Optimization Method & Optimal Guidance

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AE8120

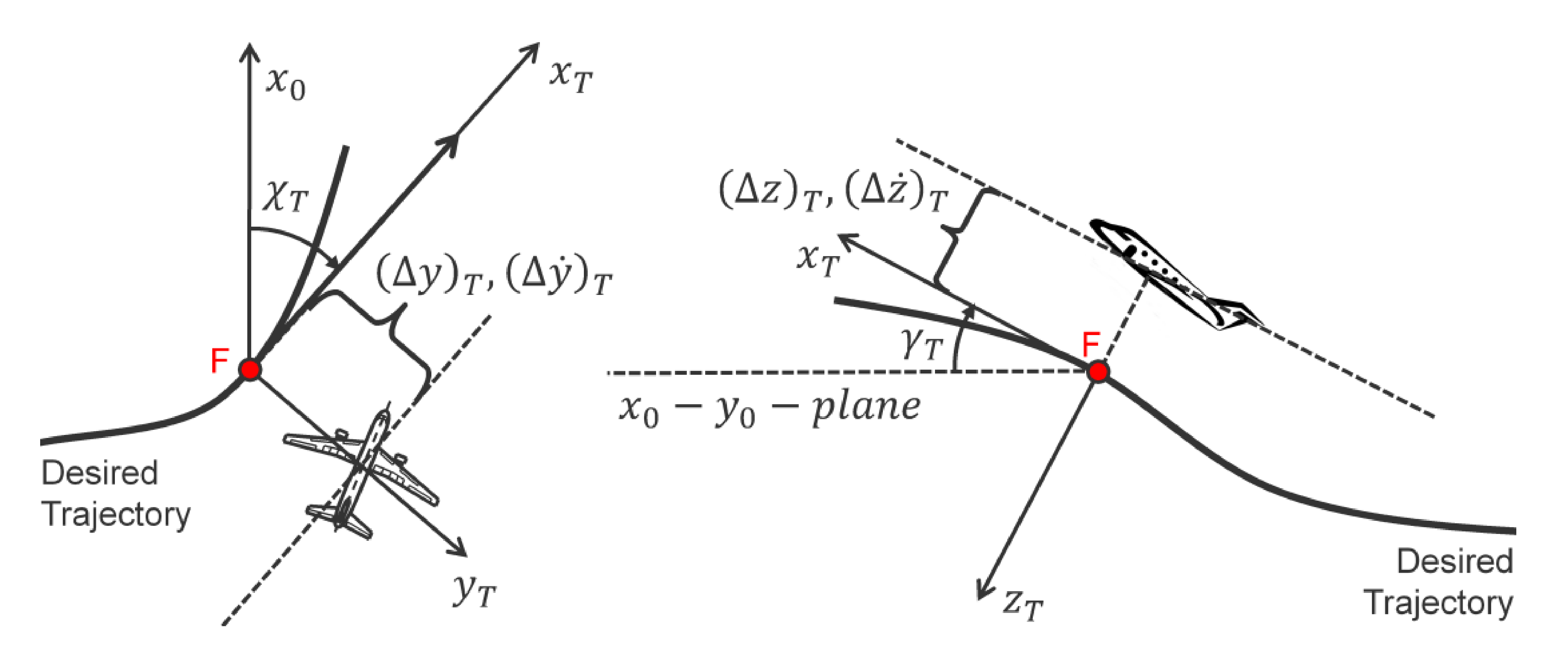
Homework

Derive the linearized dynamics for Backward Euler

- Send to haichao.hong@sjtu.edu.cn
- Include “作业3” and YOUR NAME in the subject line

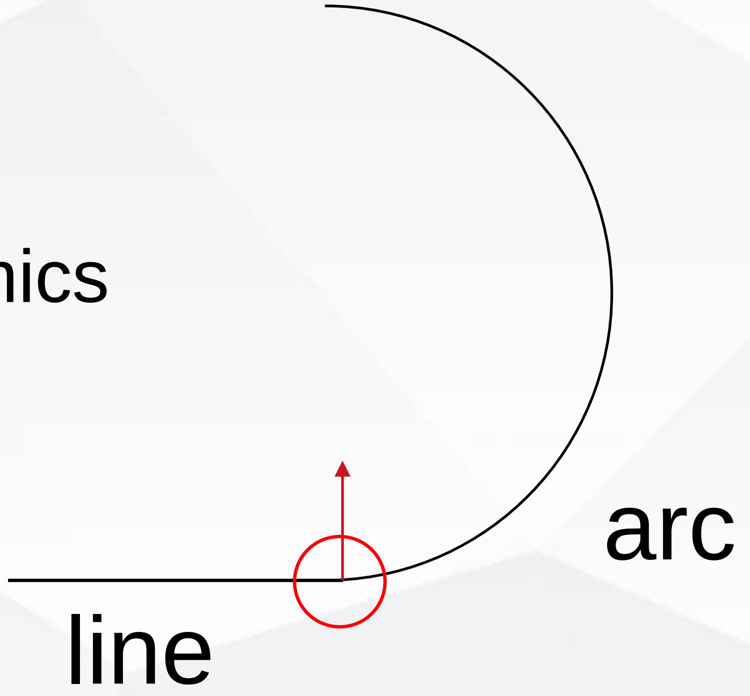
Contents

Why Smooth Trajectories



Why Smooth Trajectories

- Simple setups: lines, arcs, clothoids.
  - Simple
  - Lack of adherence to aircraft dynamics
  - Steps
- Trajectory optimization
  - Smoothness?

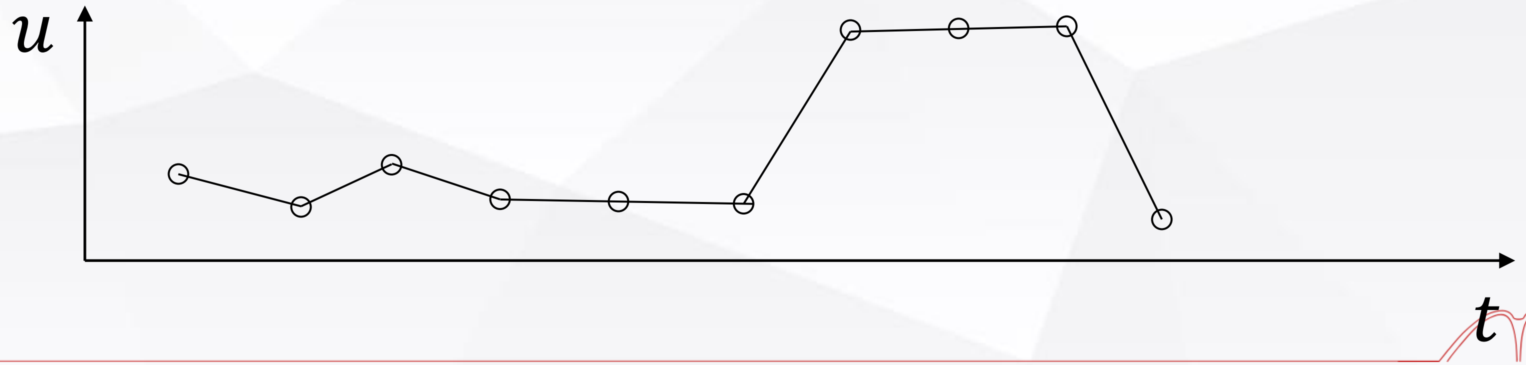


Trigonometric-polynomial Control Parameterization

A general class of constrained trajectory optimization formulations:

$$
\begin{aligned}
\underset{\boldsymbol{u}(t),\boldsymbol{p}}{\text{minimize}} \quad & J\left(\boldsymbol{x}\left(t\right),\boldsymbol{u}\left(t\right),\boldsymbol{p}\right) \\
\text{subject to} \quad & \dot{\boldsymbol{x}}(t) = \boldsymbol{f}\left(\boldsymbol{x}\left(t\right),\boldsymbol{u}\left(t\right),\boldsymbol{p}\right)\ , \\
& \boldsymbol{g}_{lb}\left(t\right) \leq \boldsymbol{g}\left(\boldsymbol{x}\left(t\right),\boldsymbol{u}\left(t\right),\boldsymbol{p}\right) \leq \boldsymbol{g}_{ub}\left(t\right)
\end{aligned}
\tag{1}
$$

Problem (1) does not always correspond to a smooth control history



Trigonometric-polynomial Control Parameterization

The i-th control component is expressed by

$$
\begin{aligned}
u^{(i)}(t) &= a_0^{(i)} + \sum_{n=1}^{N}\left(a_n^{(i)}\cos\left(\frac{n\pi}{T_f}t\right) + b_n^{(i)}\sin\left(\frac{n\pi}{T_f}t\right)\right) \\
&= \boldsymbol{s}_N^{(i)}(t)\,\boldsymbol{c}^{(i)}, \quad i = 1, 2, \ldots, m,
\end{aligned}
$$

where

$$
\begin{aligned}
\boldsymbol{s}_N^{(i)}(t) = \Bigg[&1, \cos\left(\frac{\pi}{T_f}t\right), \ldots, \cos\left(\frac{N\pi}{T_f}t\right), \\
&\sin\left(\frac{\pi}{T_f}t\right), \ldots, \sin\left(\frac{N\pi}{T_f}t\right)\Bigg]
\end{aligned}
$$

$$
\boldsymbol{c}^{(i)} = \left[a_0^{(i)}, \ldots, a_N^{(i)}, b_1^{(i)}, \ldots, b_N^{(i)}\right]^{\mathrm{T}}.
$$

Trigonometric-polynomial Control Parameterization

Defining the total coefficient vector as

$$\boldsymbol{c} = \left[\left(\boldsymbol{c}^{(1)} \right)^{\mathrm{T}}, \left(\boldsymbol{c}^{(2)} \right)^{\mathrm{T}}, \ldots, \left(\boldsymbol{c}^{(m)} \right)^{\mathrm{T}} \right]^{\mathrm{T}},$$

and the basis matrix as

$$\boldsymbol{S}_N(t) = \begin{bmatrix} \boldsymbol{s}_N^{(1)}(t) & 0 & \ldots & 0 \\ 0 & \boldsymbol{s}_N^{(2)}(t) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \boldsymbol{s}_N^{(m)}(t) \end{bmatrix},$$

(2) can be written in a more compact manner as

$$\boldsymbol{u}(t) = \boldsymbol{S}_N(t)\, \boldsymbol{c}\,. \tag{9}$$

Trigonometric-polynomial Control Parameterization

Constraints on derivatives are represented by

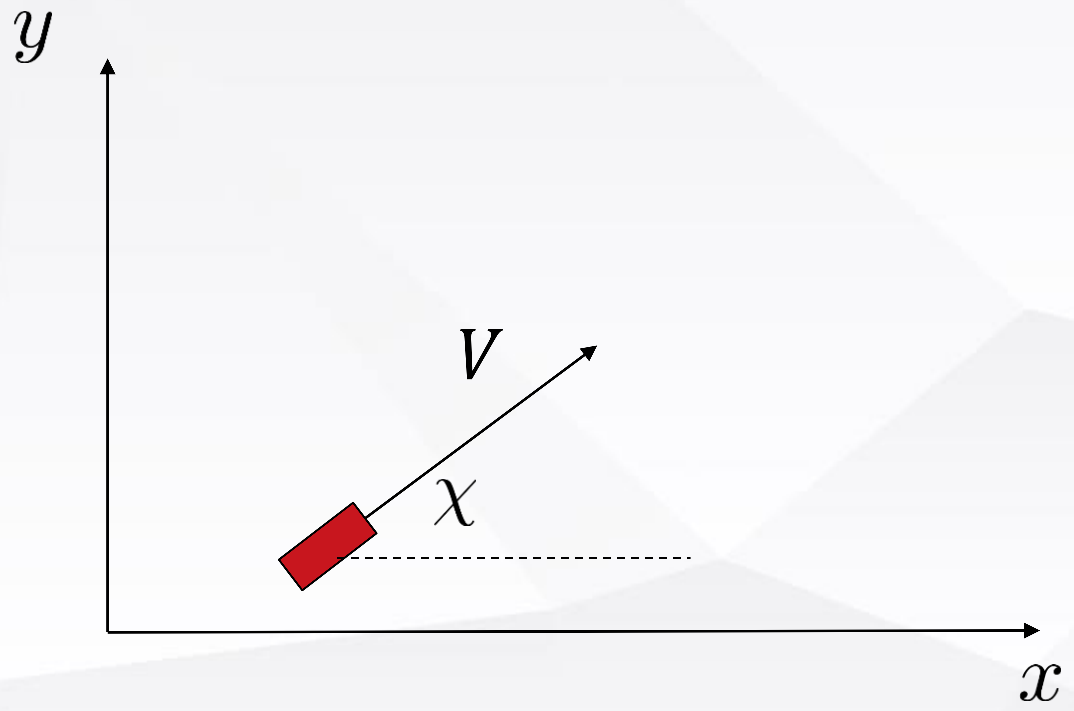
$$\boldsymbol{l}_{lb}(t) \leq \boldsymbol{l}(\boldsymbol{c}, t) \leq \boldsymbol{l}_{ub}(t) \ .$$

The original problem is re-written as

$$\begin{aligned} \underset{\boldsymbol{c},\boldsymbol{p}}{\text{minimize}} \quad & J(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{p}) \\ \text{subject to} \quad & \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{p}) \ , \\ & \boldsymbol{u}(t) = \boldsymbol{S}_N(t)\,\boldsymbol{c} , \\ & \boldsymbol{g}_{lb}(t) \leq \boldsymbol{g}(\boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{p}) \leq \boldsymbol{g}_{ub}(t) \ , \\ & \boldsymbol{l}_{lb}(t) \leq \boldsymbol{l}(\boldsymbol{c}, t) \leq \boldsymbol{l}_{ub}(t) \ . \end{aligned}$$

Benchmark – 2D Time Optimal Trajectory

Numerical Benchmark – 2D Time Optimal Trajectory



The dynamic model is given as

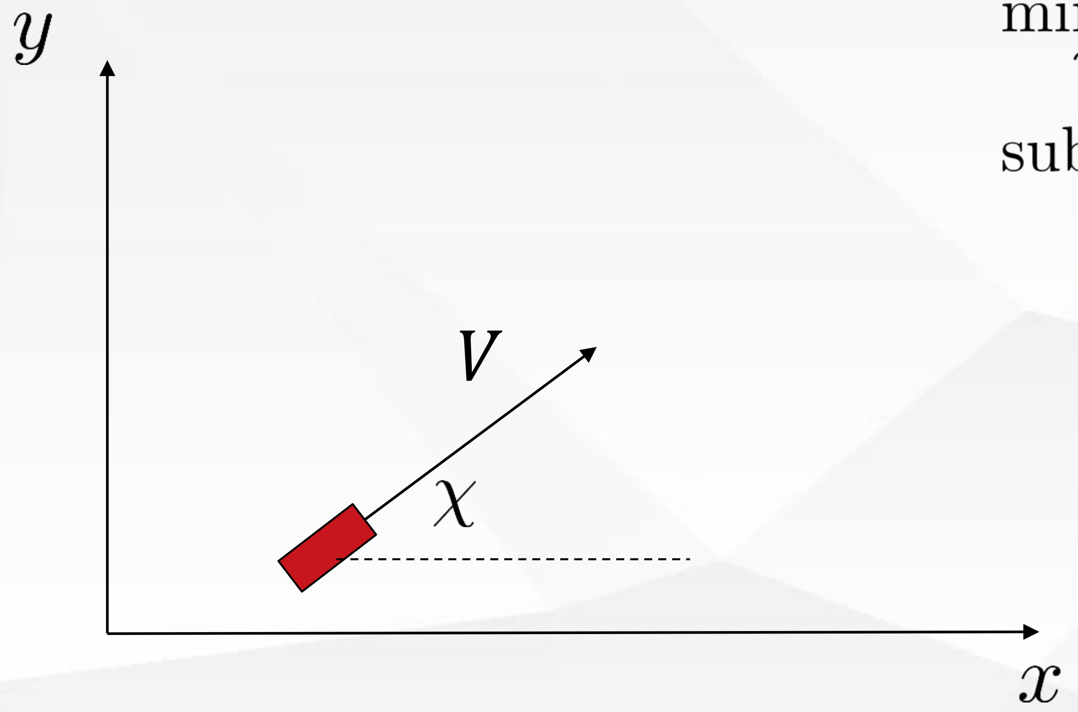
$$\begin{aligned}\dot{x}(t) &= V(t)\cos\chi(t) \ ,\\ \dot{y}(t) &= V(t)\sin\chi(t) \ ,\\ \dot{V}(t) &= \dot{V}_{cmd}(t) \ ,\\ \dot{\chi}(t) &= \dot{\chi}_{cmd}(t) \ ,\end{aligned}$$

and the state and control vectors are defined as

$$\boldsymbol{x} = [x, y, V, \chi]^{\mathrm{T}}$$

$$\boldsymbol{u} = \left[\dot{V}_{cmd}, \dot{\chi}_{cmd}\right]^{\mathrm{T}} .$$

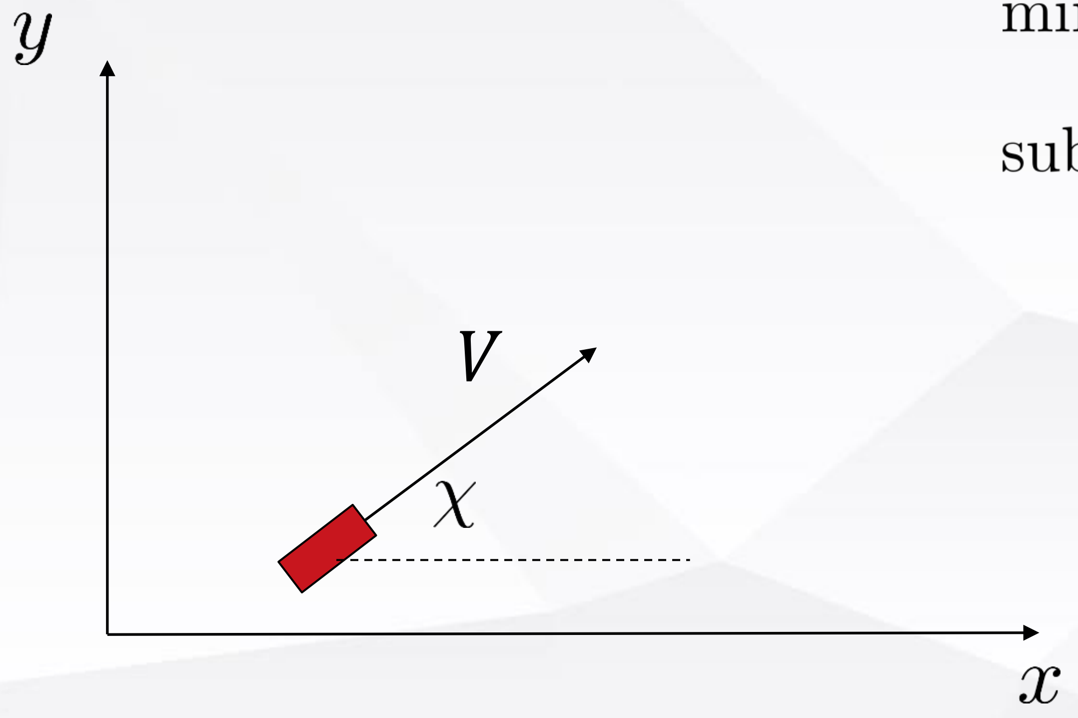
Numerical Benchmark – 2D Time Optimal Trajectory



The problem formulation is given as

$$
\begin{aligned}
\underset{\boldsymbol{u},T_f}{\text{minimize}} \quad & J = T_f \\
\text{subject to} \quad & \text{Equations of Motion}\,, \\
& \boldsymbol{x}_{min} \le \boldsymbol{x}(t) \le \boldsymbol{x}_{max}\,, \\
& \boldsymbol{u}_{min} \le \boldsymbol{u}(t) \le \boldsymbol{u}_{max}\,, \\
& \boldsymbol{x}(0) = \boldsymbol{x}_i\,, \\
& \boldsymbol{x}(T_f) = \boldsymbol{x}_f\,, \\
& -|a_c|_{max} \le \dot{\chi}_{cmd}(t) \cdot V(t) \le |a_c|_{max}\,.
\end{aligned}
$$

Numerical Benchmark – 2D Time Optimal Trajectory



The problem formulation is given as

$$
\begin{aligned}
\underset{\boldsymbol{c},T_f}{\text{minimize}} \quad & J = T_f \\
\text{subject to} \quad & \text{Equations of Motion}\,, \\
& \boldsymbol{u}(t) = \boldsymbol{S}_N(t)\,\boldsymbol{c}\,, \\
& \boldsymbol{x}_{min} \le \boldsymbol{x}(t) \le \boldsymbol{x}_{max}\,, \\
& \boldsymbol{u}_{min} \le \boldsymbol{u}(t) \le \boldsymbol{u}_{max}\,, \\
& \boldsymbol{x}(0) = \boldsymbol{x}_i\,, \\
& \boldsymbol{x}(T_f) = \boldsymbol{x}_f\,, \\
& -|a_c|_{max} \le \dot{\chi}_{cmd}(t) \cdot V(t) \le |a_c|_{max}\,.
\end{aligned}
$$

Trigonometric-polynomial Control Parameterization

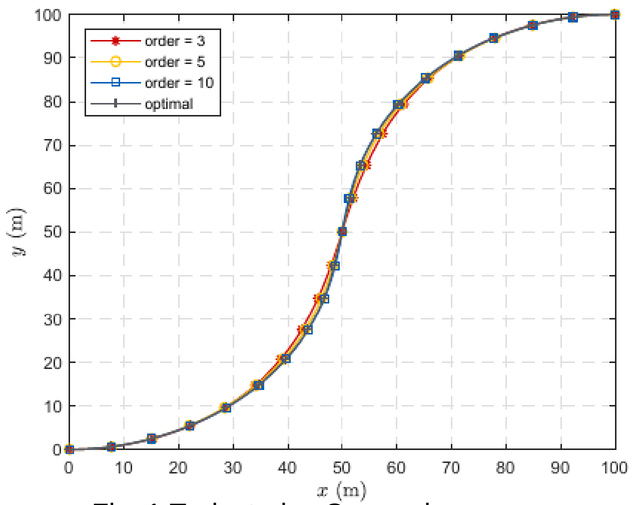


Fig. 1 Trajectories Comparison

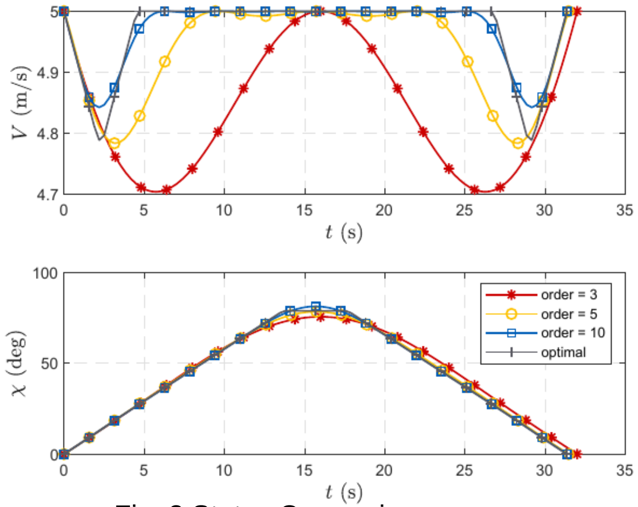


Fig. 2 States Comparison

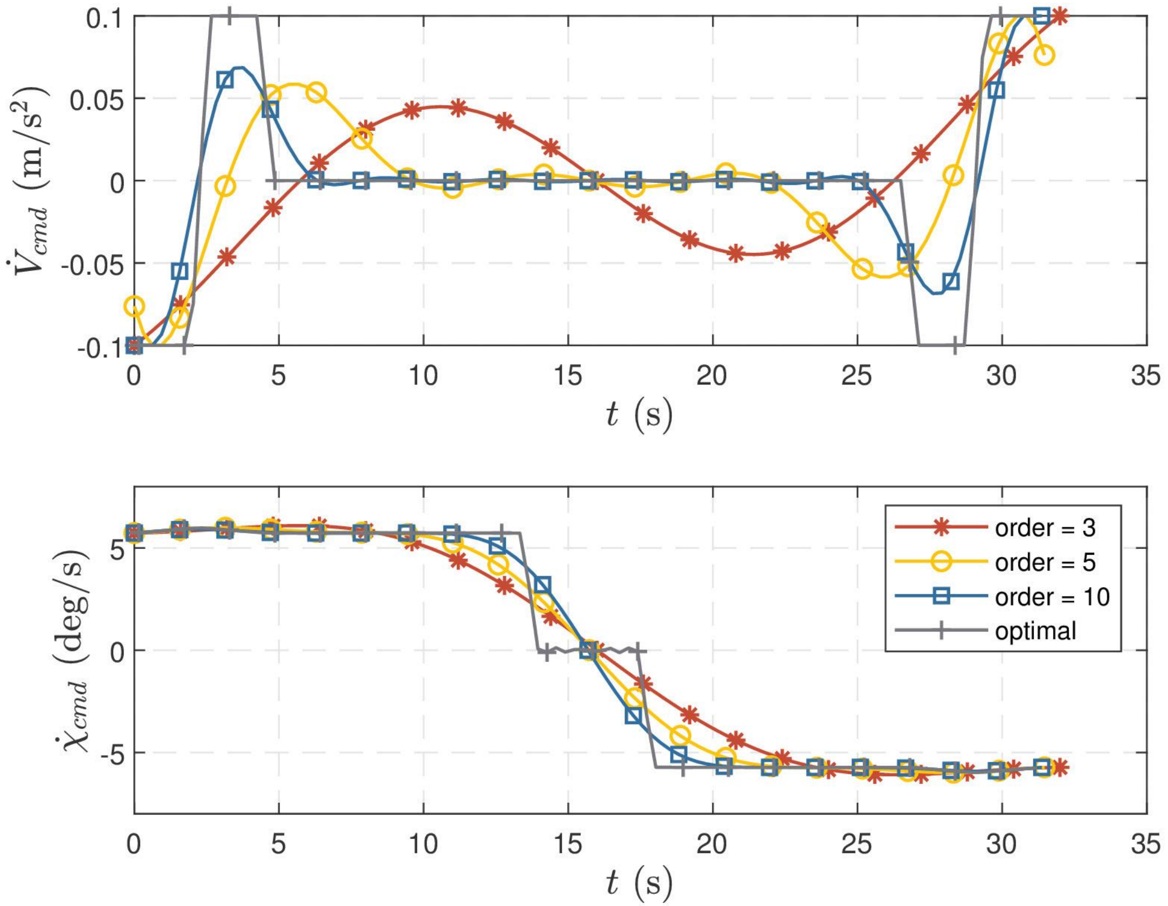
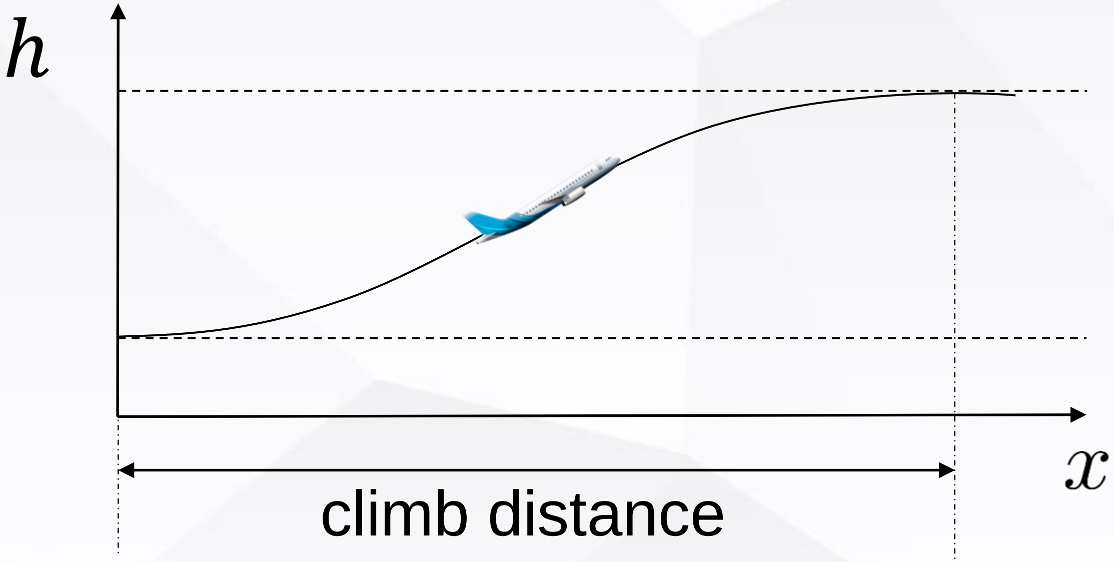


Fig. 3 Controls Comparison

Application I: With respect to an independent variable

Trigonometric-polynomial Control Parameterization

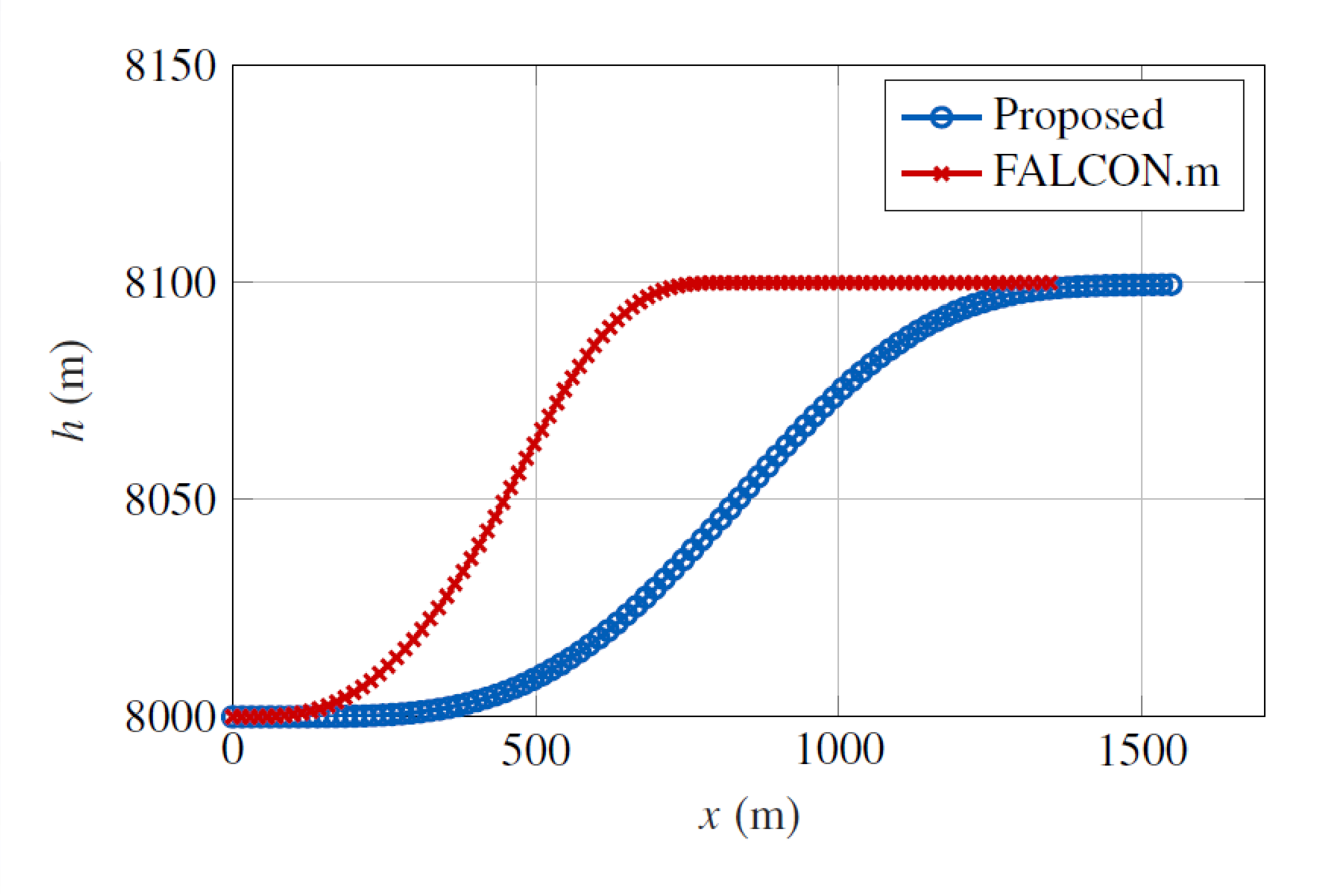


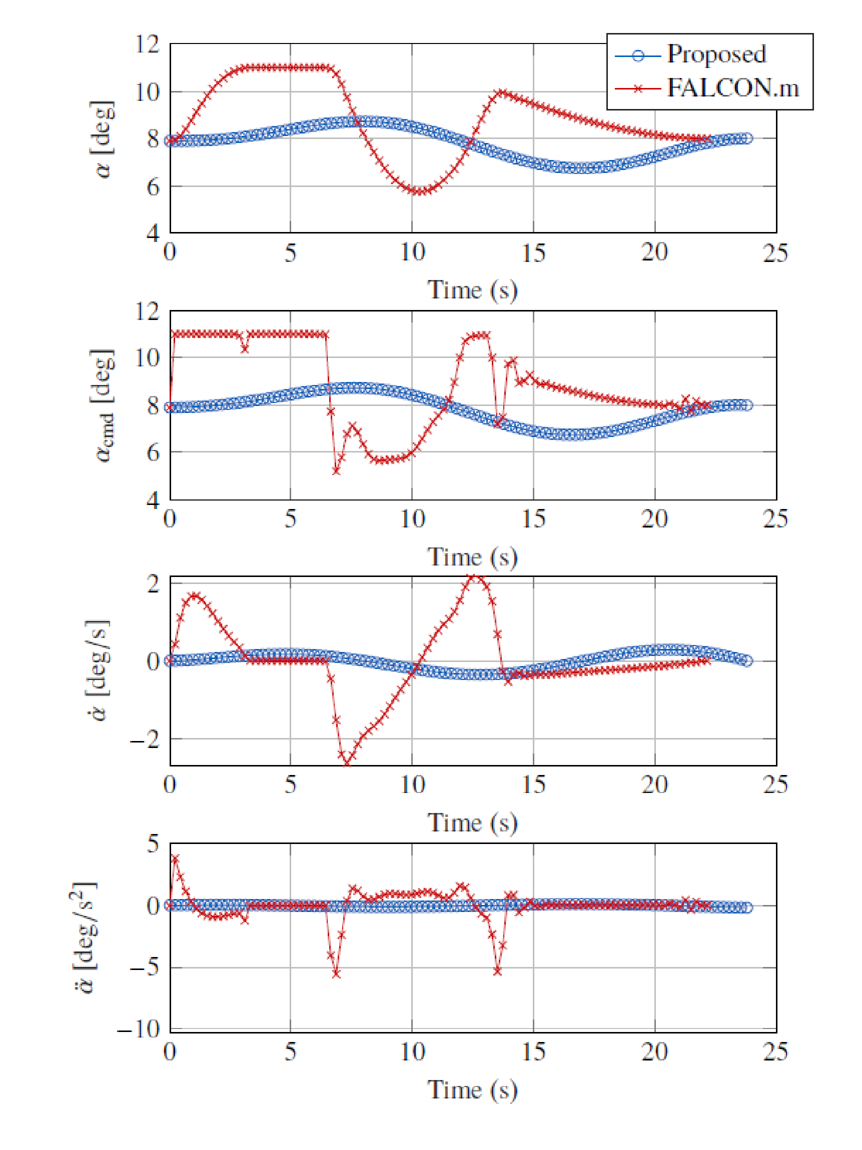
The trigonometric polynomial can be defined with respect to other independent variables.

Here, the climb distance r

$$u_i(r) = (a_0)_i + \sum_{n=1}^{N}\left[(a_n)_i \cos\left(\frac{2n\pi}{\kappa x_f} r\right) + (b_n)_i \sin\left(\frac{2n\pi}{\kappa x_f} r\right)\right]$$
$$= \boldsymbol{s}_{N\,i}(r)\,\boldsymbol{c}_i, \quad i = 1, 2, \ldots, p,$$

Trigonometric-polynomial Control Parameterization





Hong, H., Piprek, P., Gerdts, M., & Holzapfel, F. (2021). Computationally Efficient Trajectory Generation for Smooth Aircraft Flight Level Changes. *Journal of Guidance, Control, and Dynamics*.

Application II: Periodicity

Trigonometric-polynomial Control Parameterization

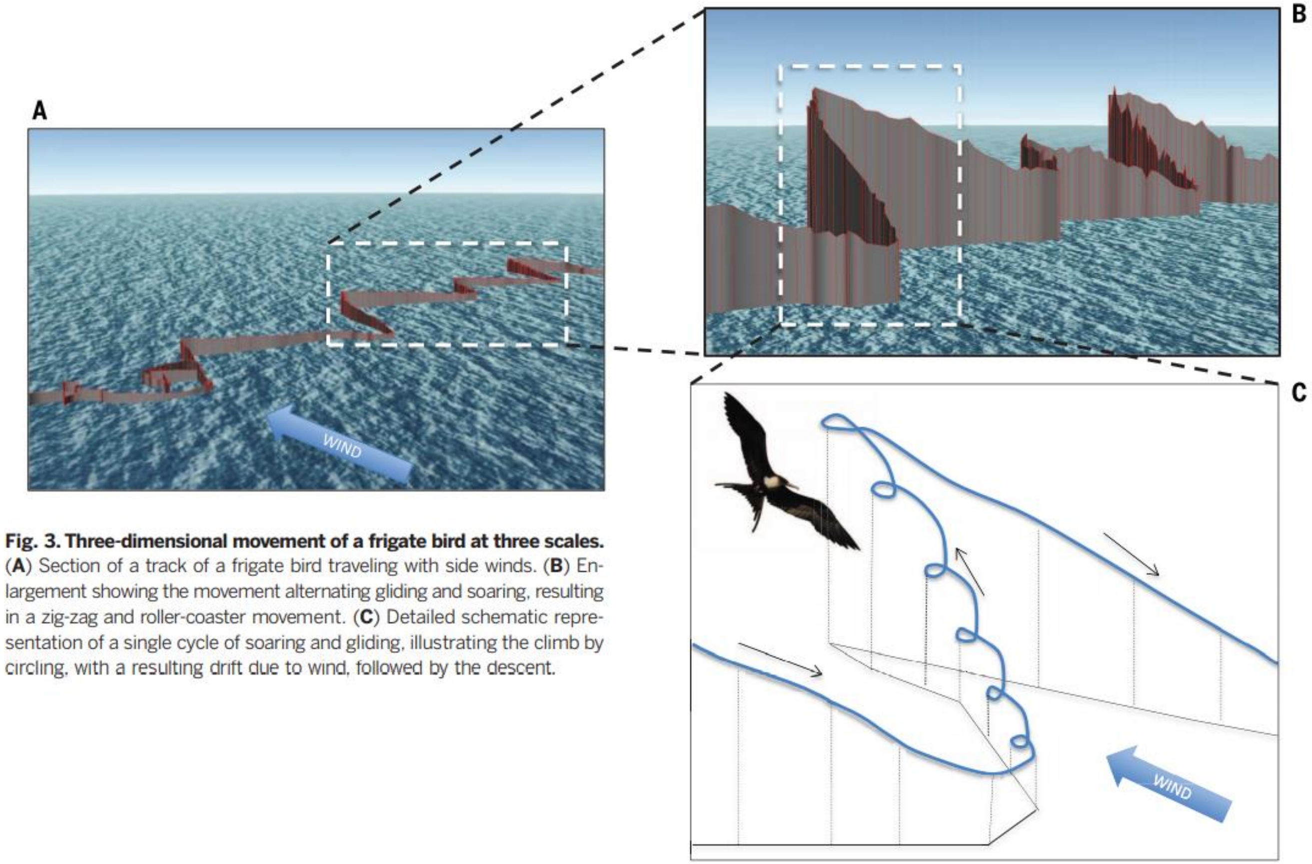
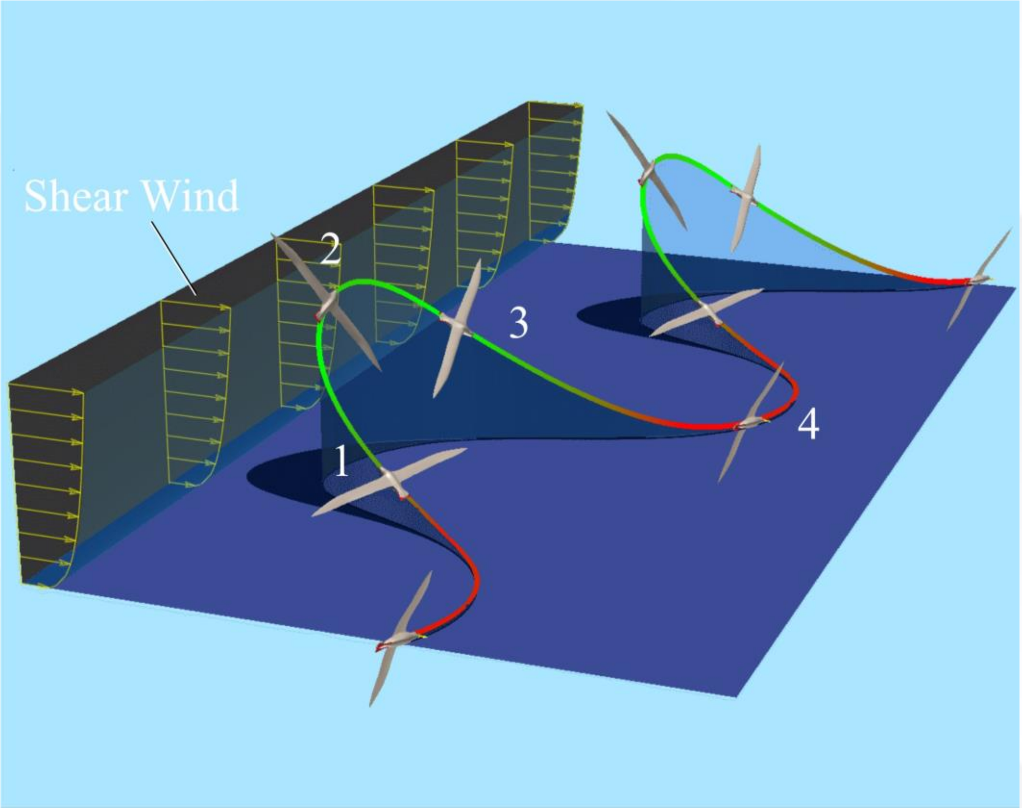


Fig. 3. Three-dimensional movement of a frigate bird at three scales. (**A**) Section of a track of a frigate bird traveling with side winds. (**B**) Enlargement showing the movement alternating gliding and soaring, resulting in a zig-zag and roller-coaster movement. (**C**) Detailed schematic representation of a single cycle of soaring and gliding, illustrating the climb by circling, with a resulting drift due to wind, followed by the descent.



Trigonometric-polynomial Control Parameterization

The original parameterization form is written as

$$u^{(i)}\left(t\right) = \boldsymbol{s}_N^{(i)}\left(t\right)\boldsymbol{c}^{(i)}$$

It is given with respect to the normalized time $\tau = \dfrac{t}{t_f} \in [0, 1]$ as

$$u^{(i)}\left(\tau\right) = \boldsymbol{s}_N^{(i)}\left(\tau\right)\boldsymbol{c}^{(i)}$$

The periodicity of the control is the same as the of the trig functions.

Trigonometric-polynomial Control Parameterization

$$u^{(i)}(\tau) = \boldsymbol{s}_N^{(i)}(\tau)\,\boldsymbol{c}^{(i)}, \quad \tau \in [0, 1]$$

$$\Downarrow$$

$$\overset{(n)}{\boldsymbol{u}}(0) = \overset{(n)}{\boldsymbol{u}}(1), \quad n \in \mathbb{N}^0$$

$$\Downarrow$$

The period of $\boldsymbol{s}_N^{(i)}(\tau)$ is 1.

Trigonometric-polynomial Control Parameterization

$$u^{(i)}(\tau) = (a_0)^{(i)} + \sum_{n=1}^{N} \left[(a_n)^{(i)} \cos(2\pi n\tau) + (b_n)^{(i)} \sin(2\pi n\tau) \right]$$

$$= \boldsymbol{s}_N{}^{(i)}(\tau)\, \boldsymbol{c}^{(i)}, \quad i = 1, 2, \ldots, q$$

$$\dot{\boldsymbol{u}}(t) = \frac{\mathrm{d}\boldsymbol{u}(\tau)}{\mathrm{d}\tau} \frac{\mathrm{d}\tau}{\mathrm{d}t} = \left[\frac{\mathrm{d}\boldsymbol{S}_N(\tau)}{\mathrm{d}\tau} \right] \frac{1}{t_f} \boldsymbol{c}$$

Also, $\boldsymbol{u}$ being a linear function of $\boldsymbol{c}$ leads that the incremental change of $\boldsymbol{u}$ and $\dot{\boldsymbol{u}}$ are linear to the increments of $\boldsymbol{c}$.

Application III: Guaranteed solution

Impact-time Control Problem

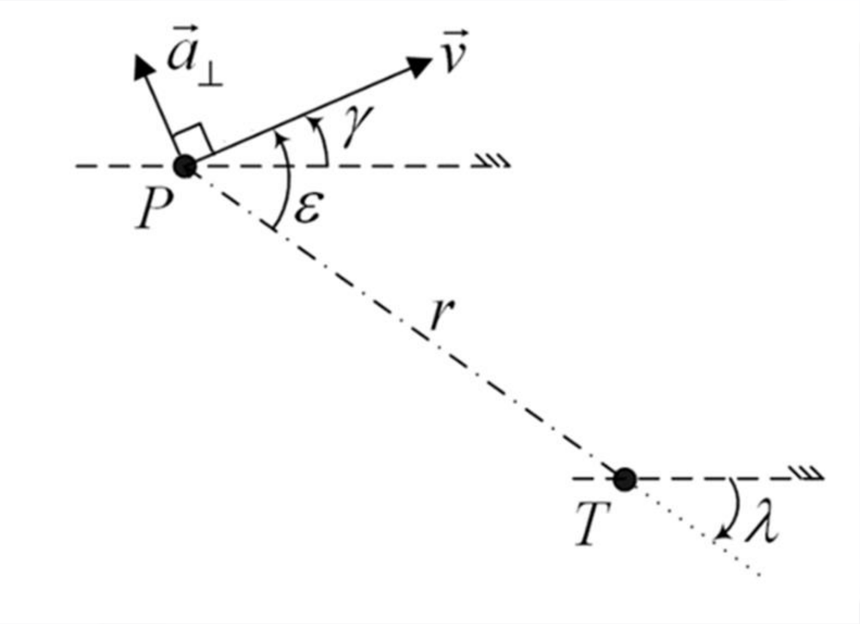
- A pursuer is required to reach the target at the desired impact time.
- Nonlinear Problem
- Constraints:

$$\varepsilon(0) = \varepsilon_i$$

$$\varepsilon(t_f) = \varepsilon_f = 0$$

$$r_i = V\int_0^{t_f} \cos\varepsilon(t)\mathrm{d}t$$

$$-|\varepsilon|_{\max} \leq \varepsilon(t) \leq |\varepsilon|_{\max}$$



Engagement geometry

Impact-time Control Problem

$$r_i = V \int_0^{t_f} \cos\varepsilon(t)\,\mathrm{d}t$$

To eliminate the nonlinearities, we parameterize

$$\cos\varepsilon(t) = \boldsymbol{s}_N(t)\,\boldsymbol{c}$$

Its definite integral is still a linear function of the coefficient vector:

$$\int_0^{t_f} \cos\varepsilon(t)\,\mathrm{d}t = \int_0^{t_f} \boldsymbol{s}_N(t)\,\boldsymbol{c}\,\mathrm{d}t = \left(\int_0^{t_f} \boldsymbol{s}_N(t)\,\mathrm{d}t\right)\boldsymbol{c} = \tilde{\boldsymbol{s}}_N\boldsymbol{c}$$

$$r_i = V \int_0^{t_f} \cos\varepsilon(t)\,\mathrm{d}t \quad \Rightarrow \quad \tilde{\boldsymbol{s}}_N\boldsymbol{c} = r_i/V$$

Impact-time Control Problem

All constraints are converted to functions of the coefficient vector

$$\varepsilon(0) = \varepsilon_i \quad \rightarrow \quad \boldsymbol{s}_N(0)\,\boldsymbol{c} = \cos\varepsilon_i$$

$$\varepsilon(t_f) = \varepsilon_f = 0 \quad \rightarrow \quad \boldsymbol{s}_N(t_f)\,\boldsymbol{c} = \cos\varepsilon_f$$

$$r_i = V\int_0^{t_f}\cos\varepsilon(t)\,\mathrm{d}t \quad \rightarrow \quad \tilde{\boldsymbol{s}}_N\boldsymbol{c} = r_i/V$$

$$-|\varepsilon|_{\max} \le \varepsilon(t) \le |\varepsilon|_{\max} \quad \rightarrow \quad \cos|\varepsilon|_{\max} \le \boldsymbol{s}_N(t)\,\boldsymbol{c} \le 1$$

Trigonometric-polynomial Control Parameterization

The i-th control component is expressed by

$$u^{(i)}(t) = a_0^{(i)} + \sum_{n=1}^{N} \left(a_n^{(i)} \cos\left(\frac{n\pi}{T_f} t\right) + b_n^{(i)} \sin\left(\frac{n\pi}{T_f} t\right) \right)$$

$$= \boldsymbol{s}_N^{(i)}(t)\, \boldsymbol{c}^{(i)}, \quad i = 1, 2, \ldots, m,$$

where

$$\boldsymbol{s}_N^{(i)}(t) = \left[1, \cos\left(\frac{\pi}{T_f} t\right), \ldots, \cos\left(\frac{N\pi}{T_f} t\right), \right.$$

$$\left. \sin\left(\frac{\pi}{T_f} t\right), \ldots, \sin\left(\frac{N\pi}{T_f} t\right) \right]$$

$$\boldsymbol{c}^{(i)} = \left[a_0^{(i)}, \ldots, a_N^{(i)}, b_1^{(i)}, \ldots, b_N^{(i)} \right]^{\mathrm{T}}.$$

Summary

- Infinite differentiability
- Linear with respect to the coefficients
- Given as function of time, or other independent variables
- Derivative / Integral

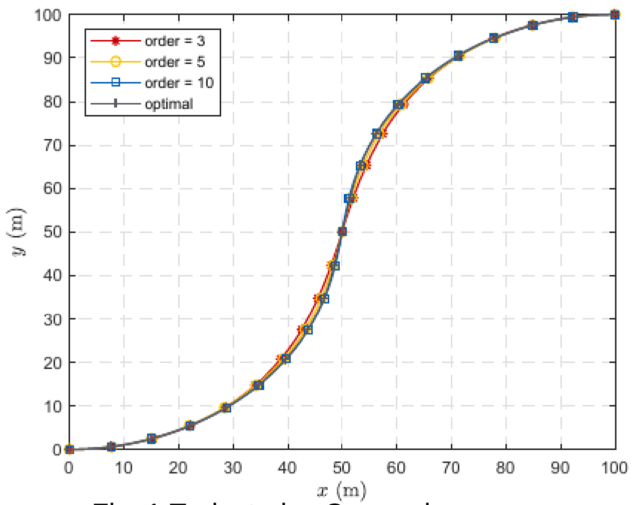


Fig. 1 Trajectories Comparison

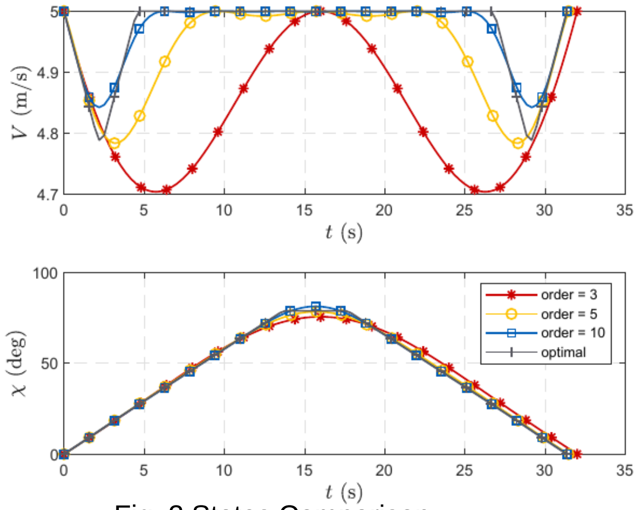


Fig. 2 States Comparison

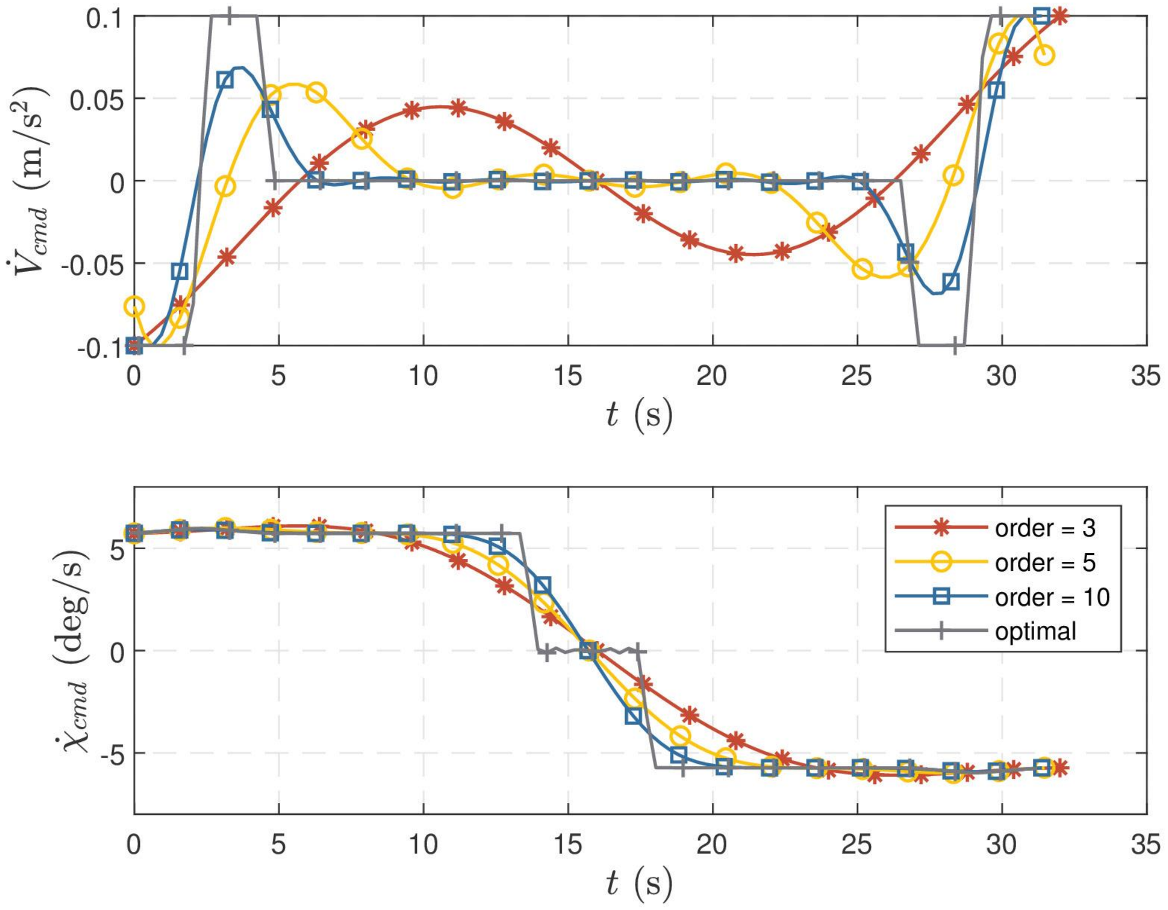


Fig. 3 Controls Comparison

Application IV: Seamless transition

Seamless Transition:

- A seamless transition satisfies the following two requirements:
  - The control values meet the steady flight (trim) conditions.
  - The control derivatives of all orders are zero.

Trigonometric-polynomial Control Parameterization

Writing the trigonometric series parameterization in an affine from as

$$u^{(i)}(t) = a_0^{(i)} + \hat{\boldsymbol{s}}_N^{(i)}(t)\,\hat{\boldsymbol{c}}^{(i)}$$

where

$$\begin{aligned}\hat{\boldsymbol{s}}_N^{(i)}(t) = \Bigg[&\cos\left(\frac{\pi}{T_f}t\right), \ldots, \cos\left(\frac{N\pi}{T_f}t\right) \\ &\sin\left(\frac{\pi}{T_f}t\right), \ldots, \sin\left(\frac{N\pi}{T_f}t\right) \Bigg]\end{aligned}$$

$$\hat{\boldsymbol{c}}^{(i)} = \left[a_1^{(i)}, \ldots, a_N^{(i)}, b_1^{(i)}, \ldots, b_N^{(i)}\right]^{\mathrm{T}}.$$

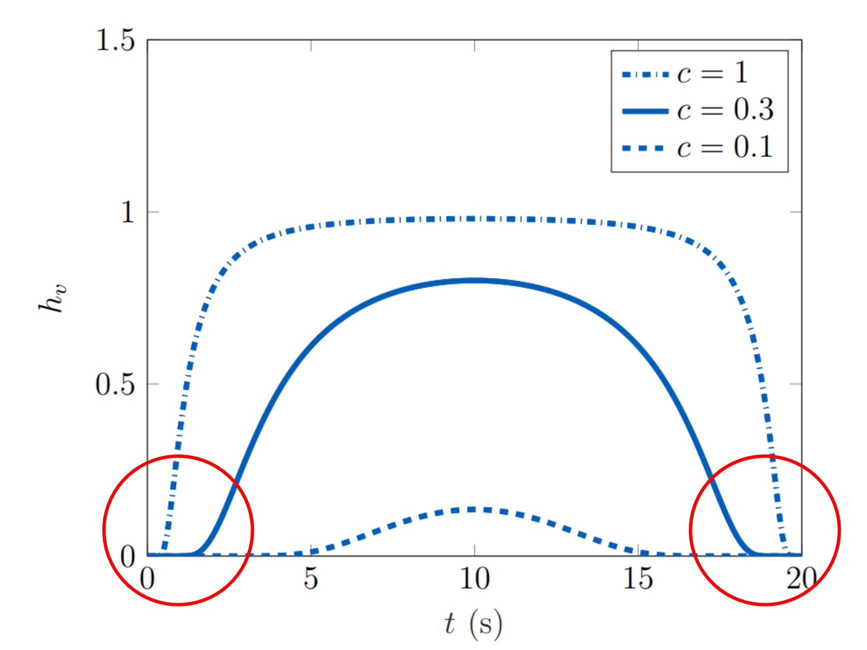
Trigonometric-polynomial Control Parameterization

The proposed hierarchical parameterization utilizes a flat function

A flat function is a smooth function, all of whose derivatives vanish at a given point.

$$h\left(t\right) = \exp\left(-1/(c \cdot t)^2 - 1/(c \cdot (t - t_f))^2\right)$$

$$h_v\left(t\right) = \begin{cases} 0 & t = 0 \\ \exp\left(-1/(c \cdot t)^2 - 1/(c \cdot (t - t_f))^2\right) & t \in (0, t_f) \\ 0 & t = t_f \end{cases}$$

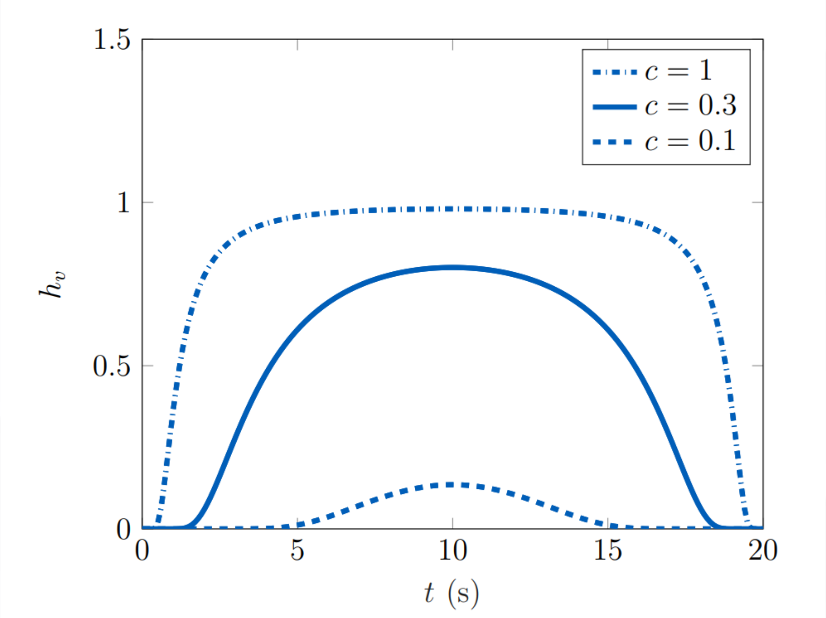


The hierarchical parameterization again parameterizes the trig series as

$$u^{(i)}(t) = a_0^{(i)} + \hat{\boldsymbol{s}}_N^{(i)}(t)\, \boldsymbol{H}_v^{(i)}(t)\, \tilde{\boldsymbol{c}}^{(i)} \,.$$

where

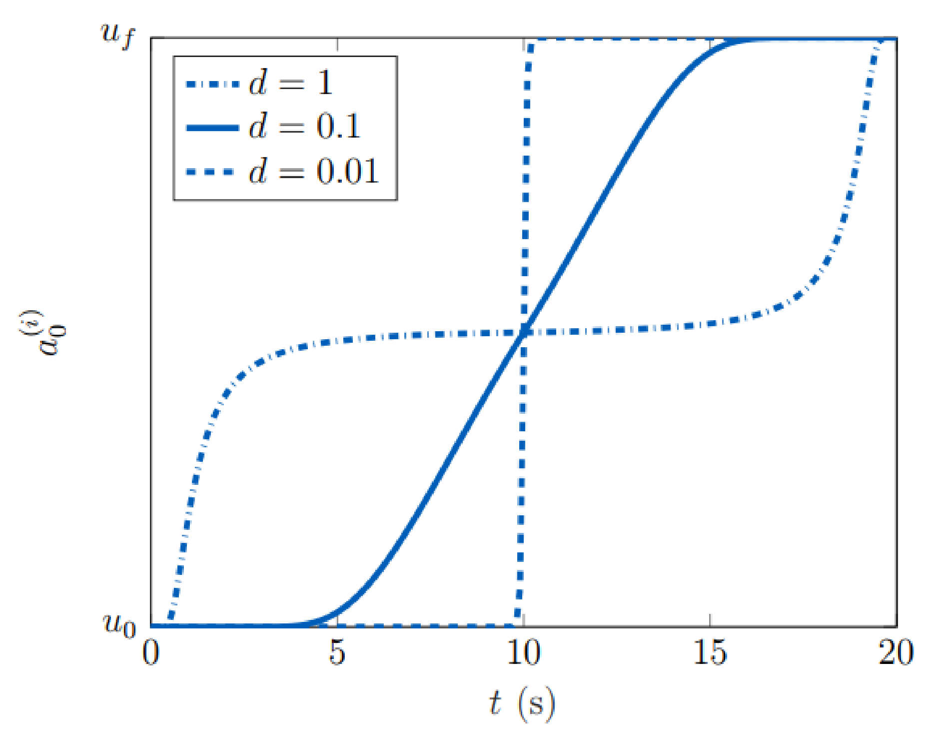
$$\boldsymbol{H}_v^{(i)}(t) = \begin{bmatrix} h_v(t) & 0 & \cdots & 0 \\ 0 & h_v(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_v(t) \end{bmatrix} .$$



Trigonometric-polynomial Control Parameterization

Therefore, a_0 also needs to be parameterized.

$$a_0^{(i)}(t) = \begin{cases} u_0^{(i)} & t = 0 \\ \frac{(u_f^{(i)} - u_0^{(i)})/2}{\exp(-1/(d_1 t_f/2)^2)} \exp\left(\frac{-1}{(d_1 t)^2}\right) + u_0^{(i)} & t \in (0, t_f/2] \\ -\frac{(u_f^{(i)} - u_0^{(i)})/2}{\exp(-1/(d_1 t_f/2)^2)} \exp\left(\frac{-1}{(d_1(t-t_f))^2}\right) + u_f^{(i)} & t \in (t_f/2, t_f) \\ u_f^{(i)} & t = t_f \end{cases}$$



The hierarchical parameterization is summarized as

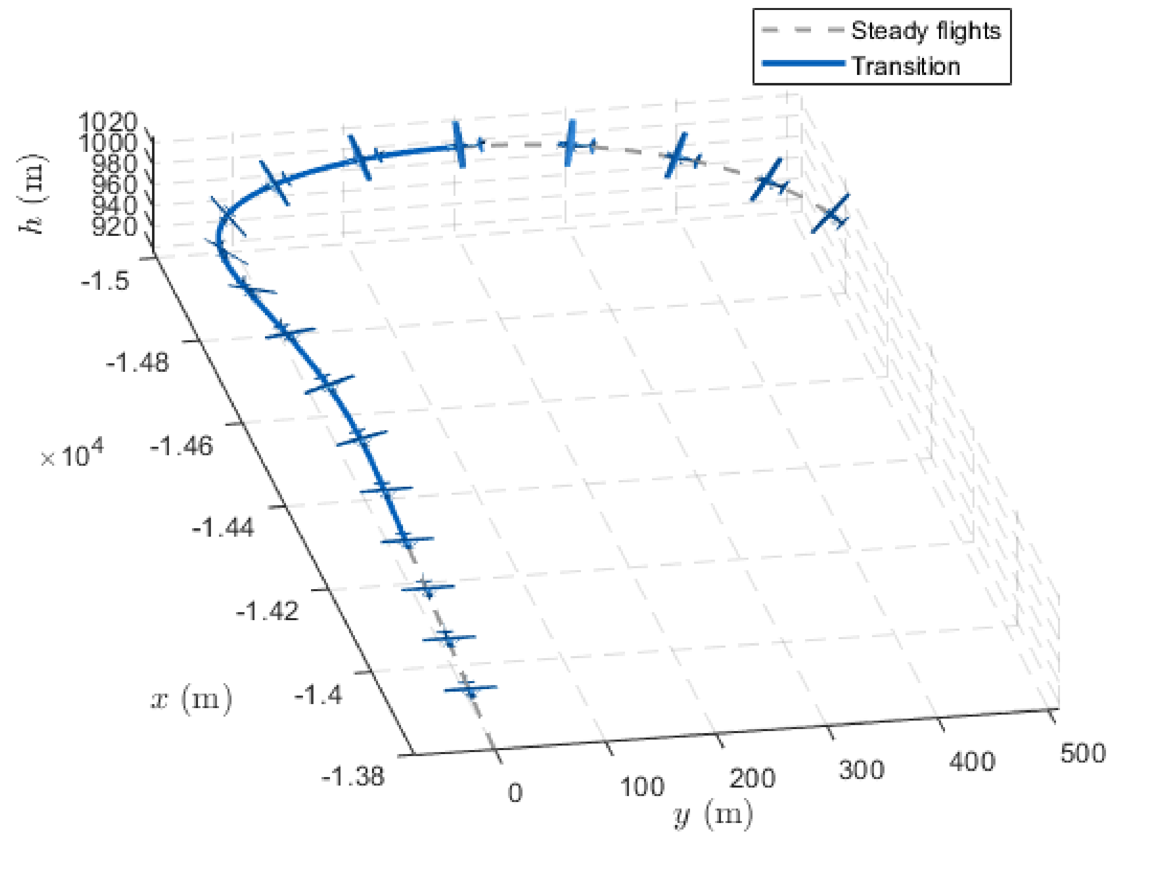
$$u^{(i)}(t) = a_0^{(i)}(t) + \hat{\boldsymbol{s}}_N^{(i)}(t)\,\boldsymbol{H}_v^{(i)}(t)\,\tilde{\boldsymbol{c}}^{(i)}$$

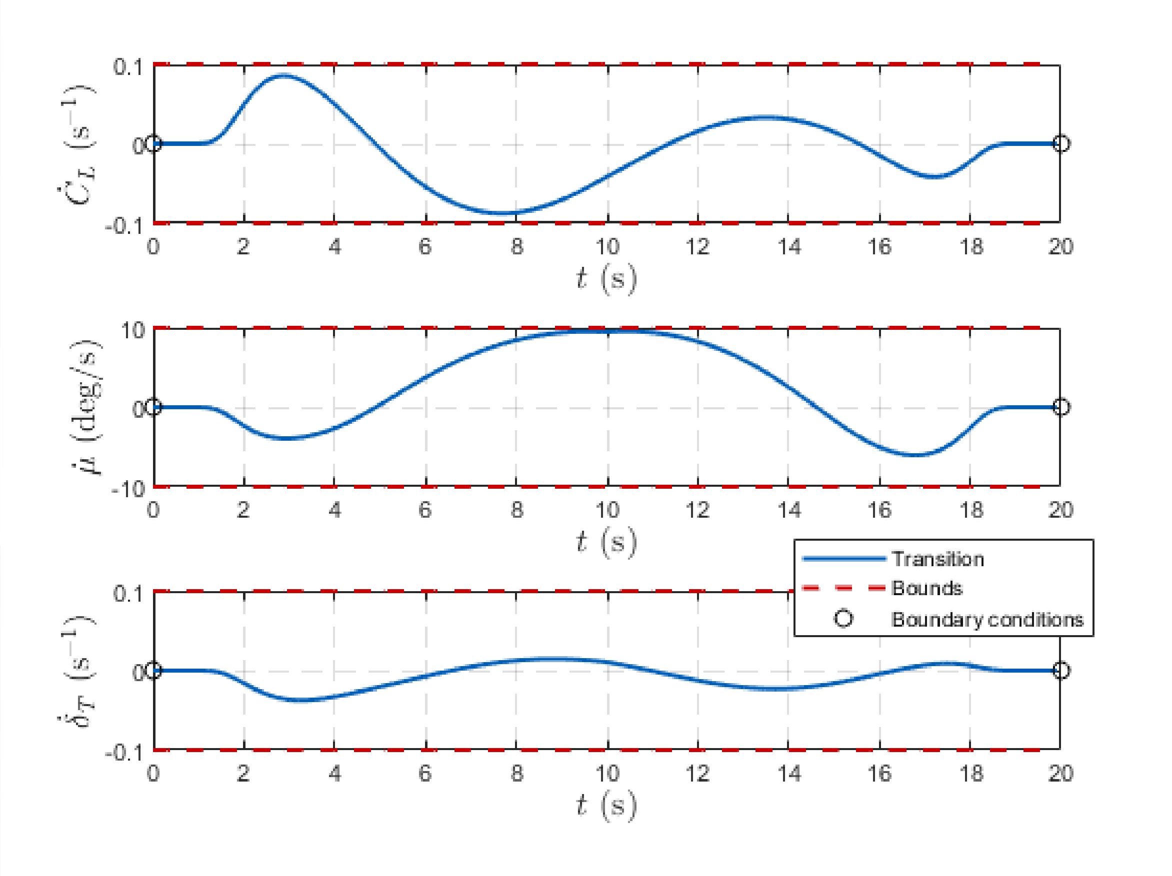
By construction, the following conditions are satisfied:

$$u^{(i)}(0) = u_0^{(i)}$$

$$u^{(i)}(t_f) = u_f^{(i)}$$

$$\overset{(r)}{u}{}^{(i)}(0) = \overset{(r)}{u}{}^{(i)}(t_f) = 0, \quad r \in \mathbb{N}^+ .$$


Steady flights
Transition
1020
1000
980
960
940
920
h (m)
-1.5
-1.48
-1.46
-1.44
-1.42
-1.4
-1.38
$\times 10^4$
x (m)
0
100
200
300
400
500
y (m)


$\dot{C}_L$ (s^{-1})
0.1
0
-0.1
$\dot{\mu}$ (deg/s)
10
0
-10
$\dot{\delta}_T$ (s^{-1})
0.1
0
-0.1
0 2 4 6 8 10 12 14 16 18 20
t (s)
Transition
Bounds
Boundary conditions

- Control parameterization is a simple and effective way to improve the smoothness in trajectory generation.
- Elegant mathematical properties of functions are found helpful in addressing real-world problems.

Solve any computational guidance or trajectory optimization problem

- Group of max. 4 people
- Presentation of 20 to 30 minutes
- Starting the 15th or the 16th week till the 17th week
- Let me know your group member by the end of next Thursday

Evaluation:

- Technical details 50%
- Presentation performance 50%