CourseProject

January 13, 2023

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0.1 Python Classes

1.1 Least square

[1]:
```
class LeastSquare:
    # constructor function maybe
    def __init__(self, points):
        # points should be a numpy array
        self.points = points
        self.x = points[:, 0]
        self.y = points[:, 1]
        # or len(points) here
        self.n = points.shape[0]
        self.m_max = self.n-2  # m < n-1

    # set m_max manually
    def set_m_max(self, max):
        self.m_max = max

    # fitting function
    # return a function
    def get_fitting_function(self, m):
        # check the m
        if m < 1:
            raise Exception("Invaild m<1, please check the m you choose!")

        A = np.ones((self.n, 1))

        for i in range(1, m+1):
            A_plus = np.power(self.x, i).reshape(self.n, 1)
            A = np.hstack((A, A_plus))

        # print(A)

        Y = self.y.reshape(self.n, 1)
```

```
        coef = np.linalg.inv(np.matmul(A.transpose(), A))
        coef = np.matmul(coef, A.transpose())
        coef = np.matmul(coef, Y)
        coef = coef.flatten()

        return coef

    def coef2function_value(self, coef, x):
        func = np.poly1d(coef)
        return func(x)

    def run(self):
        x_new = np.linspace(points[0, 0], points[self.n-1, 0], num=100)
        m_range = np.arange(1, self.m_max+1)  # from 1 to m_max, have to add 1␣
↪here

        # define labes list for matplotlib.pyplot
        m_values =  ["m value={}".format(m) for m in m_range]

        # from 1 to m_max run the fitting process
        for i in range(1, self.m_max + 1):
            coef = self.get_fitting_function(i)

            y_estimated = self.coef2function_value(np.flip(coef), x_new)
            plt.plot(x_new, y_estimated, label=m_values[i-1])

        plt.scatter(self.x, self.y, label="original data")

        plt.xlabel("x")
        plt.ylabel("y")
        plt.title("least square fitting on different m")
        plt.legend()
        plt.show()

    def fit_determined_m(self, m):
        x_new = np.linspace(points[0, 0], points[self.n-1, 0], num=100)
        coef = self.get_fitting_function(m)

        y_estimated = self.coef2function_value(np.flip(coef), x_new)

        plt.scatter(self.x, self.y, label="original data")
        plt.plot(x_new, y_estimated, label="m value=%d" %(m))

        plt.xlabel("x")
        plt.ylabel("y")
        plt.title("least square fitting on m=%d" %(m))
        plt.legend()
```

```
        plt.show()

    def get_function_m(self, m):
        coef = self.get_fitting_function(m)
        return coef
```

1.2 Numerical integration

[2]:
```
class NumericalIntegration:
    def __init__(self, function):
        self.f = function

    def closed_newton_cotes(self, x_start, x_end):
        # n from 1 to 4
        # integration from x_start to x_end

        # results
        final = []

        # Trapezoidal rule, n = 1
        final1 = (x_end - x_start) / 2 * (self.f(x_end) + self.f(x_start))
        final.append(final1)

        # Simpson's rule, n = 2
        a = x_start
        b = x_end
        h = (b - a) / 2
        final2 = h / 3 * (self.f(a) + 4 * self.f((a + b) / 2) + self.f(b))
        final.append(final2)

        # Simpson's Three Eight rule, n = 3
        h = (b - a) / 3
        final3 = 3 * h / 8 * (self.f(a) + 3 * self.f(a + h) + 3 * self.f(b - h)
↪+ self.f(b))
        final.append(final3)

        # Rank 4, n = 4
        h = (b - a) / 4
        final4 = 2 * h / 45 * (
                7 * self.f(a) + 32 * self.f(a + h) + 12 * self.f(a + 2 * h) +
↪32 * self.f(b - h) + 7 * self.f(b))
        final.append(final4)

        print(final)

    def open_newton_cotes(self, x_start, x_end):
        # n from 0 to 3
```

```
        a = x_start
        b = x_end
        final = []

        # n = 0, Midpoint rule
        n = 0
        h = (b - a) / (n + 2)
        final1 = 2 * h * self.f(a + h)
        final.append(final1)

        # n = 1
        n = 1
        h = (b - a) / (n + 2)
        final2 = 3 * h / 2 * (self.f(a + h) + self.f(b - h))
        final.append(final2)

        # n = 2
        n = 2
        h = (b - a) / (n + 2)
        final3 = 4 * h / 3 * (2 * self.f(a + h) - self.f(a + 2 * h) + 2 * self.
↪f(b - h))
        final.append(final3)

        # n = 3
        n = 3
        h = (b - a) / (n + 2)
        final4 = 5 * h / 24 * (11 * self.f(a + h) + self.f(a + 2 * h) + self.
↪f(b - 2 * h) + 11 * self.f(b - h))
        final.append(final4)

        print(final)

    def composite_newton_cotes(self, x_start, x_end, n):
        a = x_start
        b = x_end
        final = []

        # for Trapezoidal rule
        h = (b - a) / n
        sum = 0
        for i in range(1, n - 1 + 1, 1):
            # range(1, n+1) is from 1 to n
            sum += 2 * self.f(a + i * h)
        sum = sum + self.f(a) + self.f(b)
        final1 = h / 2 * sum
        final.append(final1)
```

```
        # for Simpson's rule
        if n % 2 != 0:
            raise Exception('n is not an even number, not for composite␣
↪Simpsons rule')

        h = (b - a) / n
        xi0 = self.f(a) + self.f(b)
        xi1 = 0
        xi2 = 0
        for j in range(1, int(n / 2 - 1) + 1, 1):
            xi1 += self.f(a + 2 * j * h)
        for j in range(1, int(n / 2) + 1, 1):
            xi2 += self.f(a + (2 * j - 1) * h)

        final2 = h / 3 * (xi0 + 2 * xi1 + 4 * xi2)
        final.append(final2)

        print(final)

    def romberg(self, x_start, x_end, n):
        # Romberg Integration
        romberg_matrix = np.zeros((n, n))

        # init parameters
        a = x_start
        b = x_end
        h = b - a
        temp_sum = 0

        # R11 should be defined before circle
        r11 = h / 2 * (self.f(a) + self.f(b))
        romberg_matrix[0][0] = r11

        # fill the first column of matrix
        # define k from 2 to n
        for k in range(2, n + 1, 1):
            # define i from 1 to 2^(k-2)
            for i in range(1, int(np.power(2, k - 2)) + 1, 1):
                # i = 1, temp_sum = f((a+b)/2)
                # ...
                # i = 2^(k-2), temp_sum = f(a+(k-0.5)hk)
                temp_sum += self.f(a + (2 * i - 1) * (b - a) / (np.power(2, k -␣
↪1)))

            # be advised that h is being updated with time
            rn1 = 0.5 * romberg_matrix[(k - 1) - 1][0] + 0.5 * (b - a) / (np.
↪power(2, k - 2)) * temp_sum
```

```
            romberg_matrix[(k - 1)][0] = rn1

            # update template variable to 0 for the next trial
            temp_sum = 0

        # it seems obvious that rest elements of the matrix can be calculated
        # get R_{j, k} here, noted that k and j is more than 1, k >= j
        # define j from 2 to n
        for j in range(2, n + 1, 1):
            # define k from j to n
            for k in range(j, n + 1, 1):
                r_kj = romberg_matrix[k - 1][j - 2]
                r_kj += 1 / (np.power(4, j - 1) - 1) * (romberg_matrix[k - 1][j␣
↪- 2] - romberg_matrix[k - 2][j - 2])
                # transfer data to matrix
                romberg_matrix[k - 1][j - 1] = r_kj

        print(romberg_matrix)

    def simpson(self, x_start, x_end):
        a = x_start
        b = x_end
        h = (b - a) / 2

        result = self.f(a) + self.f(b) + 4 * self.f(a + h)
        result = h / 3 * result
        return result

    def trapezoidal(self, x_start, x_end):
        a = x_start
        b = x_end
        h = (b - a) / 2

        return h * (self.f(a) + self.f(b))

    def adaptive_simpson(self, x_start, x_end, eps):
        # reference
        # https://en.wikipedia.org/wiki/Adaptive_Simpson%27s_method
        # https://cloud.tencent.com/developer/article/1637426
        # https://www.math.usm.edu/lambers/mat460/fall09/lecture30.pdf
        a = x_start
        b = x_end
        m = (a + b) / 2

        simpson_ans = self.simpson(a, b)
        left = self.simpson(a, m)
        right = self.simpson(m, b)
```

```
        print("ans of simpson's method on [%f, %f] is %f" % (a, b, simpson_ans))
        print("ans of simpson's method on [%f, %f] is %f" % (a, m, left))
        print("ans of simpson's method on [%f, %f] is %f" % (m, b, right))
        print("error is %f" % (abs(left + right - simpson_ans)))

        # a traditional way is to replace 15 by 10
        # to make the algorithm more conservative
        # seems a serious bug due to no limitation
        if abs(left + right - simpson_ans) <= 15 * eps:
            print("[%f, %f] meet the error" % (a, b))
            return left + right + (left + right - simpson_ans) / 15
        else:
            print("[%f, %f] exceed the error" % (a, b))
            return self.adaptive_simpson(a, m, eps / 2) + \
                self.adaptive_simpson(m, b, eps / 2)

    def adaptive_trapezoidal(self, x_start, x_end, eps):
        a = x_start
        b = x_end
        m = (a + b) / 2

        trapezoidal_ans = self.trapezoidal(a, b)
        left = self.trapezoidal(a, m)
        right = self.trapezoidal(m, b)
        print("ans of trapezoidal method on [%f, %f] is %f" % (a, b,␣
↪trapezoidal_ans))
        print("ans of trapezoidal method on [%f, %f] is %f" % (a, m, left))
        print("ans of trapezoidal method on [%f, %f] is %f" % (m, b, right))
        print("error is %f" % (abs(left + right - trapezoidal_ans)))

        if abs(left + right - trapezoidal_ans) <= 15 * eps:
            print("[%f, %f] meet the error" % (a, b))
            return left + right + (left + right - trapezoidal_ans) / 15
        else:
            print("[%f, %f] exceed the error" % (a, b))
            return self.adaptive_trapezoidal(a, m, eps / 2) + \
                self.adaptive_trapezoidal(m, b, eps / 2)
```

1.3 ODE solver

[3]:
```
class ODESolver:
    def __init__(self, df):
        """init function
        :param df need to be a function with 2 params, like f(t, y)"""
        self.df = df  # df is the derivative function (rank 1,   )

    def forward_euler(self, init_y, start, end, len_step):
```

```
        """compose forward euler method for a well-posed initial value problem
        :param init_y the function value at start point, namely y(start)
        :param start the value of left interval
        :param end the value of right interval
        :param len_step the length of step"""

        if end <= start:
            raise Exception('Error! Bad interval input')

        num_steps = int((end - start) / len_step)
        # init t and w in f(t, w), avoid using y for misunderstanding
        t = start
        w = init_y
        print("w0 = %f" % w)

        for i in range(num_steps):
            w += len_step * self.df(t, w)
            t += len_step
            index = int(i + 1)
            print("w%d = %f" % (index, w))

        return w

    def two_order_taylor(self, d2f, init_y, start, end, num_step):
        """compose forward euler method for a well-posed initial value problem
        :param d2f f'(t, w)
        :param init_y the function value at start point, namely y(start)
        :param start the value of left interval
        :param end the value of right interval
        :param num_step the number of step"""

        if end <= start:
            raise Exception('Error! Bad interval input')

        len_step = (end - start) / num_step
        # init t and w in f(t, w), avoid using y for misunderstanding
        t = start
        w = init_y
        print("w0 = %f" % w)

        for i in range(num_step):
            w += len_step * (self.df(t, w) + len_step / 2 * d2f(t, w))
            t += len_step
            index = i + 1
            print("w%d = %f" % (index, w))
        return w
```

```
    def modified_euler(self, init_y, start, end, num_step):
        len_step = (end - start) / num_step
        t = start
        w = init_y

        for i in range(num_step):
            k0 = len_step * self.df(t, w)
            k1 = len_step * self.df(t + len_step, w + k0)
            w = w + (k0 + k1) / 2.0
            t = t + len_step

        return w

    def runge_kutta_4(self, init_y, start, end, num_step):
        len_step = (end - start) / num_step
        t = start
        w = init_y

        for i in range(num_step):
            k0 = len_step * self.df(t, w)
            k1 = len_step * self.df(t + len_step / 2.0, w + k0 / 2.0)
            k2 = len_step * self.df(t + len_step / 2.0, w + k1 / 2.0)
            k3 = len_step * self.df(t + len_step, w + k2)
            w = w + (k0 + 2.0 * k1 + 2.0 * k2 + k3) / 6.0
            t = t + len_step

        return w
```

1.4 Non linear equations root finder

[4]:
```
class RootFinder:
    def __init__(self, func):
        self.f = func

    def bisect(self, start, end, tolerance):
        k = 1
        if self.f(start) * self.f(end) < 0:
            while abs(start - end) / 2.0 > tolerance:
                k = k + 1
                x_new = float(start + end) / 2.0
                if self.f(start) * self.f(x_new) < 0:
                    end = x_new
                else:
                    start = x_new
        else:
            print("f(a)*f(b) is non-negative, find another pair of input")
        return float(start + end) / 2.0, k
```

```
    def newton(self, df, x0, max_iter, tolerance):
        for i in range(max_iter):
            x1 = x0 - self.f(x0) / df(x0)
            if abs((x1 - x0) < tolerance):
                break
            if i == max_iter:
                print("meet max iter!")
            x0 = x1

        return x0

    def fix_point(self, x0, tolerance):
        x1 = self.f(x0)
        while abs(x0 - x1) > tolerance:
            x0 = self.f(x1)
            x1 = self.f(x0)

        return x1
```

1.5 Lagrange interpolation

[5]:
```
from fractions import Fraction

def multiply(a, b, m, n):
    prod = [0] * (m + n - 1)

    # Multiply two polynomials term by term

    # Take ever term of first polynomial
    for i in range(m):

        # Multiply the current term of first
        # polynomial with every term of
        # second polynomial.
        for j in range(n):
            prod[i + j] += a[i] * b[j]

    return prod


def print_poly(poly, n):
    for i in range(n):
        print(poly[i], end="")
        if i != 0:
            print("x^%d" % i, end="")
        if i != n - 1:
```

```
            print(" + ", end="")

class LagrangeSolver:
    def __init__(self, np_points):
        self.points = np_points
        # get rows and cols of points
        self.rows = self.points.shape[0]
        self.cols = self.points.shape[1]
        # check the dimension of the array (points)
        if self.cols != 2:
            print("considering using reshape() to revise your points")
            raise Exception("Error! The column number of your points \
            array is not 2, please check it out!")

    def change_data(self, new_np_points):
        new_cols = new_np_points.shape[1]
        # check the dimension of the array (points)
        if new_cols != 2:
            print("considering using reshape() to revise your points")
            raise Exception("Error! The column number of your points \
            array is not 2, please check it out!")
        # send data to self
        self.points = new_np_points
        self.rows = new_np_points.shape[0]
        self.cols = new_np_points.shape[1]

    def get_lagrange_den(self, index):
        point = self.points
        length = self.rows

        den = 1
        for i in range(length):
            if index != i:
                den = den * (point[index, 0] - point[i, 0])

        return den

    def get_lagrange_num(self, index):
        length = self.rows

        prod = [1]

        # start from the highest rank, which is len - 1
        for i in range(length):
            if i != index:
                m = len(prod)
                n = 2
```

```
                b = [-self.points[i, 0], 1]
                prod = multiply(prod, b, m, n)

        return prod

    def lagrange_interpolation(self):
        length = self.rows
        coefficient_list = []
        # get coefficient
        for i in range(length):
            coefficient = Fraction(1 / self.get_lagrange_den(i)).
↪limit_denominator() \
                    * Fraction(self.points[i, 1]).limit_denominator()
            coefficient_list.append(coefficient)

        print("coefficients: ", coefficient_list)

        final = [Fraction(0, 1)] * length

        # get nums
        print("each polynomial without coefficients: ")
        for i in range(length):
            # on index
            num = self.get_lagrange_num(i)
            print_poly(num, length)
            print(" ")
            for j in range(length):
                final[j] += Fraction(num[j]).limit_denominator() *␣
↪coefficient_list[i]

        return final

    def lagrange_polynomial(self):
        final = self.lagrange_interpolation()
        print("final polynomial: ")
        print_poly(final, self.rows)
```

1.6 Equations solver

[6]:
```
class EquationSolver:
    def __init__(self, A, b):
        self.A = A
        self.b = b

    def guassian_elimination(self):
        n = len(self.b)
        m = np.zeros(shape=(n, n+1))
```

```
        m[0:n, 0:n] = self.A
        m[0:n, n] = self.b.reshape(1, n)

        # start elimination
        for col in range(0, n - 1):
            for row in range(col + 1, n):
                coefficient = m[row, col] / m[col, col]
                m[row, col:n+1] -= coefficient * m[col, col:n+1]

        # back substitution
        x = np.zeros(n)
        x[n-1] = m[n-1, n] / m[n-1, n-1]
        for row in range(n-2, -1, -1):
            x[row] = (m[row, n] - m[row, row + 1:n].dot(x[row + 1:n])) / m[row,␣
↪row]

        return x

    def lu_decomposition(self):
        n = len(self.b)
        mat_l = np.zeros(shape=[n, n])
        mat_u = np.zeros(shape=[n, n])

        # set diagonals of L
        for i in range(n):
            mat_l[i, i] = 1.0

        # set first row of u
        for j in range(n):
            mat_u[0, j] = self.A[0, j]

        # set first column of L
        for i in range(1, n):
            mat_l[i, 0] = self.A[i, 0] / mat_u[0, 0]

        # interactively compute k-th row of U and k-th col of L
        for k in range(1, n):
            for j in range(k, n):
                mat_u[k, j] = self.A[k, j] - (mat_l[k, 0:k]).dot(mat_u[0:k, j])

            for i in range(k+1, n):
                mat_l[i, k] = (self.A[i, k] - (mat_l[i, 0:k]).dot(mat_u[0:k,␣
↪k])) / mat_u[k, k]

        # back substitution
        y = np.zeros(n)
        y[0] = self.b[0] / mat_l[0, 0]
```

```
        for i in range(1, n):
            y[i] = (self.b[i] - (mat_l[i, 0:i]).dot(y[0:i])) / mat_l[i, i]

        x = np.zeros(n)

        x[n - 1] = y[n - 1] / mat_u[n - 1, n - 1]
        for i in range(n - 2, -1, -1):
            x[i] = (y[i] - mat_u[i, i + 1:n].dot(x[i + 1:n])) / mat_u[i, i]

        return mat_l, mat_u, y, x

    def gauss_seidel(self, tolerance, max_iterations, x):
        # https://stackoverflow.com/questions/5622656/
↪python-library-for-gauss-seidel-iterative-solver

        # x is the initial condition
        iter1 = 0
        # Iterate
        for k in range(max_iterations):
            iter1 = iter1 + 1
            # print("The solution vector in iteration", iter1, "is:", x)
            x_old = x.copy()

            # Loop over rows
            for i in range(self.A.shape[0]):
                x[i] = (self.b[i] - np.dot(self.A[i, :i], x[:i])
                        - np.dot(self.A[i, (i + 1):], x_old[(i + 1):])) / self.
↪A[i, i]

            # Stop condition
            # L norm Inf corresponds to the absolute value of the greatest␣
↪element of the vector.

            l_norm_inf = max(abs((x - x_old))) / max(abs(x_old))
            # print("The L infinity norm in iteration", iter1, "is:",␣
↪l_norm_inf)
            if l_norm_inf < tolerance:
                break

        return x

    def sor_method(self, omega, tolerance, max_iterations, x):
        # https://stackoverflow.com/questions/53251299/
↪successive-over-relaxation

        iter1 = 0
        # Iterate
```

```
        for k in range(max_iterations):
            iter1 = iter1 + 1
            # print("The solution vector in iteration", iter1, "is:", x)
            x_old = x.copy()

            # Loop over rows
            for i in range(self.A.shape[0]):
                x[i] = x[i] * (1 - omega) + (omega / self.A[i, i]) * (
                            self.b[i] - np.dot(self.A[i, :i], x[:i]) - np.
↪dot(self.A[i, (i + 1):], x_old[(i + 1):]))

            # Stop condition
            # L norm Inf corresponds to the absolute value of the greatest␣
↪element of the vector.

            l_norm_inf = max(abs((x - x_old))) / max(abs(x_old))
            # print("The L infinity norm in iteration", iter1, "is:",␣
↪l_norm_inf)
            if l_norm_inf < tolerance:
                break

        return x
```

1.7 Eigen solver

[7]:
```
class EigenSolver:
    def __init__(self, np_matrix):
        self.mat = np_matrix

    def power_iteration(self, tolerance, max_iterations):
        n = len(self.mat)
        u = np.ones([n, 1])
        iteration = 0
        eigen_value = 0
        vector = self.mat.dot(u)
        residual = 10000

        while iteration < max_iterations and residual > tolerance:
            vector = self.mat.dot(u)
            eigen_value = max(vector, key=abs)
            vector = vector / max(abs(vector))

            residual = np.linalg.norm(u - vector, ord=2)
            u = vector
            iteration += 1

        return eigen_value, vector
```

```
  def rayleigh_quotient(self, tolerance, max_iterations, vector_guess,␣
↪value_guess):
      # https://en.wikipedia.org/wiki/Rayleigh_quotient_iteration
      n = len(self.mat)
      mat_i = np.eye(n)
      iteration = 0
      residual = 10000
      value_predict = 0

      while iteration < max_iterations and residual > tolerance:
          b = np.dot(np.linalg.inv(self.mat - value_guess * mat_i),␣
↪vector_guess)
          b = b / np.linalg.norm(b)
          # update eigen vector
          vector_guess = b

          # update eigen value
          value_predict = np.dot(b.transpose(), self.mat)
          value_predict = np.dot(value_predict, b)

          # update residual
          residual = abs(value_predict - value_guess)
          value_guess = value_predict

          # update iter
          iteration += 1

      return value_predict

  def qr_decomposition(self, tolerance, max_iterations):
      q, r = np.linalg.qr(self.mat)
      iteration = 0
      a1 = self.mat
      residual = 10000

      while iteration < max_iterations and residual > tolerance:
          q_new, r_new = np.linalg.qr(a1)
          diag_a1 = np.diag(a1)

          a1 = np.dot(r_new, q_new)
          diag_a1_new = np.diag(a1)

          residual = np.linalg.norm(diag_a1 - diag_a1_new)
          iteration += 1

      return q, r, np.diag(a1)
```

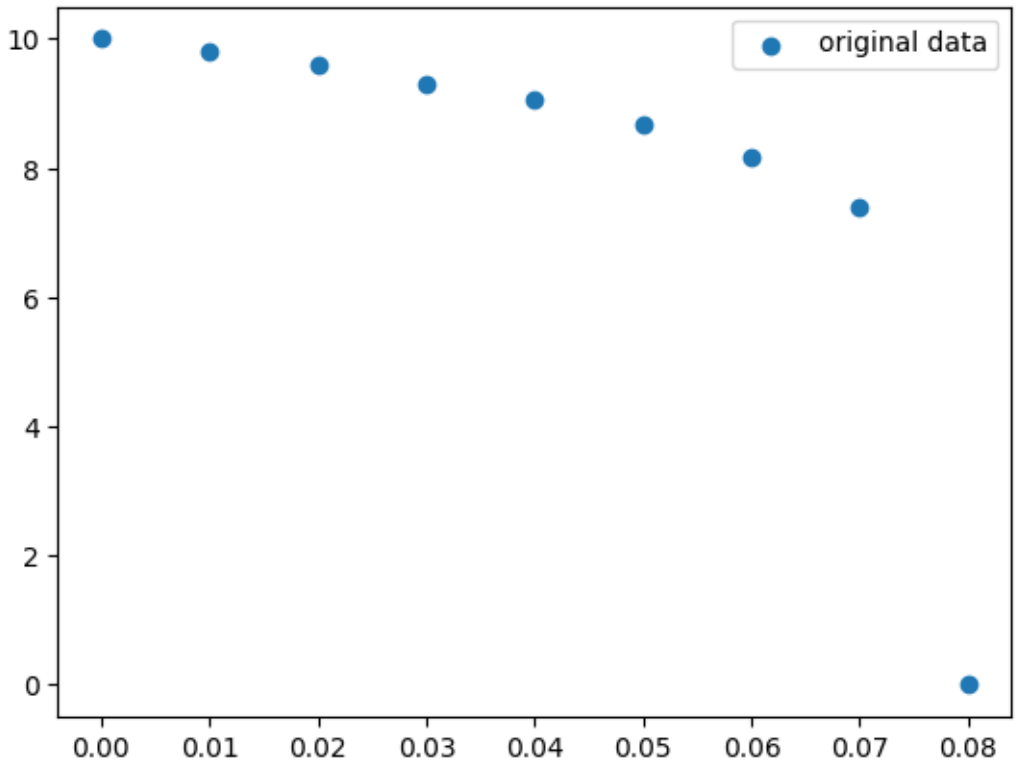
0.2 Numerical Integration

$$\int_0^R \rho v 2\pi r \mathrm{d}r$$

$\rho = 1.2 kg/m^3$

change unit to cm in r, use scatter to see the relation between r and v

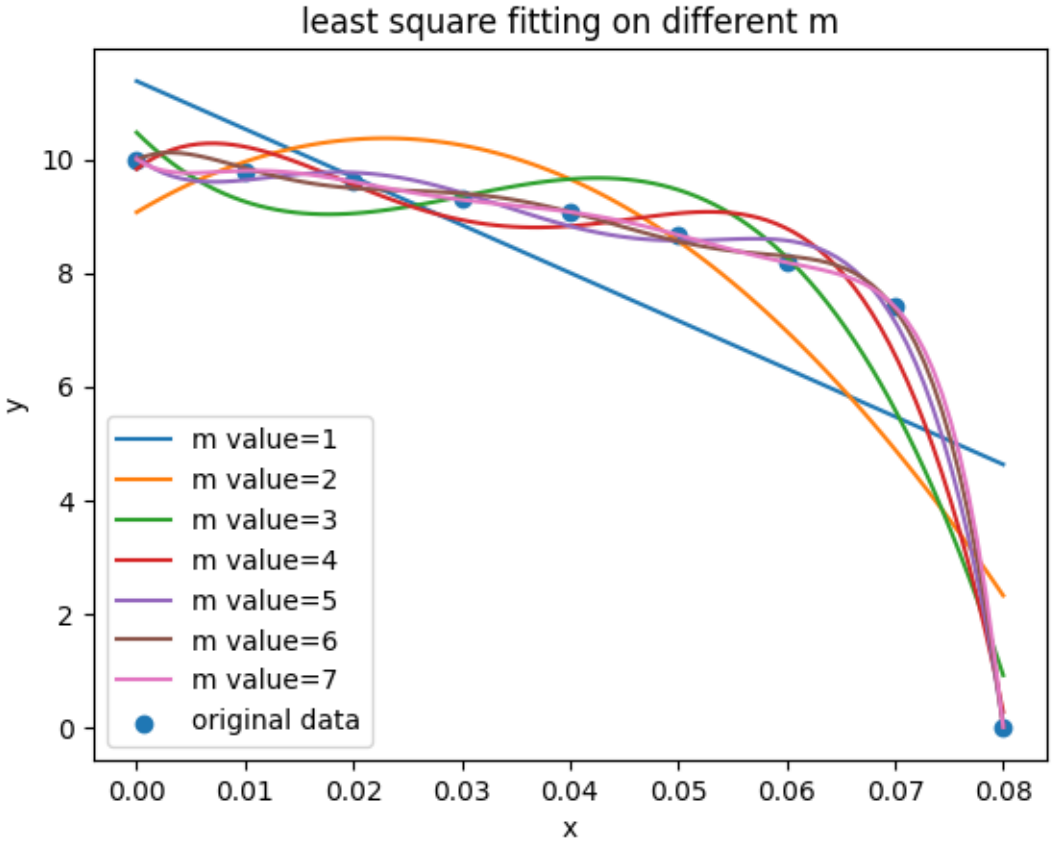
[8]:
```
import numpy as np
import matplotlib.pyplot as plt

r_sample = np.array([0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08])
v_sample = np.array([10, 9.8, 9.6, 9.3, 9.06, 8.68, 8.18, 7.41, 0])

plt.scatter(r_sample, v_sample, label="original data")
plt.legend()
plt.show()
```



then using numerical interpolation to get a polynomial

[9]:
```
points = np.vstack((r_sample, v_sample)).transpose()

# v_sample_ln = np.log(v_sample)
```

```
# points_ln = np.vstack((r_sample, v_sample_ln)).transpose()
# points_ln[8][1] = 0.001

solver = LeastSquare(points)
solver.run()

# get the v(r) function
coefficient = solver.get_function_m(5)

# apply our formula
coefficient = np.insert(coefficient, 0, 0)  # vr
coefficient = coefficient * 1.2 * 2 * 3.1415926  # vr * rho * 2 * pi
```



use the function created by LeastSquare, order 5

[10]:
```
function1 = np.poly1d(np.flip(coefficient))
integration1 = NumericalIntegration(function1)
integration1.composite_newton_cotes(0, 0.08, 50)
```

```
[0.18532954315398195, 0.18550181676722577]
```

so the answer is: * Composite Trapezoidal rule: 0.1853 * Composite Simpson's rule: 0.1855

if we use romberg method

[11]:
```
more_accurate_ans = integration1.romberg(0, 0.08, 10)
```

```
[[0.00139815 0.         0.         0.         0.         0.
  0.         0.         0.         0.        ]
 [0.10716038 0.14241446 0.         0.         0.         0.
  0.         0.         0.         0.        ]
 [0.16058675 0.17839555 0.18079428 0.         0.         0.
  0.         0.         0.         0.        ]
 [0.17888847 0.18498904 0.18542861 0.18550217 0.         0.
  0.         0.         0.         0.        ]
 [0.18382388 0.18546902 0.18550102 0.18550217 0.18550217 0.
  0.         0.         0.         0.        ]
 [0.18508103 0.18550008 0.18550215 0.18550217 0.18550217 0.18550217
  0.         0.         0.         0.        ]
 [0.18539679 0.18550204 0.18550217 0.18550217 0.18550217 0.18550217
  0.18550217 0.         0.         0.        ]
 [0.18547582 0.18550216 0.18550217 0.18550217 0.18550217 0.18550217
  0.18550217 0.18550217 0.         0.        ]
 [0.18549558 0.18550217 0.18550217 0.18550217 0.18550217 0.18550217
  0.18550217 0.18550217 0.18550217 0.        ]
 [0.18550052 0.18550217 0.18550217 0.18550217 0.18550217 0.18550217
  0.18550217 0.18550217 0.18550217 0.18550217]]
```

the answer for applying romberg method is 0.1855

0.3 ODE Initial value problem

given that

$$\frac{\partial k}{\partial t} = -\epsilon$$

and

$$\frac{\partial \epsilon}{\partial t} = -C\frac{\epsilon^2}{k}$$

where $C = 1.83$. At $t_0 = 1, \quad k = 1, \quad \epsilon = 0.2176$

predict k at $t = 5$

[12]:
```
def df_epsilon_t(epsilon, k):
    return -1.83 * epsilon * epsilon / k

def df_k_t(epsilon):
    return -epsilon
```

```
# in each iteration, first step is to get epsilon, then k
# next iteration we will use the k we get in epsilon formula
def euler(init_k, init_epsilon, start, end, num):
    step = (end - start) / num
    epsilon = init_epsilon
    k = init_k

    for i in range(num):
        epsilon += step * df_epsilon_t(epsilon, k)
        k += step * df_k_t(epsilon)

    print("Euler method: k estimated is %f" % k)

def modified_euler(init_k, init_epsilon, start, end, num):
    step = (end - start) / num
    epsilon = init_epsilon
    k = init_k

    for i in range(num):
        epsilon0 = step * df_epsilon_t(epsilon, k)
        epsilon1 = step * df_epsilon_t(epsilon + epsilon0, k)
        epsilon += (epsilon0 + epsilon1) / 2.0

        k0 = step * df_k_t(epsilon)
        k1 = step * df_k_t(epsilon)
        k += (k0 + k1) / 2.0

    print("Modified euler method: k estimated is %f" % k)

def runge_kutta4(init_k, init_epsilon, start, end, num):
    step = (end - start) / num
    epsilon = init_epsilon
    k = init_k

    for i in range(num):
        # epsilon
        kutta0 = step * df_epsilon_t(epsilon, k)
        kutta1 = step * df_epsilon_t(epsilon + kutta0 / 2.0, k)
        kutta2 = step * df_epsilon_t(epsilon + kutta1 / 2.0, k)
        kutta3 = step * df_epsilon_t(epsilon + kutta2, k)
        epsilon += (kutta0 + 2 * kutta1 + 2 * kutta2 + kutta3) / 6.0

        # k
        k += step * df_k_t(epsilon)

    print("Runge-Kutta 4 method: k estimated is %f" % k)
```

```
euler(1, 0.2176, 0, 5, 20)
modified_euler(1, 0.2176, 0, 5, 20)
runge_kutta4(1, 0.2176, 0, 5, 20)
```

```
Euler method: k estimated is 0.490872
Modified euler method: k estimated is 0.472663
Runge-Kutta 4 method: k estimated is 0.473380
```

considering both convergence and accuracy, euler's method should be the last one of the three methods, while modified euler method works better than euler's method; the most fast-converge and accurate method of the three should be 4-th runge kutta method.

0.4 Non linear equations

$$y_+ = U_+ + e^{-kB}[e^{kU_+} - 1 - kU_+ - \frac{1}{2}(kU_+)^2 - \frac{1}{6}(kU_+)^3 - \frac{1}{24}(kU_+)^4]$$

where

$$y_+ = \frac{u_r y}{v}, \quad U_+ = \frac{U}{u_r}$$

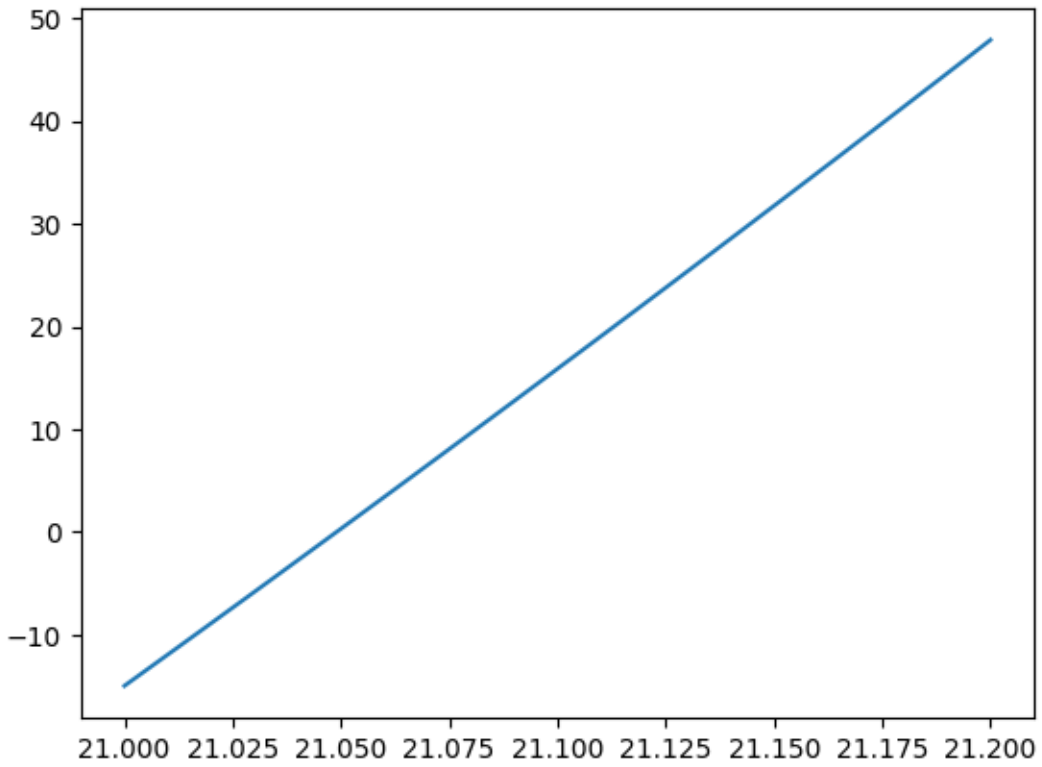
$$k = 0.41, \quad B = 5.1, \quad v = 1.5 \times 10^{-5}, \quad \rho = 1.25 kg/m^3$$

$y = 0.01m$, the velocity is $21m/s$

$$u_\tau = \sqrt{\frac{\tau_{wall}}{\rho}}$$

[13]:
```
from math import exp, log

k = 0.41
B = 5.1
rho = 1.25
u = 21
v = 1.5e-5
y = 0.01

def f_u_plus(u_plus):
    # y_plus=y * u / (v * u_plus), u_plus != 0
    return u_plus + exp(-k * B) * (exp(k * u_plus) - 1 - k * u_plus - 1/2 *␣
 ↪pow(k*u_plus, 2) -
                                   1/6 * pow(k*u_plus, 3) - 1/24 *␣
 ↪pow(k*u_plus, 4)) - y * u / (v * u_plus)

def df_u_plus(u_plus):
```

```
    return 1 + exp(-k * B) * (k * exp(k * u_plus) - k - k*k*u_plus - k/2 *␣
 ↪pow(k*u_plus, 2) -
                              k/6 * pow(k*u_plus, 3)) + y * u / (v*pow(u_plus,␣
 ↪2))

f_numpy = np.vectorize(f_u_plus)
x = np.linspace(21, 21.2, num=100)
value = f_numpy(x)

plt.plot(x, value)
plt.show()

root_finder = RootFinder(f_u_plus)
u_star, n = root_finder.bisect(21, 22, 0.001)
print("Bisection method: u plus = ", u_star, ", No. iters = ", n)
u_star = root_finder.newton(df_u_plus, 21, 22, 0.001)
print("Newton method: u plus = ", u_star)

tau_wall = pow(u/u_star, 2) * rho

def f_fix_point(u_plus):
    return log((y * u / (v * u_plus) - u_plus) / exp(-k * B) + 1 + k * u_plus -␣
 ↪1 / 2 * pow(k * u_plus, 2)
               + 1 / 6 * pow(k * u_plus, 3) + 1 / 24 * pow(k * u_plus, 4)) / k

fix_point_finder = RootFinder(f_fix_point)
u_star = fix_point_finder.fix_point(21, 0.001)
print("Fix point method: u plus = ", u_star)
print("tau_wall is ", tau_wall)
```




```
Bisection method: u plus =  21.0498046875 , No. iters =  10
Newton method: u plus =  21.0494251952156
Fix point method: u plus =  21.01908102946567
tau_wall is  1.2441367557267613
```

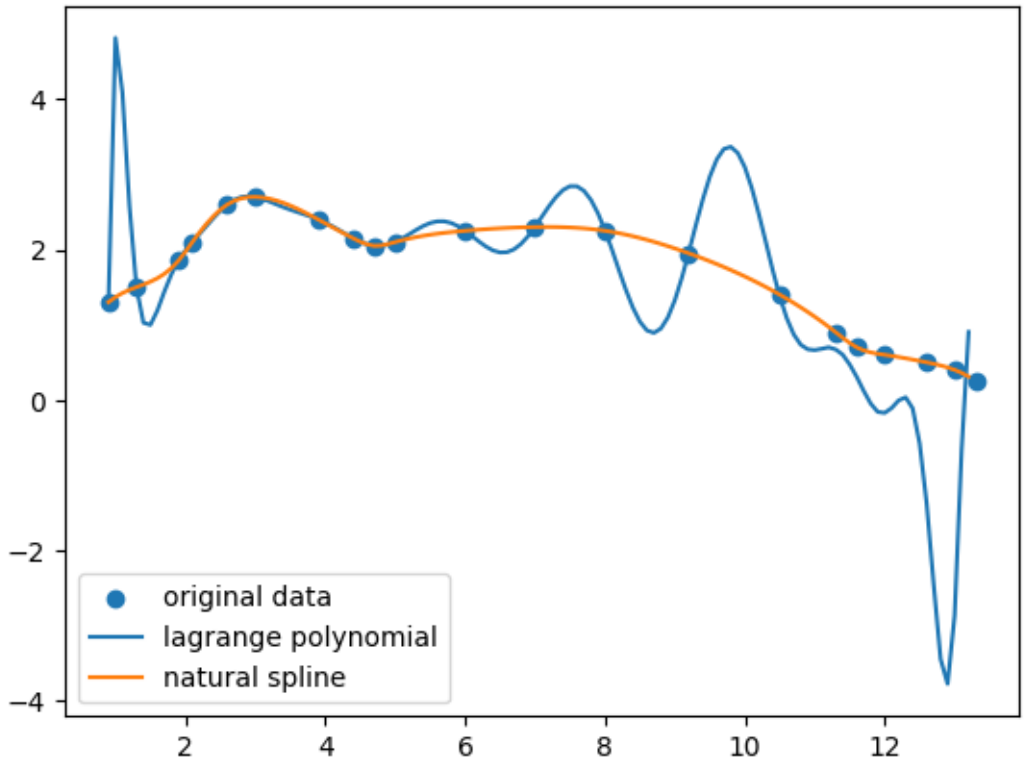
so the final $\tau_{wall} = 1.2441$, using fix point formula as below:

$$U_+ = g(U_+) = ln((y_+ + U_+)e^{kB} + 1 + kU_+ + \frac{1}{2}(kU_+)^2 + \frac{1}{6}(kU_+)^3 + \frac{1}{24}(kU_+)^4)/k$$

0.5 Interpolation

[14]:
```
from scipy.interpolate import lagrange
from scipy.interpolate import CubicSpline
from numpy.polynomial.polynomial import Polynomial

points = np.array([
    [0.9, 1.3],
    [1.3, 1.5],
    [1.9, 1.85],
    [2.1, 2.1],
    [2.6, 2.6],
    [3.0, 2.7],
```

```
    [3.9, 2.4],
    [4.4, 2.15],
    [4.7, 2.05],
    [5.0, 2.1],
    [6.0, 2.25],
    [7.0, 2.3],
    [8.0, 2.25],
    [9.2, 1.95],
    [10.5, 1.4],
    [11.3, 0.9],
    [11.6, 0.7],
    [12.0, 0.6],
    [12.6, 0.5],
    [13.0, 0.4],
    [13.3, 0.25]
])

# lagrange from scipy
poly = lagrange(points[:, 0], points[:, 1])
coefficient = Polynomial(poly.coef[::-1]).coef

# cubic spline from scipy
spline = CubicSpline(points[:, 0], points[:, 1])

x_new = np.arange(0.9, 13.3, 0.1)
plt.scatter(points[:, 0], points[:, 1], label='original data')
plt.plot(x_new, Polynomial(poly.coef[::-1])(x_new), label='lagrange polynomial')
plt.plot(x_new, spline(x_new), label='natural spline')
plt.legend()
plt.show()
```



With many as 21 points, the lagrange method could not follow the trend of original data points.

However, the natural spline function basically keeps the same changing pattern as original data points go.

0.6 Linear equations

use naive approach of polynomial interpolation to form a linear system

[15]:
```
vector_x = points[:, 0]
vector_y = points[:, 1]

def naive_creator(vec_x):
    dim = vector_x.shape[0]
    mat_x = np.ones(dim)

    for i in range(dim - 1):
        mat_x = np.vstack((mat_x, np.power(vec_x, i + 1)))

    return mat_x.transpose()

X = naive_creator(vector_x)
```

```
# initial guess
initial_guess = np.array([
    [-9.6e3], [5.2e4], [-1.3e5], [1.9e5],
    [-1.9e5], [1.4e5], [-7.4e4], [3.1e4],
    [-9.9e3], [2.6e3], [-5.2e2], [8.6e1],
    [-1.1e1], [1.2e0], [-9.7e-2], [6.3e-3],
    [-3.1e-4], [1.1e-5], [-2.8e-7], [4.3e-9],
    [-3.1e-11]
])

equation_solver = EquationSolver(X, vector_y)
gauss = equation_solver.guassian_elimination()
L, U, LU_y, LU_x = equation_solver.lu_decomposition()
seidel = equation_solver.gauss_seidel(0.01, 100, initial_guess)
sor = equation_solver.sor_method(1.1, 0.01, 100, initial_guess)

print("gaussian elimination: ", gauss)
print("LU decomposition: ", LU_x)
print("gauss-seidel: ", seidel)
print("SOR: ", sor)
```

```
gaussian elimination:  [-9.62900109e+03  5.23308734e+04 -1.28353328e+05
1.89495703e+05
 -1.89338391e+05  1.36397396e+05 -7.36542206e+04  3.05862459e+04
 -9.93812152e+03  2.55593429e+03 -5.24043290e+02  8.59478924e+01
 -1.12743546e+01  1.17827153e+00 -9.72945654e-02  6.25917429e-03
 -3.06784319e-04  1.10555985e-05 -2.75922145e-07  4.25747987e-09
 -3.05797637e-11]
LU decomposition:  [-9.60454784e+03  5.21953487e+04 -1.28013353e+05
1.88980865e+05
 -1.88809167e+05  1.36003965e+05 -7.34343002e+04  3.04914228e+04
 -9.90603943e+03  2.54731981e+03 -5.22194910e+02  8.56299984e+01
 -1.12305538e+01  1.17345714e+00 -9.68760682e-02  6.23081593e-03
 -3.05319883e-04  1.09999978e-05 -2.74460563e-07  4.23373827e-09
 -3.04003669e-11]
gauss-seidel:  [[-5.34167848e+04]
 [ 1.51689637e+05]
 [-1.81222598e+05]
 [ 1.81139810e+05]
 [-1.86383242e+05]
 [ 1.42524271e+05]
 [-7.54223407e+04]
 [ 3.04691827e+04]
 [-9.96721192e+03]
 [ 2.58850592e+03]
 [-5.25063148e+02]
 [ 8.49677686e+01]
```

```
 [-1.11779782e+01]
 [ 1.18017590e+00]
 [-9.78315352e-02]
 [ 6.30218484e-03]
 [-3.09533404e-04]
 [ 1.10590396e-05]
 [-2.74274036e-07]
 [ 4.44400390e-09]
 [-2.96942589e-11]]
SOR:  [[-5.34167848e+04]
 [ 1.51689637e+05]
 [-1.81222598e+05]
 [ 1.81139810e+05]
 [-1.86383242e+05]
 [ 1.42524271e+05]
 [-7.54223407e+04]
 [ 3.04691827e+04]
 [-9.96721192e+03]
 [ 2.58850592e+03]
 [-5.25063148e+02]
 [ 8.49677686e+01]
 [-1.11779782e+01]
 [ 1.18017590e+00]
 [-9.78315352e-02]
 [ 6.30218484e-03]
 [-3.09533404e-04]
 [ 1.10590396e-05]
 [-2.74274036e-07]
 [ 4.44400390e-09]
 [-2.96942589e-11]]
```

0.7 Eigen value

$$A = \begin{bmatrix} 52 & 30 & 49 & 28 \\ 30 & 50 & 8 & 44 \\ 49 & 8 & 46 & 16 \\ 28 & 44 & 16 & 22 \end{bmatrix}$$

[16]:
```
A = np.array([
    [52, 30, 49, 28],
    [30, 50, 8, 44],
    [49, 8, 46, 16],
    [28, 44, 16, 22]
])

vector_init = np.array([
    [1], [0.8], [0.8], [0.7]
```

```
])

example = EigenSolver(A)
power_iter_value, power_iter_vector = example.power_iteration(0.01, 100)
print("Power iteration: Max eigen value = ", power_iter_value)
rayleigh_value = example.rayleigh_quotient(0.01, 100, vector_init, 130)
print("Rayleigh quotient: Max eigen value = ", rayleigh_value)
mat_Q, mat_R, QR_diag = example.qr_decomposition(0.01, 100)
print("QR decomposition, eigen values = ", QR_diag)
```

```
Power iteration: Max eigen value =  [132.94900421]
Rayleigh quotient: Max eigen value =  [[132.62787533]]
QR decomposition, eigen values =  [132.62759169  52.4425832  -11.54109674
-3.52907815]
```

so there are two positive eigen values and two negative eigen values.