Graduate Numerical Analysis Course Project

Zifei Yin, Shuyue Lai, Hui Xu

December 17, 2022

Q. 1 (Numerical Integration) Velocity data for air are collected at different radii from the centerline of a circular 16-cm-diameter pipe as tabulated below: Use numerical integration to determine the mass

r, cm	0	1.00	2.00	3.00	4.00	5.00	6.00	7.00	8.00
V, m/s	10	9.80	9.60	9.30	9.06	8.68	8.18	7.41	0

Table 1: Sampled velocity data.

flow rate, which can be computed as

$$\int_0^R \rho v 2\pi r dr$$

where density $\rho = 1.2kg/m^3$. Express your results in kg/s.

(a) Use Composite Trapezoidal rule;

(b) Use Composite Simpson's rule.

(c) Select a method of your choice that you think is more accurate than the two methods above.

Q. 2 (ODE Initial value problem) As shown in Fig. 1, we have a cubic box containing turbulent flow. The kinetic energy per unit volume is k (dimension: m^2/s^2), and the dissipation rate of kinetic energy is ϵ (dimension m^2/s^3). The governing equations are

$$\frac{\partial k}{\partial t} = -\epsilon \quad (1)$$

$$\frac{\partial \epsilon}{\partial t} = -C\frac{\epsilon^2}{k} \quad (2)$$

where $C = 1.83$. At $t_0 = 1\ s$, $k = 1.0\ m^2/s^2$ and $\epsilon = 0.2176\ m^2/s^3$. Use the following numerical method to solve the ODE equation set and predict the kinetic energy k at $t = 5.0\ s$. Comment on the convergence and accuracy of those methods.

(a) Use the Euler method;

(b) Use the modified Euler method;

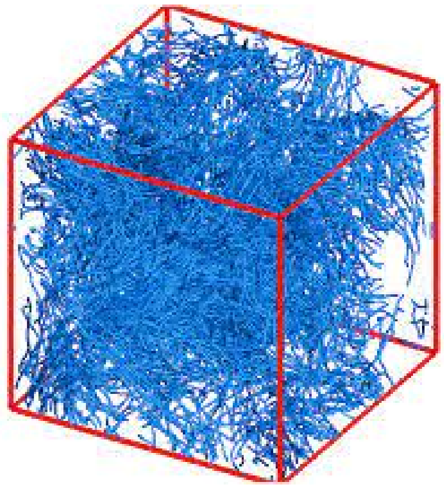

Figure 1: Homogeneous isotropic turbulence in a box

(c) Use the 4-th order Runge-Kutta method.

Q. 3 (Non-linear equations): In a turbulent boundary layer, the time-averaged velocity profile and the wall distance can be written in the following non-dimensional form

$$y_+ = U_+ + e^{-\kappa B}\left[e^{\kappa U_+} - 1 - \kappa U_+ - \frac{1}{2!}(\kappa U_+)^2 - \frac{1}{3!}(\kappa U_+)^3 - \frac{1}{4!}(\kappa U_+)^4\right] \quad (3)$$

where

$$y_+ = \frac{u_\tau y}{\nu}, \qquad U_+ = \frac{U}{u_\tau}, \quad (4)$$

$$\kappa = 0.41, \qquad B = 5.1 \quad (5)$$

In a wind tunnel experiment, we know the air viscosity $\nu = 1.5 \times 10^{-5}\ m^2/s$ and air density $\rho = 1.25\ kg/m^3$. At a certain streamwise location, someone measured that at a wall distance of $y = 0.01$ m, the velocity is 21 m/s. Given that the relation between u_τ and the wall shear stress τ_{wall} can be expressed as

$$u_\tau = \sqrt{\frac{\tau_{wall}}{\rho}} \quad (6)$$

Please try to solve the wall shear stress (τ_{wall}) corresponding to the measurement data within a tolerance of $\epsilon = 10^{-3}$.

(a) Use bisection method;

(b) Use Newton's method;

(c) Try to find a fixed point iteration formula that converges.

(Hint: the first step is to write out a non-linear equation of $f(U_+) = 0$ and solve it using the methods we learned. And then use U_+ to compute u_τ and τ_{wall}.)

Q. 4 (Interpolation) Figure 2 shows a ruddy duck in flight. To predict the flight dynamics of this duck, we need to create the geometry for computer simulations. To approximate the top profile of the duck, we have chosen points along the curve through which we want the approximating curve to pass. Table 2 lists the coordinates of 21 data points relative to the superimposed coordinate system shown in Figure 2. Notice that more points are used when the curve is changing rapidly than when it is changing more slowly. Create interpolation polynomials using the following methods and make plots to show the feasibility of each method. (Hint: matplotlib library in python can help you plot the curves.)

(a) Use Lagrange interpolating polynomial;

(b) Use natural spline;

(c) Comment on the feasibility of the above methods.

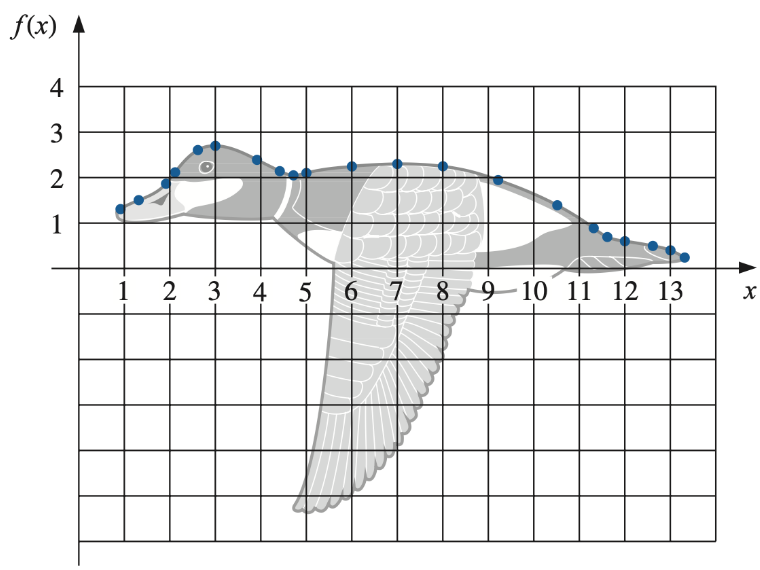

Figure 2: Ruddy duck

x	0.9	1.3	1.9	2.1	2.6	3.0	3.9	4.4	4.7	5.0	6.0	7.0	8.0
$f(x)$	1.3	1.5	1.85	2.1	2.6	2.7	2.4	2.15	2.05	2.1	2.25	2.3	2.25
x	9.2	10.5	11.3	11.6	12.0	12.6	13.0	13.3					
$f(x)$	1.95	1.4	0.9	0.7	0.6	0.5	0.4	0.25					

Table 2: The coordinates of the duck top profile (inch).

Q. 5 (Linear Equations): For the same problem of Q. 4, use the naïve approach of polynomial interpolation to form a linear system. Use the following methods to solve the linear equation.

(a) Use Gauss-Elimination;

(b) Use LU decomposition;

(c) Gauss-Seidel method, and comment on the convergence.

(d) SOR method (select your own ω), and comment on the convergence.

Q. 6 (Eigenvalue): For a dynamic system, eigenvalues can be used to determine whether a fixed point (also known as an equilibrium point) is stable or unstable. When all eigenvalues are real, positive, and distinct, the system is unstable; When all eigenvalues are real, negative, and distinct, the system is stable; If the set of eigenvalues for the system has both positive and negative eigenvalues, the fixed point is an unstable saddle point. Here is a matrix that corresponds with a system at a fixed point.

(a) Use power iteration method to determine the main eigenvalue;

(b) Use Rayleigh Quotient method to determine the main eigenvalue and comment on the convergence rate.

(c) Use QR decomposition to determine how many eigenvalues are positive and how many are negative.

$$A = \begin{pmatrix} 52 & 30 & 49 & 28 \\ 30 & 50 & 8 & 44 \\ 49 & 8 & 46 & 16 \\ 28 & 44 & 16 & 22 \end{pmatrix}$$