Chapter 2

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Exercise 2.1

Given the following 4 data points

$$\{(0,1),(1,2),(3,6),(5,7)\}$$

find a polynomial in Lagrange form to interpolate these data

we know that the formula for Lagrange interpolaiton is:

$$L_{n,k}(x) = \prod_{i=0,i\neq k}^{n} \frac{x-x_i}{x_k-x_i}$$

so the Lagrange polynomial for the 4 points should be:

$$\begin{aligned}
\mathcal{L}_0 &= -\frac{-15+23x-9x^2+x^3}{15} \\
\mathcal{L}_1 &= \frac{15x-8x^2+x^3}{4} \\
\mathcal{L}_2 &= -\frac{5x-6x^2+x^3}{2} \\
\mathcal{L}_3 &= \frac{7(3x-4x^2+x^3)}{40}
\end{aligned}$$

so the final polynomial should be sum of the 4 equations:

$$\mathcal{L} = 1+\frac{29x}{120}+\frac{9x^2}{10}-\frac{17x^3}{120}$$

Exercise 2.5

Consider a function $f(x)=\sin(\pi x)+3x$ at 6 distinct nodes in the interval $[-1,1]$ to determine the Newton's divided difference formula. The data are given in Table below:

x	**-1.0**	**-0.6**	**-0.2**	**0.2**	**0.6**	**1.0**
$f(x)$	-3.0	-2.7511	-1.1878	1.1878	2.7511	3.0

the whole coefficient matrix can be presented below:

$$\begin{bmatrix}
p=0 & p=1 & p=2 & p=3 & p=4 & p=5 \\
-3 & & & & & \\
-2.7511 & 0.62225 & & & & \\
-1.1878 & 3.90825 & 4.1075 & & & \\
1.1878 & 5.939 & 2.5384375 & -1.30755208 & & \\
2.7511 & 3.90825 & -2.5384375 & -4.23072917 & -1.82698568 & \\
3 & 0.62225 & -4.1075 & -1.30755208 & 1.82698568 & 1.82698568
\end{bmatrix}$$

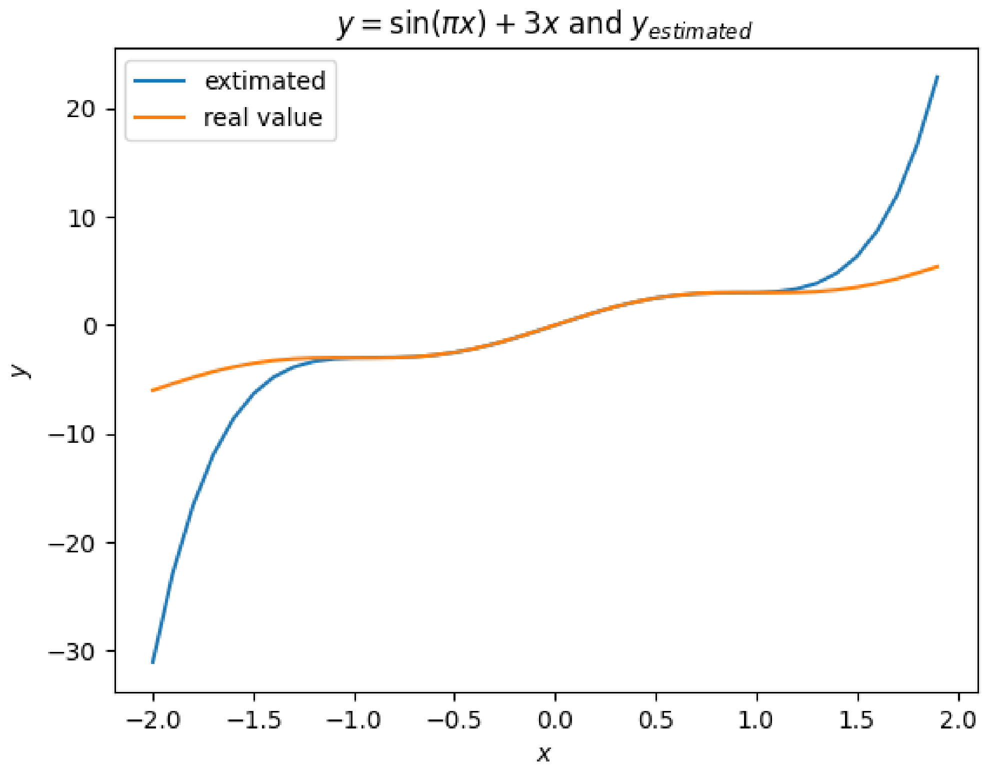
according to the Newton's divided difference formula,

$$P_n(x) = F_{0,0}+\sum_{i=1}^{n} F_i \prod_{j=0}^{i-1}(x-x_j)$$

so the polynomial should be:

$$
\begin{aligned}
P(x) = &-3 \\
&+0.62225(x+1.0) \\
&+4.1075(x+1.0)(x+0.6) \\
&-1.30755208(x+1.0)(x+0.6)(x+0.2) \\
&-1.82698568(x+1.0)(x+0.6)(x+0.2)(x-0.2) \\
&+1.82698568(x+1.0)(x+0.6)(x+0.2)(x-0.2)(x-0.6)
\end{aligned}
$$

compare the polynomial we get and the real function $f(x) = \sin(\pi x) + 3x$



we can see that **inside** the interval $[-1, 1]$, the polynomial performs well, but the error will exceed greatly **beyond** the interval.

Exercise 2.6

Construct an approximating polynomial for the following data using Hermite interpolation with Newton's forward difference

Table 1:

x	**0.1**	**0.2**	**0.3**
$f(x)$	-0.29004996	-0.56079734	-0.81401972
$f'(x)$	-2.8019975	-2.6159201	-2.9734038

Table 2:

x	**-1.0**	**-0.5**	**0.0**	**0.5**
$f(x)$	0.86199480	0.95802009	1.0986123	1.2943767
$f'(x)$	0.15536240	0.23269654	0.33333333	0.45186776

for data in Table 1:

$$
\begin{array}{llllllll}
z_0=0.1 & f[z_0]=-0.29004996 & -0.29004996 \\
z_1=0.1 & f[z_1]=-0.29004996 & -0.29004996 & -2.8019975 \\
z_2=0.2 & f[z_2]=-0.56079734 & -0.56079734 & -2.7074738 & 0.945237 \\
z_3=0.2 & f[z_3]=-0.56079734 & -0.56079734 & -2.6159201 & 0.915537 & -0.297 \\
z_4=0.3 & f[z_4]=-0.81401972 & -0.81401972 & -2.5322238 & 0.836963 & -0.39287 & -0.47935 \\
z_5=0.3 & f[z_5]=-0.81401972 & -0.81401972 & -2.9734038 & -4.4118 & -52.48763 & -260.4738 & -1299.97225
\end{array}
$$

for data in Table 2:

$$
\begin{array}{cccccccc}
0.8619948 \\
0.8619948 & 0.1553624 \\
0.95802009 & 0.19205058 & 0.07337636 \\
0.95802009 & 0.23269654 & 0.08129192 & 0.01583112 \\
1.0986123 & 0.28118442 & 0.09697576 & 0.01568384 & -0.00014728 \\
1.0986123 & 0.33333333 & 0.10429782 & 0.01464412 & -0.00103972 & -0.00089244 \\
1.2943767 & 0.3915288 & 0.11639094 & 0.01209312 & -0.002551 & -0.00100752 & -0.00007672 \\
1.2943767 & 0.45186776 & 0.12067792 & 0.00857396 & -0.00351916 & -0.00096816 & 0.00002624 & 0.00006864
\end{array}
$$

so the polynomial 1 should be:

$$
\begin{aligned}
P(x) = & -0.29004996 \\
& -2.8019975(x-0.1) \\
& +0.945237(x-0.1)^2 \\
& -0.297(x-0.1)^2(x-0.2) \\
& -0.47935(x-0.1)^2(x-0.2)^2 \\
& -1299.97225(x-0.1)^2(x-0.2)^2(x-0.3)
\end{aligned}
$$

the polynomial 2 should be:

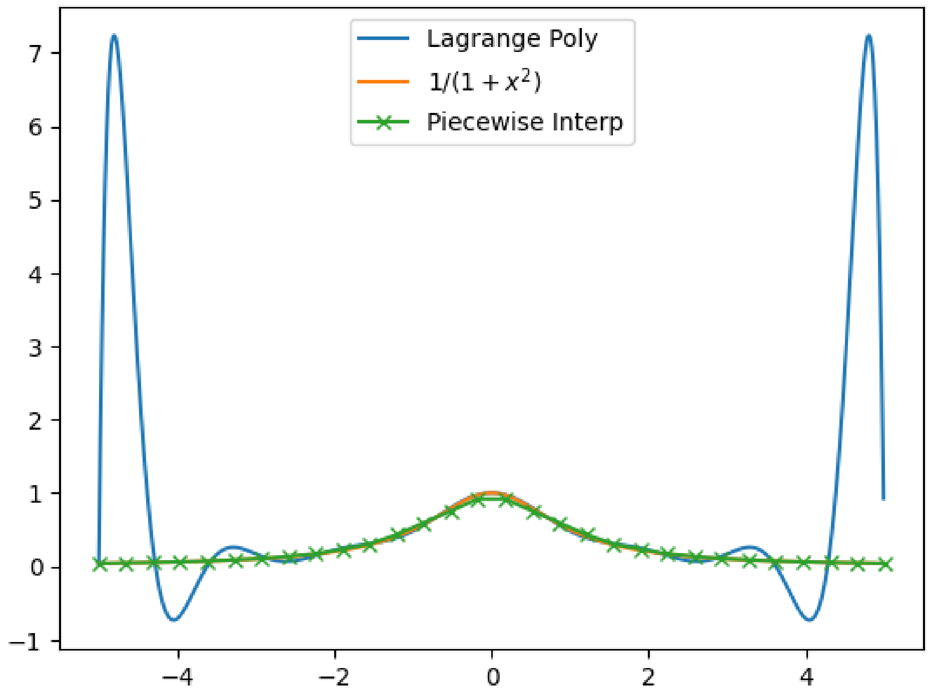
$$
\begin{aligned}
P(x) = & 0.8619948 \\
& +0.1553624(x+1) \\
& +0.07337636(x+1)^2 \\
& +0.01583112(x+1)^2(x+0.5) \\
& -0.00014728(x+1)^2(x+0.5)^2 \\
& -0.00089224(x+1)^2(x+0.5)^2 x \\
& -0.00007672(x+1)^2(x+0.5)^2 x^2 \\
& +0.00006864(x+1)^2(x+0.5)^2 x^2(x-0.5)
\end{aligned}
$$

Exercise 2.7

Choose equally-spaced 15 points in $[-5,5]$ and get the data points

$$\{x_i, f(x_i)\}_{i=0}^{14}$$

by $f(x)=1/(1+x^2)$. Using piecewise linear interpolation to get the approximation of $f(x)$ in $[-5,5]$ and, compare the result with that obtained by Lagrange interpolation.



It can be seen that Lagrange interpolation keeps a good tracing trend in the center area; Piecewise interpolation keeps a good trending in all area;

Exercise 2.8

Find the natural cubic spline through $(0,3),(1,-2),(2,1)$

for 3 points, there are 2 cubics

case 1: for $x \in [0,1)$

$$\begin{aligned} S_0(x) &= a_0 + b_0(x-0) + c_0(x-0)^2 + d_0(x-0)^3 \\ &= a_0 + b_0x + c_0x^2 + d_0x^3 \end{aligned}$$

case 2: for $x \in [1,2]$

$$S_1(x) = a_1 + b_1(x-1) + c_1(x-1)^2 + d_1(x-1)^3$$

for several conditions:

$$\begin{aligned} &S_0(0) = f(0) = 3 \\ &S_1(1) = S_0(1) = f(1) = -2 \\ &S_0'(1) = S_1'(1) \\ &S_0''(1) = S_1''(1) \\ &S_0''(0) = 0 \\ &S_1''(2) = 0 \end{aligned}$$

we can get that

$$\begin{cases} a_0 = 3 \\ b_0 = -7 \\ c_0 = 0 \\ d_0 = 2 \\ a_1 = -2 \\ b_1 = -1 \\ c_1 = 6 \\ d_1 = -2 \end{cases}$$

so the equations by cubic spline should be:

$$S(x) = \begin{cases} 3 - 7x + 2x^3 & x \in [0,1] \\ -2 - (x-1) + 6(x-1)^2 - 2(x-1)^3 & x \in [1,2] \end{cases}$$

Exercise 2.9

Decide whether or not the equations given below from a cubic spline

$$(a) \quad S(x) = \begin{cases} x^3 + x - 1 & \text{on } [0,1] \\ -(x-1)^3 + 3(x-1)^2 + 3(x-1) + 1 & \text{on } [1,2] \end{cases}$$

$$(b) \quad S(x) = \begin{cases} 2x^3 + x^2 + 4x + 5 & \text{on } [0,1] \\ (x-1)^3 + 7(x-1)^2 + 12(x-1) + 12 & \text{on } [1,2] \end{cases}$$

for equation `a`:

- $S_0(x) = x^3 + x - 1$
- $S_1(x) = -(x-1)^3 + 3(x-1)^2 + 3(x-1) + 1$

test all the conditions,

$$\begin{aligned} S_0(1) =& 1, S_1(1) = 1, S_0(1) = S_1(1) \\ S_0'(1) =& 4, S_1'(1) = 3, S_0'(1) \neq S_1'(1) \end{aligned}$$

so equation `a` is not from a cubic spline.

for equation `b`:

- $S_0(x) = 2x^3 + x^2 + 4x + 5$
- $S_1(x) = (x-1)^3 + 7(x-1)^2 + 12(x-1) + 12$

test all the conditions,

$$\begin{aligned} S_0(1) =& 12, S_1(1) = 12, S_0(1) = S_1(1) \\ S_0'(1) =& 12, S_1'(1) = 12, S_0'(1) = S_1'(1) \\ S_0''(1) =& 14, S_1''(1) = 14, S_0'(1) = S_1'(1) \end{aligned}$$

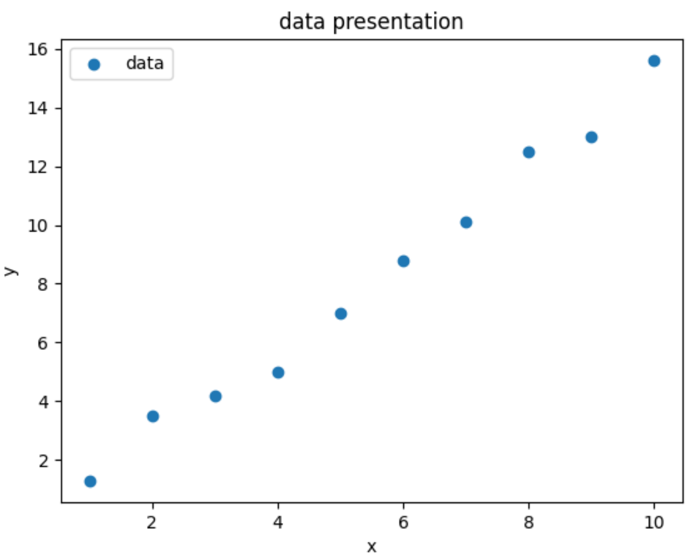
so equation `b` is from a cubic spline.

Exercise 2.12

Find the least squares line approximating the data given in Table below:

x	1	2	3	4	5	6	7	8	9	10
$f(x)$	1.3	3.5	4.2	5.0	7.0	8.8	10.1	12.5	13.0	15.6

first draw the scatter plot of points to choose model



It can be seen that these points is basically linear, according to the least square method

$$a_0 = \frac{\sum_{i=0}^m x_i^2 \sum_{i=0}^m y_i^2 - \sum_{i=0}^m x_i y_i \sum_{i=0}^m x_i}{m(\sum_{i=0}^m x_i^2) - (\sum_{i=0}^m x_i)^2}$$

$$a_1 = \frac{m \sum_{i=0}^m x_i y_i - \sum_{i=0}^m x_i \sum_{i=0}^m y_i}{m(\sum_{i=0}^m x_i^2) - (\sum_{i=0}^m x_i)^2}$$

after calculating

$$a_0 = \frac{-297}{825} = -0.36$$
$$a_1 = \frac{1269}{825} = 1.5382$$

so the line should be:

$$y = -0.36 + 1.5382x$$

Exercise 2.13

Find the discrete least square polynomial of degree at most 2 with the data given in Table below:

x	**0**	**0.25**	**0.5**	**0.75**	**1.0**
$f(x)$	1.0000	1.2840	1.6487	2.1170	2.7183

after some data process

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} 1 & x_1 & {x_1}^2 & \cdots & {x_1}^n \\ 1 & x_2 & {x_2}^2 & \cdots & x_2 \\ \vdots & \vdots & \ddots & \vdots & \\ 1 & x_m & {x_m}^2 & \cdots & {x_m}^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix}$$

according to matrix theory

$$Y = Xa$$
$$a = (X^T X)^{-1} X^T Y$$

for $n = 1$:

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 0.25 \\ 1 & 0.5 \\ 1 & 0.75 \\ 1 & 1 \end{bmatrix}, \quad a = \begin{bmatrix} 0.89968 \\ 1.70784 \end{bmatrix}$$

so the equation should be

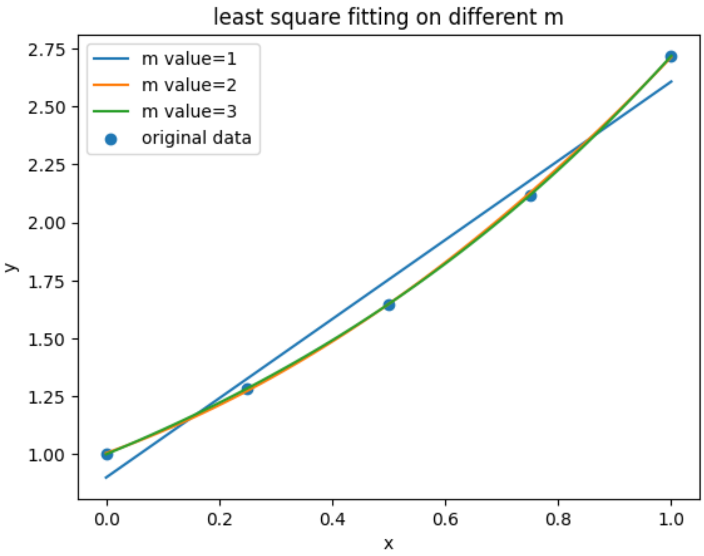
$$y = 0.88968 + 1.70784x$$

for $n = 2$:

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0.25 & 0.0625 \\ 1 & 0.5 & 0.25 \\ 1 & 0.75 & 0.5625 \\ 1 & 1 & 1 \end{bmatrix}, \quad a = \begin{bmatrix} 1.0051 \\ 0.8642 \\ 0.8437 \end{bmatrix}$$

so the equation should be

$$y = 1.0051 + 0.8642x + 0.8437x^2$$



It can be seen that the curve becomes more precise with higher m, but over-fitting is a problem

Exercise 2.14

Find linear and quadratic least-squares approximations to $f(x) = e^x$ on $(-1, 1)$ by using monomial polynomials(or naive polynomials)

for quadratic approximation:

for linear polynomial:

$$P_1(x) = a_0 + a_1 x$$

there should be

$$\begin{aligned} a_0 \int_{-1}^{1} 1 \mathrm{d}x + a_1 \int_{-1}^{1} x \mathrm{d}x &= \int_{-1}^{1} e^x \mathrm{d}x \\ a_0 \int_{-1}^{1} x \mathrm{d}x + a_1 \int_{-1}^{1} x^2 \mathrm{d}x &= \int_{-1}^{1} x e^x \mathrm{d}x \end{aligned}$$

final is

$$P_1(x) = -\frac{5e}{6} - \frac{9}{2e} + (\frac{4e}{3} - \frac{4}{e})x$$

for quadratic polynomial:

$$P_2(x) = a_0 + a_1 x + a_2 x^2$$

there should be:

$$\begin{aligned} a_0 \int_{-1}^{1} 1 \mathrm{d}x + a_1 \int_{-1}^{1} x \mathrm{d}x + a_2 \int_{-1}^{1} x^2 \mathrm{d}x &= \int_{-1}^{1} e^x \mathrm{d}x \\ a_0 \int_{-1}^{1} x \mathrm{d}x + a_1 \int_{-1}^{1} x^2 \mathrm{d}x + a_2 \int_{-1}^{1} x^3 \mathrm{d}x &= \int_{-1}^{1} x e^x \mathrm{d}x \\ a_0 \int_{-1}^{1} x^2 \mathrm{d}x + a_1 \int_{-1}^{1} x^3 \mathrm{d}x + a_2 \int_{-1}^{1} x^4 \mathrm{d}x &= \int_{-1}^{1} x^2 e^x \mathrm{d}x \end{aligned}$$

final is

$$\begin{aligned} a_0 &= -0.6045e + 4.0076e^{-1} \\ a_1 &= 0.5373e - 0.8955e^{-1} \\ a_2 &= 1.7015e - 10.8358e^{-1} \end{aligned}$$