# RL HOMEWORK ASSIGNMENT 1

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## Exercise 1 [20 points]

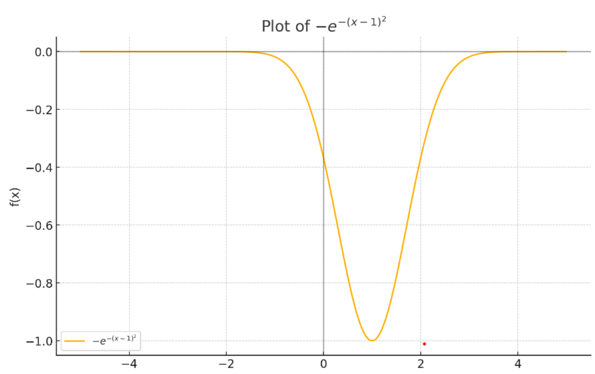
Find all the minimum(s), if they exist, of the functions below. Characterize the type of minimum (global, local, strict, etc) and justify your answers (hint: you can plot the functions in python to get an intuition of their form).

- $-\mathrm{e}^{-(x-1)^2}$, where $x \in \mathbb{R}$
- $(1-x)^2 + 100(y-x^2)^2$, where $x, y \in \mathbb{R}$
- $20x + 2x^2 + 4y - 2y^2$, where $x, y \in \mathbb{R}$
- $x^T \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} x + \begin{bmatrix} -1 & 1 \end{bmatrix} x$, where $x \in \mathbb{R}^2$
- $x^T \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 10 \end{bmatrix} x$, where $x \in \mathbb{R}^2$
- $\frac{1}{2} x^T \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} x - \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x$, where $x \in \mathbb{R}^3$

**ANSWER.**
**1)** Here to find the minimum of

$$-e^{-(x-1)^2} \text{ where } x \in R$$

we first need to plot the function which will make it easier to identify the minimum.
Consider the below-given plot of $f(x) = -e^{-(x-1)^2}$ where $x \in R$




Based on the graph we can see that the f(x) reaches its minimum at x = 1
thus

$$at\ x = 1,\ f(x) = -e^{-(x-1)^2} = -1$$

This gives us the global minimum at x = 1
Now we will find the first derivative of f(x)

$$\nabla f(x) = 2(x-1)e^{-(x-1)^2} = 0\ at\ x = 1$$

now since x = 1 also give the $\nabla f(x) = 0$ it means the its also a local minimum at x = 1

$$\nabla^2 f(x) = 2e^{-(x-1)^2} \left(2(x-1)^2 - 1\right) = 0 \ at \ x = 1$$

here since the f(x) is not greater than 0 so it is not a strict minimum

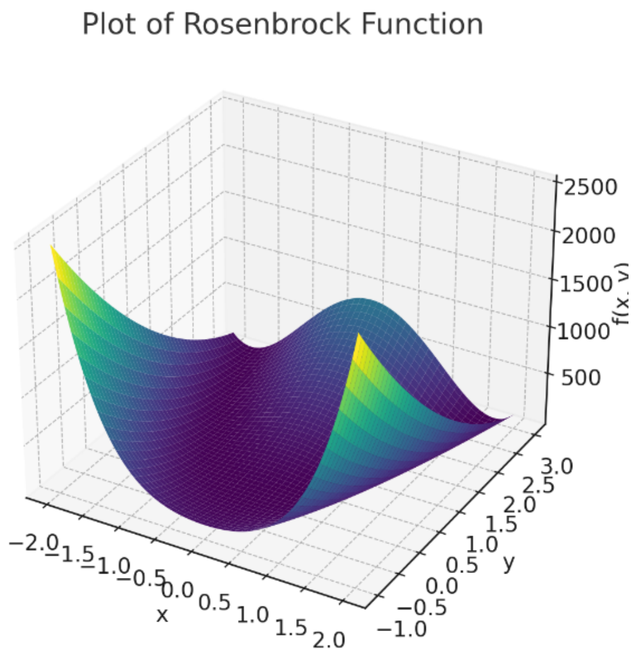
Thus there exist only global minimum and local minimum which is at point (1,0) and its value is -1 .

**2)** The given function is:

$$f(x, y) = (1 - x)^2 + 100(y - x^2)^2$$

This is the Rosenbrock function, and we will analyze its minima,including whether they are global, local, or strict.

Now to understand more about the minimum we would plot it on a graph :




here we can see that we can see from the graph that the minimum exists when the

$$f(x, y) = (1 - x)^2 + 100(y - x^2)^2 = 0$$

for that to happen we have two conditions to be satisfied :

1.

$$(1 - x)^2 = 0$$

$$(1 - x) = 0$$

$$x = 1$$

2.

$$100(y - x^2)^2 = 0$$

$$y - x^2 = 0$$

$$y = x^2$$

now from one since we got that x = 1 , on putting that in second condition , we get

$$y = x^2 = 1$$

Therefore the point of global minimum is x = 1 , y = 1

Now we need to check for the local and strict local minimum for which we need to get gradient of f(x, y) in terms of x and y

The gradient of the function $f(x,y) = (1-x)^2 + 100(y-x^2)^2$ is the vector of its partial derivatives with respect to $x$ and $y$.

The partial derivatives are:
1. Partial derivative with respect to $x$:

$$\frac{\partial f}{\partial x} = -2(1-x) - 400x(y-x^2)$$

2. Partial derivative with respect to $y$:

$$\frac{\partial f}{\partial y} = 200(y-x^2)$$

Thus, the gradient $\nabla f(x,y)$ can be written as a vector (or matrix):

$$\nabla f(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} -2(1-x) - 400x(y-x^2) \\ 200(y-x^2) \end{bmatrix}$$

This gradient vector of the above function f(x, y) at x = 1, and y = 1 is give as

$$\nabla f(x,y) = \begin{bmatrix} -2(1-x) - 400x(y-x^2) \\ 200(y-x^2) \end{bmatrix} = 0$$

which implies that we also have a local minimum at x = 1 and y = 1

Now we need to check is its a strict local minimum, we can so this by finding out $\nabla^2 f(x,y)$ at x = 1 and y = 1

Next, we calculate the second-order partial derivatives to form the $\nabla^2 f(x,y)$ .
1. Second-order derivative with respect to $x$:

$$\frac{\partial^2 f}{\partial x^2} = 2 - 400(y-x^2) + 800x^2$$

At the critical point $(1,1)$:

$$\frac{\partial^2 f}{\partial x^2} = 2 - 400(1-1^2) + 800(1^2) = 2 + 800 = 802$$

2. Second-order derivative with respect to $y$:

$$\frac{\partial^2 f}{\partial y^2} = 200$$

3. Mixed second-order derivative:

$$\frac{\partial^2 f}{\partial x \partial y} = -400x$$

At $x = 1$:

$$\frac{\partial^2 f}{\partial x \partial y} = -400(1) = -400$$

$\nabla^2 f(x,y)$ at x = 1 and y = 1 is :

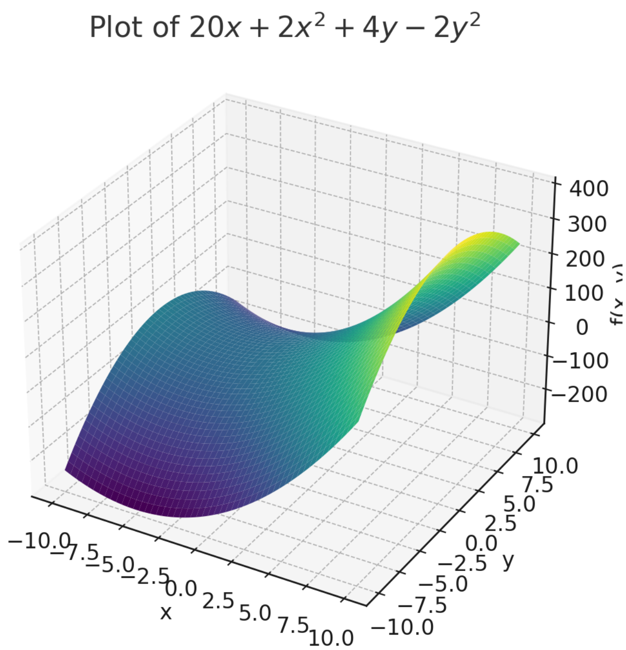
$$\nabla^2 f(x,y) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 802 & -400 \\ -400 & 200 \end{bmatrix} > 0$$

Since $\nabla^2 f(x,y)) > 0$ at x = 1 and y = 1, the minimum at this is point is a strict local minimum.
Therefore for given f(x,y) **the minimum is a global minimum and a strict local minimum** and this
point is (1,1) .

**3)**
The given function is:

$$f(x, y) = 20x + 2x^2 + 4y - 2y^2 \ where \ x, y \in R$$

To find the minimum take the partial derivatives of $f(x, y)$ with respect to $x$ and $y$ and set them equal to zero.




Partial derivative with respect to $x$:

$$\frac{\partial f(x, y)}{\partial x} = 20 + 4x = 0$$

Solving for $x$:

$$x = -5$$

Partial derivative with respect to $y$:

$$\frac{\partial f(x, y)}{\partial y} = 4 - 4y = 0$$

Solving for $y$:

$$y = 1$$

Thus, the minimum point is $(x^*, y^*) = (-5, 1)$.

Second Derivatives

Next, we use the second derivative to analyze the nature of the minimum. We compute the $\nabla^2 f(x, y)$, which is the matrix of second-order partial derivatives.

Second derivative with respect to $x$:

$$\frac{\partial^2 f(x, y)}{\partial x^2} = 4$$

Second derivative with respect to $y$:

$$\frac{\partial^2 f(x, y)}{\partial y^2} = -4$$

Mixed partial derivative:

$$\frac{\partial^2 f(x,y)}{\partial x \partial y} = 0$$

Thus, the $\nabla^2 f(x,y)$ is:

$$\nabla^2 f(x,y) = \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}$$

The determinant of the $\nabla^2 f(x,y)$ is given by:

$$\det(H) = (4)(-4) - (0)(0) = -16$$

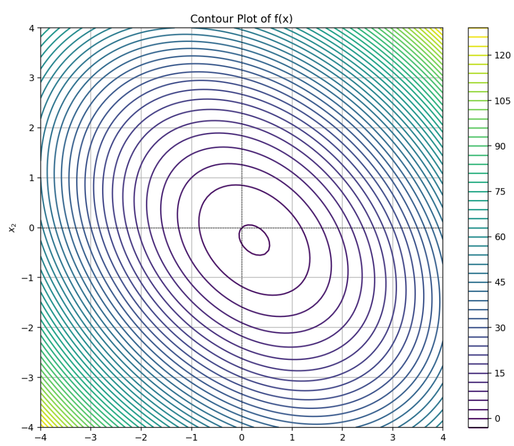
Since the determinant of the $\nabla^2 f(x,y)$ is negative, the point $(-5,1)$ is a saddle point and not a minimum.

There is **no global minima and a local minima** for this function which can be defined without taking infinity into consideration.

**4)** The given function whose minimum we have to find and classify is:

$$f(x) = x^T \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} x + \begin{pmatrix} -1 & 1 \end{pmatrix} x \ where \ x \in R^2$$

we can also plot its graph for better understanding a to determine where the minimum would be a global minimum also




Let's denote $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Then the function becomes:

$$f(x) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Expand the Function:

$$f(x) = 3x_1^2 + 2x_1x_2 + 3x_2^2 - x_1 + x_2$$

Now we need to Find the Gradient:

To find the minimum points we need to set the gradient of $f(x)$ to zero.

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 6x_1 + 2x_2 - 1 \\ 2x_1 + 6x_2 + 1 \end{pmatrix}$$

Set $\nabla f(x) = 0$:

$$\begin{cases} 6x_1 + 2x_2 - 1 = 0 \\ 2x_1 + 6x_2 + 1 = 0 \end{cases}$$

Solve the system to find the $x_1, x_2$:

$$\begin{cases} 6x_1 + 2x_2 = 1 \\ 2x_1 + 6x_2 = -1 \end{cases}$$

Multiply the second equation by 3:

$$\begin{cases} 6x_1 + 2x_2 = 1 \\ 6x_1 + 18x_2 = -3 \end{cases}$$

Subtract the first equation from the second:

$$16x_2 = -4 \implies x_2 = -\frac{1}{4}$$

Substitute $x_2$ back into the first equation:

$$6x_1 + 2\left(-\frac{1}{4}\right) = 1 \implies 6x_1 - \frac{1}{2} = 1 \implies 6x_1 = \frac{3}{2} \implies x_1 = \frac{1}{4}$$

So, the global minimum is at $x_1 = \frac{1}{4}, x_2 = -\frac{1}{4}$

Now we need to check if this global minimum is also a local minimum or a strict local minimum, for that we need to calculate $\nabla^2 f(x)$

To to calculate $\nabla^2 f(x)$ :

$$\nabla^2 f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}$$

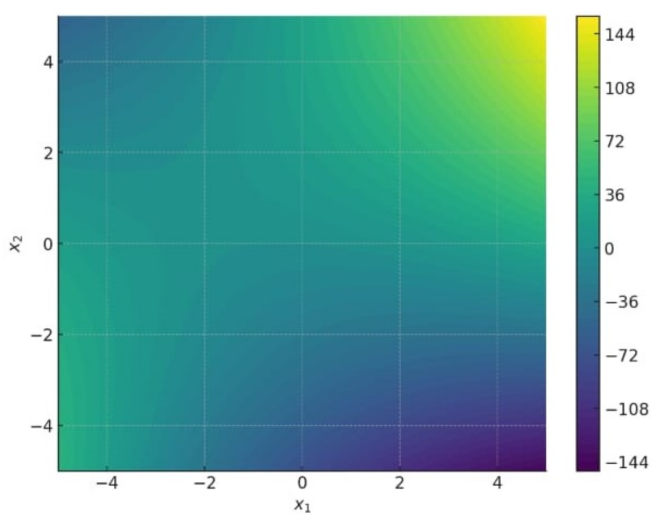
The $\nabla^2 f(x)$ is positive definite (since its eigenvalues are positive), indicating a strict local minimum.

Therefore for given f(x) **the minimum is a global minimum and a strict local minimum.**

**5)** The given function whose minimum we have to find and classify is:

$$f(x) = x^T \begin{pmatrix} 1 & 2 \\ 2 & ‘1 \end{pmatrix} x + \begin{pmatrix} 1 & 10 \end{pmatrix} x \ where\ x \in R^2$$

we can also plot its graph for better understanding a to determine where the minimum would be a global minimum also



Let's denote $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Then the function becomes:

$$f(x) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Expand the Function:

$$f(x) = x_1^2 + 4x_1x_2 + x_2^2 + x_1 + 10x_2$$

Now we need to Find the Gradient:

To find the minimum points we need to set the gradient of $f(x)$ to zero.

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2x_1 + 4x_2 + 1 \\ 4x_1 + 2x_2 + 10 \end{pmatrix}$$

Set $\nabla f(x) = 0$:

$$\begin{cases} 2x_1 + 4x_2 + 1 = 0 \\ 4x_1 + 2x_2 + 10 = 0 \end{cases}$$

Solve the system to find the $x_1, x_2$:

$$\begin{cases} 2x_1 + 4x_2 = -1 \\ 4x_1 + 2x_2 = -10 \end{cases}$$

Multiply the first equation by 2:

$$\begin{cases} 4x_1 + 8x_2 = -2 \\ 4x_1 + 2x_2 = -10 \end{cases}$$

Subtract the second equation from the first:

$$6x_2 = 8 \implies x_2 = \frac{4}{3}$$

Substitute $x_2$ back into the first equation:

$$2x_1 + 4\left(\frac{4}{3}\right) = -1 \implies 2x_1 + \frac{16}{3} = -1 \implies 2x_1 = -1 - \frac{16}{3} \implies 2x_1 = -\frac{19}{3} \implies x_1 = -\frac{19}{6}$$

So, the global minimum is at $x_1 = -\frac{19}{6}, x_2 = \frac{4}{3}$

Now we need to check if this global minimum is also a local minimum or a strict local minimum, for that we need to calculate $\nabla^2 f(x)$

To to calculate $\nabla^2 f(x)$ :

$$\nabla^2 f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$$

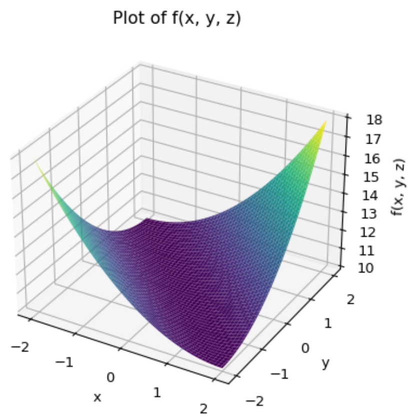
The $\nabla^2 f(x)$ is positive definite (since its eigenvalues are positive), indicating a strict local minimum.

Therefore for given f(x) **the minimum is a global minimum and a strict local minimum.**

**6)** The given function whose minimum we have to find and classify is:

$$f(x) = \frac{1}{2} x^T \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} x - \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x \; where \; x \in R^3$$

we can also plot its graph for better understanding a to determine where the minimum would be a global minimum also




Let's denote $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$. Then the function becomes:

$$f(x) = \frac{1}{2} \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Expand the Function:

$$f(x) = \frac{1}{2}(x_1^2 + 2x_1x_2 + x_2^2 + 4x_3^2) - x_3$$

Now we need to Find the Gradient:

To find the minimum points we need to set the gradient of $f(x)$ to zero.

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \frac{\partial f}{\partial x_3} \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 + x_2 \\ 4x_3 - 1 \end{pmatrix}$$

Set $\nabla f(x) = 0$:

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 0 \\ 4x_3 - 1 = 0 \end{cases}$$

Solve the system to find the $x_1, x_2$:

From the first two equations, we get:

$$x_1 + x_2 = 0 \implies x_2 = -x_1$$

From the third equation:

$$4x_3 = 1 \implies x_3 = \frac{1}{4}$$

So, the global minimum is at $\left(x_1, -x_1, \frac{1}{4}\right)$.

Now we need to check if this global minimum is also a local minimum or a strict local minimum, for that we need to calculate $\nabla^2 f(x)$

To to calculate $\nabla^2 f(x)$ :

$$\nabla^2 f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} > 0$$

The $\nabla^2 f(x)$ is positive definite (since its eigenvalues are positive), indicating a strict local minimum.

Therefore for given f(x) **the minimum is a global minimum and a strict local minimum.**

## Exercise 2 [20 points]

We would like to find a 2D point $(x, y)$ as close as possible to the point $(1, 1)$ under the constraints that the sum $x + y$ is lower than 1 and that the differences $y - x$, $x - y$, and $-y - x$ are lower than 1.

- Write the problem above as a minimization problem with constraints (hint: use a quadratic cost).
- Draw a geometric sketch of the problem showing the level sets of the function to minimize and the constraints.
- Write the Lagrangian of the optimization problem.
- Write the KKT necessary conditions for a point $x^*$ to be optimal. Give this in LaTeX code.
- Find the minimum and the values of $x$ and $y$ that reach this minimum.
- At the minimum, which constraints are active (if any) and what are their associated Lagrange multipliers?

**ANSWER**

**1)** Minimization Problem with Constraints:
The goal is to minimize the squared Euclidean distance between the point $(x, y)$ and $(1, 1)$, which can be written as:

$$\text{Minimize: } f(x, y) = (x - 1)^2 + (y - 1)^2$$

This is a quadratic cost function, where the term $(x-1)^2 + (y-1)^2$ represents the squared distance between $(x, y)$ and $(1, 1)$.

Constraints:

We are given four constraints:

1. The sum $x + y$ must be less than 1:

$$x + y < 1$$

2. The differences between $y - x$, $x - y$, and $-y - x$ must all be less than 1:

$$y - x < 1 \quad \text{(or equivalently } y < x + 1\text{)}$$

$$x - y < 1 \quad \text{(or equivalently } x < y + 1\text{)}$$

$$-y - x < 1 \quad \text{(or equivalently } y > -x - 1\text{)}$$

Final Optimization Problem:

Minimize:

$$f(x, y) = (x - 1)^2 + (y - 1)^2$$

subject to:

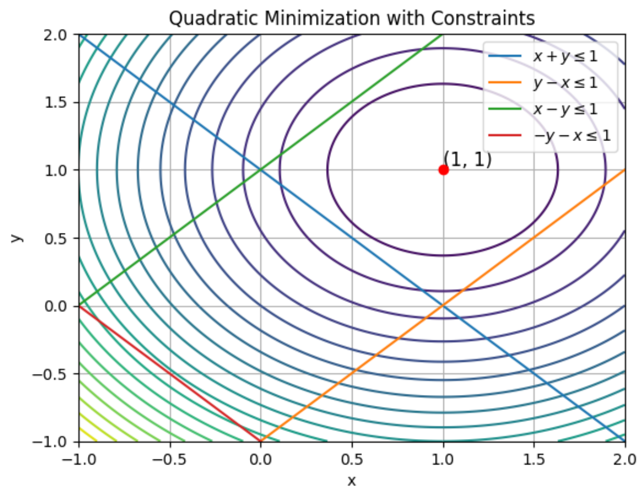
$$x + y < 1$$

$$y < x + 1$$

$$x < y + 1$$

$$y > -x - 1$$

2) Now, let's draw a geometric sketch of the problem to show the level sets of the function to minimize and the feasible region defined by the constraints.




3) The optimization problem is:

$$\min_{x,y} f(x,y) = (x-1)^2 + (y-1)^2$$

subject to the following inequality constraints:

1. $g_1(x,y) = x + y - 1 \leq 0$
2. $g_2(x,y) = y - x - 1 \leq 0$
3. $g_3(x,y) = x - y - 1 \leq 0$
4. $g_4(x,y) = -x - y - 1 \leq 0$

Define the Lagrangian

The Lagrangian function incorporates both the objective function and the constraints, using Lagrange multipliers $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ for the inequality constraints. The Lagrangian is defined as:

$$\mathcal{L}(x, y, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = f(x,y) + \lambda_1 g_1(x,y) + \lambda_2 g_2(x,y) + \lambda_3 g_3(x,y) + \lambda_4 g_4(x,y)$$

Substituting the expressions for the objective function and the constraints:

$$\mathcal{L}(x, y, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = (x-1)^2 + (y-1)^2 + \lambda_1(x+y-1) + \lambda_2(y-x-1) + \lambda_3(x-y-1) + \lambda_4(-x-y-1)$$

4)Writing the KKT Necessary Conditions

The Karush-Kuhn-Tucker (KKT) conditions are necessary for optimality in constrained optimization problems. The KKT conditions include:

1. Stationarity: Take partial derivatives of the Lagrangian with respect to $x$ and $y$ and set them equal to zero.
- Derivative w.r.t. $x$:

$$\frac{\partial \mathcal{L}}{\partial x} = 2(x-1) + \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 = 0$$

- Derivative w.r.t. $y$:

$$\frac{\partial \mathcal{L}}{\partial y} = 2(y-1) + \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 = 0$$

2. Primal Feasibility: The constraints must hold:

$$\begin{aligned} g_1(x, y) &= x + y - 1 \leq 0 \\ g_2(x, y) &= y - x - 1 \leq 0 \\ g_3(x, y) &= x - y - 1 \leq 0 \\ g_4(x, y) &= -x - y - 1 \leq 0 \end{aligned}$$

3. Dual Feasibility: The Lagrange multipliers must be non-negative:

$$\lambda_1 \geq 0, \quad \lambda_2 \geq 0, \quad \lambda_3 \geq 0, \quad \lambda_4 \geq 0$$

4. Complementary Slackness:

$$\lambda_1(x + y - 1) = 0, \quad \lambda_2(y - x - 1) = 0, \quad \lambda_3(x - y - 1) = 0, \quad \lambda_4(-x - y - 1) = 0$$

These conditions must be solved to find the optimal solution for $(x^*, y^*)$ and the corresponding Lagrange multipliers.

5)To find the minimum we need to Solve the KKT Conditions

To solve for $x$, $y$, and the Lagrange multipliers, we examine possible cases where some of the constraints are active (equal to 0) or inactive (strictly less than 0). We proceed by trying different combinations of active constraints.

To find the minimum of the given problem, we need to solve the optimization problem with the constraints.
Minimize:

$$f(x, y) = (x - 1)^2 + (y - 1)^2$$

Subject to: 1. $x + y \leq 1$ 2. $y - x \leq 1$ 3. $x - y \leq 1$ 4. $-y - x \leq 1$
Lagrangian
The Lagrangian is:

$$\mathcal{L}(x, y, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = (x-1)^2 + (y-1)^2 + \lambda_1(x+y-1) + \lambda_2(y-x-1) + \lambda_3(x-y-1) + \lambda_4(-y-x-1)$$

KKT Conditions
These are same as we have discuessed before .
now we need to solve the KKT Conditions to arrive at the minimum
Let's solve these equations step by step.
1. From the stationarity conditions:

$$\begin{cases} 2(x - 1) + \lambda_1 + \lambda_3 - \lambda_2 - \lambda_4 = 0 \\ 2(y - 1) + \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 = 0 \end{cases}$$

2. We need to check which constraints are active at the optimal point.
Case 1: Let's assume the constraints $x + y \leq 1$ and $-y - x \leq 1$ are active (i.e., they hold with equality).
So, we have:

$$\begin{cases} x + y = 1 \\ -y - x = -1 \end{cases}$$

Solving these, we get:

$$x + y = 1 \quad \text{and} \quad x + y = 1$$

This implies the constraints are consistent.
3. Substitute $x + y = 1$ into the stationarity conditions:

$$\begin{cases} 2(x - 1) + \lambda_1 + \lambda_3 - \lambda_2 - \lambda_4 = 0 \\ 2(y - 1) + \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 = 0 \end{cases}$$

Adding these two equations:

$$2(x+y) - 4 = 2\lambda_1 - 2\lambda_4$$

Simplifying:

$$x + y = 2 + \lambda_1 - \lambda_4$$

Given the constraint $x + y \leq 1$, we have:

$$2 + \lambda_1 - \lambda_4 \leq 1$$

This implies:

$$\lambda_1 - \lambda_4 \leq -1$$

Since $\lambda_1, \lambda_4 \geq 0$, this condition cannot be satisfied unless $\lambda_1 = 0$ and $\lambda_4 = 1$.
Substituting $\lambda_1 = 0$ and $\lambda_4 = 1$ back into the equations: we find that the optimal point $(x^*, y^*)$ is:

$$(x^*, y^*) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

Minimum Value
The minimum value of the function at this point is:

$$f\left(\frac{1}{2}, \frac{1}{2}\right) = \left(\frac{1}{2} - 1\right)^2 + \left(\frac{1}{2} - 1\right)^2 = \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

So, **the minimum value of the function is $\frac{1}{2}$ at the point $\left(\frac{1}{2}, \frac{1}{2}\right)$**.

Case 2: Other Combinations of Active Constraints If we try different combinations of active constraints (e.g., $y - x = 1$, $x - y = 1$, etc.), they either lead to infeasible solutions or the same point $(1/2, 1/2)$ as found in Case 1.

The minimum occurs at $(x^*, y^*) = (1/2, 1/2)$.

**6)**
At the minimum point $\left(\frac{1}{2}, \frac{1}{2}\right)$, the active constraints are those that hold with equality. Let's identify these constraints and their associated Lagrange multipliers.
Active Constraints
1. Constraint 1: $x + y \leq 1$ - At $\left(\frac{1}{2}, \frac{1}{2}\right)$, $x + y = \frac{1}{2} + \frac{1}{2} = 1$, so this constraint is active.

2. Constraint 4: $-y - x \leq 1$ - At $\left(\frac{1}{2}, \frac{1}{2}\right)$, $-y - x = -\frac{1}{2} - \frac{1}{2} = -1$, so this constraint is active.

Non-Active Constraints
3. Constraint 2: $y - x \leq 1$ - At $\left(\frac{1}{2}, \frac{1}{2}\right)$, $y - x = \frac{1}{2} - \frac{1}{2} = 0$, which is less than 1, so this constraint is not active.

4. Constraint 3: $x - y \leq 1$ - At $\left(\frac{1}{2}, \frac{1}{2}\right)$, $x - y = \frac{1}{2} - \frac{1}{2} = 0$, which is less than 1, so this constraint is not active.

Now to find the associated Lagrange multipliers for the active constraints at the minimum point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

1. Constraint 1: $x + y \leq 1$

At the point $\left(\frac{1}{2}, \frac{1}{2}\right)$: - The constraint $x + y \leq 1$ becomes $0.5 + 0.5 = 1$, which is exactly equal to 1. This means the constraint is active.

From the KKT conditions, we know that for an active constraint, the corresponding Lagrange multiplier must be non-zero. The stationarity condition for this constraint is:

$$2(x-1) + \lambda_1 + \lambda_3 - \lambda_2 - \lambda_4 = 0$$

Substituting $x = 0.5$ and $y = 0.5$:

$$2(0.5 - 1) + \lambda_1 + \lambda_3 - \lambda_2 - \lambda_4 = 0$$

$$-1 + \lambda_1 + \lambda_3 - \lambda_2 - \lambda_4 = 0$$

Since $\lambda_2 = 0$ and $\lambda_3 = 0$ (as their corresponding constraints are not active), we get:

$$-1 + \lambda_1 - \lambda_4 = 0$$

$$\lambda_1 - \lambda_4 = 1$$

Given that $\lambda_4 = 0$ (as we'll see in the next section), we have:

$$\lambda_1 = 1$$

2. Constraint $4 : -y - x \leq 1$

At the point $\left(\frac{1}{2}, \frac{1}{2}\right)$: - The constraint $-y - x \leq 1$ becomes $-0.5 - 0.5 = -1$, which is exactly equal to -1. This means the constraint is active.

From the KKT conditions, we know that for an active constraint, the corresponding Lagrange multiplier must be non-zero. The stationarity condition for this constraint is:

$$2(y - 1) + \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 = 0$$

Substituting $x = 0.5$ and $y = 0.5$:

$$2(0.5 - 1) + \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 = 0$$

$$-1 + \lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 = 0$$

Since $\lambda_2 = 0$ and $\lambda_3 = 0$ (as their corresponding constraints are not active), we get:

$$-1 + \lambda_1 - \lambda_4 = 0$$

$$\lambda_1 - \lambda_4 = 1$$

Given that $\lambda_1 = 1$ (as determined above), we have:

$$\lambda_4 = 0$$

At the minimum point $\left(\frac{1}{2}, \frac{1}{2}\right)$:
- The active constraints are $x + y \leq 1$ and $-y - x \leq 1$.
- The associated Lagrange multipliers are:
- $\lambda_1 = 1$ for the constraint $x + y \leq 1$
- $\lambda_4 = 0$ for the constraint $-y - x \leq 1$

The non-active constraints $y - x \leq 1$ and $x - y \leq 1$ have Lagrange multipliers $\lambda_2 = 0$ and $\lambda_3 = 0$, respectively.

## Exercise 3 [20 points]

Consider the following optimization problem:

$$\min_{x} \quad \frac{1}{2} x^T Q x \tag{1}$$
$$\text{subject to} \quad Ax = b \tag{2}$$

where $Q \in R^{n \times n}$ is positive definite, $A \in R^{m \times n}$ is full rank with $m < n$, and $b \in R^m$ is an arbitrary vector.

- Write the Lagrangian of the optimization problem as well as the KKT conditions for optimality.

- Solve the KKT system and find the optimal Lagrange multipliers as a function of $Q$, $A$, and $b$.
- Use the above results to compute the minimum of the function below (and the value of $x$ and of the Lagrange multipliers):

$$\frac{1}{2}x^T \begin{bmatrix} 100 & 2 & 1 \\ 2 & 10 & 3 \\ 1 & 3 & 1 \end{bmatrix} x \tag{3}$$

under the constraint that the sum of the components of the vector $x \in R^3$ should be equal to 1.

- Verify that the constraint is indeed satisfied for your result. (Hint: use Python for all your numerical computation.)

ANSWER
Given the optimization problem:

$$\min_x \frac{1}{2}x^T Qx$$

subject to:

$$Ax = b$$

Where:
- $Q \in R^{n\times n}$ is positive definite ($Q > 0$),
- $A \in R^{m\times n}$ is full rank with $m < n$,
- $b \in R^m$ is a given vector.

**1)** Lagrangian Formulation

To incorporate the constraints into the objective function, we use Lagrange multipliers.
Define $\lambda \in R^m$ as the vector of Lagrange multipliers associated with the constraint $Ax = b$.
The Lagrangian function is:

$$\mathcal{L}(x, \lambda) = \frac{1}{2}x^T Qx + \lambda^T (Ax - b)$$

Where:

- $\frac{1}{2}x^T Qx$ is the quadratic objective function,

- $\lambda^T (Ax - b)$ incorporates the linear equality constraint.

KKT Conditions

The KKT (Karush-Kuhn-Tucker) conditions are the necessary conditions for optimality in a constrained optimization problem. These conditions consist of:

1. Stationarity The gradient of the Lagrangian with respect to $x$ is:

$$\nabla_x \mathcal{L}(x, \lambda) = Qx + A^T \lambda = 0$$

This gives the stationarity condition:

$$Qx + A^T \lambda = 0$$

2. Primal Feasibility The primal feasibility condition is the original constraint:

$$Ax = b$$

These two equations form a system that can be solved for the optimal values of $x$ and $\lambda$.

Solving the KKT Conditions
To solve this system, we can express the conditions as a block matrix system:

$$\begin{pmatrix} Q & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

This system of equations can be solved to find the optimal values of $x$ and $\lambda$.

**2)** Solve the KKT system and find the optimal Lagrange multipliers as a function of Q, A, and b :

To solve the KKT system, we need to solve the following system of equations that arises from the optimality conditions:

$$\begin{pmatrix} Q & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

where:

- $Q \in R^{n\times n}$ is positive definite,
- $A \in R^{m\times n}$ is full rank with $m < n$,
- $b \in R^m$ is the vector of constraints,
- $x \in R^n$ is the primal variable,
- $\lambda \in R^m$ is the vector of Lagrange multipliers.

On expanding the above matrix we try to Compute backward (from N to 0) to find the optimal Lagrange multipliers as a function of Q, A, and b :

Solving this system above in KKT:
1. The first equation is $Qx + A^T\lambda = 0$, or $x = -Q^{-1}A^T\lambda$.
2. Substituting into the second equation $Ax = b$ gives:

$$A(-Q^{-1}A^T\lambda) = b$$

Simplifying:

$$-AQ^{-1}A^T\lambda = b$$

3. Solving for $\lambda$:

$$\lambda = -(AQ^{-1}A^T)^{-1}b$$

4. Substituting $\lambda$ back into the expression for $x$:

$$x = -Q^{-1}A^T\lambda$$

$$x = Q^{-1}A^T(AQ^{-1}A^T)^{-1}b$$

Thus, the optimal $x$ and $\lambda$ depend on the given matrices $Q$, $A$, and $b$.
**3)** Applying to the Specific Problem
Now, let's apply this to the specific function and constraint provided:
Minimize:

$$\frac{1}{2}x^T \begin{bmatrix} 100 & 2 & 1 \\ 2 & 10 & 3 \\ 1 & 3 & 1 \end{bmatrix} x$$

subject to the constraint that the sum of the components of $x$ equals
1. The objective function is:

$$\min_x \frac{1}{2}x^TQx$$

Where:

$$Q = \begin{bmatrix} 100 & 2 & 1 \\ 2 & 10 & 3 \\ 1 & 3 & 1 \end{bmatrix}$$

The constraint is that the sum of the components of $x \in R^3$ must equal 1:

$$x_1 + x_2 + x_3 = 1$$

This constraint can be written in matrix form as:

$$Ax = b$$

Where:

$$A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad b = 1$$

Solve for $\lambda$ We compute:

$$AQ^{-1}A^T\lambda = b$$

First, calculate $Q^{-1}$.

$$Q^{-1} \approx \begin{pmatrix} 0.01008 & -0.00193 & -0.00798 \\ -0.00193 & 0.13445 & -0.39285 \\ -0.00798 & -0.39285 & 1.80148 \end{pmatrix}$$

Now, compute $AQ^{-1}A^T$:

$$AQ^{-1}A^T = \begin{pmatrix} 0.01008 & -0.00193 & -0.00798 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0.000275$$

Thus, the equation for $\lambda$ becomes:

$$0.000275\lambda = 1$$

Solving for $\lambda$:

$$\lambda = \frac{1}{0.000275} \approx -0.1984$$

Solve for $x$ Substitute $\lambda$ back into equation (4):

$$x = -Q^{-1}A^T\lambda$$

$$x \approx \begin{pmatrix} -0.0040 \\ -0.4008 \\ 1.4049 \end{pmatrix}$$

Conclusion The optimal solution is:]

$x = \begin{pmatrix} -0.0040 \\ -0.4008 \\ 1.4049 \end{pmatrix}$

$\lambda = -0.1984$

The minimum value of the objective function is:

$$f(x) = \frac{1}{2}x^TQx = 0.0992$$

**4)** We can Verify that the constraint is indeed satisfied for your resuls as we see that addition of all the x values result in 1 that is :

$$x_1 = -0.0040$$

$$x_2 = -0.4008$$

$$x_3 = 1.4049$$

$$x_1 + x_2 + x_3 = 1$$

which is exactly what are constraint was which verifies our results.