, such that the two cards have identical faces.

B. The Modified Addition Rule

Our analysis in Section A of this appendix covered only situations in which a set of events could be broken down into disjoint, nonoverlapping subsets. But suppose we ask, What is the likelihood, if we draw one card from the blue deck, that the card is either a spade *or* an ace? The *or* in the question suggests, as before, that we should be adding probabilities, but in this case the two categories "spade" and "ace" are not mutually exclusive, because one card, the ace of spades, is in both subsets. Thus "spade" and "ace" are not disjoint subsets of the full deck. So, if we were to sum only the probabilities of drawing a spade (13/52) and of drawing an ace (4/52), we would get 17/52. This would suggest that we had 17 different ways of finding either an ace or a spade when in fact we have only 16—there are 13 spades (including the ace) and three additional aces from the other suits. The incorrect answer, 17/52, comes from counting the ace of spades twice. To get the correct probability in the nondisjoint case, then, we must subtract the probability associated with the overlap of the two subsets. The probability of drawing an ace or a spade is the probability of drawing an ace plus the probability of drawing a spade minus the probability of drawing the overlap, the ace of spades; that is, $13/52 + 4/52 - 1/52 = 16/52 = 0.31$.

To make this more general, if you divide the set of events, X, in which you are interested into some number of subsets Y, Z, . . . , which may overlap, then the sum of the probabilities of each subset occurring minus the probability of the overlap yields the probability of the full set of events. More formally, the **modified addition rule** states that, if the occurrence of X requires the occurrence of any one of the nondisjoint Y and Z, then the probability of X is the sum of the separate probabilities of Y and Z minus the probability that *both* Y and Z occur: $\text{Prob}(X) = \text{Prob}(Y) + \text{Prob}(Z) - \text{Prob}(Y \text{ and } Z)$.

> EXERCISE Use the modified addition rule to find the probability of drawing two cards, one from each deck, and getting at least one face card.

C. The Multiplication Rule

Now we ask, What is the likelihood that when we draw two cards, one from each deck, both of them will be spades? This event occurs if we draw a spade from the blue deck *and* a spade from the green deck. The switch from *or* to *and* in our interpretation of what we are looking for indicates a switch in mathematical operations from addition to multiplication. Thus the probability of two spades, one from each deck, is the product of the probabilities of drawing a spade from each deck, or $(13/52) \times (13/52) = 1/16 = 0.0625$, or 6.25%. Not surprisingly, we are much less likely to get two spades than we were in Section A to get one spade. (Always check to make sure that your calculations accord in this way with your intuition regarding the outcome.)

In much the same way as the addition rule requires events to be disjoint, the multiplication rules requires them to be independent; if we break down a set of events, X, into some number of subsets Y, Z, . . . , those subsets are independent if the occurrence of one does not affect the probability of the other. Our events—a spade from the blue deck and a spade from the green deck—satisfy this condition of independence; that is, drawing a spade from the blue deck does nothing to alter the probability of getting a spade from the green deck. If we were drawing both cards from the same deck, however, then after we had drawn a spade (with a probability of 13/52), the probability of drawing another spade would no longer be 13/52 (in fact, it would be 12/51); drawing one spade and then a second spade from the *same* deck are not **independent events**.

The formal statement of the **multiplication rule** tells us that, if the occurrence of X requires the simultaneous occurrence of *all* the several independent Y, Z, . . . , then the probability of X is the *product* of the separate probabilities of Y, Z, . . . : $\text{Prob}(X) = \text{Prob}(Y) \times \text{Prob}\ (Z) \times \cdots$.

EXERCISE Use the multiplication rule to find the probability of drawing two cards, one from each deck, and getting a red card from the blue deck and a face card from the green deck.

D. The Modified Multiplication Rule

What if we are asking about the probability of an event that depends on two nonindependent occurrences? For instance, suppose that we ask, What is the likelihood that with one draw we get a card that is both a spade *and* an ace? If we think about this for a moment, we realize that the probability of this event is just the probability of drawing a spade *and* the probability that our card is an ace *given that* it is a spade. The probability of drawing a spade is $13/52 = 1/4$, and the probability of drawing an ace, given that we have a spade, is 1/13. The *and* in our question tells us to take the product of these two probabilities: $(13/52)(1/13) = 1/52$.

We could have gotten the same answer by realizing that our question was the same as asking, What is the likelihood of drawing the ace of spades? The calculation of that probability is straightforward; only 1 of the 52 cards is the ace of spades, so the probability of drawing it must be 1/52. As you see, how you word the question affects how you go about looking for an answer.

In the technical language of probabilities, the probability of a particular event occurring (such as getting an ace), given that another event has already occurred (such as getting a spade) is called the **conditional probability** of drawing an ace, for example, conditioned on having drawn a spade. Then, the formal statement of the **modified multiplication rule** is that, if the occurrence of X requires the occurrence of both Y and Z, then the probability of X equals the product of two things: (1) the probability that Y alone occurs, and (2) the probability that Z occurs given that Y has already occurred, or the *conditional probability* of Z, conditioned on Y having already occurred: Prob(X) = Prob(Y alone) $\times$ Prob (Z given Y).

A third way would be to say that the probability of drawing an ace is 4/52, and the conditional probability of the suit being a spade, given that the card is an ace, is 1/4; so the overall probability of getting an ace of spades is (4/52) $\times$ 1/4. More generally, using the terminology just introduced, we have Prob(X) = Prob(Z) Prob(Y given Z).

EXERCISE Use the modified multiplication rule to find the probability that, when you draw two cards from a deck, the second card is the jack of hearts.

E. The Combination Rule

We could also ask questions of an even more complex nature than we have tried so far, in which it becomes necessary to use both the addition (or modified addition) and the multiplication (or modified multiplication) rules simultaneously. We could ask, What is the likelihood, if we draw one card from each deck, that we draw *at least one* spade? As usual, we could approach the calculation of the necessary probability from several angles, but suppose that we come at it first by considering all of the different ways in which we could draw at least one spade when drawing one card from each deck. There are three possibilities: either we could get one spade from the blue deck and none from the green deck ("spade *and* none") *or* we could get no spade from the blue deck and a spade from the green deck ("none *and* spade") *or* we could get a spade from each deck ("spade *and* spade"); our event requires that one of these three possibilities occurs, each of which entails the occurrence of both of two independent events. It should be obvious now, by using the *or*'s and *and*'s as guides, how to calculate the necessary probability. We find the probability of each of the three possible ways of getting at least one spade (which entails three products of two probabilities each) and sum these probabilities together: (1/4 $\times$ 3/4) + (3/4 $\times$ 1/4) + (1/4 $\times$ 1/4) = 7/16 = 43.75%.

The second approach entails recognizing that "at least one spade" and "not any spades" are disjoint events; together they constitute a sure thing. Therefore the probability of "at least one spade" is just 1 minus the probability of "not any spades." And the event "not any spades" occurs only if the blue card is not a spade 3/4 *and* the green card is not a spade 3/4; so its probability is 3/4 $\times$ 3/4 = 9/16.

The probability of "at least one spade" is then $1 - 9/16 = 7/16$, as we found in the preceding paragraph.

Finally, we can formally state the **combination rule** for probabilities: if the occurrence of X requires the occurrence of *exactly one* of a number of *disjoint* $Y, Z, \ldots$, the occurrence of Y requires that of *all* of a number of *independent* $Y_1, Y_2, \ldots$ the occurrence of Z requires that of *all* of a number of *independent* $Z_1, Z_2, \ldots$, and so on, then the probability of X is the sum of the probabilities of $Y, Z, \ldots$, which are the products of the probabilities $Y_1, Y_2, \ldots, Z_1, Z_2, \ldots$: or

$$\begin{aligned}\text{Prob}(X) &= \text{Prob}(Y) + \text{Prob}(Z) + \cdots \\ &= \text{Prob}(Y_1) \times \text{Prob}(Y_2) \times \cdots + \text{Prob}(Z_1) \times \text{Prob}(Z_2) \times \cdots + \cdots\end{aligned}$$

EXERCISE Suppose we now have a third (orange) deck of cards. Find the probability of drawing at least one spade when you draw one card from each of the three decks.

F. Expected Values

If a numerical magnitude (such as money winnings or rainfall) is subject to chance and can take on any one of n possible values $X_1, X_2, \ldots, X_n$ with respective probabilities $p_1, p_2, \ldots, p_n$, then the *expected value* is defined as the weighted average of all its possible values using the probabilities as weights; that is, as $p_1X_1 + p_2X_2 + \cdots + p_nX_n$. For example, suppose you bet on the toss of two fair coins. You win \$5 if both coins come up heads, \$1 if one shows heads and the other tails, and nothing if both come up tails. Using the rules for manipulating probabilities discussed earlier in this section, you can see that the probabilities of these events are, respectively, 0.25, 0.50, and 0.25. Therefore your expected winnings are $(0.25 \times \$5) + (0.50 \times \$1) + (0.25 \times \$0) = \$1.75$.

In game theory, the numerical magnitudes that we need to average in this way are payoffs, measured in numerical ratings, or money, or, as we will see later in this appendix, utilities. We will refer to the expected values in each context appropriately, for example, as *expected payoffs* or **expected utilities**.

2 ATTITUDES TOWARD RISK AND EXPECTED UTILITY

In Chapter 2, we pointed out a difficulty about using probabilities to calculate the average or expected payoff for players in a game. Consider a game where players gain or lose money, and suppose we measure payoffs simply in money amounts. If a player has a 75% chance of getting nothing and a 25% chance of getting \$100, then the expected payoff is calculated as a *probability-weighted average;* the expected payoff is the average of the different payoffs with the probabilities of each as weights. In this case, we have \$0 with a probability of 75%, which yields $0.75 \times 0 = 0$ on average, added to \$100 with a probability of 25%, which yields $0.25 \times 100 = 25$ on average. That is the same payoff as the player would get from a simple nonrandom outcome that guaranteed him \$25 every time he played. People who are indifferent between two alternatives with the same average monetary value but different amounts of risk are said to be **risk-neutral.** In our example, one prospect is riskless (\$25 for sure), while the other is risky, yielding either \$0 with a probability of 0.75 or \$100 with a probability of 0.25, for the same average of \$25. In contrast are **risk-averse** people—those who, given a pair of alternatives each with the same average monetary value, would prefer the less risky option. In our example, they would rather get \$25 for sure than face the risky \$100-or-nothing prospect and, given the choice, would pick the safe prospect. Such risk-averse behavior is quite common; we should therefore have a theory of decision making under uncertainty that takes it into account.

We also said in Chapter 2 that a very simple modification of our payoff calculation can get us around this difficulty. We said that we could measure payoffs not in money sums but by using a nonlinear rescaling of the dollar amounts. Here we show explicitly how that rescaling can be done and why it solves our problem for us.

Suppose that, when a person gets D dollars, we define the payoff to be something other than just D, perhaps $\sqrt{D}$. Then the payoff number associated with \$0 is 0, and that for \$100 is 10. This transformation does not change the way in which the person rates the two payoffs of \$0 and \$100; it simply rescales the payoff numbers in a particular way.

Now consider the risky prospect of getting \$100 with probability 0.25 and nothing otherwise. After our rescaling, the expected payoff (which is the average of the two payoffs with the probabilities as weights) is $(0.75 \times 0) + (0.25 \times 10) = 2.5$. This expected payoff is equivalent to the person's getting the dollar amount whose square root is 2.5; because $2.5 = \sqrt{6.25}$, a person getting \$6.25 for sure would also receive a payoff of 2.5. In other words, the person with our square-root payoff scale would be just as happy getting \$6.25 for sure as he would getting a 25% chance at \$100. This indifference between a guaranteed \$6.25 and a 1 in 4 chance of \$100 indicates quite a strong aversion to risk; this person is willing to give up the difference between \$25 and \$6.25 to avoid facing the risk. Figure 7A.1 shows this nonlinear scale (the square root), the expected payoff, and the person's indifference between the sure prospect and the gamble.

What if the nonlinear scale that we use to rescale dollar payoffs is the cube root instead of the square root? Then the payoff from \$100 is 4.64, and the expected payoff from the gamble is $(0.75 \times 0) + (0.25 \times 4.64) = 1.16$, which is the cube root of 1.56. Therefore a person with this payoff scale would accept only \$1.56 for sure instead of a gamble that has a money value of \$25 on average;

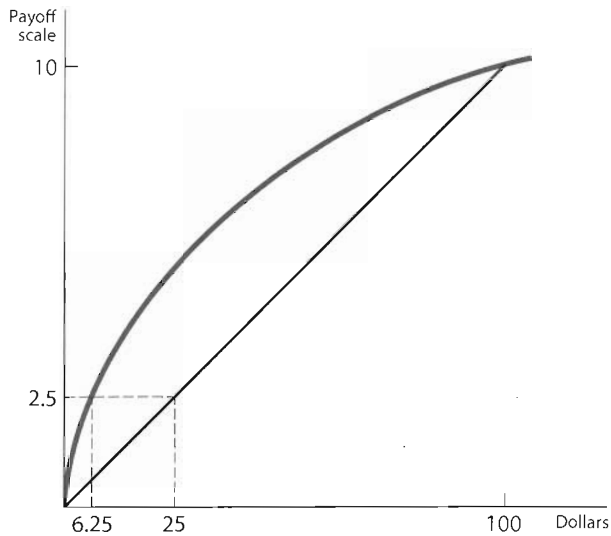


FIGURE 7A.1 Concave Scale: Risk Aversion

such a person is extremely risk-averse indeed. (Compare a graph of the cube root of x with a graph of the square root of x to see why this should be so.)

And what if the rescaling of payoffs from x dollars is done by using the function x^2? Then the expected payoff from the gamble is $(0.75 \times 0) + (0.25 \times 10{,}000) = 2{,}500$, which is the square of 50. Therefore a person with this payoff scale would be indifferent between getting \$50 for sure and the gamble with an expected money value of only \$25. This person must be a risk lover because he is not willing to give up any money to get a reduction in risk; on the contrary, he must be given an extra \$25 in compensation for the loss of risk. Figure 7A.2 shows the nonlinear scale associated with a function such as x^2.

So, by using different nonlinear scales instead of pure money payoffs, we can capture different degrees of risk-averse or risk-loving behavior. A concave scale like that of Figure 7A.1 corresponds to risk aversion, and a convex scale like that of Figure 7A.2 to risk-loving behavior. You can experiment with different simple nonlinear scales—for example, logarithms, exponentials, and other roots and powers—to see what they imply about attitudes toward risk.[4]

This method of evaluating risky prospects has a long tradition in decision theory; it is called the expected utility approach. The nonlinear scale that gives payoffs as functions of money values is called the **utility function;** the square root, cube root, and square functions referred to earlier are simple examples. Then the mathematical expectation, or probability-weighted average, of the

[4]Additional information on the use of expected utility and risk attitudes of players can be found in many intermediate microeconomic texts; for example, Hal Varian, *Intermediate Microeconomics*, 5th ed. (New York: Norton, 1999), pp. 218–227; Walter Nicholson, *Microeconomic Theory*, 7th ed. (New York: Dryden Press, 1998), pp. 211–226.

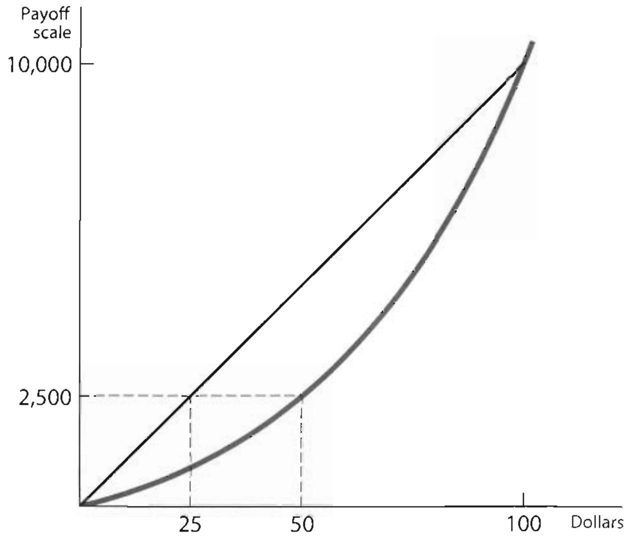


FIGURE 7A.2 Convex Scale: Risk Loving

utility values of the different money sums in a random prospect is called the expected utility of that prospect. And different random prospects are compared with one another in terms of their expected utilities; prospects with higher expected utility are judged to be better than those with lower expected utility.

Almost all of game theory is based on the expected utility approach, and it is indeed very useful, although is not without flaws. We will adopt it in this book, leaving more detailed discussions to advanced treatises.[5] However, we will indicate the difficulties that it leaves unresolved by means of a simple example in Chapter 8.

SUMMARY

The *probability* of an event is the likelihood of its occurrence by chance from among a larger set of possibilities. Probabilities can be combined by using some rules. The *addition rule* says that the probability of any one of a number of *disjoint* events occurring is the sum of the probabilities of these events; the *modified addition rule* generalizes the addition rule to overlapping events. According to the *multiplication rule,* the probability that all of a number of *independent events* will occur is the product of the probabilities of these events; the *modified*

[5]See R. Duncan Luce and Howard Raiffa, *Games and Decisions* (New York: Wiley, 1957), chap. 2 and app. 1, for an exposition; and Mark Machina, "Choice Under Uncertainty: Problems Solved and Unsolved," *Journal of Economic Perspectives,* vol. 1, no. 1 (Summer 1987), pp. 121–154, for a critique and alternatives. Although decision theory based on these alternatives has made considerable progress, it has not yet influenced game theory to any significant extent.

multiplication rule generalizes the multiplication rule to allow for lack of independence, by using *conditional probabilities.*

Judging consequences by taking expected monetary payoffs assumes *risk-neutral* behavior. *Risk aversion* can be allowed, by using the *expected-utility* approach, which requires the use of a *utility function,* which is a concave rescaling of monetary payoffs, and taking its probability-weighted average as the measure of expected payoff.

KEY TERMS

addition rule (224)
combination rule (228)
conditional probability (227)
disjoint (224)
expected utility (228)
independent events (226)
modified addition rule (225)
modified multiplication rule (227)
multiplication rule (226)
probability (223)
risk-averse (229)
risk-neutral (229)
utility function (230)

8

Simultaneous-Move Games with Mixed Strategies II: Non-Zero-Sum Games and III: General Discussion

IN CHAPTER 7, WE CONSIDERED zero-sum games in which players chose to use mixed strategies in equilibrium. The direct conflict inherent in such games makes randomized play attractive to players. And our analysis of zero-sum games in that chapter showed how mixed strategies could help players in such games by keeping the other players guessing.

We also pointed out in Section 3 of Chapter 7 that mixing can arise in non-zero-sum games as well. In those games, players do not have clearly conflicting interests and have no general reason to want to conceal their interests from others. As a result, there is no general argument in a non-zero-sum game for keeping the other player guessing. Simultaneous play can still lead players to have uncertain beliefs about the actions of a rival player and therefore to be uncertain about their own best actions. Mixed-strategy equilibria can arise in such games when players attain subjectively uncertain but correct beliefs about actions. We carefully examine this type of mixed-strategy equilibrium in this chapter.

This chapter also deals with several further topics related to mixed-strategy equilibria. In particular, we consider some important counterintuitive results that arise in games with such equilibria. We also provide a section on the general theory of mixed strategies for games in which all players have several pure strategies. These additional topics should help you to develop a better intuitive understanding of mixed-strategy equilibria and to hone your skills of analysis of games in which such equilibria arise.

1 MIXING SUSTAINED BY UNCERTAIN BELIEFS

Mixing can arise in simultaneous non-zero-sum games when, at the time each player chooses his strategy, he is unsure about what the other is choosing, and this uncertainty in turn makes him unsure about his own choice. For example, in the meeting games of Chapter 4, Section 7, Harry may be unsure about which café Sally will choose. This uncertainty may keep him dithering between his own choices. Then Sally, in turn, will be unsure about where Harry will be. Then the game may have an equilibrium in mixed strategies in which this mutual uncertainty is at just the right levels; that is, when each player has *correct* beliefs about the other's *uncertain* action.

A. Will Harry Meet Sally?

We illustrate this type of equilibrium by using the assurance version of the meeting game. For your convenience, we reproduce its table (Figure 4.12) as Figure 8.1 below. We consider the game from Sally's perspective first. If she is confident that Harry will go to Starbucks, she also should go to Starbucks. If she is confident that Harry will go to Local Latte, so should she. But, if she is unsure about Harry's choice, what is her own best choice?

To answer this question, we must give a more precise meaning to the uncertainty in Sally's mind. (The technical term for this uncertainty, in the theory of probability and statistics, is her **subjective uncertainty.**) We gain precision by stipulating the probability with which Sally thinks Harry will choose one café or the other. The probability of his choosing Local Latte can be any real number between 0 and 1 (that is, between 0% and 100%). We can cover all possible cases by using algebra, letting the symbol p denote the probability (in Sally's mind) that Harry chooses Starbucks; the variable p can take on any real value between 0 and 1. Then $(1 - p)$ is the probability (again in Sally's mind) that Harry chooses Local Latte. In other words, we describe Sally's subjective uncertainty as follows: she thinks that Harry is using a mixed strategy, mixing the two pure

		SALLY	
		Starbucks	Local Latte
HARRY	Starbucks	1, 1	0, 0
	Local Latte	0, 0	2, 2

FIGURE 8.1 Assurance

		SALLY	
		Starbucks	Local Latte
HARRY	Starbucks	1	0
	Local Latte	0	2
	p-mix	p	$2(1-p)$

FIGURE 8.2 Sally's Response to Harry's p-Mix

strategies Starbucks and Local Latte in proportions or probabilities p and $(1 - p)$ respectively. We call this mixed strategy Harry's p-mix, even though for the moment it is purely an idea in Sally's mind.

Given that Sally is thinking about Harry's choice in this way, what is her own best action? To find the answer, we must evaluate Sally's payoffs from the two choices available to her, when played against Harry's p-mix. Here we consider a modified version of the game matrix of Figure 8.1, this time showing only Sally's payoffs and adding a row for Harry's p-mix. This new matrix is shown in Figure 8.2.

Given Sally's thinking about how Harry will choose, she now thinks: "If I choose Starbucks, then with probability p, Harry will be there and I will get a payoff of 1; with probability $(1 - p)$ Harry will be at Local Latte and I will get a payoff of 0." The expected value of this prospect is $p \times 1 + (1 - p) \times 0 = p$, and this is the entry that we put into the cell corresponding to Harry's p-mix and Sally's choice of Starbucks in Figure 8.2. Similarly, her expected payoff from choosing Local Latte against Harry's p-mix is $p \times 0 + (1 - p) \times 2 = 2(1 - p)$. This value is shown in the appropriate cell in Figure 8.2.

Now we can find Sally's best response as in Chapter 7. The left-hand panel of Figure 8.3 shows Sally's expected payoffs from her two pure actions against Harry's p-mix. The rising line for Starbucks, p, meets the falling line for Local Latte, $2(1 - p)$, at $p = 2/3$. To the left of the point of intersection, Sally gets higher expected payoff from Local Latte; to the right, from Starbucks. When $p = 2/3$, Sally's expected payoffs are the same from her two pure actions and therefore also the same from any q-mix of the two, where q is her probability of choosing Starbucks.

The right-hand panel of Figure 8.3 translates the information on Sally's expected payoffs for different values of p into a best-response curve, shown in red. When $p < 2/3$, Sally's best response is pure Local Latte ($q = 0$). When $p > 2/3$, it is pure Starbucks ($q = 1$). When $p = 2/3$, all values of q are equally good for Sally; so her best response is any combination of Local Latte and Starbucks. This part of her best-response curve is shown by the vertical straight line from $q = 0$ to $q = 1$ at $p = 2/3$.

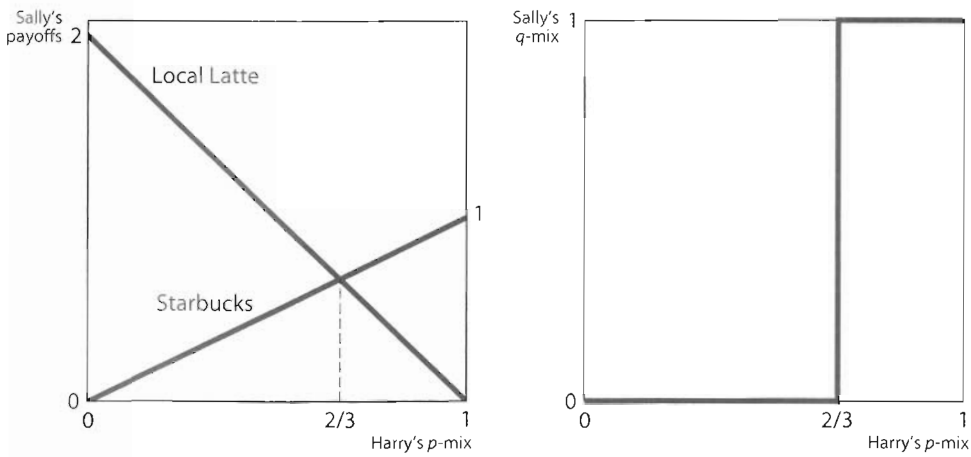


FIGURE 8.3 Sally's Best Responses

The intuition for the shape of Sally's best-response curve is simple. The value p represents Sally's estimate of the probability that Harry goes to Starbucks. If p is high, Harry is relatively more likely to be at Starbucks and so Sally should go to there, too. If p is low, she should go to Local Latte. The balancing, or breakeven, level of p is 2/3, not 1/2, because Sally's payoff from meeting at Local Latte is higher than that from meeting at Starbucks. To offset the imbalance in payoffs, a larger-than-even probability of Harry going to Starbucks is needed to tip Sally's decision that way.

The existence of a vertical segment in Sally's best-response curve has an important implication. If $p = 2/3$, all pure and mixed strategies are equally good for Sally, so she may dither between the two actions or choose randomly between them. The *subjective* uncertainty in Sally's mind about what Harry is doing can lead to a genuine or *objective* uncertainty in her own action. The same can happen to Harry, and that is how their subjective uncertainties can combine to sustain a mixed-strategy equilibrium. A Nash equilibrium in mixed strategies in this game will be a situation where each player has just the right amount of uncertainty about the other's action to sustain his own randomization.

We find the mixed-strategy equilibrium by calculating Harry's best response and then superimposing the two curves, as we did in Section 2.A of Chapter 7. Because the payoffs of the two players are symmetric, Harry's best-response curve will look just like Sally's with the two axes interchanged. Figure 8.4 shows both of the best-response curves at the same time. Sally's is the thick red curve as before; Harry's is the thick black curve.

The two best-response curves meet at three points. One is at the top right where $p = 1$ and $q = 1$. This point corresponds to each player choosing Starbucks for sure, in the correct subjective belief that the other is doing likewise.

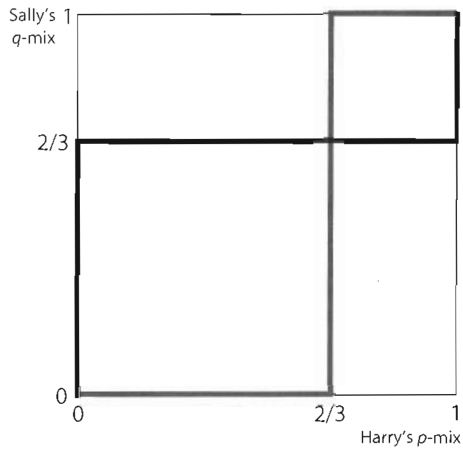


FIGURE 8.4 Mixed-Strategy Equilibrium in the Assurance Game

This is a self-sustaining state of affairs. The second meeting point is at the bottom left, where $p = 0$ and $q = 0$. Here each is not putting any probability on going to Starbucks—that is, is going to Local Latte for sure—in the correct subjective belief that the other is doing likewise. This also is a self-sustaining situation. These are just the two pure-strategy Nash equilibria for this game that we found in Chapter 4.

But there is a third meeting point in Figure 8.4, where $p = 2/3$ and $q = 2/3$. This intersection point identifies a third Nash equilibrium for this game, which our analysis in Chapter 4 could not discover. In this equilibrium, both players use mixed strategies. Each is unsure about the other's choice of action, and there is an equilibrium balance of their subjective uncertainties. Because Sally thinks that Harry is mixing between the two cafés with 2/3 probability of going to Starbucks, she cannot do any better than to mix with $q = 2/3$ herself. And the same is true for Harry. Thus the belief of each is borne out in equilibrium.

Remember that our definition of Nash equilibrium does not require a player's equilibrium strategy to be strictly better for her than her other available strategies, it merely requires that no other strategy is better than the equilibrium strategy. Here, given Sally's belief about Harry's mixed strategy, all of her strategies, pure and mixed, are equally good (or bad) for her. She has no definite reason to choose the precise proportions 2:1 in favor of Starbucks. However, her doing so is sustainable as an equilibrium; any other choice would not be an equilibrium. This may seem a lame argument to offer in support of the equilibrium, but, if you have ever dithered about a choice merely because you were unsure what choice others would be making, then you should understand how such an equilibrium can arise in practice.

The equilibrium has one major drawback. If the pair are independently making random choices with $p = 2/3$ and $q = 2/3$, then they will meet only when both happen to have chosen the same café. So the probability that they meet in Starbucks is only $pq = 4/9$, and the probability that they meet in Local Latte is $(1 - p)(1 - q) = 1/9$. They do not meet at all if they make different choices; the probability of this unfortunate turn of events is $1 - 4/9 - 1/9 = 4/9$. Therefore the expected payoff of each in the mixed-strategy equilibrium is $(4/9 \times 1) + (1/9 \times 2) + (4/9 \times 0) = 6/9 = 2/3$. Both pure-strategy Nash equilibria would give each player better payoffs; even the worse of those two equilibria ($p = 1$, $q = 1$) would give each of them the payoff of 1.

The reason for the poor expected payoff in the mixed-strategy equilibrium is obvious. When they choose randomly and independently, there is some positive and often significant probability (here 4/9) that they make mutually inconsistent choices and get a low payoff. Here it is best for them to coordinate on Starbucks with certainty.

Similar considerations arise in the pure-coordination version of the game, where the payoffs are the same from meeting in the two cafés. Again, the pair might randomize between the two cafés in equilibrium, but they would still do better by coordinating their randomization. Suppose that Harry and Sally are now in an ongoing relationship and meet at 4:30 every day for coffee. They like variety; so they would like to mix between the two cafés. But they do not like the predictability of choosing in advance, or of mechanical rotation. ("If this is Tuesday, it must be Local Latte.") And they cannot communicate with each other just before 4:30; this is before the age of instant messaging. (Only a couple of years ago, would you believe?) What might they do? One solution would be to agree in advance to decide on the basis of some random signal that is available to both. For example, as they come out of their respective classes at 4:30, if it is raining they both go to Starbucks and, if it is not raining, to Local Latte. Then they can choose randomly and yet be sure to meet.[1]

Given this analysis of the mixed-strategy equilibrium in the assurance version of the meeting game, you can now probably guess the mixed-strategy equilibria for related non-zero-sum games. In the pure-coordination version described in the preceding paragraph, the payoffs from meeting in the two cafés are the same; so the mixed-strategy equilibrium will have $p = 1/2$ and $q = 1/2$. In the battle-of-the-sexes variant, Sally prefers to meet at Local Latte because her payoff is 2 rather than the 1 that she gets from meeting at Starbucks. Her decision hinges on whether her subjective probability of Harry going to Starbucks is greater than or less than 2/3. (Sally's payoffs here are similar to those in the assurance version; so the critical p is the same.) Harry prefers to meet at Star-

[1]To analyze this scenario rigorously, we should introduce elements such as the ongoing relationship, and the desire for variety, explicitly into the structure of the game and its payoffs. We hope that you will accept the simple story for its value in illustrating the use of a randomizing device that is publicly observable by all players.

bucks; so his decision hinges on whether his subjective probability of Sally going to Starbucks is greater than or less than 1/3. Therefore the mixed-strategy Nash equilibrium has $p = 2/3$ and $q = 1/3$. To improve your algebraic skills, we ask you to work this out in detail in Exercise 4 of this chapter.

As already seen, best-response analysis with the mixture probabilities on the two axes is a comprehensive way of finding all Nash equilibria. Those in pure strategies, where the probability of one or another strategy is used with zero probability, occur along the edges of the square box. Those with genuine mixing are in the interior.

What if one wants to find only the genuinely mixed strategy equilibrium in the preceding games? This can be done more simply than by full best-response analysis. Consider the assurance game of Figures 8.1 and 8.2 again. The key to finding the mixed-strategy equilibrium is that Sally is willing to mix between her two pure strategies only if her subjective uncertainty about Harry's choice is just right; that is, if the value of p in Harry's p-mix is just right. Algebraically, this idea is borne out by solving for the equilibrium value of p by using the equation $p = 2\,(1 - p)$, which says that Sally gets the same expected payoff from her two pure strategies when each is matched against Harry's p-mix. When the equation holds, in equilibrium, it is *as if* Harry's mixture probabilities are doing the job of keeping Sally indifferent. We emphasize the "as if" because Harry has no reason in this game to keep Sally indifferent; the outcome is merely a property of the equilibrium. Still, the general idea is worth remembering: in a mixed-strategy Nash equilibrium, each person's mixture probabilities keep the other player indifferent between his pure strategies. We called this the *opponent's indifference* method in the zero-sum discussion in Chapter 7, and now we see that it remains valid even in non-zero-sum games.

B. Diced Chicken?

Finally, consider one more non-zero-sum game. Here we look at players trying to avoid a meeting in the chicken game of Chapter 4. We reproduce the game table of Figure 4.14 as Figure 8.5.

		DEAN	
		Swerve (Chicken)	Straight (Tough)
JAMES	Swerve (Chicken)	0, 0	−1, 1
	Straight (Tough)	1, −1	−2, −2

FIGURE 8.5 Chicken

Now we introduce mixed strategies. James chooses a p-mix with probability p of Swerve and $(1 - p)$ of Straight, whereas Dean chooses a q-mix with probability q of Swerve and $(1 - q)$ of Straight. To find Dean's best response, we rewrite the table as Figure 8.6, showing only Dean's payoffs and creating another row to show James's p-mix.

Dean's expected payoff from Swerve is $0 \times p + (-1) \times (1 - p) = -(1 - p) = p - 1$. His expected payoff from Straight is $1 \times p + (-2) \times (1 - p) = p - 2(1 - p) = 3p - 2$. Therefore Dean does better to choose Swerve if $p - 1 > 3p - 2$, or $2p < 1$, or $p < 1/2$. This makes intuitive sense; if James is less likely to Swerve, then Dean has a high risk of injury by going Straight; so he does better to Swerve. Conversely, Dean does better to Swerve if $p > 1/2$. If $p = 1/2$, then Dean gets equal expected payoff—namely, $(1/2) - 1 = 3 \times (1/2) - 2 = -1/2$—from both his choices; so he is indifferent between them as well as between them and any q-mix. We leave it to you to check this best-response rule graphically by showing the two lines of Dean's payoffs from his two pure choices against James's p-mix.

We show Dean's best response and a similar best response for James in the usual diagram (Figure 8.7). In the diagram, we see the two pure-strategy Nash equilibria of the game as two of the meeting points of the best-response curves. The first is in the top-left corner, where $p = 0$ and $q = 1$; so James chooses Straight and Dean chooses Swerve. The second is in the bottom-right corner, where $p = 1$ and $q = 0$; so James chooses Swerve and Dean chooses Straight. We also find a third meeting point and a third Nash equilibrium. This one is in mixed strategies, with $p = 1/2$ and $q = 1/2$. The uncertainty about each driver's choice at this point is just enough to keep the other indifferent.

From Figure 8.6 we can calculate Dean's expected payoffs from each of his pure strategies against James' equilibrium p-mix. When $p = 1/2$, Dean gets $p - 1 = -1/2$ from his Swerve, and $3p - 2 = -1/2$ from his Straight. The two are equal; this is just the "opponent's indifference" property of James's equilibrium p-mix. Thus Dean gets expected payoff $-1/2$ in the mixed-strategy equilibrium. Similarly, James gets $-1/2$. These payoffs are low, as was the case with the mixed-strategy equilibria in the coordination games considered in Section 1.A,

		DEAN	
		Swerve	Straight
JAMES	Swerve	0	1
	Straight	−1	−2
	p-mix	$p - 1$	$3p - 2$

FIGURE 8.6 Best-Response Analysis in Chicken

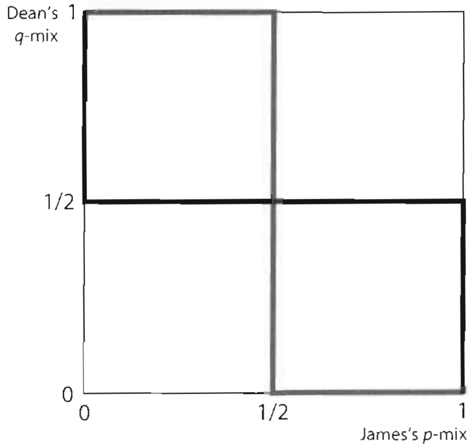


FIGURE 8.7 Best Responses and Nash Equilibria in Chicken

because independent mixing can lead to a mutually bad outcome—here a crash—with positive probability. Both drivers could do better if they could make a binding agreement for both to swerve or to alternate between the two pure-strategy equilibria in repeated play.

2 NON-ZERO-SUM MIXING WITH THREE STRATEGIES

To illustrate mixed-strategy equilibria in non-zero-sum games when each player has three (or more) strategies, we consider an expanded version of the chicken game. In the expanded version, each player has the choice not only of whether to go straight or swerve, but of whether to swerve to the right or to the left. If one player swerves to (his own) right and the other to (his own) left, then the two will crash, just as they will if both choose straight. Therefore the payoff matrix is as shown in Figure 8.8.

		DEAN		
		Left	Straight	Right
JAMES	Left	0, 0	−1, 1	−2, −2
	Straight	1, −1	−2, −2	1, −1
	Right	−2, −2	−1, 1	0, 0

FIGURE 8.8 Three-by-Three Chicken

First note that the game has the usual pure-strategy equilibria, where one player goes straight and the other swerves. In this case, the other player is indifferent about which way he swerves, so there are four such equilibria: (Straight, Left) and (Straight, Right) where Dean is the chicken, and (Left, Straight) and (Right, Straight) where James is the chicken.

Next we want to look for a *fully mixed* equilibrium, in which each player plays all three of his strategies with positive probability. Suppose James's mixture probabilities are p_1 on Left, p_2 on Straight, and $(1 - p_1 - p_2)$ on Right. Similarly, write Dean's mixture probabilities as q_1, q_2, and $(1 - q_1 - q_2)$.

Using the logic presented in the preceding section, we know that Dean mixes only when his subjective uncertainty about James's choice is just right. That is, Dean mixes only when James's p-mix just keeps Dean indifferent among his choices. To determine the equilibrium level of p, we must calculate, and equate, Dean's payoffs from each of his three pure strategies against James's mixture:

$$-p_2 - 2(1 - p_1 - p_2) = p_1 - 2p_2 + (1 - p_1 - p_2) = -2p_1 - p_2,$$

or

$$-2 + 2p_1 + p_2 = 1 - 3p_2 = -2p_1 - p_2.$$

The first two parts of this expression yield $2p_1 = 3 - 4p_2$, and the last two parts yield $2p_1 = 2p_2 - 1$. Then $3 - 4p_2 = 2p_2 - 1$, which leads to $p_2 = 4/6 = 2/3$. Using this result, we find $2p_1 = 2(2/3) - 1 = 1/3$; so $p_1 = 1/6$. Then $(1 - p_1 - p_2) = 1/6$ also. Similarly, for James to choose a mixed strategy, the equilibrium values of q_1 and q_2 in Dean's q-mix must keep James indifferent among his three pure strategies so that $-2 + 2q_1 + q_2 = 1 - 3q_2 = -2q_1 - q_2$. These equations yield $q_2 = 2/3$, $q_1 = 1/6$, and $(1 - q_1 - q_2) = 1/6$ also.

Here, both players put twice as much probability weight on Straight as on either direction of Swerve. They also choose Straight more often than in the traditional version of the game analyzed in Section 1.B. Why? In the standard version of chicken, a player could guarantee at least a "safe" outcome (and a payoff of -1) by choosing to Swerve. In this variant of chicken, however, swerving either Left or Right is not a guarantee of safety; there may still be a collision. This fact makes any sort of swerve a less attractive action than in the standard game and thereby increases each driver's desire to choose Straight in the fully mixed equilibrium.

But this is not the only mixed-strategy equilibrium for this game. There are additional ones that are only *partially mixed*. That is, there are other mixed-strategy equilibria in which one or both players use fewer than three strategies in their mixtures.

One possibility is that the players mix between Straight (S) and only one of the two directions of Swerve—say, Left (L). To see how such an equilibrium can arise, suppose that James uses only L and S in his mixture, with probability p for

L and $(1-p)$ for S. As before, Dean mixes in equilibrium only if his payoffs from his three pure strategies against James's mix are all equal. In this case, Dean's payoffs against James's p-mix are:

$$0p - 1(1-p) \quad \text{or} \quad p - 1 \text{ from L,}$$
$$1p - 2(1-p) \quad \text{or} \quad 3p - 2 \text{ from S,}$$

and

$$-2p - 1(1-p) \quad \text{or} \quad -p - 1 \text{ from R.}$$

We can see that R is dominated (by L, as well as by S) and will therefore not be used in Dean's mix. If Dean uses only L and S in his mixture, then he should be indifferent between them. Therefore it must be true that $p - 1 = 3p - 2$, or $p = 1/2$. Dean's expected payoff in equilibrium is then $-1/2$. And, when Dean is mixing between L and S, a similar calculation shows that James will not use his R, and his mixture probabilities are 50:50 between his L and his S. This equilibrium is really just like the mixed-strategy outcome in the two-by-two chicken game, which you can calculate on your own.

Yet another possibility is that only one player mixes in equilibrium. There are equilibria in which one player chooses pure S and the other mixes with arbitrary probabilities between L and R. For example, when James is playing pure S, Dean is indifferent between his own L and R and therefore willing to mix between them in any probabilities, but he does not want to play his own S. And in turn, when Dean is mixing between L and R with any probabilities, James's best response is obviously pure S. Thus we have a Nash equilibrium. And there is another such equilibrium with the players interchanged.

We see that giving players more pure strategies can lead to a wide array of possibilities for mixed-strategy equilibria, including nonuniqueness or multiple mixed-strategy equilibria for one game. In specific examples or applications, such possibilities have to be investigated individually. We consider some general theory of mixed-strategy equilibria for non-zero-sum games in the final section of this chapter.

3 GENERAL DISCUSSION OF MIXED-STRATEGY EQUILIBRIA

Now that we have seen how to find mixed-strategy equilibria in both zero-sum and non-zero-sum games, it is worthwhile to consider some additional features of these equilibria. In particular, we highlight in this section some general properties of mixed-strategy equilibria and we introduce you to some results that seem counterintuitive at first, until you fully analyze the game in question.

A. Weak Sense of Equilibrium

The best-response graphs in Figures 7.4 and 8.4 show that, when one player is choosing his equilibrium mix, the other's best response can be any mixture including the extreme cases of the two pure strategies. Thus the mixed-strategy equilibria are Nash equilibria only in a weak sense. When one player is choosing his equilibrium mix, the other has no positive reason to deviate from his own equilibrium mix. But he would not do any worse if he chose another mix or even one of his pure strategies. Each player is indifferent between his pure strategies or indeed any mixture of them so long as the other player is playing his correct (equilibrium) mix. This is also a very general property of mixed-strategy Nash equilibria.

This property seems to undermine the basis for mixed-strategy Nash equilibria as the solution concept for games. Why should a player choose his appropriate mixture when the other player is choosing his own? Why not just do the simpler thing by choosing one of his pure strategies? After all, the expected payoff is the same. The answer is that to do so would not be a Nash equilibrium; it would not be a stable outcome, because then the other player would not choose to use his mixture. For example, if Harry chooses pure Starbucks in the assurance version of the meeting game, then Sally can get a higher payoff in equilibrium (1 instead of 2/3) by switching from her 50:50 mix to her pure Starbucks as well.

B. Opponent's Indifference and Guarding Against Exploitation

The same property of player indifference between strategy choices in a mixed-strategy equilibrium has additional implications. When one player is mixing, we know that the other is indifferent among the pure strategies that appear in his mixture and we have already labeled this as the *opponent's indifference* property. This property implies that each player's equilibrium mixture probabilities can be found by solving the equations that express the opponent's indifference—namely, the equality of the opponent's expected payoffs from all of his pure strategies when played against this mixture. This then forms the basis for the general method of finding mixed-strategy equilibria discussed in Section 2.

In a zero-sum game, there are further implications. In our tennis-point game, for example, when Evert gets a higher payoff, Navratilova gets a lower payoff. Therefore another way of thinking about mixed-strategy equilibria is that, when Navratilova chooses her right mixture, she does equally well no matter what Evert chooses. Any other choice would be exploited by Evert to her own advantage and therefore to Navratilova's disadvantage. In other words, each player's equilibrium mixture has the property that it **prevents exploitation** by the opponent. This property is special to zero-sum games because in non-zero-sum games what is better for the other player is not necessarily worse for oneself. But in the zero-sum context this property gives an important positive purpose to mixing.

C. Counterintuitive Outcomes with Mixed Strategies in Zero-Sum Games

Games with mixed-strategy equilibria may exhibit some features that seem counterintuitive at first glance. The most interesting of them is the change in the equilibrium mixes that follow a change in the structure of a game's payoffs. To illustrate, we return again to Evert and Navratilova and their tennis point.

Suppose that Navratilova works on improving her skills covering down the line to the point where Evert's success using her DL strategy against Navratilova covering DL drops to 30% from 50%. This improvement in Navratilova's skill alters the payoff table, including the mixed strategies for each player, from that illustrated in Figure 7.1. We present the new table in Figure 8.9, showing only the Row player's (Evert's) payoffs.

The only change from the table in Figure 7.1 has occurred in the upper-left-hand corner of the table where our earlier 50 is now a 30. This change in the payoff table does not lead to a game with a pure-strategy equilibrium, because the players still have opposing interests; Navratilova still wants their choices to coincide, and Evert still wants their choices to differ. We still have a game in which mixing will occur.

But how will the equilibrium mixes in this new game differ from those calculated in Section 1 of Chapter 7? At first glance, many people would argue that Navratilova should cover DL more often now that she has gotten so much better at doing so. Thus, the assumption is that her equilibrium q-mix should be more heavily weighted toward DL and her equilibrium q should be higher than the 0.6 calculated before.

But, when we calculate Navratilova's q-mix by using the condition of Evert's indifference between her two pure strategies, we get $30q + 80(1 - q) = 90q + 20(1 - q)$, or $q = 0.5$. The actual equilibrium value for q, 50%, has exactly the opposite relation to the original q of 60% that the intuition of many people predicts.

Although the intuition seems reasonable, it misses an important aspect of the theory of strategy: the interaction between the two players. Evert will also be reassessing her equilibrium mix after the change in payoffs, and Navratilova must take the new payoff structure *and* Evert's behavior into account when determining her new mix. Specifically, because Navratilova is now so much better

		NAVRATILOVA		
		DL	CC	q-mix
EVERT	DL	30	80	$30q + 80(1 - q)$
	CC	90	20	$90q + 20(1 - q)$
	p-mix	$30p + 90(1 - p)$	$80p + 20(1 - p)$	

FIGURE 8.9 Counterintuitive Change in Mixture Probabilities

at covering DL, Evert uses CC more often in her mix. To counter that, Navratilova covers CC more often, too.

We can see this more explicitly by calculating Evert's new mixture. Her equilibrium p must equate $30p + 90(1 - p)$ and $80p + 20(1 - p)$; thus, her p must be 7/12, which is 0.583, or 58.3%. Comparing this new equilibrium p with the original 70% calculated in Chapter 7 shows that Evert has significantly decreased the number of times she sends her shot DL in response to Navratilova's improved skills. Evert has taken into account the fact that she is now facing an opponent with better DL coverage and so she does better to play DL less frequently in her mixture. By virtue of this behavior, Evert makes it better for Navratilova also to decrease the frequency of her DL play. Any other choice of mix by Navratilova, in particular a mix heavily favoring DL, would now be exploited by Evert.

So is Navratilova's skill improvement wasted? No, but we must judge it properly—not by how often one strategy or the other gets used but by the resulting payoffs. When Navratilova uses her new equilibrium mix with $q = 0.5$, Evert's success percentage from either of her pure strategies is $(30 \times 0.5) + (80 \times 0.5) = (90 \times 0.5) + (20 \times 0.5) = 55$. This is less than Evert's success percentage of 62 in the original example. Thus Navratilova's average payoff also rises, from 38 to 45, and she does benefit by improving her DL coverage.

Unlike the counterintuitive result that we saw when we considered Navratilova's strategic response to the change in payoffs, we see here that her response is absolutely intuitive when considered in light of her expected payoff. In fact, players' expected payoff responses to changed payoffs can never be counterintuitive, although strategic responses, as we have seen, can be.[2] The most interesting aspect of this counterintuitive outcome in players' strategic responses is the message that it sends to tennis players and to strategic game players more generally. The result here is equivalent to saying that Navratilova should improve her down-the-line coverage so that she does not have to use it so often.

There are other similar examples of possibly counterintuitive results, including one that often arises in sports. In sports, there are always some strategies that are relatively safe; they do not fail disastrously even if anticipated by the opponent but do not do very much better even if unanticipated. Other strategies are risky; they do brilliantly if the other side is not ready for them but fail miserably if the other side is ready. Thus in football, on third down with a yard to go, a run up the middle is safe and a long pass is risky. The following example incorporates this idea of safe–versus–risky strategies. In addition, although most of our examples use illustrative numbers for payoffs, here we

[2]For a general theory of the effect that changing the payoff in a particular cell has on the equilibrium mixture and the expected payoffs in equilibrium, see Vincent Crawford and Dennis Smallwood, "Comparative Statics of Mixed-Strategy Equilibria in Noncooperative Games," *Theory and Decision*, vol. 16 (May 1984), pp. 225–232.

		OPPONENT EXPECTS	
		P	R
YOU PLAY	P	c	b
	R	a	d

FIGURE 8.10 Table of Success Probabilities of Risky and Percentage Plays

change that practice to emphasize the generality of the problem. Therefore, we let the payoffs be general algebraic symbols, subject only to some conditions concerning the basic strategy being considered.

Consider any zero-sum game in which you have two pure strategies. Let us call the relatively safe strategy (the percentage play) P and the more risky strategy R. The opponent has two pure strategies that we also call P and R; his P is his best response to your P, as is his R to your R. Figure 8.10 shows the table of probabilities that your play succeeds; these are not your payoffs. The sense of "safe" and "risky" is captured by requiring $a > b > c > d$. The risky play does really well if the opponent is not prepared for it (your success probability is a) but really badly if he is (your success probability is d); the percentage play does moderately well in either case (you succeed with probability b or c) but a little worse if the opponent expects it ($c < b$).

Let your payoff or utility be W if your play succeeds and L if it fails. A "really big occasion" is when W is much bigger than L. Note that W and L are not necessarily money amounts; so they can be utilities that capture any aversion to risk as explained in the Appendix to Chapter 7. Now we can write down the table of expected payoffs from the various strategy combinations as in Figure 8.11. Note how this table is constructed. For example, if you play P and your opponent expects R, then you get utility W with probability b and utility L with probability $(1 - b)$; your expected payoff is $bW + (1 - b)L$. This game is zero-sum; so in each cell your opponent's payoffs are just the negative of yours.

		OPPONENT EXPECTS	
		P	R
YOU PLAY	P	$cW + (1 - c)L$	$bW + (1 - b)L$
	R	$aW + (1 - a)L$	$dW + (1 - d)L$

FIGURE 8.11 Payoff Table with Risky and Percentage Plays

In the mixed-strategy equilibrium, your probability p of choosing P is defined by

$$p[cW + (1 - c)L] + (1 - p)[aW + (1 - a)L] =$$
$$p[bW + (1 - b)L] + (1 - p)[dW + (1 - d)L]$$

This equation simplifies to $p = (a - d)/[(a - d) + (b - c)]$. Because $(b - c)$ is small in relation to $(a - d)$, we see that p is close to 1. That is exactly why the strategy P is called the percentage play; it is the normal play in these situations, and the risky strategy R is played only occasionally to keep the opponent guessing, or in football commentators' terminology, "to keep the defense honest."

The interesting part of this result is that the expression for p is completely independent of W and L. That is, the theory says that you should mix the percentage play and the risky play in exactly the same proportions on a big occasion as you would on a minor occasion. This runs against the intuition of many people. They think that the risky play should be engaged in less often when the occasion is more important. Throwing a long pass on third down with a yard to go may be fine on an ordinary Sunday afternoon in October, but doing so in the Super Bowl is too risky.

So which is right: theory or intuition? We suspect that readers will be divided on this issue. Some will think that the sports commentators are wrong and will be glad to have found a theoretical argument to refute their claims. Others will side with the commentators and argue that bigger occasions call for safer play. Still others may think that bigger risks should be taken when the prizes are bigger, but even they will find no support in the theory, which says that the size of the prize or the loss should make no difference to the mixture probabilities.

On many previous occasions where discrepancies between theory and intuition arose, we argued that the discrepancies were only apparent, that they were the result of failing to make the theory sufficiently general or rich enough to capture all the features of the situation that created the intuition, and that improving the theory removed the discrepancy. This one is different; the problem is fundamental to the calculation of payoffs from mixed strategies as probability-weighted averages or expected payoffs. And almost all of existing game theory has this starting point.[3]

[3]Vincent P. Crawford, "Equilibrium Without Independence," *Journal of Economic Theory*, vol. 50, no. 1 (February 1990), pp. 127–154, and James Dow and Sergio Werlang, "Nash Equilibrium Under Knightian Uncertainty," *Journal of Economic Theory*, vol. 64, no. 2 (December 1994), pp. 305–324, are among the few research papers that suggest alternative foundations for game theory. And our exposition of this problem in the first edition of this book inspired an article that uses such new methods on it: Simon Grant, Atsushi Kaji, and Ben Polak, "Third Down and a Yard to Go: Recursive Expected Utility and the Dixit-Skeath Conundrum," *Economic Letters*, vol. 73, no. 3 (December 2001), pp. 275–286. Unfortunately, it uses more advanced concepts than those available at the introductory level of this book.

D. Another Counterintuitive Result for Non-Zero-Sum Games

Here we consider a general two-by-two non-zero-sum game with the payoff table shown in Figure 8.12. In actual games, the payoffs would be actual numbers and the strategies would have particular names. In this example, we again use general algebraic symbols for payoffs so that we can examine how the probabilities of the equilibrium mixtures depend on them. Similarly, we use arbitrary generic labels for the strategies.

Suppose the game has a mixed strategy equilibrium in which Row plays Up with probability p and Down with probability $(1 - p)$. To guarantee that Column also mixes in equilibrium, Row's p-mix must keep Column indifferent between his two pure strategies, Left and Right. Equating Column's expected payoffs from these two strategies when played against Row's mixture, we have[4]

$$pA + (1 - p)C = pB + (1 - p)D, \quad \text{or}$$
$$p = (D - C)/[(A - B) + (D - C)]$$

The surprising thing about the expression for p is not what it contains, but what it does *not* contain. None of Row's own payoffs, a, b, c, or d, appear on the right-hand side. Row's mixture probabilities are totally independent of his own payoffs!

Similarly, the equilibrium probability q of Column playing Left is given by

$$q = (d - b)/[(a - c) + (d - b)].$$

Column's equilibrium mixture also is determined independently of his own payoffs.

The surprise or counterintuitive aspect of these results is resolved if you remember the general principle of the opponent's indifference. Because each player's mixture probabilities are solved by requiring the opponent to be indifferent between his pure strategies, it is natural that these probabilities should

		COLUMN	
		Left	Right
ROW	Up	a, A	b, B
	Down	c, C	d, D

FIGURE 8.12 General Algebraic Payoff Matrix for Two-by-Two Non-Zero-Sum Game

[4]For there to be a mixed-strategy equilibrium, the probability p must be between 0 and 1. This requires that $(A - B)$ and $(D - C)$ have the same sign; if A is bigger than B, then D must be bigger than C, and, if A is smaller than B, then D must be smaller than C. (Otherwise, one of the pure strategies Left and Right would dominate the other.)

depend on the opponent's payoffs, not on one's own. But remember also that it is only in zero-sum games that a player has a genuine reason to keep the opponent indifferent. There, any clear preference of the opponent for one of his pure strategies would work to one's own disadvantage. In non-zero-sum games, the opponent's indifference does not have any such purposive explanation; it is merely a logical property of equilibrium in mixed strategies.

4 EVIDENCE ON MIXING IN NON-ZERO-SUM GAMES

In Chapter 7, we noted that laboratory experiments on games with mixed strategies generally find results that do not conform to theoretical equilibria. This is especially true for non-zero-sum games. As we have seen, in such games the property that each player's equilibrium mixture keeps his opponent indifferent among his pure strategies is a logical property of the equilibrium. Unlike zero-sum games, in general each player in a non-zero-sum game has no positive or purposive reason to keep the other players indifferent. Then the reasoning underlying the mixture calculations is more difficult for players to comprehend and learn. This shows up in their behavior.

In a group of experimental subjects playing a non-zero-sum game, we may see some pursuing one pure strategy and others pursuing another. This type of mixing in the population, although it does not fit the theory of mixed-strategy equilibria, does have an interesting evolutionary interpretation, which we examine in Chapter 13.

Other experimental issues concern subjects who play the game many times. When collecting evidence on play in non-zero-sum games, it is important to rotate or randomize each subject's opponents to avoid tacit cooperation in repeated interactions. In such experiments, players change their actions from one play to the next. But, even if we interpret this as evidence of true mixing, the players' mixture probabilities change when their own payoffs are changed. According to the theory of Section 2.D, this shouldn't happen.

The overall conclusion is that mixed-strategy equilibria in non-zero-sum games should be interpreted and used, at best, with considerable caution.

5 MIXING AMONG THREE OR MORE STRATEGIES: GENERAL THEORY

We conclude this chapter with some general theory of mixed-strategy equilibria, to unify all of the ideas introduced in the various examples in Chapters 7 and 8 so far. Such general theory unavoidably requires some algebra and some abstract thinking. Readers unprepared for such mathematics or averse to it can omit this section without loss of continuity.

Suppose the Row player has available the pure strategies R_1, R_2, . . ., R_m, and the Column player has strategies C_1, C_2, . . . C_n. Write the Row player's payoff from the strategy combination (i, j) as A_{ij}, and Column's as B_{ij}, where the index i ranges from 1 to m, and the index j ranges from 1 to n. We allow each player to mix the available pure strategies. Suppose the Row player's p-mix has probabilities P_i and the Column player's q-mix has Q_j. All these probabilities must be nonnegative, and each set must add to 1; so

$$P_1 + P_2 + \cdots + P_m = 1 = Q_1 + Q_2 + \cdots + Q_n.$$

We write V_i for Row's expected payoff from using his pure strategy i against Column's q-mix. Using the reasoning that we have already seen in several examples, we have

$$V_i = A_{i1}Q_1 + A_{i2}Q_2 + \cdots + A_{in}Q_n = \sum_{j=1}^{n} A_{ij}\,Q_j,$$

where the last expression on the right uses the mathematical notation for summation of a collection of terms. When Row plays his p-mix and it is matched against Column's q-mix, Row's expected payoff is

$$P_1V_1 + \cdots + P_mV_m = \sum_{i=1}^{m} P_iV_i = \sum_{i=1}^{m}\sum_{j=1}^{n} P_iA_{ij}Q_j.$$

The Row player chooses his p-mix to maximize this expression.

Similarly, writing W_j for Column's expected payoff when his pure strategy j is pitted against Row's p-mix, we have

$$W_j = P_1B_{1j} + P_2B_{2j} + \cdots + P_mB_{mj} = \sum_{i=1}^{m} P_iB_{ij}.$$

Pitting mix against mix, we have Column's expected payoff:

$$Q_1W_1 + \cdots + Q_nW_n = \sum_{j=1}^{n} Q_jW_j = \sum_{i=1}^{m}\sum_{j=1}^{n} P_iB_{ij}Q_j,$$

and he chooses his q-mix to maximize this expression.

We have a Nash equilibrium when each player simultaneously chooses his best mix, given that of the other. That is, Row's equilibrium p-mix should be his best response to Column's equilibrium q-mix, and vice versa. Let us begin by finding Row's best-response rule. That is, let us temporarily fix Column's q-mix and consider Row's choice of his p-mix.

Suppose that, against Column's given q-mix, Row has $V_1 > V_2$. Then Row can increase his expected payoff by shifting some probability from strategy R_2 to R_1; that is, Row reduces his probability P_2 of playing R_2 and increases the probability

P_1 of playing R_1 by the same amount. Because the expressions for V_1 and V_2 do not include any of the probabilities P_i at all, this is true no matter what the original values of P_1 and P_2 were. Therefore Row should reduce the probability P_2 of playing R_2 as much as possible—namely, all the way to zero.

The idea generalizes immediately. Row should rank the V_i in descending order. At the top there may be just one strategy, in which case it should be the only one used; that is, Row should then use a pure strategy. Or there may be a tie among two or more strategies at the top, in which case Row should mix solely among these strategies and not use any of the others.[5] When there is such a tie, all mixtures between these strategies give Row the same expected payoff. Therefore this consideration alone does not serve to fix Row's equilibrium p-mix. We show later how, in a way that may seem somewhat strange at first sight, Column's indifference condition does that job.

The same argument applies to Column. He should use only that pure strategy which gives him the highest W_j or should mix only among those of his pure strategies C_j whose W_j are tied at the top. If there is such a tie, then all mixtures are equally good from Column's perspective; the probabilities of the mix are not fixed by this consideration alone.

In general, for most values of $(Q_1, Q_2, \ldots, Q_n)$ that we hold fixed in Column's q-mix, Row's $V_1, V_2, \ldots, V_m$ will not have any ties at the top, and therefore Row's best response will be a pure strategy. Conversely, Column's best response will be one of his pure strategies for most values of $(P_1, P_2, \ldots, P_m)$ that we hold fixed in Row's p-mix. We saw this several times in the examples of Chapters 7 and 8; for example, in Figures 7.4, 8.4, and 8.7, for most values of p in Row's p-mix, the best q for Column was either 0 or 1, and vice versa. For only one critical value of Row's p was it optimal for Column to mix (choose any q between 0 and 1), and vice versa.

All of these conditions—ties at the top, and worse outcomes from the other strategies—constitute the complicated set of equations and inequalities that, when simultaneously satisfied, defines the mixed-strategy Nash equilibrium of the game. To understand it better, suppose for the moment that we have done all the work and found which strategies are used in the equilibrium mix. We can always relabel the strategies so that Row uses, say, the first g pure strategies, $R_1, R_2, \ldots, R_g$, and does not use the remaining $(m - g)$ pure strategies, $R_{g+1}, R_{g+2}, \ldots, R_m$, while Column uses his first h pure strategies, $C_1, C_2, \ldots, C_h$, and does not use the remaining $(n - h)$ pure strategies, $C_{h+1}, C_{h+2}, \ldots, C_n$. Write V for the tied value of Row's top expected payoffs V_i and, similarly, W for the tied value of Column's top expected payoffs W_j. Then the equations and inequalities can be written as follows. First, for each player, we set the probabilities of

[5] In technical mathematical terms, the expression $\Sigma_i P_i V_i$ is *linear* in the P_i; therefore its maximum must be at an extreme point of the set of permissible P_i.

the unused strategies equal to zero and require those of the used strategies to sum to 1:

$$P_1 + P_2 + \cdots + P_g = 1, \quad P_{g+1} = P_{g+2} = \cdots P_m = 0 \tag{8.1}$$

and

$$Q_1 + Q_2 + \cdots + Q_h = 1, \quad Q_{h+1} = Q_{h+2} = \cdots Q_n = 0 \tag{8.2}$$

Next we set Row's expected payoffs for the pure strategies that he uses equal to the top tied value:

$$V = A_{i1}\, Q_1 + A_{i2}\, Q_2 + \cdots + A_{ih} Q_h \quad \text{for } i = 1, 2, \ldots, g, \tag{8.3}$$

and note that his expected payoffs from his unused strategies must be smaller (that is why they are unused):

$$V > A_{i1}\, Q_1 + A_{i2}\, Q_2 + \cdots + A_{ih} Q_h \quad \text{for } i = g + 1, g + 2, \ldots, n. \tag{8.4}$$

Next, we do the same for Column, writing W for his top tied payoff value:

$$W = P_1\, B_{1j} + P_2 B_{2j} + \cdots + P_g B_{gj} \quad \text{for } j = 1, 2, \ldots, h, \tag{8.5}$$

and

$$W > P_1 B_{1j} + P_2 B_{2j} + \cdots + P_g B_{gj} \quad \text{for } j = h + 1, h + 2, \ldots, n. \tag{8.6}$$

To find the equilibrium, we must take this whole system, regard the choice of g and h as well as the probabilities $P_1, P_2, \ldots, P_g$, and $Q_1, Q_2, \ldots, Q_h$ as unknowns, and attempt to solve for them.

There is always the exhaustive search method. Try a particular selection of g and h; that is, choose a particular set of pure strategies as candidates for use in equilibrium. Then take Eqs. (8.1) and (8.5) as a set of $(h + 1)$ simultaneous linear equations regarding $P_1, P_2, \ldots, P_g$ and W as $(g + 1)$ unknowns, solve for them, and check if the solution satisfies all the inequalities in Eq. (8.6). Similarly, take Eqs. (8.2) and (8.3) as a set of $(g + 1)$ simultaneous linear equations in the $(h + 1)$ unknowns $Q_1, Q_2, \ldots, Q_h$ and V, solve for them, and check if the solution satisfies all the inequalities in Eq. (8.4). If all these things check out, we have found an equilibrium. If not, take another selection of pure strategies as candidates, and try again. There is only a finite number of selections: there are $(2^m - 1)$ possible selections of pure strategies that can be used by Row in his mix and $(2^n - 1)$ possible selections of pure strategies that can be used by Column in his mix. Therefore the process must end successfully after a finite number of attempts.

When m and n are reasonably small, exhaustive search is manageable. Even then, shortcuts suggest themselves in the course of the calculation for each specific problem. Thus, in the second variant of our soccer penalty kick, the way in which the attempted solution with all strategies used failed told us which strategy to discard in the next attempt.

Even for moderately large problems, however, solution based on exhaustive search or ad hoc methods becomes too complex. That is when one must resort to more systematic computer searches or algorithms. What these computer algorithms do is to search simultaneously for solutions to two linear maximization (or linear programming in the terminology of decision theory) problems: given a q-mix, and therefore all the V_i values, choose a p-mix to maximize Row's expected payoff $\Sigma_i P_i Vi$, and, given a p-mix and therefore all the W_i values, choose a q-mix to maximize Column's expected payoff $\Sigma_j Q_j W_j$. However, for a typical q-mix, all the V_i values will be unequal. If Column were to play this q-mix in an actual game, Row would not mix but would instead play just the one pure strategy that gave him the highest V_i. But in our numerical solution method we should not adjust Row's strategy in this drastic fashion. If we did, then at the next step of our algorithm, Column's best q-mix also would change drastically, and Row's chosen pure strategy would no longer look so good. Instead, the algorithm should take a more gradual step, adjusting the p-mix a little bit to improve Row's expected payoff. Then, with the use of this new p-mix for Row, the algorithm should adjust Column's q-mix a little bit to improve his expected payoff. Then back again to another adjustment in the p-mix. The method proceeds in this way until no improvements can be found; that is the equilibrium.

We do not need the details of such procedures, but the general ideas that we have developed above already tell us a lot about equilibrium. Here are some important lessons of this kind.

1. We solve for Row's equilibrium mix probabilities $P_1, P_2, \ldots, P_g$ from Eqs. (8.1) and (8.5). The former is merely the adding-up requirement for probabilities. The more substantive equation is (8.5), which gives the conditions under which Column gets the same payoff from all the pure strategies that he uses against the p-mix. It might seem puzzling that Row adjusts his mix so as to keep Column indifferent, when Row is concerned about his own payoffs, not Column's. Actually the puzzle is only apparent. We derived those conditions [Eq.8.5] by thinking about Column's choice of his q-mix, motivated by concerns about his own payoffs. We argued that Column would use only those strategies that gave him the best (tied) payoffs against Row's p-mix. This is the requirement embodied in Eq. (8.5). Even though it appears *as if* Row is deliberately choosing his p-mix so as to keep Column indifferent, the actual force that produces this outcome is Column's own purposive strategic choice.

In Chapter 7, we gave the name "the opponent's indifference principle" to the idea that each player's indifference conditions constitute the equations that can be solved for the other player's equilibrium mix. We now have a proof of this principle for general games, zero-sum and non-zero-sum.

2. However, in the zero-sum case, the idea that each player chooses his mixture to keep the other indifferent is not just an *as if* matter; there is some

genuine reason why a player should behave in this way. When the game is zero-sum, we have a natural link between the two players' payoffs: $B_{ij} = -A_{ij}$ for all i and j, and then similar relations hold among all the combinations and expected payoffs, too. Therefore we can multiply Eqs. (8.5) and (8.6) by -1 to write them in terms of Row's payoffs rather than Column's. We write these "zero-sum versions" of the conditions as Eqs. (8.5z) and (8.6z):

$$V = P_1 A_{1j} + P_2 A_{2j} + \cdots + P_g A_{gj} \quad \text{for } j = 1, 2, \ldots, h \tag{8.5z}$$

and

$$V < P_1 A_{1j} + P_2 A_{2j} + \cdots + P_g A_{gj} \quad \text{for } j = h + 1, h + 2, \ldots, n. \tag{8.6z}$$

(Note that multiplying by -1 to go from Eq. (8.6) to Eq. (8.6z) reverses the direction of the inequality.)

Of these, Eqs. (8.5) and (8.5z) tell us that, so long as Row is using his equilibrium mix, Column (and therefore Row, too, in this zero-sum game) gets the same payoff from any of the pure strategies that he actually uses in equilibrium. Column cannot do any better for himself—and therefore in this zero-sum game cannot cause any harm to Row—by choosing one of those strategies rather than another. What is more, Eq. (8.6z) tells us that were Column to use any of the other strategies, Row would do even better. In other words, these conditions tell us that Row's equilibrium mix cannot be exploited by Column. Thus we see in a more general setting the purposive role of mixing in zero-sum games that we saw in the examples of Chapter 7; we also see more explicitly why it works only for zero-sum games.

3. Now we return to the general, non-zero-sum, case. Note that the system comprising Eqs. (8.1) and (8.5) has $(h + 1)$ linear equations and $(g + 1)$ unknowns. In general, such a system has no solution if $h > g$, has exactly one solution if $h = g$, and has many solutions if $h < g$. Conversely, the system comprising Eqs. (8.2) and (8.3) has $(g + 1)$ linear equations and $(h + 1)$ unknowns. In general, such a system has no solution if $g > h$, has exactly one solution if $g = h$, and has multiple solutions if $g < h$. Because in equilibrium we want both systems to be satisfied, in general we need $g = h$. Thus in a mixed-strategy equilibrium, the two players use equal numbers of pure strategies.

We keep on saying "in general" because exceptions can arise for fortuitous combinations of coefficients and right-hand sides of equations. In particular, it is possible for too many equations in too few unknowns to have solutions. This is just what happens in the "exceptional cases" mentioned in Section 3.B of Chapter 7.

4. We observe a very particular relation between the use of strategies and their payoffs. Row uses strategies P_1 to P_g with positive probabilities, and Eq. (8.3) shows that he gets exactly the payoff V when any one of these pure strategies is played against Column's equilibrium mix. For the remaining pure strategies in

Row's armory, Eq. (8.4) shows that they yield a lower payoff than V when played against Column's equilibrium mix, and then they are not used; that is, P_{h+1} to P_m are all zero. In other words, for any i, it is *impossible* to have

$$\text{both } V > A_{i1}Q_1 + A_{i2}Q_2 + \cdots + A_{ih}Q_h \quad \text{and} \quad P_i > 0.$$

At least one of these inequalities must collapse into equality. This is known as the principle of **complementary slackness,** and it is of great importance in the general theory of games and equilibria, as well as in mathematical optimization (programming).

5. Back to the zero-sum case. When both players choose their equilibrium mix, Row's expected payoff is

$$V = \sum_{i=1}^{m} \sum_{j=1}^{n} P_i A_{ij} Q_j,$$

and Column's is just the negative of this payoff. Moreover, the equilibrium comes about when Row, for the given q-mix, chooses his p-mix to maximize this expression, and simultaneously Column, for the given p-mix, chooses his q-mix to maximize the negative of the same expression, or to minimize the same expression. If we regard the expression as a function of all the P_i and the Q_j, therefore, and graph it in a sufficiently high-dimensional space, it will look like a saddle. The front-to-back cross section of a saddle looks like a valley or a U, with its minimum at the middle, while the side-to-side cross section looks like a peak or an inverted U, with its maximum at the middle. If each player has just two pure strategies, the p-mix and q-mix can each be described by a single number—say, the probability of choosing the first strategy. (For each player, the probability of choosing his second pure strategy is then just one minus that of choosing his first pure strategy.) We can then draw a graph in three dimensions, where the x and y axes are in a horizontal plane and the z axis points vertically upward. The p-mix is shown along the x-axis, the q-mix along the y-axis, and the value V along the z-axis. The cross section of this saddle-shaped surface along the x direction will show the maximization of V with respect to p, therefore a peak. And the cross section along the y direction will show the minimization of V with respect to q, therefore a valley. Thus the graph will look like a saddle, as illustrated in Figure 8.13. Such an equilibrium is called a **saddle point.**

The value of V in equilibrium—that is, the simultaneous maximum with respect to the P_i and the minimum with respect to the Q_j—is called the *minimax* value of the zero-sum game. The idea of such an equilibrium, as well as the formulation of the conditions such as Eqs. (8.5z) and (8.6z) that define it, was the first important achievement of game theory and appeared in the work of von Neumann and Morgenstern in the 1940s. It is called their minimax theorem.

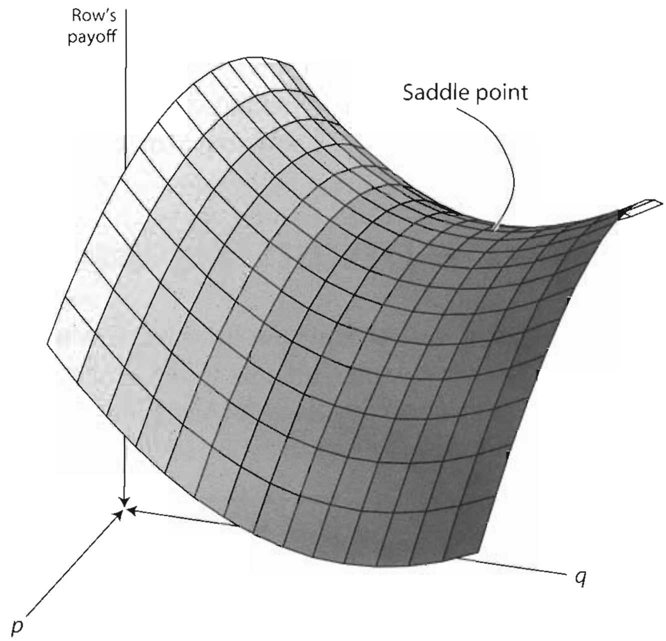


FIGURE 8.13 Saddle Point

SUMMARY

In non-zero-sum games, mixed strategies can arise when players' mutual levels of *subjective uncertainty* about each other's actions are at just the right level. In equilibrium, a player chooses his equilibrium mixture as if he is keeping his opponent indifferent between the opponent's pure strategy choices. Best-response analysis again provides the framework for determining equilibrium mixture probabilities. In large games, where players have three (or more) strategies, mixed-strategy equilibria may be fully or partially mixed, and it is possible for there to be multiple or nonunique equilibria.

Mixed-strategy equilibria are equilibria only in a weak sense because players are indifferent between all possible mixes when their opponents are mixing. In zero-sum games, the opponent's indifference property can be interpreted as helping players to *prevent exploitation* by rivals. And counterintuitive results—showing that players use less often those strategies for which payoffs have increased and that equilibrium mixtures in non-zero-sum games depend only on the *other* players' payoffs—can be found for both zero and non-zero-sum games. Experimental evidence shows that the predictions of theory should be used with caution. But the theory presented here and in Chapter 7 can be generalized for games of any size and type.

KEY TERMS

complementary slackness (256)
prevent exploitation (244)
saddle point (256)
subjective uncertainty (234)

EXERCISES

1. Find a Nash equilibrium in mixed strategies for the following game by using the method of best-response analysis. Draw a best-response diagram and show the equilibrium mixture on the diagram. Also indicate each player's expected payoff in equilibrium.

		COLUMN	
		Left	Right
ROW	Up	4, 0	−1, 2
	Down	1, 1	2, −1

2. (a) Find all pure-strategy Nash equilibria of the following non-zero-sum game.

		COLUMN			
		A	B	C	D
ROW	1	1, 1	2, 2	3, 4	9, 3
	2	2, 5	3, 3	1, 2	7, 1

(b) Now find a mixed-strategy equilibrium of the game. What are the players' expected payoffs in the equilibrium?

3. (a) The following table illustrates the money payoffs associated with a two-person simultaneous-play game. Find the Nash equilibrium in mixed strategies for this game and the players' expected payoffs in this equilibrium.

		COLUMN	
		Left	Right
ROW	Up	1, 16	4, 6
	Down	2, 20	3, 40

(b) The two players jointly get most money when Row plays Down. However, in the equilibrium, Row does not always play Down. Why not? Can you think of ways in which a more cooperative outcome can be sustained?

4. The battle-of-the-sexes version of the meeting game between Harry and Sally is illustrated in Figure 4.13.
   (a) Use best-response analysis to determine the mixed-strategy equilibrium in that game.
   (b) Determine each player's expected payoff in equilibrium and compare these payoffs with the payoffs received in the two pure-strategy Nash equilibria of the game. How could the players do better than they do when they randomize independently?

5. Recall Exercise 7 from Chapter 4 about an old lady looking for help crossing the street. Only one person is needed to help her; more are okay but no better than one. You and I are the two people in the vicinity who can help; each has to choose simultaneously whether to do so. Each of us will get pleasure worth 3 from her success (no matter who helps her). But each one who goes to help will bear a cost of 1, this being the value of our time taken up in helping. You were asked to set this up as a game and to write the payoff table in Exercise 7 of Chapter 4. If you did that exercise, you also found all of the pure-strategy Nash equilibria of the game. Now find the mixed-strategy equilibrium of this game.

6. Consider the following variant of chicken, in which James's payoff from being "tough" when Dean is "chicken" is 2, rather than 1.

		DEAN	
		Swerve	Straight
JAMES	Swerve	0, 0	−1, 1
	Straight	2, −1	−2, −2

   (a) Find the mixed-strategy equilibrium in this game, including the expected payoffs for the players.
   (b) Compare the results with those of the original game in Section 1.B of this chapter. Is Dean's probability of playing Straight (being tough) higher now than before? What about James's probability of playing Straight? What about the two players' expected payoffs? Are these differences in the equilibrium outcomes paradoxical in light of the new payoff structure? Explain how your findings can be understood in light of the opponent's indifference principle.

7. Recall the game from Exercise 11 in Chapter 4. Three contestants (A, B, and C) are participating in a game in which there is a prize worth \$30. Each can buy a ticket worth \$15 or \$30 or not buy a ticket at all. They make these choices simultaneously and independently. Then, knowing the ticket-purchase decisions, the game organizer awards the prize. If none of the contestants has bought a ticket, the prize is not awarded. If only one has bought a ticket, that player gets the prize. If two or more have bought tickets, then the one with the highest-cost ticket gets the prize if there is only one such contestant. If there are two or three contestants all having bought highest-cost tickets, the prize is split equally among them. Find a symmetric equilibrium in mixed strategies for this game, where each contestant mixes over the ticket-purchase choices and the mixture probabilities are the same for all three contestants.

8. Recall the game from Exercise 5 of Chapter 6 of ice-cream vendors on the beach. Construct the five-by-five table for the game, and find the Nash equilibrium in mixed strategies. Explain why some pure strategies are unused in the equilibrium mixtures and verify (by using the opponent's equilibrium mixture) that they should not be included.

9. **[Optional]** Recall Exercise 12 of Chapter 4 that was based on the bar scene from the film *A Beautiful Mind*. Here we consider the mixed-strategy equilibria of that game when played by $n > 2$ young men.
   (a) Begin by considering the symmetric case in which all n young men go after the solitary blonde with some probability P. This probability is determined by the condition that each young man should be indifferent between the pure strategies Blonde and Brunette, given that everyone else is mixing. What is the condition that guarantees the indifference of each player? What is the equilibrium value of P in this game?
   (b) There are asymmetric mixed-strategy equilibria in this game also. In these equilibria, $m < n$ young men each go for the blonde with probability Q and the remaining $n - m$ young men go after the brunettes. What is the condition that guarantees that each of the m young men is indifferent, given what everyone else is doing? What condition must hold so that the remaining $n - m$ players don't want to switch from the pure strategy of choosing a brunette? What is the equilibrium value of Q in the asymmetric equilibrium?

PART THREE

Some Broad Classes of Games and Strategies

9

Uncertainty and Information

UNCERTAINTY ABOUT OUTCOMES EXISTS in many of the games that we have studied so far. Sometimes this uncertainty is the result of the players' mixed strategies, but more often it is something intrinsic to the game. In tennis or football, for example, we supposed quite realistically that the players know only the probabilities of success of the various strategy combinations; they cannot forecast the exact result on any one occasion when those strategies are used. In some games, each player may be unsure about the motives or payoffs of the others; here the uncertainty is asymmetric, because the other player knows his own payoffs perfectly well. In this chapter, we develop various ideas and methods for handling games where such asymmetric uncertainties exist.

Specifically, we turn our attention to situations where different players in a game have different amounts of information. Then, as pointed out in Chapter 2, manipulation of the information becomes an important dimension of strategy. We will see when information can or cannot be communicated verbally in a credible manner. We will also examine other strategies designed to convey or conceal one's own information and to elicit another player's information. We spoke briefly of some such strategies—namely, screening and signaling—in Chapters 1 and 2; here, we study them in a little more detail.

The study of the topic of information and its manipulation in games has been very active and important in the past 25 years. It has shed new light on many previously puzzling matters in economics, such as the nature of incentive contracts, the organization of companies, markets for labor and for durable goods, government regulation of business, and myriad others.[1] More recently, the same concepts have been used by political scientists to explain phenomena such as the relation of tax and expenditures policy changes to elections, as well as the delegation of legislation to committees. The ideas have also spread to biology, where evolutionary game theory explains features such as the peacock's large and ornate tail as a signal. This chapter introduces you to some of these ideas and prepares the way for you to read more about these fascinating developments. Perhaps even more importantly, you will recognize the important role that signaling and screening plays in your daily interaction with family, friends, teachers, co-workers and so on, and will be able to improve your strategies in these games.

1 ASYMMETRIC INFORMATION: BASIC IDEAS

In many games, one or some of the players may have an advantage of knowing with greater certainty what has happened or what will happen. Such advantages, or asymmetries of information, are common in actual strategic situations. At the most basic level, each player may know his own preferences or payoffs—for example, risk tolerance in a game of brinkmanship, patience in bargaining, or peaceful or warlike intentions in international relations—quite well but those of the other players much more vaguely. The same is true for a player's knowledge of his own innate characteristics (such as the skill of an employee or the riskiness of an applicant for auto or health insurance). And sometimes the actions available to one player—for example, the weaponry and readiness of a country—are not fully known to other players. Even when the possible actions are known, the actual actions taken by one player may not be observable to others; for example, a manager may not know very precisely the effort or diligence with which the employees in his group are carrying out their tasks. Finally, some actual outcomes (such as the actual dollar value of loss to an insured homeowner in a flood or an earthquake) may be observed by one player but not by others.

By manipulating what the other players know about your abilities and preferences, you can affect the equilibrium outcome of a game. Therefore such manipulation of asymmetric information itself becomes a game of strategy. You may think that each player will always want to conceal his own information and elicit information from the others, but that is not so. Here is a list of various pos-

[1]The pioneers of the theory of asymmetric information in economics, George Akerlof, Michael Spence, and Joseph Stiglitz, received the 2001 Nobel Memorial Prize in economics for these contributions.

sibilities, with examples. The better-informed player may want to do one of the following:

1. *Conceal information* or *reveal misleading information:* When mixing moves in a zero-sum game, you don't want the other player to see what you have done; you bluff in poker to mislead others about your cards.
2. *Reveal selected information truthfully:* When you make a strategic move, you want others to see what you have done so that they may respond in the way you desire. For example, if you are in a tense situation but your intentions are not hostile, you want others to know this credibly so that there will be no unnecessary fight.

Similarly, the less-informed player may want to do one of the following:

1. *Elicit information* or *filter truth from falsehood:* An employer wants to find out the skill of a prospective employee and the effort of a current employee. An insurance company wants to know an applicant's risk class, the amount of loss of a claimant, and any contributory negligence by the claimant that would reduce its liability.
2. *Remain ignorant:* Being unable to know your opponent's strategic move can immunize you against his commitments and threats. And top-level politicians or managers often benefit from having "credible deniability."

The simplest way to convey information to others would seem to be to tell them; likewise, the simplest way to elicit information would seem to be to ask. But, in a game of strategy, players should be aware that others may not tell the truth and, likewise, that their own assertions may not be believed by others. That is, the **credibility** of mere words may be questionable. We consider when direct communication can be credible in the next section of this chapter.

We will find that, most often, words alone do not suffice to convey credible information; rather, actions speak louder than words. The less-informed players should pay attention to what a better-informed player does, not to what he says. And, knowing that the others will interpret actions in this way, the better-informed player should in turn try to manipulate his actions for their information content.

When you are playing a strategic game, you may have information that is "good" (for yourself), in the sense that, if the other players knew this information, they would alter their actions in a way that would increase your payoff. Or you may have "bad" information; its disclosure would cause others to act in a way that would hurt you. You know that others will infer your information from your actions. Therefore you try to think of, and take, actions that will induce them to believe your information is good. Such actions are called **signals,** and the strategy of using them is called **signaling.** Conversely, if others are likely to conclude that your information is bad, you may be able to stop them from making this inference by confusing them. This strategy, called **signal jamming,** is typically a mixed strategy, because the randomness of mixed strategies make inferences imprecise.

If other players know more than you do or take actions that you cannot directly observe, you can use strategies that reduce your informational disadvantage. The strategy of making another player act so as to reveal his information is called **screening,** and specific methods used for this purpose are called **screening devices.**[2] A strategy that attempts to influence an unobservable action of another player, by giving him some reward or penalty based on an observable outcome of that action, is called an **incentive scheme.** We will elaborate on and illustrate all the concepts introduced here later on, in Sections 3 through 5.

2 DIRECT COMMUNICATION, OR "CHEAP TALK"

We pointed out in the preceding section that direct assertions or responses to questions about one's private information are of doubtful credibility in a game of strategy. Other players understand one's incentive to lie when lying would bring one a higher payoff. But direct communication, which has come to be called *cheap talk* by game theorists, can be used in some cases to help players choose among multiple possible equilibria. In these games, the equilibrium achieved by using direct communication is termed a **cheap talk equilibrium.**

Direct communication of information works well if the players' interests are well aligned. The assurance game first introduced in Chapter 4 provides the most extreme example of this. We reproduce its payoff table (Figure 4.12) as Figure 9.1.

		SALLY	
		Starbucks	Local Latte
HARRY	Starbucks	1, 1	0, 0
	Local Latte	0, 0	2, 2

FIGURE 9.1 Assurance

[2]A word of warning: Don't confuse screening with signal-jamming. In ordinary language, the word "screening" can have different meanings. The one used in game theory is that of testing or scrutinizing. Thus a less-informed player uses screening to find out what a better-informed player knows. For the alternative sense of screening—namely, concealing—the game-theoretic term is signal jamming. Thus a better-informed player uses a signal-jamming action to prevent the less-informed player from correctly inferring the truth from the action.

The interests of Harry and Sally are perfectly aligned in this game; they both want to meet and prefer meeting in Local Latte. The problem is that the game is played noncooperatively; they are making their choices independently, without knowledge of what the other is choosing. But suppose that Harry is given an opportunity to send a message to Sally (or Sally is given an opportunity to ask a question and Harry replies) before their choices are made. If Harry's message (or reply; we will not keep repeating this) is: "I am going to Local Latte," Sally has no reason to think he is lying. If she believes him, she should choose Local Latte, and, if he believes she will believe him, it is equally optimal for him to choose Local Latte, making his message truthful. Thus direct communication very easily achieves the mutually preferable outcome. That is indeed the reason why, when we considered this game in Chapter 4, we had to construct an elaborate scenario in which such communication was infeasible; recall that the two were in separate classes until the last minute before their meeting and did not have each other's pager numbers.

Let us examine the outcome of allowing direct communication in the assurance game more precisely in game-theoretic terms. We have created a two-stage game. In the first stage, only Harry acts, and his action is his message to Sally. In the second stage, the original simultaneous-move game is played. In the full two-stage game, we have a rollback equilibrium where the strategies (complete plans of action) are as follows. The second-stage action plans for both players are: "If Harry's first-stage message was 'I am going to Starbucks,' then choose Starbucks; and, if Harry's first-stage message was 'I am going to Local Latte,' then choose Local Latte." (Remember that players in sequential games must specify *complete* plans of action.) The first-stage action for Harry is to send the message "I am going to Local Latte." Verification that this is indeed a rollback equilibrium of the two-stage game is easy and we leave it to you.

However, this equilibrium where cheap talk "works" is not the only rollback equilibrium of this game. Consider the following strategies: The second-stage action plan for each player is to go to Starbucks regardless of Harry's first-stage message; and Harry's first-stage message can be anything. We can verify that this also is indeed a rollback equilibrium. Regardless of Harry's first-stage message, if one player is going to Starbucks, then it is optimal for the other player to go there also. Thus, in each of the second-stage subgames that could arise—one after each of the two messages that Harry could send—both choosing Starbucks is a Nash equilibrium of the subgame. Then, in the first stage, Harry, knowing his message is going to be disregarded, is indifferent about which message he sends.

The cheap talk equilibrium—where Harry's message is not disregarded—yields higher payoffs, and we might normally think that it would be the one selected as a focal point. However, there may be reasons of history or culture that favor the other equilibrium. For example, for some reasons quite extraneous to this particular game, Harry may have a reputation for being totally unreliable. He might be a compulsive practical joker or just absent-minded. Then people might generally disregard his statements and, knowing this to be the usual state of affairs, Sally might not believe this particular one.

Such problems exist in all communication games. They always have alternative equilibria where the communication is disregarded and therefore irrelevant. Game theorists call these **babbling equilibria.** Having noted that they exist, however, we will focus on the cheap talk equilibria, where communication does have some effect.

The credibility of direct communication depends on the degree of alignment of players' interests. As a dramatic contrast with the assurance game example, consider a game where the players' interests are totally in conflict—namely, a zero-sum game. A good example is the tennis point of Figure 4.15; we reproduce its payoff matrix as Figure 9.2 below. Remember that the payoffs are Evert's success percentages. Remember also that this game has only a mixed-strategy Nash equilibrium (derived in Chapter 7); Evert's expected payoff in this equilibrium is 62.

Now suppose that we construct a two-stage game. In the first stage, Evert is given an opportunity to send a message to Navratilova. In the second stage, the simultaneous-move game of Figure 9.2 is played. What will be the rollback equilibrium?

It should be clear that Navratilova will not believe any message received from Evert. For example, if Evert's message is, "I am going to play DL," and Navratilova believes her, then Navratilova should choose to cover DL. But, if Evert thinks that Navratilova will do so, then Evert's best choice is CC. At the next level of thinking, Navratilova should see through this and not believe the assertion of DL.

But there is more. Navratilova should not believe that Evert would do exactly the opposite of what she says either. Suppose Evert's message is, "I am going to play DL," and Navratilova thinks, "She is just trying to trick me, and so I will take it that she will play CC." This will lead Navratilova to choose to cover CC. But if Evert thinks that Navratilova will disbelieve her in this simple way,

		NAVRATILOVA	
		DL	CC
EVERT	DL	50	80
	CC	90	20

FIGURE 9.2 Tennis Point

then Evert should choose CC after all. And Navratilova should see through this, too.[3]

Thus Navratilova's disbelief should mean that she should just totally disregard Evert's message. Then the full two-stage game has only the babbling equilibrium. The two players' actions in the second stage will be simply those of the original equilibrium, and Evert's first-stage message can be anything. This is true of all zero-sum games.

But what about more general games in which there is a mixture of conflict and common interest? Whether direct communication is credible in such games depends on how the two aspects of conflict and cooperation mix when players' interests are only partially aligned. Thus, we should expect to see both cheap talk and babbling equilibria in games of this type.

Consider games with multiple equilibria where one player prefers one equilibrium and the other prefers the other equilibrium, but both prefer either of the equilibria to some other outcome. One example is the battle of the sexes; we reproduce its payoff table (Figure 4.13) as Figure 9.3.

Suppose Harry is given an opportunity to send a message. Then the two-stage game has a rollback equilibrium where he sends the message "I am going to Starbucks," and the second-stage action plan for both players is to choose the location identified in Harry's message. Here, there is a cheap talk equilibrium and the opportunity to send a message can enable a player to select his preferred outcome.[4]

		SALLY	
		Starbucks	Local Latte
HARRY	Starbucks	2, 1	0, 0
	Local Latte	0, 0	1, 2

FIGURE 9.3 Battle of the Sexes

[3]When a player has three or more actions, the argument is a little trickier. If Evert has actions DL, CC, and Lob, and she says, "I am going to play DL," then Navratilova should disbelieve this but should not interpret it to mean that DL is totally ruled out either. Following through the logic of this again leads to the same conclusion—namely, that Navratilova should totally disregard Evert's message.

[4]What about the possibility that, in the second stage, the action plans are for both players to choose exactly the opposite of the location in Harry's message? That, too, is a Nash equilibrium of the second-stage subgame; so there would seem to be a "perverse" cheap talk equilibrium. In this situation, Harry's optimal first-stage action will be to say, "I am going to Local Latte." So he can still get his preferred outcome.

		THE STOCK IS	
		Good	Bad
YOU	Buy	1	−1
	Sell	−1	1

FIGURE 9.4 Your Payoffs from Your Investment Decisions

Another example of a game with partially aligned payoffs comes from a situation that you may have already experienced or, if not, soon will when you start to earn and invest. When your stockbroker recommends that you should buy a particular stock, he may be doing so as part of developing a long-run relationship with you for the steady commissions that your business will bring him or he may be touting a loser that his firm wants to get rid of for a quick profit. You have to guess the relative importance of these two possibilities in his payoffs.

Suppose there are just two possibilities: the stock may have good prospects or bad ones. If the former, you should buy it; if the latter, sell it, or *sell short.* Figure 9.4 shows your payoffs in each of the eventualities given the two possibilities, Good and Bad, and your two actions, Buy and Sell. Note that this is not a payoff matrix of a game; the columns are not the choices of a strategic player.

The broker knows whether the stock is actually Good or Bad; you don't. He can give you a recommendation of Buy or Sell. Should you follow it? That should depend on your broker's payoffs in these same situations. Suppose the broker's payoffs are a mixture of two considerations. One is the long-term relationship with you; that part of his payoffs is just a replica of yours. But he also gets an extra kickback X from his firm if he can persuade you to buy a bad stock that the firm happens to own and is eager to unload on you. Then his payoffs are as shown in Figure 9.5.

		THE STOCK IS	
		Good	Bad
YOU	Buy	1	$-1 + X$
	Sell	−1	1

FIGURE 9.5 Your Broker's Payoffs from Your Investment Decisions

Can there be a cheap talk equilibrium with truthful communication? In this example, the broker sends you a message in the first stage of the game. That message will be Buy or Sell, depending on his observation of whether the stock is Good or Bad. At the second stage, you make your choice of Buy or Sell, depending on his message. So we are looking for an equilibrium where his strategy is honesty (say Buy if Good, say Sell if Bad), and your strategy is to follow his recommendation. We have to test whether the strategy of each player is optimal given that of the other.

Given that the broker is sending honest messages, obviously it is best for you to follow the advice. Given that you are following the advice, what about the broker's strategy? Suppose he knows the stock is Good. If he sends the message Buy, you will buy the stock and his payoff will be 1; if he says Sell, you will sell and his payoff will be -1. So the "say Buy if Good" part of his strategy is indeed optimal for him. Now suppose he knows the stock to be Bad. If he says Sell, you will sell and he will get 1. If he says Buy, you will buy and he will get $-1 + X$. So honesty is optimal for him in this situation if $1 > -1 + X$, or if $X < 2$. Direct communication from your broker is credible and there is a cheap talk equilibrium in this game as long as his extra payoff from selling you a loser is not "too large."

However, if $X > 2$, then the broker's best response to your strategy of following his advice is to say Buy regardless of the truth. But, if he is doing that, then following his advice is no longer your optimal strategy. You have to disregard his message and fall back on your own prior estimate of whether the stock is Good or Bad. In this case, only the babbling equilibrium is possible.

In these examples, the available messages were simple binary ones—Starbucks or Local Latte and Buy or Sell. What happens when richer messages are possible? For example, suppose that the broker could send you a number g, representing his estimate of the rate of growth of the stock price, and this number could range over a whole continuum. Now, as long as the broker gets some extra benefit if you buy a bad stock that he recommends, he has some incentive to exaggerate g. Therefore fully accurate truthful communication is no longer possible. But partial revelation of the truth may be possible. That is, the continuous range of growth rates may split into intervals—say, from 0% to 1%, from 1% to 2%, and so on—such that the broker finds it optimal to tell you truthfully into which of these intervals the actual growth rate falls and you find it optimal to accept this advice and take your optimal action on its basis. However, we must leave further explanation of this idea to more advanced treatments.[5]

[5]The seminal paper that developed this theory of partial communication is by Vincent Crawford and Joel Sobel, "Strategic Information Transmission," *Econometrica,* vol. 50, no. 6 (November 1982), pp. 1431–1452. An elementary exposition and survey of further work is in Joseph Farrell and Matthew Rabin, "Cheap Talk," *Journal of Economic Perspectives,* vol. 10, no. 3 (Summer 1996), pp. 103–118.

3 SIGNALING AND SCREENING

When direct communication does not yield a cheap talk equilibrium, actions can speak louder than words. The reason is that mere assertions are costless to make—talk is cheap—whereas actions have a direct cost in terms of the players' payoffs. If this cost depends in a suitable way on the item of private information itself, then actions can be credible evidence of the hidden information. We now develop this idea in more detail.

Many of you expect that when you graduate you will be among the elite of America's workers, the so-called symbolic analysts. Employers of such workers want them to possess the appropriate qualities and skills—capacity for hard work, numeracy, logic, and so on. You know your own qualities and skills far better than your prospective employer does. He can test and interview you, but what he can find out by these methods is limited by the available time and resources. Your mere assertions about your qualifications are not credible; more objective evidence is needed.

What items of evidence can the employer seek, and what can you offer? Recall from Section 1 of this chapter that the former are called *screening devices*, and their use by the prospective employer to identify your qualities and skills is called *screening*. Devices that you use at your initiative are called signals, and, when you do so, you are said to engage in *signaling*. Sometimes similar or even identical devices can be used for either signaling or screening.

In this instance, if you have selected (and passed) particularly tough and quantitative courses in college, your course choices can be credible evidence of your capacity for hard work in general and of your skills in symbolic analysis in particular. Let us consider the role of course choice as a screening device.

To keep things simple, suppose college students are of just two types when it comes to the qualities most desired by employers: A (able) and C (challenged). Potential employers are willing to pay \$150,000 a year to a type A, and \$100,000 to a type C. We suppose that there are many potential employers who have to compete with one another for a limited number of job candidates; so they have to pay the maximum amount that they are willing to pay. Because employers cannot directly observe any particular job applicant's type, they have to look for other credible means to distinguish between them.

Suppose the types differ in their tolerance for taking a tough course rather than an easy one in college. Each type must sacrifice some party time or other activities to take a tougher course, but this sacrifice is less, or easier to bear, for the A types than it is for the C types. Suppose the A types regard the cost of each such course as equivalent to \$6,000 a year of salary, while the C types regard it as

$9,000 a year of salary. Can an employer use this differential to screen his applicants and tell the A types from the C types?

Consider the following policy: anyone who has taken a certain number, n, or more of the tough courses will be regarded as an A and paid $150,000, and anyone who has taken less than n will be regarded as a C and paid $100,000. The aim of this policy is to create natural incentives whereby only the A types will take the tough courses, while the C types will not. Neither wants to take more of the tough courses than he has to; so the choice is between taking n to qualify as an A or giving up and settling for being regarded as a C, in which case he may as well not take any of the tough courses and just coast through college.

The criterion that employers devise to distinguish an A from a C—namely, the number of tough courses taken—should be sufficiently strict that the C types do not bother to meet it but not so strict as to discourage even the A types from attempting it. The correct value of n must be such that the true C types prefer to settle for being revealed as such, rather than incur the extra cost of imitating the A type's behavior. That is, we need[6]

$$100{,}000 \geq 150{,}000 - 9{,}000n, \quad \text{or} \quad 9n \geq 50, \quad \text{or} \quad n \geq 5.56.$$

Similarly, the condition that the true A types prefer to prove their type by taking n tough courses is

$$150{,}000 - 6{,}000n \geq 100{,}000, \quad \text{or} \quad 6n \leq 50, \quad \text{or} \quad n \leq 8.33.$$

These constraints align the job applicant's incentives with the employer's desires, or make it optimal for the applicant to reveal the truth about his skill through his action. Therefore they are called the **incentive-compatibility constraints** for this problem. The n satisfying both constraints, because it is required to be an integer, must be either 6 or 7 or 8.[7]

What makes it possible to meet both conditions is the *difference* in the costs of taking tough courses between the two types: the cost is sufficiently lower for the "good" type that the employers wish to identify. When the constraints are met, the employer can use a policy to which the two types will respond differently, thereby revealing their types. This is called **separation of types** based on **self-selection.**

[6]We require merely that the payoff from choosing the option intended for one's type be at least as high as that from choosing a different option, not that it be strictly greater. However, it is possible to approach the outcome of this analysis as closely as one wants, while maintaining a strict inequality; so nothing substantial hinges on this assumption.

[7]If in some other context the corresponding choice variable is not required to be an integer—for example, if it is a sum of money or an amount of time—then there will be a whole continuous range satisfying both incentive compatibility constraints.

We did not assume here that the tough courses actually imparted any additional skills or work habits that might convert C types into A types. In our scenario, the tough courses serve only the purpose of identifying the persons who already possess these attributes. In other words, they have a purely screening function.

In reality, education does increase productivity. But it also has the additional screening or signaling function of the kind described here. In our example, we found that education might be undertaken solely for the latter function; in reality, the corresponding outcome is that education is carried farther than is justified by the extra productivity alone. This extra education carries an extra cost—the cost of the information asymmetry.

When the requirement of taking enough tough courses is used for screening, the A types bear the cost. Assuming that only the minimum needed to achieve separation is used—namely, $n = 6$—the cost to each A type has the monetary equivalent of $6 \times \$6{,}000 = \$36{,}000$. This is the cost, in this context, of the information asymmetry. It would not exist if a person's type could be directly and objectively identified. Nor would it exist if the population consisted solely of A types. The A types have to bear this cost *because* there are some C types in the population, from whom they (or their prospective employers) seek to distinguish themselves.[8]

Rather than having the A types bear this cost, might it be better not to bother with the separation of types at all? With the separation, A types get a salary of \$150,000 but suffer a cost, the monetary equivalent of \$36,000, in taking the tough courses; thus their net money-equivalent payoff is \$114,000. And C types get the salary of \$100,000. What happens to the two types if they are not separated?

If employers do not use screening devices, they have to treat every applicant as a random draw from the population and pay all the same salary. This is called **pooling of types.**[9] In a competitive job market, the common salary under pooling will be the population average of what the types are worth to an employer, and this average will depend on the proportions of the types in the population.

[8]As we will see in Chapter 12, Section 5, the C types in this example inflict a *negative external effect* on the A types.

[9]Note that *pooling of types* is quite different from the *pooling of risks* that is discussed in the Appendix to this chapter. The latter occurs when each person in a group faces some uncertain prospect; so they agree to put their assets into one pool from which they will share the returns. As long as their risks are not perfectly positively correlated, such an arrangement reduces the risk for each person. In pooling of types, players who differ from one another in some relevant innate characteristic (such as ability) nevertheless get the same outcome or treatment (such as allocation to a job or salary) in the equilibrium of the game. Note also that pooling of types is the opposite of *separation of types,* where players differing in their characteristics get different outcomes; so the outcome reveals the type perfectly.

For example, if 20% of the population is type A and 80% is type C, then the common salary with pooling will be

$$0.2 \times \$150{,}000 + 0.8 \times \$100{,}000 = \$110{,}000.$$

The A types will then prefer the situation with separation because it yields $114,000 instead of the $110,000 with pooling. But, if the proportions are 50–50, then the common salary with pooling will be $125,000, and the A types will be worse off under separation than they would be under pooling. The C types are always better off under pooling. The existence of the A types in the population means that the common salary with pooling will always exceed the C type's separation salary of $100,000.

However, even if both types prefer the pooling outcome, it cannot be an equilibrium when many employers or workers compete with one another in the screening or signaling process. Suppose the population proportions are 50–50 and there is an initial equilibrium with pooling where both types are paid $125,000. An employer can announce that he will pay $132,000 for someone who takes just one tough course. Relative to the initial situation, the A types will find it worthwhile because their cost of taking the course is only $6,000 and it raises their salary by $7,000, while C types will not find it worthwhile, because their cost, $9,000, exceeds the benefit, $7,000. Because this particular employer selectively attracts the A types, each of whom is worth $150,000 to him but is paid only $132,000, he makes a profit by deviating from the pooling salary package.

But his deviation starts a process of adjustment by competing employers, and that causes the old pooling situation to collapse. As A types flock to work for him, the pool available to the other employers is of lower average quality, and eventually they cannot afford to pay $125,000 anymore. As the salary in the pool is lowered, the differential between that salary and the $132,000 offered by the deviating employer widens to the point where the C types also find it desirable to take that one tough course. But then the deviating employer must raise his requirement to two courses and must increase the salary differential to the point where two courses become too much of a burden for the C types, while the A types find it acceptable. Other employers who would like to hire some A types must use similar policies if they are to attract them. This process continues until the job market reaches the separating equilibrium described earlier.

Even if the employers did not take the initiative to attract As rather than Cs, a type A earning $125,000 in a pooling situation might take a tough course, take his transcript to a prospective employer, and say: "I have a tough course on my transcript, and I am asking for a salary of $132,000. This should be convincing evidence that I am type A; no type C would make you such a proposition." Given the facts of the situation, the argument is valid, and the employer should find it very profitable to agree: the employee, being type A, will generate $150,000 for the employer but get only \$132,000 in salary. Other A types can do the same. This starts the same kind of cascade that leads to the separating equilibrium. The only difference is who takes the initiative. Now the type A workers choose to get the extra education as credible proof of their type; it becomes a case of signaling rather than screening.

The general point is that, even though the pooling outcome may be better for all, they are not choosing the one or the other in a cooperative binding process. They are pursuing their own individual interests, which lead them to the separating equilibrium. This is like a prisoners' dilemma game with many players, and therefore there is something unavoidable about the cost of the information asymmetry.

In other situations that lack such direct competition among many players, it may be possible to have pooling equilibria as well as separating ones. And that does not exhaust the possibilities. In a fully separating equilibrium, the outcome reveals the type perfectly; in a pooling equilibrium, the outcome provides no information about type. Some games can have intermediate or semiseparating equilibria, where the outcome provides partial information about type. Such games require somewhat more difficult concepts and techniques, and we postpone an example until Section 5.

We have considered only two types, but the idea generalizes immediately. Suppose there are several types: A, B, C, . . . , ranked in an order that is at the same time decreasing in their worth to the employer and increasing in the costs of extra education. Then it is possible to set up a sequence of requirements of successively higher and higher levels of education, such that the very worst type needs none, the next-worst type needs the lowest level, the type second from the bottom needs the next higher level, and so on, and the types will self-select the level that identifies them.

In this example, we developed the idea of a tough course requirement as evidence of skills seen mainly from the perspective of the employer—that is, as an example of a screening device. But at times we also spoke of a student initiating the same action as a signal. As mentioned earlier, there are indeed many parallels between signaling and screening, although in some situations the equilibrium can differ, depending on who initiates the process and on the precise circumstances of the competition among several people on the one side or the other of the transaction. We will leave such details for more advanced treatises on game theory. Here, we make a further point specific to signaling.

Suppose you are the informed party and have available an action that would credibly signal your good information (one whose credible transmission would work to your advantage). If you fail to send that signal, you will be assumed to have bad information. In this respect, signaling is like playing chicken: if you refuse to play, you have already played and lost.

You should keep this in mind when you have the choice between taking a course for a letter grade or on a pass-fail basis. The whole population in the course spans the whole spectrum of grades; suppose the average is B. A student is likely to have a good idea of his own abilities. Those reasonably confident of getting an A+ have a strong incentive to take the course for a letter grade. When they have done so, the average of the rest is less than B, say, B−, because the top end has been removed from the distribution. Now, among the rest, those expecting an A have a strong incentive to switch. That in turn lowers the average of the rest. And so on. Finally, the pass-fail option is chosen by only the Cs and Ds. And that is how the reader of a transcript (a prospective employer or the admissions officer for a professional graduate school) will interpret it.

4 INCENTIVE PAYMENTS

Suppose you are the owner of a firm, looking for a manager to undertake a project. The outcome of the project is uncertain, and the probability of success depends on the quality of the manager's effort. If successful, the project will earn \$600,000. The probability of success is 60% (0.6) if the manager's effort is of routine quality but rises to 80% if it is of high quality.

Putting out high-quality effort entails a subjective cost to the manager. For example, he may have to think about the project night and day, and his family may suffer. You have to pay him \$100,000 even for the routine effort, but he requires an extra \$50,000 for making the extra effort.

Is the extra payment in return for the extra effort worth your while? Without it, you have a 0.6 chance of \$600,000 (that is, an expected value of \$360,000), and you are paying the manager \$100,000, for an expected net profit of \$260,000. With it, the expected profit becomes $0.8 \times \$600{,}000 - \$150{,}000 = \$480{,}000 - \$150{,}000 = \$330{,}000$. Therefore the answer is yes; paying the manager \$50,000 for his extra effort is to your benefit.

How do you implement such an arrangement? The manager's contract could specify that he gets a basic salary of \$100,000 and an extra \$50,000 if his effort is of high quality. But how can you tell? The manager might simply take the extra \$50,000 and make the routine effort anyway. Much of the effort may be thinking, which may be done at home evenings and weekends. The manager can always claim to have done this thinking; you cannot check on him. If the project fails, he can attribute the failure to bad luck; after all, even with the high-quality effort, the chances of failure are 1 in 5. And if a dispute arises between you and the manager about the quality of his effort, how can a court or an arbitrator get any better information to judge the matter?

If effort cannot be observed and, in case of a dispute, verified in a court of law, then the manager's contract must be based on something that can be observed and verified. In the present instance, this is the project's success or failure. Because success or failure is *probabilistically* related to effort, it gives some information, albeit imperfect, about effort and can serve to motivate effort.

Consider a compensation package consisting of a base salary, s, along with a bonus, b, that is paid if the project succeeds. Therefore the manager's expected earnings will be $s + 0.6b$ if he makes the routine effort, and $s + 0.8b$ if he makes the high-quality effort. His expected extra earnings from making the better effort will be $(s + 0.8b) - (s + 0.6b)$ or $(0.8 - 0.6)b = 0.2b$. For the extra effort to be worth his while, it must be true that

$$0.2 \times b \geq \$50{,}000 \quad \text{or} \quad b \geq \$250{,}000.$$

The interpretation is simple: the bonus, multiplied by the *increase* in the probability of getting the bonus if he makes the extra effort, equals the manager's expected extra earnings from the extra effort. This expected increase in earnings should be at least enough to compensate him for the cost to him of that extra effort.

Thus a sufficiently high bonus for success creates enough of an incentive for the manager to put out the extra effort. As in Section 3, this is called the *incentive-compatibility condition* or *constraint* on your compensation package.

There is one other condition. The package as a whole must be good enough to get the manager to work for you at all. When the incentive-compatibility condition is met, the manager will make the high-quality effort if he works for you, and his expected earnings will be $s + 0.8b$. He also requires at least \$150,000 to work for you with high-quality effort. Therefore your offer has to satisfy the relation

$$s + 0.8b \geq \$150{,}000.$$

This relation is called the **participation condition** or **constraint.**

You want to maximize your own profit. Therefore you want to keep the manager's total compensation as low as possible, consistent with the incentive-compatibility and the participation constraints. To hold it down to its floor of \$150,000 means choosing the smallest value of s that satisfies the participation condition, or choosing $s = \$150{,}000 - 0.8b$. But, because b must be at least \$250,000, s can be no more than $\$150{,}000 - 0.8 \times \$250{,}000 = \$150{,}000 - \$200{,}000 = -\$50{,}000$.

What does a negative base salary mean? There are two possible interpretations. In one case, the negative amount might represent the amount of capital that the manager must put up for the project. Then his bonus can be interpreted as the payout from an equity stake or partnership in the enterprise. The second possibility is that the manager does not put up any capital but is fined if the project fails.

Sometimes neither of these possibilities can be done. Qualified applicants for the managerial job may not have enough capital to invest, and the law may prohibit the fining of employees. In that case, the total compensation cannot be held down to the minimum that satisfies the participation constraint. For example, suppose the basic salary, s, must be nonnegative. Even if the bonus is at the lowest level needed to meet the incentive-compatibility constraint—namely, b = \$250,000—the expected total payment to the manager when s takes on the smallest possible nonnegative value (zero) is $0 + 0.8 \times \$250{,}000 = \$200{,}000$. You are forced to overfulfill the participation constraint.

If effort could be observed and verified directly, you would write a contract where you paid a base salary of \$100,000 and an extra \$50,000 when the manager was actually seen making the high-quality effort, for a total of \$150,000. When effort is not observable or verifiable, you expect to pay \$200,000 to give the manager enough incentive to make the extra effort. The additional \$50,000 is an extra cost caused by this observation problem—that is, by the information asymmetry in the game. Such a cost generally exists in situations of asymmetric information. Game theory shows ways of coping with the asymmetry but at a cost that someone, often the less-informed player, must bear.

Is this extra cost of circumventing the information problem worth your while? In this instance, by paying it you get a 0.8 probability of \$600,000, for an expected net profit of

$$0.8 \times \$600{,}000 - \$200{,}000 = \$480{,}000 - \$200{,}000 = \$280{,}000.$$

You could have settled for the low effort, for which you need pay the manager only the basic salary of \$100,000 but get only a 60% probability of success, for an expected net profit of

$$0.6 \times \$600{,}000 - \$100{,}000 = \$360{,}000 - \$100{,}000 = \$260{,}000.$$

Thus, even with the extra cost caused by the information asymmetry, you get more expected profit by using the incentive scheme.

But that need not always be the case. Other numbers in the example can lead to different results; if the project when successful yields only \$400,000 instead of \$600,000, then you do better to settle for the low-pay, low-effort situation.

The nature of the problem here is analogous to that faced by an insurance company whose clients run the risk of a large loss—for example, through theft or fire—but can also affect the probability of the loss by taking good or poor precautions or by being more or less careful. If you are not sure that you double-locked your front door when you left your house, it is tempting not to bother to go back to check again if you are insured. As mentioned earlier, the insurance industry calls this problem moral hazard. Sometimes there may be deliberate fraud—for example, setting fire to your own house to collect insurance—but in many situations there is no necessary judgment of morality.

Economic analyses of situations of asymmetric information and incentive problems have accepted this terminology. Most generally, *any* strategic interaction where one person's action is unobservable or unverifiable is termed **moral hazard**. In the insurance context, moral hazard is controlled by requiring the insured to retain a part of the risk, by using deductibles or coinsurance. This acts as an incentive scheme; in fact, it is just a mirror image of our contract for the manager. The manager gets a bonus when the outcome, whose probability is affected by his effort, is good; the insured has to carry some loss when the outcome, whose probability is affected by his care, is bad.

Real-world moral hazard problems may be more or less severe than our simple example, and the incentive schemes to cope with them less or more feasible, depending on circumstances. Here, we supposed that the success or failure of the project was objectively observable. But suppose the manager can manipulate the situation so that the project seems to have succeeded, and the truth becomes apparent only after he has collected his bonus and disappeared. This possibility is not far-fetched; many firms that were hailed in 2000 as wonders of the new economy were seen in 2002 to have been mere shells, and their previous claims of earnings or revenues were found to have been accounting tricks. The owners (outside shareholders) suffered, while the managers took away handsome compensation packages.

On the other hand, there are ways in which the owner can get more accurate information about the manager's effort. For example, if he has several managers working in parallel on similar projects and the components of luck in their outcomes are positively correlated with one another, then the owner can judge each manager's effort by comparing his outcome with that of the other managers. One manager cannot blame his lack of success on poor luck if other managers have shown better outcomes. Only a conspiracy among all managers can defeat such a comparative evaluation, and an attempt at such a conspiracy is subject to a prisoners' dilemma where one manager can get a handsome reward by working hard while all the others slack.

Another possibility of mitigating moral hazard in our example arises if the owner and the manager interact repeatedly and the component of luck in different periods is independent. Now, although a few and occasional failures might be due to bad luck, a whole string of successive failures is less likely to have this cause. Thus the manager's persistent slackening can be detected with greater accuracy, and better incentive schemes can be designed.

5 EQUILIBRIA IN SIGNALING GAMES

In Sections 3 and 4, we explained the general concepts of screening and signaling and the possible outcomes of separation and pooling that can arise when these strategies are being used. We saw how these concepts would operate in an environment where many employers and employees meet one another, and so the salaries are determined by conditions of competition in the market. However, we have not specified and solved a game in which just two players with differential information confront each other. Here, we develop an example to show how that can be done. In particular, we will see that either separating or pooling can be an equilibrium and that a new type of **partially revealing** or **semiseparating equilibrium** can emerge. Some of the concepts and the analysis get a little intricate, and beginning readers can omit this section without loss of continuity.

Our game has two players, an Attacker and a Defender. The Defender may be Tough or Weak. The Attacker has to decide whether to Invade, without knowing the Defender's precise type. In the absence of any other information, the Attacker assumes that the probability of the Defender being Tough is 0.25.[10] The Defender knows that this is the Attacker's belief, and the Attacker knows that the Defender knows, and so on. In other words, this is common knowledge. If the Attacker invades and the Defender turns out to be Tough, there is a fight that costs both players 10, whereas, if the Defender turns out to be Weak, he gives in without a fight, and loses 5, which the Attacker gains. Finally, the Defender has an opportunity to make a first move and to take an action, which we call a Signal, and which costs 6 regardless of the Defender's type. The Attacker makes the second move after having observed whether the Defender has chosen to Signal. Our goal is to determine whether a Tough Defender can credibly signal his type by using the available action. The outcome of the game depends on our specific choices of numbers for these probabilities and payoffs. We will vary the numbers to show how the outcomes change and will later offer a general algebraic statement.

This generic example captures features of many games in reality. In international relations, one country may be contemplating aggression against another, without knowing how serious the resistance will be. The defending country can make visible preparations to meet the aggression, but, if it is fundamentally weak, the preparations will come to naught if an invasion actually occurs.[11] In economics, a firm may be considering entry into an industry that has an established monopoly, without knowing for sure how tough a competitor the established firm will be. The established firm can take actions—installing capacity to

[10]This probability is as if the actual Defender player were picked randomly from a population of potential players, of which a quarter are Tough types.

[11]Analyses of this include Robert Jervis, *The Logic of Images in International Relations* (Princeton: Princeton University Press, 1970); Barry Nalebuff, "Rational Deterrence in an Imperfect World," *World Politics*, vol. 43, no. 2 (April 1991), pp. 313–335; James Fearon, "Signaling Versus the Balance of Power and Interests," *Journal of Conflict Resolution*, vol. 38, no. 2 (June 1994), pp. 236–269; and Robert Powell, "Uncertainty, Shifting Power, and Appeasement," *American Political Science Review*, vol. 90, no. 4 (December 1996), pp. 749–764.

fight a price war or charging a low price as evidence of its ability to produce at a low cost—to convince the newcomer that entry would mean a costly price war against a tough opponent, thereby deterring entry.[12] In nature, several animals have the ability to appear bigger or stronger temporarily when threatened, even though this ability does not help them if it actually comes to a fight.[13] In tough inner-city neighborhoods, young men display expensive clothes, jewelry, and other "things that may require defending." This "presentation of self announces that [a man] can take care of himself should someone choose to tangle with him."[14]

This idea of signaling toughness is appealing but problematic. As Robert Jervis pointed out in the early 1970s, a weak defender has greater need to avoid invasion than a tough one does; so the signaling action should be interpreted as weakness, not toughness. But, if that is how potential aggressors interpret it, why would anyone bother to provide the costly signal? To sort out this confusingly circular argument, we must examine the complete game and its equilibria. We begin with the game tree in Figure 9.6.

As usual when there is uncertainty arising from something other than the players' random actions, we introduce a neutral player, Nature, who starts the game by choosing a Tough (T) Defender with probability 0.25 or a Weak (W) Defender with probability 0.75. At each of the resulting nodes, the Defender, knowing his own type, decides whether to Signal (S) or not (NS). There are now four nodes, at each of which the Attacker decides whether to Invade (I) or not (NI). But the Attacker observes only the Defender's action and not Nature's; so he can distinguish neither a Tough Signaling Defender from a Weak Signaling one nor a Tough Not Signaling Defender from a Weak Not Signaling one. Therefore there are two information sets for the Attacker: one consists of the two nodes where the Defender has chosen S, and the other consists of the two nodes where the Defender has chosen NS. At both nodes of a single information set, the Attacker must choose the same action, I or NI.

What is the appropriate notion of equilibrium in such a game? Many of the ideas of rollback in sequential-move games continue to apply. The At-

[12]For surveys of this work, see Robert Wilson, "Strategic Models of Entry Deterrence," *Handbook of Game Theory*, vol. 1, ed. Robert Aumann and Sergiu Hart (Amsterdam: North-Holland, 1994), pp. 305–329; and Kyle Bagwell and Asher Wolinsky, "Game Theory and Industrial Organization," *Handbook of Game Theory*, vol. 3, ed. Robert Aumann and Sergiu Hart (Amsterdam: North-Holland, 2002), pp. 1851–1895.

[13]For example, when a gazelle on the African savannah sees a cheetah approaching, it repeatedly leaps in place to heights of 6 feet or more. The aim is to signal to the cheetah that the gazelle is so strong that chasing it would be fruitless. See Fernando Vega-Rodando and Oren Hasson, "A Game-Theoretic Model of Predator-Prey Signaling," *Journal of Theoretical Biology*, vol. 162, no. 3 (June 1993), pp. 309–319.

[14]Elijah Anderson, *Code of the Street* (New York: Norton, 1999), pp. 23, 73.

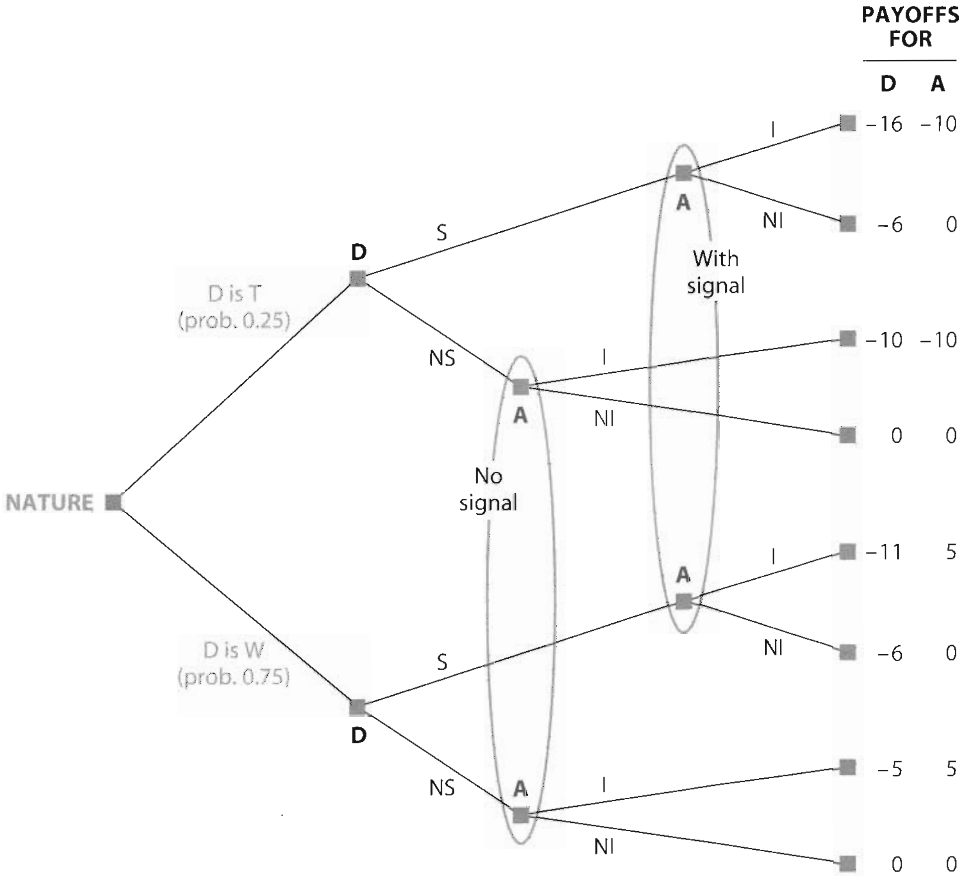


FIGURE 9.6 Game of Signaling Toughness

tacker's choice at each of his information sets should be optimal for him, and the choice of each type of Defender should be optimal for him, taking into account the Attacker's subsequent choices. But there is something more. The Attacker observes the Defender's action, S or NS, and can draw inferences from it. For example, if the equilibrium strategy for the Defender entails the T types choosing S and the W types choosing NS, then an observation of S is evidence that the Defender is of type T. That is, having observed a Defender choosing S, the Attacker will know with certainty that the Defender is a T type; the probability of the T type becomes 1, not the 25% that it was in the absence of further information. When looking for equilibria in signaling games of this type, we should allow a rational Attacker, when choosing whether to invade, to draw all such inferences and should also allow a rational Defender, when choosing whether to signal, to recognize that the Attacker is going to draw such inferences.

In probability theory, Bayes' rule governs this kind of updating of probabilities on the basis of interim observations.[15] The Attacker's optimal choice may be affected by his updating of the probabilities of meeting the two types. Therefore we must augment the rollback or subgame-perfectness notion of equilibrium and require that

1. at each information set, the player acting there takes the best action in light of the available information, including probabilities of alternatives if the information is not definite, and
2. players draw the correct inferences from their observations, as specified by Bayes' rule for updating probabilities.

An outcome satisfying these criteria is called a **perfect Bayesian equilibrium (PBE).**

In preceding chapters, we found equilibria of sequential-move games by starting at the end and working backward. But, in games such as the one we now face, the later actions require updating of information based on other players' earlier actions. This introduces an unavoidable circularity, which invalidates simple unidirectional rollback reasoning. Instead, we must now first specify a full set of strategies (complete plans of action) and belief formation (revision of probability estimates) at all stages of the game and for all players. After we have specified the strategies and belief-formation processes, we then must determine whether they meet all the requirements of a PBE. To do so for our example, we consider a variety of logically conceivable outcomes and test for the existence of a PBE.

A. Separating Equilibrium

To get a separating equilibrium, the Defender's action must fully reveal his type. The Attacker then knows with certainty the location of the decision node at which he finds himself in the second stage of the game. We will consider the case in which the Defender's action truthfully reveals his type.[16] Specifically, suppose the Defender chooses S if his type is T or chooses NS if his type is W. The Attacker infers the Defender's type from the observed actions (Defender is type T if S is observed or W if NS is observed) and chooses I if the Defender has chosen NS but NI if the Defender has chosen S.

[15]See the Appendix to this chapter for an explanation of Bayes' rule.

[16]One might also think it possible to get separation if the Defender always pretends to be the other type and the Attacker infers that he faces a T-type Defender when the action NS is observed. However, this set of strategies is not an equilibrium of the game as specified. We investigate the possibility of this equilibrium later in Section 5.D.

Are these strategies an equilibrium of the game? To answer this question, we check whether the actions and beliefs satisfy the two criteria already stated. First, if the Attacker finds himself in the S information set (having observed S), then, given the Defender's strategy of choosing S if and only if of type T, the Attacker is correct to infer with certainty that he is actually at the upper node (facing a T-type Defender with probability 1). The Attacker's best choice at this node is NI, which gets him the payoff 0; choosing I would have got him −10. Similarly, at the NS information set, the Attacker will infer with certainty that he is at the lower node (facing a W-type Defender with probability 1). Here, he chooses I which gets him 5; NI would have yielded 0. Next, we look at the optimality of actions of the two types of Defenders, given the strategies and beliefs of the Attacker. Consider a T-type Defender. If he chooses S, the Attacker will believe him to be type T and choose NI; so the Defender will get −6. If the Defender chooses NS, the Attacker will believe that he faces a W-type Defender and choose I; then the Defender, who is actually type T, gets the payoff −10. A T-type Defender prefers the stipulated action of S in this case. Following the same steps of reasoning, a W-type Defender, who gets −6 from S (the Attacker chooses NI) and −5 from NS, also prefers the stipulated action of NS.

Our analysis shows that the specified strategies and belief-formation process for the Attacker do constitute an equilibrium of this game. Each player's specified action is optimal for him, and correct inferences are drawn from observing actions. Thus, with the specified payoff structure, there is a fully separating equilibrium in which the Defender truthfully signals his type.

But what if we alter the payoff structure slightly? Suppose that the cost of signaling falls to 4, less than the cost (to a W-type Defender) of giving in without a fight. Then the game changes to the one illustrated in Figure 9.7, and the specified strategies are no longer an equilibrium. The Attacker, inferring type T from an observation of S and type W from an observation of NS, still chooses NI if S and I if NS. And the T-type Defender still prefers S (which yields −4) to NS (which yields −10). But the W-type Defender now gets −4 if he chooses S and −5 if he chooses NS. The W type's optimal choice is not NS; the requirements of a PBE are no longer met. Clearly, the existence of the separating equilibrium, and indeed any equilibrium, in this signaling game depends critically on the relative magnitudes of the payoffs associated with different strategy combinations. We examine the general implications of this conclusion in greater detail in Section 5.D, but first look specifically at two other possible types of equilibria.

B. Pooling Equilibrium

As we saw at the end of the last subsection, changing the payoffs in our game led to the disappearance of the separating equilibrium with truthful signaling, because a W-type Defender will also choose S. In this situation, therefore, it is

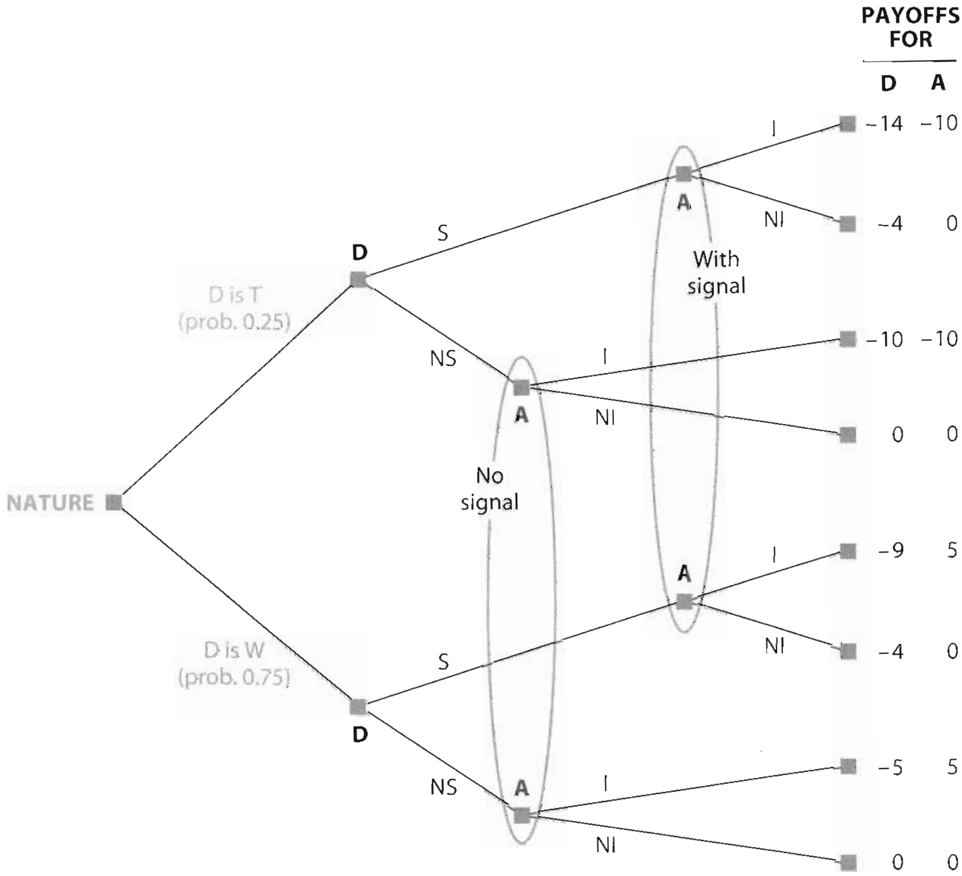


FIGURE 9.7 Signaling Toughness with Reduced Cost of Signal

conceivable to have an equilibrium in which the two types of Defenders choose the same action—that is, the types are pooled. Let us consider a possible pooling equilibrium in which all Defenders choose NS. If the Attacker observes NS, he cannot update his information and makes his optimal choice between I and NI knowing only that his opponent is Tough with 25% probability. And, if the Attacker observes S, we suppose that he attributes this type of "mistake" equally to both types of Defenders; that is, if he observes S, the Attacker also cannot update his information regarding the Defender's type and so makes his optimal choice by assuming that either type is equally likely.[17]

[17]The Attacker's complete plan of action must specify his response to all possible choices of the Defender, even those that are not expected to occur in equilibrium. Bayes' rule provides no guide on how to update probabilities after observing an event that should never have happened—that is, one with probability zero. What we do in this section is specify two reasonable conventions that the Attacker could adopt in such a situation. A large body of literature examines the reasonableness and consequences of various alternatives, but we leave that to more advanced treatments. For more on this topic, see Drew Fudenberg and Jean Tirole, *Game Theory* (Cambridge: MIT Press, 1992), Chapter 11.

If this set of strategies and information updating processes is an equilibrium, then it must be optimal for Defenders of both types to choose NS, given the Attacker's optimal choice of action. The Attacker, observing either S *or* NS and choosing I, gets a payoff of -10 if the Defender is Tough and a payoff of 5 if the Defender is Weak. Thus, the Attacker's expected payoff from choosing I is $(-10 \times 0.25) + (5 \times 0.75) = 1.25$. Similarly, if the Attacker observes either S or NS and chooses NI, his payoff is 0 against a Tough or a Weak Defender; his expected payoff from choosing NI is 0. The Attacker's preferred choice is then I whether he sees S or NS. Given this choice, the Tough Defender can determine that he gets -14 from choosing S and -10 from choosing NS; the Weak Defender gets -9 from S and -5 from NS. Both types of Defenders prefer to choose NS. Thus the specified actions satisfy the conditions for a PBE.

This equilibrium with pooling is achieved in part because of the Attacker's stated inability to discern information from an observed occurrence of (the out-of-equilibrium action) S. Suppose instead that we proposed a different outcome in which both types of Defenders continue to choose NS, but the Attacker updates his information differently when he sees S. Rather than interpreting S as a mistake equally likely to have come from either Defender type, the Attacker now interprets S as a sure sign of toughness. Then, when the Attacker observes S, he chooses NI because it yields 0 against a T-type Defender versus the -10 that he gets from choosing I. When the Attacker observes NS, he chooses I as he did before. Knowing this choice, the W-type Defender sees that he gets -4 from S and -5 from NS; so he prefers S. Similarly, the T-type Defender gets -4 from S and -10 from NS; so he, too, prefers S. Both types of Defenders choosing NS in this case cannot be an equilibrium, because both want to signal when the Attacker interprets that signal to mean that he faces a Tough Defender. This example shows the importance of specifying the Attacker's updating process when describing a pooling equilibrium.

C. Semiseparating Equilibrium

In the preceding example, we saw that it is possible with the specified payoffs and updating processes for our game to have neither a separating nor a pooling equilibrium. But one final possible type of equilibrium can arise in this asymmetric information game. That equilibrium entails mixing by the Attacker and by one of the Defender types. Such a situation could arise if one of the Defender types gets the same payoff from S and NS when the Attacker mixes and is therefore willing to mix between the two, whereas the other type clearly prefers one of S and NS. For example, consider the outcome in which all T-type defenders choose S while W-types mix. Observing S then gives the Attacker some—but not full—information about the Defender's type, hence the name *semiseparating equilibrium* that is used to describe this case. The Attacker must carefully update his probabilities by using Bayes' rule and the equilibrium will require that he mix his actions on observing S to maintain the W-type's indifference between S and NS.[18]

To determine whether the stated set of strategies is an equilibrium of the game, we need to solve for the mixture probabilities used by both the W-type Defender and the Attacker. Suppose the probability of a W type choosing S in his equilibrium mixture is x. This value of x must keep the Attacker indifferent between I and NI. But the Attacker now can use an observance of S to update his probabilities about the type of Defender he faces. Given that all T types will play S and that W types also play S with probability x, the probability that the Attacker sees S played is now $0.25 + 0.75x$. This result follows from the Combination Rule of probabilities in the Appendix to Chapter 7: the probability of a T type is 0.25 and the probability that a T type chooses S is 1; the probability of a W type is 0.75 and the probability that a W type chooses S is x. The Attacker can now revise his probability that the Defender is type T, taking into account the observation of S. The part of the probability of S that is attributed to a type-T Defender is 0.25; so, when the Attackers sees S, he assumes he is facing a Tough-type Defender with probability $0.25/(0.25 + 0.75x)$. Similarly, the revised probability that the Defender's type is W, taking into account an observation of S, is $0.75x/(0.25 + 0.75x)$.

We can now use the information on the Attacker's inferences from an observation of S to determine the value of x in the W type's mix that will keep the Attacker indifferent between I and NI. If the Attacker observes S, his expected payoff from choosing I is

$$\frac{0.25}{0.25 + 0.75x}(-10) + \frac{0.75x}{0.25 + 0.75x}5$$

His expected payoff from NI is 0. He likes the two equally and is therefore willing to mix between them if $(0.25 \times -10) + (0.75x \times 5) = 0$, or $3.75x = 2.5$, or $x = 2/3$. If the Attacker sees NS, he infers that the Defender is a W type for sure (only W types are choosing NS) and therefore chooses I with certainty.

So far, we know that a W-type Defender can keep the Attacker indifferent between I and NI by mixing with probability 2/3 on S and 1/3 on NS. But, for there to be an equilibrium, the Attacker's mixture probabilities, on observing S, must also keep the Weak Defender indifferent between S and NS. Suppose the

[18]Again, there are other possible ways to configure an equilibrium with mixing, including the possibility of mixing by both types of Defender. We focus here on the one case in which a semiseparating equilibrium exists for this game and leave the derivation of other possible equilibria, along with the conditions under which they exist, to those able to pursue more advanced treatments of the topic.

Attacker's probability of choosing I on seeing S is y. Then a W-type Defender expects a payoff of $-9y - 4(1 - y)$ from S. If he chooses NS, this choice will draw the response of I for sure from the Attacker; so the Weak Defender will get -5. These two are equal and the W type will mix if $-5y = -1$, or $y = 1/5$.

Finally, we need to confirm that the T type wants to choose S with certainty. Given $y = 1/5$, choosing S gets him $-14y - 4(1 - y) = -6$; choosing NS gets him -10. Therefore he prefers the specified pure strategy of choosing S.

Putting it all together, we do have all of the necessary ingredients for a semiseparating equilibrium in this case. The T-type Defenders choose S always. The W-type Defenders mix between S (2/3 of the time) and NS (1/3 of the time). And the Attacker, using Bayesian updating on the observation of S, mixes I (1/5 of the time) and NI (4/5 of the time) when he sees S and plays I with certainty when he sees NS.

D. General Theory

Our analysis of the Attacker-Defender game has so far been constrained by the specified payoff and probability values in Figures 9.6 and 9.7. It is possible to consider a more general form of the game, as shown in Figure 9.8, in which payoffs are shown with variable names. Here, C represents the *cost* of the signal (S); F is the cost of a *fight* (when the Attacker invades a Tough Defender); and G stands for *gain* (to the Attacker from invading a Weak Defender). Similarly, p represents the probability that the Defender is Tough. For reference, in our preceding examples, F was 10, G was 5, and p was 0.25; C was originally 6 and later 4.

Using this general payoff structure, we could redo the analysis in Sections 5.A through 5.C and derive conditions on the parameters C, F, G, and p under which the various candidate outcomes are equilibria. Recall that in Section 5.A we found a separating equilibrium in which T-type Defenders choose S, W types choose NS, and the Attacker chooses NI if he observes S and I if he observes NS. Following the same steps of reasoning, we can see that the Attacker's inferences are correct and his choice is optimal, given the Defender's strategy, because $G > 0$. A type-T Defender choosing S draws the response NI and gets $-C$; choosing NS would have drawn I and yielded $-F$. Therefore S is optimal for the type-T Defender as long as $-C > -F$, or $C < F$. A similar calculation shows that NS is optimal for a type-W Defender if $-G > -C$, or $G < C$. Putting it all together, we see that, for the general game in Figure 9.8, such a separating equilibrium is possible when $G < C < F$; note that our chosen numbers ($G = 5$, $C = 6$, $F = 10$) in Figure 9.6 do meet this restriction.

Similar calculations can be done for other conceivable types of equilibria in each category. We tabulate the results in Figure 9.9. For each type of equilibrium—separating, pooling, and semiseparating—the table shows sets of

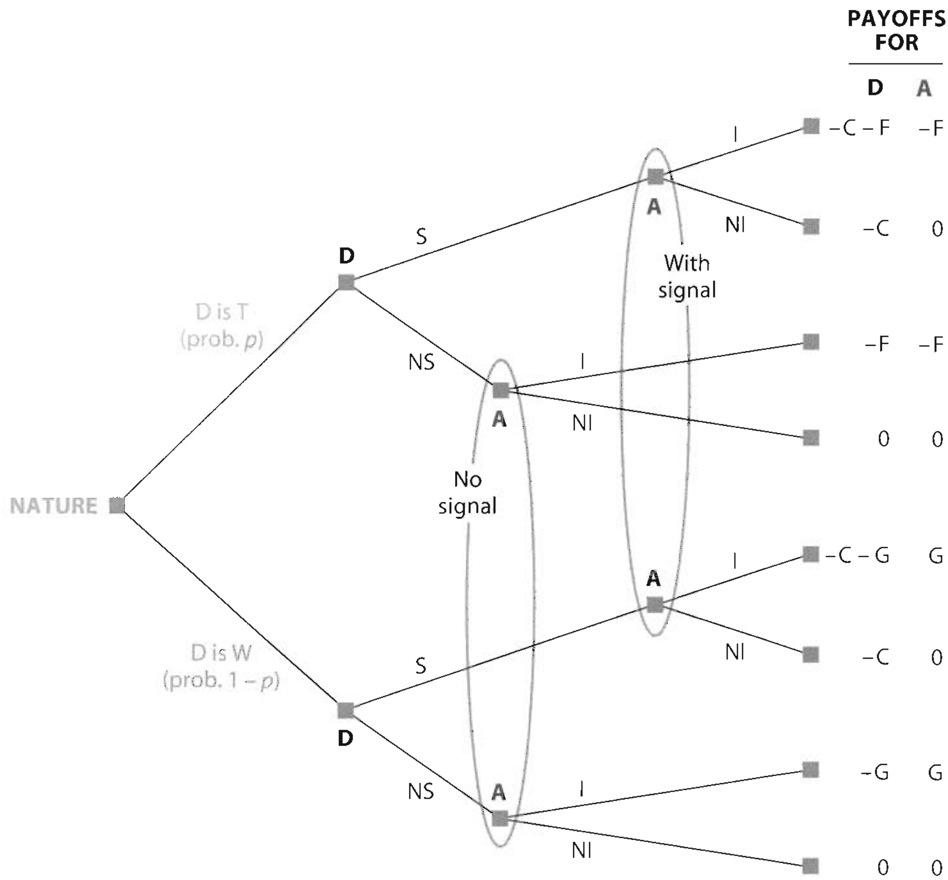


FIGURE 9.8 General Version of Signaling Game

strategies that might meet the criteria for a PBE of that type and the parameter values for which those strategies do meet those criteria. Algebraically adept readers can do the calculations as an exercise. Others can take the details of calculation for granted and merely note some useful ideas that emerge from the table.

First, the ranges of values of the payoffs and probabilities that yield the different types of equilibria are not mutually exclusive. Thus it is possible that a given combination of C, F, G, and p permits multiple equilibria. As an obvious example, a pooling equilibrium in which no type of Defender signals and the Attacker always invades is possible for any C, F, G, and p if a deviant choice of S is interpreted as a mistake that could have been made with equal likelihood by either type of Defender. Therefore such an equilibrium can coexist with any other type that might exist for particular values of C, F, G, and p. More substantively, it is possible to satisfy both G < C < F (for separation) and C < F with $(1 - p)$G < pF (for semiseparation) at the same time. And it is possible to have C < G and $(1 - p)$G > pF and get either pooling or semiseparation.

Equilibrium type	Equilibrium Strategies			Equilibrium exists
	Defender		Attacker	
	T type	W type		
Separating (fully)	S	NS	NI if S and I if NS	if $G < C < F$
	NS	S	I if S and NI if NS	never
Pooling	NS	NS	No updating if S; I if $(1 - p)G > pF$ and NI if $(1 - p)G < pF$	for any G, C, F, or p
			Assume T if S; always choose I	for $C > F$ and $C > G$, and $(1 - p)G > pF$
	S	S	No updating if NS; I if $(1 - p)G > pF$ and NI if $(1 - p)G < pF$	never
			Assume W if NS; I if $(1 - p)G > pF$ and NI if $(1 - p)G < pF$	for $C < F$ and $C < G$, and $(1 - p)G < pF$
Semiseparating	S	Mix (S with prob. x)	Update using p and x; mix if S (I with prob. y) and I if NS	if $C < G$, and $(1 - p)G > pF$
	Mix (S with prob. x)	NS	Update using p and x; mix if NS (I with prob. y) and NI if S	if $C < F$, and $(1 - p)G < pF$

FIGURE 9.9 General Results for Equilibria in Signaling Game

We came across multiplicity of equilibria in other contexts. For example, the game of chicken in Chapters 4 and 7 had two equilibria in pure strategies and one in mixed strategies. We pointed out that some other principle or procedure—focal point, commitment device, and so forth—is needed to select an outcome from the set of possible equilibria. The same is true in the present context. Numerous "refinements" of perfect Bayesian equilibrium have been proposed for this purpose, but all are mathematically difficult, and the choice among them is not always clear. Therefore we will leave the study of equilibrium refinements to more advanced levels of game theory.

Next, consider why signaling "works" in this example—fully in the separating equilibria or partially in the semiseparating ones. As usual, there must be some difference between the types. However, we have assumed that the direct cost of signaling C is the same for the two types of Defenders in this game. The difference in behavior arises in equilibrium because of the interaction of beliefs and actions. For example, in the fully separating equilibrium, S draws the response NI and NS draws I. Therefore the difference in payoffs between choosing

S and choosing NS is $(-C) - (-F) = F - C$ for the T type, and $(-C) - (-G) = G - C$ for the W type. Because $G < C < F$ in this case, S is better than NS for the T type but worse for the W type.

Finally, we can resolve Jervis's paradox in the light of our findings. He argued that a weak defender has the greater need to avoid invasion, and so perhaps the signal should be interpreted as a sign of weakness. A fully separating outcome of this kind is in the second row of the table in Figure 9.9, but we see that it cannot be an equilibrium for any combination of C, F, G, and p. The intuition is that the signal is costly and, if the Attacker is going to interpret it in a way unfavorable to the Defender, a rational Defender of either type would not find it optimal to give such a signal.

But the semiseparating equilibrium where the Defender chooses to signal if T type and not to signal if W type (the row second from the bottom in the table of Figure 9.9) has some features of Jervis's argument. The weak want to appear strong but can do so only partially. The same happens among Elijah Anderson's inner-city youth. He finds that many, but not all, of the "decent" ones who want to succeed in mainstream society nevertheless have the ability to "code-switch" and "develop a repertoire of behaviors" that provide them security in the "street."[19] This is an example of the weak partially mimicking the tough.

6 EVIDENCE ABOUT SIGNALING AND SCREENING

As we saw in Section 5, the characterization and solution of perfect Bayesian equilibria for games of signaling and screening entail some quite subtle concepts and computations. Should we expect players to perform such calculations correctly? How realistic are these equilibria as descriptions of the outcomes of these games?

There is ample evidence that people are very bad at performing calculations that include probabilities and are especially bad at conditioning probabilities on new information.[20] These calculations are exactly the ones that must be performed to update one's information on the basis of observed actions. Therefore we should be justifiably suspicious of equilibria that depend on the players doing so. Relative to this expectation, the findings of economists who have conducted laboratory ex-

[19]Anderson, *Code of the Street*, p. 36.

[20]Deborah J. Bennett, *Randomness* (Cambridge: Harvard University Press, 1998), pp. 2–3 and Chapter 10. See also Paul Hoffman, *The Man Who Loved Only Numbers* (New York: Hyperion, 1998), pp. 233–240, for an entertaining account of how several probability theorists, as well as the brilliant and prolific mathematician Paul Erdös, got a very simple probability problem wrong and even failed to understand their error when it was explained to them.

periments of signaling games are encouraging. Some surprisingly subtle refinements of perfect Bayesian equilibrium are successfully observed, even though these refinements require not only updating of information by observing actions along the equilibrium path, but also deciding how one would infer information from off-equilibrium actions that should never have been taken in the first place. However, the verdict of the experiments is not unanimous; much seems to depend on the precise details of the laboratory design of the experiment.[21]

Although the equilibria of signaling and screening games can be quite subtle and complex, the basic idea of the role of signaling or screening to elicit information is very simple—players of different "types" (that is, possessing different information about their own characteristics or about the game and its payoffs more generally) should find it optimal to take different actions so that their actions truthfully reveal their types. One can point to numerous practical applications of this idea. Here are some examples; once you start thinking along these lines you should be able to come up with dozens more.

1. Insurance companies usually offer a spectrum or menu of policies, with different provisions for deductible amounts and coinsurance. Those customers who know themselves to be particularly prone to risk prefer the policies with low deductibles and coinsurance, while those who know themselves to be less risky are more willing to take the policies stipulating that they have to bear more of the losses. Thus the different risk classes have different optimal actions, and the insurance companies use their clients' self-selection to screen and elicit the risk class of any particular client.

2. Venture capitalists are looking for inventors with good ideas, but how are they to judge the quality of any particular idea? They look for the inventor's willingness to "put his money where his mouth is"; they look for inventors who will take up as large an equity participation in the venture as their financial situation permits. Thus the willingness to bear a part of any loss from the project becomes a credible indicator of the inventor's belief that his project will turn a profit.

3. The quality of durable goods such as cars and computers is hard to evaluate for consumers, but each manufacturer has a much better idea of the quality of his own product. And, if a product is of higher quality, it is less costly for the maker to offer a longer or better warranty on it. Therefore warranties can serve as signals of quality, and consumers are intuitively quite aware of this when they make their purchase decisions.

4. Finally, an example from biology.[22] In many species of birds, the males have very elaborate and heavy plumage that females find attractive. One

[21]Douglas D. Davis and Charles A. Holt, *Experimental Economics* (Princeton: Princeton University Press, 1995), review and discuss these experiments in their Chapter 7.

[22]Matt Ridley, *The Red Queen: Sex and the Evolution of Human Nature* (New York: Penguin Books, 1995), p. 148.

should expect the females to seek genetically superior males so that their offspring will be better equipped to survive and attract mates in their turn. But why does heavy plumage indicate such desirable genetic qualities? One would think it would be a handicap, making the bird more visible to predators (including human hunters) and less easily mobile and therefore less able to evade these predators. Why do females choose these handicapped males? The answer comes from the conditions for credible signaling. Although heavy plumage is a handicap, it is less of a handicap to a male who is genetically sufficiently superior in qualities such as strength and speed. The weaker the male, the harder it is for him to produce and maintain plumage of a given quality. Thus it is precisely the heaviness of the plumage that makes it a credible signal of the male's quality.

As you can see from these examples and others, such as the dating story in Chapter 1, information asymmetry is everywhere, and strategies to deal with it are an important part not only of the science of game theory, but also of the art of strategy in everyday life. We hope that you will be intrigued by these ideas and will want to learn more than we can offer you at the elementary level of this book. For this purpose, we suggest the following additional readings:

From sociology: Erving Goffman, *The Presentation of Self in Everyday Life,* revised edition (New York: Anchor Books, 1959).

From biology: Matt Ridley, *The Red Queen: Sex and the Evolution of Human Behavior* (New York: Penguin Books, 1995).

From economics: A. Michael Spence, *Market Signaling* (Cambridge: Harvard University Press, 1974).

And, finally, also from economics, two graduate-level textbooks, strictly for the very ambitious who want to know the higher reaches of game theory: Drew Fudenberg and Jean Tirole, *Game Theory* (Cambridge: MIT Press, 1991), chapters 6–8; and David Kreps, *A Course in Microeconomic Theory* (Princeton: Princeton University Press, 1990), chapters 13, 16, and 17.

SUMMARY

Players with private information may want to conceal or reveal that information, while those without the information try to elicit it or avoid it. Actions speak louder than words in the presence of *asymmetric information.* To reveal information, a *credible signal* is required. *Signaling* works only if the signal action entails different costs to players with different information. To obtain information, when questioning is not sufficient to elicit truthful information, a *screening* scheme that looks for a specific action may be required. Screening works only if the *screening device* induces others to reveal their types truthfully; there must be *incentive compatibility* to get *separation*. Information asymmetries may be costly to resolve, and external effects may arise as a result of the resolution process.

In the equilibrium of a game with imperfect information, players must not only use their best actions given their information, but also draw correct inferences (update their information) by observing the actions of others. This type of equilibrium is known as a *perfect Bayesian equilibrium*. The outcome of such a game may entail pooling, separation, or *partial separation*, depending on the specifics of the payoff structure and the specified updating processes used by players. In some parameter ranges, such games may have multiples types of PBE.

The evidence on players' abilities to achieve perfect Bayesian equilibria seems to suggest that, despite the difficult probability calculations necessary, such equilibria are often observed. Different experimental results appear to depend largely on the design of the experiment. Many examples of signaling and screening games can be found in ordinary situations such as the provision of insurance or the issuing of government bonds.

KEY TERMS

babbling equilibrium (268)
cheap talk equilibrium (266)
credibility (265)
incentive-compatability constraint (273)
incentive scheme (266)
moral hazard (000)
partially revealing equilibrium (281)
participation condition (constraint) (278)
perfect Bayesian equilibrium (PBE) (284)
pooling of types (274)
screening (266)
screening device (266)
self-selection (273)
semiseparating equilibrium (281)
separation of types (273)
signal (265)
signaling (265)
signal jamming (266)

EXERCISES

1. In a television commercial for a well-known brand of instant cappuccino, a gentleman is entertaining a lady friend at his apartment. He wants to impress her and offers her cappuccino with dessert. When she accepts, he goes into the kitchen to make the instant cappuccino—simultaneously tossing

take-out boxes into the trash and faking the noises made by a high-class (and expensive) espresso machine. As he is doing so, a voice comes from the other room: "I want to see the machine . . ."

Use your knowledge of games of asymmetric information to comment on the actions of these two people. Pay attention to their attempts to use signaling and screening, and point out specific instances of each strategy. Offer an opinion about which player is the better strategist.

2. "Mr. Robinson pretty much concludes that business schools are a sifting device—M.B.A. degrees are union cards for yuppies. But perhaps the most important fact about the Stanford business school is that all meaningful sifting occurs before the first class begins. No messy weeding is done within the walls. 'They don't want you to flunk. They want you to become a rich alum who'll give a lot of money to the school.' But one wonders: If corporations are abdicating to the Stanford admissions office the responsibility for selecting young managers, why don't they simply replace their personnel departments with Stanford admissions officers, and eliminate the spurious education? Does the very act of throwing away a lot of money and two years of one's life demonstrate a commitment to business that employers find appealing?" (From the review by Michael Lewis of Peter Robinson's *Snapshots from Hell: The Making of an MBA*, in the *New York Times*, May 8, 1994, Book Review section.) What answer to Lewis's question can you give based on our analysis of strategies in situations of asymmetric information?

3. Firms that provide insurance to clients to protect them from the costs associated with theft or accident must necessarily be interested in the behavior of their policyholders. Sketch some ideas for the creation of an incentive scheme that such a firm might use to detect fraud or lack of care on the part of its policyholders.

4. Suppose electricians come in two types: competent and incompetent. Both types of electricians can get certified, but for the incompetent types certification takes extra time and effort. Competent ones have to spend C months preparing for the certification exam; incompetent ones take twice as long. Certified electricians can earn 100 (thousand dollars) each year working on building sites for licensed contractors. Uncertified electricians can earn only 25 (thousand dollars) each year in freelance work; licensed contractors won't hire them. Each type of electrician gets a payoff equal to $\sqrt{S} - M$, where S is the salary measured in thousands of dollars and M is the number of months spent getting certified. What is the range of values of C for which a competent electrician will choose to signal with this device but an incompetent one will not?

5. The managerial effort example of Section 4 of this chapter assumed that a successful project would earn \$600,000 for the firm. Using the same values for the rest of the example (probability of success of the project is 60% with routine quality managerial effort, manager requires \$50,000 extra payment for extra effort, and so forth), show that if a negative base salary is not feasible, then the firm does better to settle for the low-pay, low-effort situation when the value of the successful project is only \$400,000.

6. An economy has two types of jobs, Good and Bad, and two types of workers, Qualified and Unqualified. The population consists of 60% Qualified and 40% Unqualified. In a Bad job, either type of worker produces 10 units of output. In a Good job, a Qualified worker produces 100 units and an Unqualified worker produces 0. There is enough demand for workers that companies must pay for each type of job what they expect that the appointee will produce.

   Companies cannot directly observe a worker's type before hiring, but Qualified workers can signal their qualification by getting educated. For a Qualified worker, the cost of getting educated to level n is $n^2/2$; while, for an Unqualified worker, it is n^2. These costs are measured in the same units as output, and n must be an integer.

   (a) What is the minimum level of n that will achieve separation?

   (b) Now suppose the signal is made unavailable. Which kind of jobs will be filled by which kinds of workers and at what wages? Who will gain and who will lose from this change?

7. Mictel corporation has a world monopoly on the production of personal computers. It can make two kinds of computers: low-end and high-end. The total population of prospective buyers is P. There are two types of prospective buyers: casual users and intensive users. The casual ones comprise a fraction c of the population, and the rest (fraction $1 - c$) are intensive users.

   The costs of production of the two kinds of machines, as well as the benefits gained from the two by the two types of prospective buyers, are given in the following table. All figures are in thousands of dollars.

		COST	BENEFIT FOR USER TYPE	
			Casual	Intensive
PC TYPE	Low-end	1	4	5
	High-end	3	5	8

   Each type of buyer calculates the net payoff (benefit *minus* price) that he would get from each type of machine and buys the type that would give the higher net payoff, provided that this payoff is nonnegative. If both types give equal nonnegative net payoffs for a buyer, he goes for the high end; if both types have negative net payoff for a buyer, he does not purchase. Mictel wants to maximize its expected profit.

(a) If Mictel were omniscient, then, when a prospective customer came along, knowing his type, the company could offer to sell him just one type of machine at a stated price, on a take-it-or-leave-it basis. What kind of machine would Mictel offer, and at what price, to what kind of buyer?

In fact, Mictel does not know the type of any particular buyer. It just makes its "catalog" available for all buyers to choose from.

(b) First suppose the company produces just the low-end machines and sells them for price x. What value of x will maximize its profit? How does the answer depend on c, the proportion of casual users in the population?

(c) Next suppose Mictel produces just the high-end machines and sells them for price y. What value of y will maximize its profit, and how does the answer depend on c?

(d) Finally, suppose the company produces both types of machines, selling the low-end ones for price x and the high-end ones for price y. What "incentive compatibility" constraints on x and y must the company satisfy if it wants the casual users to buy the low-end machines and the intensive users to buy the high-end machines? What is the company's expected profit from this policy? What values of x and y will maximize the expected profit? How does the answer depend on c?

(e) Putting it all together, what production and pricing policy should the company pursue? How does the answer depend on c?

8. Felix and Oscar are playing a simplified version of poker. Each makes an initial bet of 8 dollars. Then each separately draws a card, which may be High or Low with equal probabilities $\frac{1}{2}$ each. Each sees his own card but not that of the other.

Then Felix decides whether to Pass or to Raise (bet an additional 4 dollars). If he chooses to pass, the two cards are revealed and compared. If the outcomes are different, the one who has the High card collects the whole pot. The pot has 16 dollars, of which the winner himself contributed 8, and so his winnings are 8 dollars; the loser's payoff is -8 dollars. If the outcomes are the same, the pot is split equally and each gets his 8 dollars back (payoff 0).

If Felix chooses Raise, then Oscar has to decide whether to Fold (concede) or See (match with his own additional 4 dollars). If Oscar chooses Fold, then Felix collects the pot irrespective of the cards. If Oscar chooses

See, then the cards are revealed and compared. The procedure is the same as that in the preceding paragraph, but the pot is now bigger.

(a) Show the game in extensive form. (Be careful about information sets.)

If the game is instead written in the normal form, Felix has four strategies: (1) Pass always (PP for short), (2) Raise always (RR), (3) Raise if his own card is High and Pass if it is Low (RP), and (4) the other way round (PR). Similarly, Oscar has four strategies: (1) Fold always (FF), (2) See always (SS), (3) See if his own card is High and Fold if it is Low (SF), and (4) the other way round (FS).

(b) Show that the table of payoffs to Felix is as follows:

		OSCAR			
		FF	SS	SF	FS
FELIX	PP	0	0	0	0
	RR	8	0	1	7
	RP	2	1	0	3
	PR	6	−1	1	4

(In each case you will have to take an expected value by averaging over the consequences for each of the four possible combinations of the card draws.)

(c) Eliminate dominated strategies as far as possible. Find the mixed-strategy equilibrium in the remaining table and the expected payoff to Felix in the equilibrium.

(d) Use your knowledge of the theory of signaling and screening to explain intuitively why the equilibrium has mixed strategies.

9. For discussion: The design of a health-care system concerns matters of information and strategy at several points. The users—potential and actual patients—have better information about their own state of health, life style, and so forth—than the insurance companies can find out. The providers—doctors, hospitals, and so forth—know more about what the patients need than do either the patients themselves or the insurance companies. Doctors also know more about their own skills and efforts, and hospitals about their own facilities. Insurance companies may have some statistical information about outcomes of treatments or surgical procedures from their past records. But outcomes are affected by many unobservable and random factors; so the underlying skills, efforts or facilities cannot be inferred perfectly from observations of the outcomes. The pharmaceutical companies know

more about the efficacy of drugs than do the others. As usual, the parties do not have natural incentives to share their information fully or accurately with others. The design of the overall scheme must try to face these matters and find the best feasible solutions.

Consider the relative merits of various payment schemes—fee for service versus capitation fees to doctors, comprehensive premiums per year versus payment for each visit for patients, and so forth—from this strategic perspective. Which are likely to be most beneficial to those seeking health care? To those providing health care? Think also about the relative merits of private insurance and coverage of costs from general tax revenues.

10. [Optional] A teacher wants to find out how confident the students are about their own abilities. He proposes the following scheme: "After you answer this question, state your estimate of the probability that you are right. I will then check your answer to the question. Suppose you have given the probability estimate x. If your answer is actually correct, your grade will be $\log(x)$. If incorrect, it will be $\log(1 - x)$." Show that the scheme will elicit the students' own estimates truthfully; that is, if the truth is p, show that a student's estimate $x = p$.

Appendix: Information and Risk

When players have different amounts of information in a game, they will try to use some device to ascertain their opponents' private information. As we saw in Section 3 of Chapter 9, it is sometimes possible for direct communication to yield a cheap talk equilibrium. But more often, players will need to determine one anothers' information by observing one anothers' actions. They then must estimate the probabilities of the underlying information by using those actions or their observed consequences. This estimation requires some relatively sophisticated manipulation of the rules of probability, and we examine this process in detail in Section 1 of this appendix.

Similarly, when the outcomes of an action are risky, there arise new aspects of strategy that have to do with manipulation of risk itself. A player who is averse to risk can benefit from devising strategies that reduce this risk. Sometimes this can be done if two or more such players can get together and combine their risk. In some games, particularly those in which the players are pitted against one another in a contest, their strategies toward risk interact in interesting ways. We consider these matters in Section 2.

1 INFERRING PROBABILITIES FROM OBSERVING CONSEQUENCES

The rules given in Section 1 of the Appendix to Chapter 7 for manipulating and calculating the probability of events, particularly the combination rule, prove useful in our calculations of payoffs when individual players are differently informed. In games of asymmetric information, players try to find out the others' information by observing their actions. Then they must draw inferences about the likelihood of—estimate the probabilities of—the underlying information by using the actions or consequences that are observed.

The best way to understand this is by example. Suppose 1% of the population has a genetic defect that can cause a disease. A test that can identify this genetic defect has a 99% accuracy: when the defect is present, the test will fail to detect it 1% of the time, and the test will also falsely find a defect when none is present 1% of the time. We are interested in determining the probability that the defect is really there in a person with a positive test result. That is, we cannot directly observe the person's genetic defect (underlying condition), but we can observe the results of the test for that defect (consequences)—except that the test is not a perfect indicator of the defect; how certain can we be, given our observations, that the underlying condition does in fact exist?

We can do a simple numerical calculation to answer the question for our particular example. Consider a population of 10,000 persons in which 100 (1%) have the defect and 9,900 do not. Suppose they all take the test. Of the 100 persons with the defect, the test will be (correctly) positive for 99. Of the 9,900 without the defect, it will be (wrongly) positive for 99. That is 198 positive test results of which one-half are right and one-half are wrong. If a random person receives a positive test result, it is just as likely to be because the test is indeed right as because the test is wrong, so the risk that the defect is truly present for a person with a positive result is only 50%. (That is why tests for rare conditions must be designed to have especially low error rates of generating "false positives.")

For general questions of this type, we use an algebraic formula called **Bayes' theorem** to help us set up the problem and do the calculations. To do so, we generalize our example, allowing for two alternative underlying conditions, A and B (genetic defect or not, for example), and two observable consequences, X and Y (positive or negative test result, for example). Suppose that, in the absence of any information (over the whole population), the probability that A exists is p; so the probability that B exists is $(1 - p)$. When A exists, the chance of observing X is a; so the chance of observing Y is $(1 - a)$. (To use the language that we developed in the Appendix to Chapter 7, a is the probability of X conditional on A, and $(1 - a)$ is the probability of Y conditional on A.) Similarly, when B exists, the chance of observing X is b; so the chance of observing Y is $(1 - b)$.

This description shows us that there are four alternative combinations of events that could arise: (1) A exists and X is observed, (2) A exists and Y is observed, (3) B exists and X is observed, and (4) B exists and Y is observed. Using the modified multiplication rule, we find the probabilities of the four combinations to be, respectively, pa, $p(1 - a)$, $(1 - p)b$, and $(1 - p)(1 - b)$.

Now suppose that X is observed; a person has the test for the genetic defect and gets a positive result. Then we restrict our attention to a subset of the four preceding possibilities—namely, the first and third, both of which include the observation of X. These two possibilities have a total probability of $pa + (1 - p)b$; this is the probability that X is observed. Within this subset of outcomes in which X is observed, the probability that A *also* exists is just pa, as we have already seen. So we know how likely we are to observe X alone and how likely it is that both X and A exist.

But we are more interested in determining how likely it is that A exists, given that we have observed X—that is, the probability that a person has the genetic defect, given that the test is positive. This calculation is the trickiest one. Using the modified multiplication rule, we know that the probability of both A and X happening equals the product of the probability that X alone happens times the probability of A conditional on X; it is this last probability that we are after. Using the formulas for "A and X" and for "X alone," which we just calculated, we get:

$$\begin{aligned}
\text{Prob}(A \text{ and } X) &= \text{Prob}(X \text{ alone}) \times \text{Prob}(A \text{ conditional on } X)\\
pa &= [\, pa + (1 - p)b \,] \times \text{Prob}(A \text{ conditional on } X)\\
\text{Prob}(A \text{ conditional on } X) &= \frac{pa}{pa + (1 - p)b.}
\end{aligned}$$

This formula gives us an assessment of the probability that A has occurred, given that we have observed X (and have therefore conditioned everything on this fact). The outcome is known as *Bayes' theorem* (or rule or formula).

In our example of testing for the genetic defect, we had $\text{Prob}(A) = p = 0.01$, $\text{Prob}(X \text{ conditional on } A) = a = 0.99$ and $\text{Prob}(Y \text{ conditional on } A) = b = 0.01$. We can substitute these values into Bayes' formula to get

$$\begin{aligned}
\text{Probability defect exists given that test is positive} &= \text{Prob}(A \text{ conditional on } X)\\
&= \frac{(0.01)(0.99)}{(0.01)(0.99) + (1 - 0.01)(0.01)}\\
&= \frac{0.0099}{0.0099 + 0.0099}\\
&= 0.5.
\end{aligned}$$

The probability algebra employing Bayes' rule confirms the arithmetical calculation that we used earlier that was based on an enumeration of all of the possible cases. The advantage of the formula is that, once we have

		OBSERVATION		Sum of row
		X	Y	
TRUE CONDITION	A	pa	$p(1-a)$	p
	B	$(1-p)b$	$(1-p)(1-b)$	$1-p$
Sum of column		$pa+(1-p)b$	$p(1-a)+(1-p)(1-b)$	

FIGURE 9A.1 Bayes' Rule

it, we can apply it mechanically; this saves us the lengthy and error-prone task of enumerating every possibility and determining each of the necessary probabilities.

We show Bayes' rule in Figure 9A.1 in tabular form, which may be easier to remember and to use than the preceding formula. The rows of the table show the alternative true conditions that might exist, for example, "genetic defect" and "no genetic defect." Here, we have just two, A and B, but the method generalizes immediately to any number of possibilities. The columns show the observed events—for example, "test positive" and "test negative."

Each cell in the table shows the overall probability of that combination of the true condition and the observation; these are just the probabilities for the four alternative combinations listed earlier. The last column on the right shows the sum across the first two columns for each of the top two rows. This sum is the total probability of each true condition (so, for instance, A's probability is p, as we have seen). The last row shows the sum of the first two rows in each column. This sum gives the probability that each observation occurs. For example, the entry in the last row of the X column is the total probability that X is observed, either when A is the true condition (a true positive in our genetic test example) or when B is the true condition (a false positive).

To find the probability of a particular condition, given a particular observation, then, Bayes' rule says that we should take the entry in the cell corresponding to the combination of that condition and that observation and divide this by the column sum in the last row for that observation. As an example, Prob (B given X) = $(1 - p)b/[pa + (1 - p)b]$.

EXERCISE In the genetic test example, suppose the test comes out negative (Y is observed). What is the probability that the person does not have the defect (B exists)? Calculate this probability by applying Bayes' rule, and then check your answer by doing an enumeration of the 10,000 members of the population.

2 CONTROLLING AND MANIPULATING RISK

A. Strategies to Reduce Risk

Imagine that you are a farmer subject to the vagaries of weather. If the weather is good for your crops, you will have an income of $15,000. If it is bad, your income will be only $5,000. The two possibilities are equally likely (probability 1/2, or 0.5, or 50% each). Therefore your average or expected income is $10,000, but there is considerable risk around this average value.

What can you do to reduce the risk that you face? You might try a crop that is less subject to the vagaries of weather, but suppose you have already done all such things that are under your individual control. Now you can reduce your income risk further only by combining it with other people's risks. There are two ways to do so: pooling and trading.

We begin with **pooling risks.** Suppose your neighbor faces a similar risk but gets good weather exactly when you get bad weather and vice versa. (Suppose you live on opposite sides of an island, and rain clouds visit one side or the other but not both.) In technical jargon, correlation is a measure of alignment between any two uncertain quantities—in this discussion, between one person's risk and another's. Thus we would say that your neighbor's risk is totally **negatively correlated** with yours. Then your combined income is $20,000, no matter what the weather: it is totally risk free. You can enter into a contract that gets each of you $10,000 for sure: you promise to give him $5,000 in years when you are lucky, and he promises to give you $5,000 in years when he is lucky. You have eliminated your risks by pooling them together.

Even without such perfect correlation, risk pooling has some benefit. Suppose the farmers' risks are independent, as if the rain clouds could toss a separate coin to decide whether to visit each of you. Then there are four possible outcomes, each with a probability of 1/4. In the first case, both of you are lucky, and your combined incomes total $30,000. In the second and third cases, one of you is lucky and the other is not (the two cases differ only in regard to which one is which); in both of these situations, your combined incomes total $20,000. In the fourth outcome, both of you are unlucky, and your combined incomes amount to only $10,000. Facing these possibilities, if the two of you make a contract to share and share alike, each of you will have $15,000 with a probability of 1/4, $5,000 with a probability of 1/4, and $10,000 with a probability of 1/2. Without the trade, each would have $15,000 or $5,000 with probabilities of 1/2 each. Thus, for each of you, the contract has reduced the probabilities of the two extreme outcomes from 1/2 to 1/4 and increased the probability of the middle outcome from 0 to 1/2. In other words, the contract has reduced the risk for each of you.

In fact, so long as the two farmers' incomes are not totally **positively correlated** —that is, their luck does not move in perfect tandem—they can reduce mutual risk by pooling. And, if there are more than two farmers with some degree of independence in their risks, then even greater reduction in the risk of each is made possible by the law of large numbers.

This risk reduction is the whole basis of insurance: by pooling together the similar but independent risks of many people, an insurance company is able to compensate any one of them when he suffers a large loss. It is also the basis of portfolio diversification: by dividing your wealth among many different assets with different kinds and degrees of risk, you can reduce your total exposure to risk.

However, several complexities limit the possibility of risk reduction by pooling. At the simplest, the pooling promise may not be credible: the farmer who gets the good luck may then refuse to share it. This is relatively easily overcome if a legal system enforces contracts, provided its officials can observe the farmers' outcomes in order to decide who should give how much to whom. The premiums that people pay to the insurance company are advance guarantors of their promise, like setting aside money in an escrow account. Then the insurance company's desire to maintain its reputation in an ongoing business ensures that it will pay out when it should. However, if outcomes cannot be observed by others, then people may try to defraud the company by pretending to have suffered a loss. Or, if people's risks can be influenced by their actions, they may become careless knowing they are insured; this is called moral hazard. Or, if the insurance company cannot know as much about the risks (say, of health) as do the insured, then those people who face the worst risks may choose to insure more, thereby worsening the average risk pool and raising premiums for all; this is called **adverse selection.** We will study some such problems in more detail soon. For now, we merely point out that, to cope with these problems, insurance companies usually provide only partial insurance, requiring you to bear part of the loss yourself. The part that you have to bear consists of a *deductible* (an initial amount of the loss) and *coinsurance* (an additional fraction of the loss beyond the deductible).

Next consider **trading risks.** Suppose you are the farmer facing the same risk as before. But now your neighbor has a sure income of $10,000. You face a lot of risk and he faces none. He may be willing to take a little of your risk, for a price that is agreeable to both of you.

To understand and calculate this price, we must allow for *risk aversion.* Suppose both of you are risk averse. As we saw in the Appendix to Chapter 5, your attitudes toward risk can be captured by using a concave scale to convert your money incomes into "utility" numbers. In the appendix, we used the square root as a simple example of such a scale; let us continue to do so here. Suppose you pay your neighbor $100 up front, and in exchange he agrees to the following deal with you: after your crop is harvested and if you have been unlucky, he will give you \$2,000, but, if you have proved to be lucky, *you* will give *him* \$2,000. In other words, he agrees to assume \$2,000 of your risk in exchange for a sure payment of \$100. The end result is that his income will be \$10,000 + \$100 + \$2,000 = \$12,100 if you are lucky, and \$10,000 + \$100 − \$2,000 = \$8,100 if you are unlucky. His expected utility is

$$0.5 \times \sqrt{12{,}100} + 0.5 \times \sqrt{8{,}100} = 0.5 \times 110 + 0.5 \times 90 = 100.$$

This is the same as the utility that he would get if he did not trade with you: $\sqrt{10{,}000} = 100$. Thus he is just willing to make this trade with you. Would you be willing to pay him the \$100 up front to have him assume \$2,000 of your risk?

Your expected utility from the risky income prospect that you face is

$$0.5 \times \sqrt{5{,}000} + 0.5 \times \sqrt{15{,}000} = 0.5 \times 70.7 + 0.5 \times 122.5 = 96.6.$$

If you pay your neighbor \$100 and he gives you \$2,000 when your luck is bad (so that he is left with \$8,100), you will have \$6,900. In return, when your luck is good, you are to give him \$2,100 (\$100 of initial payment plus \$2,000 from your high income), and so he has the \$12,100 he needs to compensate him for the risk, and you are left with \$12,900. Your expected utility with risk trading is

$$0.5 \times \sqrt{6{,}900} + 0.5 \times \sqrt{12{,}900} = 0.5 \times 83.0 + 0.5 \times 113.6 = 98.3.$$

Thus the arrangement has given you a greater expected utility while leaving your neighbor just as happy.

Your neighbor might suggest another arrangement, which would give him a higher expected utility while keeping yours at the original 96.6. There are other deals in between where both of you have a higher expected utility than you would without trading risk. Thus there is a whole range of contractual arrangements, some favoring one party and some the other, and you can bargain over them. The precise outcome of such bargains depends on the parties' bargaining power; we will study bargaining in some detail in Chapter 17. If there are many people, then an impersonal market (examined in Chapter 18) can determine the price of risk. Common to all such arrangements is the idea that mutually beneficial deals can be struck whereby, for a suitable price, someone facing less risk takes some of the risk off the shoulders of someone else who faces more.

In fact, the idea that there exists a price and a market for risk is the basis for almost all the financial arrangements in a modern economy. Stocks and bonds, as well as all the complex financial instruments such as derivatives, are just ways of spreading risk to those who are willing to bear it for the least asking price. Many people think these markets are purely forms of gambling. In a sense, they are. But the gambles are taken by those who start out with the least risk, perhaps because they have already diversified in the way that we saw earlier. And the risk is sold or shed by those who are initially most exposed to it. This enables the latter to be more adventurous in their enterprises than they would be if they had to bear all of the risk themselves. Thus financial markets promote entrepreneurship by facilitating risk trading.

These markets are subject to similar kinds of moral hazard, adverse selection, and even outright fraud, as is risk pooling. Therefore these markets are not able to do a really full job of spreading risk. For example, managers are often required to bear some of the risk created by their decisions, through equity participation (stock ownership) or some other mechanism, to give them the right incentive to make an effort.

B. Using Risk

Risk is not always bad; in some circumstances it can be used to one's advantage. Although the general theory would take us too far afield, we offer a brief example of a prominent instrument that uses risk—namely, an **option.**

An option gives the holder the right but not the obligation to take some stipulated action. For example, a call option on a stock allows the holder to buy the stock at a preset price, say 110. Suppose the stock now trades at 100, and may go to 150 or 50 with equal probability. If the former, the holder will exercise the option and make a profit of 40; if the latter, he will let the option lapse and make 0. Therefore the expected profit is

$$0.5 \times 40 + 0.5 \times 0 = 20.$$

If the uncertainty is larger—that is, the stock can go to, say, 200 or 0 with equal probabilities—the expected profit from the option is

$$0.5 \times 90 + 0.5 \times 0 = 45.$$

Thus the option is more valuable the greater the uncertainty in the underlying stock. The market recognizes this, and the price for which one can buy such an option itself fluctuates as the volatility of the market changes.

C. Manipulating Risk in Contests

Our farmers in Section 2.A faced risk, but it was due to the weather rather than to any actions of their own or of other farmers. If, instead, the players in a game can affect the risk faced by themselves or others, then they can use such manipulation of risk strategically. A prime example is contests such as R&D races between companies to develop and market new information technology or biotech products; many sporting contests have similar features.

The outcome of sporting and related contests is determined by a mixture of skill and chance. You win if

$$\text{Your skill} + \text{your luck} > \text{rival's skill} + \text{rival's luck}$$

or

$$\text{Your luck} - \text{rival's luck} > \text{rival's skill} - \text{your skill}.$$

Denote the left-hand side by the symbol L; it measures your "luck surplus." L is an uncertain magnitude; suppose its probability distribution is a normal, or bell, curve, as in Figure 9A.2. At any point on the horizontal axis, the height of the curve represents the probability that L takes on that value. Thus the area under this curve between any two points on the horizontal axis equals the probability that L lies between those points. Suppose your rival has more skill; so you are an underdog. Your "skill deficit," which equals the difference between your rival's skill and your skill, is therefore positive, as shown by the point S. You win if your luck surplus, L, exceeds your skill deficit, S. Therefore the area under the curve to the right of the point S, which is shaded in Figure 9A.2, represents your probability of winning. If you make the situation chancier, the bell curve will flatten because the probability of relatively high and low values of L increases while the probability of moderate values decreases. Then the area under the curve to the right of S also increases. In Figure 9A.2, the area under the original bell curve is shown by grey shading, and the larger area under the flatter bell curve by the red hatching. As the underdog, you should therefore adopt a strategy that flattens the curve. Conversely, if you are the favorite, you should try to reduce the element of chance in the contest.

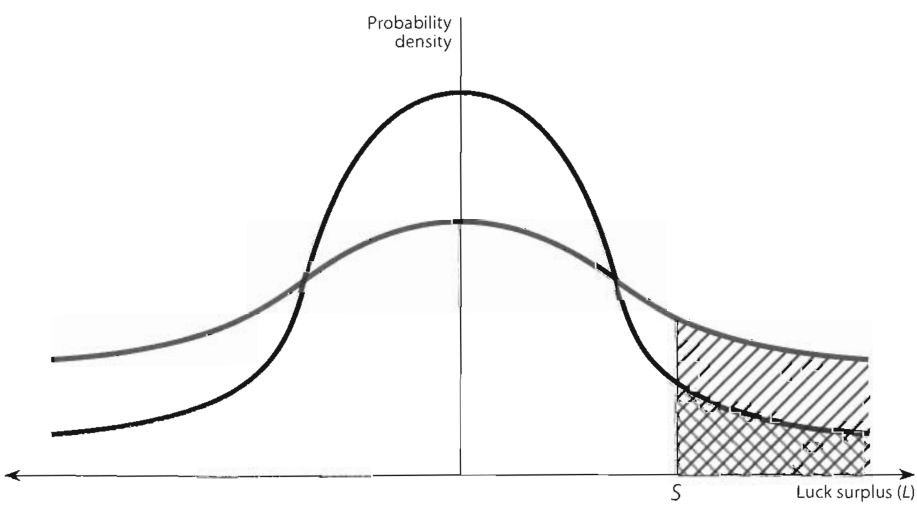


FIGURE 9A.2 The Effect of Greater Risk on the Chances of Winning

Thus we should see underdogs or those who have fallen behind in a long race try unusual or risky strategies: it is their only chance to get level or ahead. On the other hand, favorites or those who have stolen a lead will play it safe. A practical piece of advice based on this principle: if you want to challenge someone who is a better player than you to a game of tennis, choose a windy day.

You may stand to benefit by manipulating not just the amount of risk in your strategy, but also the correlation between the risks. The player who is ahead will try to choose a correlation as high and positive as possible: then, whether his own luck is good or bad, the luck of his opponent will be the same and his lead protected. Conversely, the player who is behind will try to find a risk as uncorrelated with that of his opponent as possible. It is well known that in a two-sailboat race, the boat that is behind should try to steer differently from the boat ahead, and the boat ahead should try to imitate all the tackings of the one behind.[1]

EXERCISE In the preceding risk-trading example, you had a risky income that was \$15,000 in good weather (probability 0.5) and \$5,000 in bad weather (probability 0.5). When your neighbor had a sure income of \$10,000, we derived a scheme in which you could keep him indifferent while raising your own expected utility. Construct a similar type of arrangement in which your neighbor keeps you at your original level of expected utility and he achieves a higher level of expected utility than he would in the absence of the arrangement.

SUMMARY

If players have asymmetric information in a game, they may try to infer probabilities of hidden underlying conditions from observing actions or the consequences of those actions. *Bayes' theorem* provides a formula for inferring such probabilities.

When game players have different attitudes toward risk or different amounts of information, there is scope for strategic behavior in the control and manipulation of risk and information. Players can reduce their risk through *trading* or *pooling* (combining), although the latter is complicated by *moral hazard* and *adverse selection*. Risk may be useful in some contexts, and it can be manipulated to a player's benefit, depending on the circumstances within the game.

[1]Avinash Dixit and Barry Nalebuff, *Thinking Strategically* (New York: Norton, 1991) give a famous example of the use of this strategy in sailboat racing. For a more general theoretical discussion, see Luis Cabral, "R&D Competition When the Firms Choose Variance," *Journal of Economics and Management Strategy*, vol. 12, no. 1 (Spring 2003), pp. 139–150.

KEY TERMS

adverse selection (305)
Bayes' theorem (301)
negatively correlated (304)
option (307)
pooling risk (304)
positively correlated (305)
trading risk (305)

10

Strategic Moves

A GAME IS specified by the choices or moves available to the players, the order, if any, in which they make those moves, and the payoffs that result from all logically possible combinations of all the players' choices. In Chapter 6, we saw how changing the order of moves from sequential to simultaneous or vice versa can alter the game's outcomes. Adding or removing moves available to a player or changing the payoffs at some terminal nodes or in some cells of the game table also can change outcomes. Unless the rules of a game are fixed by an outside authority, each player has the incentive to manipulate them to produce an outcome that is more to his own advantage. Devices to manipulate a game in this way are called **strategic moves,** which are the subject of this chapter.

A strategic move changes the rules of the original game to create a new two-stage game. In this sense, strategic moves are similar to the direct communications of information that we examined in Section 3 of Chapter 9. With strategic moves, though, the second stage is the original game, often with some alteration of the order of moves and the payoffs; there was no such alteration in our games with direct communication. The first stage in a game with strategic moves specifies how you will act in the second stage. Different first-stage actions correspond to different strategic moves, and we classify them into three types: *commitments, threats,* and *promises.* The aim of all three is to alter the outcome of the second-stage game to your own advantage. Which, if any, suits your purpose depends on the context. But, most importantly, any of the three works only if the other player believes that at the second stage you will indeed do what you declared at the first stage. In other words, the *credibility* of the strategic move is open to question. Only a credible strategic move will have the desired effect, and, as was often the case in Chapter 9, mere declarations are not enough. At the first stage, you must take some ancillary actions that lend credibility to your declared second-stage actions. We will study both the kinds of second-stage actions that work to your benefit and the first-stage ancillary moves that make them credible.

You are probably more familiar with the use and credibility of strategic moves than you might think. Parents, for instance, constantly attempt to influence the behavior of their children by using threats ("no dessert unless you finish your vegetables") and promises ("you will get the new racing bike at the end of the term if you maintain at least a B average in school"). And children know very well that many of these threats and promises are not credible; much bad behavior can escape the threatened punishment if the child sweetly promises not to do that again, even though the promise itself may not be credible. Furthermore, when the children get older and become concerned with their own appearance, they find themselves making commitments to themselves to exercise and diet; many of these commitments also turn out to lack credibility. All of these devices—commitments, threats, and promises—are examples of strategic moves. Their purpose is to alter the actions of another player, perhaps even your own future self, at a later stage in a game. But they will not achieve this purpose unless they are credible. In this chapter, we will use game theory to study systematically how to use such strategies and how to make them credible.

Be warned, however, that credibility is a difficult and subtle matter. We can offer you some general principles and an overall understanding of how strategic moves can work—a science of strategy. But actually making them work depends on your specific understanding of the context, and your opponent may get the better of you by having a better understanding of the concepts or the context or both. Therefore the use of strategic moves in practice retains a substantial component of art. It also entails risk, particularly when using the strategy of **brinkmanship,** which can sometimes lead to disasters. You can have success as well as fun trying to put these ideas into practice, but note our disclaimer and warning: Use such strategies at your own risk.

1 A CLASSIFICATION OF STRATEGIC MOVES

Because the use of strategic moves depends so critically on the order of moves, to study them we need to know what it means "to move first." Thus far we have taken this concept to be self-evident, but now we need to make it more precise. It has two components. First, your action must be **observable** to the other player; second, it must be **irreversible**.

Consider a strategic interaction between two players, A and B, in which A's move is made first. If A's choice is not observable to B, then B cannot respond to it, and the mere chronology of action is irrelevant. For example, suppose A and B are two companies bidding in an auction. A's committee meets in secret on Monday to determine its bid; B's committee meets on Tuesday; the bids are separately mailed to the auctioneer and opened on Friday. When B makes its decision, it does not know what A has done; therefore the moves are strategically the same as if they were simultaneous.

If A's move is not irreversible, then A might pretend to do one thing, lure B into responding, and then change its own action to its own advantage. B should anticipate this ruse and not be lured; then it will not be responding to A's choice. Once again, in the true strategic sense A does not have the first move.

Considerations of observability and irreversibility affect the nature and types of strategic moves as well as their credibility. We begin with a taxonomy of strategic moves available to players.

A. Unconditional Strategic Moves

Let us suppose that player A is the one making a strategic observable and irreversible move in the first stage of the game. He can declare: "In the game to follow, I will make a particular move, X." This declaration says that A's future move is unconditional; A will do X irrespective of what B does. Such a statement, if credible, is tantamount to changing the order of the game at stage 2 so that A moves first and B second, and A's first move is X. This strategic move is called a **commitment.**

If the previous rules of the game at the second stage already have A moving first, then such a declaration would be irrelevant. But, if the game at the second stage has simultaneous moves or if A is to move second there, then such a declaration, if credible, can change the outcome because it changes B's beliefs about the consequences of his actions. Thus a commitment is a simple seizing of the first-mover advantage when it exists.

In the street-garden game of Chapter 3, three women play a sequential-move game in which each must decide whether to contribute toward the creation of a public flower garden on their street; two or more contributors are necessary for the creation of a pleasant garden. The rollback equilibrium entails the first player (Emily) choosing not to contribute while the other players (Nina and Talia) do contribute. By making a credible commitment not to contribute, however, Talia (or Nina) could alter the outcome of the game. Even though she does not get her turn to announce her decision until after Emily and Nina have made theirs public, Talia could let it be known that she has sunk all of her savings (and energy) into a large house-renovation project, and so she will have absolutely nothing left to contribute to the street garden. Then Talia essentially commits herself to not contribute regardless of Emily's and Nina's decisions, before Emily and Nina make those decisions. In other words, Talia changes the game to one in which she is in effect the first mover. You can easily check that the new rollback equilibrium entails Emily and Nina both contributing to the garden and the equilibrium payoffs are 3 to each of them but 4 to Talia—the equilibrium outcome associated with the game when Talia moves first. Several more detailed examples of commitments are given in the following sections.

B. Conditional Strategic Moves

Another possibility for A is to declare at the first stage: "In the game to follow, I will respond to your choices in the following way. If you choose Y_1, I will do Z_1; if you do Y_2, I will do Z_2, . . . " In other words, A can use a move that is conditional on B's behavior; we call this type of move a **response rule** or *reaction function.* A's statement means that, in the game to be played at the second stage, A will move second, but how he will respond to B's choices at that point is already predetermined by A's declaration at stage 1. For such declarations to be meaningful, A must be physically able to wait to make his move at the second stage until after he has observed what B has irreversibly done. In other words, at the second stage, B should have the true first move in the double sense just explained.

Conditional strategic moves take different forms, depending on what they are trying to achieve and how they set about achieving it. When A wants to stop B from doing something, we say that A is trying to deter B, or to achieve **deterrence;** when A wants to induce B to do something, we say that A is trying to compel B, or to achieve **compellence.** We return to this distinction later. Of more immediate interest is the method used in pursuit of either of these aims. If A declares, "*Unless* your action (or inaction, as the case may be) conforms to my stated wish, I will respond in a way that will *hurt* you," that is, a **threat.** If A declares, "If your action (or inaction, as the case may be) conforms to my stated wish, I will respond in a way that will *reward* you," that is, a **promise.** "Hurt" and "reward" are measured in terms of the payoffs in the game itself. When A hurts B, A does something that lowers B's payoff; when A rewards B, A does something that leads to a higher payoff for B. Threats and promises are the two conditional strategic moves on which we focus our analysis.

To understand the nature of these strategies, consider the dinner game mentioned earlier. In the natural chronological order of moves, first the child decides whether to eat his vegetables, and then the parent decides whether to give the child dessert. Rollback analysis tells us the outcome: the child refuses to eat the vegetables, knowing that the parent, unwilling to see the child hungry and unhappy, will give him the dessert. The parent can foresee this outcome, however, and can try to alter it by making an initial move—namely, by stating a conditional response rule of the form "no dessert unless you finish your vegetables." This declaration constitutes a threat. It is a first move in a pregame, which fixes how you will make your second move in the actual game to follow. If the child believes the threat, that alters the child's rollback calculation. The child "prunes" that branch of the game tree in which the parent serves dessert even if the child has not finished his vegetables. This may alter the child's behavior; the parent hopes that it will make the child act as the parent wants him to. Similarly, in the "study game," the promise of the bike may induce a child to study harder.

2 CREDIBILITY OF STRATEGIC MOVES

We have already seen that payoffs to the other player can be altered by one player's strategic move, but what about the payoffs for the player making that move? Player A gets a higher payoff when B acts in conformity with A's wishes. But A's payoff also may be affected by his own response. In regard to a threat, A's threatened response if B does not act as A would wish may have consequences for A's own payoffs: the parent may be made unhappy by the sight of the unhappy child who has been denied dessert. Similarly, in regard to a promise, rewarding B if he does act as A would wish can affect A's own payoff: the parent who rewards the child for studying hard has to incur the monetary cost of the gift but is happy to see the child's happiness on receiving the gift and even happier about the academic performance of the child.

This effect on A's payoffs has an important implication for the efficacy of A's strategic moves. Consider the threat. If A's payoff is actually increased by carrying out the threatened action, then B reasons that A will carry out this action even if B fulfills A's demands. Therefore B has no incentive to comply with A's wishes, and the threat is ineffective. For example, if the parent is a sadist who enjoys seeing the child go without dessert, then the child thinks, "I am not going to get dessert anyway, so why eat the vegetables?"

Therefore an essential aspect of a threat is that it should be *costly* for the threatener to carry out the threatened action. In the dinner game, the parent must prefer to give the child dessert. Threats in the true strategic sense have the innate property of imposing some cost on the threatener, too; they are threats of *mutual harm.*

In technical terms, a threat fixes your strategy (response rule) in the subsequent game. A strategy must specify what you will do in each eventuality along the game tree. Thus, "no dessert if you don't finish your vegetables" is an incomplete specification of the strategy; it should be supplemented by "and dessert if you do." Threats generally don't specify this latter part. Why not? Because the second part of the strategy is automatically understood; it is implicit. And, for the threat to work, this second part of the strategy—the *implied promise* in this case—has to be automatically credible, too.

Thus the threat "no dessert if you don't finish your vegetables" carries with it an implicit promise of "dessert if you do finish your vegetables." This promise also should be credible if the threat is to have the desired effect. In our example, the credibility of the implicit promise is automatic when the parent prefers to see the child get and enjoy his dessert. In other words, the implicit promise is automatically credible precisely when the threatened action is costly for the parent to carry out.

To put it yet another way, a threat carries with it the stipulation that you will do something if your wishes are not met that, if those circumstances actually arise, you will regret having to do. Then why make this stipulation at the first stage? Why tie your own hands in this way when it might seem that leaving one's options open would always be preferable? Because, in the realm of game theory, having more options is not always preferable. In regard to a threat, your lack of freedom in the second stage of the game has strategic value. It changes other players' expectations about your future responses, and you can use this change in expectations to your advantage.

A similar effect arises with a promise. If the child knows that the parent enjoys giving him gifts, he may expect to get the racing bike anyway on some occasion in the near future—for example, an upcoming birthday. Then the promise of the bike has little effect on the child's incentive to study hard. To have the intended strategic effect, the promised reward must be so costly to provide that the other player would not expect you to hand over that reward anyway. (This is a useful lesson in strategy that you can point out to your parents: the rewards that they promise must be larger and more costly than what they would give you just for the pleasure of seeing you happy.)

The same is true of unconditional strategic moves (commitments, too). In bargaining, for example, others know that, when you have the freedom to act, you also have the freedom to capitulate; so a "no concessions" commitment can secure you a better deal. If you hold out for 60% of the pie and the other party offers you 55%, you may be tempted to take it. But, if you can credibly assert in advance that you will not take less than 60%, then this temptation does not arise and you can do better than you otherwise would.

Thus it is in the very nature of strategic moves that after the fact—that is, when the stage 2 game actually requires it—you do not want to carry out the action that you had stipulated you would take. This is true for all types of strategic moves and it is what makes credibility so problematic. You have to do something at the first stage to create credibility—something that convincingly tells the other player that you will not give in to the temptation to deviate from the stipulated action when the time comes—in order for your strategic move to work. That is why giving up your own freedom to act can be strategically beneficial. Alternatively, credibility can be achieved by changing your own payoffs in the second-stage game in such a way that it becomes truly optimal for you to act as you declare.

Thus there are two general ways of making your strategic moves credible: (1) remove from your own set of future choices the other moves that may tempt you or (2) reduce your own payoffs from those temptation moves so that the stipulated move becomes the actual best one. In the sections that follow, we first elucidate the mechanics of strategic moves, assuming them to be credible. We make some comments about credibility as we go along but postpone our general analysis of credibility until the last section of the chapter.

3 COMMITMENTS

We studied the game of chicken in Chapter 4 and found two pure-strategy Nash equilibria. Each player prefers the equilibrium in which he goes straight and the other person swerves.[1] We saw in Chapter 6 that, if the game were to have sequential rather than simultaneous moves, the first mover would choose Straight, leaving the second to make the best of the situation by settling for Swerve rather than causing a crash. Now we can consider the same matter from another perspective. Even if the game itself has simultaneous moves, if one player can make a strategic move—create a first stage in which he makes a credible declaration about his action in the chicken game itself, which is to be played at the second stage—then he can get the same advantage afforded a first mover by making a commitment to act tough (choose Straight).

Although the point is simple, we outline the formal analysis to develop your understanding and skill, which will be useful for later, more complex examples. Remember our two players, James and Dean. Suppose James is the one who has the opportunity to make a strategic move. Figure 10.1 shows the tree for the

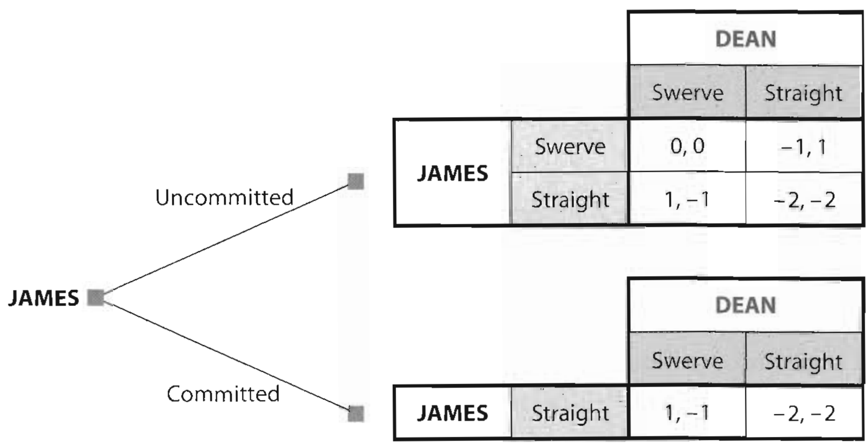


FIGURE 10.1 Chicken: Commitment by Restricting Freedom to Act

[1]We saw in Chapter 8 and will see again in Chapter 13 that the game has a third equilibrium, in mixed strategies, in which both players do quite poorly.

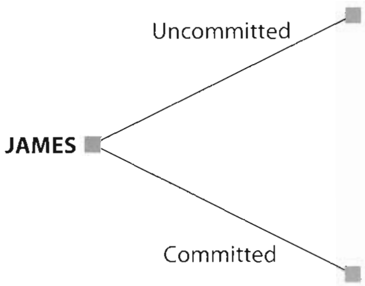
two-stage game. At the first stage, James has to decide whether to make a commitment. Along the upper branch emerging from the first node, he does not make the commitment. Then at the second stage the simultaneous-move game is played, and its payoff table is the familiar one shown in Figures 4.14 and 6.6. This second-stage game has multiple equilibria, and James gets his best payoff in only one of them. Along the lower branch, James makes the commitment. Here, we interpret this commitment to mean giving up his freedom to act in such a way that Straight is the only action available to James at this stage. Therefore the second-stage game table has only one row for James, corresponding to his declared choice of Straight. In this table, Dean's best action is Swerve; so the equilibrium outcome gives James his best payoff. Therefore, at the first stage, James finds it optimal to make the commitment; this strategic move ensures his best payoff, while not committing leaves the matter uncertain.

How can James make this commitment credibly? Like any first move, the commitment move must be (1) irreversible and (2) visible to the other player. People have suggested some extreme and amusing ideas. James can disconnect the steering wheel of the car and throw it out of the window so that Dean can see that James can no longer Swerve. (James could just tie the wheel so that it could no longer be turned, but it would be more difficult to demonstrate to Dean that the wheel was truly tied and that the knot was not a trick one that could be undone quickly.) These devices simply remove the Swerve option from the set of choices available to James in the stage 2 game, leaving Straight as the only thing he can do.

More plausibly, if such games are played every weekend, James can acquire a general reputation for toughness that acts as a guarantee of his action on any one day. In other words, James can alter his own payoff from swerving by subtracting an amount that represents the loss of reputation. If this amount is large enough—say, 3—then the second-stage game when James has made the commitment has a different payoff table. The complete tree for this version of the game is shown in Figure 10.2.

Now, in the second stage with commitment, Straight has become truly optimal for James; in fact, it is his dominant strategy in that stage. Dean's optimal strategy is then Swerve. Looking ahead to this outcome at stage 1, James sees that he gets 1 by making the commitment (changing his own stage 2 payoffs), while without the commitment he cannot be sure of 1 and may do much worse. Thus a rollback analysis shows that James should make the commitment.

Both (or all) can play the game of commitment, so success may depend both on the speed with which you can seize the first move and on the credibility with which you can make that move. If there are lags in observation, the two may even make incompatible simultaneous commitments: each disconnects his steering wheel and tosses it out of the window just as he sees the other's wheel come flying out, and then the crash is unavoidable.



Uncommitted:

		DEAN: Swerve	DEAN: Straight
JAMES	Swerve	0, 0	−1, 1
	Straight	1, −1	−2, −2

Committed:

		DEAN: Swerve	DEAN: Straight
JAMES	Swerve	−3, 0	−4, 1
	Straight	1, −1	−2, −2

FIGURE 10.2 Chicken: Commitment by Changing Payoffs

Even if one of the players has the advantage in making a commitment, the other player can defeat the first player's attempt to do so. The second player could demonstrably remove his ability to "see" the other's commitment, for example, by cutting off communication.

Games of chicken may be a 1950s anachronism, but our second example is perennial and familiar. In a class, the teacher's deadline enforcement policy can be Weak or Tough, and the students' work can be Punctual or Late. Figure 10.3 shows this game in the strategic form. The teacher does not like being tough; for him the best outcome (a payoff of 4) is when students are punctual even when he is weak; the worst (1) is when he is tough but students are still late. Of the two intermediate strategies, he recognizes the importance of punctuality and rates (Tough, Punctual) better than (Weak, Late). The students most prefer the outcome (Weak, Late), where they can party all weekend without suffering any penalty for the late assignment. (Tough, Late) is the worst for them, just as it is for the teacher. Between the intermediate ones, they prefer (Weak, Punctual) to (Tough, Punctual) because they have better self-esteem if they can think that

		STUDENT: Punctual	STUDENT: Late
TEACHER	Weak	4, 3	2, 4
	Tough	3, 2	1, 1

FIGURE 10.3 Payoff Table for Class Deadline Game

they acted punctually on their own volition rather than because of the threat of a penalty.[2]

If this game is played as a simultaneous-move game or if the teacher moves second, Weak is dominant for the teacher, and then the student chooses Late. The equilibrium outcome is (Weak, Late), and the payoffs are (2, 4). But the teacher can achieve a better outcome by committing at the outset to the policy of Tough. We do not draw a tree as we did in Figures 10.1 and 10.2. The tree would be very similar to that for the preceding chicken case, and so we leave it for you to draw. Without the commitment, the second-stage game is as before, and the teacher gets a 2. When the teacher is committed to Tough, the students find it better to respond with Punctual at the second stage, and the teacher gets a 3.

The teacher commits to a move different from what he would do in simultaneous play or indeed, his best second move if the students moved first. This is where the strategic thinking enters. The teacher has nothing to gain by declaring that he will have a Weak enforcement regime; the students expect that anyway in the absence of any declaration. To gain advantage by making a strategic move, he must commit not to follow what would be his equilibrium strategy of the simultaneous-move game. This strategic move changes the students' expectations and therefore their action. Once they believe the teacher is really committed to tough discipline, they will choose to turn in their assignments punctually. If they tested this out by being late, the teacher would like to forgive them, maybe with an excuse to himself, such as "just this once." The existence of this temptation to shift away from your commitment is what makes its credibility problematic.

Even more dramatically, in this instance the teacher benefits by making a strategic move that commits him to a dominated strategy. He commits to choosing Tough, which is dominated by Weak. The choice of Tough gets the teacher a 3 if the student chooses Punctual and a 1 if the student chooses Late, whereas, if the teacher had chosen Weak, his corresponding payoffs would have been 4 and 2. If you think it paradoxical that one can gain by choosing a dominated strategy, you are extending the concept of dominance beyond the proper scope of its validity. Dominance entails either of two calculations: (1) After the other player does something, how do I respond, and is some choice best (or worst), given all possibilities? (2) If the other player is simultaneously doing action X, what is best (or worst) for me, and is this the same for all the X actions that the other could be choosing? Neither is relevant when you are moving first.

[2]You may not regard these specific rankings of outcomes as applicable either to you or to your own teachers. We ask you to accept them for this example, whose main purpose is to convey some *general ideas* about commitment in a simple way. The same disclaimer applies to all the examples that follow.

Instead, you must look ahead to how the other will respond. Therefore the teacher does not compare his payoffs in vertically adjacent cells of the table (taking the possible actions of the students one at a time). Instead, he calculates how the students will react to each of his moves. If he is committed to Tough, they will be Punctual, but, if he is committed to Weak (or uncommitted), they will be Late, so the only pertinent comparison is that of the top-right cell with the bottom left, of which the teacher prefers the latter.

To be credible, the teacher's commitment must be everything a first move has to be. First, it must be made before the other side makes its move. The teacher must establish the ground rules of deadline enforcement before the assignment is due. Next, it must be observable—the students must know the rules by which they must abide. Finally, and perhaps the most important, it must be irreversible—the students must know that the teacher cannot, or at any rate will not, change his mind and forgive them. A teacher who leaves loopholes and provisions for incompletely specified emergencies is merely inviting imaginative excuses accompanied by fulsome apologies and assertions that "it won't happen again."

The teacher might achieve credibility by hiding behind general university regulations; this simply removes the Weak option from his set of available choices at stage 2. Or, as is true in the chicken game, he might establish a reputation for toughness, changing his own payoffs from Weak by creating a sufficiently high cost of loss of reputation.

4 THREATS AND PROMISES

We emphasize that threats and promises are *response rules*: your actual future action is conditioned on what the other players do in the meantime, but your freedom of future action is constrained to following the stated rule. Once again, the aim is to alter the other players' expectations and therefore their actions in a way favorable to you. Tying yourself to a rule, which you would not want to follow if you were completely free to act at the later time, is an essential part of this process. Thus the initial declaration of intention must be credible. Once again, we will elucidate some principles for achieving credibility of these moves, but we remind you that their actual implementation remains largely an art.

Remember the taxonomy given in Section 1. A *threat* is a response rule that leads to a bad outcome for the other players if they act contrary to your interests. A *promise* is a response rule by which you offer to create a good outcome for the other players if they act in a way that promotes your own interests. Each of these responses may aim either to stop the other players from doing something that they would otherwise do (*deterrence*) or to induce them to do something that they would otherwise not do (*compellence*). We consider these features in turn.

A. Example of a Threat: U.S.–Japan Trade Relations

Our example comes from a hardy perennial of U.S. international economic policy—namely, trade friction with Japan. Each country has the choice of keeping its own markets open or closed to the other's goods. They have somewhat different preferences regarding the outcomes.

Figure 10.4 shows the payoff table for the trade game. For the United States, the best outcome (a payoff of 4) comes when both markets are open; this is partly because of its overall commitment to the market system and free trade and partly because of the benefit of trade with Japan itself—U.S. consumers get high-quality cars and consumer electronics products, and U.S. producers can export their agricultural and high-tech products. Similarly, its worst outcome (payoff 1) occurs when both markets are closed. Of the two outcomes when only one market is open, the United States would prefer its own market to be open, because the Japanese market is smaller, and loss of access to it is less important than the loss of access to Hondas and Walkmen.

As for Japan, for the purpose of this example we accept the protectionist, producer-oriented picture of Japan, Inc. Its best outcome is when the U.S. market is open and its own is closed; its worst is when matters are the other way around. Of the other two outcomes, it prefers that both markets be open, because its producers then have access to the much larger U.S. market.[3]

Both sides have dominant strategies. No matter how the game is played—simultaneously or sequentially with either move order—the equilibrium outcome is (Open, Closed), and the payoffs are (3, 4). This outcome also fits well the

		JAPAN	
		Open	Closed
UNITED STATES	Open	4, 3	3, 4
	Closed	2, 1	1, 2

FIGURE 10.4 Payoff Table for U.S.–Japan Trade Game

[3]Again, we ask you to accept this payoff structure as a vehicle for conveying the ideas. You can experiment with the payoff tables to see what difference that would make to the role and effectiveness of the strategic moves.

common American impression of how the actual trade policies of the two countries work.

Japan is already getting its best payoff in this equilibrium and so has no need to try any strategic moves. The United States, however, can try to get a 4 instead of a 3. But in this case an ordinary unconditional commitment will not work. Japan's best response, no matter what commitment the United States makes, is to keep its market closed. Then the United States does better for itself by committing to keep its own market open, which is the equilibrium without any strategic moves anyway.

But suppose the United States can choose the following conditional response rule: "We will close our market if you close yours." The situation then becomes the two-stage game shown in Figure 10.5. If the United States does not use the threat, the second stage is as before and leads to the equilibrium in which the U.S. market is open and it gets a 3, while the Japanese market is closed and it gets a 4. If the United States does use the threat, then at the second stage only Japan has freedom of choice; given what Japan does, the United States then merely does what its response rule dictates. Therefore, along this branch of the tree, we show only Japan as an active player and write down the payoffs to the two parties: If Japan keeps its market closed, the United States closes its own, and the United States gets a 1 and Japan gets a 2. If Japan keeps its market open, then the United States threat has worked, it is happy to keep its own market open, and it gets a 4, while Japan gets a 3. Of these two possibilities, the second is better for Japan.

Now we can use the familiar rollback reasoning. Knowing how the second stage will work in all eventualities, it is better for the United States to deploy its threat at the first stage. This threat will result in an open market in Japan, and the United States will get its best outcome.

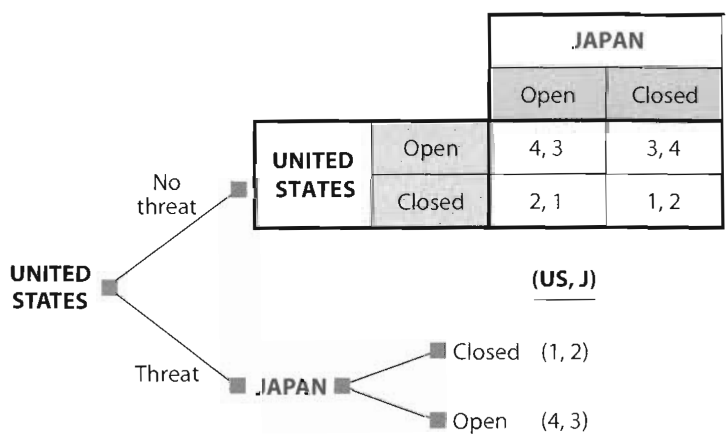


FIGURE 10.5 Tree for the U.S.–Japan Trade Game with Threat

Having described the mechanics of the threat, we now point out some of its important features.

1. When the United States deploys its threat credibly, Japan doesn't follow its dominant strategy Closed. Again, the idea of dominance is relevant only in the context of simultaneous moves or when Japan moves second. Here, Japan knows that the United States will take actions that depart from *its* dominant strategy. In the payoff table, Japan is looking at a choice between just two cells, the top left and the bottom right, and, of those two, it prefers the latter.

2. Credibility of the threat is problematic because, if Japan puts it to the test by keeping its market closed, the United States faces the temptation to refrain from carrying out the threat. In fact, if the threatened action were the best U.S. response after the fact, then there would be no need to make the threat in advance (but the United States might issue a *warning* just to make sure that the Japanese understand the situation). The strategic move has a special role exactly because it locks a player into doing something other than what it would have wanted to do after the fact. As explained earlier, a threat in the true strategic sense is necessarily costly for the threatener to carry out; the threatened action would inflict *mutual* harm.

3. The conditional rule "We will close our market if you close yours" does not completely specify the U.S. *strategy*. To be complete, it needs an additional clause indicating what the United States will do in response to an open Japanese market: "and we will keep our market open if you keep yours open." This additional clause, the implicit promise, is really part of the threat, but it does not need to be stated explicitly, because it is automatically credible. Given the payoffs of the second-stage game, it is in the best interests of the United States to keep its market open if Japan keeps its market open. If that were not the case, if the United States would respond by keeping its market closed even when Japan kept its own market open, then the implicit promise would have to be made explicit and somehow made credible. Otherwise, the U.S. threat would become tantamount to the unconditional commitment "We will keep our market closed," and that would not draw forth the desired response from Japan.

4. The threat, when credibly deployed, results in a change in Japan's action. We can regard this as deterrence or compellence, depending on the status quo. If the Japanese market is initially open, and the Japanese are considering a switch to protectionism, then the threat deters them from that action. But, if the Japanese market is initially closed, then the threat compels them to open it. Thus whether a strategic move is deterrent or compellent depends on the status quo. The distinction may seem to be a matter of semantics, but in practice the credibility of a move and the way that it works are importantly affected by this distinction. We return to this matter later in the chapter.

5. Here are a few ways in which the United States can make its threat credible. First, it can enact a law that mandates the threatened action under the right circumstances. This removes the temptation action from the set of available choices at stage 2. Some reciprocity provisions in the World Trade Organization agreements have this effect, but the procedures are very slow and uncertain. Second, it can delegate fulfillment to an agency such as the U.S. Commerce Department that is captured by U.S. producers who would like to keep our markets closed and so reduce the competitive pressure on themselves. This changes the U.S. payoffs at stage 2—replacing the true U.S. payoffs by those of the Commerce Department—with the result that the threatened action becomes truly optimal. (The danger is that the Commerce Department will then retain a protectionist stance even if Japan opens its market; gaining credibility for the threat may lose credibility for the implied promise.)

6. If a threat works, it doesn't have to be carried out. So its cost to you is immaterial. In practice, the danger that you may have miscalculated or the risk that the threatened action will take place by error even if the other player complies is a strong reason to refrain from using threats more severe than necessary. To make the point starkly, the United States could threaten to pull out of defensive alliances with Japan if it didn't buy our rice and semiconductors, but that threat is "too big" and too risky for the United States to ever carry out; therefore it is not credible.

But sometimes a range of threats is not available from which a player can choose one that is, on the one hand, sufficiently big that the other player fears it and alters his action in the way that the first player desires and, on the other hand, not so big as to be too risky for the first player to ever carry out and therefore lacking credibility. If the only available threat is too big, then a player can reduce its size by making its fulfillment a matter of chance. Instead of saying, "If you don't open your markets, we will refuse to defend you in the future," the United States can say to Japan, "If you don't open your markets, the relations between our countries will deteriorate to the point where Congress may refuse to allow us to come to your assistance if you are ever attacked, even though we do have an alliance." In fact, the United States can deliberately foster sentiments that raise the probability that Congress will do just that; so the Japanese will feel the danger more vividly. A threat of this kind, which creates a risk but not a certainty of the bad outcome, is called brinkmanship. It is an extremely delicate and even dangerous variant of the strategic move. We will study brinkmanship in greater detail in Chapter 14.

7. Japan gets a worse outcome when the United States deploys its threat than it would without this threat; so it would like to take strategic actions that defeat or disable U.S. attempts to use the threat. For example, suppose its market is currently closed, and the United States is attempting compellence. The

Japanese can accede in principle but stall in practice, pleading necessary delays for assembling the necessary political consensus to legislate the market opening, then delays for writing the necessary administrative regulations to implement the legislation, and so on. Because the United States does not want to go ahead with its threatened action, at each point it has the temptation to accept the delay. Or Japan can claim that its domestic politics makes it difficult to open all markets fully; will the United States accept the outcome if Japan keeps just a few of its industries protected? It gradually expands this list, and at any point the extra small step is not enough cause for the United States to unleash a trade war. This device of defeating a compellent threat by small steps, or "slice by slice," is called **salami tactics.**

B. Example of a Promise: The Restaurant Pricing Game

We now illustrate a promise by using the restaurant pricing game of Chapter 5. We saw in Chapter 5 that the game is a prisoners' dilemma, and we simplify it here by supposing that only two choices of price are available: the jointly best price of $26 or the Nash equilibrium price of $20. The profits for each restaurant in this version of the game can be calculated by using the functions in Section 1 of Chapter 5; the results are shown in Figure 10.6. Without any strategic moves, the game has the usual equilibrium in dominant strategies in which both stores charge the low price of 20, and both get lower profits than they would if they both charged the high price of 26.

If either side can make the credible promise "I will charge a high price if you do," the cooperative outcome is achieved. For example, if Xavier's makes the promise, then Yvonne's knows that its choice of 26 will be reciprocated, leading to the payoff shown in the lower-right cell of the table and that its choice of 20 will bring forth Xavier's usual action—namely, 20—leading to the upper-left cell. Between the two, Yvonne's prefers the first and therefore chooses the high price.

The analysis can be done more properly by drawing a tree for the two-stage game in which Xavier's has the choice of making or not making the promise at

		YVONNE'S BISTRO	
		20 (low)	26 (high)
XAVIER'S TAPAS	20 (low)	288, 288	360, 216
	26 (high)	216, 360	324, 324

FIGURE 10.6 Payoff Table for Restaurant Prisoners' Dilemma ($100s per month)

the first stage. We omit the tree, partly so that you can improve your understanding of the process by constructing it yourself and partly to show how such detailed analysis becomes unnecessary as one becomes familiar with the ideas.

The credibility of Xavier's promise is open to doubt. To respond to what Yvonne's does, Xavier's must arrange to move second in the second stage of the game; correspondingly, Yvonne's must move first in stage 2. Remember that a first move is an irreversible and observable action. Therefore, if Yvonne's moves first and prices high, it leaves itself vulnerable to Xavier's cheating, and Xavier's is very tempted to renege on its promise to price high when it sees Yvonne's in this vulnerable position. Xavier's must somehow convince Yvonne's that it will not give in to the temptation to charge a low price when Yvonne's charges a high price.

How can it do so? Perhaps Xavier's owner can leave the pricing decision in the hands of a local manager, with clear written instructions to reciprocate with the high price if Yvonne's charges the high price. Xavier's owner can invite Yvonne's to inspect these instructions, after which he leaves on a solo round-the-world sailing trip so that he cannot rescind them. (Even then, Yvonne's management may be doubtful—Xavier might secretly carry a telephone or a laptop computer on board.) This scenario is tantamount to removing the cheating action from the choices available to Xavier's at stage 2.

Or Xavier's restaurant can develop a reputation for keeping its promises, in business and in the community more generally. In a repeated relationship, the promise may work because reneging on the promise once may cause future cooperation to collapse. In essence, an ongoing relationship means splitting the game into smaller segments, in each of which the benefit from reneging is too small to justify the costs. In each such game, then, the payoff from cheating is altered by the cost of collapse of future cooperation.[4]

We saw earlier that every threat has an implicit attached promise. Similarly, every promise has an implicit attached threat. In this case, it is "I will charge the low price if you do." It does not have to be stated explicitly, because it is automatically credible—it describes Xavier's best response to Yvonne's low price.

There is also an important difference between a threat and a promise. If a threat is successful, it doesn't have to be carried out and is then costless to the threatener. Therefore a threat can be bigger than what is needed to make it effective (although making it too big may be too risky, even to the point of losing its credibility as suggested earlier). If a promise is successful in altering the other's action in the desired direction, then the promisor has to deliver what he had promised, and so it is costly. In the preceding example, the cost is simply

[4]In Chapter 11, we will investigate in great detail the importance of repeated or ongoing relationships in attempts to reach the cooperative outcome in a prisoners' dilemma.

giving up the opportunity to cheat and get the highest payoff; in other instances where the promisor offers an actual gift or an inducement to the other, the cost may be more tangible. In either case, the player making the promise has a natural incentive to keep its size small—just big enough to be effective.

C. Example Combining Threat and Promise: Joint U.S.–China Political Action

When we considered threats and promises one at a time, the explicit statement of a threat included an implicit clause of a promise that was automatically credible, and vice versa. There can, however, be situations in which the credibility of both aspects is open to question; then the strategic move has to make both aspects explicit and make them both credible.

Our example of an explicit-threat-and-promise combination comes from a context in which multiple nations must work together toward some common goal in dealing with a dangerous situation in a neighboring country. Specifically, we consider an example of the United States and China contemplating whether to take action to compel North Korea to give up its nuclear weapons programs. We show in Figure 10.7 the payoff table for the United States and China when each must choose between action and inaction.

Each country would like the other to take on the whole burden of taking action against the North Koreans; so the top-right cell has the best payoff for the United States (4), and the bottom-left cell is best for China. The worst situation for the United States is where no action is taken, because it finds the increased threat of nuclear war in that case to be unacceptable. For China, however, the worst outcome arises when it takes on the whole burden of action, because the costs of action are so high. Both regard a joint involvement as the second-best (a payoff of 3). The United States assigns a payoff of 2 to the situation in which it is the only one to act. And, for China, a payoff of 2 is assigned to the case in which no action is taken.

Without any strategic moves, the intervention game is dominance solvable. Inaction is the dominant strategy for China, and then Action is the best choice for the United States. The equilibrium outcome is the top-right cell, with payoffs of 2 for the United States and 4 for China. Because China gets its best outcome,

		CHINA	
		Action	Inaction
UNITED STATES	Action	3, 3	2, 4
	Inaction	4, 1	1, 2

FIGURE 10.7 Payoff Table for U.S.–China Political Action Game

it has no reason to try any strategic moves. But the United States can try to do better than a 2.

What strategic move will work to improve the equilibrium payoff for the United States? An unconditional move (commitment) will not work, because China will respond with "Inaction" to either first move by the United States. A threat alone ("We won't take action unless you do") does not work, because the implied promise ("We will if you do") is not credible—if China does act, the United States would prefer to back off and leave everything to China, getting a payoff of 4 instead of the 3 that would come from fulfilling the promise. A promise alone won't work: because China knows that the United States will intervene if China does not, an American promise of "We will intervene if you do" becomes tantamount to a simple commitment to intervene; then China can stay out and get its best payoff of 4.

In this game, an explicit promise from the United States must carry the implied threat "We won't take action if you don't," but that threat is not automatically credible. Similarly, America's explicit threat must carry the implied promise "We will act if you do," but that also is not automatically credible. Therefore the United States has to make both the threat and the promise explicit. It must issue the combined threat-cum-promise "We will act if, and only if, you do." It needs to make both clauses credible. Usually such credibility has to be achieved by means of a treaty that covers the whole relationship, not just with agreements negotiated separately when each incident arises.

5 SOME ADDITIONAL TOPICS

A. When Do Strategic Moves Help?

We have seen several examples in which a strategic move brings a better outcome to one player or another, compared with the original game without such moves. What can be said in general about the desirability of such moves?

An unconditional move—a commitment—need not always be advantageous to the player making it. In fact, if the original game gives the advantage to the second mover, then it is a mistake to commit oneself to move in advance, thereby effectively becoming the first mover.

The availability of a conditional move—threat or promise—can never be an actual disadvantage. At the very worst, one can commit to a response rule that would have been optimal after the fact. However, if such moves bring one an actual gain, it must be because one is choosing a response rule that in some eventualities specifies an action different from what one would find optimal at that later time. Thus whenever threats and promises bring a positive gain, they do so precisely when (one might say precisely because) their credibility is inherently questionable, and must be achieved by some specific credibility "device." We have mentioned some such devices in connection with each earlier example and will later discuss the topic of achieving credibility in greater generality.

What about the desirability of being on the receiving end of a strategic move? It is never desirable to let the other player threaten you. If a threat seems likely, you can gain by looking for a different kind of advance action—one that makes the threat less effective or less credible. We will consider some such actions shortly. However, it is often desirable to let the other player make promises to you. In fact, both players may benefit when one can make a credible promise, as in the prisoners' dilemma example of restaurant pricing earlier in this chapter, in which a promise achieved the cooperative outcome. Thus it may be in the players' mutual interest to facilitate the making of promises by one or both of them.

B. Deterrence Versus Compellence

In principle, either a threat or a promise can achieve either deterrence or compellence. For example, a parent who wants a child to study hard (compellence) can promise a reward (a new racing bike) for good performance in school or can threaten a punishment (a strict curfew the following term) if the performance is not sufficiently good. Similarly, a parent who wants the child to keep away from bad company (deterrence) can try either a reward (promise) or a punishment (threat). In practice, the two types of strategic moves work somewhat differently, and that will affect the ultimate decision regarding which to use. Generally, deterrence is better achieved by a threat and compellence by a promise. The reason is an underlying difference of timing and initiative.

A deterrent threat can be passive—you don't need to do anything so long as the other player doesn't do what you are trying to deter. And it can be static—you don't have to impose any time limit. Thus you can set a trip wire and then leave things up to the other player. So the parent who wants the child to keep away from bad company can say, "If I ever catch you with X again, I will impose a 7 P.M. curfew on you for a whole year." Then the parent can sit back to wait and watch; only if the child acts contrary to the parent's wishes does the parent have to act on her threat. Trying to achieve the same deterrence by a promise would require more complex monitoring and continual action: "At the end of each month in which I know that you did not associate with X, I will give you $25."

Compellence must have a deadline or it is pointless—the other side can defeat your purpose by procrastinating or by eroding your threat in small steps (salami tactics). This makes a compellent threat harder to implement than a compellent promise. The parent who wants the child to study hard can simply say, "Each term that you get an average of B or better, I will give you CDs or games worth $500." The child will then take the initiative in showing the parent each time he has fulfilled the conditions. Trying to achieve the same thing by a threat—"Each term that your average falls below B, I will take away one of your computer games"—will require the parent to be much more vigilant and active. The child will postpone bringing the grade report or will try to hide the games.

The concepts of reward and punishment are relative to those of some status quo. If the child has a perpetual right to the games, then taking one away is a punishment; if the games are temporarily assigned to the child on a term-by-term basis, then renewing the assignment for another term is a reward. Therefore you can change a threat into a promise or vice versa by changing the status quo. You can use this change to your own advantage when making a strategic move. If you want to achieve compellence, try to choose a status quo such that what you do when the other player acts to comply with your demand becomes a reward, and so you are using a compellent promise. To give a rather dramatic example, a mugger can convert the threat "If you don't give me your wallet, I will take out my knife and cut your throat" into the promise "Here is a knife at your throat; as soon as you give me your wallet I will take it away." But, if you want to achieve deterrence, try to choose a status quo such that, if the other player acts contrary to your wishes, what you do is a punishment, and so you are using a deterrent threat.

6 ACQUIRING CREDIBILITY

We have emphasized the importance of credibility of strategic moves throughout, and we accompanied each example with some brief remarks about how credibility could be achieved in that particular context. Devices for achieving credibility are indeed often context specific, and there is a lot of art to discovering or developing such devices. Some general principles can help you organize your search.

We pointed out two broad approaches to credibility: (1) reducing your own future freedom of action in such a way that you have no choice but to carry out the action stipulated by your strategic move and (2) changing your own future payoffs in such a way that it becomes optimal for you to do what you stipulate in your strategic move. We now elaborate some practical methods for implementing each of these approaches.

A. Reducing Your Freedom of Action

I. AUTOMATIC FULFILLMENT Suppose at stage 1 you relinquish your choice at stage 2 and hand it over to a mechanical device or similar procedure or mechanism that is programmed to carry out your committed, threatened, or promised action under the appropriate circumstances. You demonstrate to the other player that you have done so. Then he will be convinced that you have no freedom to change your mind, and your strategic move will be credible. The **doomsday device,** a nuclear explosive device that would detonate and contaminate the whole world's atmosphere if the enemy launched a nuclear attack, is the best-known example, popularized by the early 1960s movies *Fail Safe* and *Dr. Strangelove.* Luckily, it remained in the realm of fiction. But automatic procedures that retaliate with import tariffs if another country tries to subsidize its exports to your country (countervailing duties) are quite common in the arena of trade policy.

II. DELEGATION A fulfillment device does not even have to be mechanical. You could delegate the power to act to another person or to an organization that is required to follow certain preset rules or procedures. In fact, that is how the countervailing duties work. They are set by two agencies of the U.S. government—the Commerce Department and the International Trade Commission—whose operating procedures are laid down in the general trade laws of the country.

An agent should not have his own objectives that defeat the purpose of his strategic move. For example, if one player delegates to an agent the task of inflicting threatened punishment and the agent is a sadist who enjoys inflicting punishment, then he may act even when there is no reason to act—that is, even when the second player has complied. If the second player suspects this, then the threat loses its effectiveness, because the punishment becomes a case of "damned if you do and damned if you don't."

Delegation devices are not complete guarantees of credibility. Even the doomsday device may fail to be credible if the other side suspects that you control an override button to prevent the risk of a catastrophe. And delegation and mandates can always be altered; in fact, the U.S. government has often set aside the stipulated countervailing duties and reached other forms of agreements with other countries so as to prevent costly trade wars.

III. BURNING BRIDGES Many invaders, from Xenophon in ancient Greece to William the Conqueror in England to Cortes in Mexico, are supposed to have deliberately cut off their own army's avenue of retreat to ensure that it will fight hard. Some of them literally burned bridges behind them, while others burned ships, but the device has become a cliche. Its most recent users in military contexts may have been the Japanese *kamikaze* pilots in World War II, who took only enough fuel to reach the U.S. naval ships into which they were to ram their airplanes. The principle even appears in the earliest known treatise on war, in a commentary attributed to Prince Fu Ch'ai: "Wild beasts, when they are at bay, fight desperately. How much more is this true of men! If they know there is no alternative they will fight to the death."[5]

[5]Sun Tzu, *The Art of War,* trans. Samuel B. Griffith (Oxford: Oxford University Press, 1963), p. 110.

Related devices are used in other high-stakes games. Although the European Monetary Union could have retained separate currencies and merely fixed the exchange rates between them, a common currency was adopted precisely to make the process irreversible and thereby give the member countries a much greater incentive to make the union a success. (In fact, it is the extent of the necessary commitment that has kept some nations, Great Britain in particular, from agreeing to be part of the European Monetary Union.) It is not totally impossible to abandon a common currency and go back to separate national ones; it is just inordinately costly. If things get really bad inside the Union, one or more countries may yet choose to get out. As with automatic devices, the credibility of burning bridges is not an all-or-nothing matter, but one of degree.

IV. CUTTING OFF COMMUNICATION If you send the other player a message demonstrating your commitment and at the same time cut off any means for him to communicate with you, then he cannot argue or bargain with you to reverse your action. The danger in cutting off communication is that, if both players do so simultaneously, then they may make mutually incompatible commitments that can cause great mutual harm. Additionally, cutting off communication is harder to do with a threat, because you have to remain open to the one message that tells you whether the other player has complied and therefore whether you need to carry out your threat. In this age, it is also quite difficult for a person to cut himself off from all contact.

But players who are large teams or organizations can try variants of this device. Consider a labor union that makes its decisions at mass meetings of members. To convene such a meeting takes a lot of planning—reserving a hall, communicating with members, and so forth—and several weeks of time. A meeting is convened to decide on a wage demand. If management does not meet the demand in full, the union leadership is authorized to call a strike and then it must call a new mass meeting to consider any counteroffer. This process puts management under a lot of time pressure in the bargaining; it knows that the union will not be open to communication for several weeks at a time. Here, we see that cutting off communication for extended periods can establish some degree of credibility, but not absolute credibility. The union's device does not make communication totally impossible; it only creates several weeks of delay.

B. Changing Your Payoffs

I. REPUTATION You can acquire a **reputation** for carrying out threats and delivering on promises. Such a reputation is most useful in a repeated game against the same player. It is also useful when playing different games against different players, if each of them can observe your actions in the games that you play with others. The circumstances favorable to the emergence of such a reputation are the same as those for achieving cooperation in the prisoners' dilemma, and for the same reasons. The greater the likelihood that the interaction will continue and the greater the concern for the future relative to the present, the more likely the players will be to sacrifice current temptations for the sake of future gains. The players will therefore be more willing to acquire and maintain reputations.

In technical terms, this device links different games, and the payoffs of actions in one game are altered by the prospects of repercussions in other games. If you fail to carry out your threat or promise in one game, your reputation suffers and you get a lower payoff in other games. Therefore when you consider any one of these games, you should adjust your payoffs in it to take into consideration such repercussions on your payoffs in the linked games.

The benefit of reputation in ongoing relationships explains why your regular car mechanic is less likely to cheat you by doing an unnecessary or excessively costly or shoddy repair than is a random garage that you go to in an emergency. But what does your regular mechanic actually stand to gain from acquiring this reputation if competition forces him to charge a price so low that he makes no profit on any deal? His integrity in repairing your car must come at a price—you have to be willing to let him charge you a little bit more than the rates that the cheapest garage in the area might advertise.

The same reasoning also explains why, when you are away from home, you might settle for the known quality of a restaurant chain instead of taking the risk of going to an unknown local restaurant. And a department store that expands into a new line of merchandise can use the reputation that it has acquired in its existing lines to promise its customers the same high quality in the new line.

In games where credible promises by one or both parties can bring mutual benefit, the players can agree and even cooperate in fostering the development of reputation mechanisms. But, if the interaction ends at a known finite time, there is always the problem of the endgame.

In the Middle East peace process that started in 1993 with the Oslo Accord, the early steps, in which Israel transferred some control over Gaza and small isolated areas of the West Bank to the Palestinian Authority and in which the latter accepted the existence of Israel and reduced its anti-Israel rhetoric violence, continued well for a while. But, as final stages of the process approached, mutual credibility of the next steps became problematic, and by 1998 the process stalled. Sufficiently attractive rewards could have come from the outside; for example, the United States or Europe could have given to both parties contingent offers of economic aid or prospects of expanded commerce to keep the process going. The United States offered Egypt and Israel large amounts of aid in this way to achieve the Camp David Accords in 1978. But such rewards were not offered in the more recent situation and, at the date of this writing, prospects for progress do not look bright.

II. DIVIDING THE GAME INTO SMALL STEPS Sometimes a single game can be divided into a sequence of smaller games, thereby allowing the reputation mechanism to come into effect. In home-construction projects, it is customary to pay by installments as the work progresses. In the Middle East peace process, Israel would never have agreed to a complete transfer of the West Bank to the Palestinian Authority in one fell swoop in return for a single promise to recognize Israel and cease the terrorism. Proceeding in steps has enabled the process to go at least part of the way. But this again illustrates the difficulty of sustaining the momentum as the endgame approaches.

III. TEAMWORK Teamwork is yet another way to embed one game into a larger game to enhance the credibility of strategic moves. It requires that a group of players monitor one another. If one fails to carry out a threat or a promise, others are required to inflict punishment on him; failure to do so makes them in turn vulnerable to similar punishment by others, and so on. Thus a player's payoffs in the larger game are altered in a way that makes adhering to the team's creed credible.

Many universities have academic honor codes that act as credibility devices for students. Examinations are not proctored by the faculty; instead, students are required to report to a student committee if they see any cheating. Then the committee holds a hearing and hands out punishment, as severe as suspension for a year or outright expulsion, if it finds the accused student guilty of cheating. Students are very reluctant to place their fellow students in such jeopardy. To stiffen their resolve, such codes include the added twist that failure to report an observed infraction is itself an offense against the code. Even then, the general belief is that the system works only imperfectly. A poll conducted at Princeton University last year found that only a third of students said that they would report an observed infraction, especially if they knew the guilty person.

IV. IRRATIONALITY Your threat may lack credibility because the other player knows that you are rational and that it is too costly for you to follow through with your threatened action. Therefore others believe you will not carry out the threatened action if you are put to the test. You can counter this problem by claiming to be irrational so that others will believe that your payoffs are different from what they originally perceived. Apparent irrationality can then turn into strategic rationality when the credibility of a threat is in question. Similarly, apparently irrational motives such as honor or saving face may make it credible that you will deliver on a promise even though tempted to renege.

The other player may see through such **rational irrationality.** Therefore if you attempt to make your threat credible by claiming irrationality, he will not readily believe you. You will have to acquire a reputation for irrationality, for example, by acting irrationally in some related game. You could also use one of the strategies discussed in Chapter 9 and do something that is a credible signal of irrationality to achieve an equilibrium in which you can separate from the falsely irrational.

V. CONTRACTS You can make it costly to yourself to fail to carry out a threat or to deliver on a promise by signing a **contract** under which you have to pay a sufficiently large sum in that eventuality. If such a contract is written with sufficient clarity that it can be enforced by a court or some outside authority, the change in payoffs makes it optimal to carry out the stipulated action, and the threat or the promise becomes credible.

In regard to a promise, the other player can be the other party to the contract. It is in his interest that you deliver on the promise; so he will hold you to the contract if you fail to fulfill the promise. A contract to enforce a threat is more problematic. The other player does not want you to carry out the threatened action and will not enforce the contract unless he gets some longer-term benefit in associated games from being subject to a credible threat in this one. Therefore in regard to a threat, the contract has to be with a third party. But, when you bring in a third party and a contract merely to ensure that you will carry out your threat if put to the test, the third party does not actually benefit from your failure to act as stipulated. The contract thus becomes vulnerable to any renegotiation that would provide the third-party enforcer with some positive benefits. If the other player puts you to the test, you can say to the third party, "Look, I don't want to carry out the threat. But I am being forced to do so by the prospect of the penalty in the contract, and you are not getting anything out of all this. Here is a real dollar in exchange for releasing me from the contract." Thus the contract itself is not credible; therefore neither is the threat. The third party must have its own longer-term reasons for holding you to the contract, such as wanting to maintain its reputation, if the contract is to be renegotiation proof and therefore credible.

Written contracts are usually more binding than verbal ones, but even verbal ones may constitute commitments. When George Bush said, "Read my lips; no new taxes," in the presidential campaign of 1988, the American public took this promise to be a binding contract; when Bush reneged on it in 1990, the public held that against him in the election of 1992.

VI. BRINKMANSHIP In the U.S.–Japan trade-policy game, we found that a threat might be too "large" to be credible. If a smaller but effective threat cannot be found in a natural way, the size of the large threat can be reduced to a credible level by making its fulfillment a matter of chance. The United States cannot credibly say to Japan, "If you don't keep your markets open to U.S. goods, we will not defend you if the Russians or the Chinese attack you." But it can credibly say, "If you don't keep your markets open to U.S. goods, the relations between our countries will deteriorate, which will create the risk that, if you are faced with an invasion, Congress at that time will not sanction U.S. military involvement in your aid." As mentioned earlier, such deliberate creation of risk is called brinkmanship. This is a subtle idea, difficult to put into practice. Brinkmanship is best understood by seeing it in operation, and the detailed case study of the Cuban missile crisis in Chapter 14 serves just that purpose.

We have described several devices for making one's strategic moves credible and examined how well they work. In conclusion, we want to emphasize a feature common to the entire discussion. Credibility in practice is not an all-or-nothing matter but one of degree. Even though the theory is stark—rollback analysis shows either that a threat works or that it does not—practical application must recognize that between these polar extremes lies a whole spectrum of possibility and probability.

7 COUNTERING YOUR OPPONENT'S STRATEGIC MOVES

If your opponent can make a commitment or a threat that works to your disadvantage, then, before he actually does so, you may be able to make a strategic countermove of your own. You can do so by making his future strategic move less effective, for example, by removing its irreversibility or undermining its credibility. In this section, we examine some devices that can help achieve this purpose. Some are similar to devices that the other side can use for its own needs.

A. Irrationality

Irrationality can work for the would-be receiver of a commitment or a threat just as well as it does for the other player. If you are known to be so irrational that you will not give in to any threat and will suffer the damage that befalls you when your opponent carries out that threat, then he may as well not make the threat in the first place, because having to carry it out will only end up hurting him, too. Everything that we said earlier about the difficulties of credibly convincing the other side of your irrationality holds true here as well.

B. Cutting Off Communication

If you make it impossible for the other side to convey to you the message that it has made a certain commitment or a threat, then your opponent will see no point in doing so. Thomas Schelling illustrates this possibility with the story of a child who is crying too loudly to hear his parent's threats.[6] Thus it is pointless for the parent to make any strategic moves; communication has effectively been cut off.

C. Leaving Escape Routes Open

If the other side can benefit by burning bridges to prevent its retreat, you can benefit by dousing those fires or perhaps even by constructing new bridges or roads by which your opponent can retreat. This device was also known to the ancients. Sun Tzu said, "To a surrounded enemy, you must leave a way of escape." The intent is not actually to allow the enemy to escape. Rather, "show him there is a road to safety, and so create in his mind the idea that there is an alternative to death. Then strike."[7]

D. Undermining Your Opponent's Motive to Uphold His Reputation

If the person threatening you says, "Look, I don't want to carry out this threat, but I must because I want to maintain my reputation with others," you can respond, "It is not in my interest to publicize the fact that you did not punish me. I am only interested in doing well in this game. I will keep quiet; both of us will avoid the mutually damaging outcome; and your reputation with others will stay intact." Similarly, if you are a buyer bargaining with a seller and he refuses to lower his price on the grounds that, "if I do this for you, I would have to do it for everyone else," you can point out that you are not going to tell anyone else. This may not work; the other player may suspect that you would tell a few friends who would tell a few others, and so on.

E. Salami Tactics

Salami tactics are devices used to whittle down the other player's threat in the way that a salami is cut—one slice at a time. You fail to comply with the other's wishes (whether for deterrence or compellence) to a very small degree so that it is not worth the other's while to carry out the comparatively more drastic and mutually harmful threatened action just to counter that small transgression. If that works, you transgress a little more, and a little more again, and so on.

You know this perfectly well from your own childhood. Schelling[8] gives a wonderful description of the process:

> Salami tactics, we can be sure, were invented by a child. . . . Tell a child not to go in the water and he'll sit on the bank and submerge his bare feet; he is

[6]Thomas C. Schelling, *The Strategy of Conflict* (Oxford: Oxford University Press, 1960), p. 146.
[7]Sun Tzu, *The Art of War*, pp. 109–110.
[8]Thomas C. Schelling, *Arms and Influence* (New Haven: Yale University Press, 1966), pp. 66–67.

> not yet "in" the water. Acquiesce, and he'll stand up; no more of him is in the water than before. Think it over, and he'll start wading, not going any deeper. Take a moment to decide whether this is different and he'll go a little deeper, arguing that since he goes back and forth it all averages out. Pretty soon we are calling to him not to swim out of sight, wondering whatever happened to all our discipline.

Salami tactics work particularly well against compellence, because they can take advantage of the *time* dimension. When your mother tells you to clean up your room "or else," you can put off the task for an extra hour by claiming that you have to finish your homework, then for a half day because you have to go to football practice, then for an evening because you can't possibly miss the Simpsons on TV, and so on.

To counter the countermove of salami tactics you must make a correspondingly graduated threat. There should be a scale of punishments that fits the scale of noncompliance or procrastination. This can also be achieved by gradually raising the risk of disaster, another application of brinkmanship.

SUMMARY

Actions taken by players to fix the rules of later play are known as *strategic moves*. These first moves must be *observable* and *irreversible* to be true first moves, and they must be credible if they are to have their desired effect in altering the equilibrium outcome of the game. *Commitment* is an unconditional first move used to seize a first-mover advantage when one exists. Such a move usually entails committing to a strategy that would not have been one's equilibrium strategy in the original version of the game.

Conditional first moves such as *threats* and *promises* are *response rules* designed either to *deter* rivals' actions and preserve the status quo or to *compel* rivals' actions and alter the status quo. Threats carry the possibility of mutual harm but cost nothing if they work; threats that create only the risk of a bad outcome fall under the classification of *brinkmanship*. Promises are costly only to the maker and only if they are successful. Threats can be arbitrarily large, although excessive size compromises credibility, but promises are usually kept just large enough to be effective. If the implicit promise (or threat) that accompanies a threat (or promise) is not credible, players must make a move that combines both a promise and a threat and see to it that both components are credible.

Credibility must be established for any strategic move. There are a number of general principles to consider in making moves credible and a number of specific devices that can be used to acquire credibility. They generally work either by reducing your own future freedom to choose or by altering your own payoffs from future actions. Specific devices of this kind include establishing a *reputation*, using teamwork, demonstrating apparent irrationality, burning bridges, and making *contracts*, although the acquisition of credibility is often context specific. Similar devices exist for countering strategic moves made by rival players.

KEY TERMS

brinkmanship (312)
commitment (313)
compellence (314)
contract (336)
deterrence (314)
doomsday device (332)
irreversible (312)
observable (312)
promise (314)
rational irrationality (335)
reputation (333)
response rule (314)
salami tactics (000)
strategic moves (311)
threat (314)

EXERCISES

1. "One could argue that the size of a promise is naturally bounded, while in principle a threat can be arbitrarily severe so long as it is credible (and error free)." First, briefly explain why the statement is true. Despite the truth of the statement, players might find that an arbitrarily severe threat might not be to their advantage. Explain why the latter statement is also true.

2. In a scene from the movie *Manhattan Murder Mystery*, Woody Allen and Diane Keaton are at a hockey game in Madison Square Garden. She is obviously not enjoying herself, but he tells her: "Remember our deal. You stay here with me for the entire hockey game, and next week I will come to the opera with you and stay until the end." Later, we see them coming out of the Met into a deserted Lincoln Center square while inside the music is still playing. Keaton is visibly upset: "What about our deal? I stayed to the end of the hockey game, and so you were supposed to stay till the end of the opera." Allen answers: "You know I can't listen to too much Wagner. At the end of the first act, I already felt the urge to invade Poland." Comment on the strategic choices made here by using your knowledge of the theory of strategic moves and credibility.

3. Consider a game between a parent and a child. The child can choose to be good (G) or bad (B); the parent can punish the child (P) or not (N). The child gets enjoyment worth a 1 from bad behavior, but hurt worth -2 from pun-

ishment. Thus a child who behaves well and is not punished gets a 0; one who behaves badly and is punished gets $1 - 2 = -1$; and so on. The parent gets -2 from the child's bad behavior and -1 from inflicting punishment.

(a) Set up this game as a simultaneous-move game, and find the equilibrium.

(b) Next, suppose that the child chooses G or B first and that the parent chooses its P or N after having observed the child's action. Draw the game tree and find the subgame-perfect equilibrium.

(c) Now suppose that, before the child acts, the parent can commit to a strategy—for example, the threat "P if B" ("If you behave badly, I will punish you"). How many such strategies does the parent have? Write down the table for this game. Find all pure-strategy Nash equilibria.

(d) How do your answers to parts **b** and **c** differ? Explain the reason for the difference.

4. For each of the following three games, answer these questions: (i) What is the equilibrium if neither player can use any strategic moves? (ii) Can one player improve his payoff by using a strategic move (commitment, threat, or promise) or a combination of such moves? If so, which player makes what strategic move(s)?

(a)

		COLUMN	
		Left	Right
ROW	Up	0, 0	2, 1
	Down	1, 2	0, 0

(b)

		COLUMN	
		Left	Right
ROW	Up	4, 3	3, 4
	Down	2, 1	1, 2

(c)

		COLUMN	
		Left	Right
ROW	Up	4, 1	2, 2
	Down	3, 3	1, 4

5. The general strategic game in Thucydides' history of the Peloponnesian War has been expressed in game-theoretic terms by Professor William Charron of St. Louis University.[9] Athens had acquired a large empire of coastal cities around the Aegean as a part of its leadership role in defending the Greek world from Persian invasions. Sparta, fearing Athenian power, was contemplating war against Athens. If Sparta decided against war, Athens would have to decide whether to retain or relinquish its empire. But Athens in turn feared that, if it gave independence to the cities, they could choose to join Sparta in a greatly strengthened alliance against Athens and receive very favorable terms from Sparta for doing so. Thus there are three players, Sparta, Athens, and Small cities, who move in this order. There are four outcomes, and the payoffs are as follows (4 being best):

Outcome	Sparta	Athens	Small cities
War	2	2	2
Athens retains empire	1	4	1
Small cities join Sparta	4	1	4
Small cities stay independent	3	3	3

(a) Draw the game tree and find the rollback equilibrium. Is there another outcome that is better for all players?
(b) What strategic move or moves could attain the better outcome? Discuss the credibility of such moves.

6. In the classic film *Mary Poppins*, the Banks children are players in a strategic game with a number of different nannies. In their view of the world, nannies are inherently harsh, but playing tricks on such nannies is great fun. That is, they view themselves as playing a game in which the nanny moves first, showing herself to be either Harsh or Nice, and the children move second, choosing to be Good or Mischievous. The nanny prefers to have Good children to take care of but is also inherently harsh, and so she gets her highest payoff of 4 from (Harsh, Good) and her lowest payoff of 1 from (Nice, Mischievous), with (Nice, Good) yielding 3 and (Harsh, Mischievous) yielding 2. The children similarly most prefer to have a Nice nanny and then to be Mischievous; they get their highest two payoffs when the nanny is Nice (4 if Mischievous, 3 if Good) and their lowest two payoffs when the nanny is Harsh (2 if Mischievous, 1 if Good).
(a) Draw the game tree for this game and find the subgame-perfect equilibrium in the absence of any strategic moves.

[9]William C. Charron, "Greeks and Games: Forerunners of Modern Game Theory," *Forum for Social Economics*, vol. 29, no. 2 (Spring 2000), pp. 1–32.

(b) In the film, before the arrival of Mary Poppins, the children write their own ad for a new nanny in which they state: "If you won't scold and dominate us, we will never give you cause to hate us; we won't hide your spectacles so you can't see, put toads in your bed, or pepper in your tea." Use the tree from part a to argue that this statement constitutes a promise. What would the outcome of the game be if the promise works?

(c) What is the implied threat that goes with the promise in part b? Is that promise automatically credible? Explain your answer.

(d) How could the children make the promise in part b credible?

(e) Is the promise in part b compellent or deterrent? Explain your answer by referring to the status quo in the game—namely, what would happen in the absence of the strategic move.

7. It is possible to reconfigure the payoffs in the game in Exercise 6 so that the children's statement in their ad is a threat, rather than a promise.

(a) Redraw the tree from Exercise 6a and fill in payoffs for both players so that the kids' statement becomes a *threat* in the full technical sense.

(b) Define the status quo in your game, and determine whether the threat is deterrent or compellent.

(c) Explain why the threatened action is not automatically credible, given your payoff structure.

(d) Explain why the implied promise *is* automatically credible.

(e) Explain why the kids would want to make a threat in the first place, and suggest a way in which they might make their threatened action credible.

8. The following is an interpretation of the rivalry between the United States and the Soviet Union for geopolitical influence in the 1970s and 1980s.[10] Each side has the choice of two strategies: Aggressive and Restrained. The Soviet Union wants to achieve world domination, so being Aggressive is its dominant strategy. The United States wants to prevent the Soviet Union from achieving world domination; it will match Soviet aggressiveness with aggressiveness, and restraint with restraint. Specifically, the payoff table is:

		SOVIET UNION	
		Restrained	Aggressive
UNITED STATES	Restrained	4, 3	1, 4
	Aggressive	3, 1	2, 2

[10]We thank political science professor Thomas Schwartz at UCLA for the idea for this exercise.

For each player, 4 is best and 1 is worst.

(a) Consider this game when the two countries move simultaneously. Find the Nash equilibrium.

(b) Next consider three different and alternative ways in which the game could be played with sequential moves: (i) The United States moves first and the Soviet Union moves second. (ii) The Soviet Union moves first and the United States moves second. (iii) The Soviet Union moves first and the United States moves second, but the Soviet Union has a further move in which it can change its first move. For each case, draw the game tree and find the subgame-perfect equilibrium.

(c) What are the key strategic matters (commitment, credibility, and so on) for the two countries?

9. Consider the following games. In each case, (i) identify which player can benefit from making a strategic move, (ii) identify the nature of the strategic move appropriate for this purpose, (iii) discuss the conceptual and practical difficulties that will arise in the process of making this move credible, and (iv) discuss whether and how the difficulties can be overcome.

(a) The other countries of the European Monetary Union (France, Germany, and so on) would like Britain to join the common currency and the common central bank.

(b) The United States would like North Korea to stop exporting missiles and missile technology to countries such as Iran and would like China to join the United States in working toward this aim.

(c) The United Auto Workers would like U.S. auto manufacturers not to build plants in Mexico and would like the U.S. government to restrict imports of autos made abroad.

(d) The students at your university or college want to prevent the administration from raising tuition.

(e) Most participants, as well as outsiders, want to achieve a durable peace in situations such as those in Northern Ireland and the Middle East.

10. Write a brief description of a game in which you have participated, entailing strategic moves such as a commitment, threat, or promise and paying special attention to the essential aspect of credibility. Provide an illustration of the game if possible, and explain why the game that you describe ended as it did. Did the players use sound strategic thinking in making their choices?

11

The Prisoners' Dilemma and Repeated Games

IN THIS CHAPTER, we continue our study of broad classes of games with an analysis of the prisoners' dilemma game. It is probably *the* classic example of the theory of strategy and its implications for predicting the behavior of game players, and most people who learn only a little bit of game theory learn about it. Even people who know *no* game theory may know the basic story behind this game or they may have at least heard that it exists. The prisoners' dilemma is a game in which each player has a dominant strategy, but the equilibrium that arises when all players use their dominant strategies provides a worse outcome for every player than would arise if they all used their dominated strategies instead. The paradoxical nature of this equilibrium outcome leads to several more complex questions about the nature of the interactions that only a more thorough analysis can hope to answer. The purpose of this chapter is to provide that additional thoroughness.

We already considered the prisoners' dilemma in Section 3 of Chapter 4. There we took note of the curious nature of the equilibrium that is actually a "bad" outcome for the players. The "prisoners" can find another outcome that both prefer to the equilibrium outcome, but they find it difficult to bring about. The focus of this chapter is the potential for achieving that better outcome. That is, we consider whether and how the players in a prisoners' dilemma can attain and sustain their mutually beneficial cooperative outcome, overcoming their separate incentives to defect for individual gain. We first review the standard prisoners' dilemma game and then develop three categories of solutions. The first and most important method of solution consists of repetition of the standard one-shot game. Two other potential solutions rely on penalty (or reward) schemes and on the role of leadership. As we consider each potential solution, the importance of the costs of defecting and the benefits of cooperation will become clear.

This chapter concludes with a discussion of some of the experimental evidence regarding the prisoners' dilemma as well as several examples of actual dilemmas in action. Experiments generally put live players in a variety of prisoners' dilemma–type games and show some perplexing as well as some more predictable behavior; experiments conducted with the use of computer simulations yield additional interesting outcomes. Our examples of real-world dilemmas that end the chapter are provided to give a sense of the diversity of situations in which prisoners' dilemmas arise and to show how, in at least one case, players may be able to create their own solution to the dilemma.

1 THE BASIC GAME (REVIEW)

Before we consider methods for avoiding the "bad" outcome in the prisoners' dilemma, we briefly review the basics of the game. Recall our example from Chapter 4 of the husband and wife suspected of murder. Each is interrogated separately and can choose to confess to the crime or to deny any involvement. The payoff matrix that they face was originally presented as Figure 4.4 and is reproduced here as Figure 11.1. The numbers shown indicate years in jail; therefore low numbers are better for both players.

Both players here have a dominant strategy. Each does better to confess, regardless of what the other player does. The equilibrium outcome entails both players deciding to confess and each getting 10 years in jail. If they both had chosen to deny any involvement, however, they would have been better off, with only 3 years of jail time to serve.

In any prisoners' dilemma game, there is always a *cooperative strategy* and a *cheating* or *defecting strategy*. In Figure 11.1, Deny is the cooperative strategy; both players using that strategy yields the best outcome for the players. Confess is

		WIFE	
		Confess (Defect)	Deny (Cooperate)
HUSBAND	Confess (Defect)	10 yr, 10 yr	1 yr, 25 yr
	Deny (Cooperate)	25 yr, 1 yr	3 yr, 3 yr

FIGURE 11.1 Payoffs for the Standard Prisoners' Dilemma

the cheating or defecting strategy; when the players do not cooperate with each other, they choose to Confess in the hope of attaining individual gain at the rival's expense. Thus, players in a prisoners' dilemma can always be labeled, according to their choice of strategy, as either *defectors* or *cooperators*. We will use this labeling system throughout the discussion of potential solutions to the dilemma.

We want to emphasize that, although we speak of a cooperative strategy, the prisoners' dilemma game is noncooperative in the sense explained in Chapter 2—namely, the players make their decisions and implement their choices individually. If the two players could discuss, choose, and play their strategies jointly—as, for example, if the prisoners were in the same room and could give a joint answer to the question of whether they were both going to confess—there would be no difficulty about their achieving the outcome that both prefer. The essence of the questions of whether, when, and how a prisoners' dilemma can be resolved is the difficulty of achieving a cooperative (jointly preferred) outcome through noncooperative (individual) actions.

2 SOLUTIONS I: REPETITION

Of all the mechanisms that can sustain cooperation in the prisoners' dilemma, the most well known and the most natural is **repeated play** of the game. Repeated or ongoing relationships between players imply special characteristics for the games that they play against each other. In the prisoners' dilemma, this result plays out in the fact that each player fears that one instance of defecting will lead to a collapse of cooperation for the future. If the value of future cooperation is large and exceeds what can be gained in the short term by defecting, then the long-term individual interests of the players can automatically and tacitly keep them from defecting, without the need for any additional punishments or enforcement by third parties.

We consider here the meal-pricing dilemma faced by the two restaurants, Xavier's Tapas and Yvonne's Bistro, introduced in Chapter 5. For our purposes here, we have chosen to simplify that game by supposing that only two choices of price are available: the jointly best (collusive) price of \$26 or the Nash equilibrium price of \$20. The payoffs (profits measured in hundreds of dollars per month) for each restaurant can be calculated by using the quantity (demand) functions in Section 1.A of Chapter 5; these payoffs are shown in Figure 11.2. As in any prisoners' dilemma, each store has a dominant strategy to defect and price its meals at \$20, although both stores would prefer the outcome in which each cooperates and charges the higher price of \$26 per meal.

Let us start our analysis by supposing that the two restaurants are initially in the cooperative mode, each charging the higher price of \$26. If one restaurant,

		YVONNE'S BISTRO	
		20 (Defect)	26 (Cooperate)
XAVIER'S TAPAS	20 (Defect)	288, 288	360, 216
	26 (Cooperate)	216, 360	324, 324

FIGURE 11.2 Prisoners' Dilemma of Pricing ($100s per month)

say Xavier's, deviates from this pricing strategy, it can increase its profit from 324 to 360 (from $32,400 to $36,000) for 1 month. But then cooperation has dissolved and Xavier's rival, Yvonne's, will see no reason to cooperate from then on. Once cooperation has broken down, presumably permanently, the profit for Xavier's is 288 each month instead of the 324 it would have been if Xavier's had never defected in the first place. By gaining 36 ($3,600) in one month of defecting, Xavier's gives up 36 ($3,600) each month thereafter by destroying cooperation. Even if the relationship lasts as little as 3 months, it seems that defecting is not in Xavier's best interest. A similar argument can be made for Yvonne's. Thus, if the two restaurants competed on a regular basis for at least 3 months, it seems that we might see cooperative behavior and high prices rather than the defecting behavior and low prices predicted by theory for the one-shot game.

A. Finite Repetition

But the solution of the dilemma is not actually that simple. What if the relationship did last exactly 3 months? Then strategic restaurants would want to analyze the full 3-month game and choose their optimal pricing strategies. Each would use rollback to determine what price to charge each month. Starting their analyses with the third month, they would realize that, at that point, there was no future relationship to consider. Each restaurant would find that it had a dominant strategy to defect. Given that, there is effectively no future to consider in the second month either. Each player knows that there will be mutual defecting in the third month, and therefore both will defect in the second month; defecting is the dominant strategy in month 2 also. Then the same argument applies to the first month as well. Knowing that both will defect in months 2 and 3 anyway, there is no future value of cooperation in the first month. Both players defect right from the start, and the dilemma is alive and well.

This result is very general. As long as the relationship between the two players in a prisoners' dilemma game lasts a fixed and known length of time, the dominant-strategy equilibrium with defecting should prevail in the last period of play. When the players arrive at the end of the game, there is never any value to continued cooperation, and so they defect. Then rollback predicts mutual defecting all the way back to the very first play. However, in practice, players in finitely repeated prisoners' dilemma games show a lot of cooperation; more on this to come.

B. Infinite Repetition

Analysis of the finitely repeated prisoners' dilemma shows that even repetition of the game cannot guarantee the players a solution to their dilemma. But what would happen if the relationship did not have a predetermined length? What if the two restaurants expected to continue competing with each other indefinitely? Then our analysis must change to incorporate this new aspect of their interaction, and we will see that the incentives of the players change also.

In repeated games of any kind, the sequential nature of the relationship means that players can adopt strategies that depend on behavior in preceding plays of the games. Such strategies are known as **contingent strategies,** and several specific examples are used frequently in the theory of repeated game. Most contingent strategies are **trigger strategies.** A player using a trigger strategy plays cooperatively as long as her rival(s) do so, but any defection on their part "triggers" a period of **punishment,** of specified length, in which she plays noncooperatively in response. Two of the best-known trigger strategies are the grim strategy and tit-for-tat. The **grim strategy** entails cooperating with your rival until such time as she defects from cooperation; once a defection has occurred, you punish your rival (by choosing the Defect strategy) on every play for the rest of the game.[1] **Tit-for-tat (TFT)** is not as harshly unforgiving as the grim strategy and is famous (or infamous) for its ability to solve the prisoners' dilemma without requiring permanent punishment. Playing TFT means choosing, in any specified period of play, the action chosen by your rival in the preceding period of play. Thus, when playing TFT, you cooperate with your rival if she cooperated during the most recent play of the game and defect (as punishment) if your rival defected. The punishment phase lasts only as long as your rival continues to defect; you will return to cooperation one period after she chooses to do so.

Let us consider how play might proceed in the repeated restaurant pricing game if one of the players uses the contingent strategy tit-for-tat. We have already seen that, if Xavier's Tapas defects one month, it could add 36 to its profits (360 instead of 324). But if Xavier's rival is playing TFT, then such defecting would induce Yvonne's Bistro to punish Xavier's the next month in retaliation. At that point, Xavier's has two choices. One option is to continue to defect by pricing at $20, and to endure Yvonne's continued punishment according to TFT; in this case, Xavier's loses 36 (288 rather than 324) for every month thereafter in the

[1]Defecting as retaliation under the requirements of a trigger strategy is often termed *punishing,* to distinguish it from the original decision to deviate from cooperation.

foreseeable future. This option appears quite costly. But Xavier's *could* get back to cooperation, too, if it so desired. By reverting to the cooperative price of $26 after one month's defection, Xavier's would incur only one month's punishment from Yvonne's. During that month, Xavier's would suffer a loss in profit of 108 (216 rather than the 324 that would have been earned without any defection). In the second month after Xavier's defection, both restaurants could be back at the cooperative price earning 324 each month. This one-time defection yields an extra 36 in profit but costs an additional 108 during the punishment, also apparently quite costly to Xavier's.

It is important to realize here, however, that Xavier's extra $36 from defecting is gained in the first month. Its losses are ceded in the future. Therefore the relative importance of the two depends on the relative importance of the present versus the future. Here, because payoffs are calculated in dollar terms, an objective comparison can be made. Generally, money (or profit) that is earned today is better than money that is earned later because, even if you do not need (or want) the money until later, you can invest it now and earn a return on it until you need it. So Xavier's should be able to calculate whether it is worthwhile to defect, on the basis of the total rate of return on its investment (including capital gains and/or dividends and/or interest, depending on the type of investment). We use the symbol r to denote this rate of return. Thus one dollar invested generates r dollars of interest and/or dividends and/or capital gains, or 100 dollars generate $100r$, therefore the rate of return is sometimes also said to be $100r\%$.

Note that we can calculate whether it is in Xavier's interest to defect because the firms' payoffs are given in dollar terms, rather than as simple ratings of outcomes, as in some of the games in earlier chapters (the street-garden game in Chapters 3 and 6, for example). This means that payoff values in different cells are directly comparable; a payoff of 4 (dollars) is twice as good as a payoff of 2 (dollars) here, whereas a payoff of 4 is not necessarily exactly twice as good as a payoff of 2 in any two-by-two game in which the four possible outcomes are ranked from 1 (worst) to 4 (best). As long as the payoffs to the players are given in measurable units, we can calculate whether defecting in a prisoners' dilemma game is worthwhile.

I. IS IT WORTHWHILE TO DEFECT ONLY ONCE AGAINST A RIVAL PLAYING TFT? One of Xavier's options when playing repeatedly against a rival using TFT is to defect just once from a cooperative outcome and then to return to cooperating. This particular strategy gains the restaurant 36 in the first month (the month during which it defects) but loses it 108 in the second month. By the third month, cooperation is restored. Is defecting for only 1 month worth it?

We cannot directly compare the 36 gained in the first month with the 108 lost in the second month, because the additional money value of time must be incorporated into the calculation. That is, we need a way to determine how much the 108 lost in the second month is worth during the first month. Then we can compare that number with 36 to see whether defecting once is worthwhile. What we are looking for is the **present value (PV)** of 108, or how much in profit earned this month (in the present) is equivalent to (has the same value as) 108 earned next month. We need to determine the number of dollars earned this month that, with interest, would give us 108 next month; we call that number PV, the present value of 108.

Given that the (monthly) total rate of return is r, getting PV this month and investing it until next month yields a total next month of PV + rPV, where the first term is the principal being paid back and the second term is the return (interest or dividend or capital gain). When the total is exactly 108, then PV equals the present value of 108. Setting PV + rPV = 108 yields a solution for PV:

$$\mathrm{PV} = \frac{108}{1 + r}.$$

For any value of r, we can now determine the exact number of dollars that, earned this month, would be worth 108 next month.

From the perspective of Xavier's Tapas, the question remains whether the gain of 36 this month is offset by the loss of 108 next month. The answer depends on the value of PV. Xavier's must compare the gain of 36 with the PV of the loss of 108. To defect once (and then return to cooperation) is worthwhile only if $36 > 108/(1 + r)$. This is the same as saying that defecting once is beneficial only if $36(1 + r) > 108$, which reduces to $r > 2$. Thus Xavier's should choose to defect once against a rival playing TFT only if the monthly total rate of return exceeds 200%. This outcome is very unlikely; for example, prime lending rates rarely exceed 12% per year. This translates into a monthly interest rate of no more than 1% (compounded annually, not monthly), well below the 200% just calculated. Here, it is better for Xavier's to continue cooperating than to try a single instance of defecting when Yvonne's is playing TFT.

II. IS IT WORTHWHILE TO DEFECT FOREVER AGAINST A RIVAL PLAYING TFT? What about the possibility of defecting once and then continuing to defect forever? This second option of Xavier's gains the restaurant 36 in the first month but loses it 36 in every month thereafter into the future if the rival restaurant plays TFT. To determine whether such a strategy is in Xavier's best interest again depends on the present value of the losses incurred. But this time the losses are incurred over an **infinite horizon** of future months of competition.

We need to figure out the present value of all of the 36s that are lost in future months, add them all up, and compare them with the 36 gained during the month of defecting. The PV of the 36 lost during the first month of punishment and continued defecting on Xavier's part is just $36/(1 + r)$; the calculation is identical with that used in Section 2.B.I to find that the PV of 108 was $108/(1 + r)$. For the next month, the PV must be the dollar amount needed this month that, with two months of **compound interest,** would yield 36 in two months. If the PV is invested now, then in 1 month the investor would have that principal amount plus a return of rPV, for a total of PV + rPV, as before; leaving this total amount invested for the second month means that the investor, at the end of 2 months, has the amount invested at the beginning of the second month (PV + rPV) plus the return on that amount, which would be r(PV + rPV). The PV of the 36 lost 2 months from now must then solve the equation: PV + rPV + r(PV + rPV) = 36. Working out the value of PV here yields $PV(1 + r)^2 = 36$, or $PV = 36/(1 + r)^2$. You should see a pattern developing. The PV of the 36 lost in the third month of continued defecting is $36/(1 + r)^3$, and the PV of the 36 lost in the fourth month is $36/(1 + r)^4$. In fact, the PV of the 36 lost in the nth month of continued defecting is just $36/(1 + r)^n$. Xavier's loses an infinite sum of 36s, and the PV of each of them gets smaller each month.

More precisely, Xavier's loses the sum, from $n = 1$ to $n = \infty$ (where n labels the months of continued defecting after the initial month), of $36/(1 + r)^n$. Mathematically, it is written as the sum of an infinite number of terms:[2]

$$36/(1 + r) + 36/(1 + r)^2 + 36/(1 + r)^3 + 36/(1 + r)^4 + \cdots.$$

Because r is a rate of return and presumably a positive number, the ratio of $1/(1 + r)$ will be less than 1; this ratio is generally called the **discount factor** and referred to by using the Greek letter δ. With $\delta = 1/(1 + r) < 1$, the mathematical rule for infinite sums tells us that this sum converges to a specific value; in this case $36/r$.

It is now possible to determine whether Xavier's Tapas will choose to defect forever. The restaurant compares its gain of 36 with the PV of all the lost 36s, or $36/r$. Then it defects forever only if $36 > 36/r$, or $r > 1$; defecting forever is beneficial in this particular game only if the monthly rate of return exceeds 100%, an unlikely event. Thus we would not expect Xavier's to defect against a cooperative rival when both are playing tit-for-tat. When both Yvonne's Bistro and Xavier's Tapas play TFT, the cooperative outcome in which both price high is a Nash equilibrium of the game. Both playing TFT is a Nash equilibrium, and use of this contingent strategy solves the prisoners' dilemma for the two restaurants.

Remember that tit-for-tat is only one of many trigger strategies that could be used by players in repeated prisoners' dilemma. And it is one of the "nicer" ones. Thus if TFT can be used to solve the dilemma for the two restaurants, other, harsher trigger strategies should be able to do the same. The grim strategy, for instance, also can be used to sustain cooperation in this infinitely repeated game and others.

[2]The Appendix to this chapter contains a detailed discussion of the solution of infinite sums.

C. Games of Unknown Length

In addition to considering games of finite or infinite length, we can incorporate a more sophisticated tool to deal with games of unknown length. It is possible that, in some repeated games, players might not know for certain exactly how long their interaction will continue. They may, however, have some idea of the *probability* that the game will continue for another period. For example, our restaurants might believe that their repeated competition will continue only as long as their customers find *prix fixe* menus to be the dining-out experience of choice; if there were some probability each month that *à la carte* dinners would take over that role, then the nature of the game is altered.

Recall that the present value of a loss next month is already worth only $\delta = 1/(1 + r)$ times the amount earned. If in addition there is only a probability p (less than 1) that the relationship will actually continue to the next month, then next month's loss is worth only p times δ times the amount lost. For Xavier's Tapas, this means that the PV of the 36 lost with continued defecting is worth $36 \times \delta$ [the same as $36/(1 + r)$] when the game is assumed to be continuing with certainty but is worth only $36 \times p \times \delta$ when the game is assumed to be continuing with probability p. Incorporating the probability that the game may end next period means that the present value of the lost 36 is smaller, because $p < 1$, than it is when the game is definitely expected to continue (when p is *assumed* to equal 1).

The effect of incorporating p is that we now effectively discount future payoffs by the factor $p \times \delta$ instead of simply by δ. We call this **effective rate of return** R, where $1/(1 + R) = p \times \delta$, and R depends on p and δ as shown:[3]

$$1/(1 + R) = p\delta$$
$$1 = p\delta\,(1 + R)$$
$$R = \frac{1 - p\delta}{p\delta}$$

With a 5% actual rate of return on investments ($r = 0.05$, and so $\delta = 1/1.05 = 0.95$) and a 50% chance that the game continues for an additional month ($p = 0.5$), then $R = [1 - (0.5)(0.95)]/(0.5)(0.95) = 1.1$, or 110%.

Now the high rates of return required to destroy cooperation (encourage defection) in these examples seem more realistic if we interpret them as effective rather than actual rates of return. It becomes conceivable that defecting forever, or even once, might actually be to one's benefit if there is a large enough probability that the game will end in the near future. Consider Xavier's decision whether to defect forever against a TFT-playing rival. Our earlier calculations showed that permanent defecting is beneficial only when r exceeds 1, or 100%. If Xavier's faces the 5% actual rate of return and the 50% chance that the game will continue for an

[3]We could also express R in terms of r and p, in which case $R = (1 + r)/p - 1$.

additional month, as we assumed in the preceding paragraph, then the effective rate of return of 110% will exceed the critical value needed for it to continue defecting. Thus the cooperative behavior sustained by the TFT strategy can break down if there is a sufficiently large chance that the repeated game might be over by the end of the next period of play—that is, by a sufficiently small value of p.

D. General Theory

We can easily generalize the ideas about when it is worthwhile to defect against TFT-playing rivals so that you can apply them to any prisoners' dilemma game that you encounter. To do so, we use a table with general payoffs (delineated in appropriately measurable units) that satisfy the standard structure of payoffs in the dilemma as in Figure 11.3. The payoffs in the table must satisfy the relation H > C > D > L for the game to be a prisoners' dilemma, where C is the *cooperative* outcome, D is the payoff when both players *defect* from cooperation, H is the *high* payoff that goes to the defector when one player defects while the other cooperates, and L is the *low* payoff that goes to the loser (the cooperator) in the same situation.

In this general version of the prisoners' dilemma, a player's one-time gain from defecting is $(H - C)$. The single-period loss for being punished while you return to cooperation is $(C - L)$, and the per-period loss for perpetual defecting is $(C - D)$. To be as general as possible, we will allow for situations in which there is a probability $p < 1$ that the game continues beyond the next period and so we will discount payoffs using an effective rate of return of R per period. If $p = 1$, as would be the case when the game is guaranteed to continue, then $R = r$, the simple interest rate used in our preceding calculations. Replacing r with R, we find that the results attained earlier generalize almost immediately.

We found earlier that a player defects exactly once against a rival playing TFT if the one-time gain from defecting $(H - C)$ exceeds the present value of the single-period loss from being punished (the PV of $C - L$). In this general game, that means that a player defects once against a TFT-playing opponent only if $(H - C) > (C - L)/(1 + R)$, or $(1 + R)(H - C) > C - L$, or

$$R > \frac{C - L}{H - C} - 1.$$

		COLUMN	
		Defect	Cooperate
ROW	Defect	D, D	H, L
	Cooperate	L, H	C, C

FIGURE 11.3 General Version of the Prisoners' Dilemma

Similarly, we found that a player defects forever against a rival playing TFT only if the one-time gain from defecting exceeds the present value of the infinite sum of the per-period losses from perpetual defecting (where the per-period loss is $C - D$). For the general game, then, a player defects forever against a TFT-playing opponent only if $(H - C) > (C - D)/R$, or

$$R > \frac{C - D}{H - C}.$$

The three critical elements in a player's decision to defect, as seen in these two expressions, are the immediate gain from defection $(H - C)$, the future losses from punishment ($C - L$ or $C - D$ per period of punishment), and the value of the effective rate of return (R, which measures the importance of the present relative to the future). Under what conditions on these various values do players find it attractive to defect from cooperation?

First, assume that the values of the gains and losses from defecting are fixed. Then changes in R determine whether a player defects, and defection is more likely when R is large. Large values of R are associated with small values of p and small values of δ (and large values of r); so defection is more likely when the probability of continuation is low or the discount factor is low (or the interest rate is high). Another way to think about it is that defection is more likely when the future is less important than the present or when there is little future to consider; that is, defection is more likely when players are impatient or when the game is expected to end quickly.

Second, consider the case in which the effective rate of return is fixed, as is the one-period gain from defecting. Then changes in the per-period losses associated with punishment determine whether defecting is worthwhile. Here it is smaller values of $C - L$ or $C - D$ that encourage defection. In this case, defection is more likely when punishment is not very severe.[4]

Finally, assume that the effective rate of return and the per-period losses associated with punishment are held constant. Now players are more likely to defect when the gains, $H - C$, are high. This situation is more likely when defecting garners a player large and immediate benefits.

This discussion also highlights the importance of the detection of defecting. Decisions about whether to continue along a cooperative path depend on how long defecting might be able to go on before it is detected, on how accurately it is detected, and on how long any punishment can be made to last before an attempt is made to revert back to cooperation. Although our model does not incorporate

[4]The costs associated with defection may also be smaller if information transmission is not perfect, as might be the case if there are many players, and so difficulties might arise in identifying the defector and in coordinating a punishment scheme. Similarly, gains from defection may be larger if rivals cannot identify a defection immediately.

these considerations explicitly, if defecting can be detected accurately and quickly, its benefit will not last long, and the subsequent cost will have to be paid more surely. Therefore the success of any trigger strategy in resolving a repeated prisoners' dilemma depends on how well (both in speed and accuracy) players can detect defecting. This is one reason why the TFT strategy is often considered dangerous; slight errors in the execution of actions or in the perception of those actions can send players into continuous rounds of punishment from which they may not be able to escape for a long time, until a slight error of the opposite kind occurs.

You can use all of these ideas to guide you in when to expect more cooperative behavior between rivals and when to expect more defecting and cutthroat actions. If times are bad and an entire industry is on the verge of collapse, for example, so that businesses feel that there is no future, competition may become more fierce (less cooperative behavior may be observed) than in normal times. Even if times are temporarily good but are not expected to last, firms may want to make a quick profit while they can; so cooperative behavior might again break down. Similarly, in an industry that emerges temporarily because of a quirk of fashion and is expected to collapse when fashion changes, we should expect less cooperation. Thus a particular beach resort might become the place to go, but all the hotels there will know that such a situation cannot last, and so they cannot afford to collude on pricing. If, on the other hand, the shifts in fashion are among products made by an unchanging group of companies in long-term relationships with one another, cooperation might persist. For example, even if all the children want cuddly bears one year and Power Ranger action figures the next, collusion in pricing may occur if the same small group of manufacturers makes both items.

In Chapter 12, we will look in more detail at prisoners' dilemmas that arise in games with many players. We examine when and how players can overcome such dilemmas and achieve outcomes better for them all.

3 SOLUTIONS II: PENALTIES AND REWARDS

Although repetition is the major vehicle for the solution of the prisoners' dilemma, there are also several others that can be used to achieve this purpose. One of the simplest ways to avert the prisoners' dilemma in the one-shot version of the game is to inflict some direct **penalty** on the players when they defect. When the payoffs have been altered to incorporate the cost of the penalty, players may find that the dilemma has been resolved.[5]

[5]Note that we get the same type of outcome in the repeated-game case considered in Section 2.

		WIFE	
		Confess	Deny
HUSBAND	Confess	10 yr, 10 yr	21 yr, 25 yr
	Deny	25 yr, 21 yr	3 yr, 3 yr

FIGURE 11.4 Prisoners' Dilemma with Penalty for the Lone Defector

Consider the husband–wife dilemma from Section 1. If only one player defects, the game's outcome entails 1 year in jail for the defector and 25 years for the cooperator. The defector, though, getting out of jail early, might find the cooperator's friends waiting outside the jail. The physical harm caused by those friends might be equivalent to an additional 20 years in jail. If so, and if the players account for the possibility of this harm, then the payoff structure of the original game has changed.

The "new" game, with the physical penalty included in the payoffs, is illustrated in Figure 11.4. With the additional 20 years in jail added to each player's sentence when one player confesses while the other denies, the game is completely different.

A search for dominant strategies in Figure 11.4 shows that there are none. A cell-by-cell check then shows that there are now two pure-strategy Nash equilibria. One of them is the (Confess, Confess) outcome; the other is the (Deny, Deny) outcome. Now each player finds that it is in his or her best interest to cooperate if the other is going to do so. The game has changed from being a prisoners' dilemma to an assurance game, which we studied in Chapter 4. Solving the new game requires selecting an equilibrium from the two that exist. One of them—the cooperative outcome—is clearly better than the other from the perspective of both players. Therefore it may be easy to sustain it as a focal point if some convergence of expectations can be achieved.

Notice that the penalty in this scenario is inflicted on a defector only when his or her rival does *not* defect. However, stricter penalties can be incorporated into the prisoners' dilemma, such as penalties for *any* confession. Such discipline typically must be imposed by a third party with some power over the two players, rather than by the other player's friends, because the friends would have little standing to penalize the first player when their associate also defects. If both prisoners are members of a special organization (such as a gang or crime mafia) and the organization has a standing rule of never confessing to the police under penalty of extreme physical harm, the game changes again to the one illustrated in Figure 11.5.

Now the equivalent of an additional 20 years in jail is added to *all* payoffs associated with the Confess strategy. (Compare Figures 11.5 and 11.1.) In the

		WIFE	
		Confess	Deny
HUSBAND	Confess	30 yr, 30 yr	21 yr, 25 yr
	Deny	25 yr, 21 yr	3 yr, 3 yr

FIGURE 11.5 Prisoners' Dilemma with Penalty for Any Defecting

new game, each player has a dominant strategy, as in the original game. The difference is that the change in the payoffs makes Deny the dominant strategy for each player. And (Deny, Deny) becomes the unique pure-strategy Nash equilibrium. The stricter penalty scheme achieved with third-party enforcement makes defecting so unattractive to players that the cooperative outcome becomes the new equilibrium of the game.

In larger prisoners' dilemma games, difficulties arise with the use of penalties. In particular, if there are many players and some uncertainty exists, penalty schemes may be more difficult to maintain. It becomes harder to decide whether actual defecting is taking place or it's just bad luck or a mistaken move. In addition, if there really is defecting, it is often difficult to determine the identity of the defector from among the larger group. And if the game is one-shot, there is no opportunity in the future to correct a penalty that is too severe or to inflict a penalty once a defector has been identified. Thus penalties may be less successful in large one-shot games than in the two-person game we consider here. We study prisoners' dilemmas with a large number of players in greater detail in Chapter 12.

A further interesting possibility arises when a prisoners' dilemma that has been solved with a penalty scheme is considered in the context of the larger society in which the game is played. It might be the case that, although the dilemma equilibrium outcome is bad for the players, it is actually good for the rest of society or for some subset of persons within the rest of society. If so, social or political pressures might arise to try to minimize the ability of players to break out of the dilemma. When third-party penalties are the solution to a prisoners' dilemma, as is the case with crime mafias who enforce a no-confession rule, for instance, society can come up with its own strategy to reduce the effectiveness of the penalty mechanism. The Federal Witness Protection Program is an example of a system that has been set up for just this purpose. The U.S. government removes the threat of penalty in return for confessions and testimonies in court.

Similar situations can be seen in other prisoners' dilemmas, such as the pricing game between our two restaurants. The equilibrium there entailed both firms charging the low price of $20 even though they enjoy higher profits when charging the higher price of $26. Although the restaurants want to break out of this "bad" equilibrium—and we have already seen how the use of trigger strategies can help them do so—their customers are happier with the low price offered in the Nash equilibrium of the one-shot game. The customers then have an incentive to try to destroy the efficacy of any enforcement mechanism or solution process used by the restaurants. For example, because some firms facing prisoners' dilemma pricing games attempt to solve the dilemma through the use of a "meet the competition" or "price matching" campaign, customers might want to press for legislation banning such policies. We analyze the effects of such price-matching strategies in Section 6.D.

Just as a prisoners' dilemma can be resolved by penalizing defectors, it can also be resolved by rewarding cooperators. Because this solution is more difficult to implement in practice, we mention it only briefly.

The most important question is who is to pay the rewards. If it is a third party, that person or group must have sufficient interest of its own in the cooperation achieved by the prisoners to make it worth its while to pay out the rewards. A rare example of this occurred when the United States brokered the Camp David accords between Israel and Egypt by offering large promises of aid to both.

If the rewards are to be paid by the players themselves to each other, the trick is to make the rewards contingent (paid out only if the other player cooperates) and credible (guaranteed to be paid if the other player cooperates). Meeting these criteria requires an unusual arrangement; for example, the player making the promise should deposit the sum in advance in an escrow account held by an honorable and neutral third party, who will hand the sum over to the other player if she cooperates or return it to the promisor if the other defects. Exercise 11 in this chapter shows you how this can work, but we acknowledge its artificiality.

4 SOLUTIONS III: LEADERSHIP

The final method of solution for the prisoners' dilemma pertains to situations in which one player takes on the role of leader in the interaction. In most examples of the prisoners' dilemma, the game is assumed to be symmetric. That is, all the players stand to lose (and gain) the same amount from defecting (and cooperation). However, in actual strategic situations, one player may be relatively "large" (a leader) and the other "small." If the size of the payoffs is unequal enough, so much of the harm from defecting may fall on the larger player that she acts cooperatively, even while knowing that the other will defect. Saudi Arabia, for example, played such a role as the "swing producer" in OPEC (Organization of Petroleum Exporting Countries) for many years; to keep oil prices high, it cut back on its output when one of the smaller producers, such as Libya, expanded.

As with the OPEC example, **leadership** tends to be observed more often in games between nations than in games between firms or individual persons. Thus our example for a game in which leadership may be used to solve the prisoners' dilemma is one played between countries. Imagine that the populations of two countries, Dorminica and Soporia, are threatened by a disease, Sudden Acute Narcoleptic Episodes (SANE). This disease strikes 1 person in every 2,000, or 0.05% of the population, and causes the victim to fall into a deep sleep state for a year.[6] There are no aftereffects of the disease, but the cost of a worker being removed from the economy for a year is $32,000. Each country has a population of 100 million workers; so the expected number of cases in each is 50,000 ($0.0005 \times 100{,}000{,}000$) and the expected cost of the disease is $1.6 billion to each ($50{,}000 \times 32{,}000$). The total expected cost of the disease worldwide—that is, in both Dorminica and Soporia—is then $3.2 billion.

Scientists are confident that a crash research program costing $2 billion will lead to a vaccine that is 100% effective. Comparing the cost of the research program with the worldwide cost of the disease shows that, from the perspective of the entire population, the research program is clearly worth pursuing. However, the government in each country must consider whether to fund the full research program on its own. They make this decision separately, but their decisions affect the outcomes for both countries. Specifically, if only one government chooses to fund the research, the population of the other country can access the information and use the vaccine without cost. But each government's payoff depends only on the costs incurred by its own population.

The payoff matrix for the noncooperative game between Dorminica and Soporia is shown in Figure 11.6. Each country chooses from two strategies, Research and No Research; payoffs show the costs to the countries, in billions of dollars, of the various strategy combinations. It is straightforward to verify that

		SOPORIA	
		Research	No Research
DORMINICA	Research	−2, −2	−2, 0
	No Research	0, −2	−1.6, −1.6

FIGURE 11.6 Payoffs for Equal-Population SANE Research Game ($billions)

[6]Think of Rip Van Winkle or of Woody Allen in the movie *Sleeper*, but the duration is much shorter.

		SOPORIA	
		Research	No Research
DORMINICA	Research	−2, −2	−2, 0
	No Research	0, −2	−2.4, −0.8

FIGURE 11.7 Payoffs for Unequal-Population SANE Research Game ($billions)

this game is a prisoners' dilemma and that each country has a dominant strategy to do no research.

But now suppose that the populations of the two countries are unequal, with 150 million in Dorminica and 50 million in Soporia. Then, if no research is funded by either government, the cost to Dorminica of SANE will be $2.4 billion (0.0005 × 150,000,000 × 32,000) and the cost to Soporia will be $0.8 billion (0.0005 × 50,000,000 × 32,000). The payoff matrix changes to the one illustrated in Figure 11.7.

In this version of the game, No Research is still the dominant strategy for Soporia. But Dorminica's best response is now Research. What has happened to change Dorminica's choice of strategy? Clearly, the answer lies in the unequal distribution of the population in this revised version of the game. Dorminica now stands to suffer such a large portion of the total cost of the disease that it finds it worthwhile to do the research on its own. This is true even though Dorminica knows full well that Soporia is going to be a free rider and get a share of the full benefit of the research.

The research game in Figure 11.7 is no longer a prisoners' dilemma. Here we see that the dilemma has, in a sense, been "solved" by the size asymmetry. The larger country chooses to take on a leadership role and provide the benefit for the whole world.

Situations of leadership in what would otherwise be prisoners' dilemma games are common in international diplomacy. The role of leader often falls naturally to the biggest or most well established of the players, a phenomenon labeled "the exploitation of the great by the small."[7] For many decades after World War II, for instance, the United States carried a disproportionate share of the expenditures of our defense alliances such as NATO and maintained a policy of relatively free international trade even when our partners such as Japan and Europe were much more protectionist. In such situations, it might be reasonable to suggest further that a large or well-established player may accept the role of leader because its own interests are closely tied with those of the players as a whole; if the large player makes up a substantial fraction of the whole

[7]Mancur Olson, *The Logic of Collective Action* (Cambridge: Harvard University Press, 1965), p. 29.

group, such a convergence of interests would seem unmistakable. The large player would then be expected to act more cooperatively than might otherwise be the case.

5 EXPERIMENTAL EVIDENCE

Numerous people have conducted experiments in which subjects compete in prisoners' dilemma games against each other.[8] Such experiments show that cooperation can and does occur in such games, even in repeated versions of known and finite length. Many players start off by cooperating and continue to cooperate for quite a while, as long as the rival player reciprocates. Only in the last few plays of a finite game does defecting seem to creep in. Although this behavior goes against the reasoning of rollback, it can be "profitable" if sustained for a reasonable length of time. The pairs get higher payoffs than would rational calculating strategists who defect from the very beginning.

Such observed behavior can be rationalized in different ways. Perhaps the players are not sure that the relationship will actually end at the stated time. Perhaps they believe that their reputations for cooperation will carry over to other similar games against the same opponent or to other opponents. Perhaps they think it possible that their opponents are naive cooperators, and they are willing to risk a little loss in testing this hypothesis for a couple of plays. If successful, the experiment will lead to higher payoffs for a sufficiently long time.

In some laboratory experiments, players engage in multiple-round games, each round consisting of a given finite number of repetitions. All of the repetitions in any one round are played against the same rival, but each new round is played against a new opponent. Thus there is an opportunity to develop cooperation with an opponent in each round and to "learn" from preceding rounds when devising one's strategy against new opponents as the rounds continue. These situations have shown that cooperation lasts longer in early rounds than in later rounds. This result suggests that the theoretical argument on the unrav-

[8]The literature on experiments involving the prisoners' dilemma game is vast. A brief overview is given by Alvin Roth in *The Handbook of Experimental Economics* (Princeton: Princeton University Press, 1995), pp. 26–28. Journals in both psychology and economics can be consulted for additional references. For some examples of the outcomes that we describe, see Kenneth Terhune, "Motives, Situation, and Interpersonal Conflict Within Prisoners' Dilemmas," *Journal of Personality and Social Psychology Monograph Supplement*, vol. 8, no. 30 (1968), pp. 1–24; and R. Selten and R. Stoecker, "End Behavior in Sequences of Finite Prisoners' Dilemma Supergames," *Journal of Economic Behavior and Organization*, vol. 7 (1986), pp. 47–70. Robert Axelrod's *Evolution of Cooperation* (New York: Basic Books, 1984) presents the results of his computer simulation tournament for the best strategy in an infinitely repeated dilemma.

eling of cooperation, based on the use of rollback, is being learned from experience of the play itself over time as players begin to understand the benefits and costs of their actions more fully. Another possibility is that players learn simply that they want to be the first to defect, and so the timing of the initial defection occurs earlier as the number of rounds played increases.

Suppose you were playing a game with a prisoners' dilemma structure and found yourself in a cooperative mode with the known end of the relationship approaching. When should you decide to defect? You do not want to do so too early, while a lot of potential future gains remain. But you also do not want to leave it to too late in the game, because then your opponent might preempt you and leave you with a low payoff for the period in which she defects. In fact, your decision about when to defect cannot be deterministic. If it were, your opponent would figure it out and defect in the period before you planned to do so. If no deterministic choice is feasible, then the unwinding of cooperation must include some uncertainty, such as mixed strategies, for both players. Many thrillers in which there is tenuous cooperation among criminals or between informants and police acquire their suspense precisely because of this uncertainty.

Examples of the collapse of cooperation as players near the end of a repeated game are observed in numerous situations in the real world, as well as in the laboratory. The story of a long-distance bicycle (or foot) race is one such example. There may be a lot of cooperation for most of the race, as players take turns leading and letting others ride in their slipstreams; nevertheless, as the finish line looms, each participant will want to make a dash for the tape. Similarly, signs saying "no checks accepted" often appear in stores in college towns each spring near the end of the semester.

Computer simulation experiments have matched a range of very simple to very complex contingent strategies against each other in two-player prisoners' dilemmas. The most famous of them were conducted by Robert Axelrod at the University of Michigan. He invited people to submit computer programs that specified a strategy for playing a prisoners' dilemma repeated a finite but large number (200) of times. There were 14 entrants. Axelrod held a "league tournament" that pitted pairs of these programs against each other, in each case for a run of the 200 repetitions. The point scores for each pairing and its 200 repetitions were kept, and each program's scores over all its runs against different opponents were added up to see which program did best in the aggregate against all other programs. Axelrod was initially surprised when "nice" programs did well; none of the top eight programs were ever the first to defect. The winning strategy turned out to be the simplest program: Tit-for-tat, submitted by the Canadian game theorist Anatole Rapoport. Programs that were eager to defect in any particular run got the defecting payoff early but then suffered repetitions of mutual defections and poor payoffs. On the other hand, programs that were always nice and cooperative were badly exploited by their opponents. Axelrod explains the success of Tit-for-tat in terms of four properties: it is at once forgiving, nice, provocable, and clear.

In Axelrod's words, one does well in a repeated prisoners' dilemma to abide by these four simple rules: "Don't be envious. Don't be the first to defect. Reciprocate both cooperation and defection. Don't be too clever."[9] Tit-for-tat embodies each of the four ideals for a good, repeated prisoners' dilemma strategy. It is not envious; it does not continually strive to do better than the opponent, only to do well for itself. In addition, Tit-for-tat clearly fulfills the admonitions not to be the first to defect and to reciprocate, defecting only in retaliation to the opponent's preceding defection and always reciprocating in kind. Finally, Tit-for-tat does not suffer from being overly clever; it is simple and understandable to the opponent. In fact, it won the tournament not because it helped players achieve high payoffs in any individual game—the contest was not about "winner takes all"—but because it was always close; it simultaneously encourages cooperation and avoids exploitation, while other strategies cannot.

Axelrod then announced the results of his tournament and invited submissions for a second round. Here, people had a clear opportunity to design programs that would beat Tit-for-tat. The result: Tit-for-tat won again! The programs that were cleverly designed to beat it could not beat it by very much, and they did poorly against one another. Axelrod also arranged a tournament of a different kind. Instead of a league where each program met each other program once, he ran a game with a whole population of programs, with a number of copies of each program. Each type of program met an opponent randomly chosen from the population. Those programs that did well were given a larger proportion of the population; those that did poorly had their proportion in the population reduced. This was a game of evolution and natural selection, which we will study in greater detail in Chapter 13. But the idea is simple in this context, and the results are fascinating. At first, nasty programs did well at the expense of nice ones. But as the population became more and more nasty, each nasty program met other nasty programs more and more often, and they began to do poorly and fall in numbers. Then Tit-for-tat started to do well and eventually triumphed.

However, Tit-for-tat has some flaws. Most importantly, it assumes no errors in execution of the strategy. If there is some risk that the player intends to play the cooperative action but plays the defecting action in error, then this action can initiate a sequence of retaliatory defecting actions that locks two Tit-for-tat programs playing each other into a bad outcome; another error is required to rescue them from this sequence. When Axelrod ran a third variant of his tournament, which provided for such random mistakes, Tit-for-tat could be beaten by even "nicer" programs that tolerated an occasional episode of defecting to see if

[9]Axelrod, *Evolution of Cooperation*, p. 110.

it was a mistake or a consistent attempt to exploit them and retaliated only when convinced that it was not a mistake.[10]

6 REAL-WORLD DILEMMAS

Games with the prisoners' dilemma structure arise in a surprisingly varied number of contexts in the world. Although we would be foolish to try to show you every possible instance in which the dilemma can arise, we take the opportunity in this section to consider in detail four specific examples from a variety of fields of study. One example concerns policy setting on the part of state governments, another considers labor arbitration outcomes, still another comes from evolutionary biology (a field that we will study in greater detail in Chapter 13), and the final example describes the policy of "price matching" as a solution to a prisoners' dilemma pricing game.

A. Governments Competing to Attract Business

This example is based on a *Time* magazine article titled "A No-Win War Between the States."[11] The article discusses the fact that individual states have been using financial incentives to get firms to relocate to their states or to get local firms to stay in state. Such incentives, which come primarily in the form of corporate and real-estate tax breaks, can be extremely costly to the states that offer them. They can, however, bring significant benefits if they work. Large manufacturing firms bring with them the promise of increased employment opportunities for state residents and potential increases in state sales-tax revenue. The problem with the process of offering such incentives, however, is that it is a prisoners' dilemma. Each state has a dominant strategy to offer incentives, but the ultimate equilibrium outcome is worse for everyone than if they had refrained from offering incentives in the first place.

Let us consider a game played by two states, Ours and Theirs. Both states currently have the same number of firms and are considering offering identical

[10]For a description and analysis of Axelrod's computer simulations from the biological perspective, see Matt Ridley, *The Origins of Virtue* (New York: Penguin Books, 1997), pp. 61, 75. For a discussion of the difference between computer simulations and experiments using human players, see John K. Kagel and Alvin E. Roth, *Handbook of Experimental Economics* (Princeton: Princeton University Press, 1995), p. 29.

[11]John Greenwald, "A No-Win War Between the States," *Time*, April 8, 1995. Considerable interest in this topic continues within the field of state and local public finance. See, for example, Stephen Ellis and Cynthia Rogers, "Local Economic Development As a Prisoners' Dilemma: The Role of Business Climate," *Review of Regional Studies*, vol. 30, no. 3 (Winter 2000), pp. 315–330.

incentive packages to induce some of the other state's firms to move. Each state has two strategies—to offer incentives or not offer them—which we will call Incentives and None. If no incentive offer is made by either state, the status quo is maintained. Neither state incurs the cost of the incentives, but it does not get new firms to move in either; each state retains its original number of firms. If Ours offers incentives while Theirs offers none, then Ours will encourage firms to move in while Theirs loses firms; Ours gains at the expense of Theirs. (We assume that the states have done enough prior investigation to be able to construct an incentive package whose costs are offset by the benefits of the arrival of the new firms.) The opposite will be true for the situation in which Theirs offers Incentives and Ours offers none; Theirs will gain at the expense of Ours. If both states offer incentives, firms are likely to switch states to take advantage of the incentive packages, but the final number of firms in each state will be the same as in the absence of incentives. Each state has incurred a high cost to obtain an outcome equivalent to the original situation.

Given this description of the possible strategies and their payoffs, we can construct a simple payoff table for the states, as done in Figure 11.8. Each payoff number represents the rating given each outcome by each state. The highest rating of 4 goes to the state that offers incentives when its rival does not; that state pays the cost of incentives but benefits from gaining new firms. At the other end of the spectrum, the lowest rating of 1 goes to the state that does not offer incentives when its rival does; that state incurs no cost but loses firms to its rival. In between, a rating of 3 goes to the outcome in which the status quo is maintained with no cost, and a rating of 2 goes to the outcome in which the status quo is effectively maintained but the costs of incentives are incurred.

The payoff table in Figure 11.8 is a standard prisoners' dilemma. Each state has a dominant strategy to choose Incentives, and the Nash equilibrium outcome entails incentives being offered by both states. As in all prisoners' dilemmas, both states would have been better off had they agreed not to offer any incentives at all. Similar situations arise in other policy contexts as well. Governments face the same difficulties when deciding whether to provide export subsidies to domestic firms, knowing that foreign governments are simultaneously making the same choice.

		THEIRS	
		Incentives	None
OURS	Incentives	2, 2	4, 1
	None	1, 4	3, 3

FIGURE 11.8 Payoffs for the Incentive-Package Prisoners' Dilemma

Is there a solution to this dilemma? From the standpoint of the firms receiving the incentives, the equilibrium outcome is a good one, but, from the perspective of the state governments and the households that must shoulder an increased tax burden, the no-incentives outcome is preferable. Federal legislation prohibiting the use of such incentive packages could solve this particular dilemma; such a solution would fall into the category of a punishment enforced by a third party.

Similar fears have been expressed in the European Union that a "race to the bottom" may develop, where member states cut tax rates and relax regulations to attract business. This would result in less government revenue and would threaten social services in each country. Unionwide agreements would be needed to resolve this dilemma, but none have been reached yet.

B. Labor Arbitration

Our second example comes from an intriguingly titled paper by Orley Ashenfelter and David Bloom, "Lawyers As Agents of the Devil in a Prisoners' Dilemma Game."[12] The authors argue that many arbitration situations between labor unions and employers have a prisoners' dilemma structure. Given the choice of whether to retain legal counsel for arbitration, both sides find it in their interests to do so. The presence of a lawyer is likely to help the legally represented side in an arbitration process substantially if the other side has no lawyer. Hence, it is better to use a lawyer if your rival does not. But you are also sure to want a lawyer if your rival uses one. Using a lawyer is a dominant strategy. Unfortunately, Ashenfelter and Bloom go on to claim that the outcomes for these labor arbitration games change little in the presence of lawyers. That is, the probability of the union "winning" an arbitration game is basically the same when no lawyers are used as when both sides use lawyers. But lawyers are costly, just like the incentive packages that states offer to woo new firms. Players always want to use lawyers but do so at great expense and with little reward in the end. The resulting equilibrium with lawyers is worse than if both sides choose No Lawyers because each side has to pay substantial legal fees without reaping any beneficial effect on the probability of winning the game.

Figure 11.9 shows the results of Ashenfelter and Bloom's detailed study of actual arbitration cases. The values in the table are their econometric estimates of the percentage of arbitration cases won by the employer under different combinations of legal representation for the two sides. The Union sees its chances of winning against an unrepresented Employer increase (from 56% to

[12]Orley Ashenfelter and David Bloom, "Lawyers As Agents of the Devil in a Prisoners' Dilemma Game," National Bureau of Economic Research Working Paper No. 4447, September 1993.

		UNION	
		Lawyer	No Lawyer
EMPLOYER	Lawyer	46%	73%
	No Lawyer	23%	44%

FIGURE 11.9 Predicted Percentage of Employer "Wins" in Arbitration Cases

77%) when it chooses to use a lawyer, and the Employer sees a similar increase (from 44% to 73%). But when both use lawyers, the winning percentages change only slightly, by 2% (with measurement error) compared with the situation in which neither side uses a lawyer. Thus the costs incurred in hiring the lawyers are unlikely to be recouped in increased wins for either side, and each would be better off if both could credibly agree to forgo legal representation.

The search for a solution to this dilemma returns us to the possibility of a legislated penalty mechanism. Federal arbitration rules could be established that preclude the possibility of the sides bringing lawyers to the table. In this game, the likelihood of a continuing relationship between the players also gives rise to the possibility that cooperation could be sustained through the use of contingent strategies. If the union and the employer start out in a cooperative mode, the fear of never being able to return to the No Lawyer outcome might be sufficient to prevent a single occurrence of defecting—hiring a lawyer.

C. Evolutionary Biology

As our third example, we consider a game known as the bowerbirds' dilemma, from the field of evolutionary biology.[13] Male bowerbirds attract females by building intricate nesting spots called bowers, and female bowerbirds are known to be particularly choosy about the bowers built by their prospective mates. For this reason, male bowerbirds often go out on search-and-destroy missions aimed at ruining other males' bowers. When they are out, however, they run the risk of losing their own bower at the beak of another male. The ensuing competition between male bowerbirds and their ultimate choice regarding whether to maraud or guard has the structure of a prisoners' dilemma game.

Ornithologists have constructed a table that shows the payoffs in a two-bird game with two possible strategies, Maraud and Guard. That payoff table is shown in Figure 11.10. GG represents the benefits associated with Guarding when the rival bird also Guards; GM represents the payoff from Guarding when

[13] Larry Conik, "Science Classics: The Bowerbird's Dilemma," *Discover*, October 1994.

		BIRD 2	
		Maraud	Guard
BIRD 1	Maraud	MM, MM	MG, GM
	Guard	GM, MG	GG, GG

FIGURE 11.10 Bowerbird's Dilemma

the rival bird is a Marauder. Similarly, MM represents the benefits associated with Marauding when the rival bird also is a Marauder; and MG represents the payoff from Marauding when the rival bird Guards. Careful scientific study of bowerbird matings led to the discovery that MG > GG > MM > GM. In other words, the payoffs in the bowerbird game have exactly the same structure as the prisoners' dilemma. The birds' dominant strategy is to maraud, but, when both choose that strategy, they end up in an equilibrium worse off than if they had both chosen to guard.

In reality, the strategy used by any particular bowerbird is not actually the result of a process of rational choice on the part of the bird. Rather, in evolutionary games, strategies are assumed to be genetically "hardwired" into individual organisms, and payoffs represent reproductive success for the different types. Then equilibria in such games define the type of population that can be expected to be observed—all marauders, for instance, if Maraud is a dominant strategy as in Figure 11.10. This equilibrium outcome is not the best one, however, given the existence of the dilemma. In constructing a solution to the bowerbirds' dilemma, we can appeal to the repetitive nature of the interaction in the game. In the case of the bowerbirds, repeated play against the same or different opponents in the course of several breeding seasons can allow you, the bird, to choose a flexible strategy based on your opponent's last move. Contingent strategies such as tit-for-tat can be, and often are, adopted in evolutionary games to solve exactly this type of dilemma. We will return to the idea of evolutionary games and provide detailed discussions of their structure and equilibrium outcomes in Chapter 13.

D. Price Matching

Finally, we return to a pricing game, in which we consider two specific stores engaged in price competition with each other, using identical price-matching policies. The stores in question, Toys "R" Us and Kmart, are both national chains that regularly advertise prices for name-brand toys (and other items). In addition, each store maintains a published policy that guarantees customers that they will match the advertised price of any competitor on a specific item (model and item numbers must be identical) as long as the customer provides the competitor's printed advertisement.[14]

For the purposes of this example, we assume that the firms have only two possible prices that they can charge for a particular toy (Low or High). In addition, we use hypothetical profit numbers and further simplify the analysis by assuming that Toys "R" Us and Kmart are the only two competitors in the toy market in a particular city—Billings, Montana, for example.

Suppose, then, that the basic structure of the game between the two firms can be illustrated as in Figure 11.11. If both firms advertise low prices, they split the available customer demand and each earns $2,000. If both advertise high prices, they split a market with lower sales, but their markups end up being large enough to let them each earn $3,000. Finally, if they advertise different prices, then the one advertising a high price gets no customers and earns nothing while the one advertising a low price earns $4,000.

The game illustrated in Figure 11.11 is clearly a prisoners' dilemma. Advertising and selling at a low price is the dominant strategy for each firm, although both would be better off if each advertised and sold at the high price. But as mentioned earlier, each firm actually makes use of a third pricing strategy: a price-matching guarantee to its customers. How does the inclusion of such a policy alter the prisoners' dilemma that would otherwise exist between these two firms?

Consider the effects of allowing firms to choose among pricing low, pricing high, and price matching. The Match strategy entails advertising a high price but promising to match any lower advertised price by a competitor; a firm using Match then benefits from advertising high if the rival firm does so also, but it does not suffer any harm from advertising a high price if the rival advertises a low price. We can see this in the payoff structure for the new game, shown in Figure 11.12. In that table, we see that a combination of one firm playing Low while the other plays Match is equivalent to both playing Low, while a combina-

		KMART	
		Low	High
TOYS "R" US	Low	2,000, 2,000	4,000, 0
	High	0, 4,000	3,000, 3,000

FIGURE 11.11 Toys "R" Us and Kmart Toy Pricing

[14]The price-matching policy at Toys "R" Us is printed and posted prominently in all stores. A simple phone call confirmed that Kmart has an identical policy. Similar policies are appearing in many industries, including that for credit cards where "interest rate matching" has been observed. See Aaron S. Edlin, "Do Guaranteed-Low-Price Policies Guarantee High Prices, and Can Antitrust Rise to the Challenge?" *Harvard Law Review*, vol. 111, no. 2 (December 1997), pp. 529–575.

		KMART		
		Low	High	Match
TOYS "R" US	Low	2,000, 2,000	4,000, 0	2,000, 2,000
	High	0, 4,000	3,000, 3,000	3,000, 3,000
	Match	2,000, 2,000	3,000, 3,000	3,000, 3,000

FIGURE 11.12 Toys "R" Us and Kmart Price Matching

tion of one firm playing High while the other plays Match (or both playing Match) is equivalent to both playing High.

Using our standard tools for analyzing simultaneous-play games shows that High is weakly dominated by Match for both players and that, once High is eliminated, Low is weakly dominated by Match also. The resulting Nash equilibrium entails both firms using the Match strategy. In equilibrium, both firms earn $3,000—the profit level associated with both firms pricing high in the original game. The addition of the Match strategy has allowed the firms to emerge from the prisoners' dilemma that they faced when they had only the choice between two simple pricing strategies, Low or High.

How did this happen? The Match strategy acts as a penalty mechanism. By guaranteeing to match Kmart's low price, Toys "R" Us substantially reduces the benefit that Kmart achieves by advertising a low price while Toys "R" Us is advertising a high price. In addition, promising to meet Kmart's low price hurts Toys "R" Us, too, because the latter has to accept the lower profit associated with the low price. Thus the price-matching guarantee is a method of penalizing both players whenever either one defects. This is just like the crime mafia example discussed in Section 3, except that this penalty scheme—and the higher equilibrium prices that it supports—is observed in markets in virtually every city in the country.

Actual empirical evidence of the detrimental effects of these policies is available but limited, and some research has found evidence of lower prices in markets with such policies.[15] However, this result should still put all customers on alert. Even though stores that match prices promote their policies in the name of competition, the ultimate outcome when all firms use such policies can be better for the firms than if there were no price matching at all, and so customers can be the ones who are hurt.

[15]J. D. Hess and Eitan Gerstner present evidence of increased prices as a result of price-matching policies in "Price-Matching Policies: An Empirical Case," *Managerial and Decision Economics*, vol. 12 (1991), pp. 305–315. Contrary evidence is provided by Arbatskaya, Hviid, and Shaffer, who find that the effect of matching policies is to lower prices; see Maria Arbatskaya, Morten Hviid, and Greg Shaffer, "Promises to Match or Beat the Competition: Evidence from Retail Tire Prices," *Advances in Applied Microeconomics*, vol. 8: *Oligopoly* (New York: JAI Press, 1999), pp. 123–138.

SUMMARY

The prisoners' dilemma is probably the most famous game of strategy; each player has a dominant strategy (to Defect), but the equilibrium outcome is worse for all players than when each uses her dominated strategy (to Cooperate). The most well known solution to the dilemma is *repetition of play*. In a finitely played game, the *present value* of future cooperation is eventually zero and rollback yields an equilibrium with no cooperative behavior. With infinite play (or an uncertain end date), cooperation can be achieved with the use of an appropriate contingent strategy such as *tit-for-tat (TFT)* or the *grim strategy;* in either case, cooperation is possible only if the present value of cooperation exceeds the present value of defecting. More generally, the prospects of "no tomorrow" or of short-term relationships lead to decreased cooperation among players.

The dilemma can also be "solved" with *penalty* schemes that alter the payoffs for players who defect from cooperation when their rivals are cooperating or when others also are defecting. A third solution method arises if a large or strong player's loss from defecting is larger than the available gain from cooperative behavior on that player's part.

Experimental evidence suggests that players often cooperate longer than theory might predict. Such behavior can be explained by incomplete knowledge of the game on the part of the players or by their views regarding the benefits of cooperation. Tit-for-tat has been observed to be a simple, nice, provocable, and forgiving strategy that performs very well on the average in repeated prisoners' dilemmas.

Prisoners' dilemmas arise in a variety of contexts. Specific examples from the areas of policy setting, labor arbitration, evolutionary biology, and product pricing show how to explain actual behavior by using the framework of the prisoners' dilemma.

KEY TERMS

compound interest (352)
contingent strategy (349)
discount factor (352)
effective rate of return (353)
grim strategy (349)
infinite horizon (351)
leadership (360)
penalty (356)
present value (PV) (351)
punishment (349)
repeated play (347)
Tit-for-tat (TFT) (349)
trigger strategy (349)

EXERCISES

1. "If a prisoners' dilemma is repeated 100 times, the players are sure to achieve their cooperative outcome." True or false? Explain and give an example of a game that illustrates your answer.

2. Two people, Baker and Cutler, play a game in which they choose and divide a prize. Baker decides how large the total prize should be; she can choose either $10 or $100. Cutler chooses how to divide the prize chosen by Baker; Cutler can choose either an equal division or a split, where she gets 90% and Baker gets 10%. Write down the payoff table of the game and find its equilibria
   (a) when the moves are simultaneous.
   (b) when Baker moves first.
   (c) when Cutler moves first.
   (d) Is this game a prisoners' dilemma? Why or why not?

3. A firm has two divisions, each of which has its own manager. Managers of these divisions are paid according to their effort in promoting productivity in their divisions, which is judged by comparison with other managers and other divisions. If both managers are judged as having expended "high effort," each earns $150,000/year. If both are judged to have expended "low effort," each earns "only" $100,000/year. But if one of the two managers shows "high effort" while the other shows "low effort," the "high effort" manager is paid $150,000 plus a $50,000 bonus, while the second ("low effort") manager gets a reduced salary (for subpar performance in comparison with her competition) of $80,000. Managers make their effort decisions independently and without knowledge of the other manager's choice.
   (a) Assume that expending effort is costless to the managers and draw the payoff table for this game. Find the Nash equilibrium of the game and explain whether the game is a prisoners' dilemma.
   (b) Now suppose that expending high effort is costly to the managers (such as a costly signal of quality). In particular, suppose that "high effort" costs an equivalent of $60,000/year to a manager that chooses this effort level. Draw the game table for this new version of the game and find the Nash equilibrium. Explain whether the game is a prisoners' dilemma and how it has changed from the game in part **a**.
   (c) If the cost of high effort is equivalent to $80,000 a year, how does the game change from that described in part **b**? What is the new equilibrium? Explain whether the game is a prisoners' dilemma and how it has changed from the games in parts **a** and **b**.

4. Consider a small town with a population of dedicated pizza eaters but able to accommodate only two pizza shops, Donna's Deep Dish and Pierce's Pizza

Pies. Each seller has to choose a price for its pizza but, for simplicity, assume that only two prices are available: high and low. If a high price is set, the sellers can achieve a profit margin of \$12 per pie; the low price yields a profit margin of \$10 per pie. Each store has a loyal captive customer base who will buy 3,000 pies per week, no matter what price is charged in either store. There is also a floating demand of 4,000 pies per week. The people who buy these pies are price conscious and will go to the store with the lower price; if both stores are charging the same price, this demand will be split equally between them.

(a) Draw the game table for the pizza pricing game, using each store's profits per week (in thousands of dollars) as payoffs. Find the Nash equilibrium of this game and explain why it is a prisoners' dilemma.

(b) Now suppose that Donna's Deep Dish has a much larger loyal clientele, who guarantee it the sale of 11,000 (rather than 3,000) pies a week. Profit margins and the size of the floating demand remains the same. Draw the payoff table for this new version of the game and find the Nash equilibrium.

(c) How does the existence of the larger loyal clientele for Donna's Deep Dish help "solve" the pizza stores' dilemma?

5. Consider the game of chicken in Chapter 4, with slightly more general payoffs (Figure 4.14 had $k = 1$):

		DEAN	
		Swerve	Straight
JAMES	Swerve	0, 0	−1, k
	Straight	k, −1	−2, −2

Suppose this game is played repeatedly, every Saturday evening. If $k < 1$, the two players stand to benefit by cooperating to play (Swerve, Swerve) all the time; whereas, if $k > 1$, they stand to benefit by cooperating so that one plays Swerve and the other plays Straight, taking turns to go Straight in alternate weeks. Can either type of cooperation be sustained?

6. A town council consists of three members who vote every year on their own salary increases. Two Yes votes are needed to pass the increase. Each member would like a higher salary but would like to vote against it herself because that looks good to the voters. Specifically, the payoffs of each are as follows:

Raise passes, own vote is No:	10 points
Raise fails, own vote is No:	5 points
Raise passes, own vote is Yes:	4 points
Raise fails, own vote is Yes:	0 points

Voting is simultaneous. Write down the (three-dimensional) payoff table, and show that in the Nash equilibrium the raise fails unanimously. Examine how a repeated relationship among the members can secure them salary increases every year if (1) every member serves a 3-year term, and every year in rotation one of them is up for reelection, and (2) the townspeople have short memories, remembering only the votes on the salary-increase motion of the current year and not those of past years.

7. Recall the example from Exercise 3 in which two division managers' choices of High or Low effort levels determine their salary payments. In part **b** of that exercise, the cost of exerting High effort is assumed to be $60,000/year. Suppose now that the two managers play the game in part **b** of Exercise 3 repeatedly for many years. Such repetition allows scope for an unusual type of cooperation in which one is designated to choose High effort while the other chooses Low. This cooperative agreement requires that the High effort manager make a side payment to the Low effort manager so that their final salaries levels are identical.
   (a) What size side payment guarantees that the final salary levels of the two managers are identical? How much does each manager earn in a year in which the cooperative agreement is in place?
   (b) Cooperation in this repeated game entails each manager choosing her assigned effort level and the High effort manager making the designated side payment. Defection entails refusing to make the side payment. Under what values of the rate of return can this agreement sustain cooperation in the managers' repeated game?

8. Consider a two-player game between Child's Play and Kid's Korner, each of which produces and sells wooden swing sets for children. Each player can set either a high or a low price for a standard two-swing, one-slide set. If they both set a high price, each receives profits of $64,000 (per year). If one sets a low price while the other sets a high price, the low-price firm earns profits of $72,000 (per year) while the high-price firm earns $20,000. If they both set a low price, each receives profits of $57,000.
   (a) Verify that this game has a prisoners' dilemma structure by looking at the ranking of payoffs associated with the different strategy combinations (both cooperate, both defect, one defects, and so on). What are the Nash equilibrium strategies and payoffs in the simultaneous-play game if the players meet and make price decisions only once?
   (b) If the two firms decide to play this game for a fixed number of periods—say, for 4 years—what would each firm's total profits be at the end of the game? (Don't discount.) Explain how you arrived at your answer.

(c) Suppose that the two firms play this game repeatedly forever. Let each of them use a grim strategy in which they both price high unless one of them "defects," in which case they price low for the rest of the game. What is the one-time gain from defecting against an opponent playing such a strategy? How much does each firm lose, in each future period, after it defects once? If $r = 0.25$ ($\delta = 0.8$), will it be worthwhile to cooperate? Find the range of values of r (or δ) for which this strategy is able to sustain cooperation between the two firms.

(d) Suppose the firms play this game repeatedly year after year, neither expecting any change in their interaction. If the world were to end after 4 years, without either having anticipated this event, what would each firm's total profits (not discounted) be at the end of the game? Compare your answer here with the answer in part **b**. Explain why the two answers are different, if they are different, or why they are the same, if they are the same.

(e) Suppose now that the firms know that there is a 10% probability that one of them may go bankrupt in any given year. If bankruptcy occurs, the repeated game between the two firms ends. Will this knowledge change the firms' actions when $r = 0.25$? What if the probability of a bankruptcy increases to 35% in any year?

9. You have to decide whether to invest \$100 in a friend's enterprise, where in a year's time the money will increase to \$130. You have agreed that your friend will then repay you \$120, keeping \$10 for himself. But instead he may choose to run away with the whole \$130. Any of your money that you don't invest in your friend's venture, you can invest elsewhere safely at the prevailing rate of interest r, and get \$100(1 + r) next year.

(a) Draw the game tree for this situation and show the rollback equilibrium.

Next suppose this game is played repeatedly infinitely often. That is, each year you have the opportunity to invest another \$100 in your friend's enterprise, and the agreement is to split the resulting \$130 in the manner already described. From the second year onward, you get to make your decision of whether to invest with your friend in the light of whether he made the agreed repayment the preceding year. The rate of interest between any two successive periods is r, the same as the outside rate of interest and the same for you and your friend.

(b) For what values of r can there be an equilibrium outcome of the repeated game, in which each period you invest with your friend and he repays as agreed?

(c) If the rate of interest is 10% per year, can there be an alternative profit-splitting agreement, which is an equilibrium outcome of the infinitely repeated game, where each period you invest with your friend and he repays as agreed?

10. Consider the pizza stores introduced in Exercise 4: Donna's Deep Dish and Pierce's Pizza Pies. Suppose that they are not constrained to choose from only two possible prices, but that they can choose a specific value for price to maximize profits. Suppose further that it costs \$3 to make each pizza (for each store) and that experience or market surveys have shown that the relation between sales (Q) and price (P) for each firm is as follows:

$$Q_{\text{Donna}} = 12 - P_{\text{Donna}} + 0.5P_{\text{Pierce}}$$
$$Q_{\text{Pierce}} = 12 - P_{\text{Pierce}} + 0.5P_{\text{Donna}}$$

Then profits per week (Y, in thousands of dollars) for each firm are:

$$Y_{\text{Pierce}} = (P_{\text{Pierce}} - 3)\ Q_{\text{Pierce}} = (P_{\text{Pierce}} - 3)\ (12 - P_{\text{Pierce}} + 0.5P_{\text{Donna}})$$
$$Y_{\text{Donna}} = (P_{\text{Donna}} - 3)\ Q_{\text{Donna}} = (P_{\text{Donna}} - 3)\ (12 - P_{\text{Donna}} + 0.5\ P_{\text{Pierce}})$$

(a) Use these profit functions to determine each firm's best-response rule, as in Chapter 5, and use the best-response rules to find the Nash equilibrium of this pricing game. What prices do the firms choose in equilibrium and how much profit per week does each firm earn?

(b) If the firms work together and choose a jointly best price, P, then the profit of each will be:

$$Y_{\text{Donna}} = Y_{\text{Pierce}} = (P - 3)\ (12 - P + 0.5\ P) = (P - 3)\ (12 - 0.5\ P)$$

What price do they choose to maximize joint profits?

(c) Suppose the two stores are in a repeated relationship, trying to sustain the joint profit-maximizing prices calculated in part **b**. They print new menus each month and thereby commit themselves to prices for the whole month. In any one month, one of them can defect on the agreement. If one of them holds the price at the agreed level, what is the best defecting price for the other? What are its resulting profits? For what discount factors will their collusion be sustainable by using grim-trigger strategies?

11. Consider the following game, which comes from James Andreoni and Hal Varian at the University of Michigan.[16] A neutral referee runs the game. There are two players, Row and Column. The referee gives two cards to each: 2 and 7 to Row and 4 and 8 to Column. This is common knowledge. Then, playing simultaneously and independently, each player is asked to hand over to the referee either his high card or his low card. The referee hands out payoffs—which come from a central kitty, not from the players' pockets—that are measured in dollars and depend on the cards that he collects. If Row chooses his Low card 2, then Row gets \$2; if he chooses his

[16]James Andreoni and Hal Varian, "Preplay Contacting in the Prisoners' Dilemma," *Proceedings of the National Academy of Sciences*, vol. 96, no. 19 (September 14, 1999), pp. 10933–10938.

High card 7, then Column gets \$7. If Column chooses his Low card 4, then Column gets \$4; if he chooses his High card 8, then Row gets \$8.

(a) Show that the complete payoff table is as follows:

		COLUMN	
		Low	High
ROW	Low	2, 4	10, 0
	High	0, 11	8, 7

(b) What is the Nash equilibrium? Verify that this game is a prisoners' dilemma.

Now suppose the game has the following stages. The referee hands out cards as before; who gets what cards is common knowledge. Then at stage I, each player, out of his own pocket, can hand over a sum of money, which the referee is to hold in an escrow account. This amount can be zero but not negative. The rules for the treatment of these sums are as follows and are also common knowledge. If Column chooses his high card, then the referee hands over to Column the sum put by Row in the escrow account; if Column chooses Low, then Row's sum reverts back to Row. The disposition of the sum deposited by Column depends similarly on what Row does. At stage II, each player hands over one of his cards, High or Low, to the referee. Then the referee hands out payoffs from the central kitty, according to the table, and disposes of the sums (if any) in the escrow account, according to the rules just given.

(c) Find the rollback (subgame-perfect) equilibrium of this two-stage game. Does it resolve the prisoners' dilemma? What is the role of the escrow account?

Appendix: Infinite Sums

The computation of present values requires us to determine the current value of a sum of money that is paid to us in the future. As we saw in Section 2 of Chapter 11, the present value of a sum of money—say, x—that is paid to us n months from now is just $x/(1 + r)^n$, where r is the appropriate monthly rate of return. But the present value of a sum of money that is paid to us next month and every following month in the foreseeable future is more complicated to determine. In that case, the payments continue infinitely, and so there is no defined end to the sum of present values that we need to compute. To compute the present value of this flow of payments requires some knowledge of the mathematics of the summation of infinite series.

Consider a player who stands to gain \$36 this month from defecting in a prisoners' dilemma but who will then lose \$36 every month in the future as a result of her choice to continue defecting while her opponent punishes her (using the tit-for-tat, or TFT, strategy). In the first of the future months—the first for which there is a loss and the first for which values need to be discounted—the present value of her loss is $36/(1 + r)$; in the second future month, the present value of the loss is $36/(1 + r)^2$; in the third future month, the present value of the loss is $36/(1 + r)^3$. That is, in each of the n future months that she incurs a loss from defecting, that loss equals $36/(1 + r)^n$.

We could write out the total present value of all of her future losses as a large sum with an infinite number of components,

$$PV = \frac{36}{1 + r} + \frac{36}{(1 + r)^2} + \frac{36}{(1 + r)^3} + \frac{36}{(1 + r)^4} + \frac{36}{(1 + r)^5} + \frac{36}{(1 + r)^6} + \cdots,$$

or we could use summation notation as a shorthand device and instead write

$$PV = \sum_{n=1}^{\infty} \frac{36}{(1 + r)^n}.$$

This expression, which is equivalent to the preceding one, is read as "the sum, from n equals 1 to n equals infinity, of 36 over $(1 + r)$ to the nth power." Because 36 is a common factor—it appears in each term of the sum—it can be pulled out to the front of the expression. Thus we can write the same present value as

$$PV = 36 \cdot \sum_{n=1}^{\infty} \frac{36}{(1 + r)^n}.$$

We now need to determine the value of the sum within the present-value expression to calculate the actual present value. To do so, we will simplify our notation by switching to the *discount factor* δ in place of $1/(1 + r)$. Then the sum that we are interested in evaluating is

$$\sum_{n=1}^{\infty} \delta^n.$$

It is important to note here that $\delta = 1/(1 + r) < 1$ because r is strictly positive.

An expert on infinite sums would tell you, after inspecting this last sum, that it converges to the finite value $\delta/(1 - \delta)$.[17] Convergence is guaranteed because

[17]An infinite series *converges* if the sum of the values in the series approaches a specific value, getting closer and closer to that value as additional components of the series are included in the sum. The series *diverges* if the sum of the values in the series gets increasingly larger (more negative) with each addition to the sum. Convergence requires that the components of the series get progressively smaller.

increasingly large powers of a number less than 1, δ in this case, become smaller and smaller, approaching zero as n approaches infinity. The later terms in our present value, then, decrease in size until they get sufficiently small that the series approaches (but technically never exactly reaches) the particular value of the sum. Although a good deal of more sophisticated mathematics is required to deduce that the convergent value of the sum is $\delta/(1 - \delta)$, proving that this is the correct answer is relatively straightforward.

We use a simple trick to prove our claim. Consider the sum of the first m terms of the series, and denote it by S_m. Thus

$$S_m = \sum_{n=1}^{\infty} \delta^n = \delta + \delta^2 + \delta^3 + \cdots + \delta^{m-1} + \delta^m.$$

Now we multiply this sum by $(1 - \delta)$ to get

$$\begin{aligned}(1 - \delta)S_m &= \delta + \delta^2 + \delta^3 + \cdots + \delta^{m-1} + \delta^m \\ &\quad - \delta^2 - \delta^3 - \delta^4 - \cdots \quad - \delta^m - \delta^{m-1} \\ &= \delta - \delta^{m-1}.\end{aligned}$$

Dividing both sides by $(1 - \delta)$, we have

$$S_m = \frac{\delta - \delta^{m+1}}{1 - \delta}.$$

Finally we take the limit of this sum as m approaches infinity to evaluate our original infinite sum. As m goes to infinity, the value of δ^{m+1} goes to zero because very large and increasing powers of a number less than 1 get increasingly small but stay nonnegative. Thus as m goes to infinity, the right-hand side of the preceding equation goes to $\delta/(1 - \delta)$, which is therefore the limit of S^m as m approaches infinity. This completes the proof.

We need only convert back into r to be able to use our answer in the calculation of present values in our prisoners' dilemma games. Because $\delta = 1/(1 + r)$, it follows that

$$\frac{\delta}{1 - \delta} = \frac{1/(1 + r)}{r/(1 + r)} = \frac{1}{r}.$$

The present value of an infinite stream of \$36s earned each month, starting next month, is then

$$36 \cdot \sum_{n=1}^{\infty} \frac{1}{(1 + r)^n} = \frac{36}{r}.$$

This is the value that we use to determine whether a player should defect forever in Section 2 of Chapter 11. Notice that incorporating a probability of continuation, $p \leq 1$, into the discounting calculations changes nothing in the summation procedure used here. We could easily substitute R for r in the preceding calculations, and $p\delta$ for the discount factor, δ.

Remember that you need to find present values only for losses (or gains) incurred (or accrued) *in the future*. The present value of \$36 lost today is just \$36. So, if you wanted the present value of a stream of losses, all of them \$36, that begins *today*, you would take the \$36 lost today and add it to the present value of the stream of losses in the future. We have just calculated that present value as $36/r$. Thus the present value of the stream of lost \$36s, including the \$36 lost today, would be $36 + 36/r$, or $36[(r + 1)/r]$, which equals $36/(1 - \delta)$. Similarly, if you wanted to look at a player's stream of profits under a particular contingent strategy in a prisoners' dilemma, you would not discount the profit amount earned in the very first period; you would only discount those profit figures that represent money earned in future periods.

12

Collective-Action Games

THE GAMES AND STRATEGIC SITUATIONS considered in the preceding chapters have usually included only two or three players interacting with one another. Such games are common in our own academic, business, political, and personal lives and so are important to understand and analyze. But many social, economic, and political interactions are strategic situations in which numerous players participate at the same time. Strategies for career paths, investment plans, rush-hour commuting routes, and even studying have associated benefits and costs that depend on the actions of many other people. If you have been in any of these situations, you likely thought something was wrong—too many students, investors, and commuters crowding just where you want to be, for example. If you have tried to organize fellow students or your community in some worthy cause, you probably faced frustration of the opposite kind—too few willing volunteers. In other words, multiple-person games in society often seem to produce outcomes that are not deemed satisfactory by many or even all of the people in that society. In this chapter, we will examine such games from the perspective of the theory that we have already developed. We present an understanding of what goes wrong in such situations and what can be done about it.

In the most general form, such many-player games concern problems of **collective action.** The aims of the whole society or collective are best served if its members take some particular action or actions, but these actions are not in the best private interests of those individual members. In other words, the socially optimal outcome is not automatically achievable as the Nash equilibrium of the game. Therefore we must examine how the game can be modified to lead to the optimal outcome or at least to improve on the unsatisfactory Nash equilibrium. To do so, we must first understand the nature of such games. We find that they come in three forms, all of them familiar to you by now: the prisoners' dilemma, chicken, and assurance games. Although our main focus in this chapter is on situations where numerous players play such games at the same time, we build on familiar ground by beginning with games between just two players.

1 COLLECTIVE-ACTION GAMES WITH TWO PLAYERS

Imagine that you are a farmer. Your neighboring farmer and you can both benefit by constructing an irrigation and flood-control project. The two of you can join together to undertake this project or one of you might do so on your own. However, after the project has been constructed, the other automatically gets the benefit of it. Therefore each is tempted to leave the work to the other. That is the essence of your strategic interaction, and the difficulty of securing collective action.

In Chapter 4, we encountered a game of this kind: three neighbors were each deciding whether to contribute to a street garden that all of them would enjoy. That game became a prisoners' dilemma in which all three shirked; our analysis here will include an examination of a more general range of possible payoff structures. Also, in the street-garden game, we rated the outcomes on a scale of 1 to 6; when we describe more general games, we will have to consider more general forms of benefits and costs for each player.

Our irrigation project has two important characteristics. First, its benefits are **nonexcludable:** a person who has not contributed to paying for it cannot be prevented from enjoying the benefits. Second, its benefits are **nonrival:** any one person's benefits are not diminished by the mere fact that someone else is also getting the benefit. Economists call such a project a **pure public good;** national defense is often given as an example. In contrast, a pure *private* good is fully excludable and rival: nonpayers can be excluded from its benefits and, if one person gets the benefit, no one else does. A loaf of bread is a good example of a pure private good. Most goods fall somewhere on the two-dimensional spectrum of varying degrees of excludability and rivalness. We will not go any deeper into this taxonomy, but we mention it to help you relate our discussion to what you may encounter in other courses and books.[1]

[1]Public goods are studied in more detail in textbooks on *public economics* such as those by Harvey Rosen, *Public Finance,* 6th ed. (Chicago: Irwin/McGraw-Hill, 2001), and Joseph Stiglitz, *Economics of the Public Sector,* 3rd ed. (New York: Norton, 2000).

The costs and the benefits associated with building the irrigation project can depend on which players participate. Suppose each of you acting alone could complete the project in 7 weeks, whereas, if the two of you acted together, it would take only 4 weeks of time from each. The two-person project is also of better quality; each farmer gets benefits worth 6 weeks of work from a one-person project (whether constructed by you or by your neighbor) and 8 weeks' worth of benefit from a two-person project.

In the game, each farmer has to decide whether to work toward the construction of the project or not to—that is, to shirk. (Presumably, there is a short window of time in which the work must be done, and you could pretend to be called away on some very important family matter at the last minute, as could your neighbor.) Figure 12.1 shows the payoff table of the game, where the numbers measure the values in weeks of work. Given the payoff structure of this table, your best response if your neighbor does not participate is not to participate either: your benefit from completing the project by yourself (6) is less than your cost (7), whereas you can get 0 by not participating. Similarly, if your neighbor does participate, then you can reap the benefit (6) from his work at no cost to yourself; this is better for you than working yourself to get the larger benefit of the two-person project (8) while incurring the cost of the work (4). In this case, you are said to be a **free rider** on your neighbor's effort if you let the other do all the work and then reap the benefits all the same. The general feature of the game is that it is better for you not to participate no matter what your neighbor does; the same logic holds for him. Thus not building is the dominant strategy for each. But both would be better off if the two were to work together to build (payoff 4) than if neither works (payoff 0). Therefore the game is a prisoners' dilemma.

We see in this prisoners' dilemma one of the main difficulties that arises in games of collective action. Individually optimal choices—in this case, not to build regardless of what the other farmer chooses—may not be optimal from the perspective of society as a whole, even if the society is made up of just two farmers. The "social" optimum in a collective-action game is achieved when the sum total of the players' payoffs is maximized; in this prisoners' dilemma, the social opti-

		NEIGHBOR	
		Build	Not
YOU	Build	4, 4	−1, 6
	Not	6, −1	0, 0

FIGURE 12.1 Collective Action As a Prisoners' Dilemma: Version I

		NEIGHBOR	
		Build	Not
YOU	Build	2.3, 2.3	−1, 6
	Not	6, −1	0, 0

FIGURE 12.2 Collective Action As a Prisoners' Dilemma: Version II

mum is the (Build, Build) outcome. Nash equilibrium behavior of the players does not regularly bring about the socially optimal outcome, however. Hence, the study of collective-action games has focused on methods to improve on observed (generally Nash) equilibrium behavior to move outcomes toward the socially best outcomes. As we will see, the divergence between Nash equilibrium and socially optimum outcomes appears in every version of collective-action games.

Now consider what the game would look like if the numbers were to change slightly. Suppose the two-person project yields benefits that are not much better than those in the one-person project: 6.3 weeks' worth of work to each farmer. Then each of you gets $6.3 - 4 = 2.3$ when both of you build. The resulting payoff table is shown in Figure 12.2. The game is still a prisoners' dilemma and leads to the equilibrium (Not, Not). However, when both farmers build, the total payoff for both of you is only 4.6. The total payoff is maximized when one of you builds and the other does not, in which case together you get payoff $6 + (-1) = 5$. There are two possible ways to get this outcome. Achieving the social optimum in this case then poses a new problem: Who should build and suffer the payoff of -1 while the other is allowed to be a free rider and enjoy the payoff of 6?

Yet another variation in the numbers of the original prisoners' dilemma game of Figure 12.1 changes the nature of the game. Suppose the cost of the work is reduced so that it becomes better for you to build your own project if the neighbor does not. Specifically, suppose the one-person project requires 4 weeks of work and the two-person project takes 3 weeks from each; the benefits are the same as before. Figure 12.3 shows the payoff matrix resulting from these changes. Now your best response is to shirk when your neighbor works and to

		NEIGHBOR	
		Build	Not
YOU	Build	5, 5	2, 6
	Not	6, 2	0, 0

FIGURE 12.3 Collective Action As Chicken: Version I

work when he shirks. Formally, this game is just like a game of chicken, where shirking is the Straight strategy (tough or uncooperative), while working is the Swerve strategy (conciliatory or cooperative).

If this game results in one of its pure-strategy equilibria, the two payoffs sum to 8; this total is less than the total outcome that both players could get if both of them build. That is, neither of the Nash equilibria provides as much benefit to society as a whole as that of the coordinated outcome entailing both farmers choosing to build. If the outcome of the chicken game is its mixed-strategy equilibrium, the two farmers will fare even worse: their expected payoffs will add to something less than 8.

The collective-action chicken game has another possible structure if we make some additional changes to the benefits associated with the project. As with version II of the prisoners' dilemma, suppose the two-person project is not much better than the one-person project. Then each farmer's benefit from the two-person project is only 6.3, while each still gets a benefit of 6 from the one-person project. We ask you to practice your skill by constructing the payoff table for this game. You will find that it is still a game of chicken—call it chicken II. It still has two pure-strategy Nash equilibria in each of which only one farmer builds, but the sum of the payoffs when both build is only 6.6, while the sum when only one farmer builds is 8. The social optimum is for only one farmer to build. Each farmer prefers the equilibrium in which the other builds. This may lead to a new dynamic game in which each waits for the other to build. Or the original game might yield its mixed-strategy equilibrium with its low expected payoffs.

We have supposed thus far that the project yields equal benefits to both you and your neighbor farmer. But what if one farmer's activity causes harm to the other, as would happen if the only way to prevent one farm from being flooded is to divert the water to the other? Then each player's payoffs could be negative if his neighbor chose Build. Thus a third variant of chicken could arise in which each of you wants to build when the other does not, whereas it would be collectively better if neither of you did.

Finally, let us change the payoffs of the prisoners' dilemma case in a different way altogether, leaving the benefits of the two-person project and the costs of building as originally set out and reducing the benefit of a one-person project to 3. This change reduces your benefit as a free rider so much that now, if your neighbor chooses Build, your best response also is Build. Figure 12.4 shows the payoff table for this version of the game. This is now an assurance game with two pure-strategy equilibria: one where both of you participate and the other where neither of you does.

As in the chicken II version of the game, the socially optimal outcome here is one of the two Nash equilibria. But there is a difference. In chicken II, the two players differ in their preferences between the two equilibria, whereas, in the

		NEIGHBOR	
		Build	Not
YOU	Build	4, 4	−4, 3
	Not	3, −4	0, 0

FIGURE 12.4 Collective Action as an Assurance Game

assurance game, both of them prefer the same equilibrium. Therefore achieving the social optimum should be easier in the assurance game than in chicken.

Just as the problems pointed out in these examples are familiar, the various alternative ways of tackling the problems also follow the general principles discussed in earlier chapters. Before turning to solutions, let us see how the problems manifest themselves in the more realistic setting where several players interact simultaneously in such games.

2 COLLECTIVE-ACTION PROBLEMS IN LARGE GROUPS

In this section, we extend our irrigation-project example to a situation in which a population of N farmers must each decide whether to participate. If n of them participate in the construction of the project, each of the participants incurs a cost c that depends on the number n; so we write it as the function $c(n)$. Also, each person in the population, whether a contributor to the building of the project or not, enjoys a benefit from its completion that also is a function of n; we write the benefit function as $b(n)$. Thus each participant gets the payoff $p(n) = b(n) - c(n)$, whereas each nonparticipant, or shirker, gets the payoff $s(n) = b(n)$.

Suppose you are contemplating whether to participate or to shirk. Your decision will depend on what the other $(N - 1)$ farmers in the population are doing. In general, you will have to make your decision when the other $(N - 1)$ players consist of n participants and $(N - 1 - n)$ shirkers. If you decide to shirk, the number of participants in the project is still n; so you get a payoff of $s(n)$. If you decide to participate, the number of participants becomes $n + 1$; so you get $p(n + 1)$. Therefore your final decision depends on the comparison of these two payoffs; you will participate if $p(n + 1) > s(n)$, and you will shirk if $p(n + 1) < s(n)$. This comparison holds true for every version of the collective-action game analyzed in Section 1; differences in behavior in the different versions arise because the changes in the payoff structure alter the values of $p(n + 1)$ and $s(n)$.

We can relate the two-person examples of Section 1 to this more general framework. If there are just two people, then $p(2)$ is the payoff to one from

		NEIGHBOR	
		Build	Not
YOU	Build	$p(2), p(2)$	$p(1), s(1)$
	Not	$s(1), p(1)$	$s(0), s(0)$

FIGURE 12.5 General Form of a Two-Person Collective-Action Game

building when the other also builds, $s(1)$ is the payoff to one from shirking when the other builds, and so on. Therefore we can generalize the payoff tables of Figures 12.1 through 12.4 into an algebraic form. This general payoff structure is shown in Figure 12.5.

The game illustrated in Figure 12.5 is a prisoners' dilemma if the inequalities

$$p(2) < s(1), \quad p(1) < s(0), \quad p(2) > s(0)$$

all hold at the same time. The first says that the best response to Build is Not, the second says that the best response to Not also is Not, and the third says that (Build, Build) is jointly preferred to (Not, Not). The dilemma is of Type I if $2p(2) > p(1) + s(1)$; so the total payoff is higher when both build than when only one builds. You can establish similar inequalities concerning these payoffs that yield the other types of games in Section 1.

Now return to many-player games with a general n. We can use the payoff functions $p(n)$ and $s(n)$ to construct a third function showing the total payoff to society as a function of n, which we write as $T(n)$.) The total payoff to society consists of the value $p(n)$ for each of the n participants and the value $s(n)$ for each of the $(N - n)$ shirkers:

$$T(n) = np(n) + (N - n)s(n)$$

Now we can ask: What allocation of people between participants and shirkers maximizes the total payoff $T(n)$? To get a better understanding of this question, it is convenient to write $T(n)$ differently, as

$$T(n) = Ns(n) - n[s(n) - p(n)].$$

This derivation of the total social payoff shows that we can calculate it as if we gave every one of the N people the shirker's payoff but then removed the shirker's extra benefit $[s(n) - p(n)]$ from each of the n participants. We normally expect $s(n)$ to increase as n increases; therefore the first term in this expression, $Ns(n)$, also increases as n increases. If the second term does not increase too fast as n increases—as would be the case if the shirker's extra benefit, $[s(n) - p(n)]$, is small and does not increase—then the effect of the first term dominates in determining the value of $T(n)$; $T(n)$ increases steadily with n in this case and is maximized at $n = N$. This is the case in which no one shirks. But, otherwise and more generally, $T(n)$ can be maximized for some value of n less than N. That is, society's aggregate payoff may be maximized by allowing some shirking. We saw this earlier in the two-person examples, and we will encounter more examples in the multiperson situations to be examined shortly.

Now we can use graphs of the $p(n + 1)$ and $s(n)$ functions to help us determine which type of game we have encountered, its Nash equilibrium, and its socially optimal outcome. We draw the graphs by showing n over its full range from 0 to $(N - 1)$ along the horizontal axis and payoffs along the vertical axis. Two separate curves show $p(n + 1)$ and $s(n)$. Actually, n takes on only integer values, and therefore each function $p(n + 1)$ and $s(n)$ technically consists only of a discrete set of points rather than a continuous set as implied by our smooth lines. But when N is large, the discrete points are sufficiently close together that we can connect the successive points and show each payoff function as a continuous curve. We show the $p(n + 1)$ and $s(n)$ functions as straight lines to bring out the basic considerations most simply and will discuss more complicated possibilities later.

The first of our graphs, Figure 12.6, illustrates the case in which the curve $s(n)$ lies entirely above the curve $p(n + 1)$. Therefore no matter how many others participate (that is, no matter how large n gets), your payoff is higher if you shirk than if you participate; shirking is your dominant strategy. This is true for everyone, therefore the equilibrium of the game entails everyone shirking. But both curves are rising as n increases: for each action you take, you are better off if more of the others participate. And the left intercept of the $s(n)$ curve is below

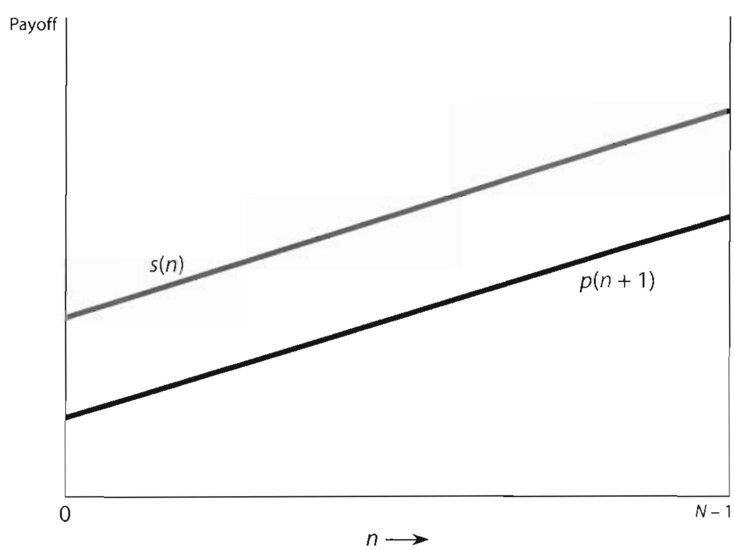


FIGURE 12.6 Multiperson Prisoners' Dilemma

the right intercept of the $p(n + 1)$ curve, or $s(0) < p(N)$. This says that, if everyone including you shirks, your payoff is less than if everyone including you participates; everyone would be better off than they are in the Nash equilibrium of the game if the outcome in which everyone participates could be sustained. This makes the game a prisoners' dilemma.

However, as we saw earlier, the fact that each person would be better off if everyone participated does not automatically imply that full participation is the best thing for society; it is not automatic that the total payoff function is maximized when n is as large as possible in the prisoners' dilemma case. In fact, if the gap between $s(n)$ and $p(n)$ widens sufficiently fast as n increases, then the negative effect of the second term in the expression for $T(n)$ outweighs the positive effect of the first term as n approaches N; then it may be best to let some people shirk—that is, the socially optimal value for n may be less than N. This type of outcome creates an inequality in the payoffs—the shirkers fare better than the participants—which adds another dimension of difficulty to society's attempts to resolve the dilemma.

Figure 12.7 shows the chicken case. Here, for small values of n, $p(n + 1) > s(n)$, and so, if few others are participating, your choice is to participate; for large values of n, $p(n + 1) < s(n)$, and so, if many others are participating, your choice is to shirk. Note the equivalence of these two statements to the idea in the two-person game that "you shirk if your neighbor works and you work if he shirks." If the two curves intersect at a point corresponding to an integer value of n, then that is the Nash equilibrium number of participants. If that is not the case, then strictly speaking the game has no Nash equilibrium. But, in practice, if the current value of n in the population is the integer just to the left of the

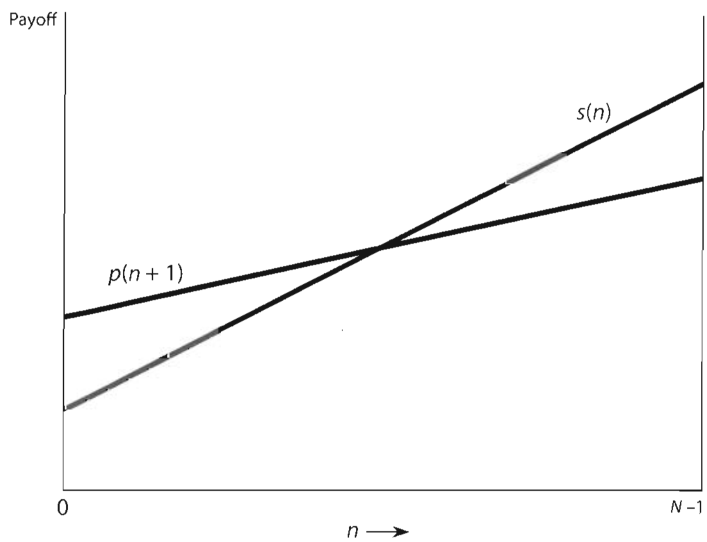


FIGURE 12.7 Multiperson Chicken

point of intersection, then one person will just want to participate, whereas, if the current value of n is the integer just to the right of the point of intersection, one person will want to switch to shirking. Therefore the number of participants will stay in a small neighborhood of the point of intersection, and we can justifiably speak of the intersection as the equilibrium in some approximate sense.

Here, we have shown both lines positively sloped, although they don't have to be; it is conceivable that the benefit for each person is smaller when more people participate. The important feature of the game is that, when there are few people taking one action, it is better for any one person to take that action; and, when there are many people taking one action, it is better for any one person to take the other action.

What is the socially optimal outcome in the chicken form of the collective-action problem? If each participant's payoff $p(n)$ increases as the number of participants increases and if each shirker's payoff $s(n)$ does not become too much greater than the $p(n)$ of each participant, then the total social payoff is maximized when everyone participates. But, more generally, it may be better to let some shirk. This is exactly the difference between versions I and II of chicken in our earlier numerical example in Section 1. As we pointed out in Section 2, in the context of the prisoners' dilemma, where the social optimum can have some participants and some shirkers, payoffs can be unequally distributed in the population, making it harder to implement the optimum. In more general games of this kind, the optimal number of participants could even be smaller than that in the Nash equilibrium. We return to examine the question of the social optimum of all of these versions of the game in greater detail in Section 5.

Figure 12.8 shows the assurance case. Here $s(n) > p(n + 1)$ for small values of n; so, if few others are participating, then you want to shirk, too, whereas $p(n + 1) > s(n)$ for large values of n, and so, if many others are participating, then you want to participate too. This game has two Nash equilibria at the two extremes: either everyone shirks or everyone participates. When the curves are rising—so each person is better off if more people participate—then clearly the right-hand extreme equilibrium is the better one for society, and the question is how to bring it about.

When the total number of people in the group, N, is very large, and any one person makes only a small difference, then $p(n + 1)$ is almost the same as $p(n)$. Thus the condition under which any one person chooses to shirk is $p(n) < s(n)$. Expressing this inequality in terms of the benefits and costs of the common project in our example—namely, $p(n) = b(n) - c(n)$ and $s(n) = b(n)$—we see that $p(n)$ (unlike $p(n + 1)$ in our preceding calculations) is always less than $s(n)$; individual persons will always want to shirk when N is very large. And that is why problems of collective provision of public projects in a large group almost always manifest themselves as prisoners' dilemmas. But, as we saw in the two-person examples discussed in Section 1, this result is not necessarily true for

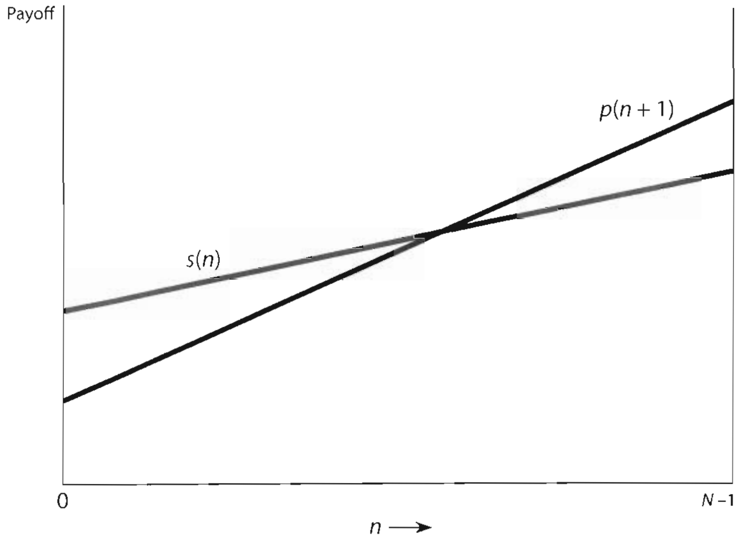


FIGURE 12.8 Multiperson Assurance Game

small groups. Morever, collective-action problems cover a broader range than that of participating in the construction of a public project whose benefits are equally available to shirkers. We will see several other examples of such games later in Section 5.

Therefore, in general, we must allow for a broader interpretation of the payoffs $p(n)$ and $s(n)$ than we did in the specific case involving the benefits and the costs of a project; we must allow $p(n)$ and $s(n)$ to be any functions of n. Then there is no automatic presumption about which of the two payoff functions is larger, and all three kinds of games—prisoners' dilemma, chicken, and assurance—and their associated graphs deserve our attention. In fact, in the most general case, $p(n)$ and $s(n)$ do not even have to be straight lines and can intersect many times. Then there can be several equilibria, although each can be thought of as representing one of the three types described so far.[2] When we make this more general interpretation, we will speak of two actions labeled P and S, which have no necessary connotation of "participation" and "shirking" but allow us to continue with the same symbols for the payoffs. Thus, when

[2]Exercises 2, 3, and 5 at the end of this chapter construct some simple situations with nonlinear curves and multiple equilibria. For a more general analysis and classification of such diagrams, see Thomas Schelling, *Micromotives and Macrobehavior* (New York: Norton, 1978), chap. 7. The theory can be taken farther by allowing each player a continuous choice (for example, the number of hours of participation) instead of just a binary choice of whether to participate. Many such situations are discussed in more specialized books on collective action; for example, Todd Sandler, *Collective Action: Theory and Applications* (Ann Arbor: University of Michigan Press, 1993), and Richard Cornes and Todd Sandler, *The Theory of Externalities, Public Goods, and Club Goods,* 2nd ed. (New York: Cambridge University Press, 1996).

there are n players taking the action P, $p(n)$ becomes the payoff of each player taking the action P, and $s(n)$ becomes that of each player taking the action S.

3 A BRIEF HISTORY OF IDEAS

The problem of collective action has been recognized by social philosophers and economists for a very long time. The 17th-century British philosopher Thomas Hobbes argued that society would break down in a "war of all against all" unless it was ruled by a dictatorial monarch or *Leviathan* (the title of his book). One hundred years later, the French philosopher Jean-Jacques Rousseau described the problem of a prisoners' dilemma in his *Discourse on Inequality*. A stag hunt needs the cooperation of the whole group of hunters to encircle and kill the stag, but any individual hunter who sees a hare may find it better for himself to leave the circle to chase the hare. But Rousseau thought that such problems were the product of civilization and that people in the natural state lived harmoniously as "noble savages." At about the same time, two Scots pointed out some dramatic solutions to such problems: David Hume in his *Treatise on Human Nature* argued that the expectations of future returns of favors can sustain cooperation. Adam Smith's *Wealth of Nations* developed a grand vision of an economy in which the production of goods and services motivated purely by private profit could result in an outcome that was best for society as a whole.[3]

The optimistic interpretation persisted, especially among many economists and even several political scientists, to the point where it was automatically assumed that, if an outcome was beneficial to a group as a whole, the actions of its members would bring the outcome about. This belief received a necessary rude shock in the mid-1960s when Mancur Olson published *The Logic of Collective Action*. He pointed out that the best outcome collectively would not prevail unless it was in each individual person's private interest to perform his assigned action—that is, unless it was a Nash equilibrium. However, he did not specify the collective-action game very precisely. Although it looked like a prisoners' dilemma, Olson insisted that it was not necessarily so, and we have already seen that the problem can also take the form of a chicken game or an assurance game.[4]

[3]The great old books cited in this paragraph have been reprinted many times in many different versions. For each, we list the year of original publication and the details of one relatively easily accessible reprint. In each case, the editor of the reprinted version provides an introduction that conveniently summarizes the main ideas. Thomas Hobbes, *Leviathan, or the Matter, Form, and Power of Commonwealth Ecclesiastical and Civil,* 1651 (Everyman Edition, London: J. M. Dent, 1973). David Hume, *A Treatise of Human Nature,* 1739 (Oxford: Clarendon Press, 1976). Jean-Jacques Rousseau, *A Discourse on Inequality,* 1755 (New York: Penguin Books, 1984). Adam Smith, *An Inquiry into the Nature and Causes of the Wealth of Nations,* 1776 (Oxford: Clarendon Press, 1976).

[4]Mancur Olson, *The Logic of Collective Action* (Cambridge: Harvard University Press, 1965).

Another major class of collective-action problems—namely, those concerning the depletion of common-access resources—received attention at about the same time. If a resource such as a fishery or a meadow is open to all, each user will exploit it as much as he can because any self-restraint on his part will merely make more available for the others to exploit. Garrett Hardin wrote a well-known article on this subject titled "The Tragedy of the Commons."[5] The problem in the common-resource game is just the reverse of our example of participation in the construction of a project that yields benefits mostly to the others. In our example, each person has a strong private incentive to refrain from participation and to enjoy a free rider's benefits. In regard to a common resource, each person has a strong private incentive to exploit it to the full, making everyone else pay the social cost that results from the degradation of the resource.

Until recently, many social scientists and most physical scientists took a Hobbesian line on the common-resource problem, arguing that it can only be solved by a government that forces everyone to behave cooperatively. Others, especially economists, retained their Smithian optimism. They argued that placing the resource in proper private ownership, where its benefits can be captured in the form of profit by the owner, will induce the owner to restrain its use in a socially optimal manner. He will realize that the value of the resource (fish or grass, for example) may be higher in the future because there will be less available, and therefore he can make more profit by saving some of it for that future.

Nowadays, thinkers from all sides have begun to recognize that collective-action problems come in diverse forms and that there is no uniquely best solution to all of them. They also understand that groups or societies do not stand helpless in the face of such problems and they devise various ways to cope with them. Much of this work has been informed by game-theoretic analysis of repeated prisoners' dilemmas and similar games.[6]

4 SOLVING COLLECTIVE-ACTION PROBLEMS

In this section, we offer a brief discussion of conceptual and practical solutions to collective-action problems, organized around the threefold classification of collective-action games as prisoners' dilemma, chicken, or assurance games. The feature common to all three types is the need to induce individual persons

[5] Garrett Hardin, "The Tragedy of the Commons," *Science*, vol. 162 (1968), pp. 1243–1248.

[6] Prominent in this literature are Michael Taylor, *The Possibility of Cooperation* (New York: Cambridge University Press, 1987), Elinor Ostrom, *Governing the Commons* (New York: Cambridge University Press, 1990), and Matt Ridley, *The Origins of Virtue* (New York: Viking Penguin, 1996).

to act cooperatively or in a manner that would be best for the group, even though the person's interests may best be served by doing something else—in particular, taking advantage of the others' cooperative behavior.[7]

Humans exhibit much in the way of cooperative behavior. The act of reciprocating gifts and skills at detecting cheating are so common in all societies and throughout history, for example, that there is reason to argue that they may be instincts.[8] But human societies generally rely heavily on purposive social and cultural customs, norms, and sanctions in inducing cooperative behavior from their individual members. These methods are conscious, deliberate attempts to design the game in order to solve the collective-action problem.[9] We approach the matter of solution methods from the perspective of the type of game being played.

A. Analysis

A solution is easiest if the collective-action problem takes the form of an assurance game. Then it is in every person's private interest to take the socially best action if he expects all other persons to do likewise. In other words, the socially optimal outcome is a Nash equilibrium; the only problem is that the same game has other, socially worse, Nash equilibria. Then all that is needed to achieve the best Nash equilibrium and thereby the social optimum is to make it a focal point—that is, to ensure the convergence of the players' expectations on it. Such a convergence can result from a social **custom,** or **convention** —namely, a

[7]The problem of the need to attain cooperation and its solutions are not unique to human societies. Examples of cooperative behavior in the animal kingdom have been explained by biologists in terms of the advantage of the gene and of the evolution of instincts. For more, see Chapter 13 and Ridley, *Origins of Virtue.*

[8]See Ridley, *Origins of Virtue,* chaps. 6 and 7.

[9]The social sciences do not have precise and widely accepted definitions of terms such as *custom* and *norm;* nor are the distinctions among such terms always clear and unambiguous. We set out some definitions in this section, but be aware that you may find a different usage in other books. Our approach is similar to those found in Richard Posner and Eric Rasmusen, "Creating and Enforcing Norms, with Special Reference to Sanctions," *International Review of Law and Economics,* vol. 19, no. 3 (September 1999), pp. 369–382, and in David Kreps, "Intrinsic Motivation and Extrinsic Incentives," *American Economic Review,* Papers and Proceedings, vol. 87, no. 2 (May 1997), pp. 359–364; Kreps uses the term "norm" for all the concepts that we classify under different names.

Sociologists have a different taxonomy of norms from that of economists; it is based on the importance of the matter (trivial matters such as table manners are called *folkways,* and more weighty matters are called *mores*), and on whether the norms are formally codified as *laws.* They also maintain a distinction between *values* and norms, recognizing that some norms may run counter to persons' values and therefore require sanctions to enforce them. This distinction corresponds to ours between customs and norms; the conflict between individual values and social goals arises for norms but not for *conventions,* as we label them. See Donald Light and Suzanne Keller, *Sociology,* 4th ed. (New York: Knopf, 1987), pp. 57–60.

mode of behavior that finds automatic acceptance because it is in everyone's interest to follow it so long as others are expected to do likewise. For example, if all the farmers, herders, weavers, and other producers in an area want to get together to trade their wares, all they need is the assurance of finding others with whom to trade. Then the custom that the market is held in village X on day Y of every week makes it optimal for everyone to be there on that day.[10]

But our analysis in Section 2 suggested that individual payoffs are often configured in such a way that collective-action problems, particularly of large groups, take the form of a prisoners' dilemma. Not surprisingly, the methods for coping with such problems have received the most attention. In Chapter 11, we described in detail three methods for achieving a cooperative outcome in prisoners' dilemma games: repetition, penalties (or rewards), and leadership. In that discussion, we were mainly concerned with two-person dilemmas. The same methods apply to collective-action problems in large groups, with some important modifications or innovations.

We saw in Chapter 11 that repetition was the most prominent of these methods; so we focus most attention on it. Repetition can achieve cooperative outcomes as equilibria of individual actions in a repeated two-person prisoners' dilemma by holding up the prospect that cheating will lead to a breakdown of cooperation. More generally, what is needed to maintain cooperation is the expectation in the mind of each player that his personal benefits of cheating are transitory and that they will quickly be replaced by a payoff lower than that associated with cooperative behavior. For players to believe that cheating is not beneficial from a long-term perspective, cheating should be detected quickly, and the punishment that follows (reduction in future payoffs) should be sufficiently swift, sure, and painful.

A group has one advantage in this respect over a pair of individual persons. The same pair may not have occasion to interact all that frequently, but each of them is likely to interact with *someone* in the group all the time. Therefore B's temptation to cheat A can be countered by his fear that others, such as C, D, and so on, whom he meets in the future will punish him for this action. An extreme case where bilateral interactions are not repeated and punishment must be inflicted on one's behalf by a third party is, in Yogi Berra's well-known saying: "Always go to other people's funerals. Otherwise they won't go to yours."

[10]Evolutionary biologist Lee Dugatkin, in his study of the emergence of cooperation, *Cheating Monkeys and Citizen Bees* (New York: Free Press, 1999), labels this case "selfish teamwork." He argues that such behavior is more likely to arise in times of crises, because each person is pivotal at those times. In a crisis, the outcome of the group interaction is likely to be disastrous for everyone if even one person fails to contribute to the group's effort to get out of the dire situation. Thus each person is willing to contribute so long as the others do. We will mention Dugatkin's full classification of alternative approaches to cooperation in Chapter 13 on evolutionary games.

But a group has some offsetting disadvantages over direct bilateral interaction when it comes to sustaining good behavior in repeated interactions. The required speed and certainty of detection and punishment suffer as the numbers in the group increase.

Start with the detection of cheating, which is never easy. In most real situations, payoffs are not completely determined by the players' actions but are subject to some random fluctuations. Even with two players, if one gets a low payoff, he cannot be sure that the other cheated; it may have been just a bad draw of the random shock. With more people, an additional question enters the picture: if someone cheated, who was it? Punishing someone without being sure of his guilt beyond a reasonable doubt is not only morally repulsive but also counterproductive—the incentive to cooperate gets blunted if even cooperative actions are susceptible to punishment by mistake.

Next, with many players, even when cheating is detected and the cheater identified, this information has to be conveyed sufficiently quickly and accurately to others. For this, the group must be small or else must have a good communication or gossip network. Also, members should not have much reason to accuse others falsely.

Finally, even after cheating is detected and the information spread to the whole group, the cheater's punishment has to be arranged. A third person often has to incur some personal cost to inflict such punishment. For example, if C is called on to punish B, who had previously cheated A, C may have to forgo some profitable business that he could have transacted with B. Then the inflicting of punishment is itself a collective-action game and suffers from the same temptation to "shirk"; that is, not to participate in the punishment. A society could construct a second-round system of punishments for shirking, but that in turn may be yet another collective-action problem! However, humans seem to have evolved an instinct whereby people get some personal pleasure from punishing cheaters even when they have not themselves been the victims of this particular act of cheating.[11]

Now consider some specific methods that society can employ to inflict punishment on "defectors" (or shirkers). One method is through **sanctions** imposed by other members of the group. Sanctions often take the form of disqualification from future games played by the group. Society can also create **norms** of behavior that change individual payoffs so as to induce cooperation. A norm changes the private payoff scale of each player, by adding an extra cost in the form of shame, guilt, or dislike of the mere disapproval by others. Society establishes norms through a process of education or culture. Norms differ from customs in that a person would not automatically follow the norm merely because

[11]For evidence of such altruistic punishment instinct, see Ernst Fehr and Simon Gachter, "Altruistic Punishment in Humans," *Nature*, vol. 415 (January 10, 2002), pp. 137–140.

of the expectation that others would follow it; the extra cost is essential. Norms also differ from sanctions in that others do not have to take any explicit actions to hurt you if you violate the norm; the extra cost becomes internalized in your own payoff scale.[12]

Society can also reward desirable actions just as it can punish undesirable ones. Again, the rewards, financial or otherwise, can be given externally to the game (like sanctions) or players' payoffs can be changed so that they take pleasure in doing the right thing (norms). The two types of rewards can interact; for example, the peerages and knighthoods given to British philanthropists and others who do good deeds for British society are external rewards, but individual persons value them only because respect for knights and peers is a British social norm.

Norms are reinforced by observation of society's general adherence to them, and they lose their force if they are frequently seen to be violated. Before the advent of the welfare state, when those who fell on hard economic times had to rely on help from family or friends or their immediate small social group, the work ethic constituted a norm that held in check the temptation to slacken one's own efforts and become a free rider on the support of others. As the government took over the supporting role and unemployment compensation or welfare became an entitlement, this norm of the work ethic weakened. After the sharp increases in unemployment in Europe in the late 1980s and early 1990s, a significant fraction of the population became users of the official support system, and the norm weakened even further.[13]

Different societies or cultural groups may develop different conventions and norms to achieve the same purpose. At the trivial level, each culture has its own set of good manners—ways of greeting strangers, indicating approval of food, and so on. When two people from different cultures meet, misunderstandings can arise. More importantly, each company or office has its own ways of getting things done. The differences between these customs and norms are subtle and difficult to pin down, but many mergers fail because of a clash of these "corporate cultures."

Successful solution of collective-action problems clearly hinges on success in detection and punishment. As a rule, small groups are more likely to have better information about their members and the members' actions and therefore are more likely to be able to detect cheating. They are also more likely to communicate the information to all members and more likely to organize when

[12]See Ostrom, *Governing the Commons*, p. 35. Our distinction between norms and sanctions is similar to Kreps's distinction between functions (iii) and (iv) of norms (Kreps, "Intrinsic Motivation and Extrinsic Incentives," p. 359).

[13]Assar Lindbeck, "Incentives and Social Norms in Household Behavior," *American Economic Review*, Papers and Proceedings, vol. 87, no. 2 (May 1997), pp. 370–377.

inflicting punishment on a cheater. One sees many instances of successful cooperation in small village communities that would be unimaginable in a large city or state.

Next, consider the chicken form of collective-action games. Here, the nature of the remedy depends on whether the largest total social payoff is attained when everyone participates (what we called "chicken version I" in Section 1) or when some cooperate and others are allowed to shirk (chicken II). For chicken I, where everyone has the individual temptation to shirk, the problem is much like that of sustaining cooperation in the prisoners' dilemma, and all the earlier remarks for that game apply here, too. Chicken II is different—easier in one respect and harder in another. Once an assignment of roles between participants and shirkers is made, no one has the private incentive to switch: if the other driver is assigned the role of going straight, then you are better off swerving and the other way around. Therefore, if a custom creates the expectation of an equilibrium, it can be maintained without further social intervention such as sanctions. However, in this equilibrium, the shirkers get higher payoffs than the participants do, and this inequality can create its own problems for the game; the conflicts and tensions, if they are major, can threaten the whole fabric of the society. Often the problem can be solved by repetition. The roles of participants and shirkers can be rotated to equalize payoffs over time.

Sometimes the problem of differential payoffs in versions II of the prisoners' dilemma or chicken is "solved," not by restoring equality but by **oppression** or **coercion,** which forces a dominated subset of society to accept the lower payoff and allows the dominant subgroup to enjoy the higher payoff. In many societies throughout history, the work of handling animal carcasses was forced on particular groups or castes in this way. The history of the maltreatment of racial and ethnic minorities and of women provides vivid examples of such practices. Once such a system becomes established, no one member of the oppressed group can do anything to change the situation. The oppressed must get together as a group and act to change the whole system, itself another problem of collective action.

Finally, consider the role of leadership in solving collective-action problems. In Chapter 11, we pointed out that, if the players are of very unequal "size," the prisoners' dilemma may disappear because it may be in the private interests of the larger player to continue cooperation and to accept the cheating of the smaller player. Here, we recognize the possibility of a different kind of bigness—namely, having a "big heart." People in most groups differ in their preferences, and many groups have one or a few who take genuine pleasure in expending personal effort to benefit the whole. If there are enough such people for the task at hand, then the collective-action problem disappears. Most schools, churches, local hospitals, and other worthy causes rely on the work of such willing volunteers. This solution, like others before it, is more likely to work in small groups, where the fruits of their actions are more closely and immediately visible to the benefactors, who are therefore encouraged to continue.

B. Applications

Elinor Ostrom, in her book *Governing the Commons,* describes several examples of resolution of common-resource problems at local levels. Most of them require taking advantage of features specific to the context in order to set up systems of detection and punishment. A fishing community on the Turkish coast, for example, assigns and rotates locations to its members; the person who is assigned a good location on any given day will naturally observe and report any intruder who tries to usurp his place. Many other users of common resources, including the grazing commons in medieval England, actually restricted access and controlled overexploitation by allocating complex, tacit but well-understood rights to individual persons. In one sense, this solution bypasses the common-resource problem by dividing up the resource into a number of privately owned subunits.

The most striking feature of Ostrom's range of cases is their immense variety. Some of the prisoners' dilemmas of the exploitation of common-property resources that she examined were solved by private initiative by the group of people actually in the dilemma; others were solved by external public or governmental intervention. In some instances, the dilemma was not resolved at all, and the group remained trapped in the all-shirk outcome. Despite this variety, Ostrom identifies several common features that make it easier to solve prisoners' dilemmas of collective action: (1) it is essential to have an identifiable and stable group of potential participants; (2) the benefits of cooperation have to be large enough to make it worth paying all the costs of monitoring and enforcing the rules of cooperation; and (3) it is very important that the members of the group can communicate with one another. This last feature accomplishes several things. First, it makes the norms clear—everyone knows what behavior is expected, what kind of cheating will not be tolerated, and what sanctions will be imposed on cheaters. Next, it spreads information about the efficacy of the detection of the cheating mechanism, thereby building trust and removing the suspicion that each participant might hold that he is abiding by the rules while others are getting away with breaking them. Finally, it enables the group to monitor the effectiveness of the existing arrangements and to improve on them as necessary. All these requirements look remarkably like those identified in Chapter 11 from our theoretical analysis of the prisoners' dilemma (Sections 2 and 3) and from the observations of Axelrod's tournaments (Section 5).

Ostrom's study of the fishing village also illustrates what can be done if the collective optimum requires different persons to do different things, in which case some get higher payoffs than others. In a repeated relationship, the advantageous position can rotate among the participants, thereby maintaining some sense of equality over time. We analyze other instances of this problem of inequality and other resolutions of it in Section 5.

Ostrom finds that an external enforcer of cooperation may not be able to detect cheating or impose punishment with sufficient clarity and swiftness. Thus the frequent reaction that centralized or government policy is needed to solve collective-action problems is often proved wrong. Another example comes from village communities or "communes" in late 19th-century Russia. These communities solved many collective-action problems of irrigation, crop rotation, management of woods and pastures, and road and bridge construction and repair in just this way. "The village . . . was not the haven of communal harmony. . . . It was simply that the individual interests of the peasants were often best served by collective activity." Reformers of early 20th-century czarist governments and Soviet revolutionaries of the 1920s alike failed, partly because the old system had such a hold on the peasants' minds that they resisted anything new, but also because the reformers failed to understand the role that some of the prevailing practices played in solving collective-action problems and thus failed to replace them by equally effective alternatives.[14]

The difference between small and large groups is well illustrated by Greif's comparison of two groups of traders in countries around the Mediterranean Sea in medieval times. The Maghribis were Jewish traders who relied on extended family and social ties. If one member of this group cheated another, the victim informed all the others by writing letters and, when guilt was convincingly proved, no one in the group would deal with the cheater. This system worked well on a small scale of trade. But, as trade expanded around the Mediterranean, the group could not find sufficiently close or reliable insiders to go to the countries with the new trading opportunities. In contrast, the Genoese traders established a more official legal system. A contract had to be registered with the central authorities in Genoa. The victim of any cheating or violation of the contract had to take a complaint to the authorities, who carried out the investigation and imposed the appropriate fines on the cheater. This system, with all its difficulties of detection, could be more easily expanded with the expansion of trade.[15] As economies grow and world trade expands, we see a similar shift from tightly linked groups to more arm's length trading relationships, and from enforcement based on repeated interactions to that of the official law.

[14]Orlando Figes, *A People's Tragedy: The Russian Revolution 1891–1924* (New York: Viking Penguin, 1997), pp. 89–90, 240–241, 729–730. See also Ostrom, *Governing the Commons*, p. 23, for other instances where external government-enforced attempts to solve common-resource problems actually made them worse.

[15]Avner Greif, "Cultural Beliefs and the Organization of Society: A Historical and Theoretical Reflection on Collectivist and Individualist Societies," *Journal of Political Economy*, vol. 102, no. 5 (October 1994), pp. 912–950.

The idea that small groups are more successful at solving collective-action problems forms the major theme of Olson's *The Logic of Collective Action* (see footnote 4) and has led to an insight important in political science. In a democracy, all voters have equal political rights, and the majority's preference should prevail. But we see many instances in which this does not happen. The effects of policies are generally good for some groups and bad for others. To get its preferred policy adopted, a group has to take political action—lobbying, publicity, campaign contributions, and so on. To do these things, the group must solve a collective-action problem, because each member of the group may hope to shirk and enjoy the benefits that the others' efforts have secured. If small groups are better able to solve this problem, then the policies resulting from the political process will reflect *their* preferences even if other groups who fail to organize are more numerous and suffer greater losses than the successful groups' gains.

The most dramatic example of this comes from the arena of trade policy. A country's import restrictions help domestic producers whose goods compete with these imports, but they hurt the consumers of the imported goods and the domestic competing goods alike, because prices for these goods are higher than they would be otherwise. The domestic producers are few in number, and the consumers are almost the whole population; the total dollar amount of the consumers' losses is typically far bigger than the total dollar amount of the producers' gains. Political considerations based on constituency membership numbers and economic considerations of dollar gains and losses alike would lead us to expect a consumer victory in this policy arena; we would expect to see at least a push for the idea that import restrictions should be abolished, but we don't. The smaller and more tightly knit associations of producers are better able to organize for political action than the numerous dispersed consumers.

More than 60 years ago, the American political scientist E. E. Schattschneider provided the first extensive documentation and discussion of how pressure politics drives trade policy. He recognized that "the capacity of a group for organization has a great influence on its activity," but he did not develop any systematic theory of what determines this capacity.[16] The analysis of Olson and others has improved our understanding of the issue, but the triumph of pressure politics over economics persists in trade policy to this day. For example, in the late 1980s, the U.S. sugar policy cost each of the 240 million people in the United States about $11.50 per year for a total of about $2.75 billion, while it increased the incomes of about 10,000 sugar-beet farmers by about $50,000 each, and the incomes of 1,000 sugarcane farms by as much as $500,000 each for a total of about $1 billion. The net loss to the U.S. economy was $1.75 billion.[17]

[16] E. E. Schattschneider, *Politics, Pressures, and the Tariff* (New York: Prentice-Hall, 1935); see especially pp. 285–286.

[17] Stephen V. Marks, "A Reassessment of the Empirical Evidence on the U.S. Sugar Program," in *The Economics and Politics of World Sugar Policies*, ed. Stephen V. Marks and Keith E. Maskus (Ann Arbor: University of Michigan Press, 1993), pp. 79–108.

Each of the unorganized consumers continues to bear his small share of the costs in silence; many of them are not even aware that each is paying \$11.50 a year too much for his sweet tooth.

If this overview of the theory and practice of solving collective-action problems seems diverse and lacking a neat summary statement, that is because the problems are equally diverse, and the solutions depend on the specifics of each problem. The one general lesson that we can provide is the importance of letting the participants themselves devise solutions by using their local knowledge of the situation, their advantage of proximity in monitoring the cooperative or shirking actions of others in the community, and their ability to impose sanctions on shirkers by using various ongoing relationships within the social group.

Finally, a word of caution. You might be tempted to come away from this discussion of collective-action problems with the impression that individual freedom leads to harmful outcomes that can and must be improved by social norms and sanctions. Remember, however, that societies face problems other than those of collective action; some of them are better solved by individual initiative than by joint efforts. Societies can often get hidebound and autocratic, becoming trapped in their norms and customs and stifling the innovation that is so often the key to economic growth. Collective action can become collective inaction.[18]

5 SPILLOVERS, OR EXTERNALITIES

Now we return to the comparison of the Nash equilibrium and the social optimum in collective-action games. The two questions that we want to answer are: What are the socially optimal numbers of people who should choose the one strategy or the other, and how will these numbers differ from the ones that emerge in a Nash equilibrium?

Remember that the total social payoff $T(n)$, when n of the N people choose the action P and $(N - n)$ choose the action S, is given by the formula

$$T(n) = np(n) + (N - n)s(n).$$

Suppose that there are initially n people who have chosen P and that one person switches from S to P. Then the number choosing P increases by 1 to $(n + 1)$, and the number choosing S decreases by 1 to $(N - n - 1)$; so the total social payoff becomes

$$T(n + 1) = (n + 1)p(n + 1) + [N - (n + 1)]s(n + 1).$$

[18]David Landes, *The Wealth and Poverty of Nations* (New York: Norton, 1998), chaps. 3 and 4, makes a spirited case for this effect.

The increase in the total social payoff is the difference between $T(n)$ and $T(n+1)$:

$$\begin{aligned} T(n+1) - T(n) &= (n+1)p(n+1) + [N-(n+1)]s(n+1) - np(n) + (N-n)s(n) \\ &= [p(n+1) - s(n)] + n\,[p(n+1) - p(n)] \\ &\quad + [N-(n+1)]\,[s(n+1) - s(n)] \end{aligned} \tag{12.1}$$

after collecting and rearranging terms.

Equation (12.1) has a very useful interpretation. It classifies the various different effects of one person's switch from S to P. Because this one person is a small part of the whole group, the change due to his switch of strategies is small, or *marginal.* The equation shows how the overall marginal change in the whole group's or whole society's payoffs—**marginal social gain** for short—is divided into the marginal change in payoffs for various subgroups of the population.

The first of the three terms in Eq. (12.1)—namely, $[p(n+1) - s(n)]$—is the change in the payoff of the person who switched; we call this the **marginal private gain** because it is "privately held" by one person. We encountered the two components of this expression in Section 2. This difference in the person's individual, or private, payoffs between what he receives when he chooses S—namely, $s(n)$—and what he receives when he chooses P—namely, $p(n+1)$—is what drives his choice, and all such individual choices then determine the Nash equilibrium.

In addition, when one person switches from S to P, there are now $(n+1)$ people choosing P; the increase from n "participants" to $(n+1)$ changes the payoffs of all the other individual members of the society, both those who play P and those who play S. When one person's action affects others, it is called a **spillover effect, external effect,** or **externality;** because we are considering a small change, we should actually call it the *marginal spillover effect.* For example, if one commuting driver switches from route A to route B, this switch decreases the congestion on route A and increases that on route B, reducing the time taken by all the other drivers on route A (improving their payoffs) and increasing the time taken by all the other drivers on route B (decreasing their payoffs). To be sure, the effect that one driver's switch has on the time for any one driver on either route is very small, but, when there are numerous other drivers (that is, when N is large), the full spillover effect can be substantial.

The second and third terms in Eq. (12.1) are just the quantifications of these effects on others. For the n other people choosing P, each sees his payoff change by the amount $[p(n+1) - p(n)]$ when one more person switches to P; this spillover effect is seen in the second group of terms in Eq. (12.1). There are also $N-(n+1)$ (or $N-n-1$) others still choosing S after the one person switches, and each of these players sees his payoff change by $[s(n+1) - s(n)]$; this spillover effect is shown in the third group of terms in the equation.

Thus we can rewrite Eq. (12.1) for a general switch of one person from either S to P or P to S as:

Marginal social gain = marginal private gain + marginal spillover effect.

For an example in which one person switches from S to P, we have

$$\begin{aligned}\text{Marginal social gain} &= T(n+1) - T(n),\\ \text{Marginal private gain} &= p(n+1) - s(n), \text{ and}\\ \text{Marginal spillover effect} &= n[p(n+1) - p(n)] + [N - (n+1)]\,[s(n+1) - s(n)].\end{aligned}$$

As a specific example of these various effects, suppose you are one of 8,000 commuters who drive every day from a suburb to the city and back; each of you takes either the expressway (action P) or a network of local roads (action S). The route by the local roads takes a constant 45 minutes, no matter how many cars are going that way. The expressway takes only 15 minutes when uncongested. But every driver who chooses the expressway increases the time for every other driver on the expressway by 0.005 minute (about one-quarter of a second).

Measure the payoffs in minutes of time saved—by how much the commute time is less than 1 hour, for instance. Then the payoff to drivers on the local roads, $s(n)$, is a constant $60 - 45 = 15$ regardless of the value of n. But the payoff to drivers on the expressway, $p(n)$, does depend on n; in particular, $p(n) = 60 - 15 = 45$ for $n = 0$, but $p(n)$ decreases by 5/1,000 (or 1/200) for every commuter on the expressway. Thus, $p(n) = 45 - 0.005n$.

Suppose that initially there are 4,000 cars on the expressway. With this many cars on that road, it takes each of them $15 + 4{,}000 \times 0.005 = 15 + 20 = 35$ minutes to commute to work; each gets a payoff of $p(n) = 25$ [which is $60 - 35$, or $p(4{,}000)$]. If you switch from a local road to the expressway, the time for each of the now 4,001 drivers there (including you) will be 35 and 1/200, or 35.005 minutes; each expressway driver now gets a payoff of $p(n+1) = p(4{,}001) = 24.995$. This payoff is higher than the 15 from driving on the local roads. Thus you have a private incentive to make the switch because, for you, $p(n+1) > s(n)$ $(24.995 > 15)$. Your marginal private gain from switching is $p(n+1) - s(n) = 9.995$ minutes. But the 4,000 other drivers on the expressway now take 0.005 minute more each as a result of your decision to switch; the payoff to each changes by $p(4{,}001) - p(4{,}000) = -0.005$. Similarly, the drivers on the local roads face a payoff change of $s(4{,}001) - s(4{,}000)$, but this is zero. The cumulative effect on all of these other drivers is $4{,}000 \times -0.005 = -20$ (minutes); this is the marginal spillover effect. The overall marginal social gain is $9.995 - 20 = -10.005$. Thus the overall social effect of your switch is bad; the social payoff is reduced by a total of just over 10 minutes and it follows that $T(n+1) - T(n) < 0$ in this case.

A. The Calculus of the General Case

Before examining some spillover situations in more detail to see what can be done to achieve socially better outcomes, we restate the general concepts of the analysis in the language of calculus. If you do not know this language, you can omit this section without loss of continuity; but, if you do know it, you will find the alternative statement much simpler to grasp and use than the algebra employed earlier.

If the total number N of people in the group is very large—say, in the hundreds or thousands—then one person can be regarded as a very small, or "infinitesimal," part of this whole. This allows us to treat the number n as a continuous variable. If $T(n)$ is the total social payoff, we calculate the effect of changing n by considering an increase of an infinitesimal marginal quantity dn, instead of a full unit increase from n to $(n + 1)$. To the first order, the change in payoff is $T'(n)dn$, where $T'(n)$ is the derivative of $T(n)$ with respect to n. Using the expression for the total social payoff,

$$T(n) = np(n) + (N - n)s(n),$$

and differentiating, we have

$$\begin{aligned} T'(n) &= p(n) + np'(n) - s(n) + (N - n)s'(n) \\ &= [p(n) - s(n)] + np'(n) + (N - n)\, s'(n). \end{aligned} \tag{12.2}$$

This is the calculus equivalent of Eq. (12.1). $T'(n)$ represents the marginal social gain. The marginal private gain is $p(n) - s(n)$, which is just the change in the payoff of the person making the switch from S to P. In Eq. (12.1), we had $p(n + 1) - s(n)$ for this change in payoff; now we have $p(n) - s(n)$. This is because the infinitesimal addition of dn to the group of the n people choosing P does not change the payoff to any one of them by a significant amount. However, the total change in their payoff, $np'(n)$, is sizable and is recognized in the calculation of the spillover effect—it is the second term in Eq. (12.2)—as is the change in the payoff of the $(N - n)$ people choosing S—namely, $(N - n)s'(n)$—the third term in Eq. (12.2). These last two terms constitute the marginal-spillover-effect part of Eq. (12.2).

In the commuting problem, we have $p(n) = 45 - 0.005n$, and $s(n) = 15$. Then, with the use of calculus, the private marginal gain for each driver who switches to the expressway when there are already n drivers using it is $p(n) - s(n) = 30 - 0.005n$. Because $p'(n) = -0.005$ and $s'(n) = 0$, the spillover effect is $n \times (-0.005) + (N - n) \times 0 = -0.005n$, which equals -20 when $n = 4{,}000$. The answer is the same as before, but calculus simplifies the derivation and helps us find the optimum directly.

B. Negative Spillovers

In the commuting example, the marginal spillover effect is negative (-20 minutes), in which case we say there is a negative externality. A *negative* externality exists when the action of one person *lowers* others' payoffs; it imposes some extra costs on the rest of society. But this person is motivated only by his own payoffs. (Remember that any guilt that he may suffer from harming others should already be reflected in his payoffs). Therefore he does not take the spillover into account when making his choice. He will change his action from S to P as long as this change has a positive marginal *private* gain. Society would be better off if the person's decision were governed by the marginal *social* gain. In our example, the marginal social gain is negative, but the marginal private gain is positive, and so the individual driver makes the switch even though society as a whole would be better off if he did not do so. More generally, in situations with negative externalities, the marginal social gain will be smaller than the marginal private gain. People will make decisions based on a cost-benefit calculation that is the wrong one from society's perspective; individual persons will choose the actions with negative spillover effects more often than society would like them to do.

We can use equation (12.1) to calculate the precise conditions under which a switch will be beneficial for a particular person versus for society as a whole. Recall that, if there are already n people using the expressway and another driver is contemplating switching from the local roads to the expressway, he stands to gain from this switch if $p(n + 1) > s(n)$, whereas the total social payoff increases if $T(n + 1) - T(n) > 0$. The private gain is positive if

$$\begin{aligned} 45 - (n + 1) \times 0.005 &> 15 \\ 44.995 - 0.005n &> 15 \\ n &< 200\,(44.995 - 15) = 5{,}999, \end{aligned}$$

whereas the condition for the social gain to be positive is

$$\begin{aligned} 45 - (n + 1) \times 0.005 - 15 - 0.005n &> 0 \\ 29.995 - 0.01n &> 0 \\ n &< 2{,}999.5. \end{aligned}$$

Thus, if given the free choice, commuters will crowd onto the expressway until there are almost 6,000 of them, but all crowding beyond 3,000 reduces the total social payoff. Society as a whole would be best off if the number of commuters on the expressway were kept down to 3,000.

We show this result graphically in Figure 12.9. On the horizontal axis from left to right, we measure the number of commuters using the expressway. On the vertical axis, we measure the payoffs to each commuter when there are n others using the expressway. The payoff to each driver on the local road is constant and equal to 15 for all n; this is shown by the horizontal line $s(n)$. The payoff to each driver who switches to the expressway is shown by the line $p(n + 1)$; it falls by 0.005 for each unit increase in n. The two lines meet at $n = 5{,}999$; that is, at the value of n for which $p(n + 1) = s(n)$ or for which the marginal private gain is just

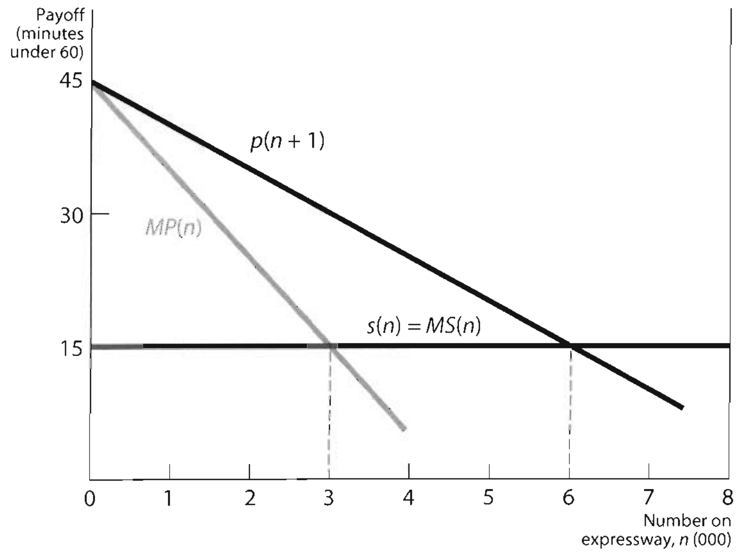


FIGURE 12.9 Equilibrium and Optimum in Route-Choice Game

zero. Everywhere to the left of this value of n, any one driver on the local roads calculates that he gets a positive gain by switching to the expressway. As some drivers make this switch, the numbers on the expressway increase—the value of n in society rises. Conversely, to the right of the intersection point (that is, for $n >$ 5,999), $s(n) > p(n + 1)$; so each of the $(n + 1)$ drivers on the expressway stands to gain by switching to the local road. As some do so, the numbers on the expressway decrease and n falls. From the left of the intersection, this process converges to $n = 5{,}999$ and, from the right, it converges to 6,000.

If we had used the calculus approach, we would have regarded 1 as a very small increment in relation to n, and graphed $p(n)$ instead of $p(n + 1)$. Then the intersection point would have been at $n = 6{,}000$ instead of at 5,999. As you can see, it makes very little difference in practice. What this means is that we can call $n =$ 6,000 the Nash equilibrium of the route-choice game when choices are governed by purely individual considerations. Given a free choice, 6,000 of the 8,000 total commuters will choose the expressway and only 2,000 will drive on the local roads.

But we can also interpret the outcome in this game from the perspective of the whole society of commuters. Society benefits from an increase in the number of commuters, n, on the expressway when $T(n + 1) - T(n) > 0$ and loses from an increase in n when $T(n + 1) - T(n) < 0$. To figure out how to show this on the graph, we express the idea somewhat differently; we take Eq. (12.1) and rearrange it into two pieces, one depending only on p and the other depending only on s:

$$\begin{aligned} T(n + 1) - T(n) &= (n + 1)p(n + 1) + [N - (n + 1)]s(n + 1) - np(n) - [N - n]s(n) \\ &= \{p(n + 1) + n[p(n + 1) - p(n)]\} \\ &\quad - \{s(n) + [N - (n + 1)][s(n + 1) - s(n)]\}. \end{aligned}$$

The expression in the first set of braces is the effect on the payoffs of the set of commuters who choose P; this expression includes the $p(n + 1)$ of the switcher and the spillover effect, $n[p(n + 1) - p(n)]$, on all the other n commuters who choose P. We call this the marginal social payoff for the P-choosing subgroup, when their number increases from n to $n + 1$, or $MP(n + 1)$ for short. Similarly, the expression in the second set of braces is the marginal social payoff for the S-choosing subgroup, or $MS(n)$ for short. Then, the full expression for $T(n + 1) - T(n)$ tells us that the total social payoff increases when one person switches from S to P (or decreases if the switch is from P to S) if $MP(n + 1) > MS(n)$, and the total social payoff decreases when one person switches from S to P (or increases when the switch is from P to S) if $MP(n + 1) < MS(n)$.

Using our expressions for $p(n + 1)$ and $s(n)$ in the commuting example, we have

$$MP(n + 1) = 45 - (n + 1) \times 0.005 + n \times (-0.005) = 44.995 - 0.01n$$

while $MS(n) = 15$ for all values of n. Figure 12.9 includes graphs of the relation $MP(n + 1)$ and $MS(n)$. Note that the $MS(n)$ coincides with $s(n)$ everywhere because the local roads are always uncongested. But the $MP(n + 1)$ curve lies below the $p(n + 1)$ curve; because of the negative spillover, the social gain from switching one person to the expressway is less than the private gain to the switcher.

The $MP(n + 1)$ and $MS(n)$ curves meet at $n = 2{,}999$, or approximately 3,000. To the left of this intersection, $MP(n + 1) > MS(n)$, and society stands to gain by allowing one more person on the expressway; to the right, the opposite is true and society stands to gain by shifting one person from the expressway to the local roads. Thus the socially optimal allocation of drivers is 3,000 on the expressway and 3,000 on the local roads.

If you wish to use calculus, you can write the total payoff for the expressway drivers as $np(n) = n[45 - 0.005n] = 45n - 0.005n^2$, and then the $MP(n + 1)$ is the derivative of this with respect to n—namely, $45 - 0.005 \times 2n = 45 - 0.01n$. The rest of the analysis can then proceed as before.

How might this society achieve the optimum allocation of its drivers? Different cultures and political groups use different systems, and each has its own merits and drawbacks. The society could simply restrict access to the expressway to 3,000 drivers. But how would it choose those 3,000? It could adopt a first-come, first-served rule, but then drivers would race one another to get there early and waste a lot of time. A bureaucratic society may set up criteria based on complex calculations of needs and merits as defined by civil servants; then everyone will undertake some costly activities to meet these criteria. In a politicized society, the important "swing voters" or organized pressure groups or contributors may be favored. In a corrupt society, those who bribe the officials or the politicians may get the preference. A more egalitarian society could allocate the rights to drive on the expressway by lottery or could rotate them from one month to the next. A scheme that lets you drive only on certain days, depending on the last digit of your car's license plate, is an example, but this scheme is not as egalitarian as it seems, because the rich can have two cars and choose license plate numbers that will allow them to drive every day.

Many economists prefer a more open system of charges. Suppose each driver on the expressway is made to pay a tax t, measured in units of time. Then the private benefit from using the expressway becomes $p(n) - t$, and the number n in the Nash equilibrium will be determined by $p(n) - t = s(n)$. (Here, we are ignoring the tiny difference between $p(n)$ and $p(n + 1)$, which is possible when N is very large.) We know that the socially optimal value of n is 3,000. Using the expressions $p(n) = 45 - 0.005n$ and $s(n) = 15$, and plugging in 3,000 for n, we find that $p(n) - t = s(n)$—that is, drivers are indifferent between the expressway and the local roads—when $45 - 15 - t = 15$, or $t = 15$. If we value time at the minimum wage of about \$5 an hour, 15 minutes comes to \$1.25. This is the tax or toll that, when charged, will keep the numbers on the expressway down to what is socially optimal.

Note that, when there are 3,000 drivers on the expressway, the addition of one more increases the time spent by each of them by 0.005 minute, for a total of 15 minutes. This is exactly the tax that each driver is being asked to pay. In other words, each driver is made to pay the cost of the negative spillover that he creates on the rest of society. This "brings home" to each driver the extra cost of his action and therefore induces him to take the socially optimal action; economists say the individual person is being made to **internalize the externality.** This idea, that people whose actions hurt others are made to pay for the harm that they cause, adds to the appeal of this approach. But the proceeds from the tax are not used to compensate the others directly. If they were, then each expressway user would count on receiving from others just what he pays, and the whole purpose would be defeated. Instead, the proceeds of the tax go into general government revenues, where they may or may not be used in a socially beneficial manner.

Those economists who prefer to rely on markets argue that, if the expressway has a private owner, his profit motive will induce him to charge for its use just enough to reduce the number of users to the socially optimal level. An owner knows that, if he charges a tax t for each user, the number of users n will be determined by $p(n) - t = s(n)$. His revenue will be $tn = n[p(n) - s(n)]$, and he will act in such a way as to maximize this revenue. In our example, the revenue is $n[45 - 0.005n - 15] = n[30 - 0.005n] = 30n - 0.005n^2$; it is easy to see this revenue is maximized when $n = 3{,}000$. But, in this case, the revenue goes into the owner's pocket; most people regard it as a bad solution.

C. Positive Spillovers

Many matters pertaining to positive spillovers or positive externalities can be understood simply as mirror images of those for negative spillovers. A person's private benefits from undertaking activities with positive spillovers are less than society's marginal benefits from such activities. Therefore such actions will be underutilized and their benefits underprovided in the Nash equilibrium. A better outcome can be achieved by augmenting people's incentives; providing those persons whose actions provide positive spillovers with a reward just equal to the spillover benefit will achieve the social optimum.

Indeed, the distinction between positive and negative spillovers is to some extent a matter of semantics. Whether a spillover is positive or negative depends on which choice you call P and which you call S. In the commuting example, suppose we called the local roads P and the expressway S. Then one commuter's switch from S to P will reduce the time taken by all the others who choose S, so this action will convey a positive spillover to them. As another example, consider vaccination against some infectious disease. Each person getting vaccinated reduces his own risk of catching the disease (marginal private gain) and reduces the risk of others getting the disease through him (spillover). If being unvaccinated is called the S action, then getting vaccinated has a positive spillover effect. If, instead, remaining unvaccinated is called the P action, then the act of remaining unvaccinated has a negative spillover effect. This has implications for the design of policy to bring individual action into conformity with the social optimum. Society can either reward those who get vaccinated or penalize those who fail to do so.

But actions with positive spillovers can have one very important new feature that distinguishes them from actions with negative spillovers—namely, **positive feedback.** Suppose the spillover effect of your choosing P is to increase the payoff to the others who are also choosing P. Then your choice increases the attraction of that action (P), and may induce some others to take it also, setting in train a process that culminates in everyone taking that action. Conversely, if very few people are choosing P, then it may be so unattractive that they, too, give it up, leading to a situation in which everyone chooses S. In other words, positive feedback can give rise to multiple Nash equilibria, which we now illustrate by using a very real example.

When you buy a computer, you have to choose between one with a Windows operating system and one with an operating system based on Unix, such as Linux. This game has the necessary payoff structure. As the number of Unix users rises, the better it will be to purchase such a computer—the system will have fewer bugs because more users will have detected those that exist, there will be more application software available, and more experts will be available to help with any problems that arise. Similarly, a Windows-based computer will be more attractive the more Windows users there are. In addition, many computing aficionados would argue that the Unix system is superior. Without necessarily taking a position on that matter, we show what will happen if that is the case. Will individual choice lead to the socially best outcome?

A diagram similar to Figures 12.6 through 12.8 can be used to show the payoffs to an individual computer purchaser of the two strategies, Unix and Windows. As shown in Figure 12.10, the Unix payoff rises as the number of Unix users rises, and the Windows payoff rises as the number of Unix owners falls (the number of Windows users rises). As already explained, the diagram is drawn so that the payoff to Unix users when everyone in the population is a Unix user (at the point labeled U) is higher than the payoff to Windows users when everyone in the population is a Windows user (at W).

If the current population has only a small number of Unix users, then the situation is represented by a point to the left of the intersection of the two payoff lines at I, and each individual user finds it better to choose Windows. When there is a larger number of Unix users in the population, placing the society to the right of I, it is better for each person to choose Unix. Thus a mixed population of Unix and Windows users is sustainable as an equilibrium only when the current population has exactly I Unix users; only then will no member of the population have any incentive to switch platforms. And even that situation is unstable. Suppose just one person by accident makes a different decision. If he switches to Windows, his choice will push the population to the left of I, in which case there will then be an incentive for others to switch to Windows, too. If he switches to Unix, the population point moves to the right of I, creating an

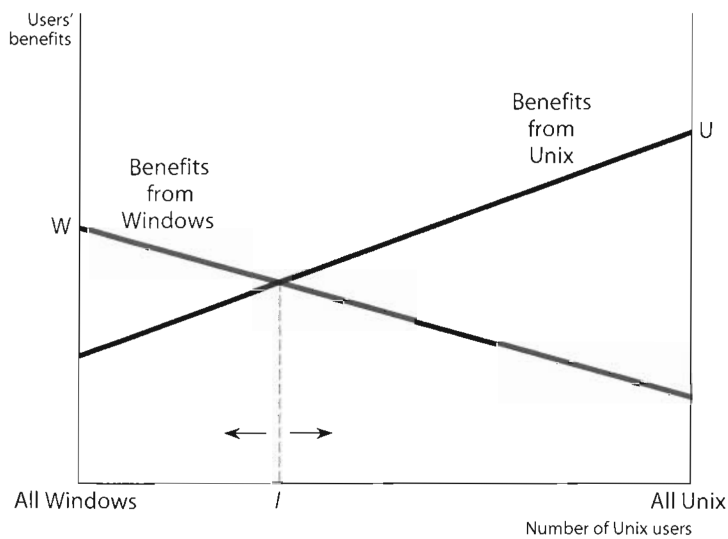


FIGURE 12.10 Payoffs in Operating-System-Choice Game

incentive for more people to switch to Unix. The cumulative effect of these switches will eventually push the society to an all-Unix or an all-Windows outcome; these are the two stable equilibria of the game.[19]

But which of the two stable equilibria will be achieved in this game? The answer depends on where the game starts. If you look at the configuration of today's computer users, you will see a heavily Windows-oriented population. Thus it seems that, because there are so few Unix users (or so many PC users), the world is moving toward the all-Windows equilibrium. Schools, businesses, and private users have become **locked in** to this particular equilibrium as a result of an accident of history. If it is indeed true that the Unix provides more benefits to society when used by everyone, then the all-Unix equilibrium should be preferred over the all-Windows one that we are approaching. Unfortunately, although society as a whole might be better off with the change, no individual computer user has an incentive to make a change from the current situation. Only coordinated action can swing the pendulum toward Unix. A critical mass of individual users, more than I in Figure 12.10, must use Unix before it becomes individually rational for others to choose the same operating system.

There are many examples of similar choices of convention being made by different groups of people. The most famous cases are those in which it has been argued, in retrospect, that a wrong choice was made. Advocates claim that steam power could have been developed for greater efficiency than gasoline; it certainly would have been cleaner. Proponents of the Dvorak typewriter/computer key configuration claim that it would be better than the QWERTY keyboard if used everywhere. Many engineers agree that Betamax had more going for it than VHS in the video recorder market. In such cases, the whims of the public or the genius of advertisers help determine the ultimate equilibrium and may lead to a "bad" or "wrong" outcome from society's perspective. Other situations do not suffer from such difficulties. Few people concern themselves with fighting for a reconfiguration of traffic-light colors, for example.[20]

The ideas of positive feedback and lock-in find an important application in macroeconomics. Production is more profitable the higher the level of demand in the economy, which happens when national income is higher. In turn, income is higher when firms are producing more and therefore hiring more workers. This positive feedback creates the possibility of multiple equilibria, of which

[19]The term "positive feedback" may create the impression that this is a good thing, but in technical language the term merely characterizes the process and includes no general value judgment about the outcome. In this example, the same positive feedback mechanism could lead to either an all-Unix outcome or an all-Windows outcome; one could be worse than the other.

[20]Not everyone agrees that the Dvorak keyboard and the Betamax video recorder were clearly superior alternatives. See two articles by S. J. Liebowitz and Stephen E. Margolis, "Network Externality: An Uncommon Tragedy," *Journal of Economic Perspectives*, vol. 8 (Spring 1994), pp. 146–149, and "The Fable of the Keys," *Journal of Law and Economics*, vol. 33 (April 1990), pp. 1–25.

the high-production, high-income one is better for society, but individual decisions may lock the economy into the low-production, low-income equilibrium. In Exercise 6 at the end of the chapter, we ask you to show this in a figure. The better equilibrium could be turned into a focal point by public declaration—"the only thing we have to fear is fear itself"—but the government can also inject demand into the economy to the extent necessary to move it to the better equilibrium. In other words, the possibility of unemployment due to a deficiency of aggregate demand, as discussed in the supply-and-demand language of economic theory by the British economist John Maynard Keynes in his well-known 1936 book titled *Employment, Interest, and Money,* can be seen from a game-theoretic perspective as the result of a failure to solve a collective-action problem.[21]

6 "HELP!": A GAME OF CHICKEN WITH MIXED STRATEGIES

In the chicken variant of collective-action problems discussed in earlier sections, we looked only at the pure-strategy equilibria. But we know from Section 1.B in Chapter 8 that such games have mixed-strategy equilibria, too. In collective-action problems, where each participant is thinking, "It is better if I wait for enough others to participate so that I can shirk; but then again maybe they won't, in which case I should participate," mixed strategies nicely capture the spirit of such vacillation. Our last story is a dramatic, even chilling application of such a mixed-strategy equilibrium.

In 1964 in New York City (in Kew Gardens, Queens), a woman named Kitty Genovese was killed in a brutal attack that lasted more than half an hour. She screamed through it all and, although her screams were heard by many people and more than 30 actually watched the attack taking place, not one went to help her or even called the police.

The story created a sensation and found several ready theories to explain it. The press and most of the public saw this episode as a confirmation of their belief that New Yorkers—or big-city dwellers or Americans or people more generally—were just apathetic or didn't care about their fellow human beings.

However, even a little introspection or observation will convince you that people do care about the well-being of other humans, even strangers. Social sci-

[21]John Maynard Keynes, *Employment, Interest, and Money* (London: Macmillan, 1936). See also John Bryant, "A Simple Rational-Expectations Keynes-type Model," *Quarterly Journal of Economics,* vol. 98 (1983), pp. 525–528, and Russell Cooper and Andrew John, "Coordination Failures in a Keynesian Model," *Quarterly Journal of Economics,* vol. 103 (1988), pp. 441–463, for formal game-theoretic models of unemployment equilibria.

entists offered a different explanation for what happened, which they labeled **pluralistic ignorance.** The idea behind this explanation is that no one can be sure about what is happening, whether help is really needed and how much. People look to one another for clues or guidance about these matters and try to interpret other people's behavior in this light. If they see that no one else is doing anything to help, they interpret it as meaning that help is probably not needed, and so they don't do anything either.

This explanation has some intuitive appeal but is unsatisfactory in the Kitty Genovese context. There is a very strong presumption that a screaming woman needs help. What did the onlookers think—that a movie was being shot in their obscure neighborhood? If so, where were the lights, the cameras, the director, other crew?

A better explanation would recognize that, although each onlooker may experience strong personal loss from Kitty's suffering and get genuine personal pleasure if she were saved, each must balance that against the cost of getting involved. You may have to identify yourself if you call the police; you may then have to appear as a witness, and so on. Thus, we see that each person may prefer to wait for someone else to call and hope to get for himself the free rider's benefit of the pleasure of a successful rescue.

Social psychologists have a slightly different version of this idea of free riding, which they label **diffusion of responsibility.** Here, the idea is that everyone might agree that help is needed, but they are not in direct communication with one another and so cannot coordinate on who should help. Each person may believe that help is someone else's responsibility. And the larger the group, the more likely it is that each person will think that someone else would probably help, and therefore he can save himself the trouble and the cost of getting involved.

Social psychologists conducted some experiments to test this hypothesis. They staged situations in which someone needed help of different kinds in different places and with different-sized crowds. Among other things, they found that, the larger the size of the crowd, the less likely was help to come forth.

The concept of diffusion of responsibility seems to explain this finding but not quite. It says that, the larger the crowd, the less likely is any one person to help. But there are more people, and only one person is needed to act and call the police to secure help. To make it less likely that even one person helps, the chance of any one person helping has to go down sufficiently fast to offset the increase in the total number of potential helpers. To find whether it does so requires game-theoretic analysis, which we now supply.[22]

[22]For a fuller account of the Kitty Genovese story and for the analysis of such situations from the perspective of social psychology, see John Sabini, *Social Psychology*, 2nd ed. (New York: Norton, 1995), pp. 39–44. Our game-theoretic model is based on Thomas Palfrey and Howard Rosenthal, "Participation and the Provision of Discrete Public Goods," *Journal of Public Economics*, vol. 24 (1984), pp. 171–193.

We consider only the aspect of diffusion of responsibility in which action is not consciously coordinated, and we leave aside all other complications of information and inference. Thus we assume that everyone believes the action is needed and is worth the cost.

Suppose there are N people in the group. The action brings each of them a benefit B. Only one person is needed to take the action; more are redundant. Anyone who acts bears the cost C. We assume that $B > C$; so it is worth the while of any one person to act even if no one else is acting. Thus the action is justified in a very strong sense.

The problem is that anyone who takes the action gets the value B and pays the cost C for a net payoff of $(B - C)$, while he would get the higher payoff B if someone else took the action. Thus each has the temptation to let someone else go ahead and to become a free rider on another's effort. When all N people are thinking thus, what will be the equilibrium or outcome?

If $N = 1$, the single person has a simple decision problem rather than a game. He gets $B - C > 0$ if he takes the action and 0 if he does not. Therefore he goes ahead and helps.

If $N > 1$, we have a game of strategic interaction with several equilibria. Let us begin by ruling out some possibilities. With $N > 1$, there cannot be a pure-strategy Nash equilibrium in which all people act, because then any one of them would do better by switching to free-ride. Likewise, there cannot be a pure-strategy Nash equilibrium in which no one acts, because, *given that no one else is acting* (remember that under the Nash assumption each player takes the others' strategies as given), it pays any one person to act.

There *are* Nash equilibria where exactly one person acts; in fact, there are N such equilibria, one corresponding to each member. But, when everyone is taking the decision individually in isolation, there is no way to coordinate and designate who is to act. Even if members of the group were to attempt such coordination, they might try to negotiate over the responsibility and not reach a conclusion, at least not in time to be of help. Therefore it is of interest to examine symmetric equilibria in which all members have identical strategies.

We already saw that there cannot be an equilibrium in which all N people follow the same pure strategy. Therefore we should see whether there can be an equilibrium in which they all follow the same mixed strategy. Actually, mixed strategies are quite appealing in this context. The people are isolated, and each is trying to guess what the others will do. Each is thinking, Perhaps I should call the police . . . but maybe someone else will . . . but what if they don't . . . ? Each breaks off this process at some point and does the last thing that he thought of in this chain, but we have no good way of predicting what that last thing is. A mixed strategy carries the flavor of this idea of a chain of guesswork being broken at a random point.

So suppose P is the probability that any one person will not act. If one particular person is willing to mix strategies, he must be indifferent between the two pure strategies of acting and not acting. Acting gets him $(B - C)$ for sure. Not acting will get him 0 if none of the other $(N - 1)$ people act and B if at least one of them does act. Because the probability that any one fails to act is P and because they are deciding independently, the probability that none of the $(N - 1)$ others acts is P^{N-1}, and the probability that at least one does act is $(1 - P^{N-1})$. Therefore the expected payoff of the one person when he does not act is

$$0 \times P^{N-1} + B(1 - P^{N-1}) = B(1 - P^{N-1}).$$

And that one person is indifferent between acting and not acting when

$$B - C = B(1 - P^{N-1}) \quad \text{or when} \quad P^{N-1} = C/B \quad \text{or} \quad P = (C/B)^{1/(N-1)}.$$

Note how this indifference condition of *one* selected player serves to determine the probability with which the *other* players mix their strategies.

Having obtained the equilibrium mixture probability, we can now see how it changes as the group size N changes. Remember that $C/B < 1$. As N increases from 2 to infinity, the power $1/(N - 1)$ decreases from 1 to 0. Then C/B raised to this power—namely, P— increases from C/B to 1. Remember that P is the probability that any one person does not take the action. Therefore the probability of action by any one person—namely, $(1 - P)$—falls from $1 - C/B = (B - C)/B$ to 0.[23]

In other words, the more people there are, the less likely is any one of them to act. This is intuitively true, and in good conformity with the idea of diffusion of responsibility. But it does not yet give us the conclusion that help is less likely to be forthcoming in a larger group. As we said before, help requires action by only one person. As there are more and more people each of whom is less and less likely to act, we cannot conclude immediately that the probability of *at least one* of them acting gets smaller. More calculation is needed to see whether this is the case.

Because the N persons are randomizing independently in the Nash equilibrium, the probability Q that *not even one* of them helps is

$$Q = P^N = (C/B)^{N/(N-1)}.$$

As N increases from 2 to infinity, $N/(N - 1)$ decreases from 2 to 1, and then Q increases from $(C/B)^2$ to C/B. Correspondingly, the probability that *at least one* person helps—namely $(1 - Q)$—decreases from $1 - (C/B)^2$ to $1 - C/B$.[24]

[23]Consider the case in which $B = 10$ and $C = 8$. Then P equals 0.8 when $N = 2$, rises to 0.998 when $N = 100$, and approaches 1 as N continues to rise. The probability of action by any one person is $1 - P$, which falls from 0.2 to 0 as N rises from 2 toward infinity.

[24]With the same sample values for B (10) and C (8), this result implies that increasing N from 2 to infinity leads to an increase in the probability that not even one person helps, from 0.64 to 0.8. And the probability that at least one person helps falls from 0.36 to 0.2.

So our exact calculation does bear out the hypothesis: the larger the group, the *less* likely is help to be given at all. The probability of provision does not, however, reduce to zero even in very large groups; instead it levels off at a positive value—namely, $(B - C)/B$—which depends on the benefit and cost of action to each individual.

We see how game-theoretic analysis sharpens the ideas from social psychology with which we started. The diffusion of responsibility theory takes us part of the way—namely, to the conclusion that any one person is less likely to act when he is part of a larger group. But the desired conclusion—that larger groups are less likely to provide help at all—needs further and more precise probability calculation based on the analysis of individual mixing and the resulting interactive (game) equilibrium.

And now we ask, did Kitty Genovese die in vain? Do the theories of pluralistic ignorance, diffusion of responsibility, and free-riding games still play out in the decreased likelihood of individual action within increasingly large cities? Perhaps not. John Tierney of the *New York Times* has publicly extolled the virtues of "urban cranks."[25] They are people who encourage the civility of the group through prompt punishment of those who exhibit unacceptable behavior—including litterers, noise polluters, and the generally obnoxious boors of society. Such people act essentially as enforcers of a cooperative norm for society. And, as Tierney surveys the actions of known "cranks," he reminds the rest of us that "[n]ew cranks must be mobilized! At this very instant, people are wasting time reading while norms are being flouted out on the street. . . . You don't live alone in this world! Have you enforced a norm today?" In other words, we need social norms and some people who get some innate payoff from enforcing these norms.

SUMMARY

Many-person games generally concern problems of *collective action.* The general structure of collective-action games may be manifested as a prisoners' dilemma, chicken, or an assurance game. The critical difficulty with such games, in any form, is that the Nash equilibrium arising from individually rational choices may not be the socially optimal outcome—the outcome that maximizes the sum of the payoffs of all the players.

Problems of collective action have been recognized for many centuries and discussed by scholars from diverse fields. Several early works professed no

[25]John Tierney, "The Boor War: Urban Cranks, Unite—Against All Uncivil Behavior. Eggs Are a Last Resort," *New York Times Magazine,* January 5, 1997.

hope for the situation, while others offered up dramatic solutions; the work of Olson in the 1960s put the problems in perspective by noting that socially optimal outcomes would not arise unless each person had a private incentive to perform his assigned task. The most recent treatments of the subject acknowledge that collective-action problems arise in diverse areas and that there is no single optimal solution.

Social scientific analysis suggests that social *custom*, or *convention*, can lead to cooperative behavior. Other possibilities for solutions come from the use of *sanctions* for the uncooperative or from the creation of *norms* of acceptable behavior. Much of the literature agrees that small groups are more successful at solving collective-action problems than large ones.

In collective-action games, when a person's action has some effect on the payoffs of all the other players, we say that there are *spillovers*, or *externalities*. They can be positive or negative and lead to individually driven outcomes that are not socially optimal; when actions create negative spillovers, they are overused from the perspective of society and, when actions create positive spillovers, they are underused. Mechanisms that induce individual members to incur the cost (or reap the benefit) of their actions can be used to achieve the social optimum. The additional possibility of *positive feedback* exists when there are positive spillovers; in such a case, there may be multiple Nash equilibria of the game.

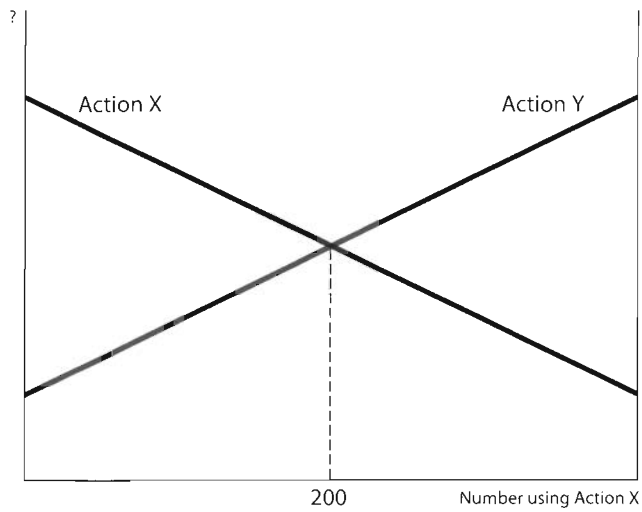
In large-group games, *diffusion of responsibility* can lead to behavior in which individual persons wait for others to take action and *free-ride* off the benefits of that action. If help is needed, it is less likely to be given at all as the size of the group available to provide it grows.

KEY TERMS

coercion (399)
collective action (382)
convention (395)
custom (395)
diffusion of responsibility (415)
external effect (404)
externality (404)
free rider (384)
internalize the externality (410)
locked in (413)
marginal private gain (404)
marginal social gain (404)
nonexcludable benefits (383)
nonrival benefits (383)
norm (397)
oppression (399)
pluralistic ignorance (415)
positive feedback (411)
pure public good (383)
sanction (397)
spillover effect (404)

EXERCISES

1. Suppose that 400 people are choosing between Action X and Action Y. The relative payoffs of the two actions depend on how many of the 400 people choose Action X and how many choose Action Y. The payoffs are as shown in the following diagram, but the vertical axis is not labeled, and so you do not know whether the lines show the benefits *or* the costs of the two actions.



   (a) You are told that the outcome in which 200 people choose Action X is an *un*stable equilibrium. Then, if 100 people are currently choosing Action X, would you expect the number of people choosing X to increase or decrease over time? Why?
   (b) For the graph to be consistent with the behavior that you described in part a, should the lines be labeled as indicating the *costs* or *benefits* of Action X and Action Y? Explain your answer.

2. Figure 12.5 illustrates the payoffs in a general two-person collective action game. There we showed various inequalities on the algebraic payoffs [$p(1)$, etc.] that made the game a prisoners' dilemma. Now you are asked to find similar inequalities corresponding to other kinds of games:
   (a) Under what condition(s) on the payoffs is the two-person game a chicken game? What further condition(s) make the game Version I of Chicken (as in Figure 12.3)?

(b) Under what condition(s) on the payoffs is the two-person game an assurance game?

3. A class with 30 students enrolled is given a homework assignment with five questions. The first four are the usual kinds of problems. But the fifth is an interactive game for the class. The question reads: "You can choose whether to answer this question. If you choose to do so, you merely write 'I hereby answer Question 5.' If you choose not to answer Question 5, your score for the assignment will be based on your performance on the first four problems. If you choose to answer Question 5, then your scoring will be as follows. If fewer than half of the students in the class answer Question 5, you get 10 points for Question 5; 10 points will be added to your score on the other four questions to get your total score for the assignment. If half or more than half of the students in the class answer Question 5, you get -10 points; that is, 10 points will be subtracted from your score on the other questions."
   (a) Draw a diagram illustrating the payoffs from the two possible strategies, "Answer Question 5" and "Don't Answer Question 5," in relation to the number of other students who answer it. Find the Nash equilibrium of the game.
   (b) What would you expect to see happen in this game if it were actually played in a college classroom and why? Consider two cases: (i) the students make their choices individually with no communication and (ii) the students make their choices individually but can discuss these choices ahead of time in a discussion forum available on the class website.

4. A group has 100 members. If n of them participate in a common project, then each participant derives the benefit $p(n) = n$, and each of the $(100 - n)$ shirkers derives the benefit $s(n) = 4 + 3n$.
   (a) What kind of a game is it?
   (b) Write the expression for the total benefit of the group.
   (c) Show that $n = 74$ yields the maximum total benefit for the group.
   (d) What difficulties will arise in trying to get exactly 74 participants and allowing the remaining 26 to shirk?
   (e) How might the group try to overcome these difficulties?

5. Consider a small geographic region with a total population of 1 million people. There are two towns, Alphaville and Betaville, in which each person can choose to live. For each person, the benefit from living in a town increases for a while with the size of the town (because larger towns have more amenities, etc.), but after a point it decreases (because of congestion and so

on). If x is the fraction of the population that lives in the same town as you do, your payoff is given by

$$\begin{array}{rl} x & \text{if} \quad 0 \le x \le 0.4 \\ 0.6 - 0.5x & \text{if} \quad 0.4 < x \le 1. \end{array}$$

(a) Draw a graph of this relation. Regard x as a continuous variable that can take any real value between 0 and 1.

(b) Equilibrium is reached either when both towns are populated and their residents have equal payoffs or when one town, say Betaville, is totally depopulated and the residents of the other town (Alphaville) get a higher payoff than would the very first person who seeks to populate Betaville. Use your graph to find all such equilibria.

(c) Now consider a dynamic process of adjustment whereby people gradually move toward the town whose residents currently enjoy a larger payoff than do the residents of the other town. Which of the equilibria identified in part **b** will be stable with these dynamics and which ones will be unstable?

6. There are two routes for driving from A to B. One is a freeway, and the other consists of local roads. The benefit of using the freeway is constant and equal to 1.8, irrespective of the number of people using it. Local roads get congested when too many people use this alternative, but, if too few people use it, the few isolated drivers run the risk of becoming victims of crimes. Suppose that, when a fraction x of the population is using the local roads, the benefit of this mode to each driver is given by

$$1 + 9x - 10x^2.$$

(a) Draw a graph showing the benefits of the two driving routes as functions of x, regarding x as a continuous variable that can range from 0 to 1.

(b) Identify all possible equilibrium traffic patterns from your graph in part **a**. Which equilibria are stable and which ones are unstable, and why?

(c) What value of x maximizes the total benefit to the whole population?

7. Suppose an amusement park is being built in a city with a population of 100. Voluntary contributions are being solicited to cover the cost. Each citizen is being asked to give \$100. The more people contribute, the larger the park will be and the greater the benefit to each citizen. But it is not possible to keep out the noncontributors; they get their share of this benefit anyway. Suppose that when there are n contributors in the population, where n can

be any whole number between 0 and 100, the benefit to each citizen in monetary unit equivalents is n^2 dollars.

(a) Suppose that initially no one is contributing. You are the mayor of the city. You would like everyone to contribute and can use persuasion on some people. What is the minimum number whom you need to persuade before everyone else will join in voluntarily?

(b) Find the Nash equilibria of the game where each citizen is deciding whether to contribute.

8. Suppose a class of 100 students is comparing the choice between two careers—lawyer or engineer. An engineer gets a take-home pay of \$100,000 per year, irrespective of the numbers who choose this career. Lawyers make work for one another; so, as the total number of lawyers increases, the income of each lawyer increases—up to a point. Ultimately, the competition between them drives down the income of each. Specifically, if there are N lawyers, each will get $100N - N^2$ thousand dollars a year. The annual cost of running a legal practice (office space, secretary, paralegals, access to online reference services, and so forth) is \$800,000. So each lawyer takes home $100\text{N} - N^2 - 800$ thousand dollars a year when there are N of them.

(a) Draw a graph showing the take-home income of each lawyer on the vertical axis and the number of lawyers on the horizontal axis. (Plot a few points—say, for 0, 10, 20, . . . , 90, 100 lawyers—and fit a curve to the points or use a computer graphics program if you have access to one.)

(b) When career choices are made in an uncoordinated way, what are the possible equilibrium outcomes?

(c) Now suppose the whole class decides how many should become lawyers, aiming to maximize the total take-home income of the whole class. What will be the number of lawyers? (Use calculus, regarding N as a continuous variable, if you can. Otherwise you can use graphical methods or a spreadsheet.)

9. Put the idea of Keynesian unemployment described at the end of Section 5.C into a properly specified game, and show the multiple equilibria in a diagram. Show the level of production (national product) on the vertical axis as a function of a measure of the level of demand (national income) on the horizontal axis. Equilibrium is reached when national product equals national income—that is, when the function relating the two cuts the 45° line. For what shapes of the function can there be multiple equilibria, and why might you expect such shapes in reality? Suppose that income increases when current production exceeds current income and that income decreases when current production is less than current income. With this dynamic process, which equilibria are stable and which ones unstable?

10. Write a brief description of a strategic game that you have witnessed or participated in that includes a large number of players, in which individual players' payoffs depend on the number of other players and their actions. Try to illustrate your game with a graph if possible. Discuss the outcome of the actual game in light of the fact that many such games have inefficient outcomes. Do you see evidence of such an outcome in your game?

13

Evolutionary Games

WE HAVE SO FAR STUDIED GAMES with many different features—simultaneous and sequential moves, zero-sum and non-zero-sum payoffs, strategic moves to manipulate rules of games to come, one-shot and repeated play, and even games of collective action in which a large number of people play simultaneously. However, one ground rule has remained unchanged in all of the discussions—namely, that all the players in all these games are rational: each player has an internally consistent value system, can calculate the consequences of her strategic choices, and makes the choice that best favors her interests.

In using this rule, we merely follow the route taken by most of game theory, which was developed mainly by economists. Economics was founded on the dual assumptions of rational behavior and equilibrium. Indeed, these assumptions have proved useful in game theory. We have obtained quite a good understanding of games in which the players participate sufficiently regularly to have learned what their best choices are by experience. The assumptions ensure that a player does not attribute any false naiveté to her rivals and thus does not get exploited by these rivals. The theory also gives some prescriptive guidance to players as to how they *should* play.

However, other social scientists are much more skeptical of the rationality assumption and therefore of a theory built on such a foundation. Economists, too, should not take rationality for granted, as pointed out in Chapter 5. The trouble is finding a feasible alternative. Although we may not wish to impose conscious and perfectly calculating rationality on players, we do not want to abandon the idea that some strategies are better than others. We want good strategies to be rewarded with higher payoffs; we want players to observe or imitate success and to experiment with new strategies; we want good strategies to be used more often and bad strategies less often, as players gain experience playing the game. We find one possible alternative to rationality in the biological theory of evolution and evolutionary dynamics and will study its lessons in this chapter.

1 THE FRAMEWORK

The process of evolution in biology offers a particularly attractive parallel to the theory of games used by social scientists. This theory rests on three fundamentals: heterogeneity, fitness, and selection. The starting point is that a significant part of animal behavior is genetically determined; a complex of one or more genes (**genotype**) governs a particular pattern of behavior, called a behavioral **phenotype.** Natural diversity of the gene pool ensures a heterogeneity of phenotypes in the population. Some behaviors are better suited than others to the prevailing conditions, and the success of a phenotype is given a quantitative measure called its **fitness.** People are used to thinking of this success as meaning the common but misleading phrase "survival of the fittest"; however, the ultimate test of biological fitness is not mere survival, but reproductive success. That is what enables an animal to pass on its genes to the next generation and perpetuate its phenotype. The fitter phenotypes then become relatively more numerous in the next generation than the less fit phenotypes. This process of **selection** is the dynamic that changes the mix of genotypes and phenotypes and perhaps leads eventually to a stable state.

From time to time, chance produces new genetic **mutations.** Many of these mutations produce behaviors (that is, phenotypes) that are ill suited to the environment, and they die out. But occasionally a mutation leads to a new phenotype that is fitter. Then such a mutant gene can successfully **invade** a population—that is, spread to become a significant proportion of the population.

At any time, a population may contain some or all of its biologically conceivable phenotypes. Those that are fitter than others will increase in proportion, some unfit phenotypes may die out, and other phenotypes not currently in the population may try to invade it. Biologists call a configuration of a population and its current phenotypes **evolutionary stable** if the population cannot be invaded successfully by any mutant. This is a static test, but often a more dynamic criterion is applied: a configuration is evolutionary stable if it is the limiting outcome of the dynamics of selection, starting from any arbitrary mixture of phenotypes in the population.[1]

The fitness of a phenotype depends on the relationship of the individual organism to its environment; for example, the fitness of a particular bird depends on the aerodynamic characteristics of its wings. It also depends on the whole complex of the proportions of different phenotypes that exist in the environment—how aerodynamic its wings are relative to the rest of its species. Thus the fitness of a particular animal, with its behavioral traits, such as aggression and sociability, depends on whether other members of its species are predominantly aggressive or passive, crowded or dispersed, and so on. For our purpose, this *interaction* between phenotypes within a species is the most interesting aspect of the story. Sometimes an individual member of a species interacts with members of another species; then the fitness of a particular type of sheep, for example, may depend on the traits that prevail in the local population of wolves. We consider this type of interaction as well, but only after we have covered the within-species case.

The biological process of evolution finds a ready parallel in game theory. The behavior of a phenotype can be thought of as a *strategy* of the animal in its interaction with others—for example, whether to fight or to retreat. The difference is that the choice of strategy is not a purposive calculation as it would be in standard game theory; rather, it is a genetically predetermined fixture of the phenotype. The interactions lead to *payoffs* to the phenotypes. In biology, the payoffs measure the evolutionary or reproductive fitness; when we apply the ideas outside of biology, they can have other connotations of success in the social, political, or economic games in question.

The payoffs or fitness numbers can be shown in a payoff table just like that for a standard game, with all conceivable phenotypes of one animal arrayed along the rows of the matrix and those of the other along the columns of the matrix. If more animals interact simultaneously—which is called **playing the field** in biology—the payoffs can be shown by functions like those for collective-action games in Chapter 12. We will consider pair-by-pair matches for most of this chapter and will look at the other case briefly in Section 9.

Because the population is a mix of phenotypes, different pairs selected from it will bring to their interactions different combinations of strategies. The actual quantitative measure of the fitness of a phenotype is the average payoff that it gets in all its interactions with others in the population. Those animals with higher fitness will have greater evolutionary success. The eventual outcome of

[1]The dynamics of phenotypes is driven by an underlying dynamics of genotypes but, at least at the elementary level, evolutionary biology focuses its analysis at the phenotype level and conceals the genetic aspects of evolution. We will do likewise in our exposition of evolutionary games. Some theories at the genotypes level can be found in the materials cited in footnote 2.

the population dynamics will be an evolutionary stable configuration of the population.

Biologists have used this approach very successfully. Combinations of aggressive and cooperative behavior, locations of nesting sites, and many more phenomena that elude more conventional explanations can be understood as the stable outcomes of an evolutionary process of selection of fitter strategies. Interestingly, biologists developed the idea of evolutionary games by using the preexisting body of game theory, drawing from its language but modifying the assumption of conscious maximizing to suit their needs. Now game theorists are in turn using insights from the research on biological evolutionary games to enrich their own subject.[2]

Indeed, the theory of evolutionary games seems a ready-made framework for a new approach to game theory, relaxing the assumption of rational behavior.[3] According to this view of games, individual players have no freedom to choose their strategy at all. Some are "born" to play one strategy, others another. The idea of inheritance of strategies can be interpreted more broadly in applications of the theory other than in biology. In human interactions, a strategy may be embedded in a player's mind for a variety of reasons—not only genetics but also (and probably more importantly) socialization, cultural background, education, or a rule of thumb based on past experience. The population can consist of a mixture of different people with different backgrounds or experiences that embed different strategies into them. Thus some politicians may be motivated to adhere to certain moral or ethical codes even at the cost of electoral success, while others are mainly concerned with their own reelection; similarly, some firms may pursue profit alone, while others are motivated by social or ecological objectives. We can call each logically conceivable strategy that can be embedded in this way a phenotype for the population of players in the context being studied.

[2]Robert Pool, "Putting Game Theory to the Test," *Science*, vol. 267, (March 17, 1995), pp. 1591–1593, is a good article for general readers and has many examples from biology. John Maynard Smith deals with such games in biology in his *Evolutionary Genetics* (Oxford: Oxford University Press, 1989), chap. 7, and *Evolution and the Theory of Games* (Cambridge: Cambridge University Press, 1982); the former also gives much background on evolution. Recommended for advanced readers are Peter Hammerstein and Reinhard Selten, "Game Theory and Evolutionary Biology," in *Handbook of Game Theory*, vol. 2, ed. R. J. Aumann and S. Hart (Amsterdam: North Holland, 1994), pp. 929–993, and Jorgen Weibull, *Evolutionary Game Theory* (Cambridge: MIT Press, 1995).

[3]Indeed, applications of the evolutionary perspective need not stop with game theory. The following joke offers an "evolutionary theory of gravitation" as an alternative to Newton's or Einstein's physical theories:

Question: Why does an apple fall from the tree to earth?

Answer: Originally, apples that came loose from trees went in all directions. But only those that were genetically predisposed to fall to the earth could reproduce.

From a population with its heterogeneity of embedded strategies, pairs of phenotypes are repeatedly randomly selected to interact (play the game) with others of the same or different "species." In each interaction, the payoff of each player depends on the strategies of both; this dependence is governed by the usual "rules of the game" and illustrated in the game table or tree. The *fitness* of a particular strategy is defined as its aggregate or average payoff in its pairings with all the strategies in the population. Some strategies have a higher level of fitness than others; in the next generation—that is, the next round of play—these higher-fitness strategies will be used by more players and will proliferate. Strategies with lower fitness will be used by fewer players and will decay or die out. Occasionally, someone may experiment or adopt a previously unused strategy from the collection of those that are logically conceivable. This corresponds to the emergence of a mutant. If the new strategy is fitter than the ones currently being used, it will start to be used by larger proportions of the population. The central question is whether this process of selective proliferation, decay, and mutation of certain strategies in the population will have an evolutionary stable outcome and, if so, what it will be. In regard to the examples just cited, will society end up with a situation in which all politicians are concerned with reelection and all firms with profit? In this chapter, we develop the framework and methods for answering such questions.

Although we use the biological analogy, the reason that the fitter strategies proliferate and the less fit ones die out in socioeconomic games differs from the strict genetic mechanism of biology: players who fared well in the last round will transmit the information to their friends and colleagues playing the next round, those who fared poorly in the last round will observe which strategies succeeded better and will try to imitate them, and some purposive thinking and revision of previous rules of thumb will take place between successive rounds. Such "social" and "educational" mechanisms of transmission are far more important in most strategic games than any biological genetics; indeed, this is how the reelection orientation of legislators and the profit-maximization motive of firms are reinforced. Finally, conscious experimentation with new strategies substitutes for the accidental mutation in biological games.

Evolutionary stable configurations of biological games can be of two kinds. First, a single phenotype may prove fitter than any others, and the population may come to consist of it alone. Such an evolutionary stable outcome is called **monomorphism**—that is, a single (mono) form (morph). In that case, the unique prevailing strategy is called an **evolutionary stable strategy (ESS).** The other possibility is that two or more phenotypes may be equally fit (and fitter than some others not played); so they may be able to coexist in certain proportions. Then the population is said to exhibit **polymorphism**—that is, a multiplicity (poly) of forms (morph). Such a state will be stable if no new phenotype or feasible mutant can achieve a higher fitness against such a population than the fitness of the types that are already present in the polymorphic population.

Polymorphism comes close to the game-theoretic notion of a mixed strategy. However, there is an important difference. To get polymorphism, no individual player need follow a mixed strategy. Each can follow a pure strategy, but the population exhibits a mixture because different individual players pursue different pure strategies.

The whole setup—the population, its conceivable collection of phenotypes, the payoff matrix in the interactions of the phenotypes, and the rule for the evolution of population proportions of the phenotypes in relation to their fitness—constitutes an evolutionary game. An evolutionary stable configuration of the population can be called an *equilibrium* of the evolutionary game.

In this chapter, we develop some of these ideas, as usual through a series of illustrative examples. We begin with symmetric games, in which the two players are on similar footing—for example, two members of the same species competing with each other for food or mates; in a social science interpretation, they could be two elected officials competing for the right to continue in public office. In the payoff table for the game, each can be designated as the row player or the column player with no difference in outcome.

2 PRISONERS' DILEMMA

Suppose a population is made up of two phenotypes. One type consists of players who are natural-born cooperators; they always work toward the outcome that is jointly best for all players. The other type consists of the defectors; they work only for themselves. As an example, we use the restaurant pricing game described in Chapter 5 and presented in a simplified version in Chapter 11. Here, we use the simpler version in which only two pricing choices are available, the jointly best price of $26 or the Nash equilibrium price of $20. A cooperator restaurateur would always choose $26, while a defector would always choose $20. The payoffs (profits) of each type in a single play of this discrete dilemma are shown in Figure 13.1, reproduced from Figure 11.2. Here we call the players simply Row and Column because each can be any individual

		COLUMN	
		20 (Defect)	26 (Cooperate)
ROW	20 (Defect)	288, 288	360, 216
	26 (Cooperate)	216, 360	324, 324

FIGURE 13.1 Prisoners' Dilemma of Pricing ($100s per Month)

restaurateur in the population who is chosen at random to compete against another random rival.

Remember that, under the evolutionary scenario, no one has the choice between defecting and cooperating; each is "born" with one trait or the other predetermined. Which is the more successful (fitter) trait in the population?

A defecting-type restaurateur gets a payoff of 288 ($28,800 a month) if matched against another defecting type and a payoff of 360 ($36,000 a month) if matched against a cooperating type. A cooperating type gets 216 ($21,600 a month) if matched against a defecting type and 324 ($32,400 a month) if matched against another cooperating type. No matter what the type of the matched rival, the defecting type does better than the cooperating type.[4] Therefore the defecting type has a better expected payoff (and is thus fitter) than does the cooperating type, irrespective of the proportions of the two types in the population.

A little more formally, let x be the proportion of cooperators in the population. Consider any one particular cooperator. In a random draw, the probability that she will meet another cooperator (and get 324) is x and that she will meet a defector (and get 216) is $(1 - x)$. Therefore a typical cooperator's expected payoff is $324x + 216(1 - x)$. For a defector, the probability of meeting a cooperator (and getting 360) is x and that of meeting another defector (and getting 288) is $(1 - x)$. Therefore a typical defector's expected profit is $360x + 288(1 - x)$. Now it is immediately apparent that

$$360x + 288(1 - x) > 324x + 216(1 - x) \quad \text{for all } x \text{ between 0 and 1.}$$

Therefore a defector has a higher expected payoff and is fitter than a cooperator. This will lead to an increase in the proportion of defectors (a decrease in x) from one "generation" of players to the next, until the whole population consists of defectors.

What if the population initially consists of all defectors? Then in this case no mutant (experimental) cooperator will survive and multiply to take over the population; in other words, the defector population cannot be invaded successfully by mutant cooperators. Even for a very small value of x—that is, when the proportion of cooperators in the population is very small—the cooperators remain less fit than the prevailing defectors and their population proportion will not increase but will be driven to zero; the mutant strain will die out.

Our analysis shows both that defectors have higher fitness than cooperators and that an all-defector population cannot be invaded by mutant cooperators. Thus the evolutionary stable configuration of the population is monomorphic, consisting of the single strategy or phenotype Defect. We therefore call Defect the evolutionary stable strategy for this population engaged in this dilemma

[4]In the rational behavior context of the preceding chapters, we would say that Defect is the strictly dominant strategy.

game. Note that Defect is a strictly dominant strategy in the rational behavior analysis of this same game. This result is very general: if a game has a strictly dominant strategy, that strategy will also be the ESS.

A. The Repeated Prisoners' Dilemma

We saw in Chapter 11 how a repetition of the prisoners' dilemma permitted consciously rational players to sustain cooperation for their mutual benefit. Let us see if a similar possibility exists in the evolutionary story. Suppose each chosen pair of players plays the dilemma three times in succession. The overall payoff to a player from such an interaction is the sum of what she gets in the three rounds.

Each individual player is still programmed to play just one strategy, but that strategy has to be a complete plan of action. In a game with three moves, a strategy can stipulate an action in the second or third play that depends on what happened in the first or second play. For example, "I will always cooperate no matter what" and "I will always defect no matter what" are valid strategies. But, "I will begin by cooperating and continue to cooperate as long as you cooperated on the preceding play; and I will defect in all later plays if you defect in an early play" is also a valid strategy; in fact, this last strategy is just tit-for-tat (TFT).

To keep the initial analysis simple, we suppose in this section that there are just two types of strategies that can possibly exist in the population: always defect (A) and tit-for-tat (T). Pairs are randomly selected from the population, and each selected pair plays the game a specified number of times. The fitness of each player is simply the sum of her payoffs from all the repetitions played against her specific opponent. We examine what happens with two, three, and more generally *n* such repetitions in each pair.

I. TWICE-REPEATED PLAY Figure 13.2 shows the payoff table for the game in which two members of the restaurateur population meet and play against each other exactly twice. If both players are A types, both defect both times, and Figure 13.1 shows that then each gets 288 each time, for a total of 576. If both are T types, defection never starts, and each gets 324 each time, for a total of 648. If one is an A type and the other a T type, then on the first play the A type defects and the T type cooperates; so the former gets 360 and the latter 216. On the second play both defect and get 288. So the A type's total payoff is 360 + 288 = 648, and the T type's total is 216 + 288 = 504.

In the twice-repeated dilemma, we see that A is only weakly dominant. It is easy to see that if the population is all A, then T-type mutants cannot invade, and A is an ESS. But, if the population is all T, then A-type mutants cannot do any better than the T types. Does this mean that T must be another ESS, just as it would be a Nash equilibrium in the rational game theoretic analysis of this game? The answer is no. If the population is initially all T and a few A mutants

		COLUMN	
		A	T
ROW	A	576, 576	648, 504
	T	504, 648	648, 648

FIGURE 13.2 Outcomes in the Twice-Repeated Prisoners' Dilemma ($100s)

entered, then the mutants would meet the predominant T types most of the time and would do as well as T does against another T. But occasionally an A mutant would meet another A mutant, and in this match she does better than would a T against an A. Thus the mutants have just *slightly* higher fitness than that of a member of the predominant phenotype. This advantage leads to an increase, albeit a slow one, in the proportion of mutants in the population. Therefore an all-T population *can* be invaded successfully by A mutants; T is not an ESS.

Our reasoning relies on two tests for an ESS. First we see if the mutant does better or worse than the predominant phenotype when each is matched against the predominant type. If this primary criterion gives a clear answer, that settles the matter. But if the primary criterion gives a tie, then we use a tie-breaking, or secondary, criterion: does the mutant fare better or worse than a predominant phenotype when each is matched against a mutant? Ties are exceptional and most of the time we do not need the secondary criterion, but it is there in reserve for situations such as the one illustrated in Figure 13.2.[5]

II. THREEFOLD REPETITION Now suppose each matched pair from the (A,T) population plays the game three times. Figure 13.3 shows the fitness outcomes, summed over the three meetings, for each type of player when matched against rivals of each type.

		COLUMN	
		A	T
ROW	A	864, 864	936, 792
	T	792, 936	972, 972

FIGURE 13.3 Outcomes in the Thrice-Repeated Prisoners' Dilemma ($100s)

[5]This game is just one example of a twice-repeated dilemma. With other payoffs in the basic game, twofold repetition may not have ties. That is so in the husband–wife jail story of Chapter 4; see Exercise 13.5

To see how these fitness numbers arise, consider a couple of examples. When two T players meet each other, both cooperate the first time, and therefore both cooperate the second time and the third time as well; both get 324 each time, for a total of 972 each over 3 months. When a T player meets an A player, the latter does well the first time (360 for the A type versus 216 for the T player), but then the T player also defects the second and third times, and each gets 288 in both of those plays (for totals of 936 for A and 792 for T).

The relative fitnesses of the two types depend on the composition of the population. If the population is almost wholly A type, then A is fitter than T (because A types meeting mostly other A types earn 864 most of the time, while T types most often get 792). But, if the population is almost wholly T type, then T is fitter than A (because T types earn 972 when they meet mostly other Ts, but A types earn 936 in such a situation). Each type is fitter when it already predominates in the population. Therefore T cannot invade successfully when the population is all A, and vice versa. Now there are two possible evolutionary stable configurations of the population; in one configuration, A is the ESS and, in the other, T is the ESS.

Next consider the evolutionary dynamics when the initial population is made up of a mixture of the two types. How will the composition of the population evolve over time? Suppose a fraction x of the population is T type and the rest, $(1 - x)$, is A type.[6] An individual A player, pitted against various opponents chosen from such a population, gets 936 when confronting a T player, which happens a fraction x of the times, and 864 against another A player, which happens a fraction $(1 - x)$ of the times. This gives an average expected payoff of

$$936x + 864(1 - x) = 864 + 72x$$

for each A player. Similarly, an individual T player gets an average expected payoff of

$$972x + 792(1 - x) = 792 + 180x.$$

Then a T player is fitter than an A player if the former earns more on average; that is, if

$$\begin{aligned} 792 + 180x &> 864 + 72x \\ 108x &> 72 \\ x &> 2/3. \end{aligned}$$

[6]Literally, the fraction of any particular type in the population is finite and can only take on values such as 1/1,000,000, 2/1,000,000, and so on. But, if the population is sufficiently large and we show all such values by points on a straight line, as in Figure 13.3, then these points are very tightly packed together, and we can regard them as forming a continuous line. This amounts to letting the fractions take on any real value between 0 and 1. We can then talk of the population *proportion* of a certain behavioral type. By the same reasoning, if one individual member goes to jail and is removed from the population, her removal does not change the population's proportions of the various phenotypes.

In other words, if more than two-thirds (67%) of the population is already T type, then T players are fitter and their proportion will grow until it reaches 100%. If the population starts with less than 67% T, then A players will be fitter, and the proportion of T players will go on declining until there are 0% of them, or 100% of the A players. The evolutionary dynamics move the population toward one of the two extremes, both of which are possible ESS. The dynamics leads to the same conclusion as the static test of mutants' invasion. This is a common, although not universal, feature of evolutionary games.

Thus we have identified two evolutionary stable configurations of the population. In each one the population is all of one type (monomorphic). For example, if the population is initially 100% T, then even after a small number of mutant A types arise, the population mix will still be more than 66.66 . . . % T; T will remain the fitter type, and the mutant A strain will die out. Similarly, if the population is initially 100% A, then a small number of T-type mutants will leave the population mix with less than 66.66 . . . % T; so the A types will be fitter and the mutant T strain will die out. And, as we saw earlier, experimenting mutants of type N can never succeed in a population of A and T types that is either largely T or largely A.

What if the initial population has exactly 66.66 . . . % T players (and 33.33 . . . % A players)? Then the two types are equally fit. We could call this polymorphism. But it is not really a suitable candidate for an evolutionary stable configuration. The population can sustain this delicately balanced outcome only until a mutant of either type surfaces. By chance, such a mutant must arise sooner or later. The mutant's arrival will tip the fitness calculation in favor of the mutant type, and the advantage will accumulate until the ESS with 100% of that type is reached. This is just an application of the secondary criterion for evolutionary stability. We will sometimes loosely speak of such a configuration as an unstable equilibrium, so as to maintain the parallel with ordinary game theory where mutations are not a consideration and a delicately balanced equilibrium can persist. But in the strict logic of the biological process, it is not an equilibrium at all.

This reasoning can be shown in a simple graph that closely resembles the graphs that we drew when calculating the equilibrium proportions in a mixed-strategy equilibrium with consciously rational players. The only difference is that, in the evolutionary context, the proportion in which the separate strategies are played is not a matter of choice by any individual player but a property of the whole population, as shown in Figure 13.4. Along the horizontal axis, we measure the proportion x of T players in the population from 0 to 1. We measure fitness along the vertical axis. Each line shows the fitness of one type. The line for the T type starts lower (at 792 compared with 864 for the A-type line) and ends higher (972 against 936). The two lines cross when $x = 0.66. \ldots$ To the right of this point, the T type is fitter; so its population proportion increases

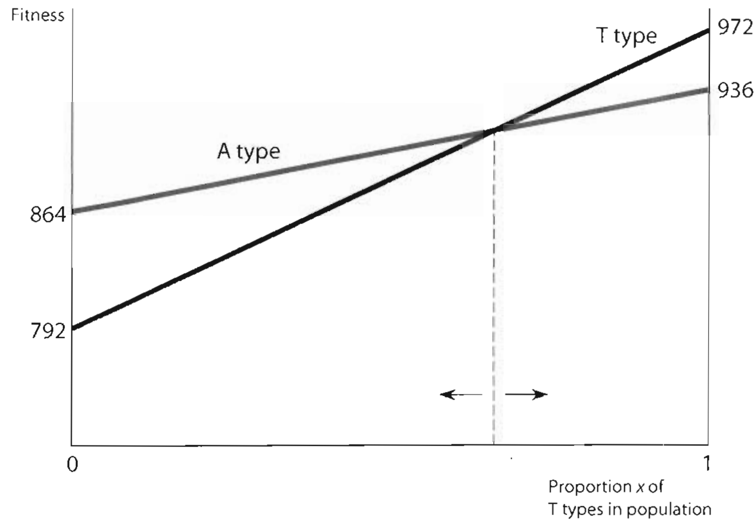


FIGURE 13.4 Fitness Graphs and Equilibria for the Thrice-Repeated Prisoners' Dilemma

over time and x *increases* toward 1. Similarly, to the left of this point, the A type is fitter; so its population proportion increases over time and x *decreases* toward 0. Such diagrams often prove useful as visual aids, and we will use them extensively.[7]

B. Multiple Repetitions

What if each pair plays some unspecified number of repetitions of the game? Let us focus on a population consisting of only A and T types in which interactions between random pairs occur n times (where $n > 2$). The table of the total outcomes from playing n repetitions is shown in Figure 13.5. When two A types meet, they always defect and earn 288 every time; so each gets $288n$ in n plays. When two T types meet, they begin by cooperating, and no one is the first to defect; so they earn 324 every time, for a total of $324n$. When an A type meets a T type, on the first play the T type cooperates and the A type defects, and so the A type gets 360 and the T type gets 216; thereafter the T type retaliates against the preceding defection of the A type for all remaining plays, and each gets 288 in all of the remaining $(n - 1)$ plays. Thus the A type earns a total of $360 + 288(n - 1) = 288n + 72$ in n plays against a T type, whereas the T type gets $216 + 288(n - 1) = 288n - 72$ in n plays against an A type.

[7]You should now draw a similar graph for the twice-repeated case. You will see that the A line is above the T line for all values of x below 1, but the two meet on the right-hand edge of the figure where $x = 1$.

		COLUMN	
		A	T
ROW	A	$288n, 288n$	$288n + 72, 288n - 72$
	T	$288n - 72, 288n + 72$	$324n, 324n$

FIGURE 13.5 Outcomes in the n-fold-Repeated Dilemma

If the proportion of T types in the population is x, then a typical A type gets $x(288n + 72) + (1 - x)288n$ on average, and a typical T type gets $x(324n) + (1 - x)(288n - 72)$ on average. Therefore the T type is fitter if

$$x(324n) + (1 - x)(288n - 72) > x(288n + 72) + (1 - x)288n$$
$$36xn > 72$$
$$x > \frac{72}{36n} = \frac{2}{n}.$$

Once again we have two monomorphic ESSs, one all T (or $x = 1$, to which the process converges starting from any $x > 2/n$) and the other all A (or $x = 0$, to which the process converges starting from any $x < 2/n$). As in Figure 13.4, there is also an unstable polymorphic equilibrium at the balancing point $x = 2/n$.

Notice that the proportion of T at the balancing point depends on n; it is smaller when n is larger. When $n = 10$, it is 2/10, or 0.2. So, if the population initially is 20% T players, in a situation where each pair plays 10 repetitions, the proportion of T types will grow until they reach 100%. Recall that, when pairs played three repetitions ($n = 3$), the T players needed an initial strength of 67% or more to achieve this outcome and only two repetitions meant that T types needed to be 100% of the population to survive. (We see the reason for this outcome in our expression for the critical value for x, which shows that, when $n = 2$, x must be above 1 before the T types are fitter.) Remember, too, that a population consisting of all T players achieves cooperation. Thus cooperation emerges from a larger range of the initial conditions when the game is repeated more times. In this sense, with more repetition, cooperation becomes more likely. What we are seeing is the result of the fact that the value of establishing cooperation increases as the length of the interaction increases.

C. Comparing the Evolutionary and Rational-Player Models

Finally, let us return to the thrice-repeated game illustrated in Figure 13.3 and, instead of using the evolutionary model, consider it played by two consciously rational players. What are the Nash equilibria? There are two in pure strategies, one in which both play A and the other in which both play T. There is also an equilibrium in mixed strategies, in which T is played 67% of the time and A 33% of the time. The first two are just the monomorphic ESSs that we found, and the third is the unstable polymorphic evolutionary equilibrium. In other words, there is a close relation between evolutionary and consciously rational perspectives on games.

That is not a coincidence. An ESS must be a Nash equilibrium of the game played by consciously rational players with the same payoff structure. To see this, suppose the contrary for the moment. If all players using some strategy, call it S, is not a Nash equilibrium, then some other strategy, call it R, must yield a higher payoff for one player when played against S. A mutant playing R will achieve greater fitness in a population playing S and so will invade successfully. Thus S cannot be an ESS. In other words, if all players using S is not a Nash equilibrium, then S cannot be an ESS. This is the same as saying that, if S is an ESS, it must be a Nash equilibrium for all players to use S.

Thus the evolutionary approach provides a backdoor justification for the rational approach. Even when players are not consciously maximizing, if the more successful strategies get played more often and the less successful ones die out and if the process converges eventually to a stable strategy, then the outcome must be the same as that resulting from consciously rational play.

Although an ESS must be a Nash equilibrium of the corresponding rational-play game, the converse is not true. We have seen two examples of this. In the twice-repeated dilemma game of Figure 13.2 played rationally, T would be a Nash equilibrium in the weak sense that, if both players choose T, neither has any positive gain from switching to A. But in the evolutionary approach A can arise as a mutation and can successfully invade the T population. And in the thrice-repeated dilemma game of Figures 13.3 and 13.4, rational play would produce a mixed-strategy equilibrium. But the biological counterpart to this mixed-strategy equilibrium, the polymorphic state, can be successfully invaded by mutants and is therefore not a true evolutionary stable equilibrium. Thus the biological concept of stability can help us select from a multiplicity of Nash equilibria of a rationally played game.

There is one limitation of our analysis of the repeated game. At the outset, we allowed just two strategies: A and T. Nothing else was supposed to exist or arise as a mutation. In biology, the kinds of mutations that arise are determined by genetic considerations. In social or economic or political games, the genesis of new strategies is presumably governed by history, culture, and experience of the players; the ability of people to assimilate and process information and to experiment with different strategies must also play a role. However, the restrictions that we place on the set of strategies that can possibly exist in a particular game have important implications for which of these strategies (if any) can be evolutionary stable. In the thrice-repeated prisoners' dilemma example, if we had allowed for a strategy S that cooperated on the first play and defected on the second and third, then S-type mutants could have successfully invaded an all-T population; so T would not have been an ESS. We develop this possibility further in Exercise 6, at the end of this chapter.

3 CHICKEN

Remember our 1950s youths racing their cars toward each other and seeing who will be the first to swerve to avoid a collision? Now we suppose the players have no choice in the matter: each is genetically hardwired to be either a Wimp (always swerve) or a Macho (always go straight). The population consists of a mixture of the two types. Pairs are picked at random every week to play the game. Figure 13.6 shows the payoff table for any two such players—say, A and B. (The numbers replicate those we used before in Figures 4.14 and 8.5.)

How will the two types fare? The answer depends on the initial population proportions. If the population is almost all Wimps, then a Macho mutant will win and score 1 lots of times, while all the Wimps meeting their own types will get mostly zeroes. But, if the population is mostly Macho, then a Wimp mutant scores −1, which may look bad but is better than the −2 that all the Machos get. You can think of this appropriately in terms of the biological context and the sexism of the 1950s: in a population of Wimps, a Macho newcomer will show all the rest to be chickens and so will impress all the girls. But, if the population consists mostly of Machos, they will be in the hospital most of the time and the girls will have to go for the few Wimps that are healthy.

In other words, each type is fitter when it is relatively rare in the population. Therefore each can successfully invade a population consisting of the other type. We should expect to see both types in the population in equilibrium; that is, we should expect an ESS with a mixture, or polymorphism.

To find the proportions of Wimps and Machos in such an ESS, let us calculate the fitness of each type in a general mixed population. Write x for the fraction of Machos and $(1 - x)$ for the proportion of Wimps. A Wimp meets another Wimp and gets 0 for a fraction $(1 - x)$ of the time and meets a Macho and gets

		B	
		Wimp	Macho
A	Wimp	0, 0	−1, 1
	Macho	1, −1	−2, −2

FIGURE 13.6 Payoff Table for Chicken

-1 for a fraction x of the time. Therefore the fitness of a Wimp is $0 \times (1 - x) - 1 \times x = -x$. Similarly, the fitness of a macho is $1 \times (1 - x) - 2x = 1 - 3x$. The macho type is fitter if

$$\begin{aligned} 1 - 3x &> -x \\ 2x &< 1 \\ x &< 1/2. \end{aligned}$$

If the population is less than half Macho, then the Machos will be fitter and their proportion will increase. On the other hand, if the population is more than half Macho, then the Wimps will be fitter and the Macho proportion will fall. Either way, the population proportion of Machos will tend toward 1/2, and this 50–50 mix will be the stable polymorphic ESS.

Figure 13.7 shows this outcome graphically. Each straight line shows the fitness (the expected payoff in a match against a random member of the population) for one type, in relation to the proportion x of Machos. For the Wimp type, this functional relation showing their fitness as a function of the proportion of the Machos is $-x$, as we saw two paragraphs earlier. This is the gently falling line, which starts at the height 0 when $x = 0$, and goes to -1 when $x = 1$. The corresponding function for the Macho type is $1 - 3x$. This is the rapidly falling line, which starts at height 1 when $x = 0$, and falls to -2 when $x = 1$. The Macho line lies above the Wimp line for $x < 1/2$ and below it for $x > 1/2$, showing that the Macho types are fitter when the value of x is small and the Wimps are fitter when x is large.

Now we can compare and contrast the evolutionary theory of this game with our earlier theory of Chapters 4 and 8, which was based on the assumption that the players were conscious rational calculators of strategies. There we found three Nash equilibria: two in pure strategies, where one player goes

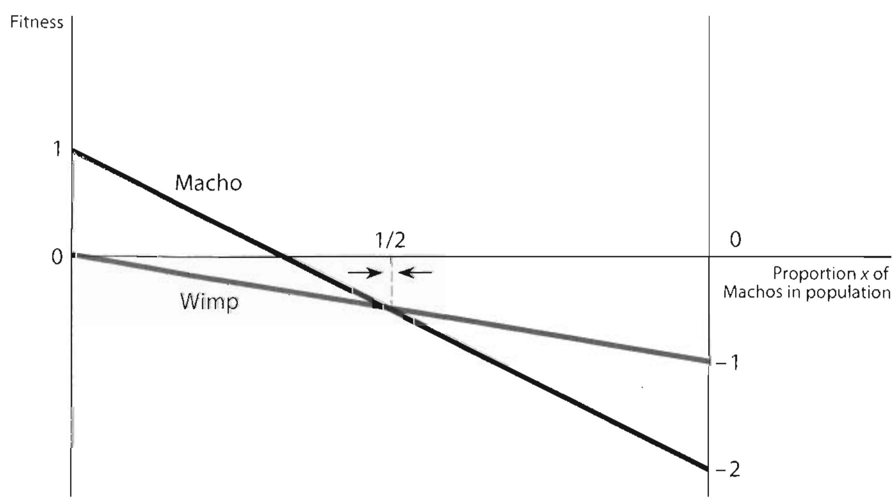


FIGURE 13.7 Fitness Graphs and Polymorphic Equilibrium for Chicken

straight and the other swerves, and one in mixed strategies, where each player goes straight with a probability of 1/2 and swerves with a probability of 1/2.

If the population is truly 100% Macho, then all players are equally fit (or equally unfit). Similarly, in a population of nothing but Wimps, all are equally fit. But these monomorphic configurations are unstable. In an all-Macho population, a Wimp mutant will outscore them and invade successfully.[8] Once some Wimps get established, no matter how few, our analysis shows that their proportion will rise inexorably toward 1/2 . Similarly, an all-Wimp population is vulnerable to a successful invasion of mutant Machos, and the process again goes to the same polymorphism. Thus the polymorphic configuration is the only true evolutionary stable outcome.

Most interesting is the connection between the mixed-strategy equilibrium of the rationally played game and the polymorphic equilibrium of the evolutionary game. The mixture proportions in the equilibrium strategy of the former are *exactly the same* as the population proportions in the latter: a 50–50 mixture of Wimp and Macho. But the interpretations differ: in the rational framework, each player mixes his own strategies; in the evolutionary framework, every member of the population uses a pure strategy, but different members use different strategies, and so we see a mixture in the population.[9]

This correspondence between Nash equilibria of a rationally played game and stable outcomes of a game with the same payoff structure when played according to the evolutionary rules is a very general proposition, and we see it in its generality later, in Section 6. Indeed, evolutionary stability provides an additional rationale for choosing one of the multiple Nash equilibria in such rationally played games.

When we looked at chicken from the rational perspective, the mixed-strategy equilibrium seemed puzzling. It left open the possibility of costly mistakes. Each player went straight one time in two; so one time in four they collided. The pure-strategy equilibria avoided the collisions. At that time, this may have led you to think that there was something undesirable about the mixed-strategy equilibrium, and you may have wondered why we were spending time on it. Now you see the reason. The seemingly strange equilibrium emerges as the stable outcome of a natural dynamic process in which each player tries to improve his payoff against the population that he confronts.

4 THE ASSURANCE GAME

Among the important classes of strategic games introduced in Chapter 4, we have studied prisoners' dilemma and chicken from the evolutionary perspective.

[8] *The Invasion of the Mutant Wimps* could be an interesting science fiction comedy movie.

[9] There can also be evolutionary stable mixed strategies in which each member of the population adopts a mixed strategy. We investigate this idea further in Section 6.E.

That leaves the assurance game. We illustrated this type of game in Chapter 4 with the story of two undergraduates, Harry and Sally, deciding where to meet for coffee. In the evolutionary context, each player is born liking either Starbucks or Local Latte and the population includes some of each type. Here we assume that pairs of the two types, which we classify generically as men and women, are chosen at random each day to play the game. We denote the strategies now by S (for Starbucks) and L (for Local Latte). Figure 13.8 shows the payoff table for a random pairing in this game; the payoffs are the same as those illustrated earlier in Figure 4.12.

If this were a game played by rational strategy-choosing players, there would be two equilibria in pure strategies: (S, S) and (L, L). The latter is better for both players. If they communicate and coordinate explicitly, they can settle on it quite easily. But, if they are making the choices independently, they need to coordinate through a convergence of expectations—that is, by finding a focal point.

The rationally played game has a third equilibrium, in mixed strategies, that we found in Chapter 8. In that equilibrium, each player chooses Starbucks with a probability of 2/3 and Local Latte with a probability of 1/3; the expected payoff for each player is 2/3. As we showed in Chapter 8, this payoff is worse than the one associated with the less attractive of the two pure-strategy equilibria, (S, S), because independent mixing leads the players to make clashing or bad choices quite a lot of the time. Here, the bad outcome (a payoff of 0) has a probability of 4/9: the two players go to different meeting places almost half the time.

What happens when this is an evolutionary game? In the large population, each member is hardwired, either to choose S or to choose L. Randomly chosen pairs of such people are assigned to attempt a meeting. Suppose x is the proportion of S types in the population and $(1 - x)$ is that of L types. Then the fitness of a particular S type—her expected payoff in a random encounter of this kind—is $x \times 1 + (1 - x) \times 0 = x$. Similarly, the fitness of each L type is $x \times 0 + (1 - x) \times 2 = 2(1 - x)$. Therefore the S type is fitter when $x > 2(1 - x)$, or for $x > 2/3$. The L type is fitter when $x < 2/3$. At the balancing point $x = 2/3$, the two types are equally fit.

		WOMEN	
		S	L
MEN	S	1, 1	0, 0
	L	0, 0	2, 2

FIGURE 13.8 Payoff Matrix for the Assurance Game

As in chicken, once again the probabilities associated with the mixed-strategy equilibrium that would obtain under rational choice seem to reappear under evolutionary rules as the population proportions in a polymorphic equilibrium. But now this mixed equilibrium is not stable. The slightest chance departure of the proportion x from the balancing point 2/3 will set in motion a cumulative process that takes the population mix farther away from the balancing point. If x increases from 2/3, the S type becomes fitter and propagates faster, increasing x even more. If x falls from 2/3, the L type becomes fitter and propagates faster, lowering x even more. Eventually x will either rise all the way to 1 or fall all the way to 0, depending on which disturbance occurs. The difference is that in chicken each type was fitter when it was rarer, so the population proportions tended to move away from the extremes and toward a midrange balancing point. In contrast, in the assurance game each type is fitter when it is more numerous; the risk of failing to meet falls when more of the rest of the population are the same type as you—so population proportions tend to move toward the extremes.

Figure 13.9 illustrates the fitness graphs and equilibria for the assurance game; this diagram is very similar to Figure 13.7. The two lines show the fitness of the two types in relation to the population proportion. The intersection of the lines gives the balancing point. The only difference is that, away from the balancing point, the more numerous type is the fitter, whereas in Figure 13.7 it was the less numerous type.

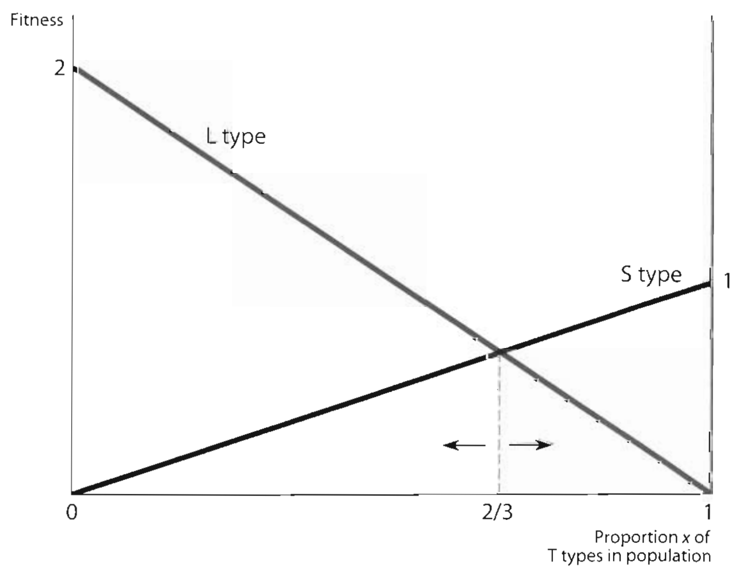


FIGURE 13.9 Fitness Graphs and Equilibria for the Assurance Game

Because each type is less fit when it is rare, only the two extreme monomorphic configurations of the population are possible evolutionary stable states. It is easy to check that both outcomes are ESS according to the static test: an invasion by a small mutant population of the other type will die out because the mutants, being rare, will be less fit. Thus in assurance or coordination games, unlike in chicken, the evolutionary process does not preserve the bad equilibrium, where there is a positive probability that the players choose clashing strategies. However, the dynamics do not guarantee convergence to the better of the two equilibria when starting from an arbitrary initial mixture of phenotypes—where the population ends up depends on where it starts.

5 INTERACTIONS ACROSS SPECIES

A final class of strategic games to consider is that of the battle of the sexes game. In Chapter 4 (Figure 4.13), we saw that the battle of the sexes game looks similar to the assurance game in some respects. We differentiate between the two by assuming here that "men" and "women" are still interested in meeting at either Starbucks or Local Latte—no meeting yields each a payoff of 0—but now each type prefers a different café. Thus there remains a premium on taking mutually consistent actions, just as in the assurance game. But the consequences of the two possible mutually consistent actions differ. The types in the assurance game do not differ in their preferences; both prefer (L, L) to (S, S). The players in the battle game differ in theirs: Local Latte gives a payoff of 2 to women and 1 to men, and Starbucks the other way around. These preferences distinguish the two types. In the language of biology, they can no longer be considered random draws from a homogeneous population of animals.[10] Effectively, they belong to different species (as indeed men and women often believe of each other).

To study such games from an evolutionary perspective, we must extend our methodology to the case in which the matches are between randomly drawn members of different species or populations. We develop the battle-of-the-sexes example to illustrate how this is done.

Suppose there is a large population of men and a large population of women. One of each "species" is picked, and the two are asked to attempt a meeting. All men agree among themselves about the valuation (payoffs) of Starbucks, Local Latte, and no meeting. Likewise, all women are agreed among themselves. But, within each population, some members are hard-liners and others are compromisers. A hard-liner will always go to his or her species' pre-

[10]In evolutionary biology, games of this type are labeled "asymmetric" games. Symmetric games are those in which a player cannot distinguish the type of another player simply from observing that player's outward characteristics; in asymmetric games, players can tell each other apart.

ferred café. A compromiser recognizes that the other species wants the opposite and goes to that location, to get along.

If the random draws happen to have picked a hard-liner of one species and a compromiser of the other, the outcome is that preferred by the hard-liner's species. We get no meeting if two hard-liners are paired and, strangely, also if two compromisers are chosen, because they go to each other's preferred café. (Remember, they have to choose independently and cannot negotiate. Perhaps even if they did get together in advance, they would reach an impasse of "No, I insist on giving way to your preference.")

We alter the payoff table in Figure 4.13 as shown in Figure 13.10; what were choices are now interpreted as actions predetermined by type (hard-liner or compromiser).

In comparison with all the evolutionary games studied so far, the new feature here is that the row player and the column player come from different species. Although each species is a heterogeneous mixture of hard-liners and compromisers, there is no reason why the proportions of the types should be the same in both species. Therefore we must introduce two variables to represent the two mixtures and study the dynamics of both.

We let x be the proportion of hard-liners among the men and y that among the women. Consider a particular hard-liner man. He meets a hard-liner woman a proportion y of the time and gets a 0, and he meets a compromising woman the rest of the time and gets a 2. Therefore his expected payoff (fitness) is $y \times 0 + (1 - y) \times 2 = 2(1 - y)$. Similarly, a compromising woman's fitness is $y \times 1 + (1 - y) \times 0 = y$. Among men, therefore, the hard-liner type is fitter when $2(1 - y) > y$, or $y < 2/3$. The hard-liner men will reproduce faster when they are fitter; that is, x increases when $y < 2/3$. Note the new, and at first sight surprising, feature of the outcome: the fitness of each type within a given species depends on the proportion of types found in other species. This is not surprising; remember that the games that each species plays are now all against the members of the other species.[11]

		WOMEN	
		Hard-liner	Compromiser
MEN	Hard-liner	0, 0	2, 1
	Compromiser	1, 2	0, 0

FIGURE 13.10 Payoffs in the Battle-of-the-Sexes Game

[11]And this finding supports and casts a different light on the property of mixed-strategy equilibria, that each player's mixture keeps the *other* player indifferent among her pure strategies. Now we can think of it as saying that, in a polymorphic evolutionary equilibrium of a two-species game, the proportion of each species' type keeps all the surviving types of the others species equally fit.

Similarly, considering the other species, we have the result that the hard-liner women are fitter; so y increases when $x < 2/3$. To understand the result intuitively, note that it says that the hard-liners of each species do better when the other species does not have too many hard-liners of its own, because then they meet compromisers of the other species quite frequently.

Figure 13.11 shows the dynamics of the configurations of the two species. Each of x and y can range from 0 to 1; so we have a graph with a unit square and x and y on their usual axes. Within that, the vertical line AB shows all points where $x = 2/3$, the balancing point at which y neither increases nor decreases. If the current population proportions lie to the left of this line (that is, $x < 2/3$), y is increasing (moving the population proportion of hard-liner women in the vertically upward direction). If the current proportions lie to the right of AB ($x > 2/3$), then y is decreasing (motion vertically downward). Similarly, the horizontal line CD shows all points where $y = 2/3$, which is the balancing point for x. When the population proportion of hard-liner women is below this line (that is, for $y < 2/3$) the proportion of hard-liner men, x, increases (motion horizontal and rightward) and decreases for population proportions above it, when $y > 2/3$ (motion horizontal and leftward).

When we combine the motions of x and y, we can follow their dynamic paths to determine the location of the population equilibrium. From a starting point in the bottom-left quadrant of Figure 13.10, for example, the dynamics en-

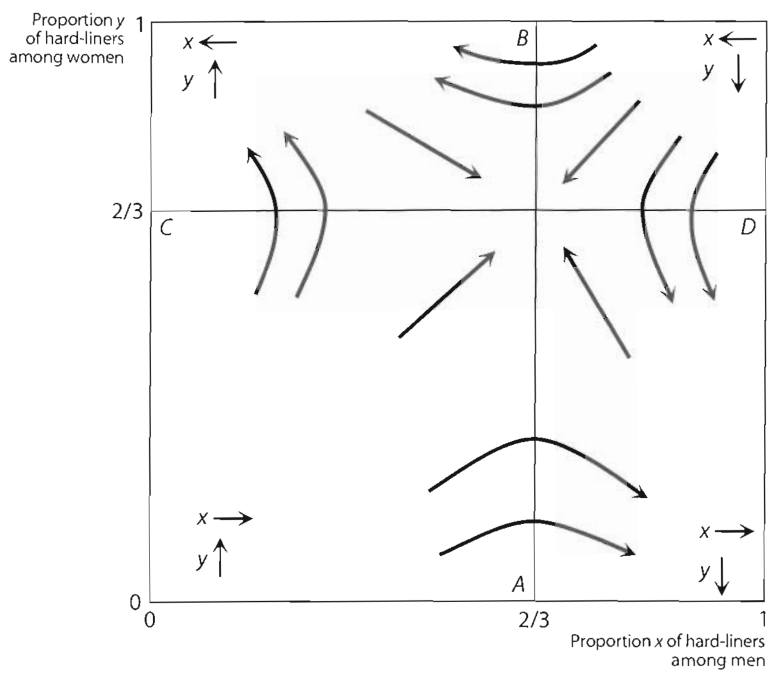


FIGURE 13.11 Population Dynamics in the Battle of the Sexes

tail both y and x increasing. This joint movement (to the northeast) continues until either $x = 2/3$ and y begins to decrease (motion now to the southeast) or $y = 2/3$ and x begins to decrease (motion now to the northwest). Similar processes in each quadrant yield the curved dynamic paths shown in the diagram. The vast majority of these paths lead to either the southeast or northwest corners of the diagram; that is, they converge either to (1, 0) or (0, 1). Thus evolutionary dynamics will lead in most cases to a configuration in which one species is entirely hard-line and the other is entirely compromising. Which species will be which type depends on the initial conditions. Note that the population dynamics starting from a situation with a small value of x and a larger value of y are more likely to cross the CD line first and head for (0, 1)—all hard-line women, $y = 1$—than to hit the AB line first and head for (1, 0); similar results follow for a starting position with a small y but a larger x. The species that starts out with more hard-liners will have the advantage of ending up all hard-line and getting the payoff of 2.

If the initial proportions are balanced just right, the dynamics may lead to the polymorphic point (2/3, 2/3). But, unlike the polymorphic outcome in chicken, the polymorphism in the battle of the sexes is unstable. Most chance departures will set in motion a cumulative process that leads to one of the two extreme equilibria; those are the two ESSs for this game. This is a general property—such multispecies games can have only ESSs that are monomorphic for each species.

6 THE HAWK-DOVE GAME

The **hawk-dove game** was the first example studied by biologists in their development of the theory of evolutionary games. It has instructive parallels with our analyses so far of the prisoners' dilemma and chicken, so we describe it here to reinforce and improve your understanding of the concepts.

The game is played not by birds of these two species, but by two animals of the same species, and Hawk and Dove are merely the names for their strategies. The context is competition for a resource. The Hawk strategy is aggressive and fights to try to get the whole resource of value V. The Dove strategy is to offer to share but to shirk from a fight. When two Hawk types meet each other, they fight. Each animal is equally likely (probability 1/2) to win and get V or to lose, be injured, and get $-C$. Thus the expected payoff for each is $(V - C)/2$. When two Dove types meet, they share without a fight; so each gets $V/2$. When a Hawk type meets a Dove type, the latter retreats and gets a 0, while the former gets V. Figure 13.12 shows the payoff table.

		B	
		Hawk	Dove
A	Hawk	$(V-C)/2, (V-C)/2$	$V, 0$
	Dove	$0, V$	$V/2, V/2$

FIGURE 13.12 Payoff Table for the Hawk-Dove Game

The analysis of the game is similar to that for the prisoners' dilemma and chicken games, except that the numerical payoffs have been replaced by algebraic symbols. We will compare the equilibria of this game when the players rationally choose to play Hawk or Dove and then compare the outcomes when players are acting mechanically and success is being rewarded with faster reproduction.

A. Rational Strategic Choice and Equilibrium

1. If $V > C$, then the game is a prisoners' dilemma in which the Hawk strategy corresponds to "defect" and Dove corresponds to "cooperate." Hawk is the dominant strategy for each, but (Dove, Dove) is the jointly better outcome.
2. If $V < C$, then it's a game of chicken. Now $(V - C)/2 < 0$ and so Hawk is no longer a dominant strategy. Rather, there are two pure-strategy Nash equilibria: (Hawk, Dove) and (Dove, Hawk). There is also a mixed-strategy equilibrium, where B's probability p of choosing Hawk is such as to keep A indifferent:

$$p(V - C)/2 + (1 - p)V = p \times 0 + (1 - p)V/2$$
$$p = V/C.$$

B. Evolutionary Stability for $V > C$

We start with an initial population predominantly of Hawks and test whether it can be invaded by mutant Doves. Following the convention used in analyzing such games, we could write the population proportion of the mutant phenotype as m, for mutant, but for clarity in our case we will use d for mutant Dove. The population proportion of Hawks is then $(1 - d)$. Then, in a match against a randomly drawn opponent, a Hawk will meet a Dove a proportion d of the time and get V on each of those occasions and will meet another Hawk a proportion $(1 - d)$ of the time and get $(V - C)/2$ on each of those occasions. Therefore the fitness of a Hawk is $[dV + (1 - d)(V - C)/2]$. Similarly, the fitness of one of the mutant doves is $[d(V/2) + (1 - d) \times 0]$. Because $V > C$, it follows that $(V - C)/2 > 0$. Also, $V > 0$ implies that $V > V/2$. Then, for any value of d between 0 and 1, we have

$$dV + (1 - d)(V - C)/2 > d(V/2) + (1 - d) \times 0,$$

and so the Hawk type is fitter. The Dove mutants cannot invade. The Hawk strategy is evolutionary stable, and the population is monomorphic (all Hawk).

The same holds true for any population proportion of Doves for all values of d. Therefore, from any initial mix, the proportion of Hawks will grow and they will predominate. In addition, if the population is initially all Doves, mutant Hawks can invade and take over. Thus the dynamics confirm that the Hawk strategy is the only ESS. This algebraic analysis affirms and generalizes our earlier finding for the numerical example of the prisoners' dilemma of restaurant pricing (Figure 13.1).

C. Evolutionary Stability for $V < C$

If the initial population is again predominantly Hawks, with a small proportion d of Dove mutants, then each has the same fitness function derived in Section 6.B. When $V < C$, however, $(V - C)/2 < 0$. We still have $V > 0$, and so $V > V/2$. But, because d is very small, the comparison of the terms with $(1 - d)$ is much more important than that of the terms with d; so

$$d(V/2) + (1 - d) \times 0 > dV + (1 - d)(V - C)/2.$$

Thus the Dove mutants have higher fitness than the predominant Hawks and can invade successfully.

But, if the initial population is almost all Doves, then we must consider whether a small proportion h of Hawk mutants can invade. (Note that, because the mutant is now a Hawk, we have used h for the proportion of the mutant invaders.) The Hawk mutants have a fitness of $[h(V - C)/2 + (1 - h)V]$ compared with $[h \times 0 + (1 - h)(V/2)]$ for the Doves. Again $V < C$ implies that $(V - C)/2 < 0$, and $V > 0$ implies that $V > V/2$. But, when h is small, we get

$$h(V - C)/2 + (1 - h)V > h \times 0 + (1 - h)(V/2).$$

This inequality shows that Hawks have greater fitness and will successfully invade a Dove population. Thus mutants of each type can invade populations of the other type. The population cannot be monomorphic, and neither pure phenotype can be an ESS. The algebra again confirms our earlier finding for the numerical example of chicken (Figures 13.6 and 13.7).

What happens in the population then when $V < C$? There are two possibilities. In one, every player follows a pure strategy, but the population has a stable mix of players following different strategies. This is the polymorphic equilibrium developed for chicken in Section 13.3. The other possibility is that every player uses a mixed strategy. We begin with the polymorphic case.

D. $V < C$: Stable Polymorphic Population

When the population proportion of Hawks is h, the fitness of a Hawk is $h(V - C)/2 + (1 - h)V$, and the fitness of a Dove is $h \times 0 + (1 - h)(V/2)$. The Hawk type is fitter if

$$h(V - C)/2 + (1 - h)V > (1 - h)(V/2),$$

which simplifies to:

$$\begin{aligned} h(V - C)/2 + (1 - h)(V/2) &> 0 \\ V - hC &> 0 \\ h &< V/C. \end{aligned}$$

The Dove type is then fitter when $h > V/C$, or when $(1 - h) < 1 - V/C = (C - V)/C$. Thus each type is fitter when it is rarer. Therefore we have a stable polymorphic equilibrium at the balancing point, where the proportion of Hawks in the population is $h = V/C$. This is exactly the probability with which each individual member plays the Hawk strategy in the mixed-strategy Nash equilibrium of the game under the assumption of rational behavior, as calculated in Section 6.A. Again, we have an evolutionary "justification" for the mixed-strategy outcome in chicken.

We leave it to you to draw a graph similar to that in Figure 13.7 for this case. Doing so will require that you determine the dynamics by which the population proportions of each type converge to the stable equilibrium mix.

E. $V < C$: Each Player Mixes Strategies

Recall the equilibrium mixed strategy of the rational-play game calculated earlier in Section 6.A in which $p = V/C$ was the probability of choosing to be a Hawk, while $(1 - p)$ was the probability of choosing to be a Dove. Is there a parallel in the evolutionary version, with a phenotype playing a mixed strategy? Let us examine this possibility. We still have H types who play the pure Hawk strategy and D types who play the pure Dove strategy. But now a third phenotype called M can exist; such a type plays a mixed strategy in which it is a Hawk with probability $p = V/C$ and a Dove with probability $1 - p = 1 - V/C = (C - V)/C$.

When an H or a D meets an M, their expected payoffs depend on p, the probability that M is playing H, and on $(1 - p)$, the probability that M is playing D. Then each player gets p times her payoff against an H, plus $(1 - p)$ times her payoff against a D. So, when an H type meets an M type, she gets the expected payoff

$$\begin{aligned} p\frac{V - C}{2} + (1 - p)V &= \frac{V}{C}\frac{V - C}{2} - \frac{C - V}{C}V \\ &= -\frac{1}{2}\frac{V}{C}(C - V) + \frac{V}{C}(C - V) \\ &= V\frac{(C - V)}{2C}. \end{aligned}$$

And, when a D type meets an M type, she gets

$$p \times 0 + (1 - p)\frac{V}{2} = \frac{C - V}{C}\frac{V}{2} = \frac{V(C - V)}{2C}.$$

The two fitnesses are equal. This should not be a surprise; the proportions of the mixed strategy are determined to achieve exactly this equality. Then an M type meeting another M type also gets the same expected payoff. For brevity of future reference, we call this common payoff K, where $K = V(C - V)/2C$.

But these equalities create a problem when we test M for evolutionary stability. Suppose the population consists entirely of M types and that a few mutants of the H type, constituting a very small proportion h of the total population, invade. Then the typical mutant gets the expected payoff $h(V - C)/2 + (1 - h)K$. To calculate the expected payoff of an M type, note that she faces another M type a fraction $(1 - h)$ of the time and gets K in each instance. She then faces an H type for a fraction h of the interactions; in these interactions she plays H a fraction p of the time and gets $(V - C)/2$, and she plays D a fraction $(1 - p)$ of the time and gets 0. Thus the M type's total expected payoff (fitness) is

$$hp(V - C)/2 + (1 - h)K.$$

Because h is very small, the fitnesses of the M types and the mutant H types are almost equal. The point is that, when there are very few mutants, both the H type and the M type meet only M types most of the time, and in this interaction the two have equal fitness as we just saw.

Evolutionary stability hinges on whether the original population M type is fitter than the mutant H when each is matched against one of the few mutants. Algebraically, M is fitter than H against other mutant H types when $pV(C - V)/2C = pK > (V - C)/2$. In our example here, this condition holds because $V < C$ (so $(V - C)$ is negative) and because K is positive. Intuitively, this condition tells us that an H-type mutant will always do badly against another H-type mutant because of the high cost of fighting, but the M type fights only part of the time and therefore suffers this cost only a fraction p of the time. Overall, the M type does better when matched against the mutants.

Similarly, the success of a Dove invasion against the M population depends on the comparison between a mutant Dove's fitness and the fitness of an M type. As before, the mutant faces another D a fraction d of the time and faces an M a fraction $(1 - d)$ of the time. An M type also faces another M type a fraction $(1 - d)$ of the time; but a fraction d of the time, the M faces a D and plays H a fraction p of these times, thereby gaining pV, and plays D a fraction $(1 - p)$ of these times, thereby gaining $(1 - p)V/2$. The Dove's fitness is then $[dV/2 + (1 - d)K]$, while the fitness of the M type is $d \times [pV + (1 - p)V/2] + (1 - d)K$. The final term in each fitness expression is the same, so a Dove invasion is successful only if $V/2$ is greater than $pV + (1 - p)V/2$. This condition does not hold; the latter expression includes a weighted average of V and $V/2$ and that must exceed $V/2$ whenever $V > 0$. Thus the Dove invasion cannot succeed either.

This analysis tells us that M is an ESS. Thus if $V < C$, the population can exhibit either of two evolutionary stable outcomes. One entails a mixture of types (a stable polymorphism) and the other entails a single type that mixes its strategies in the same proportions that define the polymorphism.

7 THREE PHENOTYPES IN THE POPULATION

If there are only two possible phenotypes (strategies), we can carry out static checks for ESS by comparing the type being considered against just one type of mutant and can show the dynamics of the population in an evolutionary game with graphs similar to those in Figures 13.4, 13.7, and 13.9. Now we illustrate how the ideas and methods can be used if there are three (or more) possible phenotypes and what new considerations arise.

A. Testing for ESS

Let us reexamine the thrice-repeated prisoners' dilemma of Section 13.2.A.II and Figure 13.3 by introducing a third possible phenotype. This strategy, labeled N, never defects. Figure 13.13 shows the fitness table with the three strategies, A, T, and N.

To test whether any of these strategies is an ESS, we consider whether a population of all one type can be invaded by mutants of one of the other types. An all-A population, for example, cannot be invaded by mutant N or T types; so A is an ESS. An all-N population can be invaded by type-A mutants, however; N lets itself get fooled thrice (shame on it). So N cannot be an ESS. What about T? An all-T population cannot be invaded by A. But it *can* be invaded by type-N mutants, so T is not an ESS either. There is thus just one monomorphic ESS, strategy A.

		COLUMN		
		A	T	N
ROW	A	864, 864	936, 792	1080, 648
	T	792, 936	972, 972	972, 972
	N	648, 1080	972, 972	972, 972

FIGURE 13.13 Thrice-Repeated Prisoners' Dilemma with Three Types ($100s)

Let us consider further the situation when an all-T population is invaded by type-N mutants. For a while the two types can coexist happily. But, if the population becomes too heavily type N, then type-A mutants can invade; A types do well against N but poorly against T. To be specific, consider a population with proportions x of N and $(1 - x)$ of T. The fitness of each of these types is 972. The fitness of a type-A mutant in this population is $936(1 - x) + 1{,}080x = 144x + 936$. This exceeds 972 if $144x > 972 - 936 = 36$, or $x > 1/4$. Thus we can have additional polymorphic equilibria with mixtures of T and N so long as the proportion of Ns is less than 25%.

B. Dynamics

To motivate our discussion of dynamics in games with three possible phenotypes, we turn to another well-known game, rock–paper–scissors (RPS). In rational game-theoretic play of this game, each player simultaneously chooses one of the three available actions, either rock (make a fist), paper (lay your hand flat), or scissors (make a scissorlike motion with two fingers). The rules of the game state that rock beats ("breaks") scissors, scissors beat ("cut") paper, and paper beats ("covers") rock; identical actions tie. If players choose different actions, the winner gets a payoff of 1 and the loser gets a payoff of -1; ties yield both players 0.

For an evolutionary example, we turn to the situation faced by the side-blotched lizards living along the California coast. That species supports three types of male mating behavior, each type associated with a particular throat color. Males with blue throats guard a small number of female mates and fend off advances made by yellow-throated males who attempt to sneak in and mate with unguarded females. The yellow-throated sneaking strategy works well against males with orange throats, who maintain large harems and are often out aggressively pursuing additional mates; those mates tend to belong to the blue-throated males who can be overpowered by the orange-throat's aggression.[12] Their interactions can be modeled by using the payoff structure of the RPS game shown in Figure 13.14. We include a column for a q-mix to allow us to consider the evolutionary equivalent of the game's mixed-strategy equilibrium, a mixture of types in the population.[13]

Suppose q_1 is the proportion of lizards in the population that are yellow throated, q_2 the proportion of blue throats, and the rest, $(1 - q_1 - q_2)$, the

[12]For more information about the side-blotched lizards, see Kelly Zamudio and Barry Sinervo, "Polygyny, Mate-Guarding, and Posthumous Fertilizations As Alternative Mating Strategies," *Proceedings of the National Academy of Sciences*, vol. 97, no. 26 (December 19, 2000), pp. 14427–14432.

[13]Exercise 4 in Chapter 7 considers the rational game-theoretic equilibrium of a version of the RPS game. You should be able to verify relatively easily that the game has no equilibrium in pure strategies.

		COLUMN			
		Yellow-throated sneaker	Blue-throated guarder	Orange-throated aggressor	q-mix
ROW	Yellow-throated sneaker	0	−1	1	$-q_2 + (1 - q_1 - q_2)$
	Blue-throated guarder	1	0	−1	$q_1 - (1 - q_1 - q_2)$
	Orange-throated aggressor	−1	1	0	$-q_1 + q_2$

FIGURE 13.14 Payoffs in the Three-Type Evolutionary Game

proportion of orange throats. The right-hand column of the table shows each Row player's payoffs when meeting this mixture of phenotypes; that is, just Row's fitness. Suppose, as has been shown to be true in the side-splotched lizard population, that the proportion of each type in the population grows when its fitness is positive and declines when it is negative.[14] Then

$$q_1 \text{ increases if and only if } \quad -q_2 + (1 - q_1 - q_2) > 0, \quad \text{or } q_1 + 2q_2 < 1.$$

The proportion of yellow-throated types in the population increases when q_2, the proportion of blue-throated types, is small or when $(1 - q_1 - q_2)$, the proportion of orange-throated types, is large. This makes sense; yellow throats do poorly against blue throats but well against orange throats. Similarly, we see that

$$q_2 \text{ increases if and only if } \quad q_1 - (1 - q_1 - q_2) > 0, \quad \text{or } 2q_1 + q_2 > 1.$$

Blue-throated males do better when the proportion of yellow-throated competitors is large or the proportion of orange-throated types is small.

Figure 13.15 shows graphically the population dynamics and resulting equilibria for this game. The triangular area defined by the axes and the line $q_1 + q_2 = 1$ contains all the possible equilibrium combinations of q_1 and q_2. There are also two straight lines within this area. The first is $q_1 + 2q_2 = 1$ (the flatter one), which is the balancing line for q_1; for combinations of q_1 and q_2 below this line, q_1 (the proportion of yellow-throated players) increases; for combinations above this line, q_1 decreases. The second, steeper line is $2q_1 + q_2 = 1$, which is the balancing line for q_2. To the right of this line (when $2q_1 + q_2 > 1$), q_2 in-

[14]A little more care is necessary to ensure that the three proportions sum to 1, but that can be done, and we hide the mathematics so as to convey the ideas in a simple way. Exercise 13.10 develops the dynamics more rigorously for readers with sufficient mathematical training.

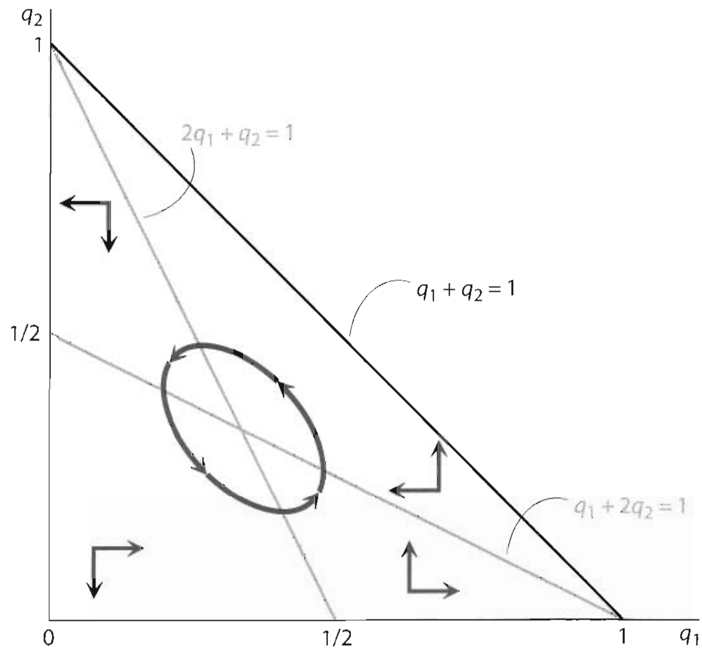


FIGURE 13.15 Population Dynamics in the Evolutionary RPS Game

creases; to the left of the line (when $2q_1 + q_2 < 1$), q_2 decreases. Arrows on the diagram show directions of motion of these population proportions; red curves show typical dynamic paths. The general idea is the same as that of Figure 13.13.

On each of the two gray lines, one of q_1 and q_2 neither increases nor decreases. Therefore the intersection of the two lines represents the point where q_1, q_2, and therefore also $(1 - q_1 - q_2)$, are all constant; this point thus corresponds to a polymorphic equilibrium. It is easy to check that here $q_1 = q_2 = 1 - q_1 - q_2 = 1/3$. These proportions are the same as the probabilities in the rational mixed-strategy equilibrium of the RPS game.

Is this polymorphic outcome stable? In general, we cannot say. The dynamics indicate paths (shown in Figure 13.15 as a single ellipse) that wind around it. Whether these paths wind in a decreasing spiral toward the intersection (in which case we have stability) or in an expanding spiral away from the intersection (indicating instability) depends on the precise response of the population proportions to the fitnesses. It is even possible that the paths circle as drawn, neither approaching nor departing from the equilibrium.

Evidence suggests that the side-splotched lizard population is cycling around the evenly split polymorphic equilibrium point, with one type being slightly more common for a period of a few years but then being overtaken by its stronger competitor. Whether the cycle is approaching the stable equilibrium remains a topic for future study. At least one other example of an RPS-type interaction in an evolutionary game entails three strains of food-poisoning-related *E. coli* bacteria. Each strain displaces one of the others but is displaced by the third, as in the three-type game described earlier. Scientists studying the competition among the three strains have shown that a polymorphic equilibrium can persist if interactions between pairs stay localized, with small clumps of each strain shifting position continuously.[15]

8 SOME GENERAL THEORY

We now generalize the ideas illustrated in Section 6 to get a theoretical framework and set of tools that can then be applied further. This generalization unavoidably requires some slightly abstract notation and a bit of algebra. Therefore we cover only monomorphic equilibria in a single species. Readers who are adept at this level of mathematics can readily develop the polymorphism cases with two species by analogy. Readers who are not prepared for this material or interested in it can omit this section without loss of continuity.[16]

We consider random matchings from a single species whose population has available strategies I, J, K . . . . Some of them may be pure strategies; some may be mixed. Each individual member is hardwired to play just one of these strategies. We let $E(\mathrm{I}, \mathrm{J})$ denote the payoff to an I player in a single encounter with a J player. The payoff of an I player meeting another of her own type is $E(\mathrm{I}, \mathrm{I})$ in the same notation. We write $W(\mathrm{I})$ for the fitness of an I player. This is just her expected payoff in encounters with randomly picked opponents, when the probability of meeting a type is just the proportion of this type in the population.

Suppose the population is all I type. We consider whether this can be an evolutionary stable configuration. To do so, we imagine that the population is invaded by a few J-type mutants; so the proportion of mutants in the population is a very small number, m. Now the fitness of an I type is

$$W(\mathrm{I}) = mE(\mathrm{I}, \mathrm{J}) + (1 - m)E(\mathrm{I}, \mathrm{I}),$$

and the fitness of a mutant is

$$W(\mathrm{J}) = mE(\mathrm{J}, \mathrm{J}) + (1 - m)E(\mathrm{J}, \mathrm{I}).$$

[15] The research on *E. coli* is reported in Martin Nowak and Karl Sigmund, "Biodiversity: Bacterial Game Dynamics," *Nature*, vol. 418 (July 11, 2002), p. 138. If the three strains were forcibly dispersed on a regular basis, a single strain could take over in a matter of days; the "winning" strain outmultiplied a second strain, which could quickly kill off the third.

[16] Conversely, readers who want more details can find them in Maynard Smith, *Evolution and the Theory of Games*, especially pp. 14–15. John Maynard Smith is a pioneer in the theory of evolutionary games.

Therefore the difference in fitness between the population's main type and its mutant type is

$$W(\mathrm{I}) - W(\mathrm{J}) = m[E(\mathrm{I}, \mathrm{J}) - E(\mathrm{J}, \mathrm{J})] + (1 - m)[E(\mathrm{I}, \mathrm{I}) - E(\mathrm{J}, \mathrm{I})].$$

Because m is very small, the main type's fitness will be higher than the mutant's if the second half of the preceding expression is positive; that is,

$$W(\mathrm{I}) > W(\mathrm{J}) \quad \text{if } E(\mathrm{I}, \mathrm{I}) > E(\mathrm{J}, \mathrm{I}).$$

Then the main type in the population cannot be invaded; it is fitter than the mutant type when each is matched against a member of the main type. This forms the **primary criterion** for evolutionary stability. Conversely, if $W(\mathrm{I}) < W(\mathrm{J})$, owing to $E(\mathrm{I}, \mathrm{I}) < E(\mathrm{J}, \mathrm{I})$, the J-type mutants will invade successfully and an all-I population cannot be evolutionary stable.

However, it is possible that $E(\mathrm{I}, \mathrm{I}) = E(\mathrm{J}, \mathrm{I})$, as indeed happens if the population initially consists of a single phenotype that plays a strategy of mixing between the pure strategies I and J (a monomorphic equilibrium with a mixed strategy), as was the case in our final variant of the Hawk-Dove game (Section 6.E). Then the difference between $W(\mathrm{I})$ and $W(\mathrm{J})$ is governed by how each type fares against the mutants.[17] When $E(\mathrm{I}, \mathrm{I}) = E(\mathrm{J}, \mathrm{I})$, we get $W(\mathrm{I}) > W(\mathrm{J})$ if $E(\mathrm{I}, \mathrm{J}) > E(\mathrm{J}, \mathrm{J})$. This is the **secondary criterion** for the evolutionary stability of I, to be invoked only if the primary one is inconclusive—that is, only if $E(\mathrm{I}, \mathrm{I}) = E(\mathrm{J}, \mathrm{I})$.

The primary criterion carries a punch. It says that, if the strategy I is evolutionary stable, then, for all other strategies J that a mutant might try, $E(\mathrm{I}, \mathrm{I}) \geq E(\mathrm{J}, \mathrm{I})$. This means that I is the best response to itself. In other words, if the members of this population suddenly started playing as rational calculators, everyone playing I would be a Nash equilibrium. *Evolutionary stability thus implies Nash equilibrium of the corresponding rationally played game!*

This is a remarkable result. If you were dissatisfied with the rational behavior assumption underlying the theory of Nash equilibria given in earlier chapters and you came to the theory of evolutionary games looking for a better explanation, you would find that it yields the same results. The very appealing biological description—fixed nonmaximizing behavior, but selection in response to resulting fitness—does not yield any new outcomes. If anything, it provides a backdoor justification for Nash equilibrium. When a game has several Nash equilibria, the evolutionary dynamics may even provide a good argument for choosing among them.

[17] If the initial population is polymorphic and m is the proportion of J types, then m may not be "very small" any more. The size of m is no longer crucial, however, because the second term in $W(\mathrm{I}) - W(\mathrm{J})$ is now assumed to be zero.

However, your reinforced confidence in Nash equilibrium should be cautious. Our definition of evolutionary stability is static rather than dynamic. It only checks whether the configuration of the population (monomorphic, or polymorphic in just the right proportions) that we are testing for equilibrium cannot be successfully invaded by a small proportion of mutants. It does not test whether, starting from an arbitrary initial population mix, all the unwanted types will die out and the equilibrium configuration will be reached. And the test is carried out for those particular classes of mutants that are deemed logically possible; if the theorist has not specified this classification correctly and some type of mutant that she overlooked could actually arise, that mutant might invade successfully and destroy the supposed equilibrium. Our remark at the end of the twice-played prisoners' dilemma in Section 2.A warned of this possibility, and Exercise 6 shows how it can arise. Finally, in Section 7 we saw how evolutionary dynamics can fail to converge at all.

9 PLAYING THE FIELD

We have thus far looked at situations where each game is played between just two players who are randomly chosen from the population. There are other situations, however, when the whole population plays at once. In biology, a whole flock of animals with a mixture of genetically determined behaviors may compete for some resource or territory. In economics or business, many firms in an industry, each following the strategy dictated by its corporate culture, may compete all with all.

Such evolutionary games stand in the same relation to the rationally played collective-action games of Chapter 12 as do the pair-by-pair played evolutionary games of the preceding sections to the rationally played two-person games of Chapters 4 through 8. Just as we converted the expected payoff graphs of those chapters into the fitness diagrams in Figures 13.4, 13.7, and 13.9, we can convert the graphs for collective-action games (Figures 12.6 through 12.8) into fitness graphs for evolutionary games. For example, consider an animal species all of whose members come to a common feeding ground. There are two phenotypes: one fights for food aggressively, and the other hangs around and sneaks what it can. If the proportion of aggressive ones is small, they will do better; but, if there are too many of them, the sneakers will do better by ignoring the ongoing fights. This will be a collective chicken game whose fitness diagram will be exactly like Figure 12.7. Because no new principles or techniques are required, we leave it to you to pursue this idea further.

10 EVOLUTION OF COOPERATION AND ALTRUISM

Evolutionary game theory rests on two fundamental ideas: first, that individual organisms are engaged in games with others in their own species or with members of other species and, second, that the genotypes that lead to higher-payoff (fitter) strategies proliferate while the rest decline in their proportions of the population. These ideas suggest a vicious struggle for survival like that depicted by some interpreters of Darwin who understood "survival of the fittest" in a literal sense and who conjured up images of a "nature red in tooth and claw." In fact, nature shows many instances of cooperation (where individual animals behave in a way that yields greater benefit to everyone in a group) and even altruism (where individual animals incur significant costs in order to benefit others). Beehives and ant colonies are only the most obvious examples. Can such behavior be reconciled with the perspective of evolutionary games?

Biologists use a fourfold classification of the ways in which cooperation and altruism can emerge among selfish animals (or phenotypes or genes). Lee Dugatkin names the four categories (1) family dynamics, (2) reciprocal transactions, (3) selfish teamwork, and (4) group altruism.[18]

The behavior of ants and bees is probably the easiest to understand as an example of family dynamics. All the individual members of an ant colony or a beehive are closely related and therefore have genes in common to a substantial extent. All worker ants in a colony are full sisters and therefore have half their genes in common; therefore the survival and proliferation of one ant's genes is helped just as much by the survival of two of its sisters as by its own survival. All worker bees in a hive are half-sisters and therefore have a quarter of their genes in common. An individual ant or bee does not make a fine calculation of whether it is worthwhile to risk its own life for the sake of two or four sisters, but the underlying genes of those groups whose members exhibit such behavior (phenotype) will proliferate. The idea that evolution ultimately operates at the level of the gene has had enormous implications for biology, although it has been misapplied by many people, just as Darwin's original idea of natural selection was misapplied.[19] The interesting idea is that a "selfish gene" may prosper by behaving unselfishly in a larger organization of genes, such as a cell.

[18]See his excellent exposition in *Cheating Monkeys and Citizen Bees: The Nature of Cooperation in Animals and Humans* (Cambridge: Harvard University Press, 1999).

[19]In this very brief account, we cannot begin to do justice to all the issues and the debates. An excellent popular account, and the source of many examples cited in this section, is Matt Ridley, *The Origins of Virtue* (New York: Penguin, 1996). We should also point out that we do not examine the connection between genotypes and phenotypes in any detail nor the role of sex in evolution. Another book by Ridley, *The Red Queen* (New York: Penguin, 1995), gives a fascinating treatment of this subject.

Similarly, a cell and its genes may prosper by participating cooperatively and accepting their allotted tasks in a body.

Reciprocal altruism can arise among unrelated individual members of the same or different species. This behavior is essentially an example of the resolution of prisoners' dilemmas through repetition in which the players use strategies that are remarkably like tit-for-tat. For example, some small fish and shrimp thrive on parasites that collect in the mouths and gills of some large fish; the large fish let the small ones swim unharmed through their mouths for this "cleaning service." A more fascinating, although gruesome, example is that of vampire bats who share blood with those who have been unsuccessful in their own searches. In an experiment in which bats from different sites were brought together and selectively starved, "only bats that were on the verge of starving (i.e., would die within twenty-four hours without a meal) were given blood by any other bat in the experiment. But, more to the point, individuals were given a blood meal only from bats they already knew from their site. . . . Furthermore, vampires were much more likely to regurgitate blood to the specific individual(s) from their site that had come to their aid when they needed a bit of blood."[20] Once again, it is not to be supposed that each animal consciously calculates whether it is in its individual interest to continue the cooperation or to defect. Instead, the behavior is instinctive.

Selfish teamwork arises when it is in the interests of each individual organism to choose cooperation when all others are doing so. In other words, this type of cooperative behavior applies to the selection of the good outcome in assurance games. Dugatkin argues that populations are more likely to engage in selfish teamwork in harsh environments than in mild ones. When conditions are bad, the shirking of any one animal in a group may bring disaster to the whole group, including the shirker. Then in such conditions each animal is crucial for survival, and none shirk so long as others are also pulling their weight. In milder environments, each may hope to become a free rider on the others' effort without thereby threatening the survival of the whole group including itself.

The next step goes beyond biology and into sociology: a body (and its cells and ultimately its genes) may benefit by behaving cooperatively in a collection of bodies—namely, a society. This brings us to the idea of group altruism which suggests that we should see some cooperation even among individual members of a group who are not close relatives. We do indeed find instances of it. Groups of predators such as wolves are a case in point, and groups of apes often behave like extended families. Even among species of prey, cooperation arises when individual fishes in a school take turns to look out for predators. And cooperation can also extend across species.

[20]Dugatkin, *Cheating Monkeys*, p. 99.

The general idea is that a group whose members behave cooperatively is more likely to succeed in its interactions with other groups than one whose members seek benefit of free-riding within the group. If, in a particular context of evolutionary dynamics, between-group selection is a stronger force than that of within-group selection, then we will see group altruism.[21]

An instinct is hardwired into an individual organism's brain by genetics, but reciprocity and cooperation can arise from more purposive thinking or experimentation within the group and can spread by socialization—through explicit instruction or observation of the behavior of elders—instead of genetics. The relative importance of the two channels—nature and nurture—will differ from one species to another and from one situation to another. One would expect socialization to be relatively more important among humans, but there are instances of its role among other animals. We cite a remarkable one. The expedition that Robert F. Scott led to the South Pole in 1911–1912 used teams of Siberian dogs. This group of dogs, brought together and trained for this specific purpose, developed within a few months a remarkable system of cooperation and sustained it by using punishment schemes. "They combined readily and with immense effect against any companion who did not pull his weight, or against one who pulled too much . . . their methods of punishment always being the same and ending, if unchecked, in what they probably called justice, and we called murder."[22]

This is an encouraging account of how cooperative behavior can be compatible with evolutionary game theory and one that suggests that dilemmas of selfish actions can be overcome. Indeed, scientists investigating altruistic behavior have recently reported experimental support for the existence of such *altruistic punishment,* or *strong reciprocity* (as distinguished from reciprocal altruism), in humans. Their evidence suggests that people are willing to punish those who don't pull their own weight in a group setting, even when it is costly to do so and when there is no expectation of future gain. This tendency toward strong reciprocity may even help to explain the rise of human civilization if groups with this trait were better able to survive the traumas of war and other catastrophic events.[23] Despite these findings, strong reciprocity may not be widespread in the animal world. "Compared to nepotism, which accounts for the cooperation of ants and every creature that cares for its young, reciprocity has proved to be scarce. This, presumably, is due to the fact that reciprocity

[21]Group altruism used to be thought impossible according to the strict theory of evolution that emphasizes selection at the level of the gene, but the concept is being revived in more sophisticated formulations. See Dugatkin, *Cheating Monkeys,* pp. 141–145 for a fuller discussion.

[22]Apsley Cherry-Garrard, *The Worst Journey in the World* (London: Constable, 1922; reprinted New York: Carroll and Graf, 1989), pp. 485–486.

[23]For the evidence on altruistic punishment, see Ernst Fehr and Simon Gachter, "Altruistic Punishment in Humans," *Nature,* vol. 415 (January 10, 2002), pp. 137–140.

requires not only repetitive interactions, but also the ability to recognize other individuals and keep score."[24] In other words, precisely the conditions that our theoretical analysis in Section 2.D of Chapter 11 identified as being necessary for a successful resolution of the repeated prisoners' dilemma are seen to be relevant in the context of evolutionary games.

SUMMARY

The biological theory of evolution parallels the theory of games used by social scientists. Evolutionary games are played by behavioral *phenotypes* with genetically predetermined, rather than rationally chosen, strategies. In an evolutionary game, phenotypes with higher *fitness* survive repeated *interactions* with others to reproduce and to increase their representation in the population. A population containing one or more phenotypes in certain proportions is called *evolutionary stable* if it cannot be *invaded* successfully by other, *mutant* phenotypes or if it is the limiting outcome of the dynamics of proliferation of fitter phenotypes. If one phenotype maintains its dominance in the population when faced with an invading mutant type, that phenotype is said to be an *evolutionary stable strategy,* and the population consisting of it alone is said to exhibit *monomorphism.* If two or more phenotypes coexist in an evolutionary stable population, it is said to exhibit *polymorphism.*

When the theory of evolutionary games is used more generally for nonbiological games, the strategies followed by individual players are understood to be standard operating procedures or rules of thumb, instead of being genetically fixed. The process of reproduction stands for more general methods of transmission including socialization, education, and imitation; and *mutations* represent experimentation with new strategies.

Evolutionary games may have payoff structures similar to those analyzed in Chapters 4 and 8, including the prisoners' dilemma and chicken. In each case, the *evolutionary stable strategy* mirrors either the pure-strategy Nash equilibrium of a game with the same structure played by rational players or the proportions of the equilibrium mixture in such a game. In a prisoners' dilemma, "always defect" is evolutionary stable; in chicken, types are fitter when rare, and so there is a polymorphic equilibrium; in the assurance game, types are less fit when rare, and so the polymorphic configuration is unstable and the equilibria are at the extremes. When play is between two different types of members of each of two different species, a more complex but similarly structured analysis is used to determine equilibria.

[24]Ridley, *Origins of Virtue,* p. 83.

The *hawk-dove game* is the classic biological example. Analysis of this game parallels that of the prisoners' dilemma and chicken versions of the evolutionary game; evolutionary stable strategies depend on the specifics of the payoff structure. The analysis can also be done when more than two types interact or in very general terms. This theory shows that the requirements for evolutionary stability yield an equilibrium strategy that is equivalent to the Nash equilibrium obtained by rational players.

KEY TERMS

evolutionary stable (426)
evolutionary stable strategy (ESS) (429)
fitness (426)
genotype (426)
hawk-dove game (447)
interaction (427)
invasion by a mutant (426)
monomorphism (429)
mutation (426)
phenotype (426)
playing the field (427)
polymorphism (429)
primary criterion (457)
secondary criterion (457)
selection (426)

EXERCISES

1. Prove the following statement: "If a strategy is strongly dominated in the payoff table of a game played by rational players, then in the evolutionary version of the same game it will die out no matter what the initial population mix. If a strategy is weakly dominated, it may coexist with some other types but not in a mixture of all types."

2. Consider a large population of players in which two possible phenotypes exist. These players meet and play a survival game in which they either fight over or share a food source. One phenotype always fights, the other always shares; for the purposes of this question, assume that no other mutant types can arise in the population. Suppose that the value of the food source is 200 calories and that caloric intake determines each player's reproductive fitness. If two sharing types meet each other, they each get half the food but, if a sharer meets a fighter, the sharer concedes immediately and the fighter gets all the food.
   (a) Suppose that the cost of a fight is 50 calories (for each fighter) and that, when two fighters meet, each is equally likely (probability $\frac{1}{2}$) to win the fight and the food or to lose and get no food. Draw the payoff table for the game played between two random players from this population. Find all of the ESSs in the population. What type of game is being played in this case?

(b) Now suppose that the cost of a fight is 150 calories for each fighter. Draw the new payoff table and find all of the ESSs for the population in this case. What type of game is being played here?

(c) Using the notation of the Hawk-Dove game of Section 13.6, indicate the values of V and C in parts a and b and confirm that your answers to those parts match the analysis presented in the text.

3. Suppose that a single play of a prisoners' dilemma has the following payoffs:

		PLAYER 2	
		Cooperate	Defect
PLAYER 1	Cooperate	3, 3	1, 4
	Defect	4, 1	2, 2

In a large population in which each member's behavior is genetically determined, each player will be either a defector (that is, always defects in any play of a prisoners' dilemma game) or a tit-for-tat player (in multiple rounds of a prisoners' dilemma, she cooperates on the first play, and on any subsequent play she does whatever her opponent did on the preceding play). Pairs of randomly chosen players from this population will play "sets" of n single plays of this dilemma (where $n \geq 2$). The payoff to each player in one whole set (of n plays) is the sum of her payoffs in the n plays.

Let the population proportion of defectors be p and the proportion of tit-for-tat players be $(1 - p)$. Each member of the population plays sets of dilemmas repeatedly, matched against a new randomly chosen opponent for each new set. A tit-for-tat player always begins each new set by cooperating on its first play.

(a) Show in a two-by-two table the payoffs to a player of each type when, in one set of plays, each player meets an opponent of each of the two types.

(b) Find the fitness (average payoff in one set against a randomly chosen opponent) for a defector.

(c) Find the fitness for a tit-for-tat player.

(d) Use the answers to parts b and c to show that, when $p > (n - 2)/(n - 1)$, the defector type has greater fitness and that, when $p < (n - 2)/(n - 1)$, the tit-for-tat type has greater fitness.

(e) If evolution leads to a gradual increase in the proportion of the fitter type in the population, what are the possible eventual equilibrium outcomes of this process for the population described in this exercise? (That is, what are the possible equilibria, and which are evolutionary stable?) Use a diagram with the fitness graphs to illustrate your answer.

(f) In what sense does greater repetition (larger values of n) facilitate the evolution of cooperation?

4. In Section 7.A, we considered testing for ESS in the thrice-repeated restaurant pricing prisoners' dilemma.
   (a) Explain completely (on the basis of the payoff structure shown in Figure 13.13) why an all-type-A population cannot be invaded by either N- or T-type mutants.
   (b) Explain also why an all-N-type population can be invaded by both type-A and type-T mutants.
   (c) Explain finally why an all-T-type population cannot be invaded by type-A mutants but can be invaded by mutants that are type N.

5. Consider a population in which there are two phenotypes: natural-born cooperators (who do not confess under questioning) and natural-born defectors (who confess readily). If two members of this population are drawn at random, their payoffs in a single play are the same as those of the husband-wife prisoners' dilemma game of Chapter 4, reproduced here:

		COLUMN	
		Confess	Not
ROW	Confess	10 yr, 10 yr	1 yr, 25 yr
	Not	25 yr, 1 yr	3 yr, 3 yr

   (a) Suppose that there are two types of strategies available in the population, as there were in the restaurant dilemma game of Section 13.2. The two strategies are A (always confess) and T (play tit-for-tat, starting with not confessing). Suppose further that each chosen pair of players plays this dilemma twice in succession. Draw the payoff table for the twice-repeated dilemma.
   (b) Find all of the ESSs in this game.
   (c) Now add a third possible strategy, N, which never confesses. Draw the payoff table for the twice-repeated dilemma with three possible strategies and find all of the ESSs of this new version of the game.

6. In the twice-repeated prisoners' dilemma of Exercise 3, suppose that there is a fourth possible type (type S) that also can exist in the population. This type does not confess on the first play and confesses on the second play of each episode of two successive plays against the same opponent.
   (a) Draw the four-by-four fitness table for this game.
   (b) If the population initially has some of each of the four types, show that the types N, T, and S will die out in that order.

(c) Show that, when a strategy (N, T, or S) has died out, as in part **b**, if a small proportion of mutants of that type reappear in the future, they cannot successfully invade the population at that time.
(d) What is the ESS in this game? Why is it different from that found in Exercise 3, part c?

7. Consider the thrice-repeated prisoners' dilemma of Section 7 in which three possible types of restaurateurs participate in a pricing game. Repeat Exercise 4 for this game, showing that all of the same results hold.

8. Recall from Exercise 2 the population of animals fighting over a food source worth 200 calories. Assume, as in part **b** of that exercise, that the cost of a fight is 150 calories per fighter. Assume also that a third phenotype exists in the population. That phenotype is a mixer; it plays a mixed strategy, sometimes fighting and sometimes sharing.
(a) Use your knowledge of how to find mixed strategies in rationally played games to posit a reasonable mixture for the mixer phenotype to use in this game.
(b) Draw the three-by-three payoff table for this game when the mixer type uses the mixed strategy that you found in part **a**.
(c) Determine whether the mixer phenotype is an ESS of this game. (Hint: You must test whether a mixer population can be invaded by either the fighting type or the sharing type.)

9. In the assurance (meeting-place) game in this chapter, the payoffs were meant to describe the value of something material gained by the players in the various outcomes; they could be prizes given for a successful meeting, for example. Then other individual persons in the population might observe the expected payoffs (fitness) of the two types, see which was higher, and gradually imitate the fitter strategy. Thus the proportions of the two types in the population would change. But a more biological interpretation can be made. Suppose the row players are always female and the column players always male. When two of these players meet successfully, they pair off, and their children are of the same type as the parents. Therefore the types would proliferate or die off as a result of meetings. The formal mathematics of this new version of the game makes it a "two-species game" (although the biology of it does not). Thus the proportion of T-type females in the population—call this proportion x—need not equal the proportion of T-type males—call this proportion y.

Examine the dynamics of x and y by using methods similar to those used in the text for the battle-of-the-sexes game. Find the stable outcome or outcomes of this dynamic process.

10. (Optional, for mathematically trained students.) In the three-type evolu-

tionary game of Section 7.B and Figure 13.14, let $q_3 = 1 - q_1 - q_2$ denote the proportion of the orange-throated aggressor types. Then the dynamics of the population proportions of each type of lizard can be stated as

$$q_1 \text{ increases} \quad \text{if and only if} \quad -q_2 + q_3 > 0$$

and

$$q_2 \text{ increases} \quad \text{if and only if} \quad q_1 - q_3 > 0.$$

We did not state this explicitly in the text, but a similar rule for q_3 is

$$q_3 \text{ increases} \quad \text{if and only if} \quad -q_1 + q_2 > 0.$$

(a) Consider the dynamics more explicitly. Let the speed of change in a variable x in time t be denoted by the derivative dx/dt. Then suppose

$$dq_1/dt = -q_2 + q_3, \quad dq_2/dt = q_1 - q_3, \quad \text{and} \quad dq_3/dt = -q_1 + q_2.$$

Verify that these derivatives conform to the preceding statements regarding the population dynamics.

(b) Define $X = (q_1)^2 + (q_2)^2 + (q_3)^2$. Using the chain rule of differentiation, show that $dX/dt = 0$; that is, show that X remains constant over time.

(c) From the definitions of the entities, we know that $q_1 + q_2 + q_3 = 1$. Combining this fact with the result from part **b**, show that over time, in three-dimensional space, the point (q_1, q_2, q_3) moves along a circle.

(d) What does the answer to part c indicate regarding the stability of the evolutionary dynamics in the colored-throated lizard population?

PART FOUR

Applications to Specific Strategic Situations

14

Brinkmanship

The Cuban Missile Crisis

In Chapter 1, we explained that our basic approach was neither pure theory nor pure case study, but a combination in which theoretical ideas were developed by using features of particular cases or examples. Thus we ignored those aspects of each case that were incidental to the concept being developed. However, after you have learned the theoretical ideas, a richer mode of analysis becomes available to you in which factual details of a particular case are more closely integrated with game-theoretic analysis to achieve a fuller understanding of what has happened and why. Such *theory-based case studies* have begun to appear in diverse fields—business, political science, and economic history.[1]

Here we offer an example from political and military history—namely, nuclear brinkmanship in the Cuban missile crisis of 1962. Our choice is motivated by the sheer drama of the episode, the wealth of factual information that has become available, and the applicability of an important concept from game theory. You may think that the risk of nuclear war died with the dissolution of the Soviet Union and that therefore our case is a historical curiosity. But nuclear arms races continue in many parts of the world, and such rivals as India and Pakistan or as Iran and Israel may find use for the lessons

[1]Two excellent examples of theory-based studies are Pankaj Ghemawat, *Games Businesses Play: Cases and Models* (Cambridge: MIT Press, 1997), and Robert H. Bates, Avner Greif, Margaret Levi, Jean-Laurent Rosenthal, and Barry Weingast, *Analytic Narratives* (Princeton: Princeton University Press, 1998).

taken from the Cuban crisis. More importantly for many of you, brinkmanship must be practiced in many everyday situations—for example, strikes and marital disputes. Although the stakes in such games are lower than those in a nuclear confrontation between superpowers, the same principles of strategy apply.

In Chapter 10, we introduced the concept of brinkmanship as a strategic move; here is a quick reminder. A *threat* is a response rule, and the threatened action inflicts a cost on both the player making the threat and the player whose action the threat is intended to influence. However, if the threat succeeds in its purpose, this action is not actually carried out. Therefore there is no apparent upper limit to the cost of the threatened action. But the risk of *errors*—that is, the risk that the threat may fail to achieve its purpose or that the threatened action may occur by accident—forces the strategist to use the minimal threat that achieves its purpose. If a smaller threat is not naturally available, a large threat can be scaled down by making its fulfillment probabilistic. You do something in advance that creates a probability, but not certainty, that the mutually harmful outcome will happen if the opponent defies you. If the need actually arose, you would not take that bad action if you had the full freedom to choose. Therefore you must arrange in advance to let things get out of your control to some extent. *Brinkmanship* is the creation and deployment of such a probabilistic threat; it consists of a deliberate loss of control.

The word *brinkmanship* is often used in connection with nuclear weapons, and the Cuban missile crisis of 1962, when the world came as close to an unaccidental nuclear war as it ever has, is often offered as the classic example of brinkmanship. We will use the crisis as an extended case study to explicate the concept. In the process, we will find that many popular interpretations and analyses of the crisis are simplistic. A deeper analysis reveals brinkmanship to be a subtle and dangerous strategy. It also shows that many conflicts in business and personal interactions—such as strikes and breakups of relationships—are examples of brinkmanship gone wrong. Therefore a clear understanding of the strategy, as well as its limitations and risks, is very important to all game players, which includes just about everyone.

1 A BRIEF NARRATION OF EVENTS

We begin with a brief story of the unfolding of the crisis. Our account draws on several books, including some that were written with the benefit of documents and statements released since the collapse of the Soviet Union.[2] We cannot hope to do justice to the detail, let alone the drama, of the events. President Kennedy said at the time of the crisis: "This is the week when I earn my salary," and much more than a president's salary stood in the balance. We urge you to read the books that tell the story in vivid detail and to talk to any relatives who lived through it to get their firsthand memories.[3]

In late summer and early fall of 1962, the Soviet Union (USSR) started to place medium- and intermediate-range ballistic missiles (MRBMs and IRBMs) in Cuba. The MRBMs had a range of 1,100 miles and could hit Washington, DC; the IRBMs, with a range of 2,200 miles, could hit most of the major U.S. cities and military installations. The missile sites were guarded by the latest Soviet SA-2-type surface-to-air missiles (SAMs), which could shoot down U.S. high-altitude U-2 reconnaissance planes. There were also IL-28 bombers and tactical nuclear weapons called Luna by the Soviets and FROG (free rocket over ground) by the United States, which could be used against invading troops.

This was the first time that the Soviets had ever attempted to place their missiles and nuclear weapons outside Soviet territory. Had they been successful, it would have increased their offensive capability against the United States manyfold. It is now believed that the Soviets had fewer than 20, and perhaps as few as "two or three," operational intercontinental ballistic missiles (ICBMs) in their own country capable of reaching the United States (*War,* 464, 509–510). Their initial placement in Cuba had about 40 MRBMs and IRBMs, which was a substantial increase. But the United States would still have retained vast superiority in the nuclear balance between the superpowers. Also, as the Soviets built up their submarine fleet, the relative importance of land-based missiles near

[2]Our sources include Robert Smith Thompson, *The Missiles of October* (New York: Simon & Schuster, 1992); James G. Blight and David A. Welch, *On the Brink: Americans and Soviets Reexamine the Cuban Missile Crisis* (New York: Hill and Wang, 1989); Richard Reeves, *President Kennedy: Profile of Power* (New York: Simon & Schuster, 1993); Donald Kagan, *On the Origins of War and the Preservation of Peace* (New York: Doubleday, 1995); Aleksandr Fursenko and Timothy Naftali, *One Hell of a Gamble: The Secret History of the Cuban Missile Crisis* (New York: Norton, 1997); and last, latest, and most direct, *The Kennedy Tapes: Inside the White House During the Cuban Missile Crisis,* ed. Ernest R. May and Philip D. Zelikow (Cambridge: Harvard University Press, 1997). Graham T. Allison's *Essence of Decision: Explaining the Cuban Missile Crisis* (Boston: Little Brown, 1971) remains important not only for its narrative, but also for its analysis and interpretation. Our view differs from his in some important respects, but we remain in debt to his insights. We follow and extend the ideas in Avinash Dixit and Barry Nalebuff, *Thinking Strategically* (New York: Norton, 1991), chap. 8.

When we cite these sources to document particular points, we do so in parentheses in the text, in each case using a key word from the title of the book followed by the appropriate page number or page range. The key words have been underlined in the sources given here.

[3]For those of you with no access to first-hand information or those who seek a beginner's introduction to both the details and the drama of the missile crisis, we recommend the film *Thirteen Days* (2000, New Line Cinema).

the United States would have decreased. But the missiles had more than mere direct military value to the Soviets. Successful placement of missiles so close to the United States would have been an immense boost to Soviet prestige throughout the world, especially in Asia and Africa, where the superpowers were competing for political and military influence. Finally, the Soviets had come to think of Cuba as a "poster child" for socialism. The opportunity to deter a feared U.S. invasion of Cuba and to counter Chinese influence in Cuba weighed importantly in the calculations of the Soviet leader and Premier, Nikita Khrushchev. (See *Gamble,* 182–183, for an analysis of Soviet motives.)

U.S. surveillance of Cuba and of shipping lanes during the late summer and early fall of 1962 had indicated some suspicious activity. When questioned about it by U.S. diplomats, the Soviets denied any intentions to place missiles in Cuba. Later, faced with irrefutable evidence, they said that their intention was defensive, to deter the United States from invading Cuba. It is hard to believe this, although we know that an offensive weapon *can* serve as a defensive deterrent threat.

An American U-2 "spy plane" took photographs over western Cuba on Sunday and Monday, October 14 and 15. When developed and interpreted, they showed unmistakable signs of construction on MRBM launching sites. (Evidence of IRBMs was found later, on October 17.) These photographs were shown to President Kennedy the following day (October 16). He immediately convened an ad hoc group of advisers, which later came to be called the Executive Committee of the National Security Council (ExComm), to discuss the alternatives. At the first meeting (on the morning of October 16), he decided to keep the matter totally secret until he was ready to act, mainly because, if the Soviets knew that the Americans knew, they might speed up the installation and deployment of the missiles before the Americans were ready to act, but also because spreading the news without announcing a clear response would create panic in the United States.

Members of ExComm who figured most prominently in the discussions were the Secretary of Defense, Robert McNamara; the National Security Adviser, McGeorge Bundy; the Chairman of the Joint Chiefs of Staff, General Maxwell Taylor; the Secretary of State, Dean Rusk, and Undersecretary George Ball; the Attorney General Robert Kennedy (who was also the President's brother); the Secretary of the Treasury, Douglas Dillon (also the only Republican in the Cabinet); and Llewellyn Thompson, who had recently returned from being U.S. Ambassador in Moscow. During the 2 weeks that followed, they would be joined by or would consult with several others, including the U.S. Ambassador to the United Nations, Adlai Stevenson; the former Secretary of State and a senior statesman of U.S. foreign policy, Dean Acheson; and the Chief of the U.S. Air Force, General Curtis Lemay.

In the rest of that week (October 16 through 21), the ExComm met numerous times. To preserve secrecy, the President continued his normal schedule, including travel to speak for Democratic candidates in the midterm Congressional elections that were to be held in November 1962. He kept in constant touch with ExComm. He dodged press questions abut Cuba and persuaded one or two trusted media owners or editors to preserve the facade of business as usual. ExComm's own attempts to preserve secrecy in Washington sometimes verged on the comic, as when almost a dozen of them had to pile into one limo, because the sight of several government cars going from the White House to the State Department in a convoy could cause speculation in the media.

Different members of ExComm had widely differing assessments of the situation and supported different actions. The military Chiefs of Staff thought that the missile placement changed the balance of military power substantially; Defense Secretary McNamara thought it changed "not at all" but regarded the problem as politically important nonetheless (*Tapes,* 89). President Kennedy pointed out that the first placement, if ignored by the United States, could grow into something much bigger and that the Soviets could use the threat of missiles so close to the United States to try to force the withdrawal of the U.S., British, and French presence in West Berlin. Kennedy was also aware that it was a part of the *geopolitical* struggle between the United States and the Soviet Union (*Tapes,* 92).

It now appears that he was very much on the mark in this assessment. The Soviets planned to expand their presence in Cuba into a major military base (*Tapes,* 677). They expected to complete the missile placement by mid-November. Khrushchev had planned to sign a treaty with Castro in late November, then travel to New York to address the United Nations, and issue an ultimatum for a settlement of the Berlin issue (*Tapes,* 679; *Gamble,* 182), by using the missiles in Cuba as a threat for this purpose. Khrushchev thought Kennedy would accept the missile placement as a *fait accompli.* Khrushchev appears to have made these plans on his own. Some of his top advisers privately thought them too adventurous, but the top governmental decision-making body of the Soviet Union, the Presidium, supported him, although its response was largely a rubber stamp (*Gamble,* 180). Castro was at first reluctant to accept the missiles, fearing that they would trigger a U.S. invasion (*Tapes,* 676–678), but in the end he, too, accepted them, and the prospect gave him great confidence and lent some swagger to his statements about the United States (*Gamble,* 186–187, 229–230).

In all ExComm meetings up to and including the one on the morning of Thursday, October 18, everyone appears to have assumed that the U.S. response would be purely military. The only options that they discussed seriously during this time were (1) an air strike directed exclusively at the missile sites and (probably) the SAM sites nearby, (2) a wider air strike including Soviet and Cuban aircraft parked at airfields, and (3) a full-scale invasion of Cuba. If anything, the attitudes hardened when the evidence of the presence of the longer-range

IRBMs arrived. In fact, at the Thursday meeting, Kennedy discussed a timetable for air strikes to commence that weekend (*Tapes*, 148).

McNamara had first mentioned a blockade toward the end of the meeting on Tuesday, October 16, and developed the idea (in a form uncannily close to the course of action actually taken) in a small group after the formal meeting had ended (*Tapes*, 86, 113). Ball argued that an air strike without warning would be a "Pearl Harbor" and that the United States should not do it (*Tapes*, 115); he was most importantly supported by Robert Kennedy (*Tapes*, 149). The civilian members of ExComm further shifted toward the blockade option when they found that what the military Joint Chiefs of Staff wanted was a massive air strike; the military regarded a limited strike aimed at only the missile sites so dangerous and ineffective that "they would prefer taking no military action than to take that limited strike" (*Tapes*, 97).

Between October 18 and Saturday, October 20, the majority opinion within ExComm gradually coalesced around the idea of starting with a blockade, simultaneously issuing an ultimatum with a short deadline (from 48 to 72 hours was mentioned), and proceeding to military action if necessary after this deadline expired. International law required a declaration of war to set up a blockade, but this problem was ingeniously resolved by proposing to call it a "naval quarantine" of Cuba (*Tapes*, 190–196).

Some people held the same positions throughout these discussions (from October 16 through 21)—for example, the military Chiefs of Staff constantly favored a major air strike—but others shifted their views, at times dramatically. Bundy initially favored doing nothing (*Tapes*, 172) and then switched toward a preemptive surprise air attack (*Tapes*, 189). President Kennedy's own positions also shifted away from an air strike toward a blockade. He wanted the U.S. response to be firm. Although his reasons undoubtedly were mainly military and geopolitical, as a good domestic politician he was also fully aware that a weak response would hurt the Democratic party in the imminent Congressional elections. On the other hand, the responsibility of starting an action that might lead to nuclear war weighed very heavily on him. He was impressed by the CIA's assessment that some of the missiles were already operational, which increased the risk that any air strike or invasion could lead to the Soviets' firing these missiles and to large U.S. civilian casualties (*Gamble*, 235). In the second week of the crisis (October 22 through 28), his decisions seemed to constantly favor the lowest-key options discussed by ExComm.

By the end of the first week's discussions, the choice lay between a blockade and an air strike, two position papers were prepared, and in a straw vote on October 20 the blockade won 11 to 6 (*War*, 516). Kennedy made the decision to start by imposing a blockade and announced it in a television address to the nation on Monday, October 22. He demanded a halt to the shipment of Soviet missiles to Cuba and a prompt withdrawal of those already there.

Kennedy's speech brought the whole drama and tension into the public arena. The United Nations held several dramatic but unproductive debates. Other world leaders and the usual busybodies of international affairs offered advice and mediation.

Between October 23 and October 25, the Soviets at first tried bluster and denial; Khrushchev called the blockade "banditry, a folly of international imperialism" and said that his ships would ignore it. The Soviets, in the United Nations and elsewhere, claimed that their intentions were purely defensive and issued statements of defiance. In secret, they explored ways to end the crisis. This exploration included some direct messages from Khrushchev to Kennedy. It also included some very indirect and lower-level approaches by the Soviets. In fact, as early as Monday, October 22—before Kennedy's TV address—the Soviet Presidium had decided not to let this crisis lead to war. By Thursday, October 25, they had decided that they were willing to withdraw from Cuba in exchange for a promise by the United States not to invade Cuba, but they had also agreed to "look around" for better deals (*Gamble,* 241, 259). The United States did not know any of the Soviet thinking about this.

In public as well as in private communications, the USSR broached the possibility of a deal concerning the withdrawal of U.S. missiles from Turkey and of Soviet ones from Cuba. This possibility had already been discussed by ExComm. The missiles in Turkey were obsolete; so the United States wanted to remove them anyway and replace them with a Polaris submarine stationed in the Mediterranean Sea. But it was thought that the Turks would regard the presence of U.S. missiles as a matter of prestige and so it might be difficult to persuade them to accept the change. (The Turks might also correctly regard missiles, fixed on Turkish soil, as a firmer signal of the U.S. commitment to Turkey's defense than an offshore submarine, which could move away on short notice; see *Tapes,* 568.)

The blockade went into effect on Wednesday, October 24. Despite their public bluster, the Soviets were cautious in testing it. Apparently, they were surprised that the United States had discovered the missiles in Cuba before the whole installation program was completed; Soviet personnel in Cuba had observed the U-2 overflights but had not reported them to Moscow (*Tapes,* 681). The Soviet Presidium ordered the ships carrying the most sensitive materials (actually the IRBM missiles) to stop or turn around. But it also ordered General Issa Pliyev, the commander of the Soviet troops in Cuba, to get his troops combat-ready and to use all means except nuclear weapons to meet any attack (*Tapes,* 682). In fact, the Presidium twice prepared (then canceled without sending) orders authorizing him to use tactical nuclear weapons in the event of a U.S. invasion (*Gamble,* 242–243, 272, 276). The U.S. side saw only that several Soviet ships (which were actually carrying oil and other nonmilitary cargo) continued to sail toward the blockade zone. The U.S. Navy showed some moderation in its enforcement of the blockade. A tanker was allowed to pass without being boarded; another tramp steamer carrying industrial cargo was boarded but allowed to proceed after only a cursory inspection. But tension was mounting, and neither side's actions were as cautious as the top-level politicians on both sides would have liked.

On the morning of Friday, October 26, Khrushchev sent Kennedy a conciliatory private letter offering to withdraw the missiles in exchange for a U.S. promise not to invade Cuba. But later that day he toughened his stance. It seems that he was emboldened by two items of evidence. First, the U.S. Navy was not being excessively aggressive in enforcing the blockade. It had let through some obviously civilian freighters; they boarded only one ship, the *Marucla,* and let it pass after a cursory inspection. Second, some dovish statements had appeared in U.S. newspapers. Most notable among them was an article by the influential and well-connected syndicated columnist Walter Lippman, who suggested the swap whereby the United States would withdraw its missiles in Turkey in exchange for the USSR's withdrawing its missiles in Cuba (*Gamble,* 275). Khrushchev sent another letter to Kennedy on Saturday, October 26, offering this swap, and this time he made the letter public. The new letter was presumably a part of the Presidium's strategy of "looking around" for the best deal. Members of ExComm concluded that the first letter was Khrushchev's own thoughts but that the second was written under pressure from hard-liners in the Presidium—or was even evidence that Khrushchev was no longer in control (*Tapes,* 498, 512–513). In fact, both of Khrushchev's letters were discussed and approved by the Presidium (*Gamble,* 263, 275).

ExComm continued to meet, and opinions within it hardened. One reason was the growing feeling that the blockade by itself would not work. Kennedy's television speech had imposed no firm deadline, and, as we know, in the absence of a deadline a compellent threat is vulnerable to the opponent's procrastination. Kennedy had seen this quite clearly and as early as Monday, October 22: in the morning ExComm meeting preceding his speech, he commented, "I don't think we're gonna be better off if they're just sitting there" (*Tapes,* 216). But a hard, short deadline was presumably thought to be too rigid. By Thursday, others in ExComm were realizing the problem; for example, Bundy said, "A plateau here is the most dangerous thing" (*Tapes,* 423). The hardening of the Soviet position, as shown by the public "Saturday letter" that followed the conciliatory private "Friday letter," was another concern. More ominously, that Friday, U.S. surveillance had discovered that there were tactical nuclear weapons (FROGs) in Cuba (*Tapes,* 475). This discovery showed the Soviet presence there to be vastly greater than thought before, but it also made invasion more dangerous to U.S. troops. Also on Saturday, a U.S. U-2 plane was shot down over Cuba. (It now appears that this was done by the local commander, who interpreted his orders more broadly than Moscow had intended (*War,* 537;

Tapes, 682). In addition, Cuban antiaircraft defenses fired at low-level U.S. reconnaissance planes. The grim mood in ExComm throughout that Saturday was well encapsulated by Dillon: "We haven't got but one more day" (*Tapes,* 534).

On Saturday, plans leading to escalation were being put in place. An air strike was planned for the following Monday, or Tuesday at the latest, and Air Force reserves were called up (*Tapes,* 612–613). Invasion was seen as the inevitable culmination of events (*Tapes,* 537–538). A tough private letter to Khrushchev from President Kennedy was drafted and was handed over by Robert Kennedy to the Soviet Ambassador in Washington, Anatoly Dobrynin. In it, Kennedy made the following offer: (1) The Soviet Union withdraws its missiles and IL-28 bombers from Cuba with adequate verification (and ships no new ones). (2) The United States promises not to invade Cuba. (3) The U.S. missiles in Turkey will be removed after a few months, but this offer is void if the Soviets mention it in public or link it to the Cuban deal. An answer was required within 12 to 24 hours; otherwise "there would be drastic consequences" (*Tapes,* 605–607).

On the morning of Sunday, October 28, just as prayers and sermons for peace were being offered in many churches in the United States, Soviet radio broadcast the text of a letter that Khrushchev was sending to Kennedy, in which he announced that construction of the missile sites was being halted immediately and that the missiles already installed would be dismantled and shipped back to the Soviet Union. Kennedy immediately sent a reply welcoming this decision, which was broadcast to Moscow by the Voice of America radio. It now appears that Khrushchev's decision to back down was made before he received Kennedy's letter through Dobrynin but that the letter only reinforced it (*Tapes,* 689).

That did not quite end the crisis. The U.S. Joint Chiefs of Staff remained skeptical of the Soviets and wanted to go ahead with their air strike (*Tapes,* 635). In fact, the construction activity at the Cuban missile sites continued for a few days. Verification by the United Nations proved problematic. The Soviets tried to make the Turkey part of the deal semipublic. They also tried to keep the IL-28 bombers in Cuba out of the withdrawal. Not until November 20 was the deal finally clinched and the withdrawal begun (*Tapes,* 663–665; *Gamble,* 298–310).

2 A SIMPLE GAME-THEORETIC EXPLANATION

At first sight, the game-theoretic aspect of the crisis looks very simple. The United States wanted the Soviet Union to withdraw its missiles from Cuba; thus the U.S. objective was to achieve compellence. For this purpose, the United States deployed a threat: Soviet failure to comply would eventually lead to a nuclear war between the superpowers. The blockade was a starting point of this inevitable process and an actual action that demonstrated the credibility of U.S. resolve. In other words, Kennedy took Khrushchev to the brink of disaster. This was sufficiently frightening to Khrushchev that he complied. The prospect of nuclear annihilation was equally frightening to Kennedy, but that is in the nature of a threat. All that is needed is that the threat be sufficiently costly to the other side to induce it to act in accordance with our wishes; then we don't have to carry out the bad action anyway.

A somewhat more formal statement of this argument proceeds by drawing a game tree like that shown in Figure 14.1. The Soviets have installed the missiles, and now the United States has the first move. It chooses between doing nothing and issuing a threat. If the United States does nothing, this is a major military and political achievement for the Soviets; so we score the payoffs as −2 for the United States and 2 for the Soviets. If the United States issues its threat, the Soviets get to move, and they can either withdraw or defy. Withdrawal is a humiliation (a substantial minus) for the Soviets and a reaffirmation of U.S. military superiority (a small plus); so we score it 1 for the United States and −4 for the Soviets. If the Soviets defy the U.S. threat, there will be a nuclear war. This outcome is terrible for both, but particularly bad for the United States, which as a democracy cares more for its citizens; so we score this −10 for the United States and −8 for the Soviets. This quantification is very rough guesswork, but the conclusions do not depend on the precise numbers that we have chosen. If you disagree with our choice, you can substitute other numbers you think to be a more accurate representation; as long as the *relative* ranking of the outcomes is the same, you will get the same subgame-perfect equilibrium.

Now we can easily find the subgame-perfect equilibrium. If faced with the U.S. threat, the Soviets get −4 from withdrawal and −8 by defiance; so they prefer to withdraw. Looking ahead to this outcome, the United States reckons on getting 1 if it issues the threat and −2 if it does not; therefore it is optimal for the

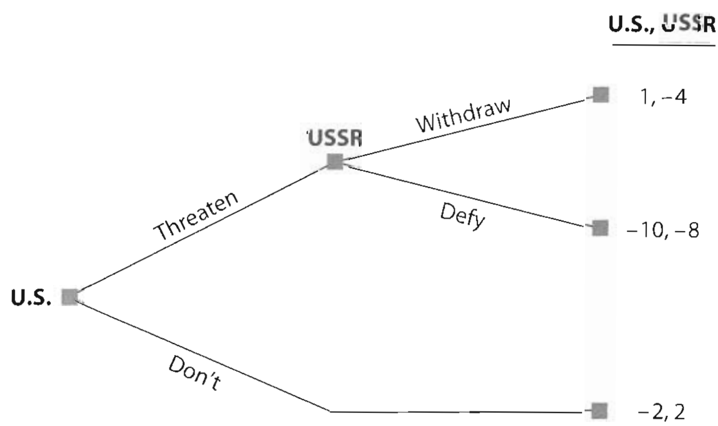


FIGURE 14.1 The Simple-Threat Model of the Crisis

United States to make the threat. The outcome gives payoffs of 1 to the United States and −4 to the Soviets.

But a moment's further thought shows this interpretation to be unsatisfactory. One might start by asking why the Soviets would deploy the missiles in Cuba at all, when they could look ahead to this unfolding of the subsequent game in which they would come out the losers. But, more importantly, several facts about the situation and several events in the course of its unfolding do not fit into this picture of a simple threat.

Before explaining the shortcomings of this analysis and developing a better explanation, however, we digress to an interesting episode in the crisis that sheds light on the requirements of successful compellence. As pointed out in Chapter 10, a compellent threat must have a deadline; otherwise the opponent can nullify it by procrastination. The discussion of the crisis at the U.N. Security Council on Tuesday, October 23, featured a confrontation between U.S. Ambassador Adlai Stevenson and Soviet Ambassador Valerian Zorin. Stevenson asked Zorin point-blank whether the USSR had placed and was placing nuclear missiles in Cuba. "Yes or no—don't wait for the translation—yes or no?" he insisted. Zorin replied: "I am not in an American courtroom. . . . You will have your answer in due course," to which Stevenson retorted, "I am prepared to wait for my answer until hell freezes over." This was dramatic debating; Kennedy watching the session on live television remarked "Terrific. I never knew Adlai had it in him" (*Profile,* 406). But it was terrible strategy. Nothing would have suited the Soviets better than to keep the Americans "waiting for their answer" while they went on completing the missile sites. "Until hell freezes over" is an unsuitable deadline for compellence.

3 ACCOUNTING FOR ADDITIONAL COMPLEXITIES

Let us return to developing a more satisfactory game-theoretic argument. As we pointed out before, the idea that a threat has only a lower limit on its size—namely, that it be large enough to frighten the opponent—is correct only if the threatener can be absolutely sure that everything will go as planned. But almost all games have some element of uncertainty. You cannot know your opponent's value system for sure, and you cannot be completely sure that the players' intended actions will be accurately implemented. Therefore a threat carries a twofold risk. Your opponent may defy it, requiring you to carry out the costly threatened action; or your opponent may comply, but the threatened action may occur by mistake anyway. When such risks exist, the cost of threatened action to oneself becomes an important consideration.

The Cuban missile crisis was replete with such uncertainties. Neither side could be sure of the other's payoffs—that is, of how seriously the other regarded the relative costs of war and of losing prestige in the world. Also, the choices of "blockade" and "air strike" were much more complex than the simple phrases suggest, and there were many weak links and random effects between an order in Washington or Moscow and its implementation in the Atlantic Ocean or in Cuba.

Graham Allison's excellent book, *Essence of Decision,* brings out all of these complexities and uncertainties. They led him to conclude that the Cuban missile crisis cannot be explained in game-theoretic terms. He considers two alternatives: one explanation based on the fact that bureaucracies have their set rules and procedures; another based on the internal politics of U.S. and Soviet governance and military apparatuses. He concludes that the political explanation is best.

We broadly agree but interpret the Cuban missile crisis differently. It is not the case that game theory is inadequate for understanding and explaining the crisis; rather, the crisis was *not a two-person game*—United States versus USSR, or Kennedy versus Khrushchev. Each of these two "sides" was itself a complex coalition of players, with differing objectives, information, actions, and means of communication. The players within each side were engaged in other games, and some members were also directly interacting with their counterparts on the other side. In other words, the crisis can be seen as a complex many-person game with alignments into two broad coalitions. Kennedy and Khrushchev can be regarded as the top-level players in this game, but each was subject to constraints of having to deal with others in his own coalition with divergent views and information, and neither had full control over the actions of these others. We argue that this more subtle game-theoretic perspective is not only a good way to look at the crisis, but also essential in understanding how to practice brinkmanship. We begin with some items of evidence that Allison emphasizes, as well as others that emerge from other writings.

First, there are several indications of divisions of opinion on each side. On the U.S. side, as already noted, there were wide differences within ExComm. In addition, Kennedy found it necessary to consult others such as former President Eisenhower and leading members of Congress. Some of them had very different views; for example, Senator Fulbright said in a private meeting that the blockade "seems to me the worst alternative" (*Tapes,* 271). The media and the political opposition would not give the President unquestioning support for too long either. Kennedy could not have continued on a moderate course if the opinion among his advisers and the public became decisively hawkish.

Individual people also *shifted* positions in the course of the 2 weeks. For example, McNamara was at first quite dovish, arguing that the missiles in Cuba were not a significant increase in the Soviet threat (*Tapes,* 89) and favoring blockade and negotiations (*Tapes,* 191), but ended up more hawkish, arguing that Khrushchev's conciliatory letter of Friday, October 26, was "full of holes" (*Tapes,* 495, 585) and arguing for an invasion (*Tapes,* 537). Most importantly, the U.S. military chiefs always advocated a far more aggressive response. Even after the crisis was over and everyone thought the United States had won a major round in the cold war, Air Force General Curtis Lemay remained dissatisfied and wanted action: "We lost! We ought to just go in there today and knock 'em off," he said (*Essence,* 206; *Profile,* 425).

Even though Khrushchev was the dictator of the Soviet Union, he was not in full control of the situation. Differences of opinion on the Soviet side are less well documented, but, for what it is worth, later memoirists have claimed that Khrushchev made the decision to install the missiles in Cuba almost unilaterally, and, when he informed them, they thought it a reckless gamble (*Tapes,* 674; *Gamble,* 180). There were limits to how far he could count on the Presidium to rubber-stamp his decisions. Indeed, 2 years later, the disastrous Cuban adventure was one of the main charges leveled against Khrushchev when the Presidium dismissed him (*Gamble,* 353–355). It has also been claimed that Khrushchev wanted to defy the U.S. blockade, and only the insistence of First Deputy Premier Anastas Mikoyan led to the cautious response (*War,* 521). Finally, on Saturday, October 27, Castro ordered his antiaircraft forces to fire on all U.S. planes overflying Cuba and refused the Soviet ambassador's request to rescind the order (*War,* 544).

Various parties on the U.S. side had very different information and a very different understanding of the situation, and at times this led to actions that were inconsistent with the intentions of the leadership or even against their explicit orders. The concept of an "air strike" to destroy the missiles is a good example. The nonmilitary people in ExComm thought this would be very narrowly targeted and would not cause significant Cuban or Soviet casualties, but the Air Force intended a much broader attack. Luckily, this difference came out in the open early, leading ExComm to decide against an air strike and the President to turn down an appeal by the Air Force (*Essence,* 123, 209). As for the blockade, the U.S. Navy had set procedures for this action. The political leadership wanted a different and softer process: form the ring closer to Cuba to give the Soviets more time to reconsider, allow the obviously nonmilitary cargo ships to pass unchallenged, and cripple but not sink the ships that defy challenge. Despite McNamara's explicit instructions, however, the Navy mostly followed its standard procedures (*Essence,* 130–132). The U.S. Air Force created even greater dangers. A U-2 plane drifted "accidentally" into Soviet air space and almost caused a serious setback. General Curtis Lemay, acting without the President's knowledge or authorization, ordered the Strategic Air Command's nuclear bombers to fly past their "turnaround" points and some distance toward Soviet air space to positions where they would be detected by Soviet radar. Fortunately, the Soviets responded calmly; Khrushchev merely protested to Kennedy.[4]

There was similar lack of information and communication, as well as weakness of the chain of command and control, on the Soviet side. For example, the construction of the missiles was left to the standard bureaucratic procedures. The Soviets, used to construction of ICBM sites in their own country where they did not face significant risk of air attack, laid out the sites in Cuba in similar way, where they would have been much more vulnerable. At the height of the crisis, when the Soviet SA-2 troops saw an overflying U.S. U-2 plane on Friday, October 26, Pliyev was temporarily away from his desk and his deputy gave the order to shoot it down; this incident created far more risk than Moscow would have wished (*Gamble*, 277–288). And at numerous other points—for example, when the U.S. Navy was trying to get the freighter *Marucla* to stop and be boarded—the people involved might have set off an incident with alarming consequences by taking some action in fear of the immediate situation. Even more dramatically, it was revealed that a Soviet submarine crew, warned to surface when approaching the quarantine line on October 27, did consider firing a nuclear-tipped torpedo that it carried on board (unbeknown to the U.S. Navy). The firing authorization rule required the approval of three officers, only two of whom agreed; the third officer, himself, may have prevented all-out nuclear war.[5]

All these factors made the outcome of any decision by the top-level commander on each side somewhat *unpredictable*. This gave rise to a substantial risk of the "threat going wrong." In fact, Kennedy thought that the chances of the blockade leading to war were "between one out of three and even" (*Essence*, 1).

As we pointed out, such uncertainty can make a simple threat too large to be acceptable to the threatener. We will take one particular form of the uncertainty—namely, U.S. lack of knowledge of the Soviets' true motives—and analyze its effect formally, but similar conclusions hold for all other forms of uncertainty.

Reconsider the game shown in Figure 14.1. Suppose the Soviet payoffs from withdrawal and defiance are the opposite of what they were before: -8 for withdrawal and -4 for defiance. In this alternative scenario, the Soviets are hardliners. They prefer nuclear annihilation to the prospect of a humiliating withdrawal and

[4]Richard Rhodes, *Dark Sun: The Making of the Hydrogen Bomb* (New York: Simon & Schuster, 1995), pp. 573–575. Lemay, renowned for his extreme views and his constant chewing of large unlit cigars, is supposed to be the original inspiration, in the 1963 movie *Dr. Strangelove*, for General Jack D. Ripper, who orders his bomber wing to launch an unprovoked attack on the Soviet Union.

[5]This story became public in a conference held in Havana, Cuba, in October 2002, to mark the 40th anniversary of the missile crisis. See Kevin Sullivan, "40 Years After Missile Crisis, Players Swap Stories in Cuba," *Washington Post*, October 13, 2002, p. A28. Vadim Orlov, who was a member of the Soviet submarine crew, identified the officer who refused to fire the torpedo as Vasili Arkhipov, who died in 1999.

the prospect of living in a world dominated by the capitalist United States; their slogan is "Better dead than red-white-and-blue." We show the game tree for this case in Figure 14.2. Now, if the United States makes the threat, the Soviets defy it. So the United States stands to get −10 from the threat but only −2 if it makes no threat and accepts the presence of the missiles in Cuba. It takes the lesser of the two evils. In the subgame-perfect equilibrium of this version of the game, the Soviets "win" and the U.S. threat does not work.

In reality, when the United States makes it move, it does not know whether the Soviets are hard-liners, as in Figure 14.2, or softer, as in Figure 14.1. The United States can try to estimate the probabilities of the two scenarios, for example, by studying past Soviet actions and reactions in different situations. We can regard Kennedy's statement that the probability of the blockade leading to war was between one-third and one-half this estimate of the probability that the Soviets are hard-line. Because the estimate is imprecise over a range, we work with a general symbol, p, for the probability, and examine the consequences of different values of p.

The tree for this more complex game is shown in Figure 14.3. The game starts with an outside force (here labeled "Nature") determining the Soviets' type. Along the upper branch of Nature's choice, the Soviets are hard-line. This leads to the upper node, where the United States makes its decision whether to issue its threat, and the rest of the tree is exactly like the game in Figure 14.2. Along the lower branch of Nature's choice, the Soviets are soft. This leads to the lower node, where the United States makes its decision whether to issue its threat, and the rest of the tree is exactly like the game in Figure 14.1. But the United States does not know from which node it is making its choice. Therefore the two U.S. nodes are enclosed in an "information set." Its significance is that the United States cannot take different actions at the nodes within the set, such as issuing the threat only if the Soviets are soft. It must take the same action at

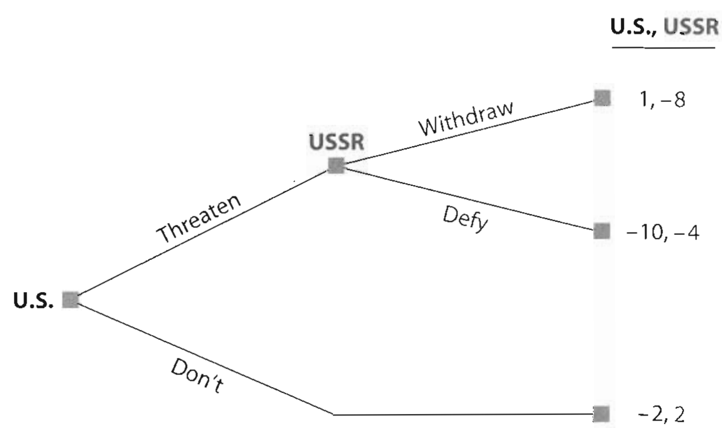


FIGURE 14.2 The Game with Hard-Line Soviets

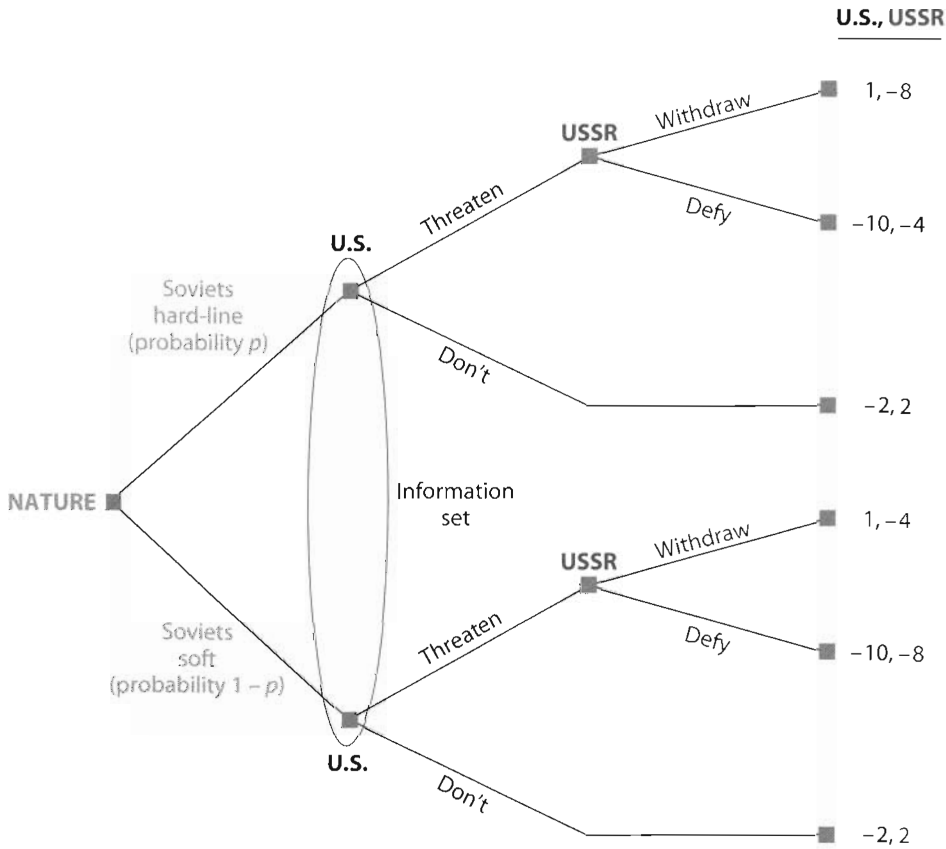


FIGURE 14.3 The Threat with Unknown Soviet Payoffs

both nodes, either threatening at both nodes or not threatening at both. It must make this decision in the light of the probabilities that the game might in truth be "located" at the one node or the other—that is, by calculating the *expected* payoffs of the two actions.

The Soviets themselves know what type they are. So we can do some rollback near the end of the game. Along the upper path, the hard-line Soviets will defy a U.S. threat and, along the lower path, the soft Soviets will withdraw in the face of the threat. Therefore the United States can look ahead and calculate that a threat will get a -10 if the game is actually moving along the upper path (a probability of p), and a 1 if it is moving along the lower path (a probability of $1 - p$). The expected U.S. payoff from making the threat is therefore $-10p + (1 - p) = 1 - 11p$.

If the United States does not make the threat, it gets a -2 along either path; so its expected payoff is also -2. Comparing the expected payoffs of the two actions, we see that the United States should make the threat if $1 - 11p > -2$, or $11p < 3$, or $p < 3/11 = 0.27$.

If the threat were sure to work, the United States would not care how bad its payoff could be if the Soviets defied it, whether -10 or even far more negative. But the risk that the Soviets might be hard-liners and thus defy a threat makes the -10 relevant in the U.S. calculations. Only if the probability, p, of the Soviets' being hard-line is small enough will the United States find it acceptable to make the threat. Thus the upper limit of 3/11 on p is also the upper limit of this U.S. tolerance, given the specific numbers that we have chosen. If we choose different numbers, we will get a different upper limit; for example, if we rate a nuclear war as -100 for the United States, then the upper limit on p will be only 3/101. But the idea of a large threat being "too large to make" if the probability of its going wrong is above a critical limit holds in general.

On this instance, Kennedy's estimate was that p lay somewhere in the range from 1/3 to 1/2. The lower end of this range, 0.33, is unfortunately just above our upper limit 0.27 for the risk that the United States is willing to tolerate. Therefore the simple bald threat "if you defy us, there will be nuclear war" is too large, too risky, and too costly for the United States to make.

4 A PROBABILISTIC THREAT

If an outright threat of war is too large to be tolerable and if you cannot find another, naturally smaller threat, then you can reduce the threat by creating merely a probability rather than a certainty that the dire consequences for the other side will occur if it does not comply. However, this does not mean that you decide after the fact whether to take the drastic action. If you had that freedom, you would shirk from the terrible consequences, your opponents would know or assume this, and so the threat would not be credible in the first place. You must relinquish some freedom of action and make a credible commitment. In this case, you must commit to a probabilistic device.

When making a *simple threat,* one player says to the other player: "If you don't comply, something will *surely* happen that will be very bad for you. By the way, it will also be bad for me, but my threat is credible because of my reputation [or through delegation or other reasons]." With a **probabilistic threat,** one player says to the other, "if you don't comply, there is a *risk* that something very bad for you will happen. By the way, it will also be very bad for me, but later I will be powerless to reduce that risk."

Metaphorically, a probabilistic threat of war is a kind of Russian roulette (an appropriate name in this context). You load a bullet into one chamber of a revolver and spin the barrel. The bullet acts as a "detonator" of the mutually costly war. When you pull the trigger, you do not know whether the chamber in the firing path is loaded. If it is, you may wish you had not pulled the trigger, but by then it will be too late. Before the fact, you would not pull the trigger if you knew that the bullet was in that chamber (that is, if the certainty of the dire action were too costly), but you are willing to pull the trigger knowing that there is only a 1 in 6 chance—in which the threat has been reduced by a factor of 6, to a point where it is now tolerable.

Brinkmanship is the creation and control of a suitable risk of this kind. It requires two apparently inconsistent things. On the one hand, you must let matters get enough out of your control that you will not have full freedom after the fact to refrain from taking the dire action, and so your threat will remain credible. On the other hand, you must retain sufficient control to keep the risk of the action from becoming too large and your threat too costly. Such "controlled lack of control" looks difficult to achieve, and it is. We will consider in Section 5 how the trick can be performed. Just one hint: All the complex differences of judgment, the dispersal of information, and the difficulties of enforcing orders, which made a simple threat too risky, are exactly the forces that make it possible to create a risk of war and therefore make brinkmanship credible. The real difficulty is not how to lose control, but how to do so in a controlled way.

We first focus on the mechanics of brinkmanship. For this purpose, we slightly alter the game of Figure 14.3 to get Figure 14.4. Here, we introduce a different kind of U.S. threat. It consists of choosing and fixing a probability, q, such that, if the Soviets defy the United States, war will occur with that probability. With the remaining probability, $(1 - q)$, the United States will give up and agree to accept the Soviet missiles in Cuba. Remember that, if the game gets to the point where the Soviets defy the United States, the latter does not have a choice in the matter. The Russian-roulette revolver has been set for the probability, q, and chance determines whether the firing pin hits a loaded chamber (that is, whether nuclear war actually happens).

Thus nobody knows the precise outcome and payoffs that will result if the Soviets defy this brinkmanship threat, but they know the probability, q, and can calculate expected values. For the United States, the outcome is -10 with the probability q and -2 with the probability $(1 - q)$; so the expected value is

$$-10q - 2(1 - q) = -2 - 8q.$$

For the Soviets, the expected payoff depends on whether they are hard-line or soft (and only they know their own type). If hard-line, they get a -4 from war, which happens with probability q, and a 2 if the United States gives up, which happens with the probability $(1 - q)$. The Soviets' expected payoff is $-4q + 2(1 - q) = 2 - 6q$. If they were to withdraw, they would get a -8, which is clearly worse no matter what value q takes between 0 and 1. Thus the hard-line Soviets will defy the brinkmanship threat.

The calculation is different if the Soviets are soft. Reasoning as before, we see that they get the expected payoff $-8q + 2(1 - q) = 2 - 10q$ from defiance, and

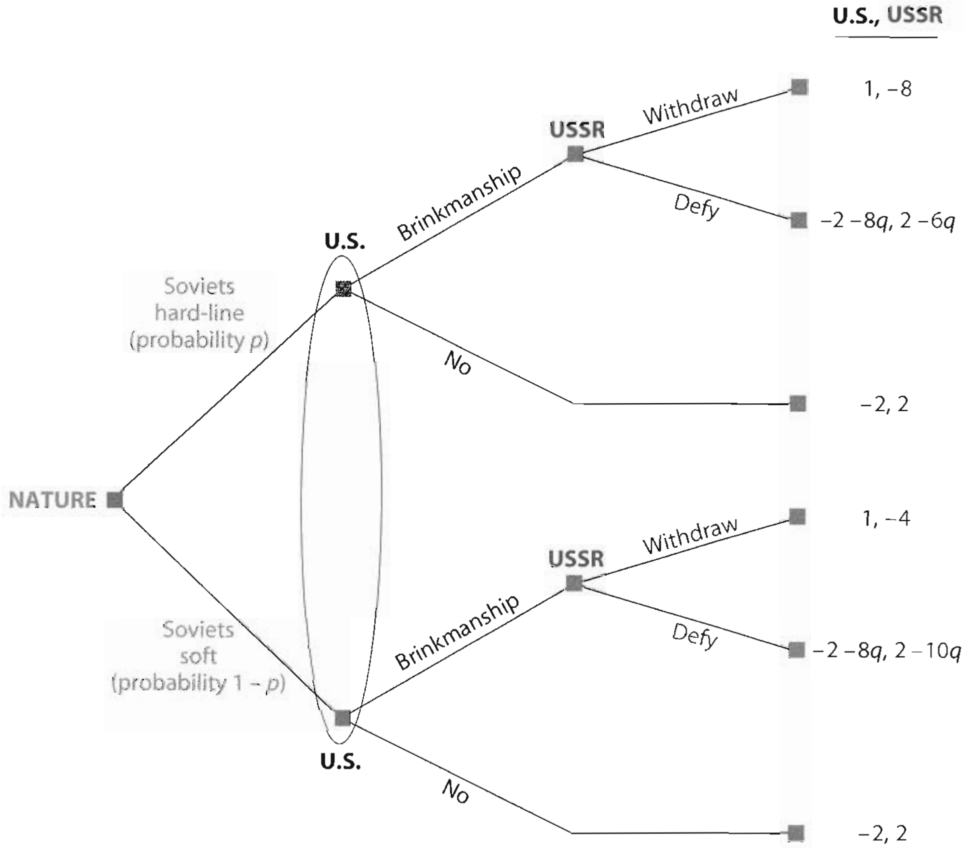


FIGURE 14.4 The Brinkmanship Model of the Crisis

the sure payoff -4 if they withdraw. For them, withdrawal is better if $-4 > 2 - 10q$, or $10q > 6$, or $q > 0.6$. Thus U.S. brinkmanship must contain at least a 60% probability of war; otherwise it will not deter the Soviets, even if they are the soft type. We call this lower bound on the probability q the **effectiveness condition.**

Observe how the expected payoffs for U.S. brinkmanship and Soviet defiance shown in Figure 14.4 relate to the simple-threat model of Figure 14.3; the latter can now be thought of as a special case of the general brinkmanship-threat model of Figure 14.4, corresponding to the extreme value $q = 1$.

We can solve the game shown in Figure 14.4 in the usual way. We have already seen that along the upper path the Soviets, being hard-line, will defy the United States and that along the lower path the soft Soviets will comply with U.S. demands if the effectiveness condition is satisfied. If this condition is not satisfied, then both types of Soviets will defy the United States; so the latter would do better never to make this threat at all. So let us proceed by assuming that the soft Soviets will comply; we look at the U.S. choices. Basically, how risky can the U.S. threat be and still remain tolerable to the United States?

If the United States makes the threat, it runs the risk, p, that it will encounter the hard-line Soviets, who will defy the threat. Then the expected U.S. payoff will be $(-2 - 8q)$, as calculated before. The probability is $(1 - p)$ that the United States will encounter the soft-type Soviets. We are assuming that they comply; then the United States gets a 1. Therefore the expected payoff to the United States from the probabilistic threat, assuming that it is effective against the soft-type Soviets, is

$$(-2 - 8q) \times p + 1 \times (1 - p) = -8pq - 3p + 1.$$

If the United States refrains from making a threat, it gets a -2. Therefore the condition for the United States to make the threat is

$$-8pq - 3p + 1 > -2$$

$$\text{or} \quad q < \frac{3}{8}\frac{1-p}{p} = 0.375(1 - p)/p.$$

That is, the probability of war must be small enough to satisfy this expression or the United States will not make the threat at all. We call this upper bound on q the **acceptability condition.** Note that p enters the formula for the maximum value of q that will be acceptable to the United States; the larger the chance that the Soviets will not give in, the smaller the risk of mutual disaster that the United States finds acceptable.

If the probabilistic threat is to work, it should satisfy both the effectiveness condition and the acceptability condition. We can determine the appropriate level of the probability of war by using Figure 14.5. The horizontal axis is the probability, p, that the Soviets are hard-line, and the vertical axis is the probability, q, that war will occur if they defy the U.S. threat. The horizontal line $q = 0.6$ gives the lower limit of the effectiveness condition; the threat should be such that its associated (p, q) combination is above this line if it is to work even against the soft-type Soviets. The curve $q = 0.375(1 - p)/p$ gives the upper limit of the acceptability condition; the threat should be such that (p, q) is below this curve if it is to be tolerable to the United States even with the assumption that it works against the soft-type Soviets. Therefore an effective and acceptable threat should fall somewhere between these two lines, above and to the left of their point of intersection, at $p = 0.38$ and $q = 0.6$ (shown as a gray "wedge" in Figure 14.5).

The curve reaches $q = 1$ when $p = 0.27$. For values of p less than this value, the dire threat (certainty of war) is acceptable to the United States and is effective against the soft-type Soviets. This just confirms our analysis in Section 3.

For values of p in the range from 0.27 to 0.38, the dire threat with $q = 1$ puts (p, q) to the right of the acceptability condition and is too large to be tolerable to the United States. But a scaled-down threat can be found. For this range of values of p, there are some values of q low enough to be acceptable to the United States and yet high enough to compel the soft-type Soviets. Brinkmanship

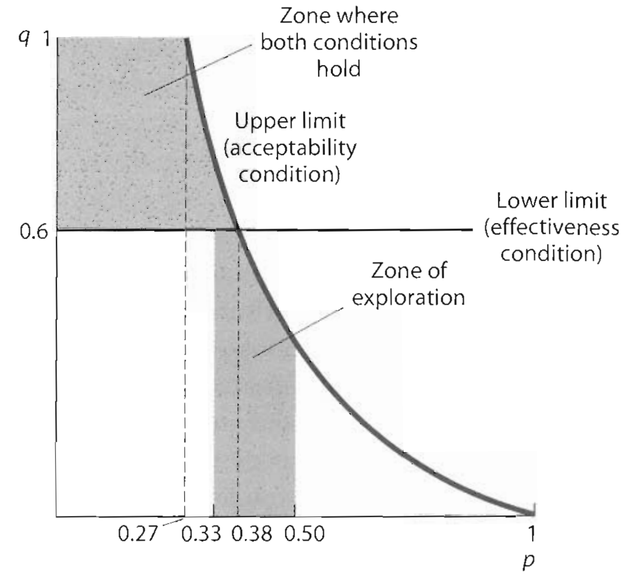


FIGURE 14.5 Conditions of Successful Brinkmanship

(using a probabilistic threat) can do the job in this situation, whereas a simple dire threat would be too risky.

If p exceeds 0.38, then there is no value of q that satisfies both conditions. If the probability that the Soviets will never give in is greater than 0.38, then any threat large enough to work against the soft-type Soviets ($q \geq 0.6$) creates a risk of war too large to be acceptable to the United States. If $p \geq 0.38$, therefore, the United States cannot help itself by using the brinkmanship strategy.

5 PRACTICING BRINKMANSHIP

If Kennedy has a very good estimate of the probability, p, of the Soviets being hard-liners, and is very confident about his ability to control the risk, q, that the blockade will lead to nuclear war, then he can calculate and implement his best strategy. As we saw in Section 3, if $p < 0.27$, the dire threat of a certainty of war is acceptable to Kennedy. (Even then he will prefer to use the smallest effective threat—namely, $q = 0.6$.) If p is between 0.27 and 0.38, then he has to use brinkmanship. Such a threat has to have the risk of disaster $0.6 < q < 0.375 (1 - p)/p$, and again Kennedy prefers the smallest of this range—namely, $q = 0.6$. If $p > 0.38$, then he should give in.

In practice Kennedy does not know p precisely; he only estimates that it lies within the range from 1/3 to 1/2. Similarly, he cannot be confident about the exact location of the critical value of q in the acceptability condition. That depends on the numbers used for the Soviet payoffs in various outcomes—for example, -8 (for war) versus -4 (for compliance)—and Kennedy can only estimate these values. Finally, he may not even be able to control the risk created by his brinkmanship action very precisely. All these ambiguities make it necessary to proceed cautiously.

Suppose Kennedy thinks that $p = 0.35$ and issues a threat backed by an action that carries the risk $q = 0.65$. The risk is greater than what is needed to be effective—namely, 0.6. The limit of acceptability is $0.375 \times (1 - 0.35)/0.35 = 0.7$, and the risk $q = 0.65$ is less than this limit. Thus, according to Kennedy's calculations, the risk satisfies both of the conditions—effectiveness and acceptability. However, suppose Kennedy is mistaken. For example, if he has not realized that Lemay might actually defy orders and take an excessively aggressive action, then q may in reality be higher than Kennedy thinks it is; for example, q may equal 0.8, which Kennedy would regard as too risky. Or suppose p is actually 0.4; then Kennedy would regard even $q = 0.65$ as too risky. Or, Kennedy's experts may have misestimated the values of the Soviet payoffs. If they rate the humiliation of withdrawal as -5 instead of -4, then the threshold of the effectiveness condition will actually be $q = 0.7$, and Kennedy's threat with $q = 0.65$ will go wrong.

All that Kennedy knows is that the general shape of the effectiveness and acceptability conditions is like that shown in Figure 14.5. He does not know p for sure. Therefore he does not know exactly what value of q to choose to fulfill both the effectiveness and the acceptability conditions; indeed, he does not even know if such a range exists for the unknown true value of p: it might be greater than or less than the borderline value of 0.38 that divides the two cases. And he is not able to fix q very precisely; therefore, even if he knew p, he would not be able to act confident of his willingness to tolerate the resulting risk.

With such hazy information, imprecise control, and large risks, what is Kennedy to do? He has to *explore* the boundaries of the Soviets' risk tolerance as well as his own. It would not do to start the exploration with a value of q that might turn out to be too high. Instead, Kennedy must explore the boundaries "from below"; he must start with something quite safe and gradually increase the level of risk to see "who blinks first." That is exactly how brinkmanship is practiced in reality.

We explain this with the aid of Figure 14.5. Observe the red-shaded area. Its left and right boundaries, $p = 1/3$ and $p = 1/2$, correspond to the limits of Kennedy's estimated range of p. The lower boundary is the horizontal axis ($q = 0$). The upper boundary is composed of two segments. For $p < 0.38$, this segment corresponds to the effectiveness condition; for $p > 0.38$, it corresponds to the acceptability condition. Remember that Kennedy does not know the precise positions of these boundaries but must grope toward them from below. Therefore the red-shaded region is where he must start the process.

Suppose Kennedy starts with a very safe action—say, q equaling approximately 0.01 (1%). In our context of the Cuban missile crisis, we can think of this as his television speech, which announced that a quarantine would soon go into effect. At this juncture, the point with coordinates (p, q) lies somewhere near the bottom edge of the shaded region. Kennedy does not know exactly where, because he does not know p for sure. But the overwhelming likelihood is that at this point the threat is quite safe but also ineffective. Therefore Kennedy escalates it a little bit. That is, he moves the point (p, q) in a vertically upward direction from wherever it was initially. This could be the actual start of the quarantine. If that proves to be still safe but ineffective, he jacks up the risk one more notch. This could be the leaking of information about bombing plans.

As he proceeds in this way, eventually his exploration will encounter one of the boundaries of the red-shaded area in Figure 14.5, and which boundary this is depends on the value of p. One of two things comes to pass. Either the threat becomes serious enough to deter the Soviets; this happens if the true value of p is less than its true critical value, here 0.38. On the diagram, we see this as a movement out of the red-shaded area and into the area in which the threat is both acceptable *and* effective. Then the Soviets concede and Kennedy has won. Or, the threat becomes too risky for the United States; this happens if $p > 0.38$. Kennedy's exploration in this case pushes him above the acceptability condition. Then Kennedy decides to concede, and Khrushchev has won. Again we point out that, because Kennedy is not sure of the true value of p, he does not know in advance which of these two outcomes will prevail. As he gradually escalates the risk, he may get some clues from Soviet behavior that enable him to make his estimate of p somewhat more precise. Eventually he will reach sufficient precision to know which part of the boundary he is headed toward and therefore whether the Soviets will concede or the United States must be the player to do so.

Actually, there are two possible outcomes only so long as the ever-present and steadily increasing mutual risk of disaster does not come to pass as Kennedy is groping through the range of ever more risky military options. Therefore there is a third possibility—namely, that the explosion occurs before either side recognizes that it has reached its limit of tolerance of risk and climbs down. This continuing and rising risk of a very bad outcome is what makes brinkmanship such a delicate and dangerous strategy.

Thus brinkmanship in practice is the **gradual escalation of the risk of mutual harm.** It can be visualized vividly as **chicken in real time.** In our analysis of chicken in Chapter 4, we gave each player a simple binary choice: either go straight or swerve. In reality, the choice is usually one of timing. The two cars are rushing toward each other, and either player can choose to swerve at any time. When the cars are very far apart, swerving ensures safety. As they get closer together, they face an ever-increasing risk that they will collide anyway, that even swerving will not avoid a collision. As the two players continue to drive toward each other, each is exploring the limit of the other's willingness to take this risk and is perhaps at the same time exploring his own limit. The one who hits that limit first swerves. But there is always the risk that they have left it long enough and are close enough that, even after choosing Swerve, they can no longer avoid the collision.

Now we see why, in the Cuban missile crisis, the very features that make it inaccurate to regard it as a two-person game make it easier to practice such brinkmanship. The blockade was a relatively small action, unlikely to start a nuclear war at once. But, once Kennedy set the blockade in motion, its operation, escalation, and other features were not totally under his control. So Kennedy was not saying to Khrushchev, "If you defy me (cross a sharp brink), I will coolly and deliberately launch a nuclear war that will destroy both out peoples." Rather, he was saying, "The wheels of the blockade have started to turn and are gathering their own momentum. The more or longer you defy me, the more likely it is that some operating procedure will slip up, the political pressure on me will rise to a point where I must give in, or some hawk will run amok. If this risk comes to pass, I will be unable to prevent nuclear war, no matter how much I may regret it at that point. Only you can now defuse the tension by complying with my demand to withdraw the missiles."

We believe that this perspective gives a much better and deeper understanding of the crisis than can most analyses based on simple threats. It tells us why the *risk* of war played such an important role in all discussions. It even makes Allison's compelling arguments about bureaucratic procedures and internal divisions on both sides an integral part of the picture: these features allow the top-level players on both sides to credibly lose some control—that is, to practice brinkmanship.

One important condition remains to be discussed. In Chapter 10, we saw that every threat has an associated implicit promise—namely, that the bad consequence will not take place if your opponent complies with your wishes. The same is required for brinkmanship. If, as you are increasing the level of risk, your opponent does comply, you must be able to "go into reverse"—begin reducing the risk immediately and quite quickly remove it from the picture. Otherwise the opponent would not gain anything by compliance. This may have been a problem in the Cuban missile crisis. If the Soviets feared that Kennedy could not control hawks such as Lemay ("We ought to just go in there today and knock 'em off"), they would gain nothing by giving in.

To reemphasize and sum up, brinkmanship is the strategy of exposing your rival and yourself to a gradually increasing risk of mutual harm. The actual occurrence of the harmful outcome is not totally within the threatener's control.

Viewed in this way, brinkmanship is everywhere. In most confrontations—for example, between a company and a labor union, a husband and a wife, a parent and a child, and the President and Congress—one player cannot be sure of the other party's objectives and capabilities. Therefore most threats carry a risk of error, and every threat must contain an element of brinkmanship. We hope that we have given you some understanding of this strategy and that we have impressed on you the risks that it carries. You will have to face up to brinkmanship or to conduct it yourself on many occasions in your personal and professional lives. Please do so carefully, with a clear understanding of its potentialities and risks.

To help you do so, we now recapitulate the important lessons learned from the handling of the Cuban missile crisis, reinterpreted as a labor union leadership contemplating a strike in pursuit of its wage demand, unsure whether this will result in the whole firm shutting down:

1. Start small and safe. Your first step should not be an immediate walkout; it should be to schedule a membership meeting at a date a few days or weeks hence, while negotiations continue.
2. Raise the risks gradually. Your public and private statements, as well as the stirring up of the sentiments of the membership, should induce management to believe that acceptance of its current low-wage offer is becoming less and less likely. If possible, stage small incidents—for example, a few one-day strikes or local walkouts.
3. As this process continues, read and interpret signals in management's actions to figure out whether the firm has enough profit potential to afford the union's high-wage demand.
4. Retain enough control over the situation; that is, retain the power to induce your membership to ratify the agreement that you will reach with management; otherwise management will think that the risk will not deescalate even if it concedes your demands.

SUMMARY

In some game situations, the risk of error in the presence of a threat may call for the use of as small a threat as possible. When a large threat cannot be reduced in other ways, it can be scaled down by making its fulfillment probabilistic. Strategic use of *probabilistic threat,* in which you expose your rival and yourself to an increasing risk of harm, is called brinkmanship.

Brinkmanship requires a player to relinquish control over the outcome of the game without completely losing control. You must create a threat with a risk level that is both large enough to be effective in compelling or deterring your rival and small enough to be acceptable to you. To do so, you must determine the levels of risk tolerance of both players through a *gradual escalation of the risk of mutual harm.*

The Cuban missile crisis of 1962 serves as a case study in the use of brinkmanship on the part of President Kennedy. Analyzing the crisis as an example of a simple threat, with the U.S. blockade of Cuba establishing credibility, is inadequate. A better analysis accounts for the many complexities and uncertainties inherent in the situation and the likelihood that a simple threat was too risky. Because the actual crisis included numerous political and military players, Kennedy was able to achieve "controlled loss of control" by ordering the blockade and gradually letting incidents and tension escalate, until Khrushchev yielded in the face of the rising risk of nuclear war.

KEY TERMS

acceptability condition (490)
chicken in real time (493)
effectiveness condition (489)
gradual escalation of the risk of mutual harm (493)
probabilistic threat (487)

EXERCISES

1. In the text, we argue that the payoff to the United States is -10 when (either type) Soviets defy the U.S. threat; these payoffs are illustrated in Figure 14.3. Suppose now that this payoff is in fact -12 rather than -10.
   (a) Incorporate this change in payoff into a game tree similar to the one in Figure 14.4.
   (b) Using the payoffs from your game tree in part **a**, find the effectiveness condition for this version of the U.S.–USSR brinkmanship game.
   (c) Using the payoffs from part **a**, find the acceptability condition for this game.
   (d) Draw a diagram similar to that in Figure 14.5, illustrating the effectiveness and acceptability conditions found in parts **b** and **c**.
   (e) For what values of p, the probability that the Soviets are hard-line, is the pure threat ($q = 1$) acceptable? For what values of p is the pure threat unacceptable but brinkmanship still possible?
   (f) If Kennedy were correct in believing that p lay between 1/3 and 1/2, does your analysis of this version of the game suggest than an effective *and* acceptable probabilistic threat existed? Use this example to explain how a game theorist's assumptions about player payoffs can have a major effect on the predictions that arise from the theoretical model.

2. Consider a game between a union and the company that employs the union membership. The union can threaten to strike (or not) to get the company to meet its wage and benefits demands. The company, when faced with a threatened strike, can choose to concede to the demands of the union or to defy its threat of a strike. The union, however, does not know the company's profit position when it decides whether to make its threat; it does not know whether the company is sufficiently profitable to meet its demands (and the company's assertions in this matter cannot be believed). Nature determines whether the company is profitable; the probability that the firm is unprofitable is p.

   The payoff structure is as follows: (i) When the union makes no threat, the union gets a payoff of 0 (regardless of the profitability of the company) and the company gets a payoff of 100 if it is profitable but a payoff of 10 if it is unprofitable. A passive union leaves more profit for the company if there is any profit to be made. (ii) When the union threatens to strike and the company concedes, the union gets 50 (regardless of the profitability of the company) and the company gets 50 if it is profitable but -40 if it is not. (iii) When the union threatens to strike and the company defies the union's threat, the union must strike and gets -100 (regardless of the profitability of the company) and the company gets -100 if it is profitable and -10 if it is not. Defiance is very costly for a profitable company but not so costly for an unprofitable one.

   (a) What happens when the union uses the pure threat to strike unless the company concedes to the union's demands?
   (b) Suppose that the union sets up a situation in which there is some risk, with probability $q < 1$, that it will strike after its threat is defied by the company. This risk may arise from the union leadership's imperfect ability to keep the membership in line. Draw a game tree similar to Figure 14.4. for this game.
   (c) What happens when the union uses brinkmanship, threatening to strike with some unknown probability q unless the company concedes to its demands?
   (d) Derive the effectiveness and acceptability conditions for this game, and determine the values for p and q for which the union can use a pure threat, brinkmanship, or no threat at all.

3. Scenes from many movies illustrate the concept of brinkmanship. Analyze the following descriptions from this perspective. What were the risks the two sides faced? How did they increase in the course of execution of the brinkmanship threat?
   (a) In a scene from the 1980 film *The Gods Must Be Crazy,* a rebel who had been captured in an attempt to assassinate the president of an African

country is being interrogated. He stands blindfolded with his back to the open door of a helicopter. Above the noise of the helicopter rotors, an officer asks him "Who is your leader? Where is your hideout?" The man does not answer, and the officer pushes him out of the door. In the next scene, we see that the helicopter, although its engine is running, is actually on the ground, and the man has fallen 6 feet on his back. The officer appears at the door and says, laughing, "Next time it will be a little higher."

(b) In the 1998 film *A Simple Plan* two brothers remove some of a $4.4 million ransom payment that they find in a crashed airplane. After many intriguing twists of fate, the remaining looter, Hank, finds himself in conference with an FBI agent. The agent, who suspects but cannot prove that Hank has some of the missing money, fills Hank in on the story of the money's origins and tells him that the FBI possesses the serial numbers of about 1 of every 10 of the bills in that original ransom payment. The agent's final words to Hank are, "Now it's simply a matter of waiting for the numbers to turn up. You can't go around passing $100 bills without eventually sticking in someone's memory."

(c) In the 1941 movie classic *The Maltese Falcon,* the hero Sam Spade (Humphrey Bogart) is the only person who knows the location of the immensely valuable gem-studded falcon figure, and the villain Caspar Gutman (Sydney Greenstreet) is threatening to torture him for that information. Spade points out that torture is useless unless the threat of death lies behind it, and Gutman cannot afford to kill Spade, because then the information dies with him; therefore he may as well not bother with the threat of torture. Gutman replies: "That is an attitude, sir, that calls for the most delicate judgment on both sides, because, as you know, sir, men are likely to forget in the heat of action where their best interests lie and let their emotions carry them away."

4. In this exercise, we provide several examples of the successful use of brinkmanship, where "success" is indicative of a mutually acceptable deal being reached by the two sides. For each example, (i) identify the interests of the parties; (ii) describe the nature of the uncertainty inherent in the situation; (iii) give the strategies the parties used to escalate the risk of disaster; (iv) discuss whether the strategies were good ones; and (v) [**Optional**] if you can, set up a small mathematical model of the kind presented in the text of this chapter. In each case, we provide a few readings to get you started; you should locate more by using the resources of your library and resources on the World Wide Web such as Lexis-Nexis.

(a) The Uruguay Round of international trade negotiations that started in 1986 and led to the formation of the World Trade Organization in 1994.

Reading: John H. Jackson, *The World Trading System,* 2nd ed. (Cambridge: MIT Press,1997), pp. 44–49 and chaps. 12 and 13.

(b) The Camp David accords between Israel and Egypt in 1978. *Reading:* William B. Quandt, *Camp David: Peacemaking and Politics* (Washington, DC: Brookings Institution, 1986).

(c) The negotiations between the South African apartheid regime and the African National Congress to establish a new constitution with majority rule, 1989 to 1994. *Reading:* Allister Sparks, *Tomorrow Is Another Country* (New York: Hill and Wang, 1995).

5. In this exercise, we provide several examples of the unsuccessful use of brinkmanship. In this case, brinkmanship is considered "unsuccessful" when the mutually bad outcome (disaster) occurred. Answer the questions outlined in Exercise 4 for the following situations:

(a) The confrontation between the regime and the student prodemocracy demonstrators in Beijing, 1989. *Readings: Massacre in Beijing: China's Struggle for Democracy,* ed. Donald Morrison (New York: Time Magazine Publications, 1989); *China's Search for Democracy: The Student and Mass Movement of 1989,* ed. Suzanne Ogden, Kathleen Hartford, L. Sullivan, and D. Zweig (Armonk, NY: M. E. Sharpe, 1992).

(b) The U.S. budget confrontation between President Clinton and the Republican-controlled Congress in 1995. *Readings:* Sheldon Wolin, "Democracy and Counterrevolution," *Nation,* April 22, 1996; David Bowermaster, "Meet the Mavericks," *U.S. News and World Report,* December 25, 1995/January 1, 1996; "A Flight that Never Seems to End," *Economist,* December 16, 1995.

(c) The Caterpillar strike, from 1991 to 1998. *Readings:* "The Caterpillar Strike: Not Over Till It's Over," *Economist,* February 28, 1998; "Caterpillar's Comeback," *Economist,* June 20, 1998; Aaron Bernstein, "Why Workers Still Hold a Weak Hand," *BusinessWeek,* March 2, 1998.

15

Strategy and Voting

VOTING IN ELECTIONS may seem outside the purview of game theory at first glance, but no one who remembers the 2000 U.S. presidential election should be surprised to discover that there are many strategic aspects to elections and voting. The presence in that election of third-party candidate Ralph Nader had a number of strategic implications for the election outcome. A group called "Citizens for Strategic Voting" took out paid advertisements in major newspapers to explain voting strategy to the general public. And analysts have suggested that Nader's candidacy may have been the element that swung the final tally in George W. Bush's favor.

It turns out that there is a lot more to participating in an election than deciding what you like and voting accordingly. Because there are many different types of voting procedures that can be used to determine a winner in an election, there can be many different election outcomes. One particular procedure might lead to an outcome that you prefer, in which case you might use your strategic abilities to see that this procedure is used. Or you might find that the procedure used for a specific election is open to manipulation; voters might be able to alter the outcome of an election by misrepresenting their preferences. If other voters behave strategically in this way, then it may be in your best interests to do so as well. An understanding of the strategic aspects of voting can help you determine your best course of action in such situations.

In the following sections, we look first at the various available voting rules and procedures and then at some nonintuitive or paradoxical results that can arise in certain situations. We consider some of the criteria used to judge the performance of the various voting methods. We also address strategic behavior of voters and the scope for outcome manipulation. Finally, we present two different versions of a well-known result known as the *median voter theorem*—as a two-person zero-sum game with discrete strategies and with continuous ones.

1 VOTING RULES AND PROCEDURES

Numerous election procedures are available to help choose from a slate of alternatives (that is, candidates or issues). With as few as three available alternatives, election design becomes interestingly complex. We describe in this section a variety of procedures from three broad classes of voting, or vote aggregation, methods. The number of possible voting procedures is enormous, and the simple taxonomy that we provide here can be broadened extensively by allowing elections based on a combination of procedures; a considerable literature in both economics and political science deals with just this topic. We have not attempted to provide an exhaustive survey but rather to give a flavor of that literature. If you are interested, we suggest you consult the broader literature for more details on the subject.[1]

A. Binary Methods

Vote aggregation methods can be classified according to the number of options or candidates considered by the voters at any given time. **Binary methods** require voters to choose between only two alternatives at a time. In elections in which there are exactly two candidates, votes can be aggregated by using the well-known principle of **majority rule,** which simply requires that the alternative with a majority of votes wins. When dealing with a slate of more than two alternatives, **pairwise voting**—a method consisting of a repetition of binary votes—can be used. Pairwise procedures are **multistage;** they entail voting on pairs of alternatives in a series of majority votes to determine which is most preferred.

One pairwise procedure, in which each alternative is put up against each of the others in a round-robin of majority votes, is called the **Condorcet method,** after the 18th-century French theorist Jean Antoine Nicholas Caritat, Marquis de Condorcet. He suggested that the candidate who defeats each of the others

[1]The classic textbook on this subject, which was instrumental in making game theory popular in political science, is William Riker, *Liberalism Against Populism* (San Francisco: W. H. Freeman, 1982). A general survey is the symposium on "Economics of Voting," *Journal of Economic Perspectives*, vol. 9, no. 1 (Winter 1995). An important early research contribution is Michael Dummett, *Voting Procedures* (Oxford: Clarendon Press, 1984). Donald Saari, *Chaotic Elections* (Providence, RI: American Mathematical Society, 2000), develops some new ideas that we use later in this chapter.

in such a series of one-on-one contests should win the entire election; such a candidate, or alternative, is now termed a **Condorcet winner.** Other pairwise procedures produce "scores" such as the **Copeland index,** which measures an alternative's win–loss record in a round-robin of contests. The first round of the World Cup soccer tournament uses a type of Copeland index to determine which teams from each group move on to the second round of play.[2]

Another well-known pairwise procedure, used when there are three possible alternatives, is the **amendment procedure,** required by the parliamentary rules of the U.S. Congress when legislation is brought to a vote. When a bill is brought before Congress, any amended version of the bill must first win a vote against the original version of the bill. The winner of that vote is then paired against the status quo and members vote on whether to adopt the version of the bill that won the first round; majority rule can then be used to determine the winner. The amendment procedure can be used to consider any three alternatives by pairing two in a first-round election and then putting the third up against the winner in a second-round vote.

B. Plurative Methods

Plurative methods allow voters to consider three or more alternatives simultaneously. One group of plurative voting methods uses information on the positions of alternatives on a voter's ballot to assign points used when tallying ballots; these voting methods are known as **positional methods.** The familiar **plurality rule** is a special-case positional method in which each voter casts a single vote for her most-preferred alternative. That alternative is assigned a single point when votes are tallied; the alternative with the most votes (or points) wins. Note that a plurality winner need *not* gain a majority, or 51%, of the vote. Thus, for instance, in the 1994 Maine gubernatorial election, independent candidate Angus King captured the governorship with only 36% of the vote; his Democratic, Republican, and Green Party opponents gained 34%, 23%, and 6% of the vote, respectively. Another special-case positional method, the **antiplurality method,** asks voters to vote against one of the available alternatives or, equivalently, to vote for all but one. For counting purposes, the alternative voted against is allocated -1 point or else all alternatives except that one receive 1 point while the alternative voted against receives 0.

One of the most well-known positional methods is the **Borda count,** named after Jean-Charles de Borda, a fellow countryman and contemporary of Condorcet. Borda described the new procedure as an improvement on plurality

[2]Note that such indices, or scores, must have precise mechanisms in place to deal with ties; World Cup soccer uses a system that undervalues a tie to encourage more aggressive play. See Barry Nalebuff and Jonathan Levin, "An Introduction to Vote Counting Schemes," *Journal of Economic Perspectives,* vol. 9, no. 1 (Winter 1995), pp. 3–26.

rule. The Borda count requires voters to rank-order all of the possible alternatives in an election and to indicate their rankings on their ballot cards. Points are assigned to each alternative on the basis of its position on each voter's ballot. In a three-person election, the candidate at the top of a ballot gets 3 points, the next candidate 2 points, and the bottom candidate 1 point. After the ballots are collected, each candidate's points are summed, and the one with the most points wins the election. A Borda count procedure is used in a number of sports-related elections, including professional baseball's Cy Young Award and college football's championship elections.

Many other positional methods can be devised simply by altering the rule used for the allocation of points to alternatives based on their positions on a voter's ballot. One system might allocate points in such a way as to give the top-ranked alternative relatively more than the others—for example, 5 points for the most-preferred alternative in a three-way election but only 2 and 1 for the second- and third-ranked options. In elections with larger numbers of candidates—say, 8—the top two choices on a voter's ballot might receive preferred treatment, gaining 10 and 9 points, respectively, while the others receive 6 or fewer.

An alternative to the positional plurative methods is the relatively recently invented **approval voting** method which allows voters to cast a single vote for each alternative of which they "approve."[3] Unlike positional methods, approval voting does not distinguish between alternatives on the basis of their positions on the ballot. Rather, all approval votes are treated equally and the alternative that receives the most approvals wins. In elections in which more than one winner can be selected (if electing a school board, for instance), a threshold level of approvals is set in advance and alternatives with more than the required minimum approvals are elected. Proponents of this method argue that it favors relatively moderate alternatives over those at either end of the spectrum; opponents claim that unwary voters could elect an unwanted novice candidate by indicating too many "encouragement" approvals on their ballots. Despite these disagreements, several professional societies have adopted approval voting to elect their officers, and some states have used or are considering using this method for public elections.

C. Mixed Methods

Some multistage voting procedures combine plurative and binary voting in **mixed methods.** The **majority runoff** procedure, for instance, is a two-stage method used to decrease a large group of possibilities to a binary decision. In a

[3]Unlike many of the other methods that have histories going back several centuries, the approval voting method was designed and named by then-graduate student Robert Weber in 1971; Weber is now a professor of managerial economics and decision sciences at Northwestern University, specializing in game theory.

first-stage election, voters indicate their most-preferred alternative, and these votes are tallied. If one candidate receives a majority of votes in the first stage, she wins. However, if there is no majority choice, a second-stage election pits the two most-preferred alternatives against each other. Majority rule chooses the winner in the second stage. French presidential elections use the majority runoff procedure, which can yield unexpected results if three or four strong candidates split the vote in the first round. In the spring of 2002, for example, far-right candidate Le Pen came in second ahead of France's socialist Prime Minister Jospin in the first-round of the presidential election. This result aroused surprise and consternation among French citizens, 30% of whom hadn't even bothered to vote in the election and some of whom had taken the first round as an opportunity to express their preference for various candidates of the far and fringe left. Le Pen's advance to the runoff election led to considerable political upheaval, although he lost in the end to sitting president Chirac.

Another mixed procedure consists of voting in successive **rounds.** Voters consider a number of alternatives in each round of voting with the worst-performing alternative eliminated after each stage. Voters then consider the remaining alternatives in a next round. The elimination continues until only two alternatives remain; at that stage, the method becomes binary and a final majority runoff determines a winner. A procedure with rounds is used to choose sites for the Olympic Games.

One could eliminate the need for successive rounds of voting by having voters indicate their preference orderings on the first ballot. Then a **single transferable vote** method can be used to tally votes in later rounds. With single transferable vote, each voter indicates her preference ordering over all candidates on a single initial ballot. If no alternative receives a majority of all first-place votes, the bottom-ranked alternative is eliminated and all first-place votes for that candidate are "transferred" to the candidate listed second on those ballots; similar reallocation occurs in later rounds as additional alternatives are eliminated until a majority winner emerges. The city of San Francisco voted in March of 2002 to use this voting method, more commonly called **instant runoff,** in all future municipal elections.

Single transferable vote is sometimes combined with **proportional representation** in an election. Proportional representation implies that a state electorate consisting of 55% Republicans, 25% Democrats, and 20% Independents, for example, would yield a body of representatives mirroring the party affiliations of that electorate. In other words, 55% of the U.S. Representatives from such a state would be Republican, and so on; this result contrasts starkly with the plurality rule method that would elect *all* Republicans (assuming that the voter mix in each district exactly mirrors the overall voter mix in the state). Candidates who attain a certain quota of votes are elected and others who fall below a certain quota are eliminated, depending on the exact specifications of the voting procedure. Votes for those candidates who are eliminated are again transferred by using the voters' preference orderings. This procedure continues until an appropriate number of candidates from each party are elected.

Clearly, there is room for considerable strategic thinking in the choice of a vote aggregation method, and strategy is also important even after the rule has been chosen. We examine some of the issues related to rule making and agenda setting in Section 2. Furthermore, strategic behavior on the part of voters, often called **strategic voting** or **strategic misrepresentation of preferences,** can also alter election outcomes under any set of rules as we will see later in this chapter.

2 VOTING PARADOXES

Even when people vote according to their true preferences, specific conditions on voter preferences and voting procedures can give rise to curious outcomes. In addition, election outcomes can depend critically on the type of procedure used to aggregate votes. This section describes some of the most famous of the curious outcomes—the so-called voting paradoxes—as well as some examples of how election results can change under different vote aggregation methods with no change in voter preferences and no strategic voting.

A. The Condorcet Paradox

The **Condorcet paradox** is one of the most famous and important of the voting paradoxes.[4] As mentioned earlier, the Condorcet method calls for the winner to be the candidate who gains a majority of votes in each round of a round-robin of pairwise comparisons. The paradox arises when no Condorcet winner emerges from this process.

To illustrate the paradox, we construct an example in which three people vote on three alternative outcomes by using the Condorcet method. Consider three city councillors (Left, Center, and Right) who are asked to rank their preferences for three alternative welfare policies, one that extends the welfare benefits currently available (call this one Generous, or G), another that decreases available benefits (Decreased, or D), and yet another that maintains the status quo (Average, or A). They are then asked to vote on each pair of policies to establish a council ranking, or a **social ranking.** This ranking is meant to describe how the council as a whole judges the merits of the possible welfare systems.

[4]It is so famous that economists have been known to refer to it as *the* voting paradox. Political scientists appear to know better, in that they are far more likely to use its formal name. As we will see, there are any number of possible voting paradoxes, not just the one named for Condorcet.

Suppose Councillor Left prefers to keep benefits as high as possible, while Councillor Center is most willing to maintain the status quo but concerned about the state of the city budget and so least willing to extend welfare benefits. Finally, Councillor Right most prefers reducing benefits but prefers an increase in benefits to the status quo; she expects that extending benefits will soon cause a serious budget crisis and turn public opinion so much against benefits that a more permanent state of low benefits will result, whereas the status quo could go on indefinitely. We illustrate these preference orderings in Figure 15.1 where the "curly" greater-than symbol, $\succ$, is used to indicate that one alternative is preferred to another. (Technically, $\succ$ is referred to as a *binary ordering relation.*)

With these preferences, if Generous is paired against Average, Generous wins. In the next pairing, of Average against Decreased, Average wins. And, in the final pairing of Generous against Decreased, the vote is again 2 to 1, this time in favor of Decreased. Therefore, if the council votes on alternative pairs of policies, a majority prefer Generous over Average, Average over Decreased, *and* Decreased over Generous. No one policy has a majority over both of the others. The group's preferences are cyclical: G $\succ$ A $\succ$ D $\succ$ G.

This cycle of preferences is an example of an **intransitive ordering** of preferences. The concept of rationality is usually taken to mean that individual preference orderings are **transitive** (the opposite of intransitive). If someone is given choices A, B, and C and you know that she prefers A to B and B to C, then transitivity implies that she also prefers A to C. (The terminology comes from the transitivity of numbers in mathematics; for instance, if $3 > 2$ and $2 > 1$, then we know that $3 > 1$.) A transitive preference ordering will not cycle as does the social ordering derived in our city council example; hence, we say that such an ordering is intransitive.

Notice that all of the *councillors* have transitive preferences over the three welfare policy alternatives but the *council* does not. This is the Condorcet paradox: even if all individual preference orderings are transitive, there is no guarantee that the social preference ordering induced by Condorcet's voting procedure also will be transitive. The result has far-reaching implications for public servants, as well as for the general public. It calls into question the basic notion of the "public interest," because such interests may not be easily defined or may not even exist. Our city council does not have any well-defined set of group preferences over the welfare policies. The lesson is that societies, institutions, or other large groups of people should not always be analyzed as if they acted like individuals.

LEFT	CENTER	RIGHT
G $\succ$ A $\succ$ D	A $\succ$ D $\succ$ G	D $\succ$ G $\succ$ A

FIGURE 15.1 Councillor Preferences over Welfare Policies

The Condorcet paradox can even arise more generally. There is no guarantee that the social ordering induced by *any* formal group voting process will be transitive just because individual preferences are. However, some estimates have shown that the paradox is most likely to arise when large groups of people are considering large numbers of alternatives. Smaller groups considering smaller numbers of alternatives are more likely to have similar preferences over those alternatives; in such situations, the paradox is much less likely to appear.[5] In fact, the paradox arose in our example because the council completely disagreed not only about which alternative was best but also about which was worst. The smaller the group, the less likely such outcomes are to occur.

B. The Agenda Paradox

The second paradox that we consider also entails a binary voting procedure, but this example considers the ordering of alternatives in that procedure. In a parliamentary setting with a committee chair who determines the specific order of voting for a three-alternative election, substantial power over the final outcome lies with the chair. In fact, the chair can take advantage of the intransitive social preference ordering that arises from some sets of individual preferences and, by selecting an appropriate agenda, manipulate the outcome of the election in any manner she desires.

Consider again the city councillors Left, Center, and Right, who must decide among Generous, Average, and Decreased welfare policies. The councillors' preferences over the alternatives were shown in Figure 15.1. Let us now suppose that one of the councillors has been appointed chair of the council by the mayor and the chair is given the right to decide which two welfare policies get voted on first and which goes up against the winner of that initial vote. With the given set of councillor preferences and common knowledge of the preference orderings, the chair can get any outcome that she wants. If Left were chosen Chair, for example, she could orchestrate a win for Generous by setting Average against Decreased in the first round with the winner to go up against Generous in round two. The result that any final ordering can be obtained by choosing an appropriate procedure is known as the **agenda paradox.**

With the set of preferences illustrated in Figure 15.1, then, we get not only the intransitive social preference ordering of the Condorcet paradox but also the result that the outcome of a binary procedure could be any of the possible alternatives. The only determinant of the outcome in such a case is the ordering of the agenda. This result implies that setting the agenda is the real game here and, because the chair sets the agenda, the appointment or election of the chair

[5]See Peter Ordeshook, *Game Theory and Political Theory* (Cambridge, UK: Cambridge University Press, 1986), p. 58.

is the true outlet for strategic behavior. Here, as in many other strategic situations, what appears to be the game (in this case, choice of a welfare policy) is not the true game at all; rather, those participating in the game engage in strategic play at an earlier point (deciding the identity of the chair) and vote according to set preferences in the eventual election.

However, the preceding demonstration of the agenda setter's power assumes that, in the first round, voters choose between the two alternatives (Average and Decreased) on the basis only of their preferences between these two alternatives, with no regard for the eventual outcome of the procedure. Such behavior is called **sincere voting;** actually, myopic or nonstrategic voting would be a better name. If Center is a strategic game player, she should realize that, if she votes for Decreased in the first round (even though she prefers Average between the pair presented at that stage), then Decreased will win the first round and will also win against Generous in the second round with support from Right. Center prefers Decreased over Generous as the eventual outcome. Therefore she should do this rollback analysis and vote strategically in the first round. But should she, if everyone else is also voting strategically? We examine the game of strategic voting and find its equilibrium in Section 4.

C. The Reversal Paradox

Positional voting methods also can lead to paradoxical results. The Borda count, for example, can yield the **reversal paradox** when the slate of candidates open to voters changes. This paradox arises in an election with at least four alternatives when one of them is removed from consideration after votes have been submitted, making recalculation necessary.

Suppose there are four candidates for a (hypothetical) special commemorative Cy Young Award to be given to a retired major-league baseball pitcher. The candidates are Steve Carlton (SC), Sandy Koufax (SK), Robin Roberts (RR), and Tom Seaver (TS). Seven prominent sportswriters are asked to rank these pitchers on their ballot cards. The top-ranked candidate on each card will get 4 points; decreasing numbers of points will be allotted to candidates ranked second, third, and fourth.

Across the seven voting sportswriters, there are three different preference orderings over the candidate pitchers; these preference orderings, with the number of writers having each ordering, are shown in Figure 15.2. When the votes are tallied, Seaver gets $(2 \times 3) + (3 \times 2) + (2 \times 4) = 20$ points; Koufax gets $(2 \times 4) + (3 \times 3) + (2 \times 1) = 19$ points; Carlton gets $(2 \times 1) + (3 \times 4) + (2 \times 2) = 18$ points; and Roberts gets $(2 \times 2) + (3 \times 1) + (2 \times 3) = 13$ points. Seaver wins the election, followed by Koufax, Carlton, and Roberts in last place.

Now suppose it is discovered that Roberts is not really eligible for the commemorative award, because he never actually won a Cy Young Award, having

ORDERING 1 (2 voters)	ORDERING 2 (3 voters)	ORDERING 3 (2 voters)
Koufax > Seaver > Roberts > Carlton	Carlton > Koufax > Seaver > Roberts	Seaver > Roberts > Carlton > Koufax

FIGURE 15.2 Sportswriter Preferences over Pitchers

reached the pinnacle of his career in the years just before the institution of the award in 1956. This discovery requires points to be recalculated, ignoring Roberts on the ballots. The top spot on each card now gets 3 points, while the second and third spots receive 2 and 1, respectively. Ballots from sportswriters with preference ordering 1, for example, now give Koufax and Seaver 3 and 2 points, respectively, rather than 4 and 3 from the first calculation; those ballots also give Carlton a single point for last place.

Adding votes with the revised point system shows that Carlton receives 15 points, Koufax receives 14 points, and Seaver receives 13 points. Winner has turned loser as the new results reverse the standings found in the first election. No change in preference orderings accompanies this result. The only difference in the two elections is the number of candidates being considered.

D. Change the Voting Method, Change the Outcome

As should be clear from the preceding discussion, election outcomes are likely to differ under different sets of voting rules. As an example, consider 100 voters who can be broken down into three groups on the basis of their preferences over three candidates (A, B, and C). Preferences of the three groups are shown in Figure 15.3. With the preferences as shown, and depending on the vote-aggregation method used, any of these three candidates could win the election.

With simple plurality rule, candidate A wins with 40% of the vote, even though 60% of the voters rank her lowest of the three. Supporters of candidate A would obviously prefer this type of election. If they had the power to choose the voting method, then plurality rule, a seemingly "fair" procedure, would win the election for A in spite of the majority's strong dislike for that candidate.

The Borda count, however, would produce a different outcome. In a Borda system with 3 points going to the most-preferred candidate, 2 points to the

GROUP 1 (40 voters)	GROUP 2 (25 voters)	GROUP 3 (35 voters)
A > B > C	B > C > A	C > B > A

FIGURE 15.3 Group Preferences over Candidates

middle candidate, and 1 to the least-preferred candidate, A gets 40 first-place votes and 60 third-place votes, for a total of 40(3) + 60(1) = 180 points. Candidate B gets 25 first-place votes and 75 second-place votes, for a total of 25(3) + 75(2) = 225 points; and C gets 35 first-place votes, 25 second-place votes, and 40 third-place votes, for a total of 35(3) + 25(2) + 40(1) = 195 points. With this procedure, B wins, with C in second place and A last. Candidate B would also win with the antiplurality vote in which electors cast votes for all but their least-preferred candidate.

And what about candidate C? She can win the election if a majority or instant runoff system is used. In either method, A and C, with 40 and 35 votes in the first round, survive to face each other in the runoff. The majority runoff system would call voters back to the polls to consider A and C; the instant runoff system would eliminate B and reallocate B's votes (from group 2 voters) to the next preferred alternative, candidate C. Then, because A is the least-preferred alternative for 60 of the 100 voters, candidate C would win the runoff election 60 to 40.

Another example of how different procedures can lead to different outcomes can be seen in the voting for the site of the Summer Olympics. The actual voting procedure for the Olympics site selection is a mixed method consisting of multiple rounds of plurality rule with elimination and then a final majority rule vote. Each country representative participating in the election casts one vote, and the candidate city with the lowest vote total is eliminated after each round. Majority rule is used to choose between the final two candidate cities. In voting for the site of both the 1996 and 2000 games, one site led through most of the early rounds and would have won a single plurality vote; for the 1996 games, Athens (Greece) led early and, for the 2000 site, Beijing led early. In both cases, these plurality winners gained little additional support as other cities were eliminated and neither front-runner was eventually chosen in either election. Each actually retained a plurality through the next-to-last round of voting but, after the final elimination round, Athens saw the needed extra votes go to Atlanta, and Beijing watched them go to Sydney. It is worth noting that the Olympics site election procedure has since been changed!

3 GENERAL IDEAS PERTAINING TO PARADOXES

The discussion of the various voting paradoxes in Section 2 suggests that voting methods can suffer from a number of faults that lead to unusual, unexpected, or even unfair outcomes. In addition, this suggestion leads us to ask: Is there one voting system that satisfies certain regularity conditions, including transitivity, and is the most "fair"—that is, most accurately captures the preferences of the electorate? Kenneth Arrow's **impossibility theorem** tells us that the answer to this question is no.[6]

The technical content of Arrow's theorem makes it beyond our scope to prove completely. But the sense of the theorem is straightforward. Arrow argued that no preference aggregation method could satisfy all six of the critical principles that he identified:

1. The social or group ranking must rank all alternatives (be complete).
2. It must be transitive.
3. It should satisfy a condition known as *positive responsiveness*, or the Pareto property. Given two alternatives, A and B, if the electorate unanimously prefers A to B, then the aggregate ranking should place A above B.
4. The ranking must not be imposed by external considerations (such as customs) independent of the preferences of individual members of the society.
5. It must not be dictatorial—no single voter should determine the group ranking.
6. And it should be independent of irrelevant alternatives; that is, no change in the set of candidates (addition to or subtraction from) should change the rankings of the unaffected candidates.

Often the theorem is abbreviated by imposing the first four conditions and focusing on the difficulty of simultaneously obtaining the last two; the simplified form states that we cannot have independence of irrelevant alternatives (IIA) without dictatorship.[7]

You should be able to see immediately that some of the voting methods considered earlier do not satisfy all of Arrow's principles. The requirement of IIA, for example, is violated by the single transferable vote procedure as well as by the Borda count. Borda's procedure is, however, nondictatorial and consistent and it satisfies the Pareto property. All of the other systems that we have considered satisfy IIA but break down on one of the other principles.

Arrow's theorem has provoked extensive research into the robustness of his conclusion to changes in the underlying assumptions. Economists, political scientists, and mathematicians have searched for a way to reduce the number of criteria or relax Arrow's principles minimally to find a procedure that satisfies the criteria without sacrificing the core principles; their efforts have been largely unsuccessful. Most economic and political theorists now accept the idea that some form of compromise is necessary when choosing a vote, or preference, aggregation method. Here are a few prominent examples,

[6]A full description of this theorem, often called "Arrow's General Possibility Theorem," can be found in Kenneth Arrow, *Social Choice and Individual Values*, 2nd ed. (New York: Wiley, 1963).

[7]See Walter Nicholson's treatment of Arrow's impossibility theorem in his *Microeconomic Theory*, 7th ed. (New York: Dryden Press, 1998), pp. 764–766, for more detail at a level appropriate for intermediate-level economics students.

each representing the approach of a particular field—political science, economics, and mathematics.

A. Black's Condition

As the discussion in Section 2.A showed, the pairwise voting procedure does not satisfy Arrow's condition on transitivity of the social ranking, even when all individual rankings are transitive. One way to surmount this obstacle to meeting Arrow's conditions, as well as a way to prevent the Condorcet Paradox, is to place restrictions on the preference orderings held by individual voters. Such a restriction, known as the requirement of **single-peaked preferences,** was put forth by political scientist Duncan Black in the late 1940s.[8] Black's seminal paper on group decision making actually predates Arrow's impossibility theorem and was formulated with the Condorcet paradox in mind, but voting theorists have since shown its relevance to Arrow's work; in fact, the requirement of single-peaked preferences is sometimes referred to as **Black's condition.**

For a preference ordering to be single peaked, it must be the case that the alternatives being considered can be ordered along some specific dimension (for example, the expenditure level associated with each policy). To illustrate this requirement, we draw a graph in Figure 15.4 with the specified dimension on the horizontal axis and a voter's preference ranking (or payoff) on the vertical axis. For the single-peaked requirement to hold, each voter must have a single ideal or most-preferred alternative, and alternatives "farther away" from the most-preferred point must provide steadily lower payoffs. The two voters in Figure 15.4, Mr. Left and Ms. Right, have different ideal points along the policy dimension but, for each, the payoff falls steadily as the policy moves away from his or her ideal point.

Black shows that, if preferences of each voter are single peaked, then the pairwise (majority) voting procedure must produce a transitive social ordering. The Condorcet paradox is prevented and pairwise voting satisfies Arrow's transitivity condition.

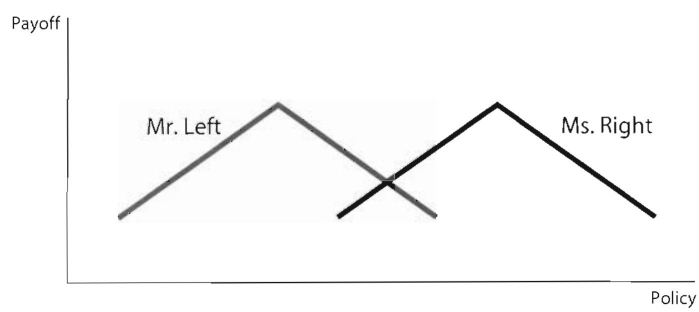


FIGURE 15.4 Single-Peaked Preferences

[8]Duncan Black, "On the Rationale of Group Decision-Making," *Journal of Political Economy*, vol. 56, no. 1 (Feb. 1948), pp. 23–34.

B. Robustness

An alternative, more recent method of compromise with Arrow comes from economic theorists Partha Dasgupta and Eric Maskin.[9] They suggest a new criterion called **robustness** by which to judge voting methods. Robustness is measured by considering how often a voting procedure that is nondictatorial and satisfies IIA as well as the Pareto property also satisfies the requirement of transitivity of its social ranking: For how many sets of voter preference orderings does such a procedure satisfy transitivity?

With the use of the robustness criterion, simple majority rule can be shown to be *maximally robust*—that is, it is nondictatorial, satisfies IIA and Pareto, and provides transitive social rankings for the largest possible set of voter preference orderings. Behind majority rule on the robustness scale lie other voting procedures, including the Borda count and plurality rule. The robustness criterion is appealing in its ability to establish one of the most commonly used voting procedures—the one most often associated with the democratic process—as a candidate for the best aggregation procedure.

C. Intensity Ranking

Another class of attempts to escape from Arrow's negative result focuses on the difficulty of satisfying Arrow's IIA requirement. A recent theory of this kind comes from mathematician Donald Saari.[10] He suggests that a vote aggregation method might use more information about voters' preferences than is contained in their mere ordering of any pair of alternatives, X and Y; rather, it could take into account each individual voter's *intensity* of preferences between that pair of alternatives. This intensity can be measured by counting the number of other alternatives, Z, W, V, . . . that a voter places between X and Y. Saari therefore replaces the IIA condition, number 6 of Arrow's principles, with a different one, which he labels IBI (intensity of binary independence) and which we will number 6′:

6′. Society's relative ranking of any two alternatives should be determined only by (1) each voter's relative ranking of the pair and (2) the intensity of this ranking.

This condition is weaker than IIA because it is like applying IIA only to such additions or deletions of "irrelevant" alternatives that do not change the intensity of people's preferences between the "relevant" ones. With this revision, the Borda

[9]See Partha Dasgupta and Eric Maskin, "On the Robustness of Majority Rule," working paper, April 2000.

[10]For more precise information about Saari's work on Arrow's theorem, see D. Saari, "Mathematical Structure of Voting Paradoxes I: Pairwise Vote," *Economic Theory*, vol. 15 (2000), pp. 1–53. Additional information on this result and on the robustness of the Borda count can be found in D. Saari, *Chaotic Elections* (Providence, RI: American Mathematical Society, 2000).

count satisfies the modified Arrow theorem and it is the only one of the positional voting methods to do so.

Saari also hails the Borda count as the only procedure that appropriately observes ties within collections of ballots, a criterion that he argues is essential for a good aggregation system to satisfy. Ties can occur two ways: through **Condorcet terms** or through **reversal terms** within voter preference orderings. In a three-candidate election among alternatives A, B, and C, the Condorcet terms are the preference orderings A > B > C, B > C > A, and C > A > B. A set of three ballots with these preferences appearing on one ballot apiece should logically offset one another, or constitute a tie. Reversal terms are preference orderings that contain a reversal in the location of a *pair* of alternatives. In the same election, two ballots with preference orderings of A > B > C and B > A > C should logically lead to a tie in a pairwise contest between A and B. Only the Borda procedure treats such collections of ballots—those with Condorcet terms or reversal terms—as tied. Although the Borda count can lead to the reversal paradox, as shown in the preceding section, it retains many proponents. The *only* time that the Borda procedure produces paradoxical results is when alternatives are dropped from consideration after ballots have been collected. Because such results can be prevented by using only ballots for the full set of candidates, the Borda procedure has gained favor in some circles as one of the best vote aggregation methods.

Other researchers have made different suggestions regarding criteria that a good aggregation system should satisfy. Some of them include the *Condorcet criterion* (that a Condorcet winner should be selected by a voting system if such a winner exists), the *consistency criterion* (that an election including all voters should elect the same alternative as would two elections held for an arbitrary division of the entire set of voters), and lack of manipulability (a voting system should not encourage manipulability—strategic voting—on the part of voters). We cannot consider each of these suggestions at length, but we do address strategic manipulation by voters in the next two sections.

4 STRATEGIC MANIPULATION OF VOTES

Several of the voting systems that we have considered yield considerable scope for strategic misrepresentation of preferences on the part of voters. In Section 2.B, we showed how the power of an agenda-setting Left chair can be countered by a Center councillor voting in the first round against her true preference, so as to knock out her least-preferred alternative and send a more preferred one into the second round. More generally, voters can choose to vote for candidates, issues, or policies that are not actually their most-preferred outcomes among the alternatives presented in an early round if such behavior can alter the final election results in their favor. In this section, we consider a number of ways in which strategic voting behavior can affect elections.

A. Plurality Rule

Plurality rule elections, often perceived as the most fair by many voters, still provide opportunities for strategic behavior. In Presidential elections, for instance, there are generally two major candidates in contention. When such a race is relatively close, there is potential for a third candidate to enter the race and divert votes away from the leading candidate; if the entry of this third player truly threatens the chances of the leader winning the election, the late entrant is called a **spoiler.**

Spoilers are generally believed to have little chance to win the whole election, but their role in changing the election outcome is undisputed. In elections with a spoiler candidate, those who prefer the spoiler to the leading major candidate but least prefer the trailing major candidate may do best to strategically misrepresent their preferences to prevent the election of their least-favorite candidate. That is, you should vote for the leader in such a case even though you would prefer the spoiler because the spoiler is unlikely to garner a plurality; voting for the leader then prevents the trailing candidate, your least favorite, from winning.[11]

Evidence shows that there has been a great deal of strategic voting in recent U.S. Presidential elections—the 1992 and 2000 races in particular. In 1992, Ross Perot ran a strong third-party campaign for much of the primary season, only to drop out of the race at midsummer and reappear in the fall. His reappearance at such a late date cast him as a spoiler in the then close race between the incumbent, President George Bush, and his Democratic challenger, Bill Clinton. Perot's role in dividing the conservative vote and clinching Clinton's victory that year has been debated, but it is certainly possible that Clinton would not have won without Perot on the ballot. A more interesting theory suggests significant misrepresentation of preferences at the polls. A *Newsweek* survey claimed that, if more voters had believed Perot was capable of winning the election, he might have done so; a plurality of 40% of voters surveyed said they would have voted for Perot (instead of Bush or Clinton) if they had thought he could have won.[12]

Ralph Nader played a similar role in the 2000 Presidential election between Democratic Vice-president Al Gore and Republican challenger, George W. Bush. Nader was more concerned about garnering 5% of the popular vote so that his Green Party could qualify for federal matching election funds than

[11]Note that an approval voting method would not suffer from this same problem.

[12]"Ross Reruns," *Newsweek*, Special Election Recap Issue, November 18, 1996, p. 104.

he was about winning the presidency, but his candidacy made an already close election even tighter. The 2000 election will be remembered more for the extraordinarily close outcome in Florida and the battle for that state's electoral votes. But Nader was pulling needed votes from Gore's camp in the run-up to the election, causing enough concern that public discussions (on radio and television and in major metropolitan newspapers) arose regarding the possibilities for strategic voting. Specifically, several groups (as well as a number of Web sites) advocated "vote swapping" schemes designed to gain Nader his needed votes without costing Gore the electoral votes of any of his key states. Nader voters in key Gore states (such as Pennsylvania, Michigan, and Maine) were asked to "swap" their votes with Gore supporters in a state destined to go to George W. Bush (such as Texas or Wyoming); a Michigan Nader supporter could vote for Gore while her Nader vote was cast in Texas. Evidence on the efficacy of these strategies is mixed. We do know that Nader failed to win his 5% of the popular vote but that Gore carried all of Pennsylvania, Michigan, and Maine.

In elections for legislatures, where many candidates are chosen, the performance of third parties is very different under a system of proportional representation of the whole population in the whole legislature from that under a system of plurality in separate constituencies. Britain has the constituency and plurality system. In the past 50 years, the Labor and Conservative parties have shared power. The Liberal Party, despite a sizable third-place support in the electorate, has suffered from strategic voting and therefore has had disproportionately few seats in Parliament. Italy has had the nationwide list and proportional representation system; there is no need to vote strategically in such a system, and even small parties can have significant presence in the legislature. Often no party has a clear majority of seats, and small parties can affect policy through bargaining for alliances.

A party cannot flourish if it is largely ineffective in influencing a country's political choices. Therefore we tend to see just two major parties in countries with the plurality system and several parties in those with the proportional representation system. Political scientists call this observation *Duverger's law.*

In the legislature, the constituency system tends to produce only two major parties—often one of them with a clear majority of seats and therefore more decisive government. But it runs the risk that the minority's interests will be overlooked—that is, of producing a "tyranny of the majority." A proportional representation system gives better voice to minority views. But it can produce inconclusive bargaining for power and legislative gridlock. Interestingly, each country seems to believe that its system performs worse and considers switching to the other; in Britain, there are strong voices calling for proportional representation, and Italy has been seriously considering a constituency system.

B. Pairwise Voting

When you know that you are bound by a pairwise method such as the amendment procedure, you can use your prediction of the second-round outcome to determine your optimal voting strategy in the first round. It may be in your interest to appear committed to a particular candidate or policy in the first round, even if it is not your most-preferred alternative, so that your least-favorite alternative cannot win the entire election in the second round.

We return here to our example of the city council with an agenda-setting chair; again, all three preference rankings are assumed to be known to the entire council. Suppose Councillor Left, who most prefers the Generous welfare package, is appointed chair and sets the Average and Decreased policies against each other in a first vote, with the winner facing off against the Generous policy in the second round. If the three councillors vote strictly according to their preferences, shown in Figure 15.1, Average will beat Decreased in the first vote and Generous will then beat Average in the second vote; the chair's preferred outcome will be chosen. The city councillors are likely to be well-trained strategists, however, who can look ahead to the final round of voting and use rollback to determine which way to vote in the opening round.

In the scenario just described, Councillor Center's least-preferred policy will be chosen in the election. Therefore, rollback analysis says that she should vote strategically in the first round to alter the election's outcome. If Center votes for her most-preferred policy in the first round, she will vote for the Average policy, which will then beat Decreased in that round and lose to Generous in round two. However, she could instead vote strategically for the Decreased policy in the first round, which would lift Decreased over Average on the first vote. Then, when Decreased is set up against Generous in the second round, Generous will lose to Decreased. Councillor Center's misrepresentation of her preference ordering with respect to Average and Decreased helps her to change the winner of the election from Generous to Decreased. Although Decreased is not her most-preferred outcome, it is better than Generous from her perspective.

This strategy works well for Center if she can be sure that no other strategic votes will be cast in the election. Thus we need to fully analyze both rounds of voting to verify the Nash equilibrium strategies for the three councillors. We do so by using rollback on the two simultaneous-vote rounds of the election, starting with the two possible second-round contests, A versus G or D versus G. In the following analysis, we use the abbreviated names of the policies, G, A, and D.

Figure 15.5 illustrates the outcomes that arise in each of the possible second-round elections. The two tables in Figure 15.5a show the winning policy (not payoffs to the players) when A has won the first round and is pitted against G; the tables in Figure 15.5b show the winning policy when L has won

(a) A versus G election

Right votes:

A

		CENTER	
		A	G
LEFT	A	A	A
	G	A	G

G

		CENTER	
		A	G
LEFT	A	A	G
	G	G	G

(b) D versus G election

Right votes:

D

		CENTER	
		D	G
LEFT	D	D	D
	G	D	G

G

		CENTER	
		D	G
LEFT	D	D	G
	G	G	G

FIGURE 15.5 Election Outcomes in Two Possible Second-Round Votes

the first round. In both cases, Councillor Left chooses the row of the final outcome, Center chooses the column, and Right chooses the actual table (left or right).

You should be able to establish that each councillor has a dominant strategy in each second-round election. In the A-versus-G election, Left's dominant strategy is to vote for G, Center's dominant strategy is to vote for A, and Right's dominant strategy is to vote for G; G will win this election. If the councillors consider D versus G, Left's dominant strategy is still to vote for G, and Right and Center both have a dominant strategy to vote for D; in this vote, D wins. A quick check shows that all of the councillors vote according to their true preferences in this round. Thus these dominant strategies are all the same: "Vote for the alternative that I prefer." Because there is no future to consider in the second-round vote, the councillors simply vote for whichever policy ranks higher in their preference ordering.[13]

We can now use the results from our analysis of Figure 15.5 to consider optimal voting strategies in the first-round of the election, in which voters choose

[13]It is a general result in the voting literature that voters faced with pairs of alternatives will always vote truthfully at the last round of voting.

between policies A and D. Because we know how the councillors will vote in the next round given the winner here, we can show the outcome of the entire election in the tables in Figure 15.6.

As an example of how we arrived at these outcomes, consider the G in the upper-left-hand cell of the right-hand table in Figure 15.6. The outcome in that cell is obtained when Left and Center both vote for A in the first round while Right votes for D. Thus A and G are paired in the second round and, as we saw in Figure 15.5, G wins. The other outcomes are derived in similar fashion.

Given the outcomes in Figure 15.6, Councillor Left (who is the chair and has set the agenda) has a dominant strategy to vote for A in this round. Similarly, Councillor Right has a dominant strategy to vote for D. Neither of these councillors misrepresent their preferences or vote strategically in either round. Councillor Center, however, has a dominant strategy to vote for D here even though she strictly prefers A to D. As the preceding discussion suggested, she has a strong incentive to misrepresent her preferences in the first round of voting; and she is the only one who votes strategically. Center's behavior changes the winner of the election from G (the winner without strategic voting) to D.

Remember that the chair, Councillor Left, set the agenda in the hope of having her most-preferred alternative chosen. Instead, her *least*-preferred alternative has prevailed. It appears that the power to set the agenda may not be so beneficial after all. But Councillor Left should anticipate the strategic behavior. Then she can choose the agenda so as to take advantage of her understanding of games of strategy. In fact, if she sets D against G in the first round and then the winner against A, the Nash equilibrium outcome is G, the chair's most-preferred outcome. With that agenda, Right misrepresents her preferences in the first round to vote for G over D to prevent A, her least-preferred outcome, from winning. You should verify that this is Councillor Left's best agenda-setting strategy. In the full voting game where setting the agenda is considered an initial, prevoting round, we should expect to see the Generous welfare policy adopted when Councillor Left is chair.

We can also see an interesting pattern emerge when looking more closely at voting behavior in the strategic version of the election. There are pairs of

Right votes:

A

		CENTER	
		A	D
LEFT	A	G	G
	D	G	D

D

		CENTER	
		A	D
LEFT	A	G	D
	D	D	D

FIGURE 15.6 Election Outcomes Based on First-Round Votes

councillors who vote "together" (the same as each other) in both rounds. Under the original agenda, Right and Center vote together in both rounds and, in the suggested alternative (D versus G in the first round), Right and Left vote together in both rounds. In other words, a sort of long-lasting coalition has formed between two councillors in each case.

Strategic voting of this type appears to have taken place in Congress on more than one occasion. One example pertains to a federal school construction funding bill considered in 1956.[14] Before being brought to a vote against the status quo of no funding, the bill was amended in the House of Representatives to require that aid be offered only to states with no racially segregated schools. Under the parliamentary voting rules of Congress, a vote on whether to accept the so-called Powell Amendment was taken first, with the winning version of the bill considered afterward. Political scientists who have studied the history of this bill argue that opponents of school funding strategically misrepresented their preferences regarding the amendment to defeat the original bill. A key group of Representatives voted for the amendment but then joined opponents of racial integration in voting against the full bill in the final vote; the bill was defeated. Voting records of this group indicate that many of them had voted against racial integration matters in other circumstances, implying that their vote for integration in this case was merely an instance of strategic voting and not an indication of their true feelings regarding school integration.

C. Strategic Voting with Incomplete Information

The preceding analysis showed that there are sometimes incentives for committee members to vote strategically to prevent their least-preferred alternative from winning an election. Our example assumed that the Council members knew the possible preference orderings and how many other councillors had those preferences. Now suppose information is incomplete; each Council member knows the possible preference orderings, her own actual ordering, and the probabilities that each of the others have a particular ordering, but not the actual distribution of the different preference orderings among the other councillors. In this situation, each councillor's strategy needs to be conditioned on her beliefs about that distribution and on their beliefs about how truthful other voters will be.[15]

For an example, suppose that we still have a three-member Council considering the three alternative welfare policies described earlier according to the

[14] A more complete analysis of the case can be found in Riker, *Liberalism Against Populism*, pp. 152–157.

[15] This result can be found in P. Ordeshook and T. Palfrey, "Agendas, Strategic Voting, and Signaling with Incomplete Information," *American Journal of Political Science*, vol. 32, no. 2 (May 1988), pp. 441–466. The structure of the example to follow is based on Ordeshook and Palfrey's analysis.

(original) agenda set by Councillor Left; that is, the Council considers policies A and D in the first-round with the winner facing G in the second-round. We assume that there are still three different possible preference orderings, as illustrated in Figure 15.1, and that the councillors know that these orderings are the only possibilities. The difference is that no one knows for sure exactly how many councillors have each set of preferences. Rather, each councillor knows her own type and she knows that there is some positive probability of observing each type of voter (Left, Center, or Right), with the probabilities p_L, p_C, and p_R summing to one.

We saw earlier that all three councillors vote truthfully in the last round of balloting. We also saw that Left- and Right-type councillors vote truthfully in the first round as well. This result remains true in the incomplete information case. Right-type voters prefer to see D win the first-round election; given this preference, Right always does at least as well by voting for D over A (if both other councillors have voted the same way) and sometimes does better by voting this way (if the other two votes split between D and A). Similarly, Left-type voters prefer to see A survive to vie against G in round two; these voters always do at least as well as otherwise—and sometimes do better—by voting for A over D.

At issue then is only the behavior of the Center-type voters. Because they do not know the types of the other councillors and because they have an incentive to vote strategically for some preference distributions—specifically the case in which it is known for certain that there is one voter of each type—their behavior will depend on the probabilities that the various voter types may occur within the council. We consider here one of two polar cases in which a Center-type voter believes that other Center types will vote truthfully and we look for a symmetric, pure-strategy Nash equilibrium. The case in which she believes that other Center types will vote strategically is taken up in Exercise 15.9.

To make outcome comparisons possible, we will specify payoffs for the Center-type voter associated with the possible winning policies. Center-type preferences are A > D > G. Suppose that, if A wins, Center types receive a payoff of 1 and, if G wins, Center types receive a payoff of 0. If D wins, Center types receive some intermediate-level payoff, call it u, where $0 < u < 1$.

Now, suppose our Center-type councillor must decide how to vote in the first round (A versus D) in an election in which she believes that both other voters vote truthfully, regardless of their type. If both voters choose either A or D, then Center's vote is immaterial to the final outcome; she is indifferent between A and D. If the other two voters split their votes, however, then Center can influence the election outcome. Her problem is that she needs to decide whether to vote truthfully herself.

If the other two voters split between A and D and if both are voting truthfully, then the vote for D must have come from a Right-type voter. But the vote for A could have come from *either* a Left type *or* a (truthful) Center type. If the A vote came from a Left-type voter, then Center knows that there is one voter of each type. If she votes truthfully for A in this situation, A will win the first round but lose to G in the end; Center's payoff will be 0. If Center votes strategically for D, D beats A and G and Center's payoff is u. On the other hand, if the A vote came from a Center-type voter, then Center knows there are two Center types and a Right type but no Left type on the Council. In this case, a truthful vote for A helps A win the first round, and then A also beats G two to one in round two; Center gets her highest payoff of 1. If Center were to vote strategically for D, D would win both rounds again and Center would get u.

To determine Center's optimal strategy, we need to compare her expected payoff from truthful voting to her expected payoff from strategic voting. With a truthful vote for A, Center's payoff depends on how likely it is that the other A vote comes from a Left type or a Center type. Those probabilities are straightforward to calculate. The probability that the other A vote comes from a Left type is just the probability of a Left type being one of the remaining voters, or $p_L/(p_L + p_C)$; similarly, the probability that the A vote comes from a Center type is $p_C/(p_L + p_C)$. Then Center's payoffs from truthful voting are 0 with probability $p_L/(p_L + p_C)$ and 1 with probability $p_C/(p_L + p_C)$, so the expected payoff is $p_C/(p_L + p_C)$. With a strategic vote for D, D wins regardless of the identity of the third voter—D wins with certainty—and so Center's expected payoff is just u. Center's final decision is to vote truthfully as long as $p_C/(p_L + p_C) > u$.

Note that Center's decision-making condition is an intuitively reasonable one. If the probability of there being more Center-type voters is large or relatively larger than the probability of having a Left-type voter, then the Center types vote truthfully. Voting strategically is useful to Center only when she is the only voter of her type on the Council.

We add two additional comments on the existence of imperfect information and its implications for strategic behavior. First, if the number of councillors, n, is larger than three but odd, then the expected payoff to a Center type from voting strategically remains equal to u and the expected payoff from voting truthfully is $[p_C/(p_L + p_C]^{(n-1)/2}$.[16] Thus, a Center type should vote truthfully only when $[p_C/(p_L + p_C)]^{(n-1)/2} > u$. Because $p_C/(p_L + p_C) < 1$ and $u > 0$, this inequality will *never* hold for large enough values of n. This result tells us that a symmetric truthful-voting equilibrium can never persist in a large enough Council! Second, imperfect information about the preferences of other voters yields additional scope for strategic behavior. With agendas that include more than two

[16]A Center type can affect the election outcome only if all other votes are split evenly between A and D. Thus, there must be exactly $(n - 1)/2$ Right-type voters choosing D in the first round and $(n - 1)/2$ other voters choosing A. If those A voters are Left types, then A won't win the second-round election and Center will get 0 payoff. For Center to get a payoff of 1, it must be true that all of the other A voters are Center types. The probability of this occurring is $[p_C/(p_L + p_C)]^{(n-1)/2}$; then Center's expected payoff from voting truthfully is as stated. See Ordeshook and Palfrey, p. 455.

rounds, voters can use their early-round votes to signal their types. The extra rounds give other voters the opportunity to update their prior beliefs about the probabilities p_C, p_L, and p_R and a chance to act on that information. With only two rounds of pairwise votes, there is no time to use any information gained during round one, because truthful voting is a dominant strategy for all voters in the final round.

5 GENERAL IDEAS PERTAINING TO MANIPULATION

The extent to which a voting procedure is susceptible to strategic misrepresentation of preferences, or strategic manipulability by voters of the types illustrated in Section 4, is another topic that has generated considerable interest among voting theorists. Arrow does not require nonmanipulability in his theorem but the literature has considered how such a requirement would relate to Arrow's conditions. Similarly, theorists have considered the scope for manipulability in various procedures, producing rankings of voting methods.

Economist William Vickrey, perhaps better known for his work on auctions (see Chapter 16), did some of the earliest work considering strategic behavior of voters. He pointed out that procedures satisfying Arrow's IIA assumption were most immune to strategic manipulation. He also set out several conditions under which strategic behavior is more likely to be attempted and successful. In particular, he noted that situations with smaller numbers of informed voters and smaller sets of available alternatives may be most susceptible to manipulation, given a voting method that is itself manipulable. This result means, however, that weakening the IIA assumption to help voting procedures satisfy Arrow's conditions makes way for more manipulable procedures. In particular, Saari's intensity ranking version of IIA (called IBI), mentioned in Section 3.C, may allow more procedures to satisfy a modified version of Arrow's theorem but may simultaneously allow more manipulable procedures to do so.

Like Arrow's general result on the impossibility of preference aggregation, the general result on manipulability is a negative one. Specifically, the **Gibbard-Satterthwaite theorem** shows that, if there are three or more alternatives to consider, the only voting procedure that prevents strategic voting is dictatorship: one voter is assigned the role of dictator and her preferences determine the election outcome.[17] Combining the Gibbard-Satterthwaite outcome with

[17]For the theoretical details on this result, see A. Gibbard, "Manipulation of Voting Schemes: A General Result," *Econometrica,* vol. 41, no. 4 (July 1973), pp. 587–601, and M. A. Satterthwaite, "Strategy-Proofness and Arrow's Conditions," *Journal of Economic Theory,* vol. 10 (1975), pp. 187–217. The theorem carries both their names because each proved the result independently of the other.

Vickrey's discussion of IIA may help the reader understand why Arrow's theorem is often reduced to a consideration of which procedures can simultaneously satisfy nondictatorship and IIA.

Finally, some theorists have argued that voting systems should be evaluated not on their ability to satisfy Arrow's conditions but on their tendency to encourage manipulation. The relative manipulability of a voting system can be determined by the amount of information about the preferences of other voters that is required by voters to successfully manipulate an election. Some research based on this criterion suggests that, of the procedures so far discussed, plurality rule is the most manipulable (that is, requires the least information). In decreasing order of manipulability are approval voting, the Borda count, the amendment procedure, majority rule, and the Hare procedure (single transferable vote).[18]

It is important to note that the classification of procedures by level of manipulability depends only on the amount of information necessary to manipulate a voting system and is not based on the ease of putting such information to good use or whether manipulation is most easily achieved by individual voters or groups. In practice, the manipulation of plurality rule by *individual* voters is quite difficult.

6 THE MEDIAN VOTER THEOREM

All of the preceding sections have focused on the behavior, strategic and otherwise, of voters in multiple alternative elections. However, strategic analysis can also be applied to *candidate* behavior in such elections. Given a particular distribution of voters and voter preferences, candidates will, for instance, need to determine optimal strategies in building their political platforms. When there are just two candidates in an election, when voters are distributed in a "reasonable" way along the political spectrum, and when each voter has "reasonably" consistent (meaning singled-peaked) preferences, the **median voter theorem** tells us that both candidates will position themselves on the political spectrum at the same place as the median voter. The **median voter** is the "middle" voter in that distribution—more precisely, the one at the 50th percentile.

The full game here has two stages. In the first stage, candidates choose their locations on the political spectrum. In the second stage, voters elect one of the candidates. The general second-stage game is open to all of the varieties of strategic misrepresentation of preferences discussed earlier. Hence we have

[18]H. Nurmi's classification can be found in his *Comparing Voting Systems* (Norwell, MA: D. Reidel, 1987).

reduced the choice of candidates to two for our analysis to prevent such behavior from arising in equilibrium. With only two candidates, second-stage votes will directly correspond to voter preferences, and the first-stage location decision of the candidates remains the only truly interesting part of the larger game. It is in that stage that the median voter theorem defines Nash equilibrium behavior.

A. Discrete Political Spectrum

Let us first consider a population of 90 million voters, each of whom has a preferred position on a five-point political spectrum: Far Left (FL), Left (L), Center (C), Right (R), and Far Right (FR). We suppose that these voters are spread symmetrically around the center of the political spectrum. The **discrete distribution** of their locations is shown by a **histogram,** or bar chart, in Figure 15.7. The height of each bar indicates the number of voters located at that position. In this example, we have supposed that, of the 90 million voters, 40 million are Left, 20 million are Far Right, and 10 million each are Far Left, Center, and Right.

Voters will vote for the candidate who publicly identifies herself as being closer to their own position on the spectrum in an election. If both candidates are politically equidistant from a group of like-minded voters, each voter flips a coin to decide which candidate to choose; this process gives each candidate one-half of the voters in that group.

Now suppose there is an upcoming Presidential election between a former First Lady (Claudia) and a former First Lady hopeful (Dolores), now running for

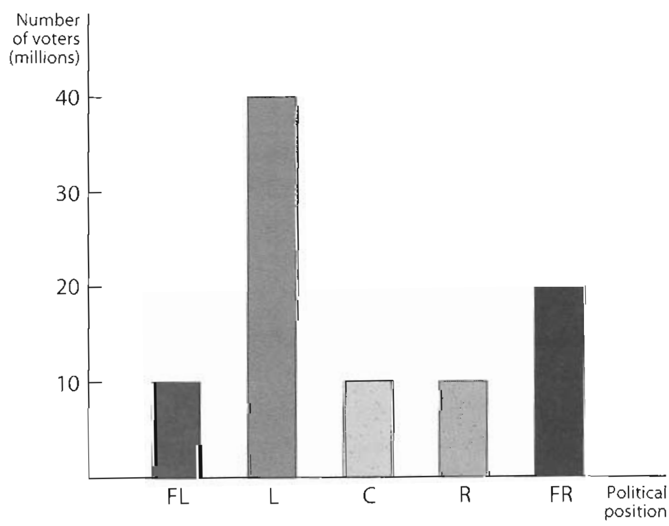


FIGURE 15.7 Discrete Distribution of Voters

office on her own.[19] Under the configuration of voters illustrated in Figure 15.7, we can construct a payoff table for the two candidates showing the number of votes that each can expect to receive under all of the different combinations of political platform choices. This five-by-five table is shown in Figure 15.8, with totals denoted in millions of votes. The candidates will choose their optimal location strategies to maximize the number of votes that they receive (and thus increase the chances of winning).[20]

Here is how the votes are allocated. When both candidates choose the *same* position (the five cells along the top-left to bottom-right diagonal of the table), each candidate gets exactly one-half of the votes; because all voters are equidistant from each candidate, all of them flip coins to decide their choices, and each candidate garners 45 million votes. When the two candidates choose *different* positions, the more-left candidate gets all the votes at or to the left of her position while the more-right candidate gets all the votes at or to the right of her position. In addition, each candidate gets the votes in central positions closer to her than to her rival, and the two of them split the votes from any voters in a central position equidistant between them. Thus, if Claudia locates herself at L while Dolores locates herself at FR, Claudia gets the 40 million votes at L, the 10 million at FL, *and* the 10 million at C (because C is closer to L than to FR). Dolores gets the 20 million votes at FR and the 10 million at R (because R is closer to FR than to L). The payoff is (60, 30). Similar calculations determine the outcomes in the rest of the table.

		DOLORES				
		FL	L	C	R	FR
CLAUDIA	FL	45, 45	10, 80	30, 60	50, 40	55, 35
	L	80, 10	45, 45	50, 40	55, 35	60, 30
	C	60, 30	40, 50	45, 45	60, 30	65, 25
	R	40, 50	35, 55	30, 60	45, 45	70, 20
	FR	35, 55	30, 60	25, 65	20, 70	45, 45

FIGURE 15.8 Payoff Table for Candidates' Positioning Game

[19]Any resemblance between our hypothetical candidates and actual past or possible future candidates in the United States is not meant to imply an analysis or prediction of their performances relative to the Nash equilibrium. Nor is our distribution of voters meant to typify U.S. voter preferences.

[20]To keep the analysis simple, we ignore the complications created by the electoral college and suppose that only the popular vote matters.

The table in Figure 15.8 is large, but the game can be solved very quickly. We begin with the now familiar search for dominant, or dominated, strategies for the two players. Immediately we see that, for Claudia, FL is dominated by L and FR is dominated by R. For Dolores, too, her FL is dominated by L and FR by R. With these extreme strategies eliminated, for each candidate her R is dominated by C. With the two R strategies gone, C is dominated by L for each candidate. The only remaining cell in the table is (L, L); this is the Nash equilibrium.

We now note three important characteristics of the equilibrium in the candidate location game. First, both candidates locate at the *same* position in equilibrium. This illustrates the **principle of minimum differentiation,** a general result in all two-player games of locational competition, whether it be political platform choice by presidential candidates, hotdog-cart location choices by street vendors, or product feature choices by electronics manufacturing firms.[21] When the persons who vote for or buy from you can be arranged on a well-defined spectrum of preferences, you do best by looking as much like your rival as possible. This explains a diverse collection of behaviors on the part of political candidates and businesses. It may help you understand, for example, why there is never just one gas station at a heavily traveled intersection or why all brands of four-door sedans (or minivans or sport utility vehicles) seem to look the same even though every brand claims to be coming out continually with a "new" look.

Second and perhaps most crucial, both candidates locate at the position of the median voter in the population. In our example, with a total of 90 million voters, the median voter is number 45 million from each end. The numbers within one location can be assigned arbitrarily, but the location of the median voter is clear; here, the median voter is located at the L position on the political spectrum. So that is where both candidates locate themselves, which is the result predicted by the median voter theorem.

Third, observe that the location of the median voter need not coincide with the geometric center of the spectrum. The two will coincide if the distribution of voters is symmetric, but the median voter can be to the left of the geometric center if the distribution is skewed to the left (as is true in Figure 15.7) and to the right if the distribution is skewed to the right. This helps explain why state political candidates in Massachusetts, for example, *all* tend to be more liberal than candidates for similar positions in Texas or South Carolina.

The median voter theorem can be expressed in different ways. One version states simply that the position of the median voter is the equilibrium location position of the candidates in a two-candidate election. Another version says that the position that the median voter most prefers will be the Condorcet winner; this position will defeat every other position in a pairwise contest. For

[21]Economists learn this result within the context of Hotelling's model of spatial location. See Harold Hotelling, "Stability in Competition," *Economic Journal*, vol. 39, no. 1 (March 1929), pp. 41–57.

example, if M is this median position and L is any position to the left of M, then M will get all the votes of people who most prefer a position at or to the right of M, plus some to the left of M but closer to M than to L. Thus M will get more than 50% of the votes. The two versions amount to the same thing because, in a two-candidate election, both seeking to win a majority will adopt the Condorcet-winner position. These interpretations are identical. In addition, to guarantee that the result holds for a particular population of voters, the theorem (in either form) requires that each voter's preferences be "reasonable," as suggested earlier. *Reasonable* here means "single peaked," as in Black's condition described in Section 3.A and Figure 15.4. Each voter has a unique most-preferred position on the political spectrum, and her utility (or payoff) decreases away from that position in either direction.[22] In actual U.S. Presidential elections, the theorem is borne out by the tendency for the main candidates to make very similar promises to the electorate.

B. Continuous Political Spectrum

The median voter theorem can also be proved for a continuous distribution of political positions. Rather than having five, three, or any finite number of positions from which to choose, a **continuous distribution** assumes there are effectively an infinite number of political positions. These political positions are then associated with locations along the real number line between 0 and 1.[23] Voters are still distributed along the political spectrum as before, but, because the distribution is now continuous rather than discrete, we use a voter **distribution function** rather than a histogram to illustrate voter locations. Two common functions—the

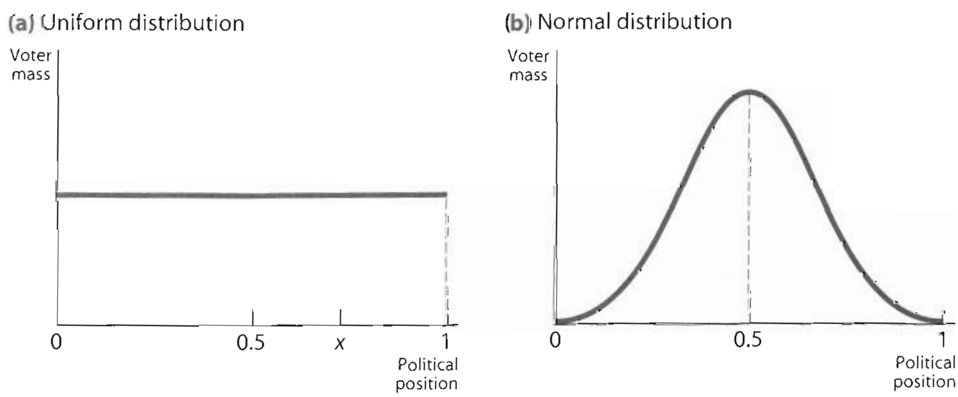


FIGURE 15.9 Continuous Voter Distributions

[22]However, the distribution of voters' ideal points along the political spectrum does not have to be single peaked, as indeed the histogram in Figure 15.7 is not—there are two peaks at L and FR.

[23]This construction is the same one used in Chapters 12 and 13 for analyzing large populations of individual members.

uniform distribution and the (symmetric) **normal distribution**—are illustrated in Figure 15.9.[24] The area under each curve represents the total number of votes available; at any given point along the interval from 0 to 1, such as x in Figure 15.9a, the number of votes up to that point is determined by finding the area under the distribution function from 0 to x. It should be clear that the median voter in each of these distributions is located at the center of the spectrum, at position 0.5.

It is not feasible to construct a payoff table for our two candidates in the continuous-spectrum case; such tables must necessarily be finitely dimensioned and thus cannot accommodate an infinite number of possible strategies for players. We can, however, solve the game by applying the same strategic logic that we used for the discrete (finite) case discussed in Section 5.A.

Consider the options of Claudia and Dolores as they contemplate the possible political positions open to them. Each knows that she must find her Nash equilibrium strategy—her best response to the equilibrium strategy of her rival. We can define a set of strategies that are best responses quite easily in this game, even though the complete set of possible strategies is impossible to delineate.

Suppose Dolores locates at a random position on the political spectrum, such as x in Figure 15.9a. Claudia can then calculate how the votes will be split for all possible positions that she might choose. If she chooses a position to the left of x, she gets all the votes to her left and half of the votes lying between her position and Dolores's. If she locates to the right of x, she gets all the votes to her right and half of the votes lying between her position and x. Finally, if she, too, locates at x, she and Dolores split the votes 50–50. These three possibilities effectively summarize all of Claudia's location choices, given that Dolores has chosen to locate at x.

But which of the response strategies just outlined is Claudia's "best" response? The answer depends on the location of x relative to the median voter. If x is to the right of the median, then Claudia knows that her best response will be to maximize the number of votes that she gains, which she can do by locating an infinitely small bit to the left of x.[25] In that case, she effectively gets all the votes from 0 to x while Dolores gets those from x to 1. When x is to the right of the median, as in Figure 15.9a, then the number of voters represented by the area under the distribution curve from 0 to x is by definition larger than the number of voters from x to 1; so Claudia would win the election. Similarly, if x

[24]We do not delve deeply into the mechanics underlying distribution theory or the integral calculus required to calculate the exact proportion of the voting population lying to the left or right of any particular position on the continuous political spectrum. Here we present only enough information to convince you that the median voter theorem continues to hold in the continuous case.

[25]Such a location, infinitesimally removed from x to the left, is feasible in the continuous case. In our discrete example, candidates had to locate at exactly the same position.

is to the left of the median, Claudia's best response will be to locate an infinitely small bit to the right of x and thus gain all the votes from x to 1. When x is exactly at the median, Claudia does best by choosing to locate at x also. The best-response strategies for Dolores are constructed exactly the same way and, given the location of her rival, are exactly the same as those described for Claudia. Graphically, these best-response curves lie just above and below the 45° line up to the position of the median voter, at which point they lie exactly on the 45° line (Claudia's best response to Dolores's location at that of the median voter is to locate in the same place; the same is true in reverse for Dolores). Beyond the position of the median voter, the best-response curves switch sides of the 45° line.

We now have complete descriptions of the best-response strategies for both candidates. The Nash equilibrium occurs at the intersection of the best-response curves; this intersection lies at the position of the median voter. You can think this through intuitively by picking any starting location for one of the candidates and applying the best-response strategies over and over until each candidate is located at a position that represents her best response to the position chosen by her rival. If Dolores were contemplating locating at x in Figure 15.9a, Claudia would want to locate just to the left of x, but then Dolores would want to locate just to the left of that, and so on. Only when the two candidates locate exactly at the median of the distribution (whether the distribution is uniform or normal or some other kind), do they find that their decisions are best responses to each other. Again we see that the Nash equilibrium is for both candidates to locate at the position of the median voter.

More complex mathematics are needed to prove the continuous version of the median voter theorem to the satisfaction of a true mathematician. For our purposes, however, the discussion given here should convince you of the validity of the theorem in both its discrete and continuous forms. The most important limitation of the median voter theorem is that it applies when there is just one issue, or a one-dimensional spectrum of political differences. If there are two or more dimensions—for example, if being conservative versus liberal on social issues does not coincide with being conservative versus liberal on economic issues—then the population is spread out in a two-dimensional "issue space" and the median voter theorem no longer holds. The preferences of every individual voter can be single peaked, in the sense that the individual voter has a most-preferred point and her payoff value drops away from this point in all directions, like the height going away from the peak of a hill. But we cannot identify a median voter in two dimensions, such that exactly the same number of voters have their most-preferred point to the one side of the median voter position as to the other side. In two dimensions, there is no unique sense of side, and the numbers of voters to the two sides can vary, depending on just how we define "side."

SUMMARY

Elections can be held with the use of a variety of different voting procedures that alter the order in which issues are considered or the manner in which votes are tallied. Voting procedures are classified as *binary, plurative,* or *mixed methods.* Binary methods include *majority rule,* as well as *pairwise* procedures such as the *Condorcet method* and the *amendment procedure. Positional methods* such as *plurality rule* and the *Borda count,* as well as *approval voting,* are plurative methods. And *majority runoffs, instant runoffs,* and *proportional representation* are mixed methods.

Voting paradoxes (such as the *Condorcet,* the *agenda,* and *the reversal paradox*) show how counterintuitive results can arise owing to difficulties associated with aggregating preferences or to small changes in the list of issues being considered. Another paradoxical result is that outcomes in any given election under a given set of voter preferences can change, depending on the voting procedure used. Certain principles for evaluating voting methods can be described, although Arrow's *impossibility theorem* shows that no one system satisfies all of the criteria at the same time. Researchers in a broad range of fields have considered alternatives to the principles identified by Arrow.

Voters have scope for strategic behavior in the game that chooses the voting procedure or in an election itself through the *misrepresentation of their own preferences.* Preferences may be strategically misrepresented to achieve the voter's most-preferred or to avoid the least-preferred outcome. In the presence of imperfect information, voters may decide whether to vote strategically on the basis of their beliefs about others' behavior and their knowledge of the distribution of preferences.

Candidates also may behave strategically in building a political platform. A general result known as the *median voter theorem* shows that, in elections with only two candidates, both locate at the preference position of the *median voter.* This result holds when voters are distributed along the preference spectrum either *discretely* or *continuously.*

KEY TERMS

agenda paradox (507)
amendment procedure (502)
antiplurality method (502)
approval voting (503)
binary method (501)
Black's condition (512)
Borda count (502)
Condorcet method (501)
Condorcet paradox (505)
Condorcet terms (514)
Condorcet winner (502)
continuous distribution (528)

EXERCISES

1. Consider a vote being taken by three roommates, A, B, and C, who share a triple dorm room. They are trying to decide which of three elective courses to take together this term. (Each roommate has a different major and is taking required courses in her major for the rest of her courses.) Their choices are Philosophy, Geology, and Sociology, and their preferences for the three courses are as shown here:

A	B	C
Philosophy	Sociology	Geology
Geology	Philosophy	Sociology
Sociology	Geology	Philosophy

They have decided to have a two-round vote and will draw straws to determine who sets the agenda. Suppose A sets the agenda and wants the Philosophy course to be chosen. How should she set the agenda to achieve this outcome if she knows that everyone will vote truthfully in all rounds? What agenda should she use if she knows that they will all vote strategically?

2. Repeat Exercise 1 for the situation in which B sets the agenda and wants to ensure that Sociology wins.

3. Suppose that voters 1 through 4 are being asked to consider three different candidates—A, B, and C—in a Borda count election. Their preferences are:

1	2	3	4
A	A	B	C
B	B	C	B
C	C	A	A

Assume that voters will cast their votes truthfully (no strategic voting). Find a Borda weighting system—a number of points to be allotted to first, second, and third preferences—in which candidate A wins.

4. Repeat Exercise 3 to find a Borda weighting system in which candidate B wins.

5. Consider a group of 50 residents attending a town meeting in Massachusetts. They must choose between three proposals for dealing with town garbage. Proposal 1 asks the town to provide garbage collection as one of its services; Proposal 2 calls for the town to hire a private garbage collector to provide collection services; and Proposal 3 calls for residents to be responsible for their own garbage. There are three types of voters. The first type prefers Proposal 1 to Proposal 2 and Proposal 2 to Proposal 3; there are 20 of these voters. The second type prefers Proposal 2 to Proposal 3 and Proposal 3 to Proposal 1; there are 15 of these voters. The third type prefers Proposal 3 to Proposal 1 and Proposal 1 to Proposal 2; there are 15 of them.
   (a) Under a plurality voting system, which proposal wins?
   (b) Suppose voting proceeds with the use of a Borda count in which voters list the proposals, in order of preference, on their ballots. The proposal listed first (or at the top) on a ballot gets three points; the proposal listed second gets two points; and the proposal listed last gets one point. In this situation, with no strategic voting, how many points are gained by each proposal? Which proposal wins?
   (c) What strategy can the second and third types of voters use to alter the outcome of the Borda count vote in part **b** to one that both types prefer? If they use this strategy, how many points does each proposal get, and which wins?

6. Olympic skaters complete two programs in their competition, one short and one long. In each program, the skaters are scored and then ranked by a panel of nine judges and their positions in the rankings are used to

determine their final scores. A skater's ranking depends on the number of judges placing her first (or second or third); the skater judged to be best by the most judges is ranked number one and so on. In calculating a skater's final score, the short program gets half the weight of the long program. That is, Final score = 0.5 (Rank in short program) + Rank in long program. The skater with the lowest final score wins the gold medal. In the event of a tie, the skater judged best in the long program by the largest number of judges takes the gold. In 2002 in Salt Lake City, the women's individual figure skating competition had Michelle Kwan in first place after the short program. She was followed by Irina Slutskaya, Sasha Cohen, and Sarah Hughes, who were in second, third, and fourth places, respectively. In the long program, the judges' cards for these four skaters were as follows:

		JUDGE NUMBER								
		1	2	3	4	5	6	7	8	9
KWAN	**Points**	11.3	11.5	11.7	11.5	11.4	11.5	11.4	11.5	11.4
	Rank	2	3	2	2	2	3	3	2	3
SLUTSKAYA	**Points**	11.3	11.7	11.8	11.6	11.4	11.7	11.5	11.4	11.5
	Rank	3	1	1	1	4	1	2	3	2
COHEN	**Points**	11.0	11.6	11.5	11.4	11.4	11.4	11.3	11.3	11.3
	Rank	4	2	4	3	3	4	4	4	4
HUGHES	**Points**	11.4	11.5	11.6	11.4	11.6	11.6	11.3	11.6	11.6
	Rank	1	4	3	4	1	2	1	1	1

(a) At the Olympics, Slutskaya skated last of the top skaters. Use the information from the judges' cards to determine the judges' long-program ranks for Kwan, Cohen, and Hughes *before* Slutskaya skated. Then, using the standings already given for the short program in conjunction with your calculated ranks for the long program, determine the final scores, and standings, among these three skaters *before* Slutskaya skated. (Note that Kwan's rank in the short program was 1, and so her partial score after the short program is 0.5.)

(b) Given your answer to part **a**, what would have been the final outcome of the competition if the judges had ranked Slutskaya's long program above all three of the others?

(c) Use the judges' cards to determine the actual final scores for all four skaters *after* Slutskaya skated. Who won each medal?

(d) What important principle, of those identified by Arrow, does the Olympic figure-skating scoring system violate? Explain.

7. During the Cuban missile crisis, serious differences of opinion arose within the ExComm group advising President John Kennedy, which we summarize here. There were three options: Soft (a blockade), Medium (a limited air strike), and Hard (a massive air strike or invasion). There were also three groups in ExComm. The civilian doves ranked the alternatives Soft best, Medium next, and Hard last. The civilian hawks preferred Medium best, Hard next, and Soft last. The military preferred Hard best, but they felt "so strongly about the dangers inherent in the limited strike that they would prefer taking no military action rather than to take that limited strike." [*The Kennedy Tapes: Inside the White House During the Cuban Missile Crisis*, ed. Ernest R. May and Philip D. Zelikow (Cambridge, MA: Harvard University Press, 1997), p.97.] In other words, they ranked Soft second and Medium last. Each group constituted about one-third of ExComm, and so any two of the groups would form a majority.
   (a) If the matter were to be decided by majority vote in ExComm and the members voted sincerely, which alternative, if any, would win?
   (b) What outcome will arise if members vote strategically? What outcome will arise if one group has agenda-setting power? (Model your discussion in these two cases after the analysis found in Sections 2.B and 4.B.)

8. John Paulos, in his book titled *A Mathematician Reads the Newspaper*, gives the following caricature based on the 1992 Democratic Presidential primary caucuses. There are five candidates: Jerry Brown, Bill Clinton, Tom Harkin, Bob Kerrey, and Paul Tsongas. There are 55 voters, with different preference orderings concerning the candidates. There are six different orderings, which we label I through VI. The preference orderings (1 for best to 5 for worst) along with the numbers of voters with each ordering, are shown in the following table; the candidates are identified by the first letters of their last names.[26]

		GROUPS AND THEIR SIZES					
		I 18	II 12	III 10	IV 9	V 4	VI 2
RANKING	1	T	C	B	K	H	H
	2	K	H	C	B	C	B
	3	H	K	H	H	K	K
	4	B	B	K	C	B	C
	5	C	T	T	T	T	T

[26]John Allen Paulos, *A Mathematician Reads the Newspaper* (New York: Basic Books, 1995), pp. 104–106.

(a) First suppose that all voters vote sincerely, and consider the outcomes of each of several different election rules. Show each of the following outcomes: (i) Under the plurality method (the one with the most first preferences), Tsongas wins. (ii) Under the runoff method (the top two first preferences go into a second round), Clinton wins. (iii) Under the elimination method (at each round, the one with the least first preferences at that round is eliminated, and the rest go into the next round), Brown wins. (iv) Under the Borda count method (five points for first preference, four for second, and so on; the candidate with the most points wins), Kerrey wins. (v) Under the Condorcet method (pairwise comparisons), Harkin wins.

(b) Suppose that you are a Brown, Kerrey, or Harkin supporter. Under the plurality method, you would get your worst outcome. Can you benefit by voting strategically? If so, how?

(c) Are there opportunities for strategic voting under each of the other methods as well? If so, explain who benefits from voting strategically and how they can do so.

9. Recall the three-member Council considering three alternative welfare policies in Section 4.C. There, three Councillors (Left, Center, and Right) considered policies A and D in a first-round vote with the winner facing policy G in a second-round election, but no one knows for sure exactly how many councillors have each set of possible preferences; the possible preference orderings are shown in Figure 15.1. Each councillor knows her own type and she knows the probabilities of observing each type of voter, p_L, p_C, and p_R (with $p_L + p_C + p_R = 1$). The behavior of the Center-type voters in the first-round election is the only unknown in this situation and will depend on the probabilities that the various preference types occur. Suppose here that a Center-type voter believes (in contrast with the case considered in the text) that other Center types will vote strategically; suppose further that the Center type's payoffs are as in Section 4.C: 1 if A wins, 0 if G wins, and $0 < u < 1$ if D wins.

(a) Under what configuration of the other two votes does the Center-type voter's first-round vote matter to the outcome of the election? Given her assumption about the behavior of other Center-type voters, how would she identify the source of the first-round votes?

(b) Following the analysis in Section 4.C, determine the expected payoff to the Center type when she votes truthfully. Compare this with her expected payoff when she votes strategically. What is the condition under which the Center type votes strategically?

10. An election has three candidates and takes place under the plurality rule. There are numerous voters, spread along an ideological spectrum from left to right. Represent this spread by a horizontal straight line whose extreme

points are 0 (left) and 1 (right). Voters are uniformly distributed along this spectrum; so the number of voters in any segment of the line is proportional to the length of that segment. Thus a third of the voters are in the segment from 0 to 1/3, a quarter in the segment from 1/2 to 3/4, and so on. Each voter votes for the candidate whose declared position is closest to the voter's own position. The candidates have no ideological attachment and take up any position along the line, each seeking only to maximize her share of votes.

(a) Suppose you are one of the three candidates. The left-most of the other two is at point x, and the right-most is at the point $(1 - y)$, where $x + y < 1$ (so the right-most candidate is a distance y from 1). Show that your best response is to take up the following positions under the given conditions:
  (i) just slightly to the left of x if $x > y$ and $3x + y > 1$,
  (ii) just slightly to the right of $(1 - y)$ if $y > x$ and $x + 3y > 1$,
  (iii) and exactly halfway between the other candidates if $3x + y < 1$ and $x + 3y < 1$.

(b) In a graph with x and y along the axes, show the areas (the combination of x and y values) where each of the response rules (i to iii in part a) is best for you.

(c) From your analysis, what can you conclude about the Nash equilibrium of the game where the three candidates each choose positions?

16

Bidding Strategy and Auction Design

IN THIS CHAPTER, we consider a topic that is becoming increasingly relevant in many of our lives—bidding strategy and auction design. Although you might argue that you have never been, nor do you expect ever to be, at Christie's or Sotheby's or an antique auction in the wilds of Vermont or a livestock auction in Illinois, you probably participate in more auctions than you imagine. With the phenomenal recent growth in online auction sites, for example, millions of people now buy regularly at auctions. And other transactions in which you participate may also be classified as auctions.

Auctions entail the transfer of a particular object from a seller to a buyer (or bidder) for a certain price (or bid). Considered in this simple form, many market transactions resemble auctions. For example, stores such as Filene's Basement in Boston use a clever pricing strategy to keep customers coming back for more: they reduce the prices on items remaining on the racks successively each week until either the goods are purchased or the price gets so low that they donate the items to charity. Shoppers love it. Little do they realize that they are participating in what is known as a descending, or Dutch, auction—one of the types of auctions described in detail in this chapter.

Even if you do not personally participate in many auctions, your life is greatly influenced by them. The Federal Communications Commission (FCC) since 1994 has auctioned off large parts of the electromagnetic broadcasting spectrum in more than 40 different auctions. The kind of television that you watch for the next few decades and the kinds of cellular phones that you use will be affected by this process and its outcomes. These auctions have already raised some $41 billion in government revenues, as of June 2003, and there are four auctions scheduled for the summer and fall of 2003. Because these revenues have made significant contributions to the federal budget, they have affected important macroeconomic magnitudes, such as interest rates. International variables also have been affected by not only the U.S. spectrum auctions, but also similar auctions in at least six European countries as well as in Australia and New Zealand. Understanding how auctions work will help you understand these important events and their implications.

From a strategic perspective, auctions have several characteristics of interest. Most crucial is the existence of asymmetric information between seller and bidders, as well as among bidders. Thus, signaling and screening can be important components of strategy for both bidders and sellers. In addition, optimal strategies for both bidders and sellers will depend on their levels of aversion to risk. We will also see that under some specific circumstances expected payoffs to the seller as well as to the winning bidder are the same across auction types.

This chapter explores the various types of auctions, as well as the strategies that you might want to employ as either the buyer or the seller in such situations. The formal theory of auctions relies on advanced calculus to derive its results, but we eschew most of this difficult mathematics in favor of more intuitive descriptions of optimal behavior and strategy choice.[1] As an example of how auction design and bidding theory influence modern commerce, we provide a detailed discussion of the use of auctions on the Internet.

1 TYPES OF AUCTIONS

Auctions differ in the methods used for the submission of bids and for the determination of the final price paid by the winner. In addition, auctions can be classified according to the way in which buyers might value the object being auctioned. Here we categorize the various auction types, describing their characteristics and mechanics.

The four major categories of auctions can be divided into two groups. The first group is known as **open outcry.** In this type of auction, bidders call out or otherwise make their bids in public. All bidders are able to observe bids as they are made. This type perhaps best fits the popular vision of the way in which auctions work—an image that includes feverish bidders and an auctioneer. But open outcry auctions can be organized in two ways. Only one of them would ever demonstrate "feverish" bidding.

[1]A reference list of sources for additional information on the theory and practice of auctions can be found in the final section of the chapter.

The **ascending,** or **English,** version of an open-outcry auction conforms best to this popular impression of auctions. Ascending auctions were and still are the norm at English auction houses such as Christie's and Sotheby's from which they take their alternate name. The auction houses have a conventional auctioneer who starts at a low price and calls out successively higher prices for an item, waiting to receive a bid at each price before going on. When no further bids can be obtained, the item goes to the most recent, highest, bidder. Thus, any number of bidders can take part in English auctions, although only the top bidder gains the item up for sale. And the bidding process may not literally entail the actual outcry of bids, because the mere nod of a head or the flick of a wrist is common bidding behavior in such auctions. A large majority of the existing Internet auction sites now run what are essentially ascending auctions (in virtual, rather than real, time) for almost any item imaginable.

The other type of open outcry auction is the **Dutch,** or **descending,** auction. Dutch auctions, which get their name from the way in which tulips and other flowers are auctioned in the Netherlands, work in the opposite direction from that of English auctions. The auctioneer starts at an extremely high price and calls out successively lower prices until one of the assembled potential bidders accepts the price, makes a bid, and takes the item. Because of the desire or need for speed, Dutch flower auctions, as well as auctions for other agricultural or perishable goods (such as the daily auction at the Sydney Fish Market), use a "clock" that ticks down (counterclockwise) to ever lower prices until one bidder "stops the clock" and collects her merchandise. In many cases, the auction clock displays considerable information about the lot of goods currently for sale in addition to the falling price of those goods. And, unlike the English auction, there is no feverish bidding in a Dutch auction, because only the one person who "stops the clock" takes any action.

The second group of auctions are those that are conducted by **sealed bid.** In these auctions, bidding is done privately and bidders cannot observe any of the bids made by others; in many cases, only the winning bid is announced. Bidders in such auctions, as in Dutch auctions, have only one opportunity to bid. (Technically, you could submit multiple bids, but only the highest one would be relevant to the auction outcome.) Sealed-bid auctions have no need for an auctioneer. They require only an overseer who opens the bids and determines the winner.

Within sealed-bid auctions, there are two methods for determining the price paid by the high bidder. In a **first-price** sealed-bid auction, the highest bidder wins the item and pays a price equal to her bid. In a **second-price** sealed-bid auction, the highest bidder wins the item but pays a price equal to the bid of the second-highest bidder. A second-price structure can be extremely useful for eliciting truthful bids, as we will see in Section 4, and such auctions are often termed **Vickrey auctions** after the economist who first noted this particular characteristic. We will also see that the sealed-bid auctions are each similar, in regard to bidding strategy and expected payoffs, to one of the open-outcry auctions; first-price sealed-bid auctions are similar to Dutch auctions, and second-price sealed-bid auctions are similar to English auctions.

Finally, there are two ways in which bidders may value an item up for auction. In a **common-value,** or **objective-value,** auction, the value of the object is the same for all the bidders, but each bidder generally knows only an imprecise estimate of it. Bidders may have some sense of the distribution of possible values, but each must form her own estimate before bidding. For example, an oil-drilling tract has a given amount of oil that should produce the same revenue for all companies, but each company has only its own expert's estimate of the amount of oil contained under the tract. Similarly, each bond trader has only an estimate of the future course of interest rates. In such auctions, signaling and screening can play an important role. Each bidder should be aware of the fact that other bidders possess some (however sketchy) information about an object's value, and she should attempt to infer the contents of that information from the actions of rival bidders. In addition, she should be aware of how her own actions might signal her private information to those rival bidders.

The second type of valuation is the **private-value,** or **subjective-value,** auction. In this case, bidders place different values on an object. For example, a gown worn by Princess Diana or a necklace worn by Jacqueline Bouvier Kennedy Onassis may have sentimental value to some bidders. Bidders know their own private valuations in such auctions but do not know one another's valuations of an object. Similarly, the seller does not know any of the bidders' valuations. Bidders and sellers may each be able to formulate rough estimates of others' valuations and, as above, can use signals and screens to attempt to improve their final outcomes. The information problem is relevant, then, not only to bidding strategies, but also to the seller's strategy in designing the form of auction to identify the highest valuation and to extract the best price.

Other, less common, configurations also can be used to sell goods at auction. For example, you could set up an auction in which the highest bidder wins but the top two bidders pay their bids or one in which the high bidder wins but all bidders pay their bids, a procedure discussed in Section 5. We do not attempt to consider all possible combinations here. Rather, we analyze several of the most common auction schemes by using examples that bring out important strategic concepts.

2 THE WINNER'S CURSE

A standard but often ignored outcome arises in common-value auctions. Recall that such auctions entail the sale of an object whose value is fixed and identical for all bidders, although each bidder can only estimate it. The **winner's curse** is a warning to bidders that, if they win the object in the auction, they are likely to have paid more than it is worth.

Suppose you are a corporate raider bidding for Targetco. Your experts have studied this company and produced estimates that, in the hands of the current management, it is worth somewhere between 0 and \$10 billion, all values in this range being equally likely. The current management knows the precise figure, but of course it is not telling you. You believe that, whatever Targetco is worth under existing management, it will be worth 50% more under your control. What should you bid?

You might be inclined to think that, on average, Targetco is worth \$5 billion under existing management and thus \$7.5 billion, on average, under yours. If so, then a bid somewhere between \$5 billion and \$7.5 billion should be profitable. But such a bidding strategy reckons without the response of the existing management to your bid. If Targetco is actually worth more than your bid, the current owners are not going to accept the bid. You are going to get the company only if its true worth is toward the lower end of the range.

Suppose you bid amount b. Your bid will be accepted and you will take over the management of Targetco if it is worth somewhere between 0 and b under the current management; on average, you can expect the company to be currently worth $b/2$ if your bid is accepted. In your hands, the average worth will be 50% more than the current worth, or $(1.5)(b/2) = 0.75b$. Because this value is always less than b, you would win the takeover battle only when it was not worth winning! Many raiders seem to have discovered this fact too late.

But corporate raiders, often engaged in one-on-one negotiations with target firms resembling auctions with only one bidder, are not the only ones affected by the winner's curse. Similar problems arise when you are competing with other bidders in a common-value auction and all of you have separate estimates for the object's value.

Consider a lease for the oil- or gas-drilling rights on a tract of land (or sea).[2] At the auction for this lease, you win only if your rivals make estimates of the value of the lease that are lower than your estimate. You should recognize this fact and try to learn from it.

Suppose the true value of the lease, unknown to any of the bidders, is \$1 billion. (In this case, the seller probably does not know the true value of the tract either.) Suppose there are 10 oil companies in the bidding. Each company's experts estimate the value of the tract with an error of \$100 million, all numbers in this range being equally likely. If all 10 of the estimates could be pooled, their arithmetic average would be an unbiased and much more accu-

[2]For example, the United States auctions leases for offshore oil-drilling rights, including rights in the Gulf of Mexico and off the coast of Alaska. The state of Pennsylvania is contemplating the auction of leases for natural gas–drilling rights on almost half a million acres of state forest land.

rate indicator of the true value than any single estimate. But, when each bidder sees only one estimate, the largest of these estimates is biased: on average, it will be $1.08 billion, right near the upper end of the range.[3] Thus, the winning company is likely to pay too much, unless it recognizes the problem and adjusts its bid downward to compensate for this bias. The exact calculation required to determine how far to shade down your bid without losing the auction is difficult, however, because you must also recognize that all the other bidders will be making the same adjustment.

We do not pursue the advanced mathematics required to create an optimal bidding strategy in the common-value auction. However, we can provide you with some general advice. If you are bidding on an item, the question "Would I be willing to purchase the lease for $1.08 billion, given what I know before submitting my bid?" is very different from the question "Would I still be willing to purchase the lease for $1.08 billion, given what I know before submitting my bid and given the knowledge that I will be able to purchase the lease only if no one else is willing to bid $1.08 billion for it?"[4] Even in a sealed-bid auction, it is the second question that reveals correct strategic thinking, because you win with any given bid only when all others bid less—only when all other bidders have a lower estimate of the value of the object than you do.

If you do not take the winner's curse into account in your bidding behavior, you should expect to lose substantial amounts, as indicated by the numerical calculations done earlier for bidding on the hypothetical Targetco. How real is this danger in practice? Richard Thaler has marshaled a great deal of evidence to show that the danger is very real indeed.[5]

The simplest experiment to test the winner's curse is to auction a jar of pennies. The prize is objective, but each bidder forms a subjective estimate of how many pennies there are in the jar and therefore of the size of the prize; this experiment is a pure example of a common-value auction. Most teachers have conducted such experiments with students and found significant overbidding. In a similar but related experiment, M.B.A. students were asked to bid for a hypothetical company instead of a penny jar. The game was repeated, with feedback after each round on the true value of the company. Only 5 of 69 students learned to bid less over time; the average bid actually went up in the later rounds.

Observations of reality confirm these findings. There is evidence that winners of oil- and gas-drilling leases at auctions take substantial losses on their leases. Baseball players who as free agents went to new teams were found to be overpaid in comparison with those who re-signed with their old teams.

[3]The 10 estimates will, on average, range from $0.9 billion to $1.1 billion ($100 million on either side of $1 billion). The low and high estimates will, on average, be at the extremes of the distribution.

[4]See Steven Landsburg, *The Armchair Economist* (New York: Free Press, 1993), p. 175.

[5]Richard Thaler, "Anomalies: The Winner's Curse," *Journal of Economic Perspectives*, vol. 2, no. 1 (Winter 1988), pp. 191–201.

We repeat: the precise calculations that show how much you should shade down your bidding to take into account the winner's curse are beyond the scope of this text; the articles cited in Section 9 contain the necessary mathematical analysis. Here we merely wish to point out the problem and emphasize the need for caution. When your willingness to pay depends on your expected ability to make a profit from your purchase or on the expected resale value of the item be wary.

This analysis shows the importance of the prescriptive role of game theory. From observational and experimental evidence, we know that many people fall prey to the winner's curse. By doing so, they lose a lot of money. Learning the basics of game theory would help them anticipate the winner's curse and prevent attendant losses.

3 BIDDING STRATEGIES

We turn now to private-value auctions and a discussion of optimal bidding strategies. Suppose you are interested in purchasing a particular lot of Chateau Margaux 1952 Bordeaux wine. Consider some of the different possible auction procedures that could be used to sell the wine.

Suppose first that you are participating in a standard, English auction. Your optimal bidding strategy is straightforward, given that you know your valuation V. Start at any step of the bidding process. If the last bid made by a rival bidder, call it r, is at or above V, you are certainly not willing to bid higher; so you need not concern yourself with any further bids. Only if the last bid is still below V do you bid at all. In that case, you can add a penny (or the smallest increment allowed by the auction house) and bid r plus one cent. If the bidding ends there, you get the wine for r (or virtually r), and you make an effective profit of $V - r$. If the bidding continues, you repeat the process, substituting the value of the new last bid for r. In this type of auction, the high bidder gets the wine for (virtually) the valuation of the second-highest bidder. How close the final price is to the second-highest valuation will be determined by the minimum bid increment defined in the auction rules.

Now suppose the wine auction is first-price sealed-bid and you suspect that you are a very high value bidder. You need to decide whether to bid V or something other than V. Should you put in a bid equal to the full value V that you place on the object?

Remember that the high bidder in this auction will be required to pay her bid. In that case, you should not in fact bid V. Such a bid would be sure to give you zero profit, and you could do better by reducing your bid somewhat. If you bid a little less than V, you run the risk of losing the object should a rival bidder make a bid above yours but below V. But, as long as you do not bid so low that this outcome is guaranteed, you have a positive probability of making a positive profit. Your optimal bidding strategy entails **shading** your bid. Calculus would be required to describe the actual strategy required here, but an intuitive understanding of the result is simple. An increase in shading (a lowering of your bid from V) provides both an advantage and a disadvantage to you; it increases your profit margin if you obtain the wine, but it also lowers your chances of being the high bidder and therefore of actually obtaining the wine. Your bid is optimal when the last bit of shading just balances these two effects.

What about a Dutch auction? Your bidding strategy in this case is similar to that for the first-price sealed-bid auction. Consider your bidding possibilities. When the price called out by the auctioneer is above V, you choose not to bid. If no one has bid by the time the price gets down to V, you may choose to do so. But, again, as in the sealed-bid case, you have two options. You can bid now and get zero profit or wait for the price to drop lower. Waiting a bit longer will increase the profit that you take from the sale, but it also increases your risk of losing the wine to a rival bidder. Thus shading is in your interest here as well, and the precise amount of shading depends on the same cost-benefit analysis described in the preceding paragraph.

Finally, there is the second-price sealed-bid auction. In that auction, the cost-benefit analysis regarding shading is different from that in the preceding three types of auctions. This result is due to the fact that the advantage gained from shading, the increase in your profit margin, is zero in this auction. You do not improve your profit by shading your bid, because your profit is determined by the second-highest bid, not your own. Because second-price sealed-bid auctions have this interesting property, all of the next section deals with the analysis of bidding strategies in such auctions.

4 VICKREY'S TRUTH SERUM

We have just seen that bidding strategies in private-value English auctions differ from those for first-price sealed-bid auctions. The high bidder in the English auction can get the object for essentially the valuation of the second-highest bidder. In the sealed-bid auction, the high bidder pays her bid, regardless of the distance between it and the next highest bid. Strategic bidders in a sealed-bid auction recognize this fact and attempt to retain some profit (surplus) for themselves by shading their bids. All else being equal, sellers would prefer bids that were not shaded downward. Let us see how sellers might induce bidders to reveal their true valuations with their bids.

Rather than using a first-price auction when selling a subjectively (private) valued object with sealed bids, William Vickrey showed that truthful revelation of valuations from bidders would arise if the seller used a modified version of the sealed-bid scheme; his suggestion was to modify the sealed-bid auction so that it more closely resembles its open-outcry counterpart.[6] That is, the highest bidder should get the object for a price equal to the second-highest bid—a second-price sealed-bid auction. Vickrey showed that, with these rules, every bidder has a dominant bidding strategy to bid her true valuation. Thus we facetiously dub it **Vickrey's truth serum.**

However, we saw in Chapter 9 that there is a cost to extracting information. Auctions are no exception. Buyers reveal the truth about their valuations in an auction using Vickrey's scheme only because it gives them some profit from doing so. It reduces the profit for the seller, just as the shading of bids does in a first-price auction. The relative merit of the two procedures from the seller's point of view therefore depends on which one entails a greater reduction in her profit. We consider this matter later in Section 6; but first we explain how Vickrey's scheme works.

Suppose you are an antique china collector and you have discovered that a local estate auction will be selling off a 19th century Meissen "Blue Onion" tea set in a sealed-bid second-price auction. As someone experienced with vintage china but lacking this set for your collection, you value it at $3,000, but you do not know the valuations of the other bidders. If they are inexperienced, they may not realize the considerable value of the set. If they have sentimental attachments to Meissen or the "Blue Onion" pattern, they may value it more highly than the value that you have calculated.

The rules of the auction allow you to bid any real-dollar value for the tea set. We will call your bid b and consider all of its possible values. Because you are not constrained to a small specific set of bids, we cannot draw a finite payoff matrix for this bidding game, but we can logically deduce the optimal bid.

The success of your bid will obviously depend on the bids submitted by others interested in the tea set, primarily because you need to consider whether your bid will win. The outcome thus depends on all rival bids, but only the largest bid among them will affect your outcome. We call this largest bid r and disregard all bids below r.

What is your optimal value of b? We will look at bids both above and below $3,000 to determine whether any option other than exactly $3,000 can yield you a better outcome than bidding your true valuation.

We start with $b > 3{,}000$. There are three cases to consider. First, if your rival bids less than $3,000 (so $r < 3{,}000$), then you get the tea set at the price r. Your

[6]Vickrey was one of the most original minds in economics in the past four decades. In 1996, he won the Nobel Prize for his work on auctions and truth-revealing procedures. Sadly, he died just 3 days after the prize was announced.

profit, which depends only on what you pay relative to your true valuation, is (3,000 − r), which is what it would have been had you simply bid \$3,000. Second, if your rival's bid falls between your actual bid and your true valuation (so $3{,}000 < r < b$), then you are forced to take the tea set for more than it is worth to you. Here you would have done better to bid \$3,000; you would not have gotten the tea set, but you would not have given up the (r − 3,000) in lost profit either. Third, your rival bids even more than you do (so $b < r$). You still do not get the tea set, but you would not have gotten it even had you bid your true valuation. Putting together the reasoning of the three cases, we see that bidding your true valuation is never worse, and sometimes better, than bidding something higher.

What about the possibility of shading your bid slightly and bidding $b <$ 3,000? Again, there are three situations. First, if your rival's bid is lower than yours (so $r < b$), then you are the high bidder, and you get the tea set for r. Here you could have gotten the same result by bidding \$3,000. Second, if your rival's bid falls between 3,000 and your actual bid (so $b < r < 3{,}000$), your rival gets the tea set. If you had bid \$3,000 in this case, you would have gotten the tea set, paid r, and still made a profit of (3,000 − r). Third, your rival's bid could have been higher than \$3,000 (so $3{,}000 < r$). Again, you do not get the tea set but, if you had bid \$3,000, you still would not have gotten it, so there would have been no harm in doing so. Again, we see that bidding your true valuation, then, is no worse, and sometimes better, than bidding something lower.

If truthful bidding is never worse and sometimes better than bidding either above or below your true valuation, then you do best to bid truthfully. That is, no matter what your rival bids, it is always in your best interest to be truthful. Put another way, bidding your true valuation is your dominant strategy whether you are allowed discrete or continuous bids.

Vickrey's remarkable result that truthful bidding is a dominant strategy in second-price sealed-bid auctions has many other applications. For example, if each member of a group is asked what she would be willing to pay for a public project that will benefit the whole group, each has an incentive to understate her own contribution—to become a "free rider" on the contributions of the rest. We have already seen examples of such effects in the collective-action games of Chapter 12. A variant of the Vickrey scheme can elicit the truth in such games as well.

5 ALL-PAY AUCTIONS

We have considered most of the standard auction types discussed in Section 1 but none of the more creative configurations that might arise. Here we consider a common-value sealed-bid first-price auction in which every bidder, win or lose, pays to the auctioneer the amount of her bid. An auction where the losers also pay may seem strange. But, in fact, many contests result in this type of outcome. In political contests, all candidates spend a lot of their own money and a lot of time and effort for fund raising and campaigning. The losers do not get any refunds on all their expenditures. Similarly, hundreds of competitors spend 4 years of their lives preparing for an event at the next Olympic games. Only one wins the gold medal and the attendant fame and endorsements; two others win the far less valuable silver and bronze medals; the efforts of the rest are wasted. Once you start thinking along these lines, you will realize that such all-pay auctions are, if anything, more frequent in real life than situations resembling the standard formal auctions where only the winner pays.

How should you bid (that is, what should your strategy be for expenditure of time, effort, and money) in an **all-pay auction?** Once you decide to participate, your bid is wasted unless you win, so you have a strong incentive to bid very aggressively. In experiments, the sum of all the bids often exceeds the value of the prize by a large amount, and the auctioneer makes a handsome profit.[7] In that case, everyone submitting extremely aggressive bids cannot be the equilibrium outcome; it seems wiser to stay out of such destructive competition altogether. But, if everyone else did that, then one bidder could walk away with the prize for next to nothing; thus, not bidding cannot be an equilibrium strategy either. This analysis suggests that the equilibrium lies in mixed strategies.

Consider a specific auction with n bidders. To keep the notation simple, we choose units of measurement so that the common-value object (prize) is worth 1. Bidding more than 1 is sure to bring a loss, and so we restrict bids to those between 0 and 1. It is easier to let the bid be a continuous variable x, where x can take on any (real) value in the interval [0, 1]. Because the equilibrium will be in mixed strategies, each person's bid, x, will be a continuous random variable. Because you win the object only if all other bidders submit bids below yours, we can express your equilibrium mixed strategy as $P(x)$, the probability that your bid takes on a value less than x; for example, $P(1/2) = 0.25$ would mean that your equilibrium strategy entailed bids below 1/2 one-quarter of the time (and bids above 1/2 three-quarters of the time).[8]

As usual, we can find the mixed-strategy equilibrium by using an indifference condition. Each bidder must be indifferent about the choice of any particular value of x, given that the others are playing their equilibrium mixes.

[7]One of us (Dixit) has auctioned \$10 bills to his Games of Strategy class and made a profit of as much as \$60 from a 20-student section. At Princeton there is a tradition of giving the professor a polite round of applause at the end of a semester. Once Dixit offered \$20 to the student who kept applauding continuously the longest. This is an open-outcry all-pay auction with payments in kind (applause). Although most students dropped out between 5 and 20 minutes, three went on for $4\frac{1}{2}$ hours!

[8]$P(x)$ is called the *cumulative probability distribution function* for the random variable x. The more familiar probability density function for x is its derivative, $P'(x) = p(x)$. Then $p(x)dx$ denotes the probability that the variable takes on a value in a small interval from x to $x + dx$.

Suppose you, as one of the n bidders, bid x. You win if all of the remaining $(n - 1)$ are bidding less than x. The probability of anyone else bidding less than x is $P(x)$; the probability of two others bidding less than x is $P(x) \times P(x)$, or $[P(x)]^2$; the probability of all $(n - 1)$ of them bidding less than x is $P(x) \times P(x) \times P(x) \cdots$ multiplied $(n - 1)$ times, or $[P(x)]^{n-1}$. Thus with a probability of $[P(x)]^{n-1}$, you win 1. Remember that you pay x no matter what happens. Therefore, your net expected payoff for any bid of x is $[P(x)]^{n-1} - x$. But you could get 0 for sure by bidding 0. Thus, because you must be indifferent about the choice of any particular x, including 0, the condition that defines the equilibrium is $[P(x)]^{n-1} - x = 0$. In a full mixed-strategy equilibrium, this condition must be true for all x. Therefore the equilibrium mixed-strategy bid is $P(x) = x^{1/(n-1)}$.

A couple of sample calculations will illustrate what is implied here. First, consider the case in which $n = 2$; then $P(x) = x$ for all x. Therefore the probability of bidding a number between two given levels x_1 and x_2 is $P(x_2) - P(x_1) = x_2 - x_1$. Because the probability that the bid lies in any range is simply the length of that range, any one bid must be just as likely as any other bid. That is, your equilibrium mixed-strategy bid should be random and uniformly distributed over the whole range from 0 to 1.

Next let $n = 3$. Then $P(x) = \sqrt{x}$. For $x = 1/4$, $P(x) = 1/2$; so the probability of bidding 1/4 or less is 1/2. The bids are no longer uniformly distributed over the range from 0 to 1; they are more likely to be in the lower end of the range.

Further increases in n reinforce this tendency. For example, if $n = 10$, then $P(x) = x^{1/9}$, and $P(x)$ equals 1/2 when $x = (1/2)^9 = 1/512 = 0.00195$. In this situation, your bid is as likely to be smaller than 0.00195 as it is to be anywhere within the whole range from 0.00195 to 1. Thus your bids are likely to be very close to 0.

Your average bid should correspondingly be smaller the larger the number n. In fact a more precise mathematical calculation shows that, if everyone bids according to this strategy, the average or expected bid of any one player will be just $(1/n)$.[9] With n players bidding, on average, $1/n$ each, the total expected bid is 1, and the auctioneer makes zero expected profit. This calculation provides more precise confirmation that the equilibrium strategy eliminates overbidding.

The idea that your bid should be much more likely to be close to 0 when the total number of bidders is large makes excellent intuitive sense, and the finding that equilibrium bidding eliminates overbidding lends further confidence to the theoretical analysis. Unfortunately, many people in actual all-pay auctions either do not know or forget this theory and bid to excess.

[9]The expected bid of any one player is calculated as the expected value of x, by using the probability density function, $p(x)$. In this case, $p(x) = P'(x) = (1/n - 1)x^{(2-n)/(n-1)}$, and the expected value of x is the sum from 0 to 1 of $x\, p(x)\, dx = 1/n$.

6 HOW TO SELL AT AUCTION

Bidders are not the only auction participants who need to carefully consider their optimal strategies. An auction is really a sequential-play game in which the first move is the setting of the rules and bidding starts only in the second round of moves. It falls to the sellers, then, to determine the path that later bidding will follow by choosing a particular auction structure.

As a seller interested in auctioning off your prized art collection or even your home, you must decide on the best type of auction to use. To guarantee yourself the greatest profit from your sale, you must look ahead to the predicted outcome of the different types of auctions before making a choice. One concern of many sellers is that an item will go to a bidder for a price lower than the value that the seller places on the object. To counter this concern, most sellers insist on setting a **reserve price** for auctioned objects; they reserve the right to withdraw the object from the sale if no bid higher than the reserve price is obtained.

Beyond setting a reserve price, however, what can sellers do to determine the type of auction that might net them the most profit possible? One possibility is to use Vickrey's suggested scheme, a second-price sealed-bid auction. According to him, this kind of auction elicits truthful bidding from potential buyers. Does this effect make it a good auction type from the seller's perspective?

In a sense, the seller in such a second-price auction is giving the bidder a profit margin to counter the temptation to shade down the bid in the hope of a larger profit. But this outcome then reduces the seller's revenue, just as shading down in a first-price sealed-bid auction would. Which type of auction is ultimately better for the seller actually turns out to depend on the bidders' attitudes toward risk and their beliefs about the value of the object for sale.

A. Risk-Neutral Bidders and Independent Estimates

The least-complex configuration of bidder risk attitudes and beliefs is when there is risk neutrality (no risk aversion) and when bidder estimates about the value of the object for sale remain independent of one another. As we said in the Appendix to Chapter 5, risk-neutral people care only about the expected monetary value of their outcomes, regardless of the level of uncertainty associated with those outcomes. Independence in estimates means that a bidder is not influenced by the estimates of other bidders when determining how much an object is worth to her; the bidder has decided independently exactly how much the object is worth to her. In this case, there can be no winner's curse. If these conditions for bidders hold, sellers can expect the same average revenue (over a large number of trials) from any of the four primary types of auction: English, Dutch, and first- and second-price sealed-bid.

This **revenue equivalence** result implies not that all of the auctions will yield the same revenue for every item sold, but that the auctions will yield the same selling price on average in the course of numerous auctions. We can see the equivalence quite easily between second-price sealed-bid auctions and English auctions. We have already seen that, in the second-price auction, each bidder's dominant strategy is to bid her true valuation. The highest bidder gets the object for the second-highest bid, and the seller gets a price equal to the valuation of the second-highest bidder. Similarly, in an English auction, bidders drop out as the price increases beyond their valuations, until only the first- and second-highest-valuation bidders remain. When the price reaches the valuation of the second-highest bidder, that bidder also will drop out, and the remaining (highest-valuation) bidder will take the object for just a cent more than the second-highest bid. Again, the seller gets a price (essentially) equivalent to the valuation of the second-highest bidder.

More advanced mathematical techniques are needed to prove that revenue equivalence can be extended to Dutch and first-price sealed-bid auctions as well, but the intuition should be clear. In all four types of auctions, in the absence of any risk aversion on the part of bidders, the highest-valuation bidder should win the auction and pay on average a price equal to the second-highest valuation. If the seller is likely to use a particular type of auction repeatedly, she need not be overly concerned about her choice of auction structure; all four would yield her the same expected price.

Experimental and field evidence has been collected to test the validity of the revenue-equivalent theorem in actual auctions. The results of laboratory experiments tend to show Dutch auction prices lower, on average, than first-price sealed-bid auction prices for the same items being bid on by the same group of bidders, possibly owing to some positive utility associated with the suspense factor in Dutch auctions. These experiments also find evidence of overbidding (bidding above your known valuation) in second-price sealed-bid auctions but not in English auctions. Such behavior suggests that bidders go higher when they have to specify a price, as they do in sealed-bid auctions; these auctions seem to draw more attention to the relationship between the bid price and the probability of ultimately winning the item. Field evidence from Internet auctions finds literally opposite results, with Dutch auction revenue as much as 30% higher, on average, than first-price sealed-bid revenue. Additional bidder interest in the Dutch auctions or impatience in the course of a 5-day auction could explain the anomaly. The Internet-based field evidence did find near revenue equivalence for the other two auction types.

B. Risk-Averse Bidders

Here we continue to assume that bids and beliefs are uncorrelated but incorporate the possibility that auction outcomes could be affected by bidders' attitudes toward risk. In particular, suppose bidders are risk averse. They may be much more concerned, for example, about the losses caused by underbidding—losing the object—than by the costs associated with bidding at or close to their true valuations. Thus, risk-averse bidders generally want to win if possible without ever overbidding.

What does this preference structure do to the types of bids that they submit in first-price versus second-price (sealed-bid) auctions? Again, think of the first-price auction as being equivalent to the Dutch auction. Here risk aversion leads bidders to bid earlier rather than later. As the price drops to the bidder's valuation and beyond, there is greater and greater risk in waiting to bid. With risk-averse bidders, we expect them to bid quickly, not to wait just a little bit longer in the hope of gaining those extra few pennies of profit. Applying this reasoning to the first-price sealed-bid auction, we expect bidders to shade down their bids by less than they would if they were not risk averse: too much shading actually increases the risk of not gaining the object, which risk-averse bidders would want to avoid.

Compare this outcome with that of the second-price auction, where bidders pay a price equal to the second-highest bid. Bidders bid their true valuations in such an auction but pay a price less than that. If they shade their bids only slightly in the first-price auction, then those bids will tend to be close to the bidders' true valuations—and bidders pay their bids in such auctions. Thus bids will be shaded somewhat, but the price ultimately paid in the first-price auction will probably exceed what would be paid in the second-price auction. When bidders are risk averse, the seller then does better to choose a first-price auction rather than a second-price auction.

The seller does better with the first-price auction in the presence of risk aversion only in the sealed-bid case. If the auction were English, the bidders' attitudes toward risk are irrelevant to the outcome. Thus, risk aversion does not alter the outcome for the seller in these auctions.

C. Correlated Estimates

Now suppose that, in determining their own valuations of an object, bidders are influenced by the estimates (or by their beliefs about the estimates) of other bidders. Such a situation is relevant for common-value auctions, such as those for oil or gas exploration considered in Section 2. Suppose your experts have not presented a glowing picture of the future profits to be gleaned from the lease on a specific tract of land. You are therefore pessimistic about its potential benefits, and you have constructed an estimate V of its value that you believe corresponds to your pessimism.

Under the circumstances, you may be concerned that your rival bidders also have received negative reports from their experts. When bidders believe their valuations are all likely to be similar, either all relatively low or all relatively high, for example, we say that those beliefs or estimates of value are positively correlated. Recall that we introduced the concept of correlation in the appendix to Chapter 9, where we considered correlated risks and insurance. In the current context, the likelihood that your rivals' estimates also are unfavorable may magnify the effect of your pessimism on your own valuation. If you are participating in a first-price sealed-bid auction, you may be tempted to shade down your bid even more than you would in the absence of correlated beliefs. If bidders are optimistic and valuations generally high, correlated estimates may lead to less shading than when estimates are independent.

However, the increase in the shading of bids that accompanies correlated low (or pessimistic) bids in a first-price auction should be a warning to sellers. With positively correlated bidder beliefs, the seller may want to avoid the first-price auction and take advantage of Vickrey's recommendation to use a second-price structure. We have just seen that this auction type encourages truthful revelation and, when correlated estimates are possible, the seller does even better to avoid auctions in which there might be any additional shading of bids.

An English auction will have the same ultimate outcome as the second-price sealed-bid auction, and a Dutch auction will have the same outcome as a first-price sealed-bid auction. Thus a seller facing bidders with correlated estimates of an object's value also should prefer the English to the Dutch version of the open-outcry auction. If you are bidding on the oil land lease in an English auction and the price is nearing your estimate of the lease's value but your rivals are still bidding feverishly, you can infer that their estimates are at least as high as yours—perhaps significantly higher. The information that you obtain from observing the bidding behavior of your rivals may convince you that your estimate is too low. You might even increase your own estimate of the land's value as a result of the bidding process. Your continuing to bid may provide an impetus for further bidding by other bidders, and the process may continue for a while. If so, the seller reaps the benefits. More generally, the seller can expect a higher selling price in an English auction than in a first-price sealed-bid auction when bidder estimates are correlated. For the bidders, however, the effect of the open bidding is to disperse additional information and to reduce the effect of the winner's curse.

The discussion of correlated estimates assumes that a fairly large number of bidders take part in the auction. But an English auction can be beneficial to the seller if there are only two bidders, both of whom are particularly enthusiastic about the object for sale. They will bid against each other as long as possible, pushing the price up to the lower of the valuations, both of which were high from the start. The same auction can be disastrous for the seller, however, if one of the bidders has a very low valuation; the other is then quite likely to have a valuation considerably higher than the first. In this case, we say that bidder valuations are negatively correlated. We encourage any seller facing a small number of bidders with potentially very different valuations to choose a Dutch or first-price sealed-bid structure. Either of them would reduce the possibility of the high-valuation bidder gaining the object for well under her true valuation; that is, either type would transfer the available profit from the buyer to the seller.

7 SOME ADDED TWISTS TO CONSIDER

A. Multiple Objects

When you think about an auction of a group of items, such as a bank's auctioning repossessed vehicles or estate sales auctioning the contents of a home, you probably envision the auctioneer bringing each item to the podium individually and selling it to the highest bidder. This process is appropriate when each bidder has independent valuations for each item. However, independent valuations may not always be an appropriate way to model bidder estimates. Then, if bidders value specific groups or whole packages of items higher than the sum of their values for the component items, the choice of auctioning the lots separately or together makes a big difference to bidding strategies as well as to outcomes.

Consider a real-estate developer named Red who is interested in buying a very large parcel of land on which to build a townhouse community for professionals. Two townships, Cottage and Mansion, are each auctioning a land parcel big enough to suit her needs. Both parcels are essentially square in shape and encompass 4 square acres. The mayor of Cottage has directed that the auctioneer sell the land as quarter-acre blocks, one at a time, starting with the perimeter of the land and working inward, selling the corner lots first and then the lots on the north, south, east, and west borders in that order. The mayor of Mansion has at the same time directed that the auctioneer attempt to sell the land in her town first as a full 4-acre block, then as two individual 2-acre lots, and then as four 1-acre lots after that, if no bids exceed the set reserve prices.

Red has determined by virtue of extensive market analyses that the blocks of land in Cottage and Mansion would provide the same value to her. However, she has to obtain the full 4 acres of land in either town to have enough room for her planned development. The auctions are being held on the same day at the same time. Which should she attend?

It should be clear that her chances of acquiring a 4-acre block of land for a reasonable price—less than or equal to her valuation—are much better in Mansion than in Cottage. In the Mansion auction, she would simply wait to see how bidding proceeded, submitting a final high bid if the second-highest offer fell below her valuation of the property. In the Cottage auction, she would need to win each and every one of the 16 parcels up for sale. Under the circumstances, she should expect rival bidders interested in owning land in Cottage to become more intent on their goals—perhaps even joining forces—as the number of available parcels decreases in the course of the auction. Red would have to bid aggressively enough to win parcels in the early rounds while being conservative enough to ensure that she did not exceed her total valuation by the end of the auction. The difficulties in crafting a bidding strategy for such an auction are numerous, and the probability of being unable to profitably obtain every parcel is quite large—hence Red's preference for the Mansion auction.

Note that, from the seller's point of view, the Cottage auction is likely to bring in greater revenue than the Mansion auction if there is an adequate number of bidders interested in small pieces of land. If the only bidders are all developers like Red, however, they might be hesitant even to participate in the Cottage auction for fear of being beaten in just one round. In that case, the Mansion-type auction is better for the seller.

The township of Cottage could allay the fears of developers by revising the rules for its auction. In particular, it would not need to auction each parcel individually. Instead it could use a single auction in which all parcels would be available simultaneously. Such an auction could be run so that each bidder could specify the number of parcels that she wanted and the price that she was willing to pay per parcel. The bidder with the highest total-value bid, determined by multiplying the number of parcels desired by the price for each, would win the desired number of parcels. If parcels remained after the high bidder took her land, additional parcels would be won in a similar way until all the land was sold. This mechanism gives bidders interested in larger parcels an opportunity to bid, potentially against one another, for blocks of the land. Thus Cottage might find this type of auction more lucrative in the end.

B. Defeating the System

We saw earlier which auction structure is best for the seller, given different assumptions about how bidders felt toward risk and whether their estimates were correlated. There is always an incentive for bidders, though, to come up with a bidding strategy that defeats the seller's efforts. The best-laid plans for a profitable auction can almost always be defeated by an appropriately clever bidder or, more often, group of bidders.

Even the Vickrey second-price sealed-bid auction can be defeated if there are only a few bidders in the auction, all of whom can collude among themselves. By submitting one high bid and a lowball second-highest bid, collusive bidders can obtain an object for the second-bid price. This outcome relies on the fact that no other bidders submit intermediate bids or that the collusive group is able to prevent such an occurrence. The possibility of collusion highlights the need for the seller's reserve prices, although they only partly offset the problem in this case.

First-price sealed-bid auctions are less vulnerable to bidder collusion for two reasons. The potential collusive group engages in a multiperson prisoners' dilemma game in which each bidder has a temptation to cheat. In such cheating, an individual bidder might submit her own high bid so as to win the object for herself, reneging on any obligation to share profits with group members. Collusion among bidders in this type of auction is also difficult to sustain because cheating (that is, making a different bid from that agreed to within the collusive group) is easy to do but difficult for other buyers to detect. Thus the sealed-bid nature of the auction prevents detection of a cheater's behavior, and hence punishment, until the bids are opened and the auction results announced; at that point, it is simply too late. However, there may be more scope for sustaining collusion if a particular group of bidders participates in a number of similar auctions over time, so that they engage in the equivalent of a repeated game.

Other tricky bidding schemes can be created to meet the needs of specific individual bidders or groups of bidders in any particular type of auction. One very clever example of bid rigging arose in an early U.S. Federal Communications Commission auction of the U.S. airwave spectrum, specifically for personal cellular service (Auction 11, August 1996–January 1997). After watching prices soar in some of the earlier auctions, bidders were apparently eager to reduce the price of the winning bids. The solution, used by three firms (later sued by the Department of Justice), was to signal their intentions to go after licenses for certain geographic locations by using the FCC codes or telephone area codes for those areas as the last three digits of their bids. The FCC has claimed that this practice significantly reduced the final prices on these particular licenses. In addition, other signaling devices were apparently used in earlier broadband auctions. While some firms literally announced their intentions to win a particular license, others used a variety of strategic bidding techniques to signal their interest in specific licenses or to dissuade rivals from horning in on their territories. In the first broadband auction, for example, GTE and other firms apparently used the code-bidding technique of ending their bids with the numbers that spelled out their names on a touch-tone telephone keypad!

We note briefly here that fraudulent behavior is not merely the territory of bidders at auction. Sellers also can use underhanded practices to inflate the final bid price of pieces that they are attempting to auction. **Shilling,** for example, occurs when a seller is able to plant false bids at her own auction. Possible only in English auctions, shilling can be done with the use of an agent working for the seller who pretends to be a regular bidder. On Internet auction sites, shilling is actually easier, because a seller can register a second identity and log in and bid in her own auction; all Internet auctions have rules and oversight mechanisms designed to prevent such behavior. Sellers in second-price sealed-bid auctions can also benefit if they inflate the level of the (not publicly known) second-highest bid.

C. Information Disclosure

Finally, we consider the possibility that the seller has some private information about an object that might affect the bidders' valuations of that object. Such a situation arises when the quality or durability of a particular object, such as an automobile, a house, or a piece of electronic equipment, is of great importance to the buyers. Then the seller's experience with the object in the past may be a good predictor of the future benefits that will accrue to the winning bidder.

Under such circumstances, the seller must carefully consider any temptation to conceal information. If the bidders know that the seller has some private information, they are likely to interpret any failure to disclose that information as a signal that the information is unfavorable. Even if the seller's information is unfavorable, she may be better off to reveal it; bidders' beliefs might be worse than the actual information. Thus honesty is often the best policy.

Honesty can also be in the seller's interests for another reason. When she has private information about a common-value object, she should disclose that information to sharpen the bidders' estimates of the value of the object. The more confident the bidders are that their valuations are correct, the more likely they are to bid up to those valuations. Thus disclosure of private seller information in a common-value auction can help not only the seller by reducing the amount of shading done by bidders but also the bidders by reducing the effects of the winner's curse.

8 AUCTIONS ON THE INTERNET

Auctions have become a significant presence in the world of e-commerce. As many as 40 million people are making purchases at online auction sites each month, with as much as $60 billion per year in sales being transacted. Auction sites themselves, which make money on listing fees, seller commissions, and advertising revenues, are making $1 billion to $2 billion per year. With the Internet auction traffic and revenue increasing annually, this particular area of e-commerce is of considerable interest to many industry analysts as well as to consumers and to auction theorists.

Internet auction sites have been in existence less than a decade. The eBay site began operation in September 1995, shortly after the advent of Onsale.com in May of that year.[10] A large number of auction sites now exist, with between 100 and 150 different sites available; precise numbers change frequently as new sites are created, as mergers are consummated between existing sites, and as smaller, unprofitable sites shut down. These sites, both small and large, sell an enormous variety of items in many different ways.

The majority of items on the larger, most well used sites, such as eBay and uBid, are goods that are classified as "collectibles," which can mean anything from an antique postcard to a melamine bowl acquired by saving cereal-box tops. But there are also specialty auction sites that deal with items ranging from postage stamps, wine, and cigars to seized property from police raids, medical equipment, and large construction equipment (scissorlift, anyone?). Most of these items, regardless of the type of site, would be considered "used." Thus, consumers have access to what might be called the world's largest garage sale, all at their fingertips.

This information regarding items sold is consistent with one hypothesis in the literature that suggests that Internet auctions are most useful for selling goods available in limited quantity, for which there is unknown demand, and for which the seller cannot easily determine an appropriate price. The auction process can effectively find a "market price" for these goods. Sellers of such goods then have their best profit opportunity online where a broad audience can supply formerly unknown demand parameters. And consumers can obtain desired but obscure items, presumably with profit margins of their own. Economists would call this matching process, in which each agent receives a positive profit from a trade, *efficient*. (Efficient mechanisms are discussed in greater detail in Chapter 18.)

In addition to selling many different categories of goods, Internet auctions employ a variety of auction rules. Many sites actually offer several auction types and allow a seller to choose her auction's rules when she lists an item for sale. The most commonly used rules are those for English and second-price sealed-bid auctions; one or both of them are offered by the majority of auction sites.

The sites that offer true English auctions post the high bid as soon as it is received and, at the end of the auction, the winner pays her bid. In addition, there are many sites that appear to use the English auction format but allow what is known as **proxy-bidding.** The proxy-bidding process actually makes the auction second-price sealed-bid rather than English.

With proxy-bidding, a bidder enters the maximum price that she is willing to pay (her **reservation price**) for an item. Rather than displaying this maximum price, the auction site displays only one bid increment above the most-recent

[10]Onsale merged with Egghead.com in 1999. Amazon bought the assets of the merged company late in 2001. The three auction sites originally available as Onsale, Egghead, and Amazon are now simply "Amazon.com Auctions."

high bid. The proxy-bidding system then bids for the buyer, outbidding others by a single bid increment, until the buyer's maximum price is reached. This system allows the auction winner to pay just one bid increment over the second highest bid rather than paying her own bid. Most of the large auction sites, including eBay, uBid, Amazon, and Yahoo, offer proxy-bidding. In addition, some sites offer explicit second-price auctions in which actual high bids are posted, but the winner pays a price equal to the second-highest bid.

The prevalence of true Dutch auctions at online auction sites is quite rare. Some sites that have been reported in the literature as having descending price auctions, some with clocks, are no longer in existence. Only a few retail sites now offer the equivalent of Dutch auctions. Land's End, for example, posts some overstocked items each weekend in a special area of its website; it then reduces the prices on these items three times in the next week, removing unsold items at week's end. CNET Shopper posts recent price decreases for electronic goods but does not promise additional reductions within a specified time period. One suspects that the dearth of Dutch auctions can be attributed to the real-time nature of such auctions that make them difficult to reproduce online.

There are, however, several sites that offer auctions *called* Dutch auctions. These auctions, along with a companion type known as **Yankee auctions,** actually offer multiple (identical) units in a single auction. Similar to the auction described in Section 7.A for the available land parcels in the township of Cottage, these auctions offer bidders the option to bid on one or more of the units. Online sites use the terminology "Yankee auction" to refer to the system that we described for the Cottage auction; bidders with the highest total-value bid(s) win the items and each bidder pays her bid price per unit. The "Dutch auction" moniker is reserved for auctions in which bids are ranked by total value but, at the end of the auction, all bidders pay the lowest winning bid price for their units.

Compared with live auctions, the Internet versions tend to be quite similar in their rules and outcomes. Strategic issues such as those considered in Sections 6 and 7 are relevant to both. There are some benefits to online auctions as well as costs. Online auctions are good for buyers because they are easy to "attend" and they provide search engines that make it simple to identify items of interest. Similarly, sellers are able to reach a wide audience and often have the convenience of choosing the rules of their own auctions. On the other hand, online auction sales can suffer from the fact that buyers cannot inspect goods before they must bid and because both buyer and seller must trust the other to pay or deliver as promised.

The most interesting difference between live and online auctions, though, is the way in which the auctions must end. Live (English and Dutch) auctions cease when no additional bids can be obtained. Online auctions need to have specific auction-ending rules. In virtually all cases, online auctions end after a specified period of time, often 7 days. Different sites, though, use different rules about how hard a line they take on the ending time. Some sites, such as Amazon, allow an extension of the auction if a new bid is received within 10 minutes of the posted auction end time. Others, such as eBay, end their auctions at the posted ending time, regardless of how many bids are received in the closing minutes.

Considerable interest has been generated in the auction literature by the differences in bidder behavior under the two ending-time rules. Evidence has been gathered by Alvin Roth and Axel Ockenfels that shows that eBay's hard end time makes it profitable for bidders to bid late. This behavior, referred to as *sniping,* is found in both private-value and common-value auctions.

Strategically, late bidders on eBay gain by avoiding bidding wars with others who do not use the proxy-bidding system and update their own bids throughout the auction. In addition, they gain by protecting any private information that they hold regarding the common valuation of a good. (Even on eBay, frequent users can identify bidders who seem to know a particularly good item when they see it, particularly in the antiques category. Bidders with this information may want to hide it from others to keep the final sale price down.) Thus, buyers of goods that require special valuation skills, such as antiques, may prefer eBay to Amazon. By the same token, sellers of such goods may get better prices at Amazon—but only if the expert buyers are willing to bid there. Over time, the evidence suggests that repeat bidders on eBay tend to bid nearer the end in later auctions; bidders on Amazon learn that they can bid earlier. This finding is evidence of movement toward equilibrium bidding behavior in the same way that bidder strategy in the different auctions conforms to our expectations about payoff-maximizing behavior.

9 ADDITIONAL READING ON THE THEORY AND PRACTICE OF AUCTIONS

Much of the literature on the theory of auctions is quite mathematically complex. Some general insights into auction behavior and outcomes can be found in Paul Milgrom, "Auctions and Bidding: A Primer," Orley Ashenfelter, "How Auctions Work for Wine and Art," and John G. Riley, "Expected Revenues from Open and Sealed Bid Auctions," all in the *Journal of Economic Perspectives,* vol. 3, no. 3 (Summer 1989), pp. 3–50. These papers should be readable by those of you with a reasonably strong background in calculus.

More complex information on the subject also is available. R. Preston McAfee and John McMillan have an overview paper, "Auctions and Bidding," in the *Journal of Economic Literature,* vol. 25 (June 1987), pp. 699–738. A more recent review of the literature can be found in Paul Klemperer, "Auction Theory: A Guide to the Literature," in the *Journal of Economic Surveys,* vol. 13, no. 3 (July 1999), pp. 227–286. Both of these pieces contain some of the high-level mathematics associated with auction theory but also give comprehensive references to the rest of the literature.

Vickrey's original article containing the details on truthful bidding in second-price auctions is "Counterspeculation, Auctions, and Competitive Sealed Tenders," *Journal of Finance,* vol. 16, no. 1 (March 1961), pp. 8–37. This paper was one of the first to note the existence of revenue equivalence. A more recent study gathering a number of the results on revenue outcomes for various auction types is J. G. Riley and W. F. Samuelson, "Optimal Auctions," *American Economic Review,* vol. 71, no. 3 (June 1981), pp. 381–392. A very readable history of the "Vickrey" second-price auction is David Lucking-Reiley, "Vickrey Auctions in Practice: From Nineteenth Century Philately to Twenty-First Century E-commerce," *Journal of Economic Perspectives,* vol. 14, no. 3 (Summer 2000), pp. 183–192.

Some of the experimental evidence on auction behavior is reviewed in John H. Kagel, "Auctions: A Survey of Experimental Research" in John Kagel and Alvin Roth, eds., *The Handbook of Experimental Economics* (Princeton: Princeton University Press, 1995), pp. 501–535. More recent evidence on revenue equivalence is provided in David Lucking-Reiley, "Using Field Experiments to Test Equivalence Between Auction Formats: Magic on the Internet," *American Economic Review,* vol. 89, no. 5 (December 1999), pp. 1063–1080. Other evidence on behavior in online auctions is presented in Alvin Roth and Axel Ockenfels, "Last-Minute Bidding and the Rules for Ending Second-Price Auctions: Evidence from eBay and Amazon Auctions on the Internet," *American Economic Review,* vol. 92, no. 4 (September 2002), pp. 1093–1103.

For information specific to bid rigging in the Federal Communications Commission's airwave spectrum auctions and auction design matters, see Peter Cramton and Jesse Schwartz, "Collusive Bidding: Lessons from the FCC Spectrum Auctions," *Journal of Regulatory Economics,* vol. 17 (May 2000), pp. 229–252, and Paul Klemperer, "What Really Matters in Auction Design," *Journal of Economic Perspectives,* vol. 16, no. 1 (Winter 2002), pp. 169–189. A survey of Internet auctions can be found in David Lucking-Reiley, "Auctions on the Internet: What's Being Auctioned, and How?" *Journal of Industrial Economics,* vol. 48, no. 3 (September 2000), pp. 227–252.

SUMMARY

In addition to the standard *first-price open-outcry ascending,* or *English,* auction, there are also *Dutch,* or *descending* auctions as well as *first-price* and *second-price, sealed-bid* auctions. Objects for bid may have a single *common value* or many *private values* specific to each bidder. With common-value auctions, bidders often win only when they have overbid, falling prey to the *winner's curse.* In private-value auctions, optimal bidding strategies, including decisions about when to *shade* down bids from your true valuation, depend on the auction type used. In the familiar first-price auction, there is a strategic incentive to underbid.

Vickrey showed that sellers can elicit true valuations from bidders by using a second-price sealed-bid auction. Generally, sellers will choose the mechanism that guarantees them the most profit; this choice will depend on bidder risk attitudes and bidder beliefs about an object's value. If bidders are risk neutral and have independent valuation estimates, all auction types will yield the same outcome. Decisions regarding how to auction a large number of objects, individually or as a group, and whether to disclose information are nontrivial. Sellers must also be wary of bidder collusion or fraud.

The Internet is currently one of the largest venues for items sold at auction. Online auction sites sell many different types of goods and use a variety of different types of auctions. The main strategic difference between live and online auctions arises owing to the hard ending times imposed at some sites, particularly eBay. Late bidding and sniping have been shown to be rational responses to this end-of-auction rule.

KEY TERMS

all-pay auction (548)
ascending auction (540)
common value (541)
descending auction (540)
Dutch auction (540)
English auction (540)
first-price auction (540)
objective value (541)
open outcry (539)
private value (541)
proxy-bidding (558)
reservation price (558)
reserve price (550)
revenue equivalence (551)
sealed bid (540)
second-price auction (540)
shading (545)
shilling (556)
subjective value (541)
Vickrey auction (540)
Vickrey's truth serum (546)
winner's curse (541)
Yankee auction (559)

EXERCISES

1. A house painter has a regular contract to work for a builder. On these jobs, her cost estimates are generally right: sometimes a little high, sometimes a little low, but correct on average. When the regular work is slack, she bids

competitively for other jobs. "Those are different," she says. "They almost always end up costing more than I estimate." Assuming that her estimating skills do not differ between the jobs, what can explain the difference?

2. "In the presence of very risk averse bidders, a person selling her house in an auction will have a higher expected profit by using a first-price sealed-bid auction." True or false? Explain your answer.

3. Consider an auction where n identical objects are offered, and there are $(n + 1)$ bidders. The actual value of an object is the same for all bidders and equal for all objects, but each bidder gets only an independent estimate, subject to error, of this common value. The bidders put in sealed bids. The top n bidders get one object each, and each pays what she had bid. What considerations will affect your bidding strategy and how?

4. Suppose that there are three risk-neutral bidders interested in purchasing a Princess Beanie Baby. The bidders (numbered 1 through 3) have valuations of \$12, \$14, and \$16, respectively. The bidders will compete in auctions as described in parts a through d; in each case, bids can be made in \$1 increments at any value from \$5 to \$25.
   (a) Which bidder wins an open-outcry English auction? What are the final price paid and the profit to the winning bidder?
   (b) Which bidder wins a second-price sealed-bid auction? What are the final price paid and the profit to the winning bidder? Contrast your answer here with that for part **a**. What is the cause of the difference in profits in these two cases?
   (c) In a sealed-bid first-price auction, all the bidders will bid a positive amount (at least \$1) less than their true valuations. What is the likely outcome in this auction? Contrast your answer with those for parts **a** and **b**. Does the seller of the Beanie Baby have any clear reason to choose one of these auction mechanisms over the other?
   (d) Risk-averse bidders would reduce the shading of their bids in part **c**; assume, for the purposes of this question, that they do not shade at all. If that were true, what would be the winning price (and profit for the bidder) in part **c**? Does the seller care about which type of auction she uses? Why?

5. The idea of the winner's curse can be expressed slightly differently from its usage in the text: "The only time your bid matters is when you win, which happens when your estimate is higher than the estimates of all the other bidders. Therefore you should focus on this case. That is, you should always act as if all the others have received estimates lower than yours, and use this 'information' to revise your own estimate." Here we ask you to apply this idea to a very different situation.

A jury consists of 12 people who hear and see the same evidence presented, but each of them interprets this evidence in accord with her own thinking and experience and arrives at an estimate of the guilt or the innocence of the accused. Then each is asked to vote: Guilty or Not guilty. The accused is convicted if all 12 vote Guilty and is acquitted if one or more vote Not guilty. Each juror's objective is to arrive at a verdict that is the most accurate verdict in light of the evidence. Each juror votes strategically, seeking to maximize this objective and using all the devices of information inference that we have studied. Will the equilibrium outcome of their voting be optimal?

6. You are in the market for a used car and see an ad for the model that you like. The owner has not set a price but invites potential buyers to make offers. Your prepurchase inspection gives you only a very rough idea of the value of the car; you think it is equally likely to be anywhere in the range of \$1,000 to \$5,000 (so your calculation of the average of this value is \$3,000). The current owner knows the exact value and will accept your offer if it exceeds that value. If your offer is accepted and you get the car, then you will find out the truth. But you have some special repair skills and know that, when you own the car, you will be able to work on it and increase its value by a third (33.3 . . . %) of whatever it is worth.
   (a) What is your expected profit if you offer \$3,000? Should you make such an offer?
   (b) What is the highest offer that you can make without losing money on the deal?

7. [**Optional**] The mathematical analysis of bidding strategies and equilibrium is unavoidably difficult. If you know some simple calculus, we lead you here through a very simple example.

   Consider a first-price sealed-bid private-value auction. There are two bidders. Each knows her own value but not that of the other. The two values are equally likely to lie anywhere between 0 and 1; that is, each bidder's value is uniformly distributed over the interval [0, 1]. The strategy of each can be expressed as a complete plan of action drawn up in advance: "If my value is v, I will bid b." In other words, the strategy expresses the bid as a function of value, $b = B(v)$. We construct an equilibrium in such strategies. The players are identical, and we look for a symmetric equilibrium in which their strategies are the same function B. We must use the equilibrium conditions to solve for the function B.

   Suppose the other player is using the function B. Your value is v. You are contemplating your best response. You can always act as if your value were x—that is, bid $B(x)$. With this bid you will win if the other player's value is less than x. Because all values on the interval [0,1] are equally likely, the probability of finding a value less than x is just x. If you win, your surplus is your true value minus your bid—that is, $v - B(x)$. Thus your expected payoff from the strategy of bidding as if your value were x is given by $x\,[v - B(x)]$.

(a) Write down the first-order condition for x to maximize your expected payoff.

(b) In equilibrium, you will follow the strategy of the function B; that is, you will choose x equal to your true value v. Rewrite the first-order condition of part **a** using this information.

(c) Substitute $B(v) = v/2$ into the equation that you obtained in part **b** and see that it holds. Assuming that the equation in part **b** has a unique solution, this is the equilibrium strategy that we seek. Interpret it intuitively.

8. [**Optional**] Repeat Exercise 7, but with n bidders instead of two, and verify that the function $B(v) = v(n - 1)/n$ gives a common equilibrium strategy for all bidders. Explain intuitively why $B(v)$ increases for any v as n increases.

17

Bargaining

PEOPLE ENGAGE IN BARGAINING throughout their lives. Children start by negotiating to share toys and to play games with other children. Couples bargain about matters of housing, child rearing, and the adjustments that each must make for the other's career. Buyers and sellers bargain over price, workers and bosses over wages. Countries bargain over policies of mutual trade liberalization; superpowers negotiate mutual arms reduction. And the two authors of this book had to bargain with each other—generally very amicably—about what to include or exclude, how to structure the exposition, and so forth. To get a good result from such bargaining, the participants must devise good strategies. In this chapter, we raise and explicate some of these basic ideas and strategies.

All bargaining situations have two things in common. First, the total payoff that the parties to the negotiation are capable of creating and enjoying as a result of reaching an agreement should be greater than the sum of the individual payoffs that they could achieve separately—the whole must be greater than the sum of the parts. Without the possibility of this excess value, or "surplus," the negotiation would be pointless. If two children considering whether to play together cannot see a net gain from having access to a larger total stock of toys or from each other's company in play, then it is better for each to "take his toys and play by himself." The world is full of uncertainty and the expected benefits may not materialize. But, when engaged in bargaining, the parties must at least perceive some gain therefrom: Faust, when he agreed to sell his soul to the Devil, thought the benefits of knowledge and power that he gained were worth the price that he would eventually have to pay.

The second important general point about bargaining follows from the first: it is not a zero-sum game. When a surplus exists, the negotiation is about how to divide it up. Each bargainer tries to get more for himself and leave less for the others. This may appear to be zero-sum, but behind it lies the danger that, if the agreement is not reached, no one will get any surplus at all. This mutually harmful alternative, as well as *both* parties' desire to avoid it, is what creates the potential for the threats—explicit and implicit—that make bargaining such a strategic matter.

Before the advent of game theory, one-on-one bargaining was generally thought to be a difficult and even indeterminate problem. Observation of widely different outcomes in otherwise similar-looking situations lent support to this view. Management–union bargaining over wages yields different outcomes in different contexts; different couples make different choices. Theorists were not able to achieve any systematic understanding of why one party gets more than another and attributed this result to vague and inexplicable differences in "bargaining power."

Even the simple theory of Nash equilibrium does not take us any farther. Suppose two people are to split $1. Let us construct a game in which each is asked to announce what he would want. The moves are simultaneous. If their announcements x and y add up to 1 or less, each gets what he announced. If they add up to more than 1, neither gets anything. Then *any* pair (x, y) adding to 1 constitutes a Nash equilibrium in this game: *given* the announcement of the other, each player cannot do better than to stick to his own announcement.[1]

Further advances in game theory have brought progress along two quite different lines, each using a distinct mode of game-theoretic reasoning. In Chapter 2, we distinguished between cooperative game theory, in which the players decide and implement their actions jointly, and noncooperative game theory, in which the players decide and take their actions separately. Each of the two lines of advance in bargaining theory uses one of these two approaches. One approach views bargaining as a *cooperative* game, in which the parties find and implement a solution jointly, perhaps, using a neutral third party such as an arbitrator for enforcement. The other approach views bargaining as a *noncooperative* game, in which the parties choose strategies separately and we look for an equilibrium. However, unlike our earlier simple game of simultaneous announcements, whose equilibrium was indeterminate, here we impose more structure and specify a sequential-move game of offers and counteroffers, which leads to a determinate equilibrium. As in Chapter 2, we emphasize that the labels "cooperative" and "noncooperative"

[1]As we saw in Chapter 5 (Section 3.B), this type of game can be used as an example to bolster the critique that the Nash equilibrium concept is too imprecise. In the bargaining context, we might say that the multiplicity of equilibria is just a formal way of showing the indeterminacy that previous analysts had claimed.

refer to joint versus separate actions, not to nice versus nasty behavior or to compromise versus breakdown. The equilibria of noncooperative bargaining games can entail a lot of compromise.

1 NASH'S COOPERATIVE SOLUTION

In this section we present Nash's cooperative-game approach to bargaining. First we present the idea in a simple numerical example; then we develop the more general algebra.[2]

A. Numerical Example

Imagine two Silicon Valley entrepreneurs, Andy and Bill. Andy produces a microchip set which he can sell to any computer manufacturer for $900. Bill has a software package that can retail for $100. The two meet and realize that their products are ideally suited to each other and that, with a bit of trivial tinkering, they can produce a combined system of hardware and software worth $3,000 in each computer. Thus together they can produce an extra value of $2,000 per unit, and they expect to sell millions of these units each year. The only obstacle that remains on this path to fortune is to agree to a division of the spoils. Of the $3,000 revenue from each unit, how much should go to Andy and how much to Bill?

Bill's starting position is that, without his software, Andy's chip set is just so much metal and sand; so Andy should get only the $900 and Bill himself should get $2,100. Andy counters that, without his hardware, Bill's programs are just symbols on paper or magnetic signals on a diskette; so Bill should get only $100, and $2,900 should go to him, Andy.

Watching them argue, you might suggest they "split the difference." But that is not an unambiguous recipe for agreement. Bill might offer to split the profit on each unit equally with Andy. Under this scheme each will get a profit of $1,000, meaning that $1,100 of the revenue goes to Bill and $1,900 to Andy. Andy's response might be that they should have an equal percentage of profit on their contribution to the joint enterprise. Thus Andy should get $2,700 and Bill $300.

The final agreement depends on their stubbornness or patience if they negotiate directly with each other. If they try to have the dispute arbitrated by a third party, the arbitrator's decision depends on his sense of the relative value of

[2]John F. Nash, Jr., "The Bargaining Problem," *Econometrica*, vol. 18, no. 2 (1950), pp. 155–162.

hardware and software and on the rhetorical skills of the two principals as they present their arguments before the arbitrator. For the sake of definiteness, suppose the arbitrator decides that the division of the profit should be 4 : 1 in favor of Andy; that is, Andy should get four-fifths of the surplus while Bill gets one-fifth, or Andy should get four times as much as Bill. What is the actual division of revenue under this scheme? Suppose Andy gets a total of x and Bill gets a total of y; thus Andy's profit is $(x - 900)$ and Bill's is $(y - 100)$. The arbitrator's decision implies that Andy's profit should be four times as large as Bill's; so $x - 900 = 4(y - 100)$, or $x = 4y + 500$. The total revenue available to both is \$3,000; so it must also be true that $x + y = 3{,}000$, or $x = 3{,}000 - y$. Then $x = 4y + 500 = 3{,}000 - y$, or $5y = 2{,}500$, or $y = 500$, and thus $x = 2{,}500$. This division mechanism leaves Andy with a profit of $2{,}500 - 900 = \$1{,}600$ and Bill with $500 - 100 = \$400$, which is the 4 : 1 split in favor of Andy that the arbitrator wants.

We now develop this simple data into a general algebraic formula that you will find useful in many practical applications. Then we go on to examine more specifics of what determines the ratio in which the profits in a bargaining game get split.

B. General Theory

Suppose two bargainers, A and B, seek to split a total value v, which they can achieve if and only if they agree on a specific division. If no agreement is reached, A will get a and B will get b, each by acting alone or in some other way acting outside of this relationship. Call these their *backstop* payoffs or, in the jargon of the Harvard Negotiation Project, their **BATNAs (best alternative to a negotiated agreement).**[3] Often a and b are both zero, but, more generally, we only need to assume that $a + b < v$, so that there is a positive **surplus** $(v - a - b)$ from agreement; if this were not the case, the whole bargaining would be moot because each side would just take up its outside opportunity and get its BATNA.

Consider the following rule: each player is to be given his BATNA plus a share of the surplus, a fraction h of the surplus for A and a fraction k for B, such that $h + k = 1$. Writing x for the amount that A finally ends up with, and similarly y for B, we translate these statements as

$$x = a + h(v - a - b) = a(1 - h) + h(v - b)$$
$$x - a = h(v - a - b)$$

and

$$y = b + k(v - a - b) = b(1 - k) + k(v - a)$$
$$y - b = k(v - a - b)$$

[3]See Roger Fisher and William Ury, *Getting to Yes*, 2nd ed. (New York: Houghton Mifflin, 1991).

We call these expressions the Nash formulas. Another way of looking at them is to say that the surplus $(v - a - b)$ gets divided between the two bargainers in the proportions of $h : k$, or

$$\frac{y - b}{x - a} = \frac{k}{h}$$

or, in slope-intercept form,

$$y = b + \frac{k}{h}(x - a) = \left(b - \frac{ak}{h}\right) + \frac{k}{h}x$$

To use up the whole surplus, x and y must also satisfy $x + y = v$. The Nash formulas for x and y are actually the solutions to these last two simultaneous equations.

A geometric representation of the **Nash cooperative solution** is shown in Figure 17.1. The backstop, or BATNA, is the point P, with coordinates (a, b). All points (x, y) that divide the gains in proportions $h : k$ between the two players lie along the straight line passing through P and having slope k/h; this slope is just the line $y = b + (k/h)(x - a)$ that we derived earlier. All points (x, y) that use up the whole surplus lie along the straight line joining $(v,0)$ and $(0,v)$; this line is the second equation that we derived—namely, $x + y = v$. The Nash solution is at the intersection of the lines, at the point Q. The coordinates of this point are the parties' payoffs after the agreement.

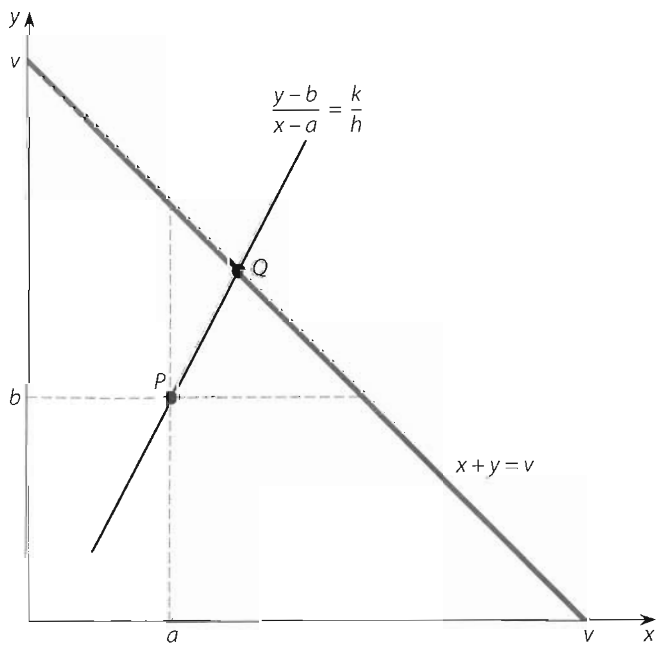


FIGURE 17.1 The Nash Bargaining Solution in the Simplest Case

The Nash formula says nothing about how or why such a solution might come about. And this vagueness is its merit—it can be used to encapsulate the results of many different theories taking many different perspectives.

At the simplest, you might think of the Nash formula as a shorthand description of the outcome of a bargaining process that we have not specified in detail. Then h and k can stand for the two parties' relative bargaining strengths. This shorthand description is a cop-out; a more complete theory should explain where these bargaining strengths come from and why one party might have more than the other. We do so in a particular context later in the chapter. In the meantime, by summarizing any and all of the sources of bargaining strength in these numbers h and k, the formula has given us a good tool.

Nash's own approach was quite different—and indeed different from the whole approach to game theory that we have taken thus far in this book. Therefore it deserves more careful explanation. In all the games that we have studied so far, the players chose and played their strategies separately from one another. We have looked for equilibria in which each player's strategy was in his own best interests, given the strategies of the others. Some such outcomes were very bad for some or even all of the players, the prisoners' dilemma being the most prominent example. In such situations, there was scope for the players to get together and agree that all would follow some particular strategy. But, in our framework, there was no way in which they could be sure that the agreement would hold. After reaching an agreement, the players would disperse, and, when it was each player's turn to act, he would actually take the action that served his own best interests. The agreement for joint action would unravel in the face of such separate temptations. True, in considering repeated games in Chapter 11, we found that the implicit threat of the collapse of an ongoing relationship might sustain an agreement, and, in Chapter 9, we did allow for communication by signals. But individual action was of the essence, and any mutual benefit could be achieved only if it did not fall prey to the selfishness of separate individual actions. In Chapter 2, we called this approach to game theory *noncooperative,* emphasizing that the term signified how actions are taken, not whether outcomes are jointly good. The important point, again, is that any joint good has to be an equilibrium outcome of separate action in such games.

What if joint action *is* possible? For example, the players might take all their actions immediately after the agreement is reached, in one another's presence. Or they might delegate the implementation of their joint agreement to a neutral third party, or to an arbitrator. In other words, the game might be *cooperative* (again in the sense of joint action). Nash modeled bargaining as a cooperative game.

The thinking of a collective group that is going to implement a joint agreement by joint action can be quite different from that of a set of individual people who know that they are *interacting* strategically but are *acting* noncooperatively.

Whereas the latter set will think in terms of an equilibrium and then delight or grieve, depending on whether they like the results, the former can think first of what is a good outcome and then see how to implement it. In other words, the theory defines the outcome of a cooperative game in terms of some general principles or properties that seem reasonable to the theorist.

Nash formulated a set of such principles for bargaining and proved that they implied a unique outcome. His principles are roughly as follows: (1) the outcome should be invariant if the scale in which the payoffs are measured changes linearly; (2) the outcome should be **efficient;** and (3) if the set of possibilities is reduced by removing some that are irrelevant, in the sense that they would not be chosen anyway, then the outcome should not be affected.

The first of these principles conforms to the theory of expected utility, which was discussed briefly in the Appendix to Chapter 7. We saw there that a nonlinear rescaling of payoffs represents a change in a player's attitude toward risk and a real change in behavior; a concave rescaling implies risk aversion, and a convex rescaling implies risk preference. A linear rescaling, being the intermediate case between these two, represents no change in the attitude toward risk. Therefore it should have no effect on expected payoff calculations and no effect on outcomes.

The second principle simply means that no available mutual gain should go unexploited. In our simple example of A and B splitting a total value of v, it would mean that x and y has to sum to the full amount of v available, and not to any smaller amount; in other words, the solution has to lie on the $x + y = v$ line in Figure 17.1. More generally, the complete set of logically conceivable agreements to a bargaining game, when plotted on a graph as was done in Figure 17.1, will be bounded above and to the right by the subset of agreements that leave no mutual gain unexploited. This subset need not lie along a straight line such as $x + y = v$ (or $y = v - x$); it could lie along any curve of the form $y = f(x)$.

In Figure 17.2, all of the points on and below (that is, "south" and to the "west" of) the thick red curve labeled $y = f(x)$ constitute the complete set of conceivable outcomes. The curve itself consists of the efficient outcomes; there are no conceivable outcomes that include more of both x and y than the outcomes on $y = f(x)$; so there are no unexploited mutual gains left. Therefore we call the curve $y = f(x)$ the **efficient frontier** of the bargaining problem. As an example, if the frontier is a quarter circle, $x^2 + y^2 = v^2$, then $y = \sqrt{v^2 - x^2}$, and the function $f(x)$ is defined by $f(x) = \sqrt{v^2 - x^2}$.

The third principle also seems appealing. If an outcome that a bargainer wouldn't have chosen anyway drops out of the picture, what should it matter? This assumption is closely connected to the "independence of irrelevant alternatives" assumption of Arrow's impossibility theorem, which we met in Chapter 15, Section 3, but we must leave the development of this connection to more advanced treatments of the subject.

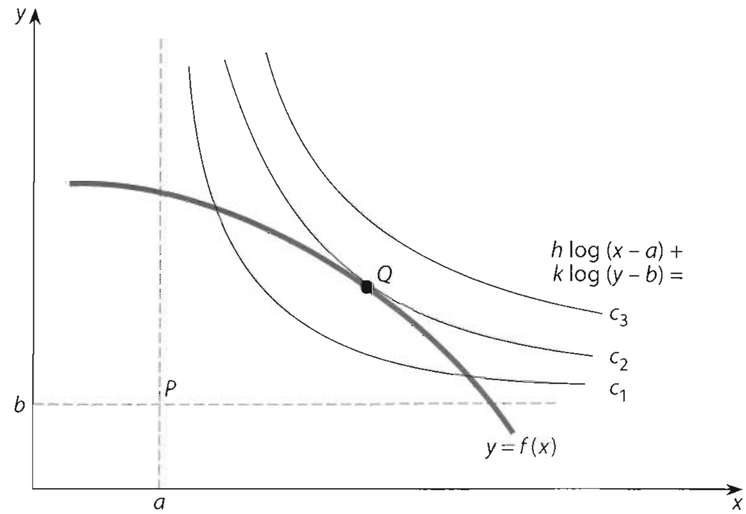


FIGURE 17.2 The General Form of the Nash Bargaining Solution

Nash proved that the cooperative outcome that satisfied all three of these assumptions could be characterized by the mathematical maximization problem: choose x and y to

$$\text{maximize } (x - a)^h(y - b)^k \quad \text{subject to } y = f(x).$$

Here x and y are the outcomes, a and b the backstops, and h and k two positive numbers summing to 1, which are like the bargaining strengths of the Nash formula. The values for h and k cannot be determined by Nash's three assumptions alone; thus they leave a degree of freedom in the theory and in the outcome. Nash actually imposed a fourth assumption on the problem—that of symmetry between the two players; this additional assumption led to the outcome $h = k = 1/2$ and fixed a unique solution. We have given the more general formulation that subsequently became common in game theory and economics.

Figure 17.2 also gives a geometric representation of the objective of the maximization. The black curves labeled c_1, c_2, and c_3 are the level curves, or contours, of the function being maximized; along each such curve, $(x - a)^h(y - b)^k$ is constant and equals c_1, c_2, or c_3 (with $c_1 < c_2 < c_3$) as indicated. The whole space could be filled with such curves, each with its own value of the constant, and curves farther to the "northeast" would have higher values of the constant.

It is immediately apparent that the highest-possible value of the function is at that point of tangency, Q, between the efficient frontier and one of the level curves.[4] The location of Q is defined by the property that the contour passing

[4]One and only one of the (convex) level curves can be tangential to the (concave) efficient frontier; in Figure 17.2, this level curve is labeled c_2. All lower-level curves (such as c_1) cut the frontier in two points; all higher-level curves (such as c_3) do not meet the frontier at all.

through Q is tangent to the efficient frontier. This tangency is the usual way to illustrate the Nash cooperative solution geometrically.[5]

In our example of Figure 17.1, we can also derive the Nash solution mathematically; to do so requires calculus, but the ends here are more important—at least to the study of games of strategy—than the means. For the solution, it helps to write $X = x - a$ and $Y = y - b$; thus X is the amount of the surplus that goes to A, and Y is the amount of the surplus that goes to B. The efficiency of the outcome guarantees that $X + Y = x + y - a - b = v - a - b$, which is just the total surplus and which we will write as S. Then $Y = S - X$, and

$$(x - a)^h(y - b)^k = X^hY^k = X^h(S - X)^k.$$

In the Nash solution, X takes on the value that maximizes this function. Elementary calculus tells us that the way to find X is to take the derivative of this expression with respect to X and set it equal to zero. Using the rules of calculus for taking the derivatives of powers of X and of the product of two functions of X, we get

$$hX^{h-1}(S - X)^k - X^hk(S - X)^{k-1} = 0.$$

When we cancel the common factor $X^{h-1}(S - X)^{k-1}$, this equation becomes

$$\begin{aligned} h(S - X) - kX &= 0 \\ hY - kX &= 0 \\ kX &= hY \\ \frac{X}{h} &= \frac{Y}{k}. \end{aligned}$$

Finally, expressing the equation in terms of the original variables x and y, we have $(x - a)/h = (y - b)/k$, which is just the Nash formula. The punch line: Nash's three conditions lead to the formula we originally stated as a simple way of splitting the bargaining surplus.

The three principles, or desired properties, that determine the Nash cooperative bargaining solution are simple and even appealing. But, in the absence of a good mechanism to make sure that the parties take the actions stipulated by the agreement, these principles may come to nothing. A player who can do better by strategizing on his own than by using the Nash solution may simply reject the principles. If an arbitrator can enforce a solution, the player may simply refuse to go to arbitration. Therefore Nash's cooperative solution will seem more compelling if it can be given an alternative interpretation—namely,

[5]If you have taken an elementary microeconomics course, you will have encountered the concept of social optimality, illustrated graphically by the tangent point between the production possibility frontier of an economy and a social indifference curve. Our Figure 17.2 is similar in spirit; the efficient frontier in bargaining is like the production possibility frontier, and the level curves of the objective in cooperative bargaining are like social indifference curves.

as the Nash equilibrium of a noncooperative game played by the bargainers. This can indeed be done, and we will develop an important special case of it in Section 5.

2 VARIABLE-THREAT BARGAINING

In this section, we embed the Nash cooperative solution within a specific game—namely, as the second stage of a sequential-play game. We assumed in Section 1 that the players' backstops (BATNAs) a and b were fixed. But suppose there is a first stage to the bargaining game in which the players can make strategic moves to manipulate their BATNAs within certain limits. After they have done so, the Nash cooperative outcome starting from those BATNAs will emerge in a second stage of the game. This type of game is called **variable-threat bargaining.** What kind of manipulation of the BATNAs is in a player's interest in this type of game?

We show the possible outcomes from a process of manipulating BATNAs in Figure 17.3. The originally given backstops (a and b) are the coordinates for the game's backstop point P; the Nash solution to a bargaining game with these backstops is at the outcome Q. If player A can increase his BATNA to move the game's backstop point to P_1, then the Nash solution starting there leads to the outcome Q', which is better for player A (and worse for B). Thus a strategic

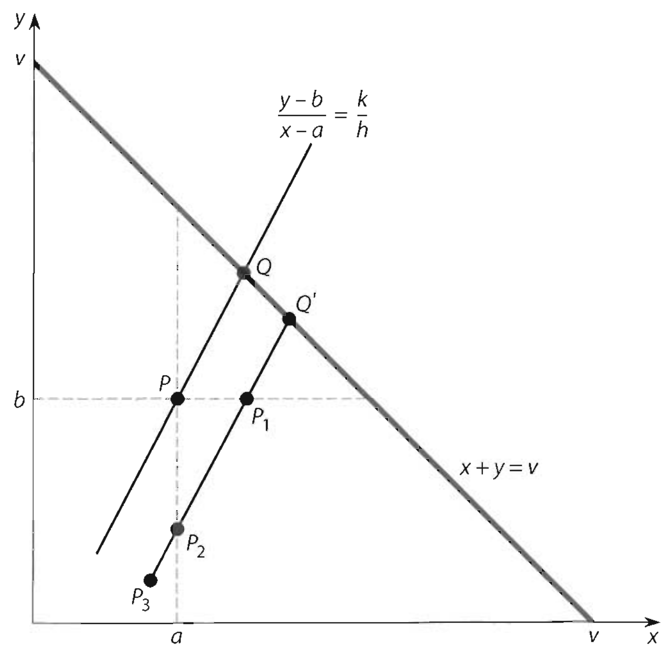


FIGURE 17.3 Bargaining Game of Manipulating BATNAs

move that improves one's own BATNA is desirable. For example, if you have a good job offer in your pocket—a higher BATNA—when you go for an interview at another company, you are likely to get a better offer from that employer than you would if you did not have the first alternative.

The result that improving your own BATNA can improve your ultimate outcome is quite obvious, but the next step in the analysis is less so. It turns out that, if player A can make a strategic move that *reduces* player B's BATNA and moves the game's backstop point to P_2, the Nash solution starting there leads to the *same* outcome Q' that was achieved after A increased his own BATNA to get to the backstop point P_1. Therefore this alternative kind of manipulation is equally in player A's interest. As an example of decreasing your opponent's BATNA, think of a situation in which you are already working and want to get a raise. Your chances are better if you can make yourself indispensable to your employer so that without you his business has much worse prospects; his low outcome in the absence of an agreement—not offering you a raise and your leaving the firm—may make him more likely to accede to your wishes.

Finally and even more dramatically, if player A can make a strategic move that lowers *both players' BATNAs* so that the game's backstop point moves to P_3, that again has the same result as each of the preceding manipulations. This particular move is like using a threat that says, "This will hurt you more than it hurts me."

In general, the key for player A is to shift the game's BATNA point to somewhere below the line PQ. The farther southeast the BATNA point is moved, the better it is for player A in the eventual outcome. As is usual with threats, the idea is not to actually suffer the low payoff but merely to use its prospect as a lever to get a better outcome.

The possibility of manipulating BATNAs in this way depends on the context. We offer just one illustration. In 1980 there was a baseball players' strike. It took a very complicated form. The players struck in spring training, then resumed working (playing, really) when the regular season began in April, and went on strike again starting on Memorial Day. A strike is costly to both sides, employers and employees, but the costs differ. During spring training the players do not have salaries, but the owners make some money from vacationing spectators. At the start of the regular season, in April and May, the players get salaries but the weather is cold and the season is not yet exciting; therefore the crowds are small, and so the cost of a strike to the owners is low. The crowds start to build up from Memorial Day onward, which raises the cost of a strike to the owners, while the salaries that the players stand to lose stay the same. So we see that the two-piece strike was very cleverly designed to lower the BATNA of the owners *relative to* that of the players as much as possible.[6]

[6]See Larry DeBrock and Alvin Roth, "Strike Two: Labor-Management Negotiations in Major League Baseball," *Bell Journal of Economics*, vol. 12, no. 2 (Autumn 1981), pp. 413–425.

One puzzle remains: Why did the strike occur at all? According to the theory, everyone should have seen what was coming; a settlement more favorable to the players should have been reached so that the strike would have been unnecessary. A strike that actually happens is a threat that has "gone wrong." Some kind of uncertainty—asymmetric information or brinkmanship—must be responsible.

3 ALTERNATING-OFFERS MODEL I: TOTAL VALUE DECAYS

Here we move back to the more realistic noncooperative game theory and think about the process of individual strategizing that may produce an equilibrium in a bargaining game. Our standard picture of this process is one of **alternating offers.** One player—say, A—makes an offer. The other—say, B—either accepts it or makes a counteroffer. If he does the latter, then A can either accept it or come back with another offer of his own. And so on. Thus we have a sequential-move game and look for its rollback equilibrium.

To find a rollback equilibrium, we must start at the end and work backward. But where is the end point? Why should the process of offers and counteroffers ever terminate? Perhaps more drastically, why would it ever start? Why would the two bargainers not stick to their original positions and refuse to budge? It is costly to both if they fail to agree at all, but the benefit of an agreement is likely to be smaller to the one who makes the first or the larger concession. The reason that anyone concedes must be that continuing to stand firm would cause an even greater loss of benefit. This loss takes one of two broad forms. The available pie, or surplus, may **decay** (shrink) with each offer, a possibility that we consider in this section. The alternative possibility is that time has value and **impatience** is important, and so a delayed agreement is worth less; we examine this possibility in Section 5.

Consider the following story of bargaining over a shrinking pie. A fan arrives at a professional football (or basketball) game without a ticket. He is willing to pay as much as $25 to watch each quarter of the game. He finds a scalper who states a price. If the fan is not willing to pay this price, he goes to a nearby bar to watch the first quarter on the big-screen TV. At the end of the quarter, he comes out, finds the scalper still there, and makes a counteroffer for the ticket. If the scalper does not agree, the fan goes back to the bar. He comes out again at the end of the second quarter, when the scalper makes him yet another offer. If that offer is not acceptable to the fan, he goes back into the bar, emerging at the end of the third quarter to make yet another counteroffer. The value of watching the rest of the game is declining as the quarters go by.[7]

[7]Just to keep the argument simple, we imagine this process as one-on-one bargaining. Actually, there may be several fans and several scalpers, turning the situation into a *market*. We analyze interactions in markets in detail in Chapter 18.

Rollback analysis enables us to predict the outcome of this alternating-offers bargaining process. At the end of the third quarter, the fan knows that, if he does not buy the ticket then, the scalper will be left with a small piece of paper of no value. So the fan will be able to make a very small offer that, for the scalper, will still be better than nothing. Thus, on his last offer, the fan can get the ticket almost for free. Backing up one period, we see that, at the end of the second quarter, the scalper has the initiative in making the offer. But he must look ahead and recognize that he cannot hope to extract the whole of the remaining two quarters' value from the fan. If the scalper asks for more than \$25—the value of the *third* quarter to the fan—the fan will turn down the offer because he knows that he can get the fourth quarter later for almost nothing; so the scalper can at most ask for \$25. Now consider the situation at the end of the first quarter. The fan knows that, if he does not buy the ticket now, the scalper can expect to get only \$25 later, and so \$25 is all that the fan needs to offer now to secure the ticket. Finally, before the game even begins, the scalper can look ahead and ask for \$50; this \$50 includes the \$25 value of the *first* quarter to the fan plus the \$25 for which the fan can get the remaining three quarters' worth. Thus the two will strike an immediate agreement, and the ticket will change hands for \$50, but the price is determined by the full forward-looking rollback reasoning.[8]

This story can be easily turned into a more general argument for two bargainers, A and B. Suppose A makes the first offer to split the total surplus, which we call v (in some currency—say, dollars). If B refuses the offer, the total available drops by x_1 to $(v - x_1)$; B offers a split of this amount. If A refuses B's offer, the total drops by a further amount x_2 to $(v - x_1 - x_2)$; A offers a split of this amount. This offer and counteroffer process continues until finally—say, after 10 rounds—$v - x_1 - x_2 - \cdots - x_{10} = 0$; so the game ends. As usual with sequential-play games, we begin our analysis at the end.

If the game has gone to the point where only x_{10} is left, B can make a final offer whereby he gets to keep "almost all" of the surplus, leaving a measly cent or so to A. Left with the choice of that or absolutely nothing, A should accept the offer. To avoid the finicky complexity of keeping track of tiny cents, let us call this outcome "x_{10} to B, 0 to A." We will do the same in the other (earlier) rounds.

Knowing what is going to happen in round 10, we turn to round 9. Here A is to make the offer, and $(x_9 + x_{10})$ is left. A knows that he must offer at least x_{10} to B or else B will refuse the offer and take the game to round 10, where he can get that much. Bargainer A does not want to offer any more to B. So, on round 9, A will offer a split where he keeps x_9 and leaves x_{10} to B.

[8]To keep the analysis simple, we omitted the possibility that the game might get exciting, and so the value of the ticket might actually increase as the quarters go by. The uncertainty makes the problem much more complex but also more interesting. The ability to deal with such possibilities should serve as an incentive for you to go beyond this book or course to study more advanced game theory.

Then on the round before, when $x_8 + x_9 + x_{10}$ is left, B will offer a split where he gives x_9 to A and keeps $(x_8 + x_{10})$. Working backward, on the very first round, A will offer a split where he keeps $(x_1 + x_3 + x_5 + x_7 + x_9)$ and gives $(x_2 + x_4 + x_6 + x_8 + x_{10})$ to B. This offer will be accepted.

You can remember these formulas by means of a simple trick. *Hypothesize* a sequence in which all offers are refused. (This sequence is *not* what actually happens.) Then add up the amounts that would be destroyed by the refusals of one player. This total is what the other player gets in the actual equilibrium. For example, when B refused A's first offer, the total available surplus dropped by x_1, and x_1 became part of what went to A in the equilibrium of the game.

If each player has a positive BATNA, the analysis must be modified somewhat to take them into account. At the last round, B must offer A at least the BATNA a. If x_{10} is greater than a, B is left with $(x_{10} - a)$; if not, the game must terminate before this round is reached. Now at round 9, A must offer B the larger of the two amounts—the $(x_{10} - a)$ that B can get in round 10 or the BATNA b that B can get outside this agreement. The analysis can proceed all the way back to round 1 in this way; we leave it to you to complete the rollback reasoning for this case.

We have found the rollback equilibrium of the alternating-offers bargaining game, and, in the process of deriving the outcome, we have also described the full strategies—complete contingent plans of action—behind the equilibrium—namely, what each player *would* do if the game reached some later stage. In fact, actual agreement is immediate on the very first offer. The later stages are not reached; they are off-equilibrium nodes and paths. But, as usual with rollback reasoning, the foresight about what rational players would do at those nodes if they were reached is what informs the initial action.

The other important point to note is that *gradual decay* (several potential rounds of offers) leads to a more even or fairer split of the total than does *sudden decay* (only one round of bargaining permitted). In the latter, no agreement would result if B turned down A's very first offer; then, in a rollback equilibrium, A would get to keep (almost) the whole surplus, giving B an "ultimatum" to accept a measly cent or else get nothing at all. The subsequent rounds give B the credible ability to refuse a very uneven first offer.

4 EXPERIMENTAL EVIDENCE

The theory of this particular type of bargaining process is fairly simple, and many people have staged laboratory or classroom experiments that create such conditions of decaying totals, to observe what the experimental subjects actually do. We mentioned some of them briefly in Chapter 3 when considering the validity of rollback reasoning; now we examine them in more detail in the context of bargaining.[9]

The simplest bargaining experiment is the **ultimatum game,** in which there is only one round: player A makes an offer and, if B does not accept it, the bargaining ends and both get nothing. The general structure of these games is as follows. A pool of players is brought together, either in the same room or at computer terminals in a network. They are paired; one person in the pair is designated to be the *proposer* (the one who makes the offer or is the seller who posts a price), and the other to be the *chooser* (the one who accepts or refuses the offer or is the customer who decides whether to buy at that price). The pair is given a fixed surplus, usually $1 or some other sum of money, to split.

Rollback reasoning suggests that A should offer B the minimal unit—say, 1 cent out of a dollar—and that B should accept such an offer. Actual results are dramatically different. In the case in which the subjects are together in a room and the assignment of the role of proposer is made randomly, the most common offer is a 50:50 split. Very few offers worse than 75:25 are made (with the proposer to keep 75% and the chooser to get 25%) and, if made, they are often rejected.

This finding can be interpreted in one of two ways. Either the players cannot or do not perform the calculation required for rollback or the payoffs of the players include something other than what they get out of this round of bargaining. Surely the calculation in the ultimatum game is simple enough that anyone should be able to do it, and the subjects in most of these experiments are college students. A more likely explanation is the one that we put forth in Chapters 3 (Section 6) and 5 (Section 3)—that the theory, which assumed payoffs to consist only of the sum earned in this one round of bargaining, is too simplistic.

Participants can have payoffs that include other things. They may have self-esteem or pride that prevents them from accepting a very unequal split. Even if the proposer A does not include this consideration in his own payoff, if he thinks that B might, then it is a good strategy for A to offer enough to make it likely that B will accept. Proposer A balances his higher payoff with a smaller offer to B against the risk of getting nothing if B rejects an offer deemed too unequal.

A second possibility is that, when the participants in the experiment are gathered in a room, the anonymity of pairing cannot be guaranteed. If the participants come from a group such as college students or villagers who have ongoing relationships outside this game, they may value those relationships. Then the proposers fear that, if they offer too unequal a split in this game, those relationships may suffer. Therefore they would be more generous in their offers

[9]For more details, see Douglas D. Davis and Charles A. Holt, *Experimental Economics* (Princeton, NJ: Princeton University Press, 1993), pp. 263–269, and *The Handbook of Experimental Economics*, ed. John H. Kagel and Alvin E. Roth (Princeton, NJ: Princeton University Press, 1995), pp. 255–274.

than the simplistic theory suggests. If this is the explanation, then ensuring greater anonymity should enable the proposers to make more unequal offers, and experiments do find this to be the case.

Finally, people may have a sense of fairness drilled into them during their nurture and education. This sense of fairness may have evolutionary value for society as a whole and may therefore have become a social norm. Whatever its origin, it may lead the proposers to be relatively generous in their offers, quite irrespective of the fear of rejection. One of us (Skeath) has conducted classroom experiments of several different ultimatum games. Students who had partners previously known to them with whom to bargain were noticeably "fairer" in their split of the pie. In addition, several students cited specific cultural backgrounds as explanations for behavior that was inconsistent with theoretical predictions.

Experimenters have tried variants of the basic game to differentiate between these explanations. The point about ongoing relationships can be handled by stricter procedures that visibly guarantee anonymity. Doing so by itself has some effect on the outcomes but still does not produce offers as extreme as those predicted by the purely selfish rollback argument of the theory. The remaining explanations—namely, "fear of rejection" and the "ingrained sense of fairness"—remain to be sorted out.

The fear of rejection can be removed by considering a variant called the *dictator game.* Again, the participants are matched in pairs. One person—say, A—is designated to determine the split, and the other—say, B—is simply a passive recipient of what A decides. Now the split becomes decidedly more uneven, but even here a majority of As choose to keep no more than 70%. This result suggests a role for an ingrained sense of fairness.

But such a sense has its limits. In some experiments, a sense of fairness was created simply when the experimenter randomly assigned roles of proposer and chooser. In one variant, the participants were given a simple quiz, and those who performed best were made proposers. This created a sense that the role of proposer had been earned, and the outcomes did show more unequal splits. When the dictator game was played with earned rights and with stricter anonymity conditions, most As kept everything, but some (about 5%) still offered a 50:50 split.

One of us (Dixit) carried out a classroom experiment in which students in groups of 20 were gathered together in a computer cluster. They were matched randomly and anonymously in pairs, and each pair tried to agree on how to split 100 points. Roles of proposer and chooser were not assigned; either could make an offer or accept the other's offer. Offers could be made and changed at any time. The pairs could exchange messages instantly with their matched opponent on their computer screens. The bargaining round ended at a random time between 3 and 5 minutes; if agreement was not reached in time by a pair, both got zero. There were 10 such rounds with different random opponents each time. Thus the game itself offered no scope for cooperation through repetition. In a classroom context, the students had ongoing relationships outside the game, but they did not generally know or guess with whom they were playing in any round, even though no great attempt was made to enforce anonymity. Each student's score for the whole game was the sum of his point score for the 10 rounds. The stakes were quite high, because the score accounted for 5% of the course grade!

The highest total of points achieved was 515. Those who quickly agreed on 50:50 splits did the best, and those who tried to hold out for very uneven scores or who refused to split a difference of 10 points or so between the offers and ran the risk of time running out on them did poorly.[10] It seems that moderation and fairness do get rewarded, even as measured in terms of one's own payoff.

5 ALTERNATING-OFFERS MODEL II: IMPATIENCE

Now we consider a different kind of cost of delay in reaching an agreement. Suppose the actual monetary value of the total available for splitting does not decay, but players have a "time value of money" and therefore prefer early agreement to later agreement. They make offers alternately as described in Section 3, but their time preferences are such that money now is better than money later. For concreteness, we will say that both bargainers believe that having only 95 cents right now is as good as having $1 one round later.

A player who prefers having something right away to having the same thing later is impatient; he attaches less importance to the future relative to the present. We came across this idea in Chapter 11 (Section 2) and saw two reasons for it. First, player A may be able to invest his money—say, $1—now and get his principal back along with interest and capital gains at a rate of return r, for a total of $(1 + r)$ in the next period (tomorrow, next week, next year, or whatever is the length of the period). Second, there may be some risk that the game will end between now and the next offer (like the sudden end at a time between 3 and 5 minutes in the classroom game described earlier). If p is the probability that the game continues, then the chance of getting a dollar next period has an expected value of only p now.

Suppose we consider a bargaining process between two players with zero BATNAs. Let us start the process with one of the two bargainers—say, A—making an offer to split $1. If the other player, B, rejects A's offer, then B will

[10] Those who were best at the mathematical aspects of game theory, such as problem sets, did a little worse than the average, probably because they tried too hard to eke out an extra advantage and met resistance. And women did slightly better than men.

have an opportunity to make his own offer one round later. The two bargainers are in identical situations when each makes his offer, because the amount to be split is always \$1. Thus in equilibrium the amount that goes to the person currently in charge of making the offer (call it x) is the same, regardless of whether that person is A or B. We can use rollback reasoning to find an equation that can be solved for x.

Suppose A starts the alternating offer process. He knows that B can get x in the next round when it is B's turn to make the offer. Therefore, A must give B at least an amount that is equivalent, in B's eyes, to getting x in the next round; A must give B at least $0.95x$ now. (Remember that, for B, 95 cents received now is equivalent to \$1 received in the next round; so $0.95x$ now is as good as x in the next round.) Player A will not give B any more than is required to induce B's acceptance. Thus A offers B exactly $0.95x$ and is left with $(1 - 0.95x)$. But the amount that A gets when making the offer is just what we called x. Therefore $x = 1 - 0.95x$, or $(1 + 0.95)x = 1$, or $x = 1/1.95 = 0.512$.

Two things about this calculation should be noted. First, even though the process allows for an unlimited sequence of alternating offers and counteroffers, in the equilibrium the very first offer A makes gets accepted and the bargaining ends. Because time has value, this outcome is efficient. The cost of delay governs how much A must offer B to induce acceptance; it thus affects A's rollback reasoning. Second, the player who makes the first offer gets more than half of the pie—namely, 0.512 rather than 0.488. Thus each player gets more when he makes the first offer than when the other player makes the first offer. But this advantage is far smaller than that in an ultimatum game with no future rounds of counteroffers.

Now suppose the two players are not equally patient (or impatient, as the case may be). Player B still regards \$1 in the next round as being equivalent to 95 cents now, but A regards it as being equivalent to only 90 cents now. Thus A is willing to accept a smaller amount to get it sooner; in other words, A is more impatient. This inequality in rates of impatience can translate into unequal equilibrium payoffs from the bargaining process. To find the equilibrium for this example, we write x for the amount that A gets when he starts the process and y for what B gets when he starts the process.

Player A knows that he must give B at least $0.95y$; otherwise B will reject the offer in favor of the y that he knows he can get when it becomes his turn to make the offer. Thus the amount that A gets, x, must be $1 - 0.95y$; $x = 1 - 0.95y$. Similarly, when B starts the process, he knows that he must offer A at least $0.90x$, and then $y = 1 - 0.90x$. These two equations can be solved for x and y:

$$\begin{array}{c} x = 1 - 0.95(1 - 0.9x) \\ [1 - 0.95(0.9)]x = 1 - 0.95 \\ 0.145x = 0.05 \\ x = 0.345 \end{array} \quad \text{and} \quad \begin{array}{c} y = 1 - 0.9(1 - 0.95y) \\ [1 - 0.9(0.95)]y = 1 - 0.9 \\ 0.145y = 0.10 \\ y = 0.690 \end{array}$$

Note that x and y do not add up to 1, because each of these amounts is the payoff to a given player when he makes the first offer. Thus, when A makes the first offer, A gets 0.345 and B gets 0.655; when B makes the first offer, B gets 0.69 and A gets 0.31. Once again, each player does better when he makes the first offer than when the other player makes the first offer, and once again the difference is small.

The outcome of this case with unequal rates of impatience differs from that of the preceding case with equal rates of impatience in a major way. With unequal rates of impatience, the more impatient player, A, gets a lot less than B even when he is able to make the first offer. We expect that the person who is willing to accept less to get it sooner ends up getting less, but the difference is very dramatic. The proportions of A's and B's shares are almost 1:2.

As usual, we can now build on these examples to develop the more general algebra. Suppose A regards \$1 immediately as being equivalent to \$$(1 + r)$ one offer later or, equivalently, A regards \$$1/(1 + r)$ immediately as being equivalent to \$1 one offer later. For brevity, we substitute a for $1/(1 + r)$ in the calculations that follow. Likewise, suppose player B regards \$1 today as being equivalent to \$$(1 + s)$ one offer later; we use b for $1/(1 + s)$. If r is high (or, equivalently, if a is low), then player A is very impatient. Similarly, B is impatient if s is high (or if b is low).

Here we look at bargaining that takes place in alternating rounds, with a total of \$1 to be divided between two players, both of whom have zero BATNAs. (You can do the even more general case easily once you understand this one.) What is the rollback equilibrium?

We can find the payoffs in such an equilibrium by extending the simple argument used earlier. Suppose A's payoff in the rollback equilibrium is x when he makes the first offer; B's payoff in the rollback equilibrium is y when he makes the first offer. We look for a pair of equations linking the values x and y and then solve these equations to determine the equilibrium payoffs.[11]

When A is making the offer, he knows that he must give B an amount that B regards as being equivalent to y one period later. This amount is $by = y/(1 + s)$. Then, after making the offer to B, A can keep only what is left: $x = 1 - by$.

Similarly, when B is making the offer, he must give A the equivalent of x one period later—namely, ax. Therefore $y = 1 - ax$. Solving these two equations is now a simple matter. We have $x = 1 - b(1 - ax)$, or $(1 - ab)x = 1 - b$. Expressed in terms of r and s, this equation becomes

$$x = \frac{1 - b}{1 - ab} = \frac{s + rs}{r + s + rs}.$$

[11]We are taking a shortcut; we have simply assumed that such an equilibrium exists and that the payoffs are uniquely determined. More rigorous theory proves these things. For a step in this direction, see John Sutton, "Non-Cooperative Bargaining: An Introduction," *Review of Economic Studies*, vol. 53, no. 5 (October 1986), pp. 709–724. The fully rigorous (and quite difficult) theory is given in Ariel Rubinstein, "Perfect Equilibrium in a Bargaining Model," *Econometrica*, vol. 50, no. 1 (January 1982), pp. 97–109.

Similarly, $y = 1 - a(1 - by)$, or $(1 - ab)y = 1 - a$. This equation becomes

$$y = \frac{1 - a}{1 - ab} = \frac{r + rs}{r + s + rs}.$$

Although this quick solution might seem a sleight of hand, it follows the same steps used earlier, and we soon give a different reasoning yielding exactly the same answer. First, let us examine some features of the answer.

First note that, as in our simple unequal-impatience example, the two magnitudes x and y add up to more than 1:

$$x + y = \frac{r + s + 2rs}{r + s + rs} > 1$$

Remember that x is what A gets when he has the right to make the first offer, and y is what B gets when he has the right to make the first offer. When A makes the first offer, B gets $(1 - x)$, which is less than y; this just shows A's advantage from being the first proposer. Similarly, when B makes the first offer, B gets y and A gets $(1 - y)$, which is less than x.

However, usually r and s are small numbers. When offers can be made at short intervals such as a week or a day or an hour, the interest that your money can earn from one offer to the next or the probability that the game ends precisely within the next interval is quite small. For example, if r is 1% (0.01) and s is 2% (0.02), then the formulas yield $x = 0.668$ and $y = 0.337$; so the advantage of making the first offer is only 0.005. (A gets 0.668 when making the first offer, but $1 - 0.337 = 0.663$ when B makes the first offer; the difference is only 0.005.) More formally, when r and s are each small compared with 1, then their product rs is very small indeed; thus we can ignore rs to write an approximate solution for the split that does not depend on which player makes the first offer:

$$x = \frac{s}{r + s} \qquad \text{and} \qquad y = \frac{r}{r + s}.$$

Now $x + y$ is approximately equal to 1.

Most importantly, x and y in the approximate solution are the shares of the surplus that go to the two players, and $y/x = r/s$; that is, the shares of the players are inversely proportional to their rates of impatience as measured by r and s. If B is twice as impatient as A, then A gets twice as much as B; so the shares are 1/3 and 2/3, or 0.333 and 0.667, respectively. Thus we see that patience is an important advantage in bargaining. Our formal analysis supports the intuition that, if you are very impatient, the other player can offer you a quick but poor deal, knowing that you will accept it.

This effect of impatience hurts the United States in numerous negotiations that our government agencies and diplomats conduct with other countries. The American political process puts a great premium on speed. The media, interest groups, and rival politicians all demand results and are quick to criticize the administration or the diplomats for any delay. With this pressure to deliver, the negotiators are always tempted to come back with results of any kind. Such results are often poor from the long-run U.S. perspective; the other countries' concessions often have loopholes, and their promises are less than credible. The U.S. administration hails the deals as great victories, but they usually unravel after a few years. A similar example, perhaps closer to people's everyday lives, refers to insurance companies, which often make lowball offers of settlement to people who have suffered a major loss; the insurers know that their clients urgently want to make a fresh start and are therefore very impatient.

As a conceptual matter, the formula $y/x = r/s$ ties our noncooperative game approach to bargaining to the cooperative approach of the Nash solution discussed in Section 1. The formula for shares of the available surplus that we derived there becomes, with zero BATNAs, $y/x = k/h$. In the cooperative approach, the shares of the two players stood in the same proportions as their bargaining strengths, but these strengths were assumed to be given somehow from the outside. Now we have an explanation for the bargaining strengths in terms of some more basic characteristics for the players—h and k are inversely proportional to the players' rates of impatience r and s. In other words, Nash's cooperative solution can also be given an alternative and perhaps more satisfactory interpretation as the rollback equilibrium of a noncooperative game of offers and counteroffers, if we interpret the abstract bargaining-strength parameters in the cooperative solution correctly in terms of the players' characteristics such as impatience.

Finally, note that agreement is once again immediate—the very first offer is accepted. The whole rollback analysis as usual serves the function of disciplining the first offer, by making the first proposer recognize that the other would credibly reject a less adequate offer.

To conclude this section, we offer an alternative derivation of the same (precise) formula for the equilibrium offers that we derived earlier. Suppose this time that there are 100 rounds; A is the first proposer and B the last. Start the backward induction in the 100th round; B will keep the whole dollar. Therefore in the 99th round, A will have to offer B the equivalent of \$1 in the 100th round—namely, b, and A will keep $(1 - b)$. Then proceed backwards:

In round 98, B offers $a(1 - b)$ to A and keeps $1 - a(1 - b) = 1 - a + ab$.

In round 97, A offers $b(1 - a + ab)$ to B and keeps $1 - b(1 - a + ab) = 1 - b + ab - ab^2$.

In round 96, B offers $a(1 - b + ab - ab^2)$ to A and keeps $1 - a + ab - a^2b + a^2b^2$.

In round 95, A offers $b(1 - a + ab - a^2b + a^2b^2)$ to B and keeps $1 - b + ab - ab^2 + a^2b^2 - a^2b^3$.

Proceeding in this way and following the established pattern, we see that, in round 1, A gets to keep

$$1 - b + ab - ab^2 + a^2b^2 - a^2b^3 + \cdots + a^{49}b^{49} - a^{49}b^{50} = (1 - b)[1 + ab + (ab)^2 + \cdots + (ab)^{49}]$$

The consequence of allowing more and more rounds is now clear. We just get more and more of these terms, growing geometrically by the factor ab for every two offers. To find A's payoff when he is the first proposer in an infinitely long sequence of offers and counteroffers, we have to find the limit of the infinite geometric sum. In the Appendix to Chapter 11 we saw how to sum such series. Using the formula obtained there, we get the answer

$$(1 - b)[1 + ab + (ab)^2 + \cdots + (ab)^{49} + \cdots] = \frac{1 - b}{1 - ab}$$

This is exactly the solution for x that we obtained before. By a similar argument, you can find B's payoff when he is the proposer and, in doing so, improve your understanding and technical skills at the same time.

6 MANIPULATING INFORMATION IN BARGAINING

We have seen that the outcomes of a bargain depend crucially on various characteristics of the parties to the bargain, most importantly their BATNAs and their impatience. We proceeded thus far by assuming that the players knew each other's characteristics as well as their own. In fact, we assumed that each player knew that the other knew, and so on; that is, the characteristics were common knowledge. In reality, we often engage in bargaining without knowing the other side's BATNA or degree of impatience; sometimes we do not even know our own BATNA very precisely.

As we saw in Chapter 9, a game with such uncertainty or informational asymmetry has associated with it an important game of signaling and screening of strategies for manipulating information. Bargaining is replete with such strategies. A player with a good BATNA or a high degree of patience wants to signal this fact to the other. However, because someone without these good attributes will want to imitate them, the other party will be skeptical and will examine the signals critically for their credibility. And each side will also try screening, by using strategies that induce the others to take actions that will reveal its characteristics truthfully.

In this section, we look at some such strategies used by buyers and sellers in the housing market. Most Americans are active in the housing market several times in their lives, and many people are professional estate agents or brokers, who have even more extensive experience in the matter. Moreover, housing is one of the few markets in the United States where haggling or bargaining over price is accepted and even expected. Therefore considerable experience of strategies is available. We draw on this experience for many of our examples and interpret it in the light of our game-theoretic ideas and insights.[12]

When you contemplate buying a house in a new neighborhood, you are unlikely to know the general range of prices for the particular type of house in which you are interested. Your first step should be to find this range so that you can then determine your BATNA. And that does not mean looking at newspaper ads or realtors' listings, which indicate only asking prices. Local newspapers and some Internet sites list recent actual transactions and the actual prices; you should check them against the asking prices of the same houses to get an idea of the state of the market and the range of bargaining that might be possible.

Next comes finding out (screening) the other side's BATNA and level of impatience. If you are a buyer, you can find out why the house is being sold and how long it has been on the market. If it is empty, why? And how long has it been that way? If the owners are getting divorced or have moved elsewhere and are financing another house on an expensive bridge loan, it is likely that they have a low BATNA or are rather impatient.

You should also find out other relevant things about the other side's preferences, even though these preferences may seem irrational to you. For example, some people consider an offer too far below the asking price an insult and will not sell at any price to someone who makes such an offer. Norms of this kind vary across regions and times. It pays to find out what the common practices are.

Most importantly, the *acceptance* of an offer more accurately reveals a player's true willingness to pay than anything else and therefore is open to exploitation by the other player. A brilliant game-theorist friend of ours tried just such a ploy. He was bargaining for a floor lamp. Starting with the seller's asking price of $100, the negotiation proceeded to a point where our friend made an offer to buy the lamp for $60. The seller said yes, at which point our friend thought: "This guy is willing to sell it for $60, so his true rock-bottom price must be even lower. Let me try to find out whether it is." So our friend said, "How about $55?" The seller got very upset, refused to sell for any price, and asked our friend to leave the store and never come back.

The seller's behavior conformed to the norm that it is utmost bad faith in bargaining to renege on an offer once it is accepted. It makes good sense as a norm in the whole context of all bargaining games that take place in society. If an offer on the table cannot be accepted in good faith by the other player without fear of the kind of exploitation attempted by our friend, then each bargainer

[12]We have taken the insights of practitioners from Andrée Brooks, "Honing Haggling Skills," *New York Times*, December 5, 1993.

will wait to get the other to accept an offer, thereby revealing the limit of his true rock-bottom acceptance level, and the whole process of bargains will grind to a halt. Therefore such behavior has to be disallowed, and making it a social norm to which people adhere instinctively, as the seller in the example did, is a good way for society to achieve this aim.

The offer may explicitly say that it is open only for a specified and limited time; this stipulation can be part of the offer itself. Job offers usually specify a deadline for acceptance; stores have sales for limited periods. But in that case the offer is truly a *package* of price and time, and reneging on either dimension provokes a similar instinctive anger. For example, customers get quite angry if they arrive at a store in the sale period and find an advertised item unavailable. The store must offer a rain check, which allows the customer to buy the item at its sale price when next available at the regular price; even this offer causes some inconvenience to the customer and risks some loss of goodwill. The store can specify "limited quantities, no rain checks" very clearly in its advertising of the sale; even then, many customers get upset if they find that the store has run out of the item.

Next on our list of strategies to use in one-on-one bargaining, like the housing market, comes signaling your own high BATNA or patience. The best way to signal patience is to *be* patient. Do not come back with counteroffers too quickly. "Let the sellers think they've lost you." This signal is credible because someone not genuinely patient would find it too costly to mimic the leisurely approach. Similarly, you can signal a high BATNA by starting to walk away, a tactic that is common in negotiations at bazaars in other countries and some flea markets and tag sales in the United States.

Even if your BATNA is low, you may commit yourself to not accepting an offer below a certain level. This constraint acts just like a high BATNA, because the other side cannot hope to get you to accept anything less. In the housing context, you can claim your inability to concede any further by inventing (or creating) a tightwad parent who is providing the down payment or a spouse who does not really like the house and will not let you offer any more. Sellers can try similar tactics. A parallel in wage negotiations is the *mandate.* A meeting is convened of all the workers who pass a resolution—the mandate—authorizing the union leaders to represent them at the negotiation but with the constraint that the negotiators must not accept an offer below a certain level specified in the resolution. Then, at the meeting with the management, the union leaders can say that their hands are tied; there is no time to go back to the membership to get their approval for any lower offer.

Most of these strategies entail some risk. While you are signaling patience by waiting, the seller of the house may find another willing buyer. As employer and union wait for each other to concede, tensions may mount so high that a strike that is costly to both sides nevertheless cannot be prevented. In other words, many strategies of information manipulation are instances of brinkmanship. We saw in Chapter 14 how such games can have an outcome that is bad for both parties. The same is true in bargaining. *Threats* of breakdown of negotiations or of strikes are strategic moves intended to achieve quicker agreement or a better deal for the player making the move; however, an *actual* breakdown or strike is an instance of the threat "gone wrong." The player making the threat—initiating the brinkmanship—must assess the risk and the potential rewards when deciding whether and how far to proceed down this path.

7 BARGAINING WITH MANY PARTIES AND ISSUES

Our discussion thus far has been confined to the classic situation where two parties are bargaining about the split of a given total surplus. But many real-life negotiations include several parties or several issues simultaneously. Although the games get more complicated, often the enlargement of the group or the set of issues actually makes it easier to arrive at a mutually satisfactory agreement. In this section, we take a brief look at such matters.[13]

A. Multi-Issue Bargaining

In a sense, we have already considered multi-issue bargaining. The negotiation over price between a seller and a buyer always comprises *two* things: (1) the object offered for sale or considered for purchase and (2) money. The potential for mutual benefit arises when the buyer values the object more than the seller does—that is, when the buyer is willing to give up more money in return for getting the object than the seller is willing to accept in return for giving up the object. Both players can be better off as a result of their bargaining agreement.

The same principle applies more generally. International trade is the classic example. Consider two hypothetical countries, Freedonia and Ilyria. If Freedonia can convert 1 loaf of bread into 2 bottles of wine (by using less of its resources such as labor and land in the production of bread and using them to produce more wine instead) and Ilyria can convert 1 bottle of wine into 1 loaf of bread (by switching its resources in the opposite direction), then between them they can create more goods "out of nothing." For example, suppose that Freedonia can produce 200 more bottles of wine if it produces 100 fewer loaves of bread and that Ilyria can produce 150 more loaves of bread if it produces 150 fewer bottles of wine. These switches in resource utilization create an extra 50

[13]For a more thorough treatment, see Howard Raiffa, *The Art and Science of Negotiation* (Cambridge: Harvard University Press, 1982), parts III and IV.

loaves of bread and 50 bottles of wine relative to what the two countries produced originally. This extra bread and wine is the "surplus" that they can create if they can agree on how to divide it between them. For example, suppose Freedonia gives 175 bottles of wine to Ilyria and gets 125 loaves of bread. Then each country will have 25 more loaves of bread and 25 more bottles of wine than it did before. But there is a whole range of possible exchanges corresponding to different divisions of the gain. At one extreme, Freedonia may give up all the 200 extra bottles of wine that it has produced in exchange for 101 loaves of bread from Ilyria, in which case Ilyria gets almost all the gain from trade. At the other extreme, Freedonia may give up only 151 bottles of wine in exchange for 150 loaves of bread from Ilyria, and so Freedonia gets almost all the gain from trade.[14] Between these limits lies the frontier where the two can bargain over the division of the gains from trade.

The general principle should now be clear. When two or more issues are on the bargaining table at the same time and the two parties are willing to trade more of one against less of the other at different rates, then a mutually beneficial deal exists. The mutual benefit can be realized by trading at a rate somewhere between the two parties' different rates of willingness to trade. The division of gains depends on the choice of the rate of trade. The closer it is to one side's willingness ratio, the less that side gains from the deal.

Now you can also see how the possibilities for mutually beneficial deals can be expanded by bringing more issues to the table at the same time. With more issues, you are more likely to find divergences in the ratios of valuation between the two parties and are thereby more likely to locate possibilities for mutual gain. In regard to a house, for example, many of the fittings or furnishings may be of little use to the seller in the new house to which he is moving, but they may be of sufficiently good fit and taste that the buyer values having them. Then, if the seller cannot be induced to lower the price, he may be amenable to including these items in the original price to close the deal.

However, the expansion of issues is not an unmixed blessing. If you value something greatly, you may fear putting it on the bargaining table; you may worry that the other side will extract big concessions from you, knowing that you want to protect that one item of great value. At the worse, a new issue on the table may make it possible for one side to deploy threats that lower the other

[14]Economics uses the concept *ratio of exchange*, or price, which here is expressed as the number of bottles of wine that trade for each loaf of bread. The crucial point is that the possibility of gain for both countries exists with any ratio that lies between the 2:1 at which Freedonia can just convert bread into wine and the 1:1 at which Ilyria can do so. At a ratio close to 2:1, Freedonia gives up almost all of its 200 extra bottles of wine and gets little more than the 100 loaves of bread that it sacrificed to produce the extra wine; thus Ilyria has almost all of the gain. Conversely, at a ratio close to 1:1, Freedonia has almost all of the gain. The issue in the bargaining is the division of gain and therefore the ratio or the price at which the two should trade.

side's BATNA. For example, a country engaged in diplomatic negotiations may be vulnerable to an economic embargo; then it would much prefer to keep the political and economic issues distinct.

B. Multiparty Bargaining

Having many parties simultaneously engaged in bargaining also may facilitate agreement, because, instead of having to look for pairwise deals, the parties can seek a circle of concessions. International trade is again the prime example. Suppose the United States can produce wheat very efficiently but is less productive in cars, Japan is very good at producing cars but has no oil, and Saudi Arabia has a lot of oil but cannot grow wheat. In pairs, they can achieve little, but the three together have the potential for a mutually beneficial deal.

As with multiple issues, expanding the bargaining to multiple parties is not simple. In our example, the deal would be that the United States would send an agreed amount of wheat to Saudi Arabia, which would send its agreed amount of oil to Japan, which would in turn ship its agreed number of cars to the United States. But suppose that Japan reneges on its part of the deal. Saudi Arabia cannot retaliate against the United States, because, in this scenario, it is not offering anything to the United States that it can potentially withhold. Saudi Arabia can only break its deal to send oil to Japan, an important party. Thus enforcement of multilateral agreements may be problematic. The General Agreement on Tariffs and Trade (GATT) between 1946 and 1994, as well as the World Trade Organization (WTO) since then, have indeed found it difficult to enforce their agreements and to levy punishments on countries that violate the rules.

SUMMARY

Bargaining negotiations attempt to divide the *surplus* (excess value) that is available to the parties if an agreement can be reached. Bargaining can be analyzed as a *cooperative* game in which parties find and implement a solution jointly or as a (structured) *noncooperative* game in which parties choose strategies separately and attempt to reach an equilibrium.

Nash's cooperative solution is based on three principles of the outcomes' invariance to linear changes in the payoff scale, *efficiency,* and invariance to removal of irrelevant outcomes. The solution is a rule that states the proportions of division of surplus, beyond the backstop payoff levels (also called *BATNAs* or *best alternatives to a negotiated agreement*) available to each party, based on relative bargaining strengths. Strategic manipulation of the backstops can be used to increase a party's payoff.

In a noncooperative setting of *alternating offer and counteroffer*, rollback reasoning is used to find an equilibrium; this reasoning generally includes a first-round offer that is immediately accepted. If the surplus value *decays* with refusals, the sum of the (hypothetical) amounts destroyed owing to the refusals of a single player is the payoff to the other player in equilibrium. If delay in agreement is costly owing to *impatience*, the equilibrium offer shares the surplus roughly in inverse proportion to the parties' rates of *impatience*. Experimental evidence indicates that players often offer more than is necessary to reach an agreement in such games; this behavior is thought to be related to player anonymity as well as beliefs about fairness.

The presence of information asymmetries in bargaining games makes signaling and screening important. Some parties will wish to signal their high BATNA levels or extreme patience; others will want to screen to obtain truthful revelation of such characteristics. When more issues are on the table or more parties are participating, agreements may be easier to reach but bargaining may be riskier or the agreements more difficult to enforce.

KEY TERMS

alternating offers (577)
best alternative to a negotiated agreement (BATNA) (569)
decay (577)
efficient frontier (572)
efficient outcome (572)
impatience (577)
Nash cooperative solution (570)
surplus (569)
ultimatum game (580)
variable-threat bargaining (575)

EXERCISES

1. Consider the bargaining situation between Compaq Computer Corporation and the California businessman who owned the Internet address *www.altavista.com*.[15] Compaq, which had recently taken over Digital Equipment Corporation, wanted to use this man's Web address for Digital's Internet search engine, which was then accessed at *www.altavista.digital.com*. Compaq and the businessman apparently negotiated long and hard during the summer of 1998 over a selling price for the latter's address. Although the businessman was the "smaller" player in this game, the final agreement

[15]Details regarding this bargaining game were reported in "A Web Site by Any Other Name Would Probably Be Cheaper," *Boston Globe*, July 29, 1998, and in Hiawatha Bray's "Compaq Acknowledges Purchase of Web Site," *Boston Globe*, August 12, 1998.

appeared to entail a $3.35 *million* price tag for the Web address in question. Compaq confirmed the purchase in August and began using the address in September but refused to divulge any of the financial details of the settlement. Given this information, comment on the likely values of the BATNAs for these two players, their bargaining strengths or levels of impatience, and whether a cooperative outcome appears to have been attained in this game.

2. Ali and Baba are bargaining to split a total that starts out at $100. Ali makes the first offer, stating how the $100 will be divided between them. If Baba accepts this offer, the game is over. If Baba rejects it, a dollar is withdrawn from the total, and so it is now only $99. Then Baba gets the second turn to make an offer of a division. The turns alternate in this way, a dollar being removed from the total after each rejection. Ali's BATNA is $2.25 and Baba's BATNA is $3.50. What is the rollback equilibrium outcome of the game?

3. Recall the variant of the pizza pricing game in Exercise 4 of Chapter 11, in which one store (Donna's Deep Dish) was much larger than the other (Pierce's Pizza Pies). The payoff table for that version of the game is:

		PIERCE'S PIZZA PIES	
		High	Low
DONNA'S DEEP DISH	High	156, 60	132, 70
	Low	150, 36	130, 50

The noncooperative dominant strategy equilibrium is (High, Low), yielding profits of 132 to Donna's and 70 to Pierce's, for a total of 202. If the two could achieve (High, High), their total profit would be $156 + 60 = 216$, but Pierce's would not agree to this pricing. Suppose the two stores can reach an enforceable agreement where both charge High and Donna's pays Pierce's a sum of money. The alternative to this agreement is simply the noncooperative dominant-strategy equilibrium. They bargain over this agreement, and Donna's has 2.5 times as much bargaining power as Pierce's. In the resulting agreement, what sum will Donna's pay to Pierce's?

4. Two hypothetical countries, Euphoria and Militia, are holding negotiations to settle a dispute. They meet once a month, starting in January, and take turns making offers. Suppose the total at stake is 100 points. The government of Euphoria is facing reelection in November. Unless the government produces an agreement at the October meeting, it will lose the election, which it regards as being just as bad as getting zero points from an agreement. The government of Militia does not really care about reaching an agreement; it is

just as happy to prolong the negotiations or even to fight, because it would be settling for anything significantly less than 100. What will be the outcome of the negotiations? What difference will the identity of the first mover make?

5. In light of your answer to Exercise 4, discuss why actual negotiations often continue right down to the deadline.

6. Let x be the amount that player A asks for and let y be the amount that B asks for, when making the first offer in an alternating-offers bargaining game with impatience. Their rates of impatience are r and s, respectively.
   (a) If we use the approximate formulas $x = s/(r + s)$ for x and $y = r/(r + s)$ for y and if B is twice as impatient as A, then A gets two-thirds of the surplus and B gets one-third. Verify that this result is correct.
   (b) Let $r = 0.01$ and $s = 0.02$, and compare the x and y values found by using the approximation method with the more exact solutions for x and y found by using the formulas $x = (s + rs)/(r + s + rs)$ and $y = (r + rs)/(r + s + rs)$ derived in the text.

18

Markets and Competition

AN ECONOMY IS A SYSTEM ENGAGED in production and consumption. Its primary resources, such as labor, land, and raw materials, are used in the production of goods and services, including some that are then used in the production of other goods and services. The end points of this chain are the goods and services that go to the ultimate consumers. The system has evolved differently in different places and times, but always and everywhere it must create methods of linking its component parts—that is to say, a set of institutions and arrangements that enable various suppliers, producers, and buyers to deal with one another.

In Chapters 16 and 17, we examined two such institutions: auctions and bargaining. In auctions, one seller generally deals with several actual or potential buyers, although the reverse arrangement—in which one buyer deals with several actual or potential sellers—also exists; an example is a construction or supply project being offered for tender. Bargaining generally confronts one buyer with one seller.

That leaves perhaps the most ubiquitous economic institution of all—namely, the *market,* where several buyers and several sellers can deal simultaneously. A market can conjure up the image of a bazaar in a foreign country, where several sellers have their stalls, several buyers come, and much bargaining is going on at the same time between pairs of buyers and sellers. But most markets operate differently. A town typically has two or more supermarkets, and the households in the town decide to shop at one (or perhaps buy some items at one and some at another) by comparing availability, price, quality, and so on. Supermarkets are generally not located next to one another; nevertheless, they know that they have to compete with one another for the townspeople's patronage, and this competition affects their decisions on what to stock, what prices to charge, and so forth. There is no overt bargaining between a store and a customer; prices are posted, and one can either pay that price or go elsewhere. But the existence of another store limits the price that each store can charge.

In this chapter, we will briefly examine markets from a strategic viewpoint. How should, and how do, a group of sellers and buyers act in this environment? What is the equilibrium of their interaction? Are buyers and sellers efficiently matched to one another in a market equilibrium?

We have already seen a few examples of strategies in markets. Remember the pricing game of the two restaurants introduced in Chapter 4? Because the strategic interaction pits two sellers against each other, we call it a *duopoly* (from the Greek *duo,* meaning "two," and *polein,* "to sell"). We found that the equilibrium was a prisoners' dilemma, in which both players charge prices below the level that would maximize their joint profit. In Chapter 11, we saw some ways, most prominently through the use of a repeated relationship, whereby the two restaurants can resolve the prisoners' dilemma and sustain tacit collusion with high prices; they thus become a *cartel,* or in effect a *monopoly.*

But, even when the restaurants do not collude, their prices are still higher than their costs because each restaurant has a somewhat captive clientele whom it can exploit and profit from. When there are more than two sellers in a market—and as the number of sellers increases—we would expect the prisoners' dilemma to worsen, or the competition among the sellers to intensify, to the point where prices are as low as they can be. If there are hundreds or thousands of sellers, each might be too small to have a truly strategic role, and the game-theoretic approach would have to be replaced with one or more suitable for dealing with many individually small buyers and sellers. In fact, that is what the traditional economic "apparatus" of *supply and demand* does—it assumes that each such small buyer or seller is unable to affect the market price. Each party simply decides how much to buy or sell at the prevailing price. Then the price adjusts, through some invisible and impersonal market, to the level that equates supply and demand and thus "clears" the market. Remember the distinction that we drew in Chapter 2 between an individual decision-making situation and a game of strategic interaction. The traditional economic approach regards the market as a collection of individual decision makers who interact not strategically but through the medium of the price set by the market.

We can now go beyond this cursory treatment to see precisely how an increase in the number of sellers and buyers intensifies the competition between them and leads to an outcome that looks like an impersonal market in which the price adjusts to equate demand and supply. That is to say, we can derive our supply-and-demand analysis from the more basic strategic considerations of bargaining and competition. This approach will deepen your understanding of the market mechanism even if you have already studied it in elementary economics courses and textbooks. If you have not, this chapter will allow you to leapfrog ahead of others who have studied economics but not game theory.

In addition to the discussion of the market mechanism, we will show when and how competition yields an outcome that is socially desirable; we also consider what is meant by the phrase "socially desirable." Finally, in Sections 4 and 5, we examine some alternative mechanisms that attempt to achieve efficient or fair allocation where markets cannot or do not function well.

1 A SIMPLE TRADING GAME

We begin with a situation of pure bargaining between two people, and then we introduce additional bargainers and competition. Imagine two people, a seller S and a buyer B, negotiating over the price of a house. Seller S may be the current owner who gets a payoff with a monetary equivalent of \$100,000 from living in the house. Or S may be a builder whose cost of constructing the house is \$100,000. In either interpretation, this amount represents the seller's BATNA, or, in the language of auctions, his reserve price. From living in the same house, B can get a payoff of \$300,000, which is the maximum that he would be willing to pay. Right now, however, B does not own a house; so his BATNA is 0. Throughout this section, we assume that the sellers' reserve prices and the buyers' willingness to pay are common knowledge for all the participants; so the negotiation games are not plagued by the additional difficulty of information manipulation. We consider the latter in connection with the market mechanism later in the chapter.

If the house is sold for a price p, the seller gets p but gives up the house; so his surplus, measured in thousands of dollars, is $(p - 100)$. The buyer gets to live in the house, which gives him a benefit of 300, for which he pays p; so his surplus is $(300 - p)$. The two surpluses add up to

$$(300 - p) + (p - 100) = 300 - 100 = 200$$

regardless of the specific value of p. Thus 200 is the total surplus that the two can realize by striking a deal—the extra value that exists when S and B reach an agreement with each other.

The function of the price in this agreement is to divide up the available surplus between the two. For S to agree to the deal, he must get a nonnegative surplus from it; therefore p must be at least 100. Similarly, for B to agree to the deal, p cannot exceed 300. The full range of the negotiation for p lies between these two limits, but we cannot predict p precisely without specifying more details of the bargaining process. In Chapter 17, we saw two ways to model this process. The Nash cooperative model determined an outcome, given the relative bargaining strengths of S and B; the noncooperative game of alternating offers and counteroffers fixed an outcome, given the relative patience of the two.

Here we develop a different graphical illustration of the BATNAs, the surplus, and the range of negotiation from the approach used in Chapter 17. This device will help us to relate ideas based on bargaining to those of trading in markets. Figure 18.1 shows quantity on the horizontal axis and money measured in thousands of dollars on the vertical axis. The curve *S*, which represents the seller's behavior, consists of three vertical and horizontal red lines that together are shaped like a rising step.[1] The curve *B*, which represents buyer's behavior, consists of three vertical and horizontal gray lines, shaped like a falling step.

Here's how to construct the seller's curve. First, locate the point on the vertical axis corresponding to the seller's reserve price, here 100. Then draw a line from 0 to 100 on the vertical axis, indicating that no house is available for prices below 100. From there, draw a horizontal line (shown in red in Figure 18.1) of length 1; this line shows that the quantity that the seller wants to sell is exactly one house. Finally, from the right-hand end of this line, draw a line vertically upward to the region of very high money amounts. This line emphasizes that, in this example, there is no other seller, and so no quantity greater than 1 is available for sale, no matter how much money is offered for it.

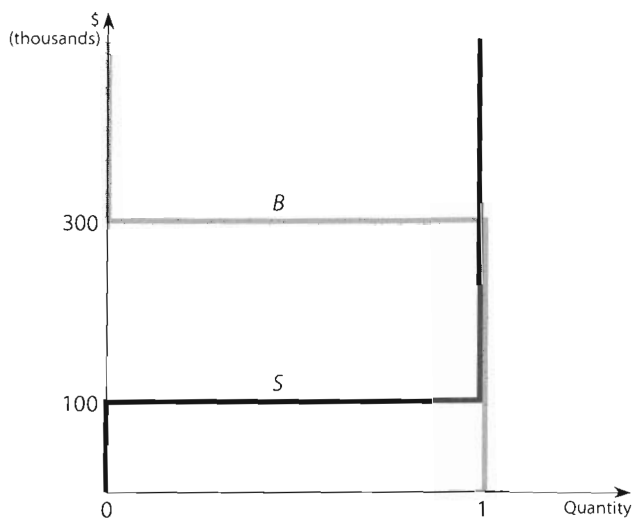


FIGURE 18.1 One Buyer, One Seller

[1]The word *curve* is more commonly used for a smoothly curving shape; therefore you may think it strange to have a "curve" consisting of three (or later, more) straight line segments joined end to end at right angles. But we use it, for want of a better word.

In a similar manner, the buyer's bargaining curve B is constructed. As before, first locate the point on the vertical axis representing the buyer's willingness to pay, here 300. From there, draw a vertical line going upward along the axis, indicating that, for prices above 300, the buyer is not willing to purchase any house at all. Next, draw a horizontal line (shown in gray in Figure 18.1) of length 1, representing the quantity (one house) that the buyer wants to buy. Finally, from the right end of this horizontal line, draw a line going vertically downward to the horizontal axis, suggesting that, in this example, there is no other buyer, and so no quantity greater than 1 finds any customers, no matter how low the asking price.

Note that the two step-shaped curves S and B overlap along their vertical segments, at quantity 1 and at money amounts ranging between 100 and 300. The overlap indicates the range of negotiation for bargaining between buyer and seller. The horizontal coordinate of the overlapping segment (here 1) is the amount traded; the vertical segment of overlap (here from 100 to 300) represents the range of prices at which the final bargaining agreement can take place.

Those of you who have had an introductory course in economics will recognize the S as the seller's **supply curve** and B as the buyer's **demand curve.** For those unfamiliar with these concepts, we can explain them here briefly. Suppose the trades are initiated and negotiated not by the buyers and sellers themselves, as in the two-person game described in Figure 18.1, but in an organized market. A neutral "market maker," whose job it is to bring buyers and sellers together by determining a price at which they can trade, finds out how much will be on offer for sale and how much will be demanded at any price and then adjusts the price so that the market clears—that is, the quantity demanded equals the quantity supplied. This market maker is sometimes a real person—for example, the manager of the trading desk for a particular security in financial markets; at other times he is a hypothetical construct in economic theorizing. The same conclusions follow in either case.

Let us see how we can apply the market concept to the outcome in our bargaining example. First, consider the supply side. Curve S in Figure 18.1 tells us that, for any price below 100, nothing is offered for sale and that, for any price above 100, exactly one house is offered. This curve also shows that, when the price is exactly 100, the seller would be just as happy selling the house as he would be remaining in it as the original owner (or not building it at all if he is the builder); so the quantity offered for sale could be 0 or 1. Thus this curve indicates the quantity offered for sale at each possible price. In our example, the fractional quantities between 0 and 1 do not have any significance, but, for other objects of exchange such as wheat or oil, it is natural to think of the supply curve as showing continuously varying quantities as well.

Next consider the demand side. The graph of the buyer's willingness to pay, labeled B in Figure 18.1, indicates that, for any price above 300, there is no willing buyer, but, for any price below 300, there is a buyer willing to buy exactly one house. At a price of exactly 300, the buyer is just as happy buying as not; so he may purchase either 0 or 1 item.

The demand curve and the supply curve coincide at quantity 1 over the range of prices from 100 to 300. When the market maker chooses any price in this range, exactly one buyer and exactly one seller will appear. The market maker can then bring them together and consummate the trade; the market will have cleared. We explained this outcome earlier as arising from a bargaining agreement, but economists would call it a **market equilibrium.** Thus the market is an institution or arrangement that brings about the same outcome as would direct bargaining between the buyer and the seller. Moreover, the range of prices that clears the market is the same as the range of negotiation in bargaining.

The institution of a market, where each buyer and seller responds passively to a price set by a market maker, seems strange when there is just one buyer and one seller. Indeed, the very essence of markets is competition, which demands the presence of several sellers and several buyers. Therefore we now gradually extend the scope of our analysis by introducing more of each type of agent, and at each step we examine the relationship between direct negotiation and market equilibrium.

Let us consider a situation in which there is just the one original buyer, *B*, who is still willing to pay a maximum of \$300,000 for a house. But now we introduce a second seller, S_2, who has a house for sale identical with that offered by the first seller, whom we now call S_1. Seller S_2 has a BATNA of \$150,000. It may be higher than S_1's because S_2 is an owner who places a higher subjective value on living in his house than S_1 does or S_2 is a builder with a higher cost of construction than S_1's cost.

The existence of S_2 means that S_1 cannot hope to strike a deal with B for any price higher than 150. If S_1 and B try to make a deal at a price of even 152, for example, S_2 could undercut this price with an offer of 151 to the buyer. S_2 would still get a positive surplus instead of being left out of the bargain and getting 0. Similarly, S_2 cannot hope to charge more than 150, because S_1 could undercut that price even more eagerly. Thus the presence of the competing seller narrows the range of bargaining in this game from (100, 300) to (100, 150). The equilibrium trade is still between the original two players, S_1 and B, because it will take place at a price somewhere *between* 100 and 150, which cuts S_2 out of the trade. The presence of the second seller thus drives down the price that the original seller can get. Where the price settles between 100 and 150 depends on the relative bargaining powers of B and S_1. Even though S_2 drops out of the picture at any price below 150, if B has a lot of bargaining power for other reasons not mentioned here (for example, relative patience), then he may refuse an offer from S_1 that is close to 150 and hold out for a figure much closer to 100. If S_1 has relatively little bargaining power, he may have to accept such a counteroffer.

We show the demand and supply curves for this case in Figure 18.2a. The demand curve is the same as before, but the supply curve has two steps. For any price below 100, neither seller is willing to sell, and so the quantity is zero; this is the vertical line along the axis from 0 to 100. For any price from 100 to 150, S_1 is willing to sell, but S_2 is not. Thus the quantity supplied to the market is 1, and the vertical line at this quantity extends from 100 to 150. For any price above 150, both sellers are willing to sell and the quantity is 2; the vertical line at this quantity, from the price of 150 upward, is the last segment of the supply curve. Again we draw horizontal segments from quantity 0 to quantity 1 at price 100 and from quantity 1 to quantity 2 at price 150 to show the supply curve as an unbroken entity.

The overlap of the demand and supply curves is now a vertical segment representing a quantity of 1 and prices ranging from 100 to 150. This is exactly the range of negotiation that we found earlier when considering bargaining instead of a market. Thus the two approaches predict the same range of outcomes: one house changes hands, the seller with the lower reserve price makes the sale, and the final price is somewhere between \$100,000 and \$150,000.

What if both sellers had the same reserve price of \$100,000? Then neither would be able to get any price above 100. For example, if S_1 asked for \$101,000, then S_2, facing the prospect of making no sale at all, could counter by asking only \$100,500. The range of negotiation would be eliminated entirely. One seller would succeed in selling his house for \$100,000. The other would be left out, but at this price he is just as happy not selling the house as he is selling it.

We show this case in the context of market supply and demand in Figure 18.2b. Neither seller offers his house for sale for any price below 100, and both

FIGURE 18.2 One Buyer, Two Sellers

sellers offer houses as soon as the price rises above 100. Therefore the supply curve has a horizontal segment of length 2 at height 100; at its right-hand end starts the terminal vertical upward line of indefinite length. The demand curve has not changed, and so now the two curves overlap only at the point (1,100). The range of prices is now reduced to one value, and this price represents the market equilibrium. One house changes hands, one of the two sellers (it does not matter which) makes the sale, and the price is exactly 100. The competition between the two sellers with identical reserve prices prevents either from getting more than the reserve price; the buyer reaps the benefit of the competition between the sellers in the form of a large surplus—namely, $300 - 100 = 200$.

The situation in the market could be reversed so that there is just one seller, S, but a second buyer, B_2, who is willing to pay \$200,000 for a house. In this case, the first buyer, now labelled B_1, cannot hope to acquire the house for any less than 200. You should be able to figure out why: B_2 will bid the price up to at least 200; so, if B_1 is to get the house, he will have to offer more than that amount. The range of possible prices on which S and B_1 could agree now extends from 200 to 300. This range is again narrower than the original 100 to 300 that we had with only one buyer and one seller. We leave you to do the explicit analysis by using the negotiation approach and the market supply-and-demand curves.

Proceeding with our pattern of introducing more buyers and sellers into our bargaining or market, we next consider a situation with two of each type of trader. Suppose the sellers, S_1 and S_2, have reserve prices of 100 and 150, respectively, and the buyers, B_1 and B_2, have a maximum willingness to pay of 300 and 200, respectively. We should expect the range of negotiation in the bargaining game to extend from 150 to 200 and can prove it as follows.

Each buyer and each seller is looking for the best possible deal for himself. They can make and receive tentative proposals while continuing to look around for better ones, until no one can find anything better, at which point the whole set of deals is finalized. Consider one such tentative proposal, in which one house—say, S_1's—would sell at a price $p_1 < 150$ and the other, S_2's, at $p_2 > 150$. The buyer who would pay the higher price—say, B_1—can then approach S_1 and offer him $(p_1 + p_2)/2$. This offer is greater than p_1 and so is better for S_1 than the original tentative proposal and is less than p_2 and so is better for B_1 as well. This offer would be worse for the other two traders but, because B_1 and S_1 are free to choose their partners and trades, B_2 and S_2 can do nothing about it. Thus any price less than 150 cannot prevail. Similarly, you can check that any price higher than 200 cannot prevail either.

All of this is a fairly straightforward continuation of our reasoning in the earlier cases with fewer traders. But, with two of each type of trader, an interesting new feature emerges: the two houses (assumed to be identical) must both sell for the same price. If they did not, the buyer who was scheduled to pay more in the original proposal could strike a mutually better deal with the seller who was getting less.

Note the essential difference between competition and bargaining here. You *bargain* with someone on the "other side" of the deal; you *compete* with someone on the "same side" of the deal for an opportunity to strike a deal with someone on the other side. Buyers *bargain* with sellers; buyers *compete* with other buyers, and sellers *compete* with other sellers.

We draw the supply and demand curves for the two-seller, two-buyer case in Figure 18.3a. Now each curve has three vertical segments and two horizontal lines joining the vertical segments at the reserve prices of the two traders on that side of the market. The curves overlap at the quantity 2 and along the range of prices from 150 to 200. This overlap is where the market equilibrium must lie. The two houses will change hands at the same price; it does not matter whether S_1 sells to B_1 and S_2 to B_2, or S_1 sells to B_2 and S_2 to B_1.

We move finally to the case of three buyers and three sellers. Suppose the first two sellers, S_1 and S_2, have the same respective reserve prices of 100 and 150 as before, but the third, S_3, has a higher reserve price of 220. Similarly, the first two buyers B_1 and B_2 have the same willingness to pay as before, 300 and 200, respectively, but the third buyer's willingness to pay is lower, only 140. The new buyer and seller do not alter the negotiation or its range at all; only two houses get traded, between sellers S_1 and S_2 and buyers B_1 and B_2, for a price somewhere between 150 and 200 as before. B_3 and S_3 do not get to make a trade.

To see why B_3 and S_3 are left out of the trading, suppose S_3 does come to an agreement to sell his house. Only B_1 could be the buyer because neither B_2 nor B_3 is willing to pay S_3's reserve price. Suppose the agreed-on price is 221; so B_1

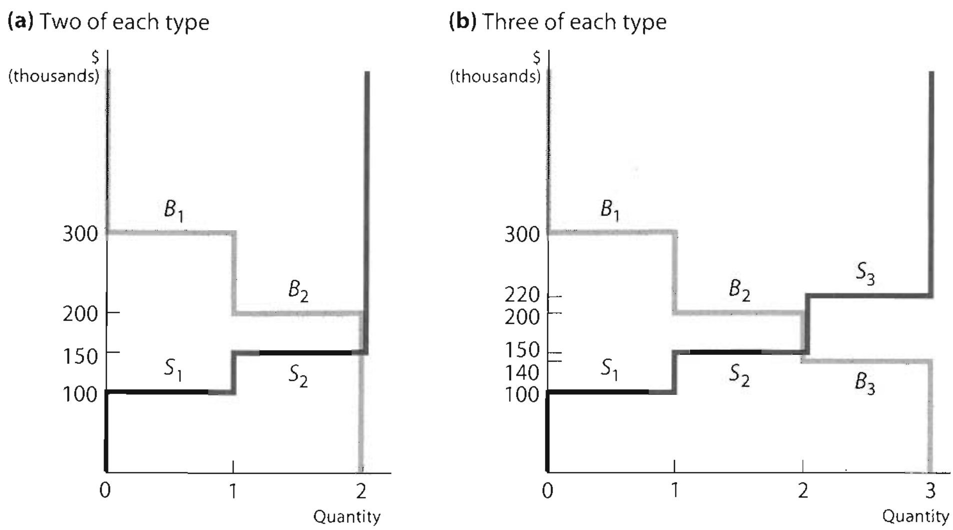


FIGURE 18.3 More Buyers and Sellers

gets a surplus of 79 and S_3 gets only 1. Then consider how the others must pair off. S_1, with his reserve price of 100, must sell to B_3, with his willingness to pay of 140; and S_2, with his reserve price of 150, must sell to B_2, with his willingness to pay of 200. The surplus from the S_1 to B_3 sale is 40, and the surplus from the S_2 to B_2 sale is 50. Therefore S_1 can get at most almost 40 in surplus from his sale, and S_2 can get at most 50 from his; either seller could benefit by approaching B_1 and striking a mutually beneficial alternative deal. For example, S_2 could approach B_1 and say, "With your willingness to pay of 300 and my reserve price of 150, we have a total available surplus of 150 between us. You are getting 79 in your deal with S_3; I am getting less than 50 in my deal with B_2. Let us get together; you could have 90 and I could have 60, and we both would be better off." B_1 would accept this deal, and his tentative arrangement with S_3 would fall through.

The numbers derived here are specific to our example, but the argument itself is perfectly general. For any number of buyers and sellers, first rank sellers by increasing reserve prices and buyers by decreasing willingness to pay. Then pair buyers and sellers of the same rank. When you find the pairing number n such that the nth buyer's willingness to pay is less than the nth seller's reserve price, you know that all buyers and sellers ranked n or more will not get to make a trade.

We can see this argument even more clearly by using market supply-and-demand-curve analysis as in Figure 18.3b. Construction of the supply-and-demand curves follows the same steps as before; so we do not describe it in detail here. The two curves overlap at the quantity 2, and the prices in the overlap range from 150 to 200. Therefore, in the market equilibrium, two houses change hands at a price somewhere between these two limits. Given the range of possible prices, we know that B_3's willingness to pay, 140, will be less than the market price, and S_3's reserve price, 220, will be higher than the market price. Therefore, just as we could say in the general case that the nth buyer and seller do not get to trade, we can assert that these two agents will not be active in the market equilibrium. Once again, it does not matter whether S_1 sells to B_1 and S_2 to B_2, or S_1 sells to B_2 and S_2 to B_1.

Suppose for the sake of definiteness that the market price in this case is 175. Then the surpluses for the traders are 75 for S_1, 25 for S_2, 125 for B_1, 25 for B_2, and 0 for S_3 and B_3; the total surplus is 250. You can check that the total surplus will be the same no matter where the price ends up within the range from 150 to 200.

But couldn't we pair differently and do better—especially if all three houses get sold? The answer is no. The negotiation process or the market mechanism produces the largest possible surplus. As before, to come up with an alternate solution we have to pair B_1 with S_3, B_3 with S_1, and B_2 with S_2. Then the B_1 and S_3 pairing generates a surplus of $300 - 200 = 80$ and the B_3 and S_1 pairing generates $140 - 100 = 40$, while the remaining pairing between B_2 and S_2 generates $200 - 150 = 50$. The total surplus in this alternative scenario comes to only 170, which is 80 less than the surplus of the 250 associated with the negotiation outcome or the market equilibrium. The problem is that B_3 is willing to pay only 140, a price below the equilibrium range, but S_3 has a reserve price of 220, above the equilibrium price range; between them, they bring to the game a negative surplus of $140 - 220 = -80$. If they get an opportunity to trade, there is no escaping the fact that the total surplus will go down by 80.

Although the outcome of the negotiation process or the market equilibrium yields the highest total surplus, it does not necessarily guarantee the most equal distribution of surplus: B_1, B_2, S_1, and S_2 share the whole 250, while B_3 and S_3 get nothing. In our alternate outcome, everyone got something, but the pairings could not survive the process of renegotiation. If you are sufficiently concerned about equity, you might prefer the alternative outcome despite the level of surplus generated. You could then restrict the process of renegotiation to bring about your preferred outcome. Better still, you could let the negotiation or market mechanisms go their way and generate the maximum surplus; then you could take some of that surplus away from those who enjoy it and redistribute it to the others by using a tax-and-transfer policy. The subject of public economics considers such issues in depth.

Let us pause here to sum up what we have accomplished in this section. We have developed a method of thinking about how negotiations between buyers and sellers proceed to secure mutually advantageous trades. We also linked the outcomes of this bargaining process with those of the market equilibrium based on supply-and-demand analysis. Although our analysis was of simple numerical examples with small numbers of buyers and sellers, the ideas have broad-ranging implications, which we develop in a more general way in the sections that follow. In the process, we also take the negotiation idea beyond that of the trade context. As we saw in Section 7 of Chapter 17, with many agents and many goods and services available to trade, the best deals may be multilateral. A good general theory must allow not just pairs, but also triplets or more general groupings of the participants. Such groups—sometimes called *coalitions*—can get together to work out tentative deals as the individual people and groups continue the search for better alternatives. The process of deal making stops only when no person or coalition can negotiate anything better for itself. Then the tentative deals become final and are consummated. We take up the subject of creating mutually advantageous deals in the next section.

2 THE CORE

We now introduce some notation and a general theory for analyzing a process of negotiation and renegotiation for mutually advantageous deals. Then we apply it in a simple context. We will find that, as the number of participants becomes large, the complex process of coalition formation, re-formation, and deal making produces an outcome very close to what a traditional market would produce. Thus we will see the market is an institution that can simplify the complex process of deal making among large numbers of traders.

Consider a general game with n players, labeled simply 1, 2, . . . n, such that the set of all players is $N = \{1, 2, 3, \ldots, n\}$. Any subset of players in N is a **coalition** C. When it is necessary to specify the members of a coalition in full, we list them in braces; $C = \{1, 2, 7\}$ might be one such coalition. We allow the N to be a subset of itself and call it the **grand coalition** G; thus $G = N$. A simple formula in combinatorial mathematics gives the total number of subsets of N that include one or two or three or . . . n people out of the n members of N. This number of possible subsets is $(2^n - 1)$; there are thus $(2^n - 1)$ possible coalitions altogether.

For the simple two-person trading game discussed in Section 1, $n = 2$, and so we can imagine three coalitions: two single-member ones of one buyer and one seller, respectively, and the grand coalition of the two together. When there are two sellers and one buyer, we have seven $(2^3 - 1)$ coalitions: three trivial ones consisting of one player each; three pairs, $\{S_1, B\}$, $\{S_2, B\}$, and $\{S_1, S_2\}$; and the grand coalition, $\{S_1, S_2, B\}$.

A coalition can strike deals among its own members to exploit all available mutual advantage. For any coalition C, we let the function $v(C)$ be the total surplus that its members can achieve on their own, regardless of what the players not in the coalition (that is, in the set $N - C$) do. This surplus amount is called the **security level** of the coalition. When we say "regardless of what the players not in the coalition do," we visualize the worst-case scenario—namely, what would happen to members of C if the others (those not in C) took the action that was the worst from C's perspective. This idea is the analogue of an individual player's minimax, which we studied in Chapters 7 and 8.

The calculation of $v(C)$ comes from the specific details of a game. As an example, consider our simple game of house trading from Section 1. No individual player can achieve any surplus on his own without some cooperation from a player on the other side, and so the security levels of all single-member coalitions are zero; $v(\{S_1\}) = v(\{S_2\}) = v(\{B\}) = 0$. The coalition $\{S_1, B\}$ can realize a total surplus of $300 - 100 = 200$, and S_2 on his own cannot do anything to reduce the surplus that the other two can enjoy (arson being outside the rules of the game); so $v(\{S_1, B\}) = 200$. Similarly, $v(\{S_2, B\}) = 300 - 150 = 150$. The two sellers on their own cannot achieve any extra payoff without including the buyer; so $v(\{S_1, S_2\}) = 0$.

In a particular game, then, for any coalition C, we can find its security level, $v(C)$, which is also called the **characteristic function** of the game. We assume that the characteristic function is common knowledge, that there is no asymmetry of information.

Next, we consider possible outcomes of the game. An **allocation** is a list of surplus amounts, $(x_1, x_2, \ldots x_n)$, for the players. An allocation is **feasible** if it is logically conceivable within the rules of the game—that is, if it arises from any deal that could in principle be made. Any feasible allocation may be proposed as a tentative deal, but the deal can be upset if some coalition can break away and do better for itself. We say that a feasible allocation is **blocked** by a coalition C if

$$v(C) > \sum_{i \text{ in } C} x_i.$$

This inequality says that the security level of the coalition—the total surplus that it can minimally guarantee for its members—exceeds the sum of the surpluses that its members get in the proposed tentative allocation. If this is true, then the coalition can form and will strike a bargain among its members that leaves all of them better off than in the proposed allocation. Therefore the implicit threat by the coalition to break away (back out of the deal if the surplus gained is not high enough) is credible; that is how the coalition can upset (block) the proposed allocation.

The set of allocations that cannot be blocked by any coalition contains all of the possible stable deals, or the bargaining range of the game. This set cannot be reduced further by any groups searching for better deals; we call this set of allocations the **core** of the game.

A. Numerical Example

We have already the blocking in the simple two-person trading game in Section 1, although we did not use this terminology. However, we can now see that what we called the "range of negotiation" corresponds exactly to the core. So we can translate, or restate, our results as follows. With one buyer and one seller, the range of negotiation is from 100 to 300. This outcome corresponds to a core consisting of all allocations ranging from the one (with price 100) where the buyer gets surplus 200 and the seller gets 0 to the one (with price 300) where the buyer gets 0 and the seller gets 200.

Now we use our new analytical tools to consider the case in which one buyer, willing to pay as much as 300, meets two sellers having equal reserve prices of 100. What is the core of the game among these three participants?

Consider an allocation of surpluses—say, x_1 and x_2—to the two sellers and y to the buyer. For this allocation to be in the core, no coalition should be able to achieve more for its members than what they receive here. We know that each single-member coalition can achieve only zero; therefore a core allocation must satisfy the conditions

$$x_1 \geq 0, \quad x_2 \geq 0, \quad \text{and} \quad y \geq 0.$$

Next, the coalitions of one buyer and one seller can each achieve a total of 300 − 100 = 200, but the coalition of the two sellers can achieve only zero. Therefore, for the proposed allocation to survive being blocked by a two-person coalition, it must satisfy

$$x_1 + y \geq 200, \quad x_2 + y \geq 200, \quad \text{and} \quad x_1 + x_2 \geq 0.$$

Finally, all three participants together can achieve a surplus of 200; therefore, if the proposed allocation is not to be blocked by the grand coalition, we must have

$$x_1 + x_2 + y \geq 200.$$

But, because the total surplus available is only 200, $x_1 + x_2 + y$ cannot be greater than 200. Therefore we get the equation $x_1 + x_2 + y = 200$.

Given this last equation, we can now argue that x_1 must be zero. If x_1 were strictly positive and we combined the equation $x_1 > 0$ with one of the two-person inequalities, $x_2 + y \geq 200$, then we would get $x_1 + x_2 + y > 200$, which is impossible. Similarly, we find $x_2 = 0$ also. Then it follows that $y = 200$. In other words, the sellers get no surplus—it all goes to the buyer. To achieve such an outcome, one of the houses must sell for its reserve price $p = 100$; it does not matter which.

We conducted the analysis in Section 2.A by using our intuition about competition. When both sellers have the same reserve price, the competition between them becomes very fierce. If one has negotiated a deal to sell at a price of 102, the other is ready to undercut and offer the buyer a deal at 101 rather than go without a trade. In this process, the sellers "beat each other down" to 100. In this section, we have developed some general theory about the core and applied it in a mechanical and mathematical way to get a result that leads back to the same intuition. We hope this two-way traffic between mathematical theory and intuitive application reinforces both in your mind. The purely intuitive thinking becomes harder as the problem grows more complex, with many objects, many buyers, and many sellers. The mathematics provides a general algorithm that we can apply to all of these problems; you will have ample opportunities to use it. But, having solved the more complex problems by using the math, you should also pause and relate the results to intuition, as we have done here.

B. Some Properties of the Core

EFFICIENCY We saw earlier that an allocation is in the core if it cannot be blocked by *any* coalition—including the grand coalition. Therefore, if a core allocation gives n players surpluses of $(x_1, x_2, \ldots x_n)$, the sum of these surpluses must be at least as high as the security level that the grand coalition can achieve:

$$x_1 + x_2 + \cdots + x_n \geq v(\{1, 2, \ldots, n\})$$

But the grand coalition can achieve anything that is feasible. Therefore for *any* feasible allocation of surpluses—say, $(z_1, z_2, \ldots z_n)$—it must be true that

$$v(\{1, 2, \ldots, n\}) \geq z_1 + z_2 + \cdots + z_n.$$

These two inequalities together imply that it is impossible to find another feasible allocation that will give everyone a higher payoff than a core allocation; we *cannot* have

$$z_1 > x_1, \quad z_2 > x_2, \ldots, \quad \text{and} \quad z_n > x_n$$

simultaneously. For this reason, the core allocation is said to be an **efficient allocation.** (In the jargon of economists, it is called *Pareto efficient,* after Wilfredo Pareto, who first introduced this concept of efficiency.)

Efficiency sounds good, but it is only one of a number of good properties that we might want a core allocation to have. Efficiency simply says that no other allocation will improve everyone's payoff simultaneously; the core allocation could still leave some people with very low payoffs and some with very high payoffs. That is, efficiency says nothing about the equity or fairness or distributive justice of a core allocation. In our trading game, a seller got zero surplus if there was another seller with the same reserve price; otherwise he had hope of some positive surplus. This is a matter of luck, not fairness. And we saw that some sellers with high reserve prices and some buyers with a low willingness to pay could get cut out of the deals altogether because the others could get greater total surplus; this outcome also sacrifices equity for the sake of efficiency.

A core allocation will always yield the maximum total surplus, but it does not regulate the distribution of that surplus. In our two-buyer, two-seller example, the buyer with the higher willingness to pay (B_1) gets more surplus because competition from the other buyer (B_2) means that the two pay the same price. Similarly, the lower-reserve-price seller (S_1) gets the most surplus—or, equivalently, the lower-cost builder (S_1) makes the most profit.

In our simple trading game, the core was generally a range rather than a single point. Similarly, in other games there may be many allocations in the core. All will be efficient but will entail different distributions of the total payoff among the players, some fairer than others. The concept of efficiency says nothing about how we might choose one particular core allocation when there are many. Some other theory or mode of analysis is needed for that. Later in the chapter, we take a brief look at other mechanisms that do pay attention to the distribution of payoffs among the players.

EXISTENCE Is there a guarantee that all games will have a core? Alas, no. In games where the players have negative spillover effects (externalities) on one another, it may be possible for every tentatively proposed allocation to be upset by a coalition of players who gang together to harm the others. Then the core may be empty.

An extreme example is the *garbage game.* There are n players, each with one bag of garbage. Each can dump it in his own yard or in a neighbor's yard; he can even divide it up and dump some in one yard and some in others. The payoff from having b bags of garbage in your yard is $-b$, where b does not have to be an integer. Then a coalition of m people has the security level $-(n - m)$ as long as $m < n$, because they get rid of their m bags in nonmembers' yards but, in the worst-case scenario that underlies the calculation of the security level, all $(n - m)$ nonmembers dump their bags in members' yards. Therefore the characteristic function of the game is given by $v(m) = -(n - m) = m - n$ so long as $m < n$. But the grand-coalition members have no nonmembers on whom to dump their garbage; so $v(n) = -n$.

Now it is easy to see that any proposed allocation can be blocked. The most severe test is for a coalition of $(n - 1)$ people, which has a very high security level of -1 because they are dumping all their bags on the one unfortunate person excluded from the coalition and suffer only his one bag among the $(n - 1)$ of them. Suppose person n is currently the excluded one. Then he can get together with any $(n - 2)$ members of the coalition—say, the last $(n - 2)$—and offer them an even better deal. The new coalition of $(n - 1)$—namely, 2 through n—should dump its bags on 1. Person n will accept the whole of 1's bag. Thus persons 2 through $(n - 1)$ will be even better off than in the original coalition of the first $(n - 1)$ because they have 0 surplus to share among them rather than -1. And person n will be dramatically better off with 1 bag on his lawn than in the original situation in which he was the excluded person and suffered $(n - 1)$ bags. Thus the original allocation can be blocked. The same is true of every allocation; therefore the core is nonexistent, or empty.

The problem arises because of the nature of "property rights," which is implicit in the game. People have the right to dump their garbage on others, just as in some places and at some times firms have a right to pollute the air or the water or people have the right to smoke. In such a regime of "polluter's rights," negative externalities go unchecked and an unstable process of coalition formation and reformation can result, leading to an empty core. In an alternative system where people have the right to remain free from the pollution caused by others, this problem does not arise. In our game, if the property right is changed so that no one can dump garbage on anyone else without that person's consent, then the core is not empty; the allocation in which every person keeps his own bag in his own yard and gets a payoff of -1 cannot be blocked.

C. Discussion

COOPERATIVE OR NONCOOPERATIVE GAME? The concept of the core was developed for an abstract formulation of a game, starting with the characteristic function. No details are provided about just *how* a coalition C can achieve total surplus equal to its security level $v(C)$. Those details are specific to each application. In the same way, when we find a core allocation, nothing is said about how the individual actions that give rise to that allocation are implemented. Implicitly, some external referee is assumed to see to it that all players carry out their assigned roles and actions. Therefore the core must be understood as a concept within the realm of cooperative game theory, where actions are jointly chosen and implemented.

But the freedom of individual players to join coalitions or to break up existing coalitions and form new ones introduces a strong noncooperative element into the picture. That is why the core is often a good way to think about competition. However, for competition to take place freely, it is important that all buyers and sellers have equal and complete freedom to form and re-form coalitions. What if only the sellers are active players in the game, and the buyers must passively accept their stated prices and cannot search for better deals? Then the core of the sellers' game that maximizes their total profit entails forming the grand coalition of sellers, which is a cartel or a monopoly. This problem is often observed in practice. There are numerous buyers in the market at any given time but, because each individual buyer is small and in the market infrequently, they find it difficult to get together or to negotiate with sellers for a better deal. Sellers are fewer in number and have repeated relationships among themselves; they are more likely to form coalitions. That is why antitrust laws try to prevent cartels and to preserve competition, whereas similar laws to prevent combination on the buyers' side are almost nonexistent.[2]

ADDITIVE AND NONADDITIVE PAYOFFS We defined the security level of a coalition as the minimum sum total of payoffs that could be guaranteed for its members. When we add individual payoffs in this way, we are making two assumptions: (1) that the payoffs are measured in some comparable units, money being the most frequently used yardstick of this kind; and (2) that the coalition somehow solves its own internal bargaining problem. Even when it has a high total payoff, a coalition can break up if some members get a lot while others get very little—unless the former can compensate the latter in a way that keeps everyone better off by being within the coalition than outside it. Such compensation must take the form of transfers of the unit in which payoffs are measured, again typically money.

If money (or something else that is desired by all and measured in comparable units for all) is not transferable or if people's payoffs are nonlinear rescalings of money amounts—for example, because of risk aversion—then a coalition's

[2]In labor markets in some European countries such as Sweden, the coalition of all buyers (owners or management) deals with the coalition of all sellers (unions), but that is not the way that the concept of core assumes the process will work. The core would allow coalitions of smaller subsets of companies and workers to form and re-form, constantly seeking a better deal for themselves. In practice there is pure bargaining between two fixed coalitions.

security level cannot be described by its total payoff alone. We then have to keep track of the separate payoffs of all individual members. In other words, the characteristic function $v(C)$ is no longer a number but a set of vectors listing all members' payoff combinations that are possible for this coalition regardless of what others do. The theory of the core for this situation, technically called the case of *nontransferable utility,* can be developed, and you will find it in more advanced treatments of cooperative game theory.[3]

3 THE MARKET MECHANISM

We have already looked at supply and demand curves for markets with small numbers of traders. Generalizing them for more realistic markets is straightforward, as we see in Figure 18.4. There can be hundreds or thousands of potential buyers and sellers of each commodity; on each side of the market, they span a range of reserve prices and willingness to pay. For each price p, measured on the vertical axis, the line labeled S shows the quantity of all sellers whose reserve price is below p—those who are willing to sell at this price. When we increase p enough that it exceeds a new seller's reserve price, that seller's quantity comes onto the market, and so the quantity supplied increases; we show this quantity as a horizontal segment of the curve at the relevant value of

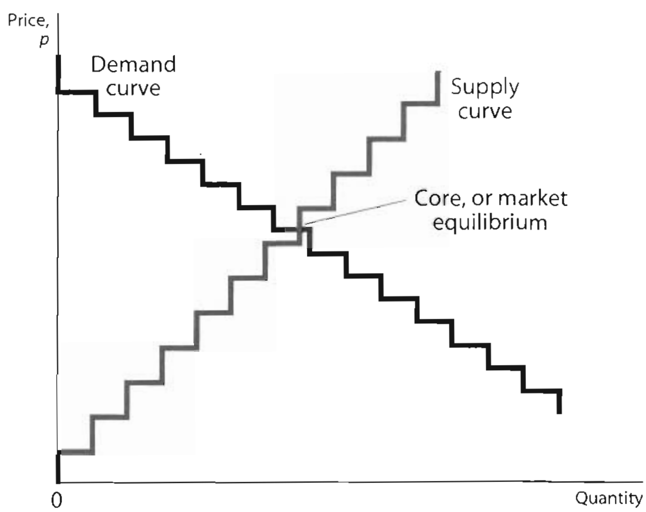


FIGURE 18.4 Market Equilibrium with Many Buyers and Sellers

[3] R. Duncan Luce and Howard Raiffa, *Games and Decisions* (New York: Wiley, 1957), especially chaps. 8 and 11, remains a classic treatment of cooperative games with transferable utility. An excellent modern general treatment that also allows nontransferable utility is Roger B. Myerson, *Game Theory* (Cambridge: Harvard University Press, 1991), chap. 9.

p. The rising-staircase form that results from this procedure is the supply curve. Similarly, the descending-staircase graph on the buyers' side shows, for each price, the quantity that the buyers are willing to purchase (demand); therefore it is called the demand curve.

If all the sellers have access to the same inputs to production and the same technology of production, they may have the same costs and therefore the same reserve price. In that case, the supply curve may be horizontal rather than stepped. However, among the buyers there are generally idiosyncratic differences of taste, and so the demand curve is always a step-shaped curve.

Market equilibrium occurs where the two curves meet, which can happen in three possible ways. One is a vertical overlap at one quantity. This overlap indicates that exactly that quantity is the amount traded, and the price (common to all the trades) can be anything in the range of overlap. The second possibility is that the two curves overlap on a horizontal segment. Then the price is uniquely determined, but the quantity can vary over the range shown by the overlap. A horizontal overlap further means that the unique price is simultaneously the reserve price of one or more sellers and the willingness to pay of one or more buyers; at this price, the two types are equally happy making or not making a trade, and so it does not matter where along the overlap the quantity traded "settles." The third possibility is that a horizontal segment of one curve cuts a vertical segment of the other curve. Then there is only one point common to both curves, and the price and the quantity are both uniquely determined.

In each case, the price (even if not uniquely determined) is one at which, if our neutral market maker were to call it out, the quantity demanded would equal the quantity supplied, and the market would clear. Any such price at which the market clears is called a market equilibrium price. It is the market maker's purpose to find such a price, and he does so by a process of trial and error. He calls out a price and asks for offers to buy and sell. All of these offers are tentative at first. If the quantity demanded at the current price is greater than the quantity supplied, the market maker tries out a slightly higher price; if demand is less than supply, he tries a slightly lower price. The market maker adjusts the trial price in this way until the market clears. Only then are the tentative offers to buy and sell made actual, and they are put into effect at that market equilibrium price.

When thousands of buyers are ranked according to their willingness to pay and hundreds or thousands of sellers according to their reserve prices, the differences between two successive trades on each side are quite small. Therefore each step of the demand curve falls only a little and that of the supply curve rises only a little. With this type of supply and demand curves, even when there is a vertical overlap, the range of indeterminacy of the price in the market equilibrium is small, and the market-clearing price will be almost uniquely determined.

We showed graphically in simple examples that the core of our trading game, where each seller has one unit to offer and each buyer wants to buy just one unit, coincides with the market equilibrium. Similar results are true for any number of buyers and sellers, each of whom may want to buy or sell multiple units at different prices, but the algebra is more complex than is worthwhile, and so we will omit it. If some sellers or some buyers want to trade more than one unit each, then the link between the core and the market equilibrium is slightly weaker. A market equilibrium always lies in the core, but the core may have other allocations that are not market equilibria. However, in large markets, with numerous buyers and sellers, there are relatively few such extra allocations, and the core and the market outcomes virtually coincide. Because this result comes from rather advanced microeconomic theory, we must leave it as an assertion, hoping that the simple examples worked out here give you confidence in its truth.[4]

A. Properties of the Market Mechanism

EFFICIENCY The correspondence between the outcomes of the core and of the market equilibrium has an immediate and important implication. We saw in Section 2 that a core allocation is efficient; it maximizes the total possible surplus. Therefore the market equilibrium also must be efficient, a fact that accounts in part for the conceptual appeal of the market.

As with the core, the market does not guarantee a fair or just distribution of income or wealth. A seller's profit depends on how far the market price happens to be above his cost, and a buyer's surplus depends on how far the market price happens to be below his willingness to pay.

INFORMATION AND MANIPULABILITY The market maker typically does not know the reserve price of any seller or the willingness to pay of any buyer. He sets a price and asks traders to disclose some of this information through their offers to sell and buy. Do they have the incentive to reply truthfully? In other words, is the market mechanism manipulable? This question is quite central to the strategic perspective of this book and relates to the discussion of information in Chapter 9.

First consider the case in which each seller and each buyer wants to trade just one unit. For simplicity, consider the two-buyer, two-seller example from Section 1, in which seller S_1 has a reserve price of 100. He could pretend to have a higher reserve price (by not offering to sell his house until the market maker calls out a price appropriately higher than 100) or a lower one. That is, he could

[4]For a discussion of the general theory about the relationship between the core and the market equilibrium, see Andreu Mas-Colell, Michael D. Whinston, and Jerry R. Green, *Microeconomic Theory* (New York: Oxford University Press, 1995), pp. 652–660.

pretend that his supply curve is different from what it is. Would that be in his interest?

Just as the market maker does not know anyone's reserve price or willingness to pay, S_1 does not know this information about anyone else. So he does not know what the market-clearing price will be. Suppose he pretends that his reserve price is 120. It might be that the market-clearing price exceeds 120, in which case he would get to make the sale, as he would have anyway, and would get the market price as he would have had he shown his reserve price to be 100; his exaggeration would make no difference. But the market-clearing price might be somewhere between 100 and 120, in which case the market maker would not take up S_1's offer to sell, and S_1 would lose a deal that would have been profitable. In this instance, exaggeration can only hurt him. Similarly, he might understate his reserve price and offer to sell when the market maker calls out a price of 80. If the market-clearing price ended up below 80, the understatement would make no difference. But if it ended up somewhere between 80 and 100, then the market maker would hold S_1 to his offer, and the sale would net a loss to S_1. Thus understatement, like exaggeration, can only hurt him. In other words, truthful revelation is the dominant strategy for S_1. The same is true for all buyers and sellers.

It is no coincidence that this assertion is similar to the argument in Chapter 16 that truthful bidding is the dominant strategy in a private-value second-price auction. A crucial thing about the second-price auction is that what you pay when you win depends not on what you bid, but on what someone else bid. Over- or underbidding can affect your chances of winning only in an adverse way; you might fail to win when it would have been good and might win when it would be bad. Similarly, in the market, if you get to make a sale or a purchase, the price depends not on your reserve price or your willingness to pay, but on that of someone else. Misstating your values can affect your chances of making a trade only in an adverse way.

This argument does not work if some person has several units to trade and is a large trader who can affect the price. For example, a seller might have a reserve price of 100 and 15 units for sale. At a price of 100, there might be 15 willing buyers, but the seller might pretend to have only 10 units. Then the market maker would believe there was some scarcity, and he would set a higher market-clearing price at which the number of willing buyers matched the number of units the seller chose to put up for sale. If the buyers' willingness to pay rises sufficiently rapidly as the quantity goes down (that is, if the demand curve is steep), then the seller in question might make more profit by selling 10 units at the higher price than he would by selling 15 units at a lower price. Then the misstatement would work to his advantage, and the market mechanism would be manipulable. In the jargon of economics, this is an instance of the seller exercising some **monopoly power.** By offering less than he actually has (or can produce) at a particular price, the seller is driving up the market equilibrium price. Similarly, a multiunit buyer can understate his demand to drive the price down and thereby get greater surplus on the units that he does buy; he is exercising **monopsony power.**

However, if the market is so large that any one seller's attempt to exercise monopoly power can alter the price only very slightly, then such actions will not be profitable to the seller. When each seller and each buyer is small in relation to the market, each has negligible monopoly and monopsony power. In the jargon of economics, competition becomes *perfect;* in our jargon of game theory, the market mechanism becomes **nonmanipulable,** or **incentive compatible.**

B. Experimental Evidence

We have told a "story" of the process by which a market equilibrium is reached. However, as noted earlier, the market maker who played the central role in that story, calling out prices, collecting offers to buy and sell, and adjusting the price until the market cleared, does not actually exist in most markets. Instead, the buyers and sellers have to search for trading partners and attempt to strike deals on their own. The process is somewhat like the formation and re-formation of coalitions in the theory of the core but even more haphazard and unorganized. Can such a process lead to a market-clearing outcome?

Through observation of actual markets, we can readily find many instances where markets seem to function well and many where they fail. But situations of the former kind differ in many ways from those of the latter, and many of these differences are not easily observable. Therefore the evidence does not help us understand whether and how markets can solve the problems of coordinating the actions of many buyers and sellers to arrive at an equilibrium price. Laboratory experiments, on the other hand, allow us to observe outcomes under controlled conditions, varying just those conditions whose effects we want to study. Therefore they are a useful approach and a valuable item of evidence for judging the efficacy of the market mechanism.

In the past half-century, many such experiments have been carried out. Their objectives and methods vary, but in some form all bring together a group of subjects who are asked to trade a tangible or intangible "object." Each is informed of his own valuation and given real incentives to act on this valuation. For example, would-be sellers who actually consummate a trade are given real monetary rewards equal to their profit or surplus, which is equivalent to the price that they succeed in obtaining *minus* the cost that they are assigned.

The results are generally encouraging for the theory. In experiments with only two or three traders on one side of the market, attempts to exercise monopoly or monopsony power are observed. But otherwise, even with relatively small numbers of traders (say, five or six on each side), trading prices generally converge quite quickly to the market equilibrium that the experimenter has calculated, knowing the valuations that he has assigned to the traders. If anything, the experimental results are "too good." Even when the traders know nothing about the valuations of other traders, have no understanding of the theory of market equilibrium, and have no prior experience in trading, they arrive at the equilibrium configuration relatively quickly. In recent work, the focus has shifted away from asking *whether* the market mechanism works to asking *why* it works so much better than we have any reason to expect.[5]

There is one significant exception—namely, markets for assets. The decision to buy or sell a long-lived asset must take into account not merely the relation between the price and your own valuation, but also the expected movement of the price in the future. Even if you don't expect your dot.com stock ever to pay any dividend, you might buy and hold it for a while if you believe that you can then sell it to someone else for a sufficiently higher price, thereby making a capital gain. You might come to this belief because you have observed rising prices for a while. But prices might be rising because others are buying dot.com stock, too, similarly expecting to resell it with a capital gain. If many people simultaneously try to sell the stocks to obtain those expected gains, the price will fall, the gains will fail to be realized, and the expectations will collapse. Then we will say that the dot.com market has experienced a *speculative bubble* that has burst. Such bubbles are observed in experimental markets, as they are in reality.[6]

4 THE SHAPLEY VALUE

The core has an important desirable property of efficiency, but it also has some undesirable ones. Most basically, there are games that have no core; many others have very large cores consisting of whole ranges of outcomes, and so the concept does not determine an outcome uniquely. Other concepts that do better in these respects have been constructed. The best known is the Shapley value, named after Lloyd Shapley of UCLA.

Like Nash's bargaining solution, discussed in Chapter 17, the Shapley value is a cooperative concept. It is similarly grounded in a set of axioms or assumptions that, it is argued, a solution should satisfy. And it is the unique solution that conforms to all these axioms. The desirability of the properties is a matter

[5]Douglas D. Davis and Charles A. Holt, *Experimental Economics* (Princeton, NJ: Princeton University Press, 1993), chap. 3.

[6]Shyam Sunder, "Experimental Asset Markets: A Survey," *Handbook of Experimental Economics*, ed. John H. Kagel and Alvin E. Roth (Princeton, NJ: Princeton University Press, 1995), pp. 468–474.

of judgment; other game theorists have specified other combinations of properties, each of which leads to its own unique solution. But the solutions for different combinations of properties are different, and not all the properties can be equally desirable. Perhaps more importantly, each player will judge the properties not by their innate attractiveness, but by what the outcome gives him. If an outcome is not satisfactory for some players and they believe they can do better by insisting on a different game, the proposed solution may be a nonstarter.

Therefore any cooperative solution should not be accepted as the likely actual outcome of the game based solely on the properties that seem desirable to some game theorists or in the abstract. Rather, it should be judged by its actual performance for prediction or analysis. From this practical point of view, the Shapley value turns out to do rather well. We do not go into the axiomatic basis of the concept here. We simply state its formula, briefly interpret and motivate it, and demonstrate it in action.

Suppose a game has n players and its characteristic function is $v(C)$, which is the minimum total payoff that coalition C can guarantee to its members. The **Shapley value** is an allocation of payoffs u_i to each player i as defined by

$$u_i = \sum_{C} \frac{(n-k)!(k-1)!}{n!} [v(C) - v(C - \{i\})], \tag{18.1}$$

where $n!$ denotes the product $1 \times 2 \times \cdots \times n$; where the sum is taken over all the coalitions C that have i as a member, and where, in each term of the sum, k is the size of the coalition C.

The idea is that each player should be given a payoff equal to the average of the "contribution" that he makes to each coalition to which he could belong, where all coalitions are regarded as equally likely in a suitable sense. First consider the size k of the coalition—from 1 to n. Because all sizes are equally likely, a particular size coalition occurs with probability $1/n$. Then the $(k-1)$ partners of i in a coalition of size k can be chosen from among the remaining $(n-1)$ players in any of

$$\frac{(n-1)!}{[(n-1)-(k-1)]!\,(k-1)!} = \frac{(n-1)!}{(n-k)!\,(k-1)!}$$

ways. The reciprocal of this expression is the probability of any one such choice. Combining that reciprocal with $(1/n)$ gives us the probability of a particular coalition C of size k containing member i, and that is what appears as the built-up fraction in each term of the sum on the right-hand side of the formula for u_i. What multiplies the fraction is the difference between the security level of the coalition C—namely, $v(C)$—and the security level that the remaining people would have if i were removed from the coalition—namely, $v(C - \{i\})$. This term measures the contribution that i makes to C.

The idea that each player's payoff should be commensurate with his contribution has considerable appeal. Most importantly, if a player must make some effort to generate this contribution, then such a payment scheme gives him the correct incentive to make that effort. In the jargon of economics, each person's incremental contribution to the economy is called his *marginal product;* therefore the concept of tying payoffs to contributions is called the *marginal productivity theory of distribution.* A market mechanism would automatically reward each participant for his contribution. The Shapley value can be understood as a way of implementing this principle—of achieving a marketlike outcome—at least approximately when an actual market cannot be arranged, such as within units of a single larger organization.

The formula embodies much more than the general principle of marginal productivity payment. Averaging over all coalitions to which a player could belong and regarding all coalitions as equally likely are very specific procedures that may or may not have any counterpart in reality. Nevertheless, the Shapley value often produces outcomes that have some realistic features. We now examine two specific examples of the Shapley value in action, one from politics and one from economics.

A. Power in Legislatures and Committees

Suppose a 100-member legislature consists of four parties, Red, Blue, Green, and Brown. The Reds have 43 seats, the Blues 33, the Greens 16, and the Browns 8. Each party is a coherent block that votes together, and so each can be regarded as one player. A majority is needed to pass any legislation; therefore no party can get legislation through without the help of another block, and so no party can govern on its own. In this situation the power of a party will depend on how crucial that party is to the formation of a majority coalition. The Shapley value provides a measure of just that power. Therefore in this context it is called the **power index.** The index was used in this way by Shapley and Martin Shubik. A similar index, but with a different assumption about coalition formation, was devised by John Banzhaf.[7]

Give the value 1 to any coalition (including the grand coalition) that has a majority and 0 to any coalition without a majority. Then, in the formula for the Shapley value of party i (Eq. 18.1), the contribution that it makes to a coalition C—namely, $v(C) - v(C - \{i\})$—is 1 if the coalition C has a majority but $C - \{i\}$ if it does not. When party i's contribution to C is 1, we say that party i is a *pivotal*

[7]See Lloyd S. Shapley and Martin Shubik, "A Method for Evaluating the Distribution of Power in a Committee System," *American Political Science Review,* vol. 48, no. 3 (September 1954), pp. 787–792, and John Banzhaf, "Mathematical Analysis of Voting Power and Effective Representation," *George Washington Law Review,* vol. 36, no. 4 (1968), pp. 808–823.

member of the majority coalition C. In all other cases—namely, if coalition C does not have a majority at all or if C would have a majority even without i—the contribution of i to C is zero.

We can now list all the majority coalitions and which party is pivotal in each. None of the four possible (albeit trivial) one-party "coalitions" has a majority. Three of the six possible two-party coalitions have a majority—namely, {Red, Blue}, {Red, Green}, and {Red, Brown}—and in each case both members are pivotal because the loss of either would mean the loss of the majority. Of the four possible three-party coalitions, in three of them—namely, {Red, Blue, Green}, {Red, Blue, Brown}, and {Red, Green, Brown}—only Red is pivotal. In the fourth three-party coalition—namely, {Blue, Green, Brown}—all three parties are pivotal. In the grand coalition, no party is pivotal.

In the Shapley value formula, the term corresponding to each two-party coalition takes the value $(4 - 2)!(2 - 1)!/4! = 2! \times 1!/4! = 2/24 = 1/12$, and each three-party coalition gets $(4 - 3)!(3 - 1)! = 1! \times 2!/4! = 1/12$ also.

With this information we can calculate the Shapley value of each party. We have

$$u_{\text{Red}} = \frac{1}{12} \times 3 + \frac{1}{12} \times 3 = \frac{1}{2}$$

because the Red party is pivotal in three two-party coalitions and three three-party ones and each such coalition gets a weight of 1/12. Similarly

$$u_{\text{Blue}} = \frac{1}{12} \times 1 + \frac{1}{12} \times 1 = \frac{1}{6}$$

and $u_{\text{Green}} = u_{\text{Brown}} = 1/6$ likewise, because Blue, Green, and Brown are each pivotal in exactly one two-party and one three-party coalition.

Even though the Blues have almost twice as many members as the Greens, who in turn have twice as many as the Browns, the three parties have equal power. The reason is that they are all equally crucial in the formation of a majority coalition, either one at a time with Red, or the three of them joined together against Red.

We do observe in reality that small parties in multiparty legislatures enjoy far more power than their number proportions in the legislature might lead us to believe, which the Shapley value index shows in a dramatic way. It does make some unappealing assumptions; for example, it takes all coalitions to be equally likely and all contributions to have the same value 1 if they create a majority and the same value 0 if they do not. Therefore we should interpret the power index not as a literal or precise quantitative measure but as a rough indication of political power levels. More realistic analysis can sometimes show that small parties have even greater power than the Shapley measure might suggest. For example, the two largest parties often have ideological antagonisms that rule out a big coalition between them. If coalitions that include both Red and Blue together are not possible, then Green and Brown will have even greater power because of their ability to construct a majority—one of them together with Red or both of them together with Blue.

B. Allocation of Joint Costs

Probably the most important practical use of the Shapley value is for allocating the costs of a joint project among its several constituents. Suppose a town is contemplating the construction of a multipurpose auditorium building. The possible uses are as a lecture hall, a theater, a concert hall, and an opera house. A single-purpose lecture hall would cost \$1 million. A theater would need more sophisticated stage and backstage facilities and would cost \$4 million. A concert hall would need better acoustics than the lecture hall but less sophisticated staging than the theater, and so the cost of a pure concert hall would be \$3 million. An opera house would need both staging and acoustics, and the cost of the two together, whether for a theater–concert combination or for opera, would be \$6 million. If the opera house is built, it can also be used for any of the three other purposes, and a theater or a concert hall can also be used for lectures.

The town council would like to recoup the construction costs by charging the users. The tickets for each type of activity will therefore include an amount that corresponds to an appropriate share of the building cost attributable to that activity. How are these shares to be apportioned? This is a mirror image of the problem of distributing payoffs according to contributions—we want to charge each use for the extra cost that it entails. And the Shapley value offers an answer to the problem. For each activity—say, the theater—we list all the combinations in which it could take place and what it would cost to construct the facility for that combination, as well as for that combination minus the theater. This analysis tells us the contribution of the theater to the cost of building for that combination. Assuming all combinations for this activity to be equally likely and averaging, we get the overall cost share of the theater.

First, we calculate the probabilities of each combination of uses. The theater could be on its own with a probability of 1/4. It could be in a combination of two activities with a probability of 1/4, and there are three ways to combine the theater with another activity; so the probability of each such combination is 1/12. Similarly, the probability of each three-activity combination including the theater is also 1/12. Finally, the probability of the theater being in the grand coalition of all four activities is 1/4.

Next, we calculate the contributions that the theater makes to the cost of each combination. Here we just give an example. The theater–concert combination costs \$6 million, and the concert hall alone would cost \$3 million; therefore $6 - 3 = 3$ is the contribution of the theater to this combination. The complete list of combinations and contributions is shown in Figure 18.5. Each row focuses on one activity. The left-most column simply labels the activity: L for lecture, T for theater, C for concert, and O for opera. The successive columns to the right show one-, two-, three-, and four-activity combinations. The heading of each column shows the size and the probability of each combination of that size. The actual cells for each row in these columns list the combinations of that size that include the activity of that row (when there is only one activity we omit the listing), followed by the contribution of the particular activity to that combination. The right-most column gives the average of all these contributions with the use of the appropriate probabilities as weights—that is, the Shapley value.

It is interesting to note that, though the cost of building an opera facility is *6* times that of building a lecture hall (6 versus 1 in the bottom and top cells of the "Size 1" column), the cost share of opera in the combined facility is *13* times that of lectures (2.75 versus 0.25 in the "Shapley Value" column). This is because providing for lectures adds nothing to the cost of building for any other use, and so the contribution of lectures to all other combinations is zero. On the other hand, providing for opera raises the cost of *all* combinations except those that also include both theater and concert, of which there is only one.

Note also that the cost shares add up to 6, the total cost of building the combined facility—another useful property of the Shapley value. If you have sufficient mathematical facility, you can use Eq. (18.1) to show that the payoffs u_i of all the players sum to the characteristic function v of the grand coalition.

To sum up, the Shapley value allocates the costs of a joint project by calculating what the presence of each participant adds to the total cost. Each participant

ACTIVITY	COMBINATIONS				Shapley Value
	Size 1 Prob. = 1/4	Size 2 Prob. = 1/12	Size 3 Prob. = 1/12	Size 4 Prob. = 1/4	
L	1	LT: $4-4=0$ LC: $3-3=0$ LO: $6-6=0$	LTC: $6-6=0$ LTO: $6-6=0$ LCO: $6-6=0$	$6-6=0$	0.25
T	4	LT: $4-1=3$ TC: $6-3=3$ TO: $6-6=0$	TLC: $6-3=3$ TLO: $6-6=0$ TCO: $6-6=0$	$6-6=0$	1.75
C	3	CL: $3-1=2$ CT: $6-4=2$ CO: $6-6=0$	CLT: $6-4=2$ CLO: $6-6=0$ CTO: $6-6=0$	$6-6=0$	1.25
O	6	OL: $6-1=5$ OT: $6-4=2$ OC: $6-3=3$	OLT: $6-4=2$ OLC: $6-3=3$ OTC: $6-6=0$	$6-6=0$	2.75

FIGURE 18.5 Cost Allocation for Auditorium Complex

is then charged this addition to the cost (or in the jargon of economics his *marginal cost*).

The principle can be applied in reverse. Suppose some firms could create a larger total profit by cooperating than they could by acting separately. They could bargain over the division of the surplus (as we saw in Chapter 17), but the Shapley value provides a useful alternative, in which each firm receives its marginal contribution to the joint enterprise. Adam Brandenburger and Barry Nalebuff, in their book *Co-opetition*, develop this idea in detail.[8]

5 FAIR DIVISION MECHANISMS

Neither the core nor the market mechanism nor the Shapley value guarantee any fairness of payoffs. Game theorists have examined other mechanisms that focus on fairness and have developed a rich line of theory and applications. Here we give you just a brief taste.[9]

The oldest and best-known **fair division mechanism** is "one divides, the other chooses." If A divides the cake into two pieces or the collection of objects available into two piles and B chooses one of the pieces or piles, the outcome will be fair. The idea is that, if A were to produce an unequal division, then B would choose the larger. Knowing this, A will divide equally. Such behavior then constitutes a rollback equilibrium of this sequential-move game.

A related mechanism has a referee who slowly moves a knife across the face of the cake—say, from left to right. At any point, either player can say "stop." The referee cuts the cake at that point and gives the piece to the left of the knife to the player who spoke.

We do not require the cake to be homogeneous—one side may have more raisins and another more walnuts—or all objects in the pile to be identical. The participants can differ in their preferences for different parts of the cake or different objects, and therefore they may have different perceptions about the relative values of portions and about who got a better deal. Then we can distinguish two concepts of fairness. First, the division should be *proportional*, meaning that each of n people believes that his share is at least $1/n$ of the total. Second, the division should be **envy-free,** which means that no participant believes someone else got a better deal. With two people, the two concepts are equivalent but, with more, envy-freeness is a stronger requirement, because each may think he got $1/n$ while also thinking that someone else got more than he did.

With more than two people, simple extensions of the "one cuts, the other chooses" procedure ensure proportionality. Suppose there are three people, A,

[8]Adam Brandenburger and Barry Nalebuff, *Co-opetition* (New York: Doubleday, 1996).

[9]For a thorough treatment, see Steven J. Brams and Alan D. Taylor, *Fair Division: From Cake-Cutting to Dispute Resolution* (New York: Cambridge University Press, 1996).

B and C. Let any one person—say, A—divide the total into three parts. Ask B and C which portions they regard as acceptable—that is, of size at least 1/3. If there is only one piece that both reject, then give that to A, leaving each of B and C a piece that he regards as acceptable. If they reject two pieces, give one of the rejected pieces to A and reassemble the remaining two pieces. Both B and C must regard this total as being of size at least 2/3 in size. Then use the "one cuts, the other chooses" procedure to divide this piece between them. Each gets what he regards as at least 1/2 of this piece, which is at least 1/3 of the initial whole. And A will make the initial cuts to create three pieces that are equal in his own judgment, to ensure himself of getting 1/3. With more people, the same idea can be extended by using mathematical induction. Similar, and quite complex, generalizations exist for the moving-knife procedure.

Ensuring envy-freeness with more than two people is harder. Here is how it can be done with three. Let A cut the cake into three pieces. If B regards one of the pieces as being larger than the other two, he "trims" the piece just enough so that there is at least a two-way tie for largest. The trimming is set aside and C gets to choose from the three pieces. If C chooses the trimmed piece, then B can choose either of the other two, and A gets the third. If C chooses an untrimmed piece, then B must take the piece that he trimmed, and A gets the third.

Leaving the trimming out of consideration for the moment, this division is envy-free. Because C gets to choose any of the three pieces, he does not envy anyone. B created at least a two-way tie for what he believed to be the largest and gets one of the two (the trimmed piece if C does not choose it) or the opportunity to choose one of the two, and so B does not envy anyone. Finally, A will create an initial cut that he regards as equal, and so he will not envy anyone.

Then whoever of B or C did *not* get the trimmed piece divides the trimming into three portions. Whoever of B or C *did* get the trimmed piece takes the first pick, A gets the second pick, and the divider of the trimming gets the residual piece of it. A similar argument shows that the division of the whole cake is envy-free. For examples of even more people, and for many other fascinating details and applications, read Brams and Taylor's *Fair Division* (cited earlier).

We do not get fairness without paying a price in the form of the loss of some other desirable property of the mechanism. In realistic environments with heterogeneous cakes or other objects and with diverse preferences, these fair division mechanisms generally are not efficient; they are also manipulable.

SUMMARY

We consider a market where each buyer and each seller attempts to make a deal with someone on the other side, in competition with others on his own side looking for similar deals. This is an example of a general game where *coalitions* of players can form to make tentative agreements for joint action. An outcome can be prevented or blocked if a coalition can form and guarantee its members better payoffs. The *core* consists of outcomes that cannot be blocked by any coalition.

In the trading game with identical objects for exchange, the core can be interpreted as a *market equilibrium* of *supply and demand*. When there are several buyers and sellers with slight differences in the prices at which they are willing to trade, the core shrinks and a competitive market equilibrium is determinate. This outcome is *efficient*. With few sellers or buyers, preference manipulation can occur and amounts to the exercise of *monopoly* or *monopsony power*. Laboratory experiments find that, with as few as five or six participants, trading converges quickly to the market equilibrium.

The *Shapley value* is a mechanism for assigning payoffs to players based on an average of their contributions to possible coalitions. It is useful for allocating costs of a joint project and is a source of insight into the relative power of parties in a legislature.

Neither the core nor the Shapley value guarantees fair outcomes. Other *fair division mechanisms* exist; they are often variants or generalizations of the "one cuts, the other chooses" idea. But they are generally inefficient and manipulable.

KEY TERMS

allocation (607)
blocking (608)
characteristic function (607)
coalition (607)
core (608)
demand curve (600)
efficient allocation (610)
envy-free allocation (624)
fair division mechanism (624)
feasible allocation (607)
grand coalition (607)
incentive-compatible market mechanism (617)
market equilibrium (601)
monopoly power (616)
monopsony power (617)
nonmanipulable market mechanism (617)
power index (620)
security level (607)
Shapley value (619)
supply curve (600)

EXERCISES

1. Consider the two buyers, B_1 and B_2, and one seller, S, in Section 1. The seller's reserve price is 100; B_1 has a willingness to pay of 300 and B_2 a willingness to pay of 200. (All numbers are dollar amounts in the thousands.)
   (a) Use the negotiation approach to describe the equilibrium range of negotiation and the trade that takes place in this situation.

(b) Construct and illustrate the market supply and demand curves for this case. Show on your diagram the equilibrium range of prices and the equilibrium quantity traded.

2. For two sellers with unequal reserve prices, 100 for S_1 and 150 for S_2, and one buyer B with a willingness to pay of 300, show that, in a core allocation, the respective surpluses x_1, x_2, and y must satisfy $x_1 \leq 50$, $x_2 = 0$, and $y \geq 150$. Verify that this outcome means that S_1 sells his house for a price between 100 and 150.

3. There are four sellers and three buyers. Each seller has one unit to sell and a reserve price of 100. Each buyer wishes to buy one unit. One buyer is willing to pay as much as 400; each of the other two, as much as 300. Find the market equilibrium by drawing a figure. Find the core by setting up and solving all the no-blocking inequalities.

4. In Exercise 3, suppose the four sellers get together and decide that only two of them will go to the market and that all four will share any surplus that they get there. Can they benefit by doing so? Can they benefit even more if one or three of them go to the market? What is the intuition for these results?

5. An airport runway is going to be used by four types of planes: small corporate jets and commercial jets of the narrow-body, wide-body, and jumbo varieties. A corporate jet needs a runway 2,000 feet long, a narrow-body 6,000 feet, a wide-body 8,000 feet, and a jumbo 10,000 feet. The cost of constructing a runway is proportional to its length. Considering each *type* of aircraft as a player, use the Shapley value to allocate the costs of constructing the 10,000-foot runway among the four types. Is this a reasonable way to allocate the costs?

Glossary

Here we define the key terms that appear in the text. We aim for verbal definitions that are logically precise, but not mathematical or detailed like those found in more advanced textbooks.

acceptability condition (490) An upper bound on the probability of fulfillment in a brinkmanship threat, expressed as a function of the probability of error, showing the upper limit of risk that the player making the threat is willing to tolerate.

action node (46) A node at which one player chooses an action from two or more that are available.

addition rule (224) If the occurrence of X requires the occurrence of *any one* of several disjoint $Y, Z, \ldots$, then the probability of X is the sum of the separate probabilities of $Y, Z, \ldots$

adverse selection (305) A form of information asymmetry where a player's type (available strategies, payoffs . . .) is his private information, not directly known to others.

agenda paradox (507) A voting situation where the order in which alternatives are paired when voting in multiple rounds determines the final outcome.

allocation (607) A list of payoffs (sometimes also details of the underlying outcomes such as the quantities of commodities being consumed), one for each player in a game.

all-pay auction (548) An auction in which each person who submits a bid must pay her highest bid amount at the end of the auction, even if she does not win the auction.

alternating offers (577) A sequential move procedure of bargaining in which, if the offer made by one player is refused by the other, then the refuser gets the next turn to make an offer, and so on.

amendment procedure (502) A procedure in which any amended version of a proposal must win a vote against the original version before the winning version is put to a vote against the status quo.

antiplurality method (502) A positional voting method in which the electorate is asked to vote against one item on the slate (or to vote for all but one).

approval voting (503) A voting method in which voters cast votes for all alternatives of which they approve.

ascending auction (540) An open outcry auction in which the auctioneer accepts ascending bids during the course of the auction; the highest bid wins. Also called **English auction.**

assurance game (107) A game where each player has two strategies, say cooperate and not, such that the best response of each is to cooperate if the other cooperates, not if not, and the outcome from (cooperate, cooperate) is better for both than the outcome of (not, not).

asymmetric information (23) Information is said to be asymmetric in a game if some aspects of it—rules about what actions are permitted and the order of moves if any, payoffs as functions of the players strategies, outcomes of random choices by "nature," and of previous actions by the actual players in the game—are known to some of the players but are not common knowledge among all players.

babbling equilibrium (268) In a game where communication among players (which does not affect their payoffs directly) is followed by their choices of actual strategies, a babbling equilibrium is one where the strategies are chosen ignoring the communication, and the communication at the first stage can be arbitrary.

backward induction (54) Same as **rollback.**

battle of the sexes (108) A game where each player has two strategies, say Hard and Soft, such that [1] (Hard, Soft) and (Soft, Hard) are both Nash equilibria, [2] of the two Nash equilibria, each player prefers the outcome where he is Hard and the other is Soft, and [3] both prefer the Nash equilibria to the other two possibilities, (Hard, Hard) and (Soft, Soft).

Bayes' theorem (301) An algebraic formula for estimating the probabilities of some underlying event, by using knowledge of some consequences of it that are observed.

belief (89) The notion held by one player about the strategy choices of the other players and used when choosing his own optimal strategy.

best alternative to a negotiated agreement (BATNA) (569) In a bargaining game, this is the payoff a player would get from his other opportunities if the bargaining in question failed to reach an agreement.

best response (87) The strategy that is optimal for one player, given the strategies actually played by the other players, or the belief of this player about the other players' strategy choices.

best-response analysis (99) Finding the Nash equilibria of a game by calculating the best response functions or curves of each player, and solving them simultaneously for the strategies of all the players.

best-response curve (127) A graph showing the best strategy of one player as a function of the strategies of the other player(s) over the entire range of those strategies.

best-response rule (124) A function expressing the strategy that is optimal for one player, for each of the strategy combinations actually played by the other players, or the belief of this player about the other players' strategy choices.

binary method (501) A class of voting methods in which voters choose between only two alternatives at a time.

Black's condition (512) Same as the condition of **single-peaked preferences.**

blocking (608) A coalition of players blocks an allocation if it can, using the strategies feasible for its members, ensuring a better outcome for all of its members, regardless of the strategy choices of nonmembers (players outside the coalition).

Borda count (502) A positional voting method in which the electorate indicates its order of preference over a slate of alternatives. The winning alternative is determined by allocating points based on an alternative's position on each ballot.

branch (46) Each branch emerging from a node in a game tree represents one action that can be taken at that node.

brinkmanship (312) A threat that creates a risk but not certainty of a mutually bad outcome if the other player defies your specified wish as to how he should act, and then gradually increases this risk until one player gives in or the bad outcome happens.

cell-by-cell inspection (89) Finding the Nash equilibria of a game by examining each cell in turn to see if any one player can do better by moving to another cell along his dimension of choice (row or column). Also called **enumeration.**

characteristic function (607) A function that shows, for each coalition, the aggregate payoff its members can ensure for themselves, regardless of the strategy choices of nonmembers.

cheap talk equilibrium (266) In a game where communication among players (which does not affect their payoffs directly) is followed by their choices of actual strategies, a cheap-talk equilibrium is one where the strategies are chosen optimally given the players' interpretation of the communication, and the communication at the first stage is optimally chosen by calculating the actions that will ensue.

chicken (109) A game where each player has two strategies, say Macho and Wimp, such that [1] both (Macho, Wimp) and (Wimp, Macho) are Nash equilibria, [2] of the two, each prefers the outcome where he plays Macho and the other plays Wimp, and [3] the outcome (Macho, Macho) is the worst for both.

chicken in real time (493) A game of chicken in which the choice to swerve is not once and for all, but where a decision must be made at any time, and as time goes on while neither driver has swerved, the risk of a crash increases gradually.

coalition (607) In a game, a subset of the players that coordinates the strategy choices of the members of the subset.

coercion (399) In this context, forcing a player to accept a lower payoff in an asymmetric equilibrium in a collective action game, while other favored players are enjoying higher payoffs. Also called **oppression** in this context.

collective action problem (382) A problem of achieving an outcome that is best for society as a whole, when the interests of some or all individuals will lead them to a different outcome as the equilibrium of a noncooperative game.

combination rule (228) A formula for calculating the probability of occurrence of *X*, which requires any one of several disjoint *Y*, *Z* etc., each of which in turn requires two or more other events to occur. A combination of the addition and multiplication rules defined in the glossary.

commitment (313) An action taken at a pregame stage, stating what action you would take unconditionally in the game to follow.

common value (541) An auction is called a common-value auction when the object up for sale has the same value to all bidders, but each bidder knows only an imprecise estimate of that value. Also called **objective value.**

compellence (314) An attempt to induce the other player(s) to act to change the status quo in a specified manner.

complementary slackness (256) A property of a mixed strategy equilibrium, saying that for each player, against the equilibrium mixture of the other players, all the strategies that are used in this player's mixture yield him equal payoff, and all the strategies that would yield him lower payoff are not used.

compound interest (352) When an investment goes on for more than one period, compound interest entails calculating interest in any one period on the whole accumulation up to that point, including not only the principal initially invested but also the interest earned in all previous periods, which itself involves compounding over the period previous to that.

conditional probability (227) The probability of a particular event X occurring, given that another event Y has already occurred, is called the conditional probability of X given (or conditioned on) Y.

Condorcet method (501) A voting method in which the winning alternative must beat each of the other alternatives in a round-robin of pairwise contests.

Condorcet paradox (505) Even if all individual voter preference orderings are transitive, there is no guarantee that the social preference ordering generated by Condorcet's voting method will also be transitive.

Condorcet terms (514) A set of ballots that would generate the Condorcet paradox and that should together logically produce a tied vote among three possible alternatives. In a three-candidate election among A, B, and C, the Condorcet terms are three ballots that show A preferred to B preferred to C; B preferred to C preferred to A; C preferred to A preferred to B.

Condorcet winner (502) The alternative that wins an election run using the **Condorcet method.**

constant-sum game (85) A game in which the sum of all players' payoffs is a constant, the same for all their strategy combinations. Thus there is a strict conflict of interests among the players—a higher payoff to one must mean a lower payoff to the collectivity of all the other players. If the payoff scales can be adjusted to make this constant equal to zero, then we have a **zero-sum game.**

contingent strategy (349) In repeated play, a plan of action that depends on other players' actions in previous plays. (This is implicit in the definition of a strategy; the adjective "contingent" merely reminds and emphasizes.)

continuation (175) The continuation of a strategy from a (noninitial) node is the remaining part of the plan of action of that strategy, applicable to the subgame that starts at this node.

continuous distribution (528) A probability distribution in which the random variables may take on a continuous range of values.

continuous strategy (124) A choice over a continuous range of real numbers available to a player.
contract (336) In this context, a way of achieving credibility for one's strategic move by entering into a legal obligation to perform the committed, threatened, or promised action in the specified contingency.
convention (395) A mode of behavior that finds automatic acceptance as a focal point, because it is in each individual's interest to follow it when others are expected to follow it too (so the game is of the Assurance type). Also called **custom.**
convergence of expectations (107) A situation where the players in a noncooperative game can develop a common understanding of the strategies they expect will be chosen.
cooperative game (26) A game in which the players decide and implement their strategy choices jointly, or where joint-action agreements are directly and collectively enforced.
coordination game (105) A game with multiple Nash equilibria, where the players are unanimous about the relative merits of the equilibria, and prefer any equilibrium to any of the nonequilibrium possibilities. Their actions must somehow be coordinated to achieve the preferred equilibrium as the outcome.
Copeland index (502) An index measuring an alternative's record in a round-robin of contests where different numbers of points are allocated for wins, ties, and losses.
core (608) An allocation that cannot be blocked by any coalition.
credible (175, 265) A strategy is credible if its continuation at all nodes, on or off the equilibrium path, is optimal for the subgame that starts at that node.
custom (395) Same as **convention.**
decay (577) Shrinkage over time of the total surplus available to be split between the bargainers, if they fail to reach an agreement for some length of time during the process of their bargaining.
decision (18) An action situation in a passive environment where a person can choose without concern for the reactions or responses of others.
decision node (46) A decision node in a decision or game tree represents a point in a game where an action is taken.
decision tree (46) Representation of a sequential decision problem facing one person, shown using nodes, branches, terminal nodes, and their associated payoffs.
demand curve (600) A graph with quantity on the horizontal axis and price on the vertical axis, showing for each price the quantity that one buyer in a market (or the aggregate of all buyers depending on the context) will choose to buy.
descending auction (540) An open outcry auction in which the auctioneer announces possible prices in descending order. The first person to accept the announced price makes her bid and wins the auction. Also called **Dutch auction.**
deterrence (314) An attempt to induce the other player(s) to act to maintain the status quo.
diffusion of responsibility (415) A situation where action by one or a few members of a large group would suffice to bring about an outcome that all

regard as desirable, but each thinks it is someone else's responsibility to take this action.

discount factor (352) In a repeated game, the fraction by which the next period's payoffs are multiplied to make them comparable with this period's payoffs.

discrete distribution (525) A probability distribution in which the random variables may take on only a discrete set of values such as integers.

disjoint (224) Events are said to be disjoint if two or more of them cannot occur simultaneously.

distribution function (528) A function that indicates the probability that a variable takes on a value less than or equal to some number.

dominance solvable (96) A game where iterated elimination of dominated strategies leaves a unique outcome, or just one strategy for each player.

dominant strategy (92) A strategy X is dominant for a player if, for each permissible strategy configuration of the other players, X gives him a higher payoff than any of his other strategies. (That is, his best response function is constant and equal to X.)

dominated strategy (92) A strategy X is dominated for a player if there is another strategy Y such that, for each permissible strategy configuration of the other players, Y gives him a higher payoff than X.

doomsday device (332) An automaton that will under specified circumstances generate an outcome which is very bad for all players. Used for giving credibility to a severe threat.

Dutch auction (540) Same as a **descending auction.**

effectiveness condition (489) A lower bound on the probability of fulfillment in a brinkmanship threat, expressed as a function of the probability of error, showing the lower limit of risk that will induce the threatened player to comply with the wishes of the threatener.

effective rate of return (353) Rate of return corrected for the probability of noncontinuation of an investment to the next period.

efficient allocation (610) An allocation is called efficient if there is no other allocation that is feasible within the rules of the game, and if it yields a higher payoff to at least one player without giving a lower payoff to any player.

efficient frontier (572) This is the north-east boundary of the set of feasible payoffs of the players, such that in a bargaining game it is not possible to increase the payoff of one person without lowering that of another.

efficient outcome (572) An outcome of a bargaining game is called efficient if there is no feasible alternative that would leave one bargainer with a higher payoff without reducing the payoff of the other.

English auction (540) Same as an **ascending auction.**

enumeration (89) Same as **cell-by-cell inspection.**

envy-free allocation (624) An allocation is called envy-free if no player would wish to have the outcome (typically quantities of commodities available for consumption) that someone else is getting.

equilibrium (33) A configuration of strategies where each player's strategy is his best response to the strategies of all the other players.

equilibrium path of play (58) The **path of play** actually followed when players choose their rollback equilibrium strategies in a sequential game.

evolutionary game (35) A situation where the strategy of each player in a population is fixed genetically, and strategies that yield higher payoffs in random matches with others from the same population reproduce faster than those with lower payoffs.

evolutionary stable (426) A population is evolutionary stable if it cannot be successfully invaded by a new mutant phenotype.

evolutionary stable strategy (429) A phenotype or strategy which can persist in a population, in the sense that all the members of a population or species are of that type; the population is evolutionary stable (static criterion). Or, starting from an arbitrary distribution of phenotypes in the population, the process of selection will converge to this strategy (dynamic criterion).

expected payoff (29, 186) The probability-weighted average (statistical mean or expectation) of the payoffs of one player in a game, corresponding to all possible realizations of a random choice of nature or mixed strategies of the players.

expected utility (228) The probability-weighted average (statistical mean or expectation) of the utility of a person, corresponding to all possible realizations of a random choice of nature or mixed strategies of the players in a game.

expected value (197) The probability-weighted average of the outcomes of a random variable, that is, its statistical mean or expectation.

extensive form (46) Representation of a game by a game tree.

external effect (404) When one person's action alters the payoff of another person or persons. The effect or spillover is **positive** if one's action raises others' payoffs (for example, network effects), and **negative** if it lowers others' payoffs (for example, pollution or congestion). Also called **externality** or **spillover.**

externality (404) Same as **external effect.**

fair division mechanism (624) A procedure for dividing some total quantity between two or more people according to some accepted notion of "fairness" such as equality or being envy-free.

feasible allocation (607) An allocation that is permissible within the rules of the game: being within the constraints of resource availability and production technology, and perhaps also constraints imposed by incomplete information.

first-mover advantage (60) This exists in a game if, considering a hypothetical choice between moving first and moving second, a player would choose the former.

first-price auction (540) An auction in which the highest bidder wins and pays the amount of her bid.

fitness (426) The expected payoff of a phenotype in its games against randomly chosen opponents from the population.

focal point (106) A configuration of strategies for the players in a game, which emerges as the outcome because of the convergence of the players' expectations on it.

free rider (384) A player in a collective action game who intends to benefit from the positive externality generated by others' efforts without contributing any effort of his own.

game (18) An action situation where there are two or more mutually aware players, and the outcome for each depends on the actions of all.

game matrix (84) A spreadsheetlike table whose dimension equals the number of players in the game; the strategies available to each player are arrayed along one of the dimensions (row, column, page, . . .); and each cell shows the payoffs of all the players in a specified order, corresponding to the configuration of strategies which yield that cell. Also called **game table** or **payoff table.**

game table (84) Same as **game matrix.**

game tree (46) Representation of a game in the form of nodes, branches, and terminal nodes and their associated payoffs.

genotype (426) A gene or a complex of genes, which give rise to a phenotype and which can breed true from one generation to another. (In social or economic games, the process of breeding can be interpreted in the more general sense of teaching or learning.)

Gibbard-Satterthwaite theorem (523) With three or more alternatives to consider, the only voting method that prevents strategic voting is dictatorship; one person is identified as the dictator and her preferences determine the outcome.

gradual escalation of the risk of mutual harm (493) A situation where the probability of having to carry out the threatened action in a probabilistic threat increases over time, the longer the opponent refuses to comply with what the threat is trying to achieve.

grand coalition (607) The set of all players in the game, acting as a coalition.

grim strategy (349) A strategy of noncooperation forever in the future, if the opponent is found to have cheated even once. Used as a threat of punishment in an attempt to sustain cooperation.

hawk-dove game (447) An evolutionary game where members of the same species or population can breed to follow one of two strategies, Hawk and Dove, and depending on the payoffs, the game between a pair of randomly chosen members can be either a prisoners' dilemma or chicken.

histogram (525) A bar chart; data is illustrated by way of bars of a given height (or length).

impatience (577) Preference for receiving payoffs earlier rather than later. Quantitatively measured by the discount factor.

imperfect information (23) A game is said to have perfect information if each player, at each point where it is his turn to act, knows the full history of the game up to that point, including the results of any random actions taken by nature or previous actions of other players in the game, including pure actions as well as the actual outcomes of any mixed strategies they may play. Otherwise, the game is said to have imperfect information.

impossibility theorem (511) A theorem that indicates that no preference aggregation method can satisfy the six critical principles identified by Kenneth Arrow.

incentive-compatibility constraint (273) In the case of moral hazard, this is the requirement on your incentive scheme that other players' optimal response is the action you want them to take, even though you cannot observe that action directly. In the case of adverse selection, this is the constraint on your screening device that it should appeal only to the types you want to select and not to others, even though you cannot observe the type directly.

incentive-compatible market mechanism (617) Same as **nonmanipulable market mechanism.**

incentive scheme (266) This is a schedule of payments offered by a "principal" to an "agent," as a function of an observable outcome, that induces the agent in his own interests to choose the underlying unobservable action at a level the principal finds optimal (recognizing the cost of the incentive scheme).

incomplete information (23) A game is said to have incomplete information if rules about what actions are permitted and the order of moves if any, payoffs as functions of the players' strategies, outcomes of random choices by "nature" and of previous actions by the actual players in the game are not common knowledge among the players. This is essentially a more formal term for **asymmetric information.**

independent events (226) Events Y and Z are independent if the actual occurrence of one does not change the probability of the other occurring. That is, the conditional probability of Y occurring given that Z has occurred is the same as the ordinary or unconditional probability of Y.

infinite horizon (351) A repeated decision or game situation that has no definite end at a fixed finite time.

information set (172) A set of nodes among which a player is unable to distinguish when taking an action. Thus his strategies are restricted by the condition that he should choose the same action at all points of an information set. For this, it is essential that all the nodes in an information set have the same player designated to act, with the same number and similarly labeled branches emanating from each of these nodes.

initial node (46) The starting point of a sequential-move game. (Also called the **root** of the tree.)

instant runoff (504) Same as **single transferable vote.**

intermediate (65) **valuation function** A rule assigning payoffs to nonterminal nodes in a game. In many complex games, this must be based on knowledge or experience of playing similar games, instead of explicit rollback analysis.

internalize the externality (410) To offer an individual a reward for the external benefits he conveys on the rest of society, or to inflict a penalty for the external costs he imposes on the rest, so as to bring his private incentives in line with social optimality.

intransitive ordering (506) A preference ordering that cycles and is not **transitive.** For example, a preference ordering over three alternatives A, B, and C is intransitive if A is preferred to B and B is preferred to C but it it not true that A is preferred to C.

invasion by a mutant (426) The appearance of a small proportion of mutants in the population.

irreversible action (312) An action that cannot be undone by a later action. Together with observability, this is an important condition for a game to be sequential-move.

iterated elimination of dominated strategies (96) Considering the players in turns and repeating the process in rotation, eliminating all strategies that are dominated for one at a time, and continuing doing so until no such further elimination is possible.

leadership (360) In a prisoners' dilemma with asymmetric players, this is a situation where a large player chooses to cooperate even though he knows that the smaller players will cheat.

locked in (413) A situation where the players persist in a Nash equilibrium that is worse for everyone than another Nash equilibrium.

majority rule (501) A voting method in which the winning alternative is the one that garners a majority (more than 50%) of the votes.

majority runoff (503) A two-stage voting method in which a second round of voting ensues if no alternative receives a majority in the first round. The top two vote-getters are paired in the second round of voting to determine a winner.

marginal private gain (404) The change in an individual's own payoff as a result of a small change in a continuous strategy variable that is at his disposal.

marginal social gain (404) The change in the aggregate social payoff as a result of a small change in a continuous strategy variable chosen by one player.

market equilibrium (601) An outcome in a market where the price is such that the aggregate quantity demanded equals the aggregate quantity supplied. Graphically, this is represented by a point where the demand and supply curves intersect.

maximin (100) In a zero-sum game, for the player whose strategies are arrayed along the rows, this is the maximum over the rows of the minimum of his payoffs across the columns in each row.

median voter (524) The voter in the middle—at the 50th percentile—of a distribution.

median voter theorem (524) If the political spectrum is one-dimensional and every voter has single-peaked preferences, then [1] the policy most preferred by the median voter will be the Condorcet winner, and [2] power-seeking politicians in a two-candidate election will choose platforms that converge to the position most preferred by the median voter. (This is also known as the **principle of minimum differentiation.**)

minimax (100) In a zero-sum game, for the player whose strategies are arrayed along the columns, this is the minimum over the columns of the maximum of the other player's payoffs across the rows in each column.

minimax method (99) Searching for a Nash equilibrium in a zero-sum game by finding the maximin and the minimax and seeing if they are equal and attained for the same row-column pair.

mixed method (503) A multistage voting method that uses plurative and binary votes in different rounds.

mixed strategy (84) A mixed strategy for a player consists of a random choice, to be made with specified probabilities, from his originally specified pure strategies.

modified addition rule (225) If the occurrence of X requires the occurrence of one or both of Y and Z, then the probability of X is the sum of the separate probabilities of Y and Z minus the probability that *both* Y and Z occur.

modified multiplication rule (227) If the occurrence of X requires the occurrence of both Y and Z, then the probability of X equals the product of two things: [1] the probability that Y alone occurs, and [2] the probability that Z occurs given that Y has already occurred, or the *conditional probability* of Z, conditioned on Y having already occurred.

monomorphism (429) All members of a given species or population exhibit the same behavior pattern.

monopoly power (616) A large seller's ability to raise the price by reducing the quantity offered for sale.

monopsony power (617) A large buyer's ability to lower the price by reducing the quantity purchased.

moral hazard (280) A situation of information asymmetry where one player's actions are not directly observable to others.

move (48) An action at one node of a game tree.

multiplication rule (226) If the occurrence of X requires the simultaneous occurrence of *all* the several independent $Y, Z, \ldots$, then the probability of X is the *product* of the separate probabilities of $Y, Z, \ldots$

multistage procedure (501) A voting procedure in which there are multiple rounds of voting. Also called **rounds.**

mutation (426) Emergence of a new genotype.

Nash cooperative solution (570) This outcome splits the bargainers' surpluses in proportion to their bargaining powers.

Nash equilibrium (87) A configuration of strategies (one for each player) such that each player's strategy is best for him, given those of the other players. (Can be in pure or mixed strategies.)

negatively correlated (304) Two random variables are said to be negatively correlated if, as a matter of probabilistic average, when one is above its expected value, the other is below its expected value.

never a best response (145) A strategy is never a best response for a player if, for each list of strategies that the other players choose (or for each list of strategies that this player believes the others are choosing), some other strategy is this player's best response. (The other strategy can be different for different lists of strategies of the other players.)

node (46) This is a point from which branches emerge, or where a branch terminates, in a decision or game tree.

noncooperative game (26) A game where each player chooses and implements his action individually, without any joint-action agreements directly enforced by other players.

nonexcludable benefits (383) Benefits that are available to each individual, regardless of whether he has paid the costs that are necessary to secure the benefits.

nonmanipulable market mechanism (617) A market mechanism is called nonmanipulable if no buyer has monopsony power and no seller has monopoly power, so no player can change the outcome by misrepresenting his true preferences in his buying or selling decision. Also called **incentive-compatible market mechanism.**

nonrival benefits (383) Benefits whose enjoyment by one person does not detract anything from another person's enjoyment of the same benefits.

norm (397) A pattern of behavior that is established in society by a process of education or culture, to the point that a person who behaves differently experiences a negative psychic payoff.

normal distribution (529) A commonly used statistical distribution for which the **distribution function** looks like a bell-shaped curve.

normal form (84) Representation of a game in a game matrix, showing the strategies (which may be numerous and complicated if the game has several moves) available to each player along a separate dimension (row, column, etc.) of the matrix and the outcomes and payoffs in the multidimensional cells. Also called **strategic form.**

objective value (541) Same as **common value.**

observable action (312) An action that other players know you have taken before they make their responding actions. Together with irreversibility, this is an important condition for a game to be sequential-move.

off-equilibrium path (174) A path of play that does not result from the players' choices of strategies in a subgame perfect equilibrium.

off-equilibrium subgame (174) A subgame starting at a node that does not lie on the equilibrium path of play.

open outcry (539) An auction mechanism in which bids are made openly for all to hear or see.

opponent's indifference property (194) An equilibrium mixed strategy of one player in a two-person game has to be such that the other player is indifferent among all the pure strategies that are actually used in his mixture.

oppression (399) In this context, same as **coercion.**

option (307) A right, but not an obligation, to take an action such as buying something, after some information pertinent to the action has been revealed.

pairwise voting (501) A voting method in which only two alternatives are considered at the same time.

partially revealing equilibrium (281) A perfect Bayesian equilibrium in a game of incomplete information, where the actions in the equilibrium convey some additional information about the players' types, but some ambiguity about these types remains. Also called **semi-separating equilibrium.**

participation condition (constraint) (278) A constraint on an incentive scheme or a screening device that it should give the more-informed player an expected payoff at least as high as he can get outside this relationship.

path of play (58) A route through the game tree (linking a succession of nodes and branches) that results from a configuration of strategies for the players that are within the rules of the game. (See also **equilibrium path of play.**)

payoff (28) The objective, usually numerical, that a player in a game aims to maximize.

payoff table (84) Same as **game matrix.**

penalty (356) We reserve this term for one-time costs (such as fines) introduced into a game to induce the players to take actions that are in their joint interests.

perfect Bayesian equilibrium (PBE) (284) An equilibrium where each player's strategy is optimal at all nodes given his beliefs, and beliefs at each node are updated using Bayes' rule in the light of the information available at that point including other players' past actions.

phenotype (426) A specific behavior or strategy, determined by one or more genes. (In social or economic games, this can be interpreted more generally as a customary strategy or a rule of thumb.)

playing the field (427) A many-player evolutionary game where all animals in the group are playing simultaneously, instead of being matched in pairs for two-player games.

pluralistic ignorance (415) A situation of collective action where no individual knows for sure what action is needed, so everyone takes the cue from other people's actions or inaction, possibly resulting in persistence of wrong choices.

plurality rule (502) A voting method in which two or more alternatives are considered simultaneously and the winning alternative is the one that garners the largest number of votes; the winner needs only gain more votes than each of the other alternatives and does not need 50% of the vote as would be true in **majority rule.**

plurative method (502) Any voting method that allows voters to consider a slate of three or more alternatives simultaneously.

polymorphism (429) An evolutionary stable equilibrium in which different behavior forms or phenotypes are exhibited by subsets of members of an otherwise identical population.

pooling of types (274) An outcome of a signaling or screening game in which different types follow the same strategy and get the same payoffs, so types cannot be distinguished by observing actions.

pooling risk (304) Combining two or more random outcomes that are less than perfectly positively correlated, so as to reduce the total risk to which one is exposed.

positional method (502) A voting method that determines the identity of the winning alternative using information on the position of alternatives on a voter's ballot to assign points used when tallying ballots.

positive feedback (411) When one person's action increases the payoff of another person or persons taking the same action, thus increasing their incentive to take that action too.

positively correlated (305) Two random variables are said to be positively correlated if, as a matter of probabilistic average, when one is above its expected value, the other is also above its expected value, and vice versa.

power index (620) A measure of an individual's ability to make a difference to the payoffs of coalitions which he could join, averaged across these coalitions.

present value (351) The total payoff over time, calculated by summing the payoffs at different periods each multiplied by the appropriate discount factor to make them all comparable with the initial period's payoffs.

prevent exploitation (244) A method of finding mixed strategy equilibria in zero-sum games. A player's mixture probabilities are found by equating his own expected payoffs from the mixture against all the pure strategies used in the opponent's mixture. This method is not valid for non-zero-sum games.

primary criterion (457) Comparison of the fitness of a mutant with that of a member of the dominant population, when each plays against a member of the dominant population.

principle of minimum differentiation (527) Same as part [2] of the **median voter theorem.**

prisoners' dilemma (90) A game where each player has two strategies, say Cooperate and Defect, such that [1] for each player, Defect dominates Cooperate, and [2] the outcome (Defect, Defect) is worse for both than the outcome (Cooperate, Cooperate).

private value (541) A bidder's individual valuation of an object available at auction. Also called **subjective value.**

probabilistic threat (487) A strategic move in the nature of a threat, but with the added qualification that if the event triggering the threat (the opponent's action in the case of deterrence, or inaction in the case of compellence) comes about, a chance mechanism is set in motion, and if its outcome so dictates, the threatened action is carried out. The nature of this mechanism and the probability with which it will call for the threatened action must both constitute prior commitments.

probability (223) The probability of a random event is a quantitative measure of the likelihood of its occurrence. For events that can be observed in repeated trials, it is the long-run frequency with which it occurs. For unique events or other situations where uncertainty may be in the mind of a person, other measures are constructed, such as subjective probability.

promise (314) An action by one player, say A, in a pre-game stage, establishing a response rule that, if the other player B chooses an action specified by A, then A will respond with a specified action that is costly to A and rewards B (gives him a higher payoff). (For this to be feasible, A must have the ability to move second in the actual game.)

proportional representation (504) This voting system requires that the number of seats in a legislature be allocated in proportion to each party's share of the popular vote.

proxy-bidding (558) A process by which a bidder submits her maximum bid (**reservation price**) for an item up for auction and a third party takes over bidding for her; the third party bids only the minimum increment above any existing bids and bids no higher than the bidder's specified maximum.

prune (52) To use rollback analysis to identify and eliminate from a game tree those branches that will not be chosen when the game is rationally played.

punishment (349) We reserve this term for costs that can be inflicted on a player in the context of a repeated relationship (often involving termination of the relationship) to induce him to take actions that are in the joint interests of all players.

pure coordination game (105) A coordination game where the payoffs of each player are the same in all the Nash equilibria. Thus all players are indifferent among all the Nash equilibria, and coordination is needed only to ensure avoidance of a non-equilibrium outcome.

pure public good (383) A good or facility that benefits all members of a group, when these benefits cannot be excluded from a member who has not contributed efforts or money to the provision of the good, and the enjoyment of the benefits by one person does not significantly detract from their simultaneous enjoyment by others.

pure strategy (84) A rule or plan of action for a player that specifies without any ambiguity or randomness the action to take in each contingency or at each node where it is that player's turn to act.

rational behavior (30) Perfectly calculating pursuit of a complete and internally consistent objective (payoff) function.

rational irrationality (335) Adopting a strategy that is not optimal after the fact, but serves a rational strategic purpose of lending credibility to a threat or a promise.

rationalizability (145) A solution concept for a game. A list of strategies, one for each player, is a rationalizable outcome of the game if each strategy in the list is rationalizable for the player choosing it.

rationalizable (145) A strategy is called rationalizable for a player if it is his optimal choice given some belief about what (pure or mixed strategy) the other player(s) would choose, provided this belief is formed recognizing that the other players are making similar calculations and forming beliefs in the same way. (This concept is more general than that of the Nash equilibrium and yields outcomes that can be justified on the basis only of the players' common knowledge of rationality.)

refinement (143) A restriction that narrows down possible outcomes when multiple Nash equilibria exist.

repeated play (347) A situation where a one-time game is played repeatedly in successive periods. Thus the complete game is mixed, with a sequence of simultaneous-move games.

reputation (333) Relying on the effect on payoffs in future or related games to make threats or promises credible, when they would not have been credible in a one-off or isolated game.

reservation price (558) The maximum amount that a bidder is willing to pay for an item.

reserve price (550) The minimum price set by the seller of an item up for auction; if no bids exceed the reserve, the item is not sold.

response rule (314) A rule that specifies how you will act in response to various actions of other players.

revenue equivalence (551) In the equilibrium of a private value auction where all bidders are risk-neutral and have independent valuations, all auction forms will yield the seller the same expected revenue.

reversal paradox (508) This paradox arises in an election with at least four alternatives when one of these is removed from consideration after votes have been submitted and the removal changes the identity of the winning alternative.

reversal terms (514) A set of ballots that would generate the **reversal paradox** and that should together logically produce a tied vote between a pair of alternatives. In a three-candidate election among A, B, and C, the reversal terms are two ballots that show a reversal in the location of a pair of alternatives. For example, one ballot with A preferred to B preferred to C and another with B preferred to A preferred to C should produce a tie between A and B.

risk-averse (229) A decision-maker (or a player in a game) is called risk-averse if he prefers to replace a lottery of monetary amounts by the expected monetary value of the same lottery, but now received with certainty.

risk-neutral (229) A decision-maker (or a player in a game) is called risk-neutral if he is indifferent between a lottery of monetary amounts and the expected monetary value of the same lottery, but now received with certainty.

robustness (513) A measure of the number of sets of voter preference orderings that are nondictatorial, satisfy independence of irrelevant alternatives and the Pareto property, and also produce a transitive **social ranking.**

rollback (54) Analyzing the choices that rational players will make at all nodes of a game, starting at the terminal nodes and working backward to the initial node.

rollback equilibrium (54) The strategies (complete plans of action) for each player that remain after rollback analysis has been used to prune all the branches that can be pruned.

root (46) Same as **initial node.**

rounds (504) A voting situation in which votes take place in several stages. Also called **multistage.**

saddle point (256) In this context, an equilibrium of a two-person zero-sum game where the payoff of one player is simultaneously maximized with respect to his own strategy choice and minimized with respect to the strategy choice of the other player.

salami tactics (326) A method of defusing threats by taking a succession of actions, each sufficiently small to make it non-optimal for the other player to carry out his threat.
sanction (397) Punishment approved by society and inflicted by others on a member who violates an accepted pattern of behavior.
screening (24, 266) Strategy of a less-informed player to elicit information credibly from a more-informed player.
screening devices (24, 266) Methods used for screening.
sealed bid (540) An auction mechanism in which bids are submitted privately in advance of a specified deadline, often in sealed envelopes.
secondary criterion (457) Comparison of the fitness of a mutant with that of a member of the dominant population, when each plays against a mutant.
second-mover advantage (59) A game has this if, considering a hypothetical choice between moving first and moving second, a player would choose the latter.
second-price auction (540) An auction in which the highest bidder wins the auction but pays a price equal to the value of the second-highest bid; also called a **Vickrey auction.**
security level (607) The security level of a coalition is the total payoff or surplus its members can ensure for themselves, no matter what strategies other players who are not members of the coalition may choose.
selection (426) The dynamic process by which the proportion of fitter phenotypes in a population increases from one generation to the next.
self-selection (273) Where different types respond differently to a screening device, thereby revealing their type through their own action.
semiseparating equilibrium (281) Same as **partially revealing equilibrium.**
separation of types (273) An outcome of a signaling or screening game in which different types follow different strategies and get the different payoffs, so types can be identified by observing actions.
sequential moves (20) The moves in a game are sequential if the rules of the game specify a strict order such that at each action node only one player takes an action, with knowledge of the actions taken (by others or himself) at previous nodes.
shading (545) A strategy in which bidders bid slightly below their true valuation of an object.
Shapley value (619) A solution concept for a cooperative game, where all coalitions to which an individual may belong are regarded equally likely, and each individual is awarded the average of his contributions to raising the aggregate payoffs of all these coalitions.
shilling (556) A practice used by sellers at auction by which they plant false bids for an object they are selling.
signaling (24, 265) Strategy of a more-informed player to convey his "good" information credibly to a less-informed player.
signal jamming (266) A situation in a signaling game where an informed player of a "bad" type mimics the strategy of a "good" type, thereby preventing separation or achieving pooling. This term is used particularly if the action in question is a mixed strategy.
signals (24) Devices used for signaling.
simultaneous moves (20) The moves in a game are simultaneous if each player must take his action without knowledge of the choices of others.

sincere voting (508) Voting at each point for the alternative that you like best among the ones available at that point, regardless of the eventual outcome.

single-peaked preferences (512) A preference ordering in which alternatives under consideration can be ordered along some specific dimension and each voter has a single ideal or most-preferred alternative with alternatives farther away from the most-preferred point providing steadily lower payoffs.

single transferable vote (504) A voting method in which each voter indicates her preference ordering over all candidates on a single initial ballot. If no alternative receives a majority of all first-place votes, the bottom-ranked alternative is eliminated and all first-place votes for that candidate are "transferred" to the candidate listed second on those ballots; this process continues until a majority winner emerges. Also called **instant runoff.**

social ranking (505) The preference ordering of a group of voters that arises from aggregating the preferences of each member of the group.

spillover effect (404) Same as **external effect.**

spoiler (515) Refers to a third candidate who enters a two-candidate race and reduces the chances that the leading candidate actually wins the election.

strategic form (84) Same as **normal form.**

strategic game (18) See **game.**

strategic misrepresentation of preferences (505) Refers to strategic behavior of voters when they use rollback to determine that they can achieve a better outcome for themselves by not voting strictly according to their preference orderings.

strategic moves (311) Actions taken at a pregame stage that change the strategies or the payoffs of the subsequent game (thereby changing its outcome in favor of the player[s] making these moves).

strategic voting (505) Voting in conformity with your optimal rational strategy found by doing rollback analysis on the game tree of the voting procedure.

strategy (27) A complete plan of action for a player in a game, specifying the action he would take at all nodes where it is his turn to act according to the rules of the game (whether these nodes are on or off the equilibrium path of play). If two or more nodes are grouped into one information set, then the specified action must be the same at all these nodes.

subgame (160) A game comprising a portion or remnant of a larger game, starting at a noninitial node of the larger game.

subgame-perfect equilibrium (175) A configuration of strategies (complete plans of action) such that their continuation in any subgame remains optimal (part of a rollback equilibrium), whether that subgame is on- or off-equilibrium. This ensures credibility of all the strategies.

subjective uncertainty (234) A situation where one person is unsure in his mind about which of a set of events will occur, even though there may not be any chance mechanism such as a coin toss with objectively calculable probabilities that governs the outcome.

subjective value (541) Same as **private value.**

successive elimination of dominated strategies (96) Same as **iterated elimination of dominated strategies.**

supply curve (600) A graph with quantity on the horizontal axis and price on the vertical axis, showing for each price the quantity that one seller in a market (or the aggregate of all sellers, depending on the context) will choose to buy.

surplus (569) A player's surplus in a bargaining game is the excess of his payoff over his BATNA.

terminal node (48) This represents an end point in a game tree, where the rules of the game allow no further moves, and payoffs for each player are realized.

threat (314) An action by one player, say A, in a pre-game stage, establishing a response rule that, if the other player B chooses an action specified by A, then A will respond with a specified action that is damaging to B (gives him a lower payoff), and also costly to A to carry out after the fact. (For this to be possible, A must have the ability to be the second mover in the actual game.)

Tit-for-tat (349) In a repeated prisoners' dilemma, this is the strategy of [1] cooperating on the first play and [2] thereafter doing each period what the other player did the previous period.

trading risk (305) Inducing someone else to bear some of the risk to which one is exposed, in return for a suitable monetary or other compensation.

transitive ordering (506) A preference ordering for which it is true that if option A is preferred to B and B is preferred to C, then A is also preferred to C.

trigger strategy (349) In a repeated game, this strategy cooperates until and unless a rival chooses to defect, and then switches to noncooperation for a specified period.

ultimatum game (580) A form of bargaining where one player makes an offer of a particular split of the total available surplus, and the other has only the all-or-nothing choice of accepting the offer or letting the game end without agreement, when both get zero surplus.

uniform distribution (529) A common statistical distribution in which the **distribution function** is horizontal; data is distributed uniformly at each location along the range of possible values.

utility function (230) In this context, a nonlinear scaling of monetary winnings or losses, such that its expected value (the expected utility) accurately captures a person's attitudes toward risk.

variable-threat bargaining (575) A two-stage game where at the first stage you can take an action that will alter the BATNAs of both bargainers (within certain limits), and at the second stage bargaining results in the Nash solution on the basis of these BATNAs.

Vickrey auction (540) Same as **sealed bid** auction.

Vickrey's truth serum (546) Our name for the result that, in a second-price sealed bid auction, it is every bidder's dominant strategy to bid her true valuation.

winner's curse (541) A situation in a common value auction, where although each person may make an unbiased estimate of the value, only the one with the highest estimate will bid high and win the object, and is therefore likely to have made an upward-biased (too high) estimate. A rational calculation of your bidding strategy will take this into account and lower your bid appropriately to counter this effect.

Yankee auction (559) An auction in which multiple units of a particular item are available for sale; bidders can bid on one or more units at the same time.

zero-sum game (85) A game where the sum of the payoffs of all players equals zero for every configuration of their strategy choices. (This is a special case of a **constant-sum game,** but in practice no different because adding a constant to all the payoff numbers of any one player makes no difference to his choices.)

Index