IMPLEMENTATION OF QUANTUM ERROR CORRECTION USING THE FAMILY OF $[[n, 1, 3]]$ QECCS

Naman Jain
University of Southern California
jainnama@usc.edu

ABSTRACT

This project explores the implementation of $[[n, 1, 3]]$ Quantum Error Correcting Codes (QECCs) for n = 3, 5, 7, utilizing Qiskit's dynamic circuits for hardware execution and AerSimulator for noise simulation, along with using pre-loaded noise models from earlier devices. Inspired by classical repetition codes, the bit-flip and phase-flip codes are scrutinized for their efficacy in mitigating errors. The study encompasses theoretical foundations, practical methodology, and results analysis on simulators and actual quantum hardware. The effects of noise and the importance of fault tolerance are also discussed at the end.

1 Introduction

Quantum computers represent a revolutionary paradigm in computation, offering unprecedented speedups for specific problems in domains such as chemistry, machine learning, biology, and finance. These quantum advantages stem from the exploitation of quantum mechanical phenomena, during computational processes. However, the same properties that make quantum computing so powerful also make it extremely prone to errors. The delicate nature of quantum information renders quantum systems susceptible to errors induced by various environmental factors. The advent of Quantum Error Correcting Codes (QECCs) addresses this inherent vulnerability, providing a crucial framework to navigate the delicate landscape of quantum information. In contrast to classical error correction, which typically involves duplicating information to detect and rectify errors, QECCs harness quantum principles to shield quantum information against the detrimental effects of noise and decoherence.

QECCs, such as the $[[n, 1, 3]]$ family under investigation in this project, play a crucial role in realizing the full potential of quantum computers. These codes employ techniques to encode quantum information in a manner that allows for the detection and correction of errors without directly measuring the quantum state. In the notation $[[n, k, d]]$, the code corrects or detects errors of weight $(d-1)/2$ or $d-1$, respectively, and utilizes n physical qubits to encode k logical qubits. The significance of QECCs extends beyond theoretical advancements, as they are indispensable for achieving fault-tolerant quantum computation. The intrinsic fragility of quantum bits, or qubits, necessitates the development and implementation of robust error correction strategies to enable practical and scalable quantum computing.

This project focuses on implementing the $[[n, 1, 3]]$ QECCs for $n = 3, 5, 7$ aiming to explore their efficacy in mitigating errors in quantum systems. In the case of $n = 3$, both the bit-flip and phase-flip codes have been implemented. The $n = 5$ scenario showcases the implementation of the perfect code, while the $n = 7$ case features the implementation of the Steane code. The subsequent sections of this report delve into the theoretical foundations of these codes, outlining the methodology employed for their implementation, and presenting the results obtained through experiments conducted on both quantum simulators and actual hardware devices.

2 Theoretical formalism

2.1 3 qubit code

2.1.1 3 qubit bit-flip code

The 3 qubit bit-flip and phase-flip codes are inspired from the classical repetition codes. Consider the single qubit bit-flip noise channel $\mathcal{N}_{\mathcal{X}}(\rho) = (1-p)\rho + pX\rho X$. To protect against this noise channel, a simple strategy is to take advantage of the fact that orthogonal quantum states may be cloned to create an encoding of the form,

$$|0\rangle \mapsto |000\rangle = |\bar{0}\rangle\,, |1\rangle \mapsto |111\rangle = |\bar{1}\rangle$$

so that any one qubit state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ can be encoded as $|\overline{\psi}\rangle = \alpha\,|000\rangle + \beta\,|111\rangle$. A unitary that does this encoding is shown in Fig. 1 in circuit notation.

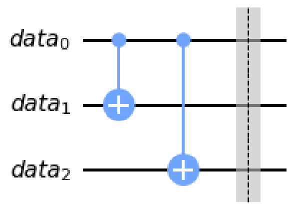

Figure 1: Enoding circuit for 3 qubit bit-flip code. Note that $data_0$ holds $|\psi\rangle$.

On measuring the parity of the bits $(1, 2)$ and $(2, 3)$ via the stabilizers ZZI, IZZ, any single qubit error on one of the 3 qubits can be detected and subsequently recovered from by applying the suitable recovery operator. Table 1 lists the error syndrome look up and the corresponding recovery operator. The circuit implementation is discussed in 3.

Error	**Syndrome (Classical register value)**	**Recovery**
X_1	1	X_1
X_2	3	X_2
X_3	2	X_3

Table 1: Look up table for 3 qubit bit-flip code

2.1.2 3 qubit phase-flip code

The 3 qubit phase-flip code is obtained naturally by changing from the computational basis to the X basis via Hadamards. The action of the single-qubit phase-flip channel on a state ρ is described by $\mathcal{N}_{\mathcal{Z}}(H\rho H) = H\mathcal{N}_{\mathcal{X}}(\rho)H$. In other words, by changing the encoding to

$$|0\rangle \mapsto |\bar{0}\rangle = |+++\rangle\,, |1\rangle \mapsto |\bar{1}\rangle = |---\rangle$$

any one qubit state $|\overline{\psi}\rangle = \alpha\,|+++\rangle + \beta\,|+++\rangle$ can be protected against single qubit phase-flip errors. A unitary that does this encoding is shown in Fig. 2 below.

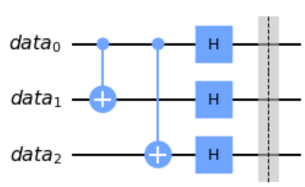

Figure 2: Enoding circuit for 3 qubit phase-flip code. Note that $data_0$ holds $|\psi\rangle$.

To correctly identify and rectify the phase-flip error, the parity of bits $(1, 2)$ and $(2, 3)$ are now measured in the $|\pm\rangle$ basis via the stabilizers XXI, IXX. The error syndrome and recovery operators are indicated in Table 2.

Error	Syndrome (Classical register value)	Recovery
Z_1	1	Z_1
Z_2	3	Z_2
Z_3	2	Z_3

Table 2: Look up table for 3 qubit phase-flip code

2.2 5 qubit code

The minimum size for a quantum code that encodes a single qubit so that any error on a single qubit in the encoded state can be detected and corrected is five qubits [1]. The 5 qubit "perfect" code is represented via its stabilizer generators

$$
\begin{gathered}
ZXXZI \\
IZXXZ \\
ZIZXX \\
XZIZX
\end{gathered}
$$

The encoded logical codewords for the five qubit code are

$$
\begin{aligned}
|\bar{0}\rangle = \frac{1}{4}\Big[& |00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle + |01010\rangle - |11011\rangle - |00110\rangle - |11000\rangle \\
& - |11101\rangle - |00011\rangle - |11110\rangle - |01111\rangle - |10001\rangle - |01100\rangle - |10111\rangle + |00101\rangle \Big] \\
|\bar{1}\rangle = \frac{1}{4}\Big[& |11111\rangle + |01101\rangle + |10110\rangle + |01011\rangle + |10101\rangle - |00100\rangle - |11001\rangle - |00111\rangle \\
& - |00010\rangle - |11100\rangle - |00001\rangle - |10000\rangle - |01110\rangle - |10011\rangle - |01000\rangle + |11010\rangle \Big]
\end{aligned}
$$

and the encoding circuit is shown in Fig. 3

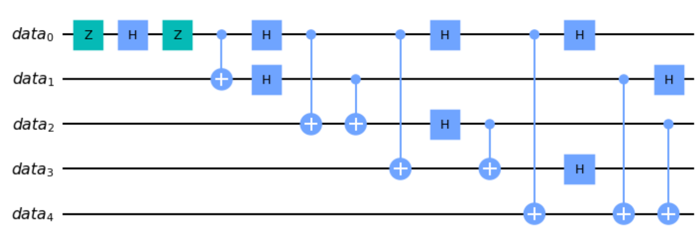

Figure 3: Enoding circuit for 5 qubit perfect code. Note that $data_0$ holds $|\psi\rangle$.

The error syndrome and recovery operators are indicated in Table 3 below.

X_1	1001	Z_1	0110	Y_1	1111
X_2	0010	Z_2	1100	Y_2	1110
X_3	0101	Z_3	1000	Y_3	1101
X_4	1010	Z_4	0001	Y_4	1011
X_5	0100	Z_5	0011	Y_5	0111

Table 3: Look up table for 5 qubit code. The recovery operators are inverses of the errors, that is, $R = E_i^\dagger$ for error E_i

2.3 7 qubit

The 7 qubit Steane code is yet another distance 3 stabilizer code that encodes 1 logical quibt in 7 physical qubits. The stabilizers of the Steane code are

$$
\begin{array}{c}
IIIXXXX \\
IXXIIXX \\
XIXIXIX \\
IIIZZZZ \\
IZZIIZZ \\
ZIZIZIZ
\end{array}
$$

and the encoded codewords are

$$
\begin{aligned}
|\bar{0}\rangle = \frac{1}{\sqrt{8}}\Big[& |0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle \\
& + |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle \Big] \\
|\bar{1}\rangle = \frac{1}{\sqrt{8}}\Big[& |1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle \\
& + |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle \Big]
\end{aligned}
$$

The encoding circuit is shown in Fig. 4.

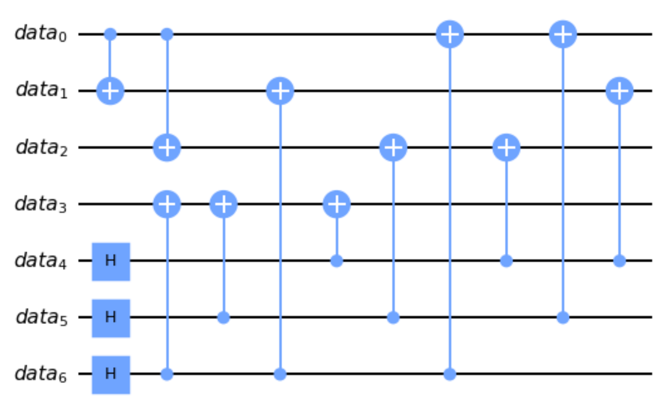

Figure 4: Enoding circuit for 7 qubit Steane code. Note that $data_0$ holds $|\psi\rangle$.

The error syndrome and recovery operators are indicated in Table 4 below.

Error	bit_syn	Error	phase_syn
X_1	1	Z_1	1
X_2	2	Z_2	2
X_3	3	Z_3	3
X_4	4	Z_4	4
X_5	5	Z_5	5
X_6	6	Z_6	6
X_7	7	Z_7	7

Table 4: Look up table for 7 qubit code. The recovery operators are inverses of the errors, that is, $R = E_i^\dagger$. Although the table doesn't explicitly list the Y errors, it is understood that whenever both bit and phase flip errors occur

3 Implementation methodology

The implementation was carried out using Qiskit [2], a comprehensive quantum computing SDK. To implement the stabilizer measurements in error syndrome identification, additional ancilla qubits are used. The circuits for syndrome measurement are shown in Fig. 5.

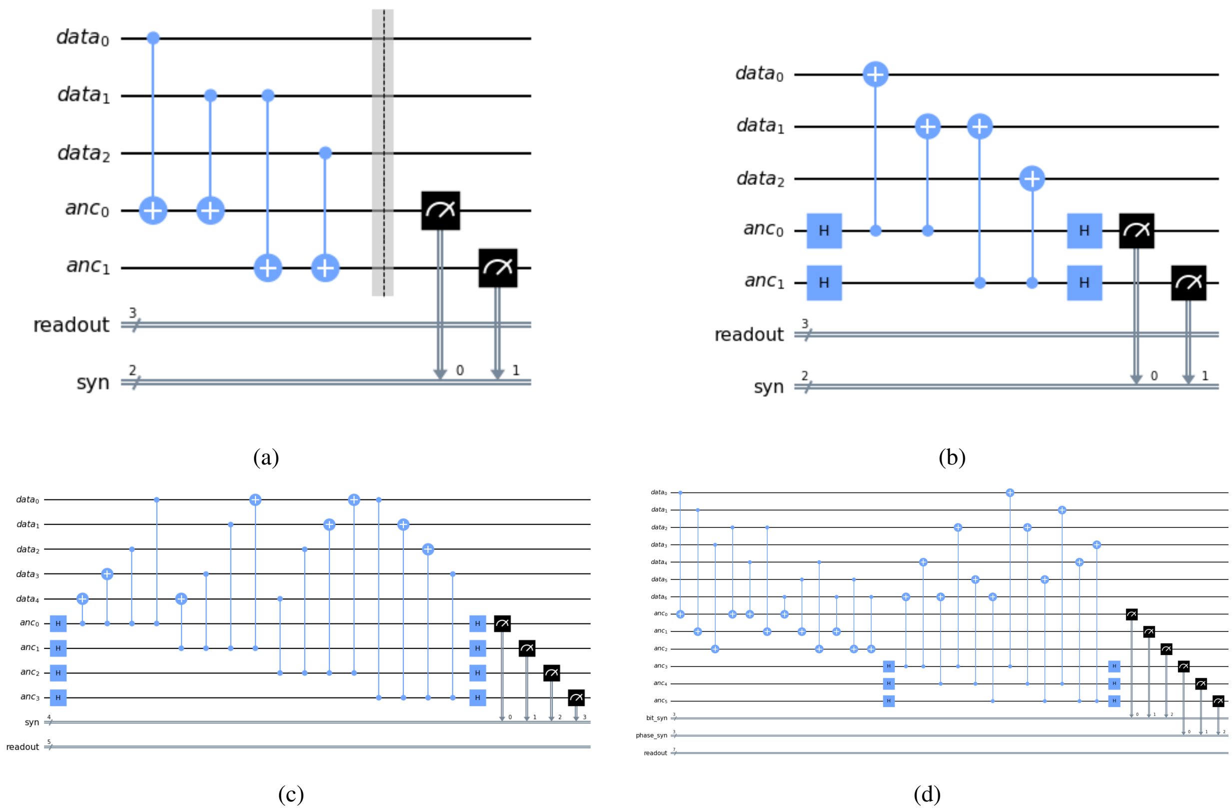

Figure 5: Syndrome measurement circuits for the QECCs implemented in this work. a) 3 qubit bit-flip code b) 3 qubit phase-flip code c) 5 qubit "perfect" code d) 7 qubit Steane code

Recently, Qiskit added a feature known as Dynamic circuits. A quantum circuit is a sequence of quantum operations — including gates, measurements, and resets — acting on qubits. In static circuits, none of those operations depend on data produced at run time. Dynamic circuits, on the other hand, incorporate classical processing within the coherence time of the qubits. This means that dynamic circuits can make use of mid-circuit measurements and perform feed-forward operations, using the values produced by measurements to determine what gates to apply next. This feature enables performing conditional mid-circuit measurements to apply suitable recovery operators once the error syndrome is identified.

The current hardware capabilities of IBM's quantum computing processors are limited. Hence, a fake 27 qubit backend known as *FakeMontreal* has also been used to simulate the results and observe the effects of using QECCs. All circuits created were executed on the backends with 1024 shots and the resulting probability distributions obtained were analysed.

A full circuit snapshot of the 7 qubit code is shown in Fig. 6

The entire code implementation is available on GitHub.[1].

[1]https://github.com/Jain-Naman/qecc_codes

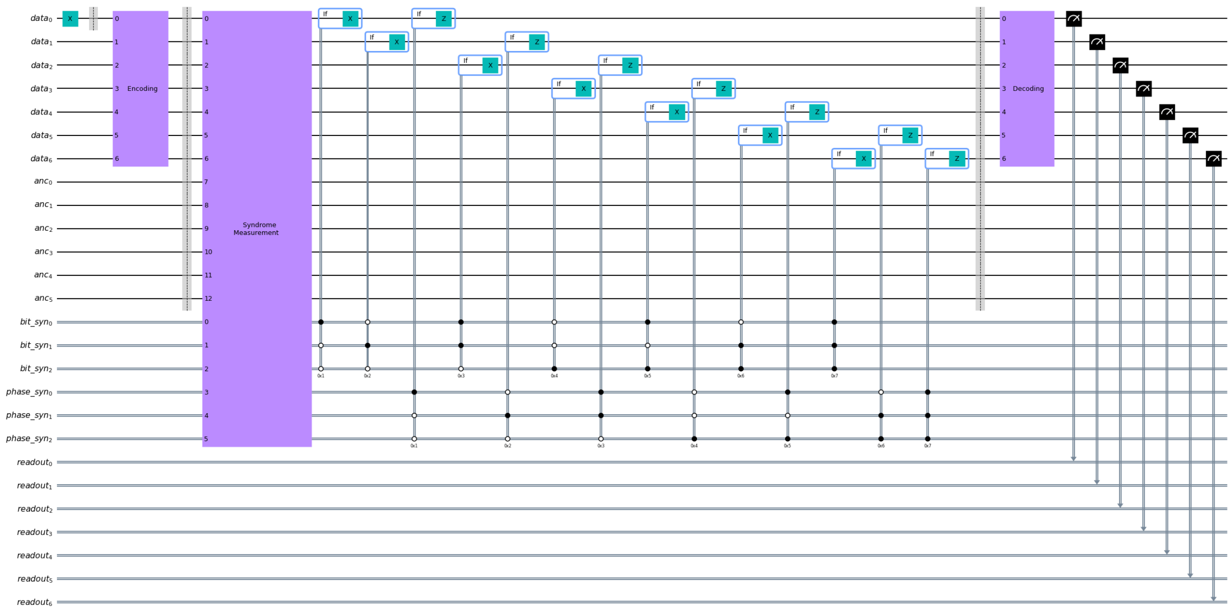

Figure 6: Complete circuit for the 7-qubit Steane code. The initial state is $|\psi\rangle = |1\rangle$. The mid-circuit conditional values are shown in hexadecimal. The decoding circuit is inverse of the encoding

Code	with correction		without correction	
	fidelity	**error outcomes**	**fidelity**	**error outcomes**
$[[3, 1, 3]]$ bit-flip	0.898	104/1024	0.902	100/1024
$[[3, 1, 3]]$ phase-flip	0.918	82/1024	0.933	68/1024
$[[5, 1, 3]]$	0.775	230/024	0.784	221/1024
$[[7, 1, 3]]$	0.729	277/1024	0.778	227/1024

Table 5: The results obtained for all the codes implemented.

4 Results and Discussion

To compare the results obtained with and without error correction, a metric known as the Hellinger fidelity is employed. The fidelity is defined as $(1 - H)^2$ where H is the Hellinger distance. This value is bounded in the range $[0, 1]$. For two discrete probability distributions $P = (p_1, p_2 \dots p_k)$ and $Q = (q_1, q_2 \dots q_k)$ their Hellinger distance is defined as

$$H(P, Q) = \frac{1}{\sqrt{2}} \sqrt{\sum_{i=1}^{k} (\sqrt{p_i} - \sqrt{q_i})^2}$$

.

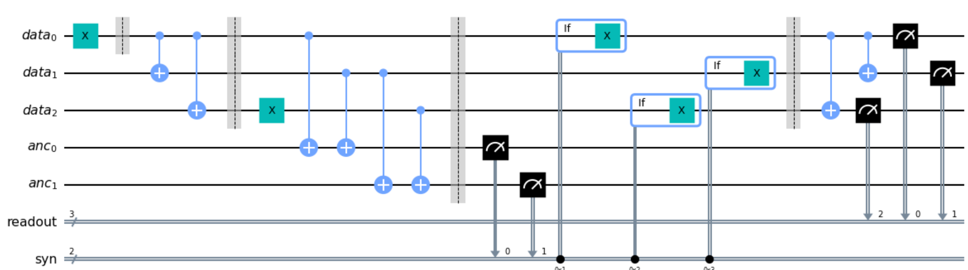

Figure 7: A circuit for 3 qubit bit-flip code. The X gate on the 3rd qubit after the second barrier is applied deliberately to simulate a bit-flip noise.

The results obtained on actual hardware are extremely poor. For instance, for the 7 qubit Steane code executed on the *ibm_brisbane* device with 127 qubits, the fidelity obtained is as low as 0.0097 with error correction and 0.0078 without error correction. Similar trends were obtained on executing the 5 qubit code on *ibm_kyoto* machine.

The results obtained on *FakeMontreal* are shown in Table 5. The table indicates that the results with error correction are worse than those without error correction. This is due to the fact that the syndrome measurements performed in this implementation are not fault-tolerant which leads to noise from the ancilla qubits creeping into the data qubits. However, to check the accuracy of the implementation, deliberate bit and phase flips were applied, like the one shown in Fig. 7. The results in such cases showed that the circuit without error correction had a fidelity of 0 while those with error correction were perfectly preserved.

References

[1] Daniel Gottesman. An introduction to quantum error correction and fault-tolerant quantum computation, 2009.

[2] Qiskit contributors. Qiskit: An open-source framework for quantum computing, 2023.