Fourier Transform Notes

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Appendix C – The Fourier Transform

Pages 11

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https://github.com/JaimeRMC/Sien

Sources

- Text: Computational Physics (Giordano)
- New markdown techniques: markdownguide.org

Section C.1 – Theoretical Background

Purpose

The Fourier Transform allows a signal \y(t)\ to be decomposed into its frequency components. This is especially useful in physics and signal processing, where analyzing the behavior of a signal in the frequency domain is often more insightful than viewing it in the time domain.

Simple Representation

A time-domain signal can be approximated by a sum of sinusoids:

$$\text{(C.1)}\quad y(t) = \sum_j A_j \sin(2\pi f_j t + \phi_j)$$

Where:

- A_j: amplitude of the jth frequency component
- f_j: frequency of the jth component
- \ \phi_j : phase offset of the jth component

Continuous Fourier Transform

To capture all possible frequencies, we generalize the representation using integrals over all \f:

Forward Fourier Transform:

$$\text{(C.2)}\ \ Y(f) = \int_{-\infty}^{\infty} y(t)e^{-2\pi i f t}dt$$

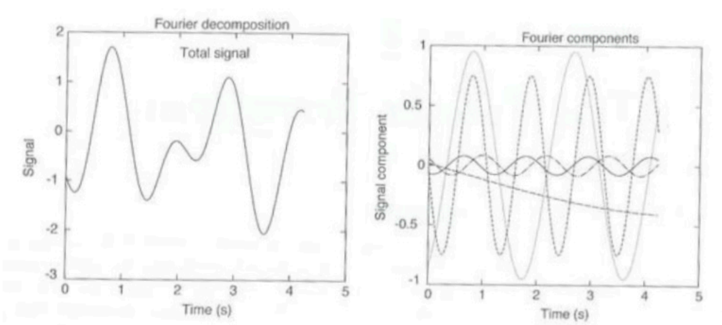
Inverse Fourier Transform:

$$\text{(C.3)}\ \ y(t) = \int_{-\infty}^{\infty} Y(f)e^{2\pi i f t}df$$

Where:

- Y(f) is the frequency-domain representation of the signal
- y(t) is the time-domain signal
- f: frequency (Hz)
- i: imaginary unit

Figure C.1



- Left: Hypothetical time-domain signal
- Right: Superposition of sine wave components that recreate the signal

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Section C.2 – Discrete Fourier Transform (DFT)

Discretizing the Signal

In practice, signals are sampled at discrete intervals:

$$t_n = n\Delta t, \quad y_n = y(t_n), \quad n = 0, 1, \ldots, N-1$$

Where:

- \Delta t: time step between samples
- \N: number of total samples

Discrete Fourier Transform

Forward DFT:

$$\text{(C.6)} \;\; Y_k = \sum_{n=0}^{N-1} y_n e^{-2\pi ikn/N}$$

Inverse DFT:

$$\text{(C.7)} \;\; y_n = \frac{1}{N} \sum_{k=0}^{N-1} Y_k e^{2\pi ikn/N}$$

Where:

- Y_k: Fourier coefficient for frequency index k
- y_n: time-domain sample at n

Section C.3 – Fast Fourier Transform (FFT)

Purpose

Calculating the DFT directly takes O(N^2) operations. The Fast Fourier Transform (FFT) is an algorithm that reduces this to O(Nlog(N)), enabling fast analysis for large datasets.

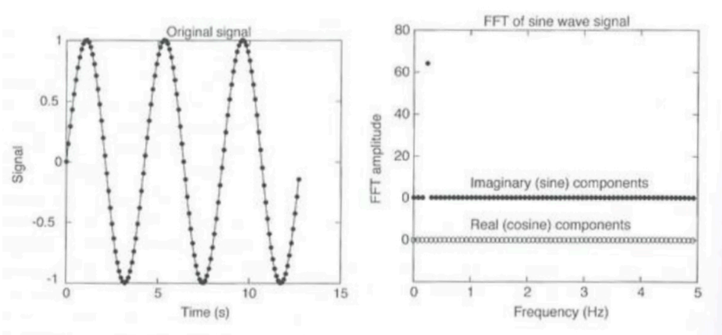
Requirements

- Most FFT algorithms assume N = 2^m for some integer m
- Uses divide-and-conquer and symmetry properties of exponentials

FFT Algorithm Steps (Example C.1)

1. Read in data
2. Check that N is a power of 2
3. Rearrange input in bit-reversed order
4. Recursively combine results using butterfly operations
5. Normalize if needed

Figure C.2



Illustrates signal and how it is decomposed and reconstructed using the FFT

Section C.4 – Sampling Interval and Number of Data Points

Sampling Effects

The duration and resolution of the FFT are affected by:

- Number of samples: (N)
- Time interval between samples: (\Delta t)

These determine:

- **Total signal duration:**

$$T = N\Delta t$$

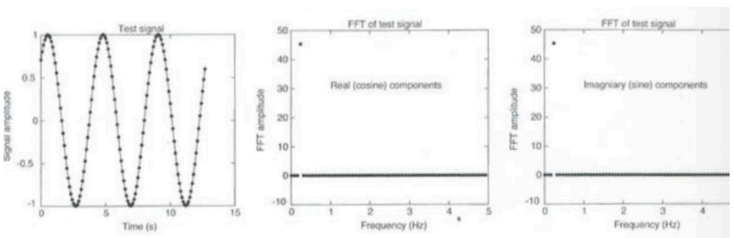
- **Frequency resolution:**

$$\Delta f = \frac{1}{T}$$

- **Nyquist frequency:**

$$f_{\text{Nyquist}} = \frac{1}{2\Delta t}$$

Figure C.3



Shows effects of sampling interval and number of samples on the accuracy of FFT output

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Section C.5 – Examples: Aliasing

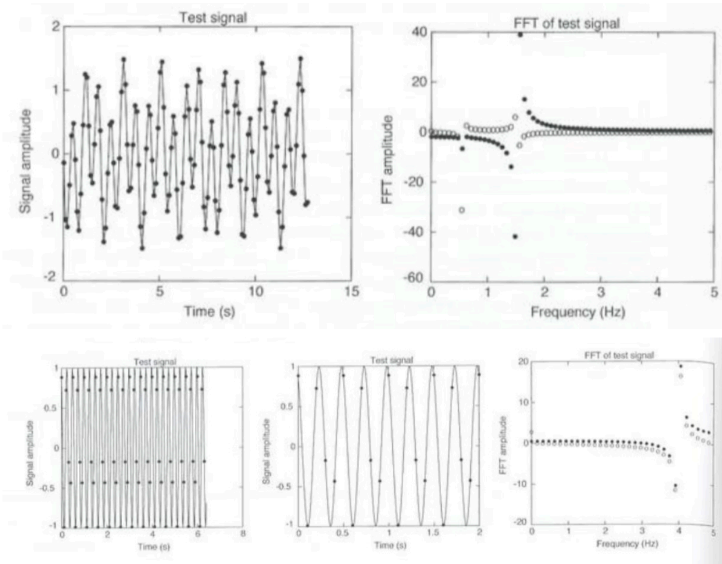
Concept

Aliasing occurs when the sampling frequency is too low to capture the highest frequency in the signal. Frequencies above the Nyquist limit are folded into lower frequencies and misrepresented in the FFT.

To avoid aliasing:

$$f_{\text{sample}} > 2f_{\text{max}} \Rightarrow \Delta t < \frac{1}{2f_{\text{max}}}$$

Figure C.4 and C.5



Demonstrate aliasing effect with different sampling rates and distorted FFT results caused by undersampling

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Section C.6 – Power Spectrum

Purpose

The power spectrum shows how the power (energy per unit time) of a signal is distributed over different frequency components.

Given the Fourier Transform (Y(f)), the power spectrum is:

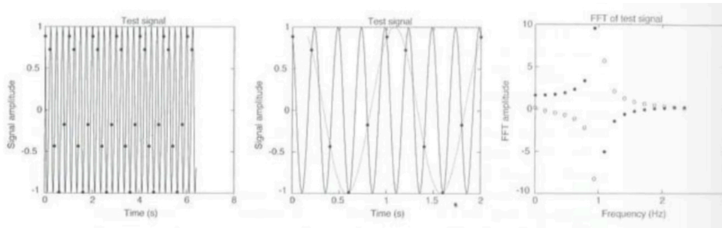
$$\text{(C.13)}\quad P(f) = \frac{1}{T}|Y(f)|^2$$

Parseval's Theorem

Connects total energy in time and frequency domains:

$$\text{(C.15)}\quad \int_{-\infty}^{\infty} |y(t)|^2 dt = \int_{-\infty}^{\infty} |Y(f)|^2 df$$

Figure C.6



Power spectrum plot with peaks at dominant signal frequencies

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Summary of Key Equations

Continuous Fourier Transform:

$$Y(f) = \int_{-\infty}^{\infty} y(t) e^{-2\pi i f t} dt$$

$$y(t) = \int_{-\infty}^{\infty} Y(f) e^{2\pi i f t} df$$

Discrete Fourier Transform:

$$Y_k = \sum_{n=0}^{N-1} y_n e^{-2\pi i k n / N}$$

$$y_n = \frac{1}{N} \sum_{k=0}^{N-1} Y_k e^{2\pi i k n / N}$$

Nyquist Frequency:

$$f_{\text{Nyquist}} = \frac{1}{2\Delta t}$$

Power Spectrum:

$$P(f) = \frac{1}{T} |Y(f)|^2$$

Parseval's Theorem:

$$\int |y(t)|^2 dt = \int |Y(f)|^2 df$$

Questions

1. Why does the FFT assume periodic extension of the signal? What implications does this have for edge behavior?
2. In what cases would the FFT be inappropriate or misleading?
3. How are FFTs adapted for non-uniformly sampled data?

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