刘昭宏 122431910061

Exercise 4.1

$y' = 1 + t\sin(ty)$, $t \in [0,2]$

$y(0) = 0$

holding t constant and applying the Mean Value Theorem to the function

$f(t,y) = 1 + t\sin(ty)$

when $y_1 < y_2$, a number $\xi \in (y_1, y_2)$ exists with

$$\frac{f(t,y_2) - f(t,y_1)}{y_2 - y_1} = \frac{\partial}{\partial y} f(t,\xi)$$

$$= t^2\cos(\xi t)$$

Thus

$|f(t,y_2) - f(t,y_1)| = |y_2 - y_1|$

$|t^2\cos(\xi t)| \le 4|y_2 - y_1|$

and f satifies a Lipschitz condition in the variable y with Lipschitz constant $L = 4$

$f(t,y)$ is contimus when $t \in [0,2]$ and $y \in (-\infty, +\infty)$ so theorem 4.4 implies that a unique solution exists to this initial value problem.

Exercise 4.2

$y'(t) = y(t) - t^2 + 1$ $t \in [0,2]$

$y(0) = 0.5$ $h = 0.2$

set $f(t,w) = w - t^2 + 1$

$w_0 = y(0) = 0.5$

$w_1 = w_0 + hf(0, w_0) = 0.8$

$w_2 = w_1 + hf(0.2, w_1) = 1.152$

$w_3 = w_2 + hf(0.4, w_2) = 1.5504$

$w_4 = w_3 + hf(0.6, w_3) = 1.98848$

$w_5 = w_4 + hf(0.8, w_4) = 2.458176$

$w_6 = w_5 + hf(1, w_5) = 2.949811$

$w_7 = w_6 + hf(1.2, w_6) = 3.451773$

$w_8 = w_7 + hf(1.4, w_7) = 3.950128$

$w_9 = w_8 + hf(1.6, w_8) = 4.428154$

$w_{10} = w_9 + hf(1.8, w_9) = 4.865785$

for $y' = y - t^2 + 1$ use x for t

$\frac{dy}{dx} - y = -x^2 + 1$

$y = Ce^{-\int - dx} + e^{-\int - dx}\int(-x^2+1)e^{\int dx}\,dx$

$= Ce^x + e^x\int(1-x^2)e^{-x}\,dx$

$= Ce^x + (x+1)^2$

$x=0$, $y = C + 1 = 0.5$ $C = -0.5$

$y = -0.5e^x + (x+1)^2$

$y' = -0.5e^x + 2x + 2$

$y'' = -0.5e^x + 2$

clear that $y'' \ge \underbrace{-0.5e^2 + 2}_{<0}$

$y'' \le 1$

$|y''| \le 0.5e^2 - 2$

using the inequality in the error bound for Euler's method which $h = 0.2$, $L = 1$, and $M = 0.5e^2 - 2$ gives

$|y_i - w_i| \le 0.1(0.5e^2 - 2)(e^{t_i} - 1)$

eg: $|y(0.2) - w_1| \le 0.1(0.5e^2 - 2)(e^{0.2} - 1)$

Bound Error Table

t_i	Error bound
0.2	0.03752
0.4	0.08334
0.6	0.13931
0.8	0.20767
1.0	0.29117
1.2	0.39315
1.4	0.51771
1.6	0.66985
1.8	0.85568
2.0	1.08264

$y(2) \approx w_{10} = 4.865785$

Exercise 4.3

Taylor method of order n

$w_0 = \alpha$

$w_{i+1} = w_i + h\,T^{(n)}(t_i, w_i)$

$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \dots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i)$

(1) order = 2

$f'(t, y(t)) = \frac{d}{dt}(y - t^2 + 1) = y' - 2t$

$= y - t^2 + 1 - 2t$

$T^{(2)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i)$

$= (1 + \frac{h}{2})(w_i - t_i^2 + 1) - ht_i$

$w_{i+1} = w_i + hT^{(2)}(t_i, w_i)$

$= 1.22w_i - 0.0088i^2 - 0.008i + 0.22$

t_i	w_i (order 2)
0.2	0.83
0.4	1.2158
0.6	1.652076
0.8	2.132333
1.0	2.648646
1.2	3.191348
1.4	3.748645
1.6	4.306146
1.8	4.846299
2.0	5.347684

(2) order 4

$f''(t, y(t)) = y^2 - t^2 - 2t - 1$

$f'''(t, y(t)) = y^2 - t^2 - 2t - 1$

$\Rightarrow T^{(4)}(t_i, w_i)$

$= (1 + \frac{h}{2} + \frac{h^2}{6} + \frac{h^3}{24})(w_i - t_i^2)$

$- (1 + \frac{h}{3} + \frac{h^2}{12})(ht_i) + 1 + \frac{h}{2} - \frac{h^2}{6} - \frac{h^3}{24}$

$\Rightarrow w_{i+1} = 1.2214w_i - 0.008856i^2$

$- 0.00856i + 0.2186$

t_i	w_i (order 4)
0.2	0.8293
0.4	1.214091
0.6	1.648947
0.8	2.127240
1.0	2.640874
1.2	3.179964
1.4	3.732432
1.6	4.283529
1.8	4.815238
2.0	5.305555

order 2 $\quad y(2) \approx 5.347684$

order 4 $\quad y(2) \approx 5.305555$

Exercise 4.4