

Chapter 2

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Exercise 2.1

Given the following 4 data points

$$\{(0, 1), (1, 2), (3, 6), (5, 7)\}$$

find a polynomial in Lagrange form to interpolate these data

we know that the formula for Lagrange interpolation is:

$$L_{n,k}(x) = \prod_{i=0, i \neq k}^n \frac{x - x_i}{x_k - x_i}$$

so the Lagrange polynomial for the 4 points should be:

$$\begin{aligned}\mathcal{L}_0 &= -\frac{-15 + 23x - 9x^2 + x^3}{15} \\ \mathcal{L}_1 &= \frac{15x - 8x^2 + x^3}{4} \\ \mathcal{L}_2 &= -\frac{5x - 6x^2 + x^3}{2} \\ \mathcal{L}_3 &= \frac{7(3x - 4x^2 + x^3)}{40}\end{aligned}$$

so the final polynomial should be sum of the 4 equations:

$$\mathcal{L} = 1 + \frac{29x}{120} + \frac{9x^2}{10} - \frac{17x^3}{120}$$

Exercise 2.5

Consider a function $f(x) = \sin(\pi x) + 3x$ at 6 distinct nodes in the interval $[-1, 1]$ to determine the Newton's divided difference formula. The data are given in Table below:

x	-1.0	-0.6	-0.2	0.2	0.6	1.0
$f(x)$	-3.0	-2.7511	-1.1878	1.1878	2.7511	3.0

the whole coefficient matrix can be presented below:

$$\begin{bmatrix} p=0 & p=1 & p=2 & p=3 & p=4 & p=5 \\ -3 & & & & & \\ -2.7511 & 0.62225 & & & & \\ -1.1878 & 3.90825 & 4.1075 & & & \\ 1.1878 & 5.939 & 2.5384375 & -1.30755208 & & \\ 2.7511 & 3.90825 & -2.5384375 & -4.23072917 & -1.82698568 & \\ 3 & 0.62225 & -4.1075 & -1.30755208 & 1.82698568 & 1.82698568 \end{bmatrix}$$

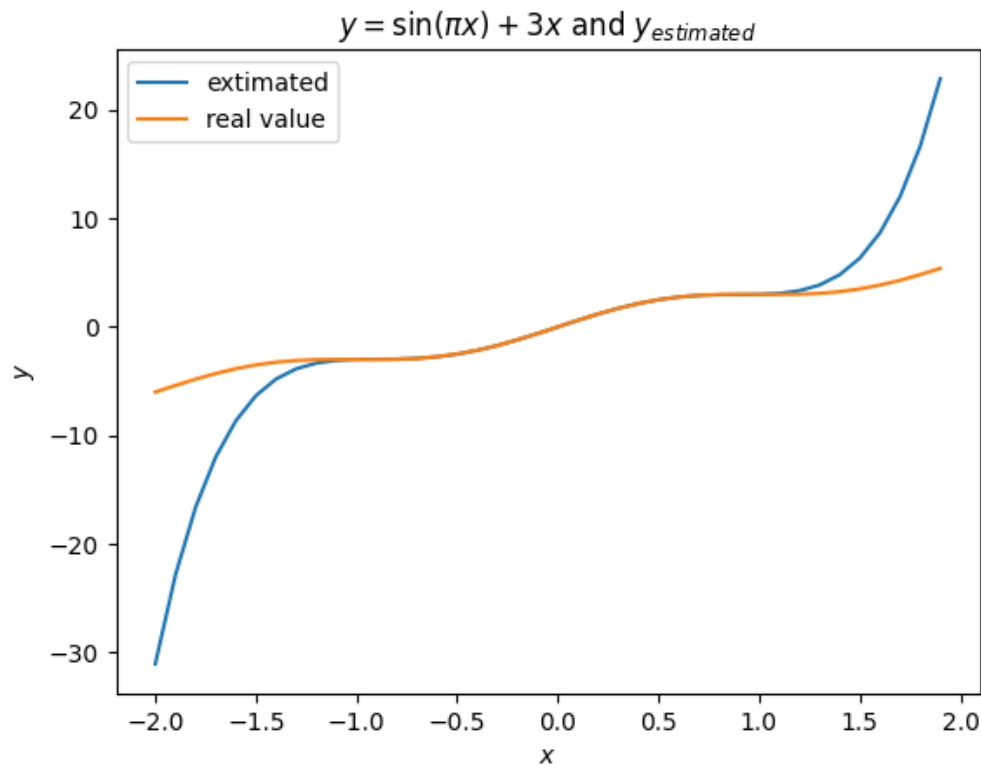
according to the Newton's divided difference formula,

$$P_n(x) = F_{0,0} + \sum_{i=1}^n F_i \prod_{j=0}^{i-1} (x - x_j)$$

so the polynomial should be:

$$\begin{aligned}
 P(x) = & -3 \\
 & + 0.62225(x + 1.0) \\
 & + 4.1075(x + 1.0)(x + 0.6) \\
 & - 1.30755208(x + 1.0)(x + 0.6)(x + 0.2) \\
 & - 1.82698568(x + 1.0)(x + 0.6)(x + 0.2)(x - 0.2) \\
 & + 1.82698568(x + 1.0)(x + 0.6)(x + 0.2)(x - 0.2)(x - 0.6)
 \end{aligned}$$

compare the polynomial we get and the real function $f(x) = \sin(\pi x) + 3x$



we can see that **inside** the interval $[-1, 1]$, the polynomial performs well, but the error will exceed greatly **beyond** the interval.

Exercise 2.6

Construct an approximating polynomial for the following data using Hermite interpolation with Newton's forward difference

Table 1:

x	0.1	0.2	0.3
$f(x)$	-0.29004996	-0.56079734	-0.81401972
$f'(x)$	-2.8019975	-2.6159201	-2.9734038

Table 2:

x	-1.0	-0.5	0.0	0.5
$f(x)$	0.86199480	0.95802009	1.0986123	1.2943767
$f'(x)$	0.15536240	0.23269654	0.33333333	0.45186776

for data in Table 1:

$z_0 = 0.1$	$f[z_0] = -0.29004996$	-0.29004996					
$z_1 = 0.1$	$f[z_1] = -0.29004996$	-0.29004996	-2.8019975				
$z_2 = 0.2$	$f[z_2] = -0.56079734$	-0.56079734	-2.7074738	0.945237			
$z_3 = 0.2$	$f[z_3] = -0.56079734$	-0.56079734	-2.6159201	0.915537	-0.297		
$z_4 = 0.3$	$f[z_4] = -0.81401972$	-0.81401972	-2.5322238	0.836963	-0.39287	-0.47935	
$z_5 = 0.3$	$f[z_5] = -0.81401972$	-0.81401972	-2.9734038	-4.4118	-52.48763	-260.4738	-1299.97225

for data in Table 2:

0.8619948							
0.8619948	0.1553624						
0.95802009	0.19205058	0.07337636					
0.95802009	0.23269654	0.08129192	0.01583112				
1.0986123	0.28118442	0.09697576	0.01568384	-0.00014728			
1.0986123	0.33333333	0.10429782	0.01464412	-0.00103972	-0.00089244		
1.2943767	0.3915288	0.11639094	0.01209312	-0.002551	-0.00100752	-0.00007672	
1.2943767	0.45186776	0.12067792	0.00857396	-0.00351916	-0.00096816	0.00002624	0.00006864

so the polynomial 1 should be:

$$\begin{aligned}
 P(x) = & -0.29004996 \\
 & -2.8019975(x - 0.1) \\
 & + 0.945237(x - 0.1)^2 \\
 & - 0.297(x - 0.1)^2(x - 0.2) \\
 & - 0.47935(x - 0.1)^2(x - 0.2)^2 \\
 & - 1299.97225(x - 0.1)^2(x - 0.2)^2(x - 0.3)
 \end{aligned}$$

the polynomial 2 should be:

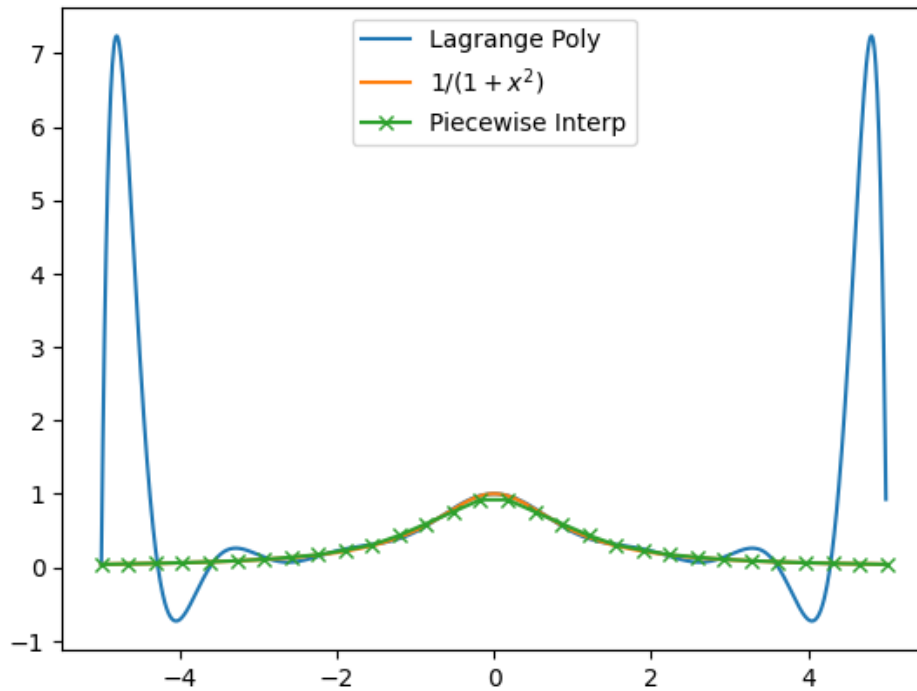
$$\begin{aligned}
 P(x) = & 0.8619948 \\
 & + 0.1553624(x + 1) \\
 & + 0.07337636(x + 1)^2 \\
 & + 0.01583112(x + 1)^2(x + 0.5) \\
 & - 0.00014728(x + 1)^2(x + 0.5)^2 \\
 & - 0.00089224(x + 1)^2(x + 0.5)^2x \\
 & - 0.00007672(x + 1)^2(x + 0.5)^2x^2 \\
 & + 0.00006864(x + 1)^2(x + 0.5)^2x^2(x - 0.5)
 \end{aligned}$$

Exercise 2.7

Choose equally-spaced 15 points in $[-5, 5]$ and get the data points

$$\{x_i, f(x_i)\}_{i=0}^{14}$$

by $f(x) = 1/(1 + x^2)$. Using piecewise linear interpolation to get the approximation of $f(x)$ in $[-5, 5]$ and, compare the result with that obtained by Lagrange interpolation.



It can be seen that Lagrange interpolation keeps a good tracing trend in the center area; Piecewise interpolation keeps a good trending in all area;

Exercise 2.8

Find the natural cubic spline through $(0, 3)$, $(1, -2)$, $(2, 1)$

for 3 points, there are 2 cubics

case 1: for $x \in [0, 1)$

$$\begin{aligned} S_0(x) &= a_0 + b_0(x - 0) + c_0(x - 0)^2 + d_0(x - 0)^3 \\ &= a_0 + b_0x + c_0x^2 + d_0x^3 \end{aligned}$$

case 2: for $x \in [1, 2]$

$$S_1(x) = a_1 + b_1(x - 1) + c_1(x - 1)^2 + d_1(x - 1)^3$$

for several conditions:

$$\begin{aligned} S_0(0) &= f(0) = 3 \\ S_1(1) &= S_0(1) = f(1) = -2 \\ S'_0(1) &= S'_1(1) \\ S''_0(1) &= S''_1(1) \\ S''_0(0) &= 0 \\ S''_1(2) &= 0 \end{aligned}$$

we can get that

$$\begin{cases} a_0 = 3 \\ b_0 = -7 \\ c_0 = 0 \\ d_0 = 2 \\ a_1 = -2 \\ b_1 = -1 \\ c_1 = 6 \\ d_1 = -2 \end{cases}$$

so the equations by cubic spline should be:

$$S(x) = \begin{cases} 3 - 7x + 2x^3 & x \in [0, 1] \\ -2 - (x - 1) + 6(x - 1)^2 - 2(x - 1)^3 & x \in [1, 2] \end{cases}$$

Exercise 2.9

Decide whether or not the equations given below from a cubic spline

$$(a) \quad S(x) = \begin{cases} x^3 + x - 1 & \text{on } [0,1] \\ -(x-1)^3 + 3(x-1)^2 + 3(x-1) + 1 & \text{on } [1,2] \end{cases}$$

$$(b) \quad S(x) = \begin{cases} 2x^3 + x^2 + 4x + 5 & \text{on } [0,1] \\ (x-1)^3 + 7(x-1)^2 + 12(x-1) + 12 & \text{on } [1,2] \end{cases}$$

for equation **a**:

- $S_0(x) = x^3 + x - 1$
- $S_1(x) = -(x-1)^3 + 3(x-1)^2 + 3(x-1) + 1$

test all the conditions,

$$S_0(1) = 1, S_1(1) = 1, S_0(1) = S_1(1)$$

$$S'_0(1) = 4, S'_1(1) = 3, S'_0(1) \neq S'_1(1)$$

so equation **a** is not from a cubic spline.

for equation **b**:

- $S_0(x) = 2x^3 + x^2 + 4x + 5$
- $S_1(x) = (x-1)^3 + 7(x-1)^2 + 12(x-1) + 12$

test all the conditions,

$$S_0(1) = 12, S_1(1) = 12, S_0(1) = S_1(1)$$

$$S'_0(1) = 12, S'_1(1) = 12, S'_0(1) = S'_1(1)$$

$$S''_0(1) = 14, S''_1(1) = 14, S''_0(1) = S''_1(1)$$

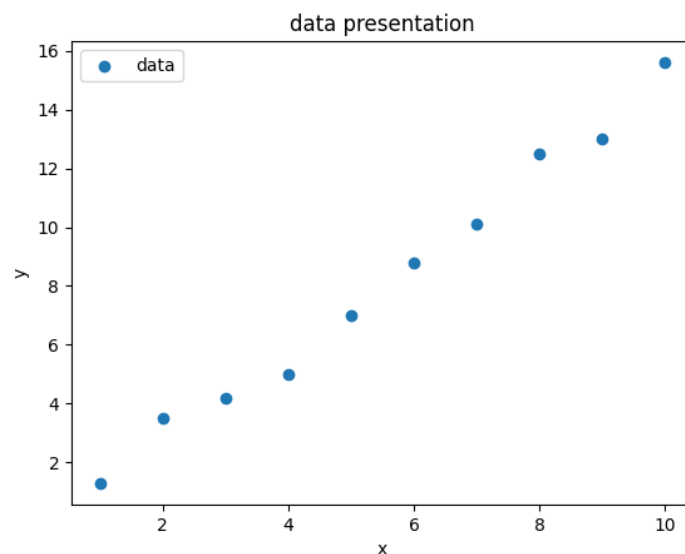
so equation **b** is from a cubic spline.

Exercise 2.12

Find the least squares line approximating the data given in Table below:

x	1	2	3	4	5	6	7	8	9	10
$f(x)$	1.3	3.5	4.2	5.0	7.0	8.8	10.1	12.5	13.0	15.6

first draw the scatter plot of points to choose model



It can be seen that these points are basically linear, according to the least square method

$$a_0 = \frac{\sum_{i=0}^m x_i^2 \sum_{i=0}^m y_i^2 - \sum_{i=0}^m x_i y_i \sum_{i=0}^m x_i}{m(\sum_{i=0}^m x_i^2) - (\sum_{i=0}^m x_i)^2}$$

$$a_1 = \frac{m \sum_{i=0}^m x_i y_i - \sum_{i=0}^m x_i \sum_{i=0}^m y_i}{m(\sum_{i=0}^m x_i^2) - (\sum_{i=0}^m x_i)^2}$$

after calculating

$$a_0 = \frac{-297}{825} = -0.36$$

$$a_1 = \frac{1269}{825} = 1.5382$$

so the line should be:

$$y = -0.36 + 1.5382x$$

Exercise 2.13

Find the discrete least square polynomial of degree at most 2 with the data given in Table below:

x	0	0.25	0.5	0.75	1.0
$f(x)$	1.0000	1.2840	1.6487	2.1170	2.7183

after some data process

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

according to matrix theory

$$Y = Xa$$

$$a = (X^T X)^{-1} X^T Y$$

for $n = 1$:

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 0.25 \\ 1 & 0.5 \\ 1 & 0.75 \\ 1 & 1 \end{bmatrix}, \quad a = \begin{bmatrix} 0.89968 \\ 1.70784 \end{bmatrix}$$

so the equation should be

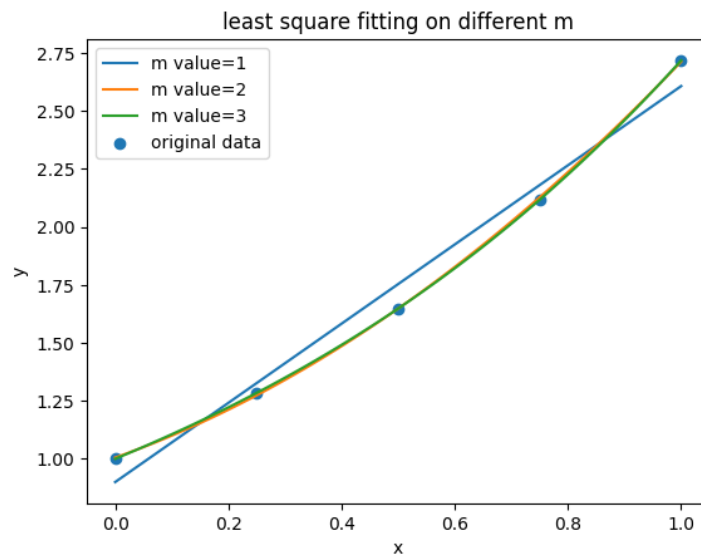
$$y = 0.88968 + 1.70784x$$

for $n = 2$:

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0.25 & 0.0625 \\ 1 & 0.5 & 0.25 \\ 1 & 0.75 & 0.5625 \\ 1 & 1 & 1 \end{bmatrix}, \quad a = \begin{bmatrix} 1.0051 \\ 0.8642 \\ 0.8437 \end{bmatrix}$$

so the equation should be

$$y = 1.0051 + 0.8642x + 0.8437x^2$$



It can be seen that the curve becomes more precise with higher m , but over-fitting is a problem

Exercise 2.14

Find linear and quadratic least-squares approximations to $f(x) = e^x$ on $(-1, 1)$ by using monomial polynomials(or naive polynomials)

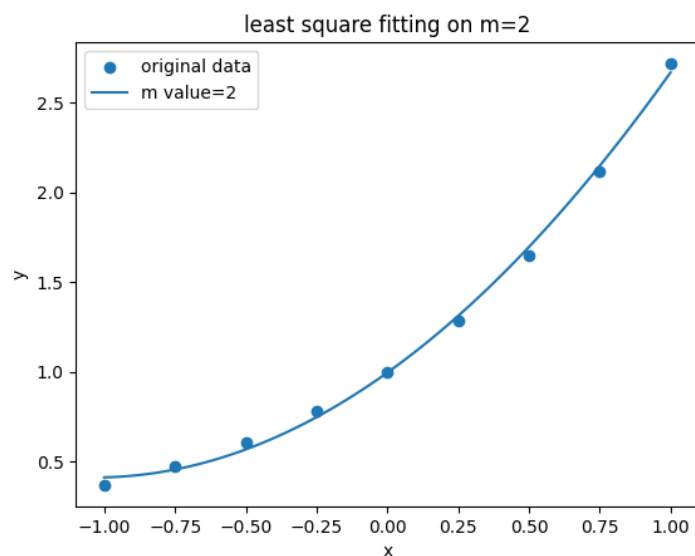
for quadratic approximation:

using 9 points from -1 to 1, and the matrix X will be

$$X = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -0.75 & 0.5625 \\ 1 & -0.5 & 0.25 \\ 1 & -0.25 & 0.0625 \\ 1 & 0 & 0 \\ 1 & 0.25 & 0.0625 \\ 1 & 0.5 & 0.25 \\ 1 & 0.75 & 0.5625 \\ 1 & 1 & 1 \end{bmatrix}$$

the equation should be:

$$y = 0.9948 + 1.1283x + 0.5442x^2$$



for linear approximation:

the matrix X will be

$$X = \begin{bmatrix} 1 & -1 \\ 1 & -0.75 \\ 1 & -0.5 \\ 1 & -0.25 \\ 1 & 0 \\ 1 & 0.25 \\ 1 & 0.5 \\ 1 & 0.75 \\ 1 & 1 \end{bmatrix}$$

the equation should be:

$$y = 1.2215 + 1.1283x$$

